Generic arm separation for planar models

Loïc Gassmann (Fribourg)

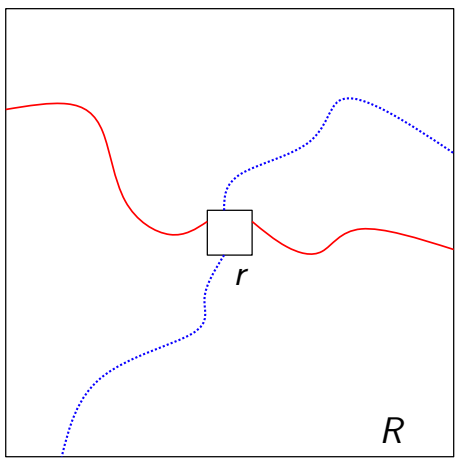
Joint work with Ioan Manolescu and Maran Mohanarangan

Generic arm separation

Theorem (Manolescu, Mohanarangan, G, 26+)

Take any planar percolation $\mathbb{P}$. *Fix* $\sigma \in \{0,1\}^N$ *and* $\eta > 0$.
There exists $c > 0$, *such that for any* $R \geq 2r$

$$\mathbb{P}[\text{"arms are } \eta\text{-well-separated"} \mid A_\sigma(r,R)] \geq c.$$



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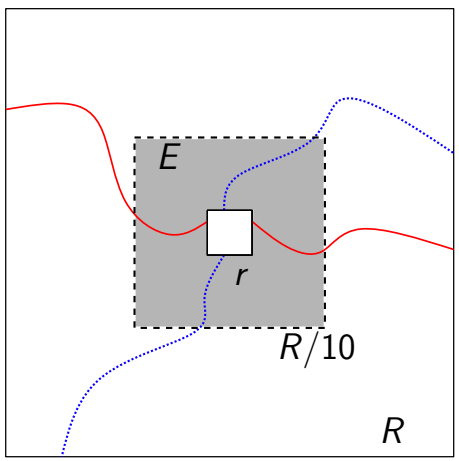
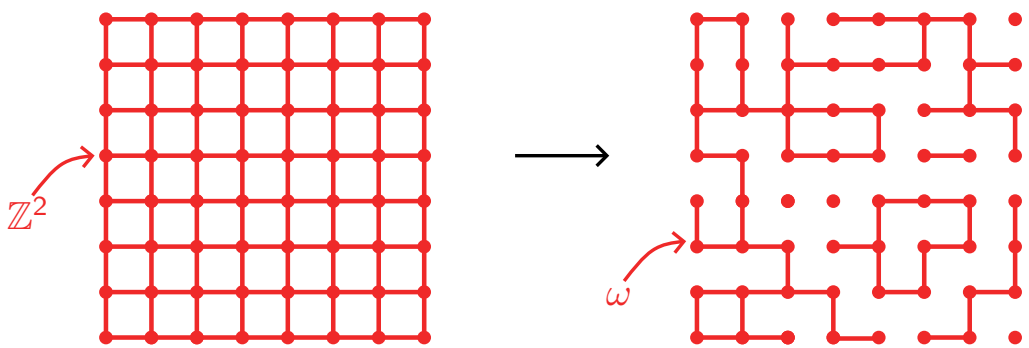


Classical arm separation [Kesten 87]

Setup: Take $\mathbb{P}$ **critical** Bernoulli percolation on $\mathbb{Z}^2$ ($p = p_c = 1/2$).



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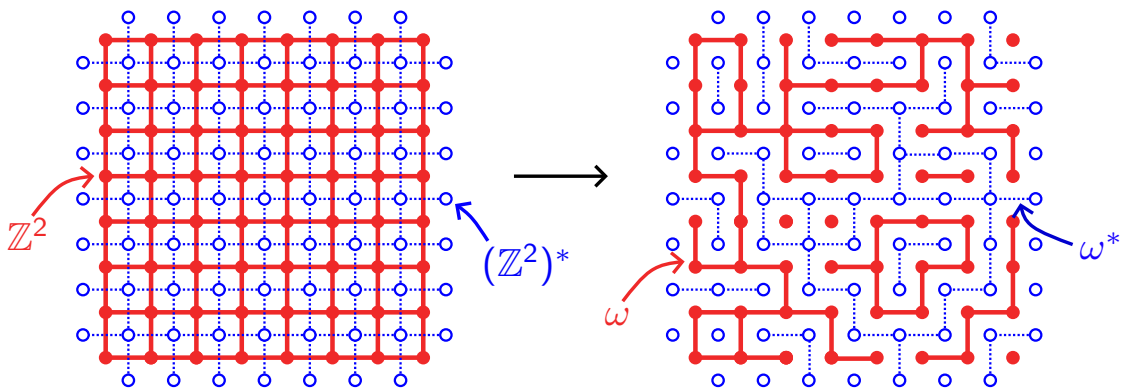

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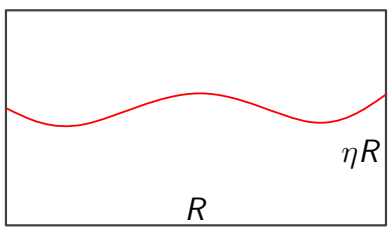
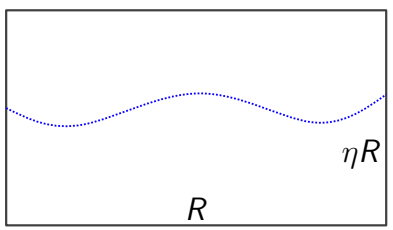
$$\forall R > 0, \quad \mathbb{P}\left(\right) = \mathbb{P}\left(\right) \geq C_\eta$$





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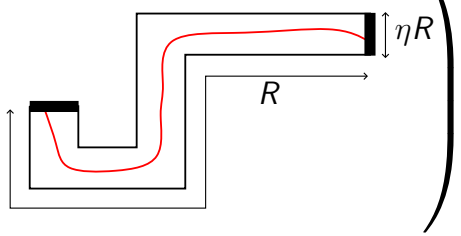
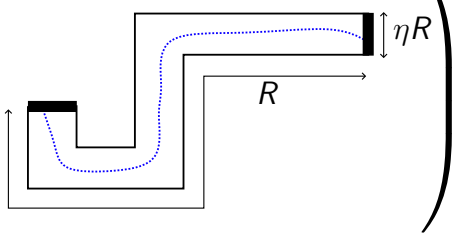
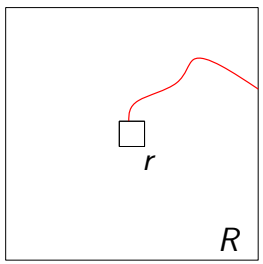
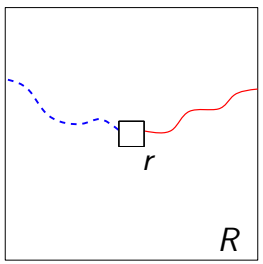


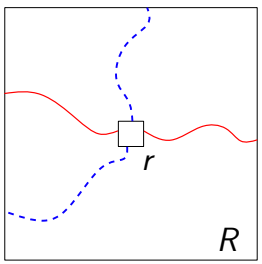


Arm events

For $0 \leq r < R$,

$A_1(r,R) =$ 

$A_{01}(r,R) =$ 

$A_{0101}(r,R) =$ 

More generally: $A_\sigma(r,R)$ with $\sigma \in \{0,1\}^N$ ($0 \to$ dual, $1 \to$ primal)

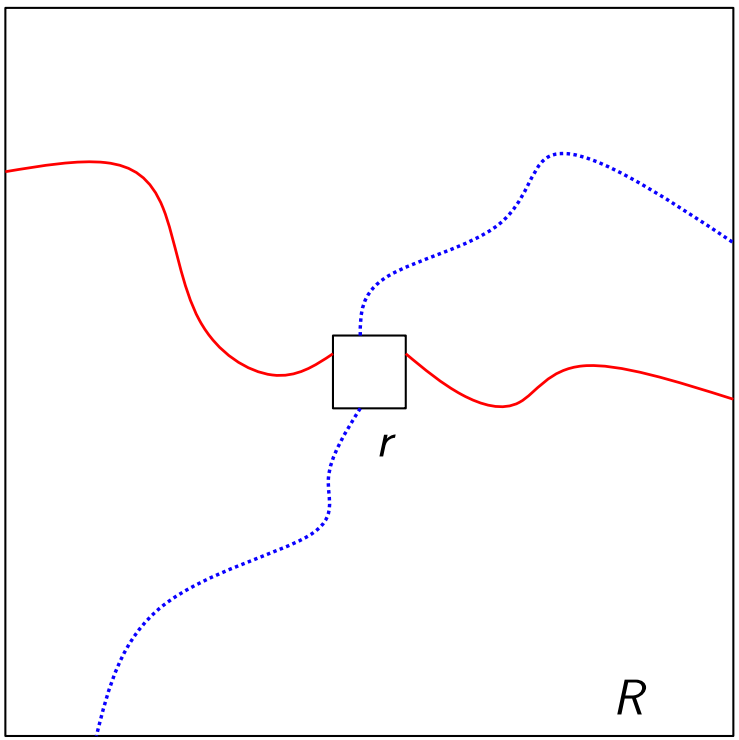
Keep in mind that

$$r \ll R$$

Classical arm separation [Kesten 87]

Fix a color sequence σ (e.g. $\sigma = 0101$) and $\eta > 0$.

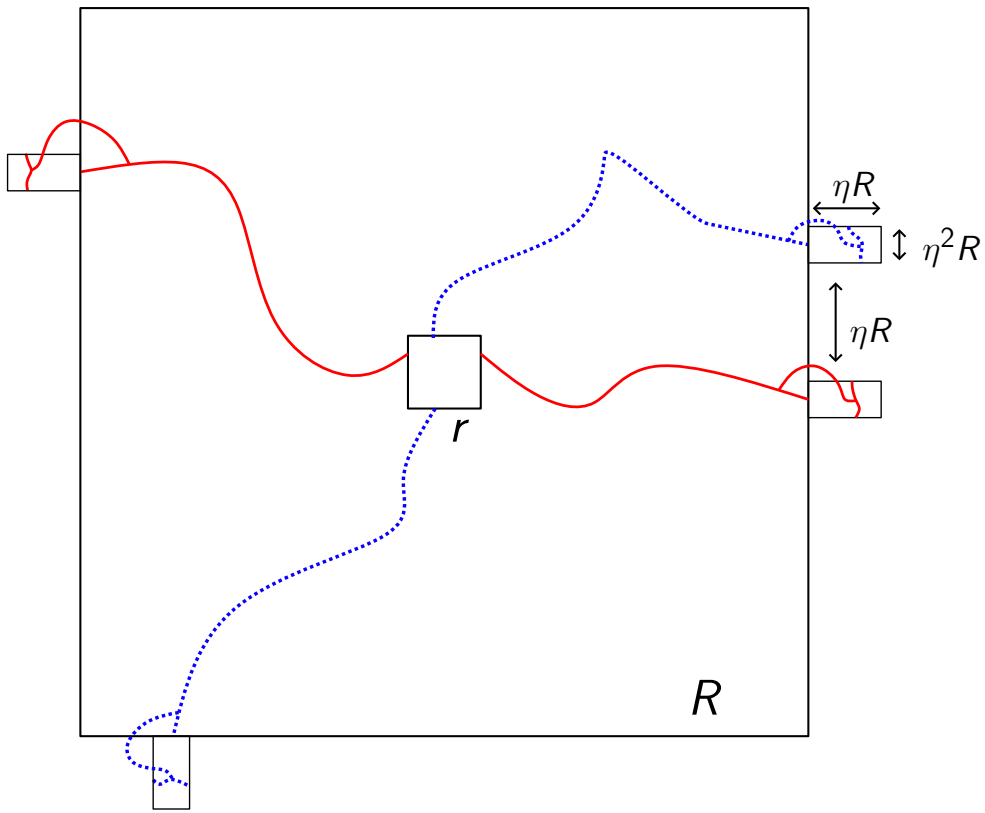
$A_\sigma(r, R)$ $=$



Classical arm separation [Kesten 87]

Fix a color sequence σ (e.g. $\sigma = 0101$) and $\eta > 0$.

$A_\sigma^\eta(r, R)$ $=$

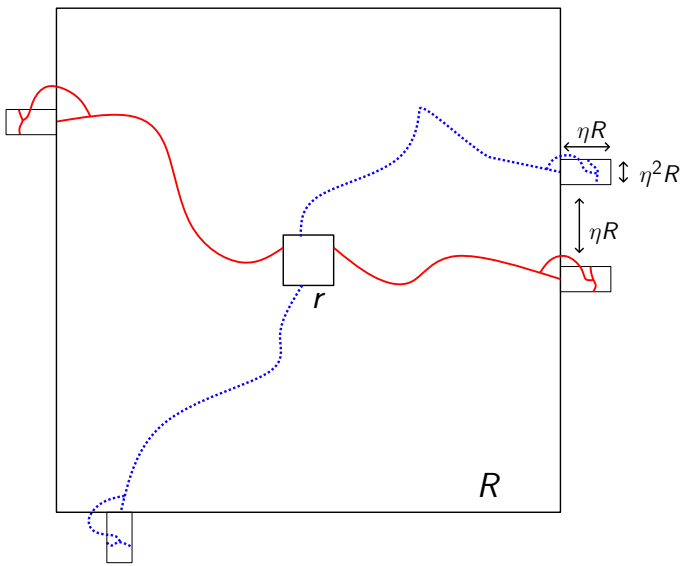


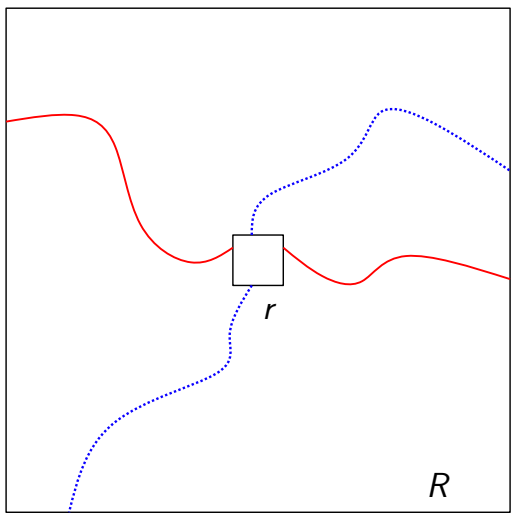
Classical arm separation [Kesten 87]

Theorem (Kesten 87, see also Nolin 07)

Fix a color sequence σ and $\eta > 0$ (not too large).
There exists $C > 0$ such that, for all $R \geq 2r \geq 2r_\sigma$,

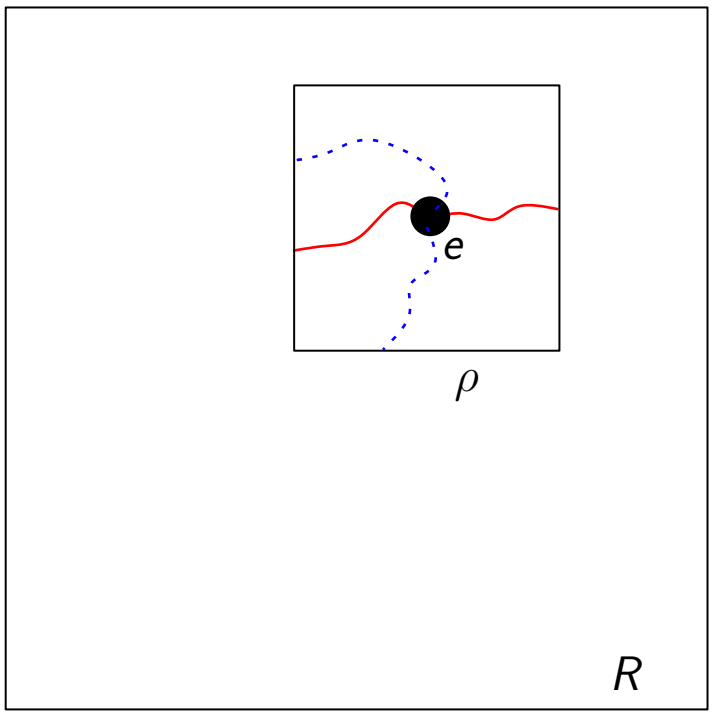
$$\mathbb{P}(A_\sigma^\eta(r,R) \,|\, A_\sigma(r,R)) \geq C$$





How to use it?

Pivotal edges



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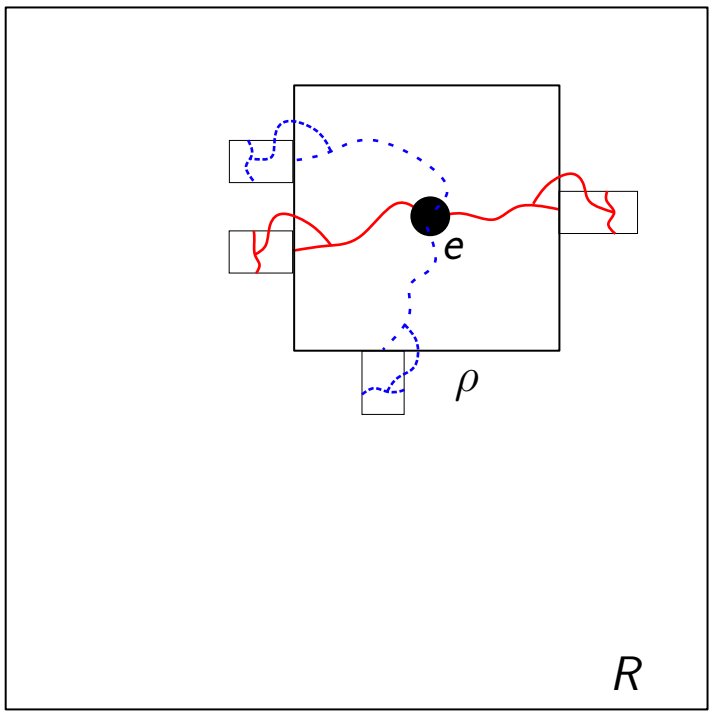

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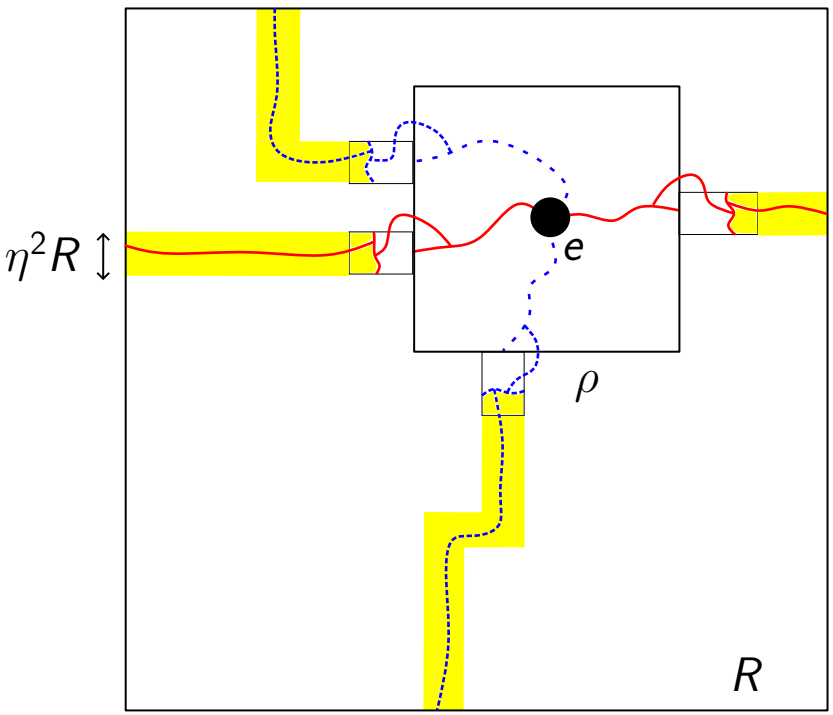

Disjoint macroscopic tubes + RSW $\rightsquigarrow$ arms can be reconnected

How to use it?

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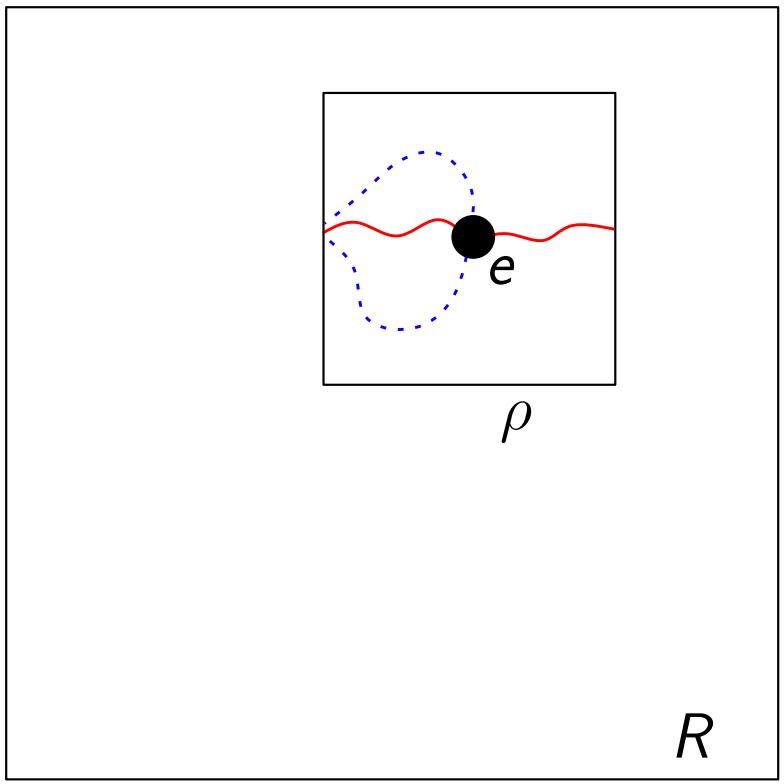

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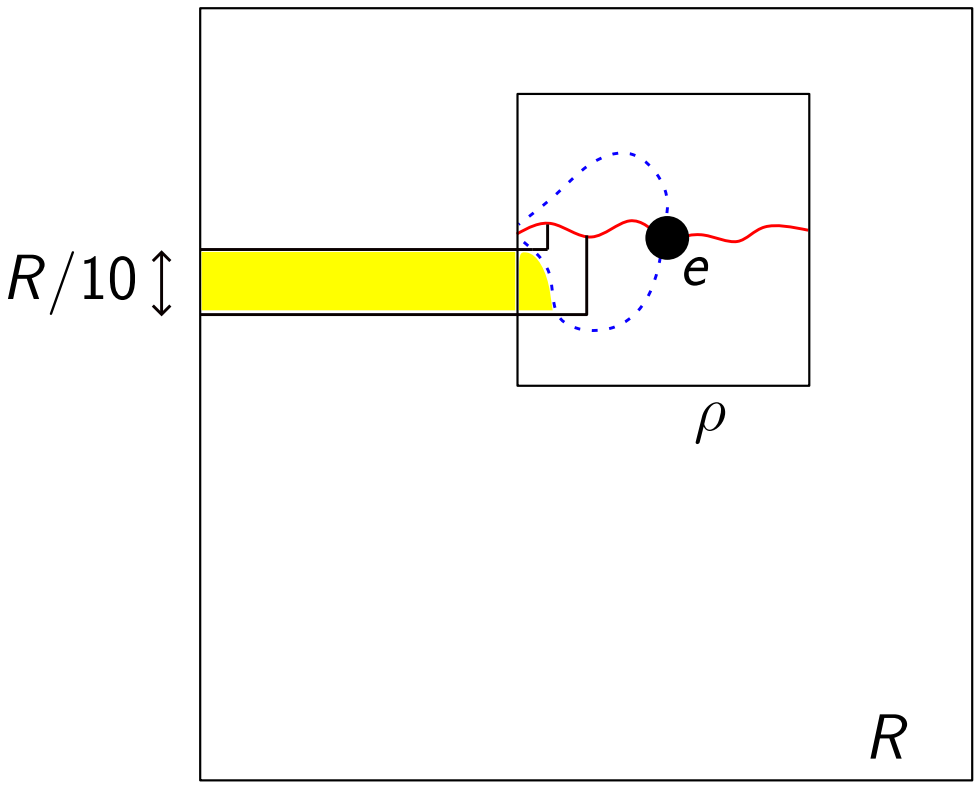

Generic arm separation: Setup

Setup: Take $\mathbb{P}$ a percolation on $\mathbb{Z}^2$.

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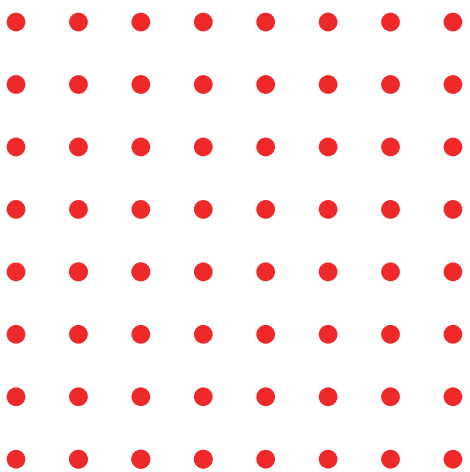
Generic arm separation: Exploration algorithm

Exploration algorithm

- Reveals edges of ω one by one
- Choice of the next edge depends on the already sampled edges



$\mathcal{E}$ = set of explored edges

$(\omega_e)_{e\in\mathcal{E}}$ state of explored edges

Open current edge e_n with probability

$$\mathbb{P}(\omega_{e_n} = 1 | (\omega_e)_{e\in\mathcal{E}})$$

Stopping time T

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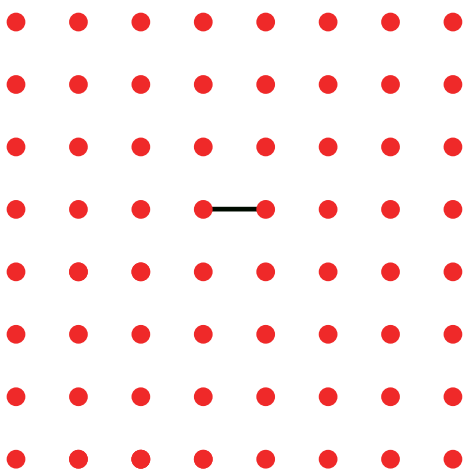

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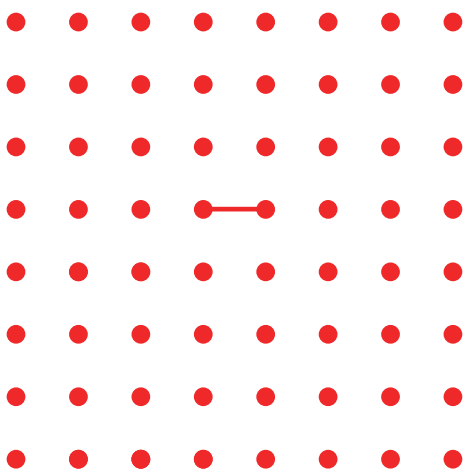

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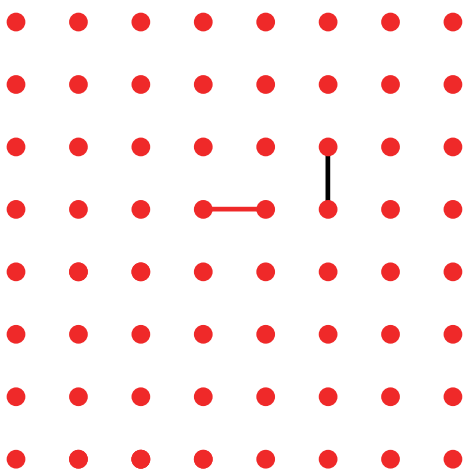

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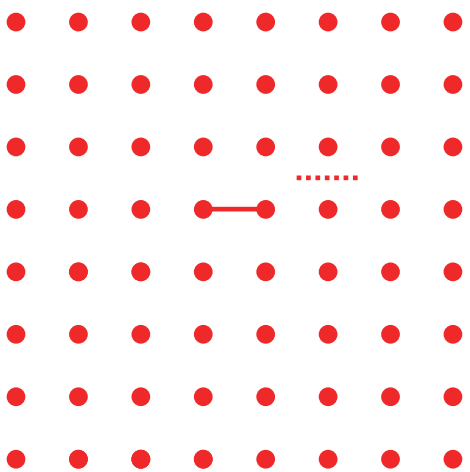

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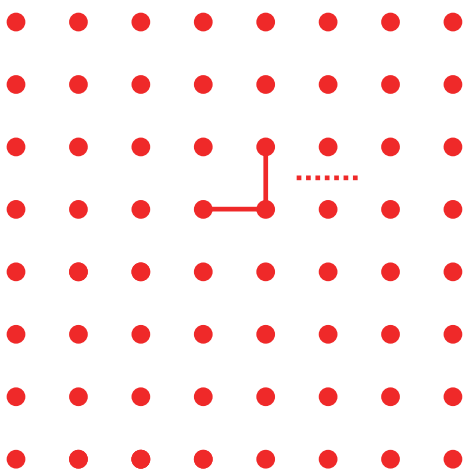

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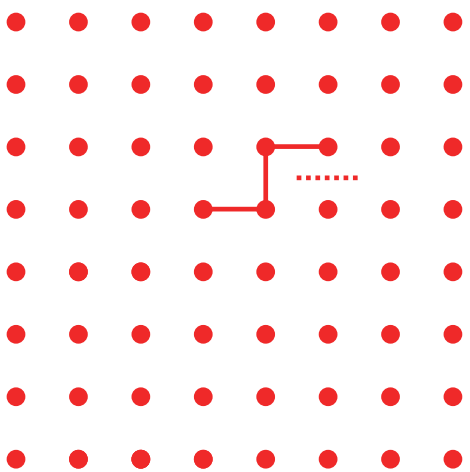

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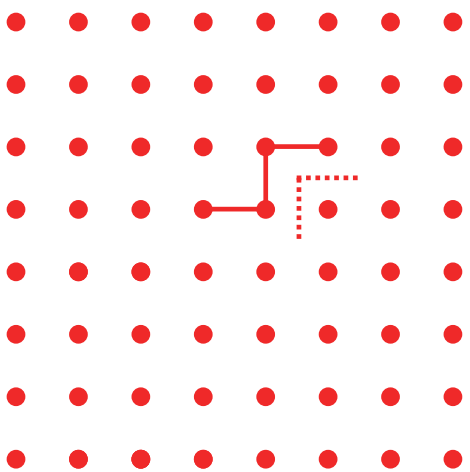

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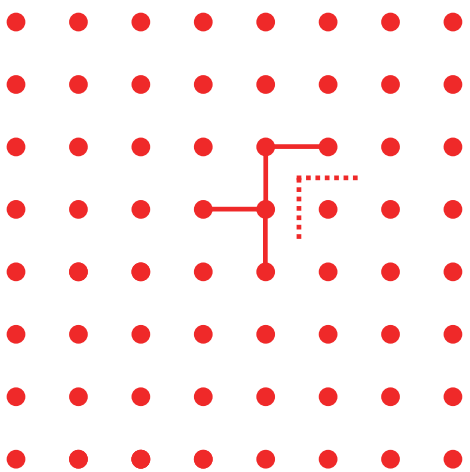

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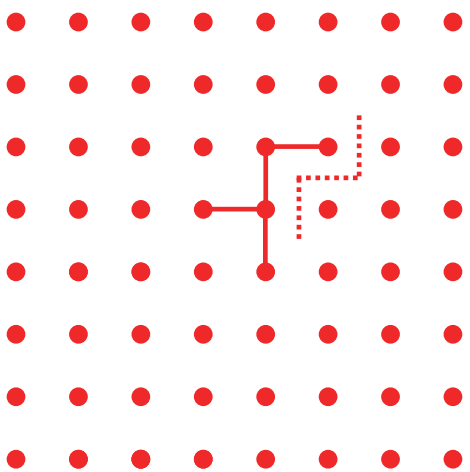

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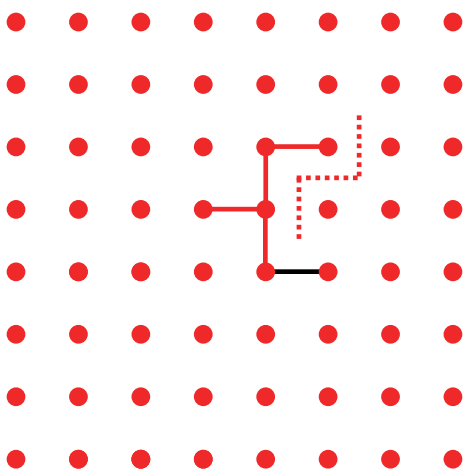

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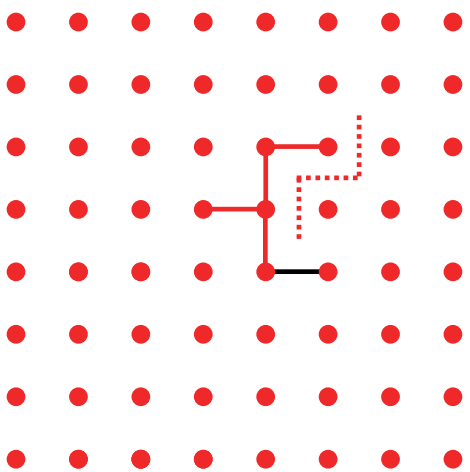

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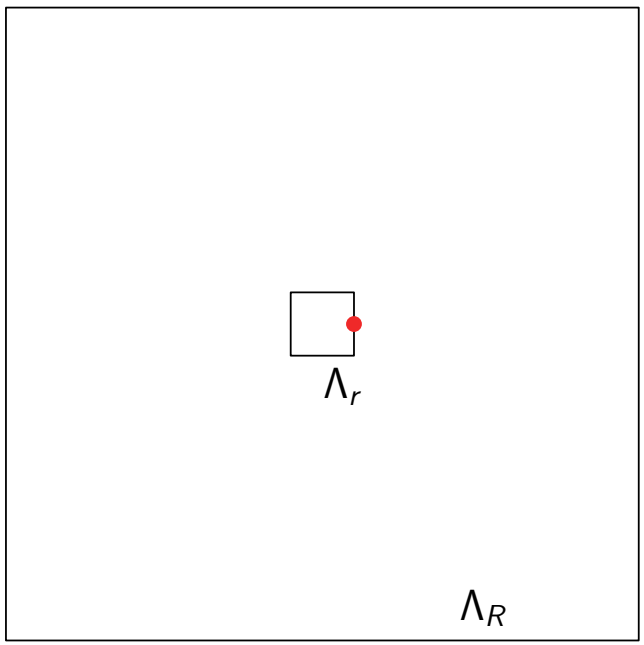
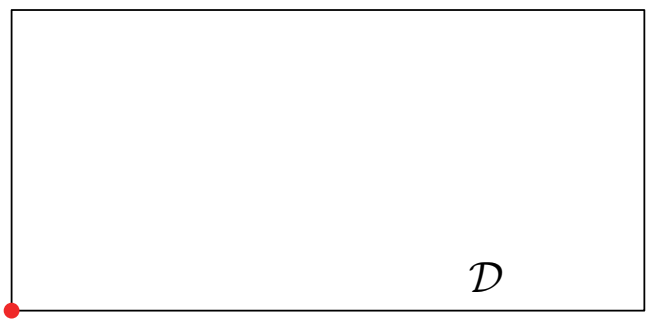
One can add extra randomness !

Two classical explorations



Interface



Left most open path

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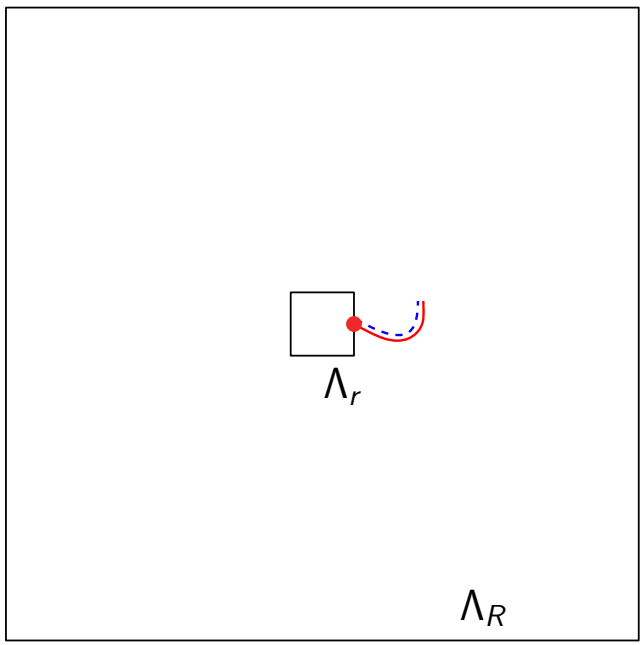

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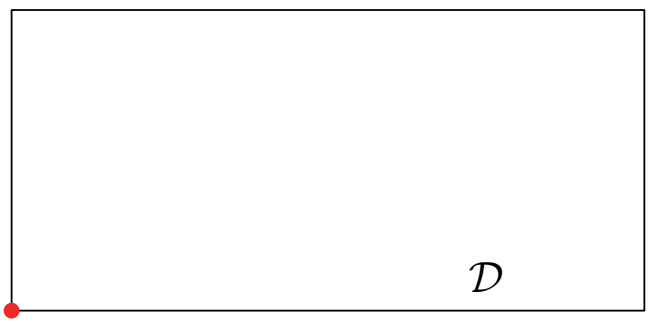

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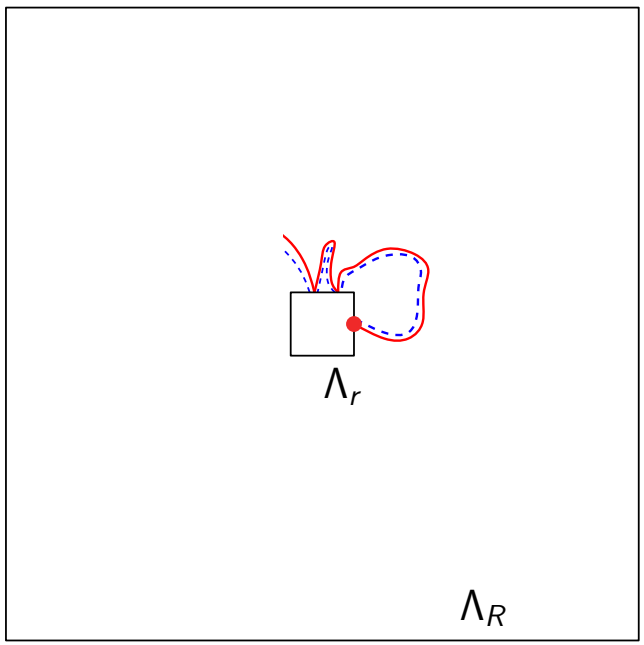

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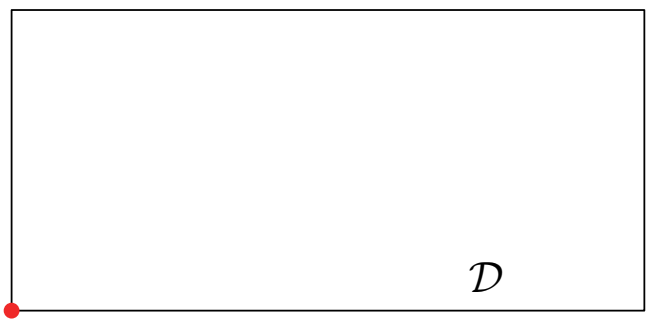

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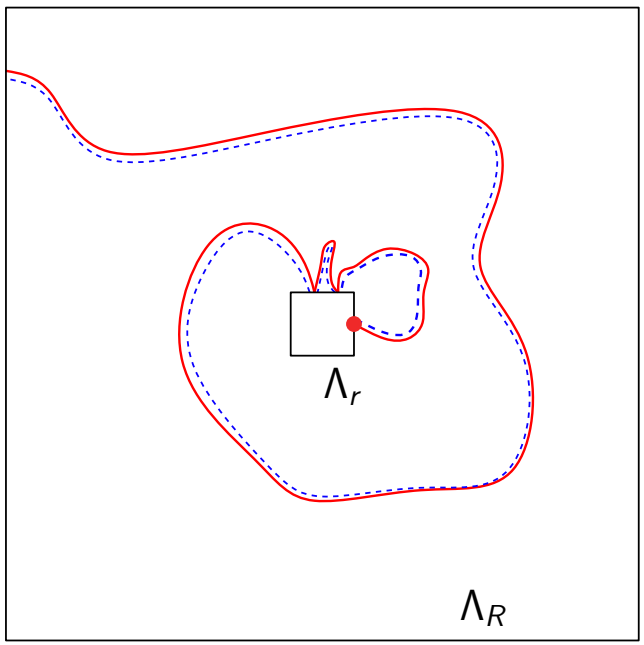

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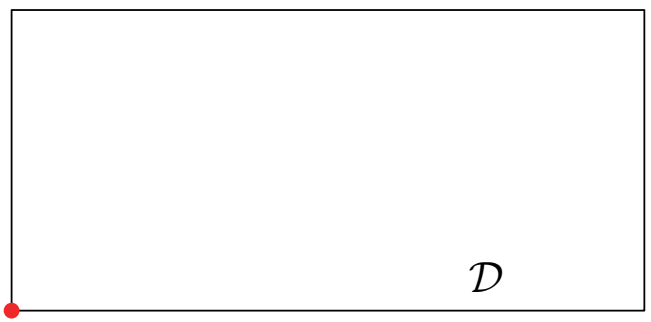

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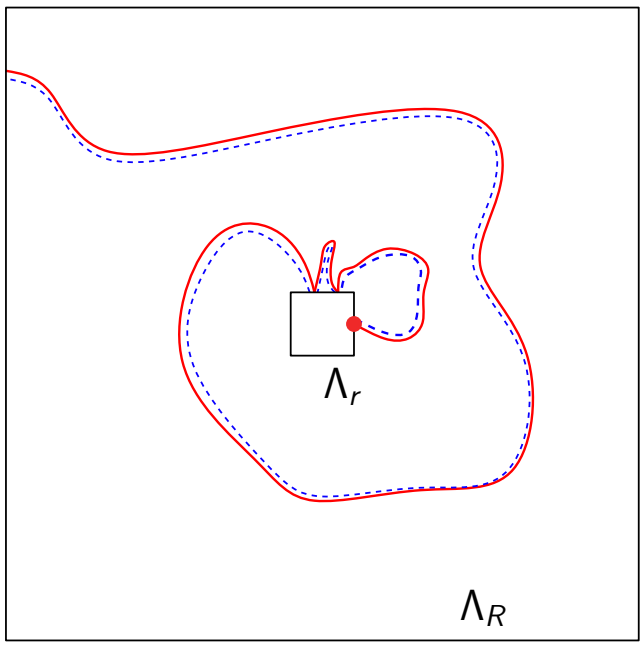

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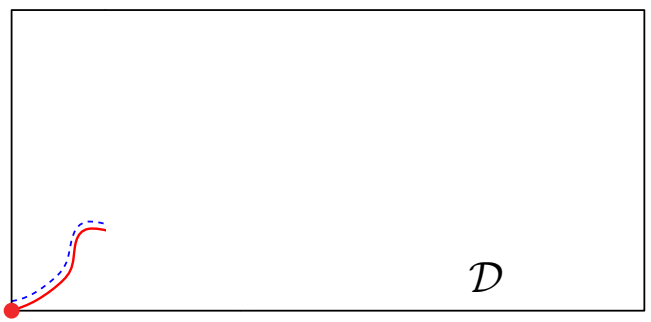

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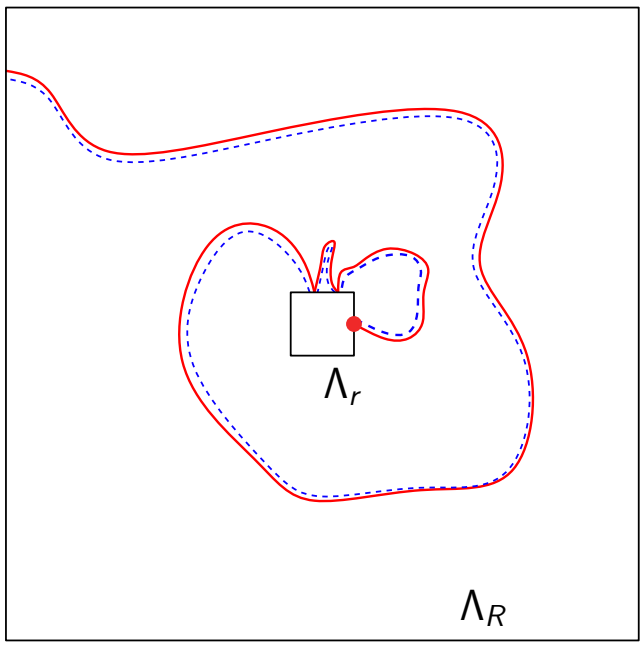
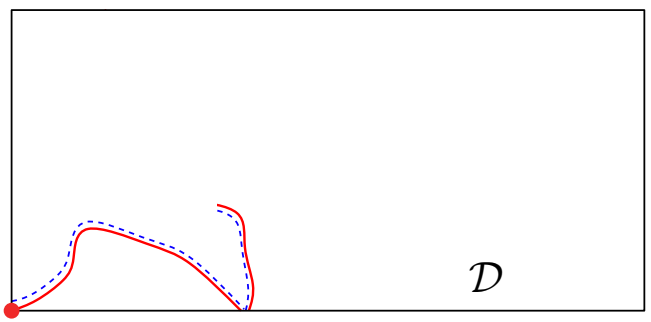
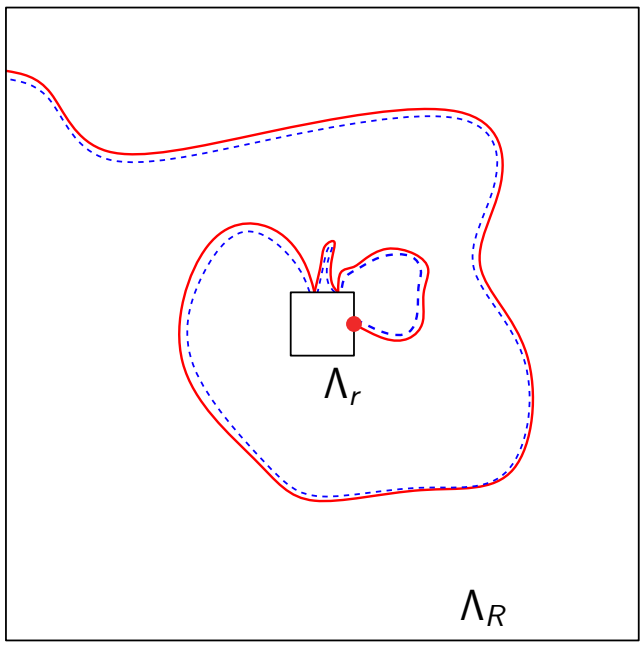


Interface

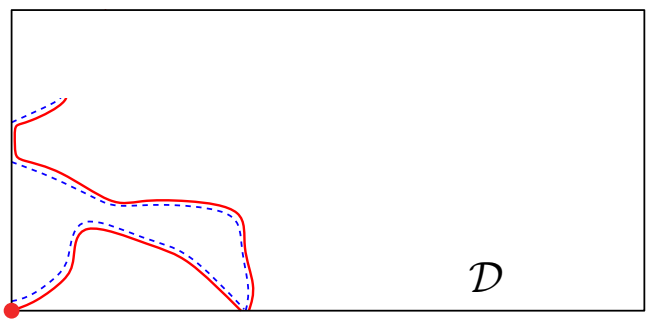


Left most open path

Two classical explorations

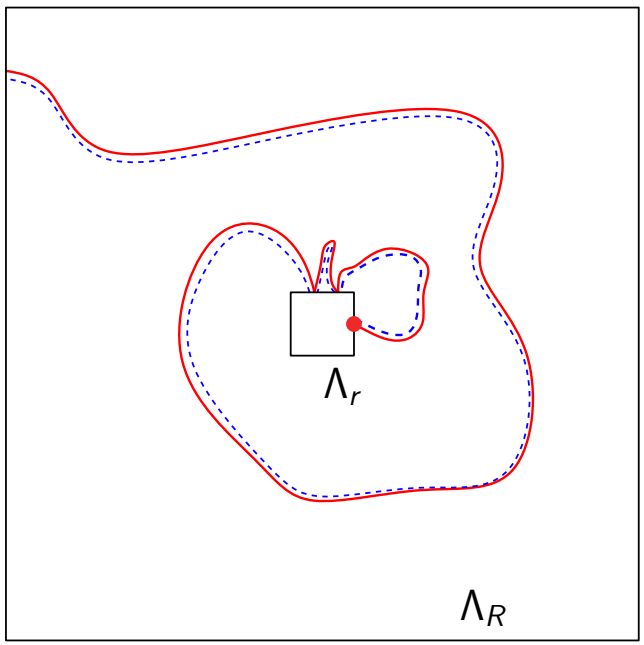


Interface

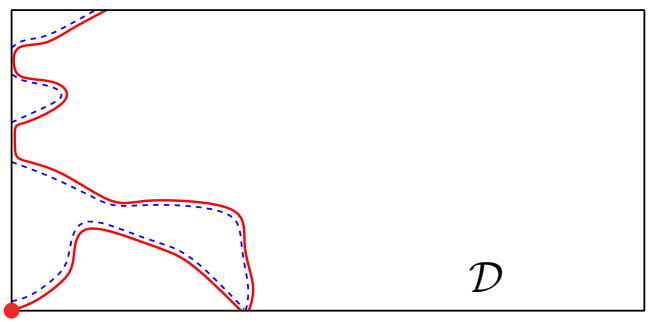


Left most open path

Two classical explorations



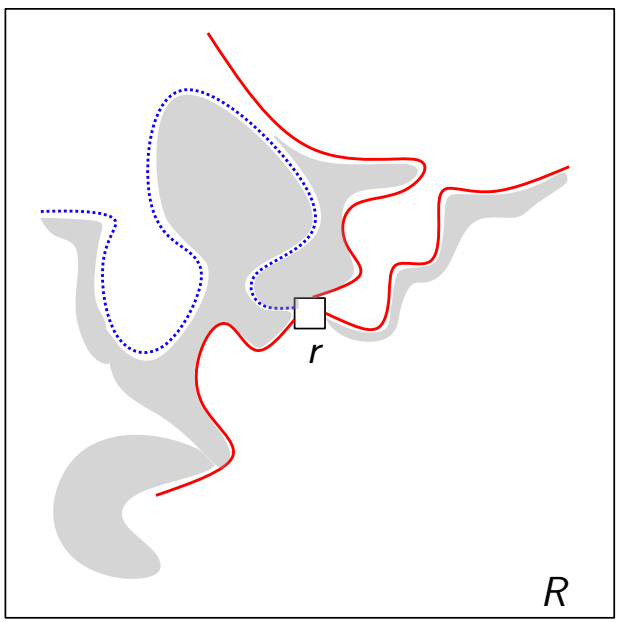
Interface



Left most open path

Arm exposition

Fix $\eta > 0$ and consider the **final state** of an exploration :



The arms of $A_{0111}(r, R)$ are η-exposed (w.r.t. exploration)

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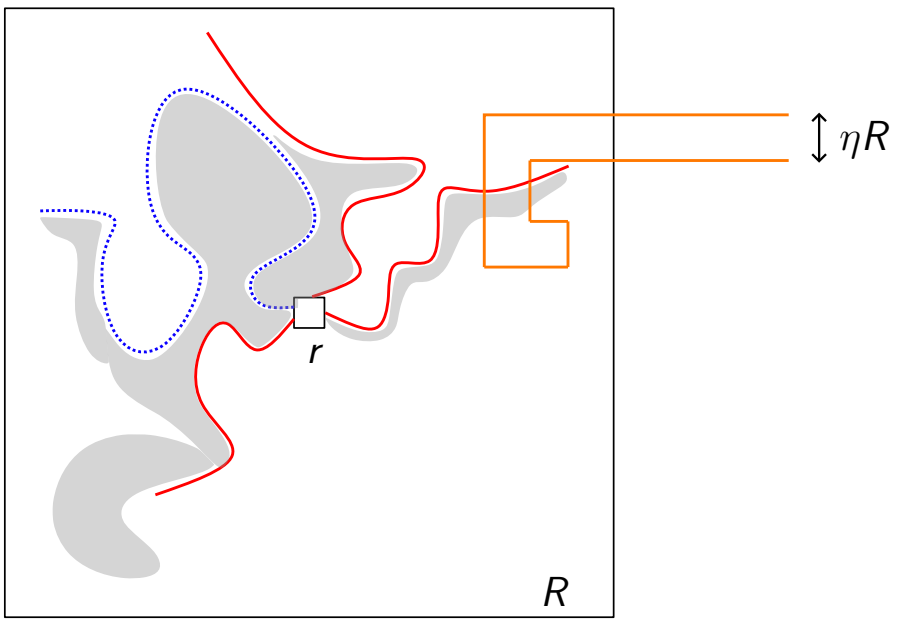

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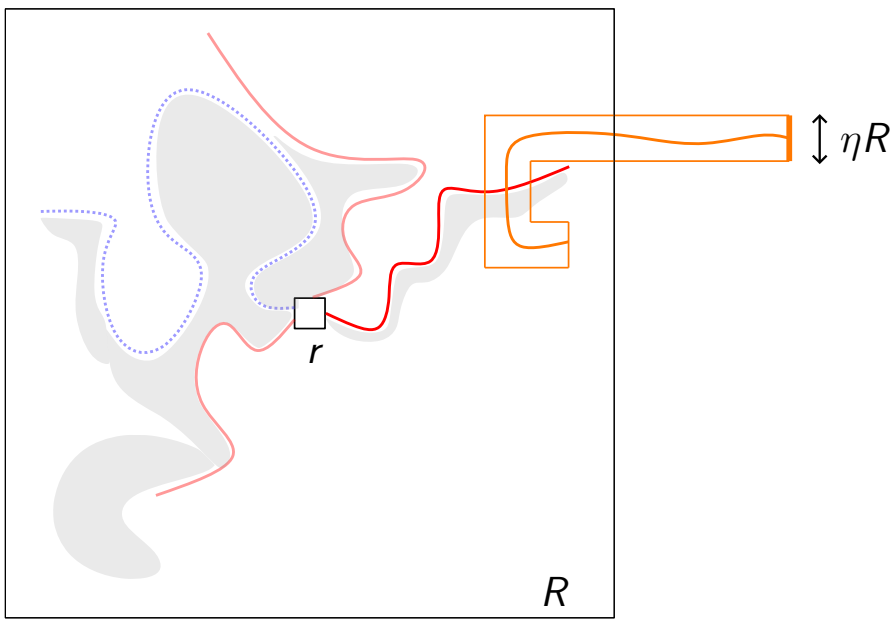

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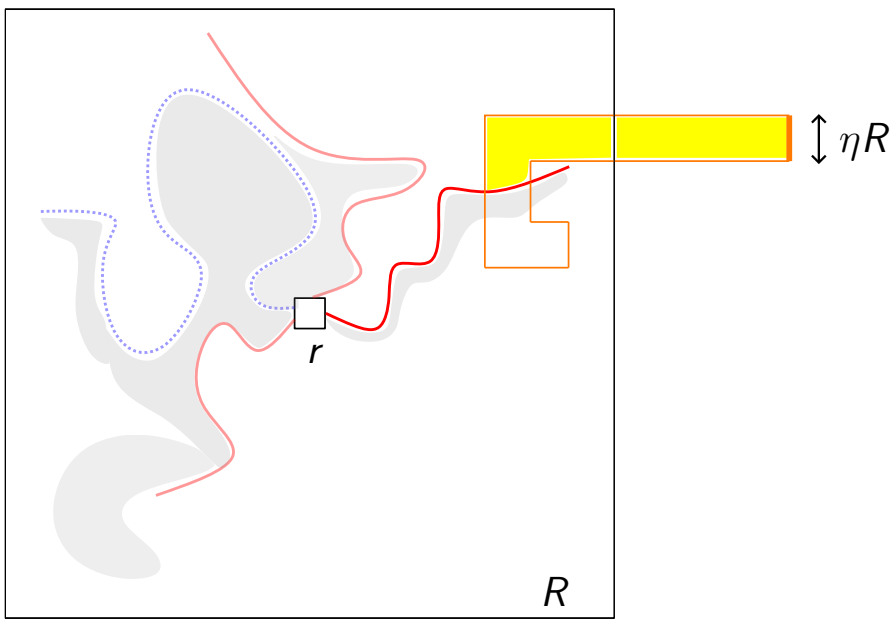

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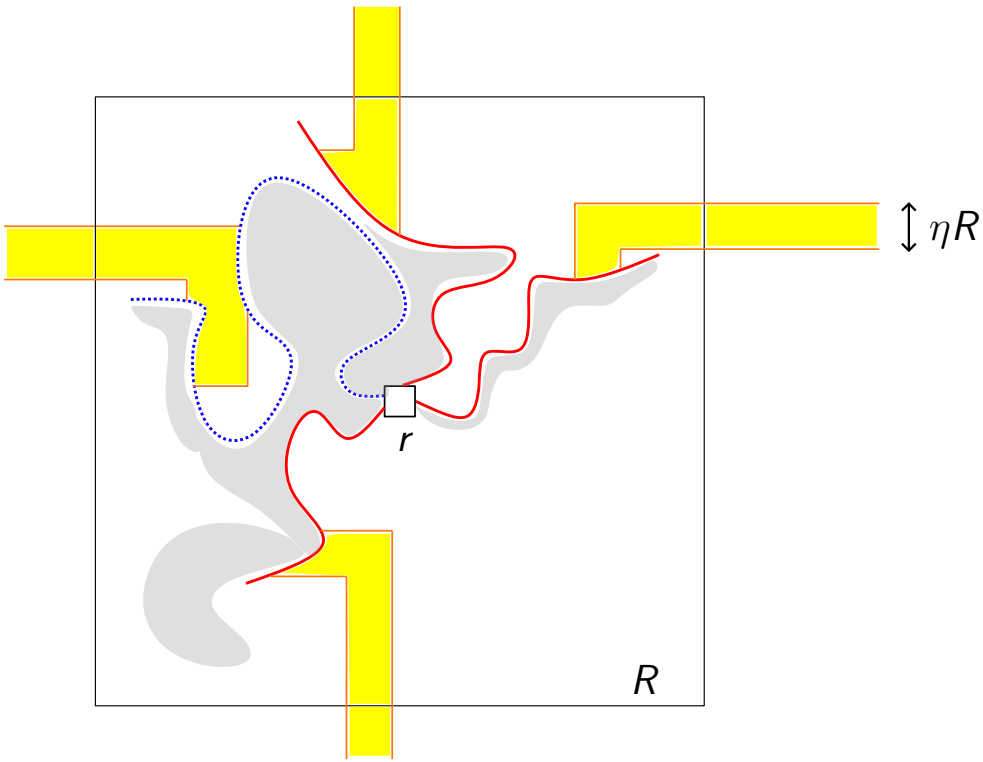


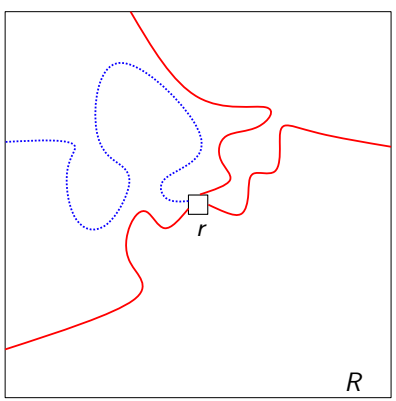
The arms of $A_{0111}(r, R)$ are η-exposed (w.r.t. exploration)

Arm exposition

Theorem (Manolescu, Mohanarangan, G, 26+)

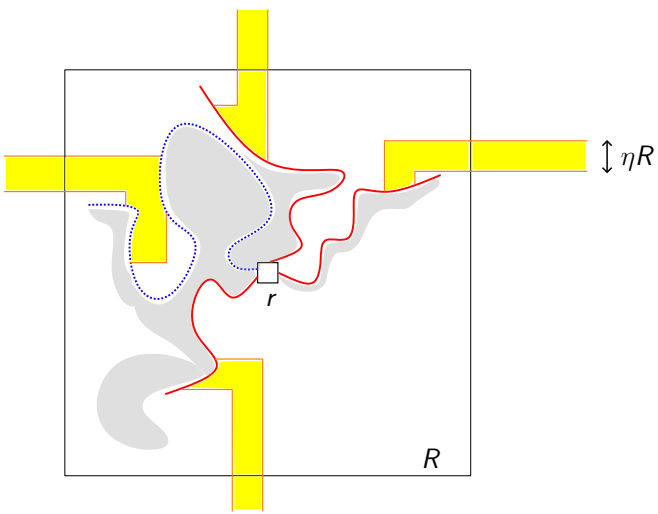
Fix $N \in \mathbb{N}$ and $\sigma \in \{0,1\}^N$ polychromatic. Take $\eta > 0$ sufficiently small. There exists an exploration algorithm $\mathcal{A}$ such that, for all $R \geq 2r$,

$$\mathbb{P}[\mathcal{A} \text{ } \eta\text{-exposes arms} \,|\, A_\sigma(r,R)] \geq C_N$$



$A_\sigma(r,R)$

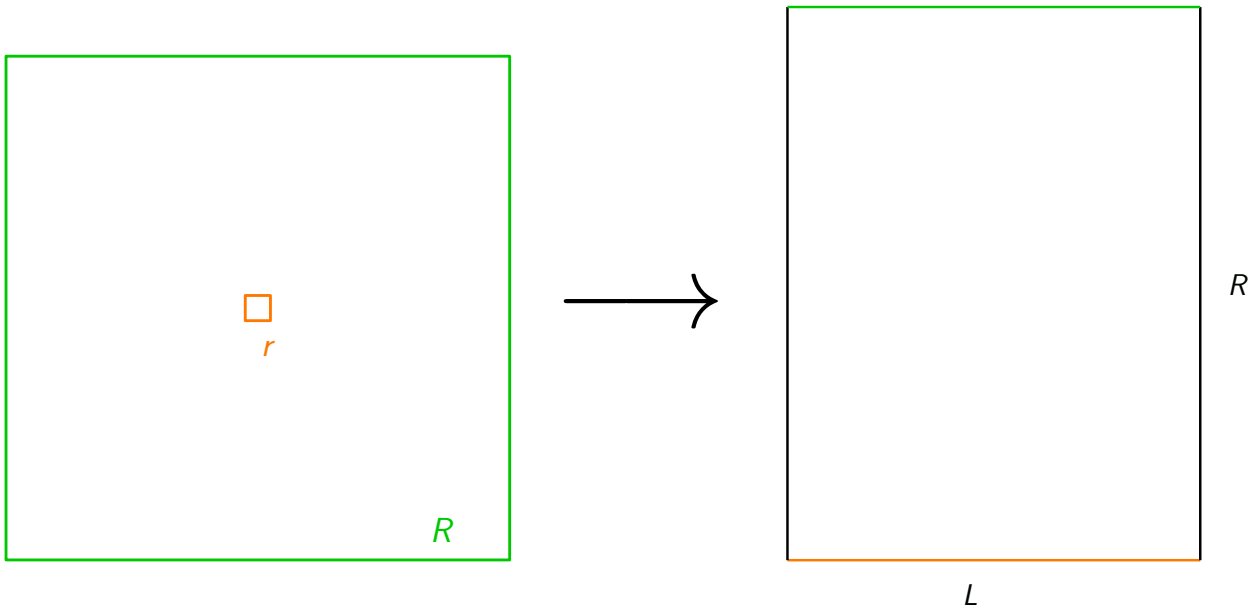
$\mathcal{A}$



Revealement of $\mathcal{A}$

Uniform in $\mathbb{P}$: same algorithm for all $\mathbb{P}$

Exposing arms in a simply connected domain



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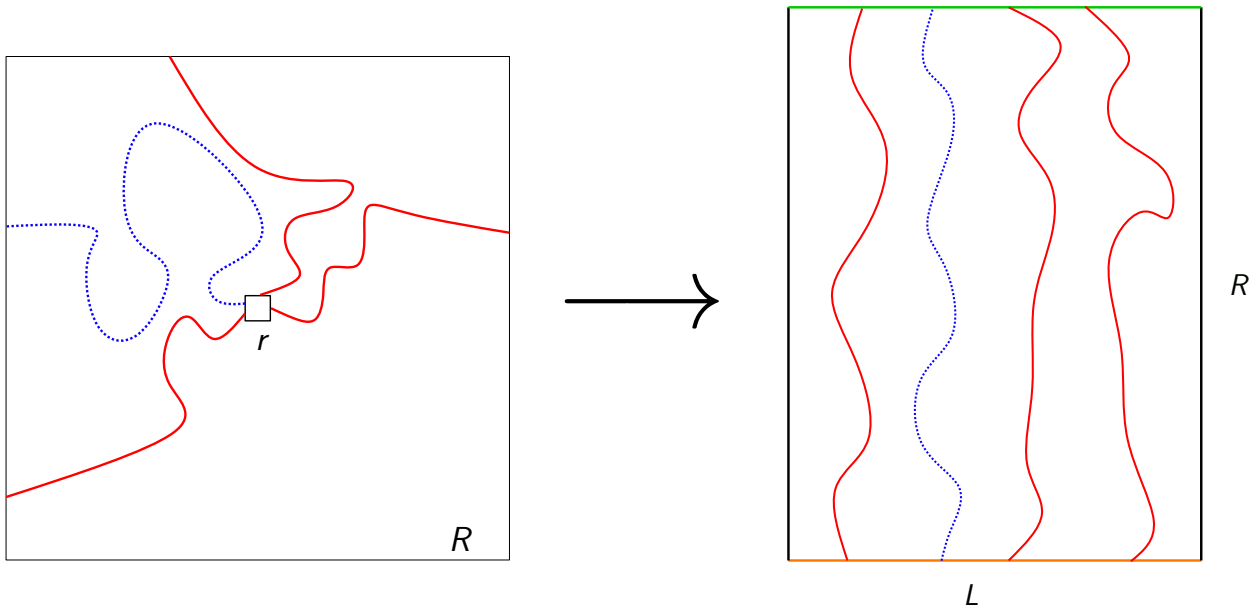

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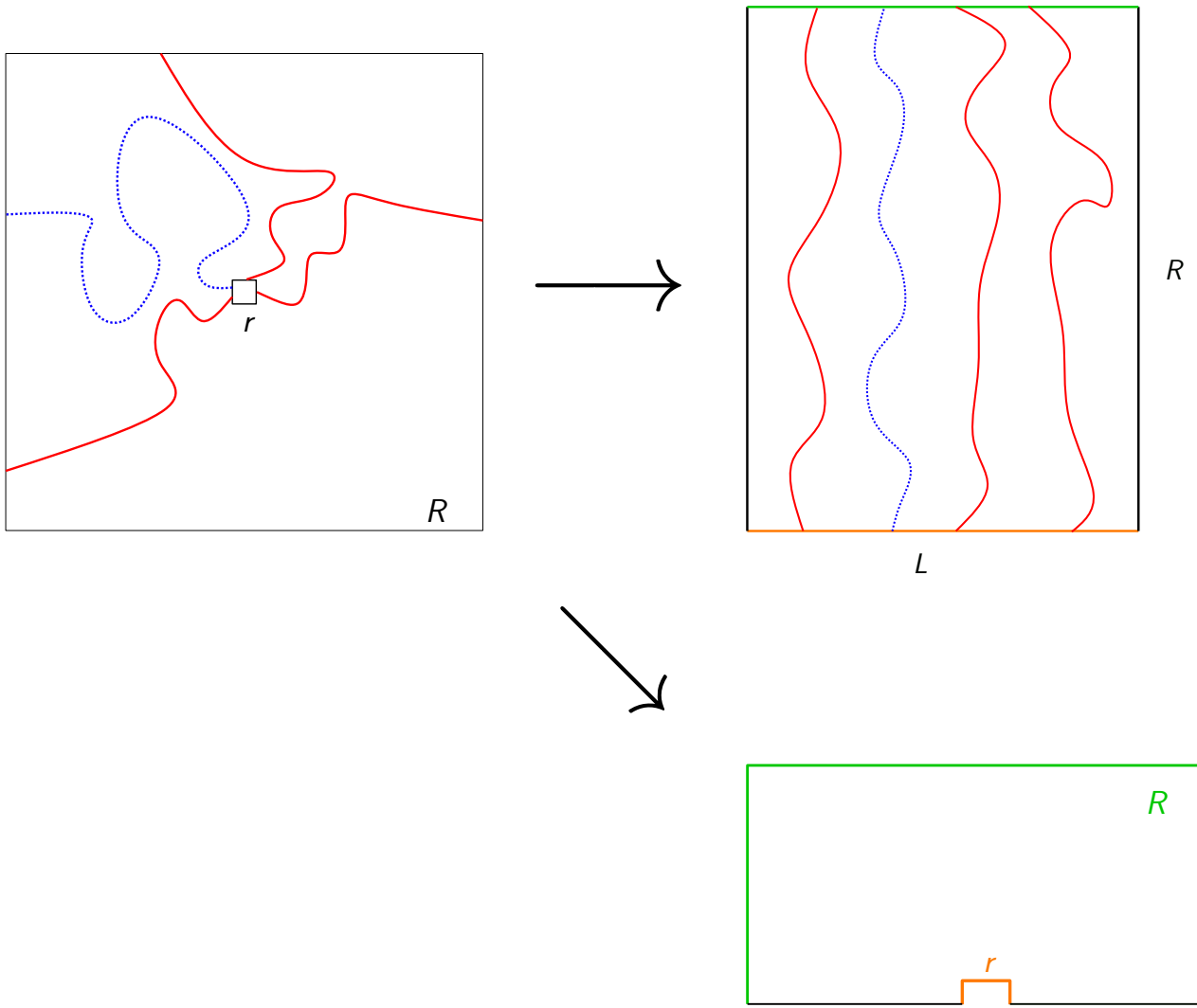

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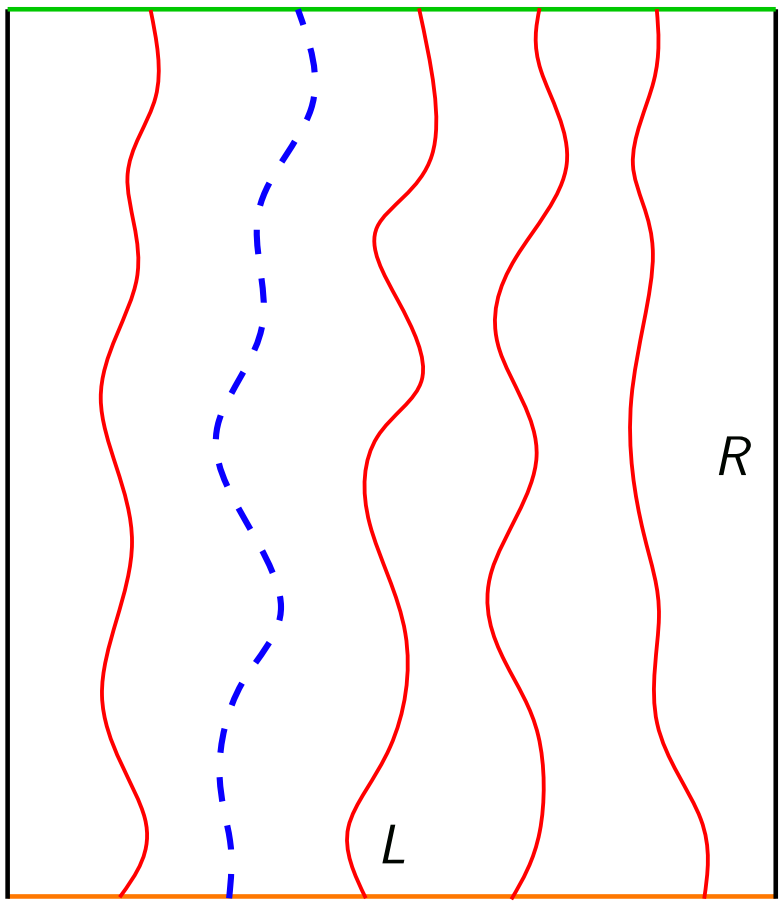
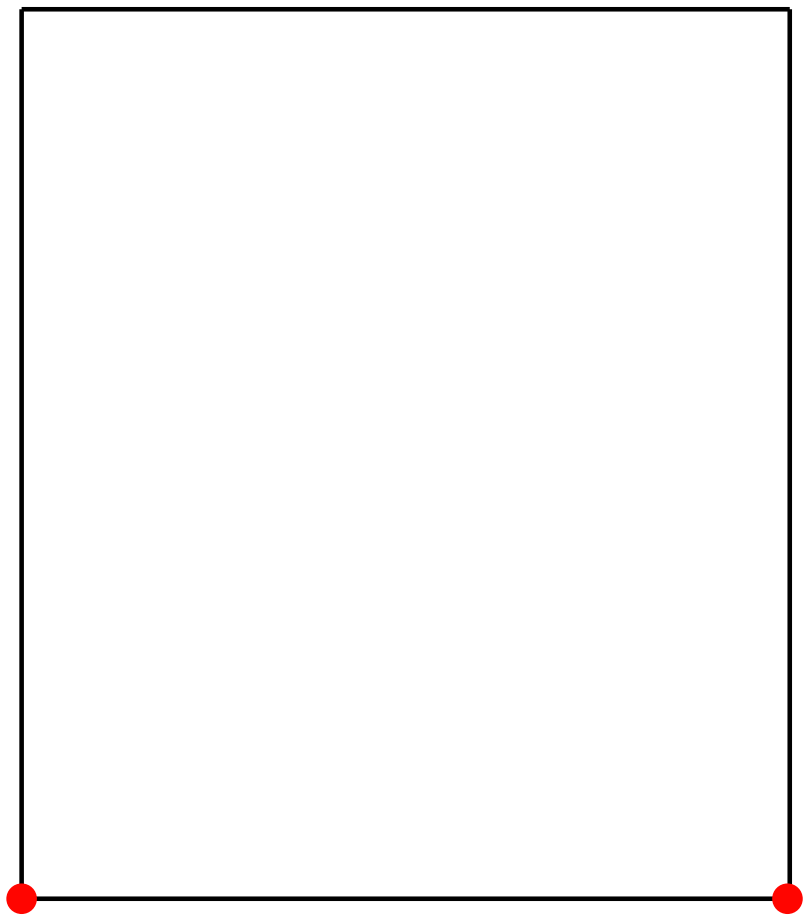


1	0	1	1	1

Exposing arms in a simply connected domain



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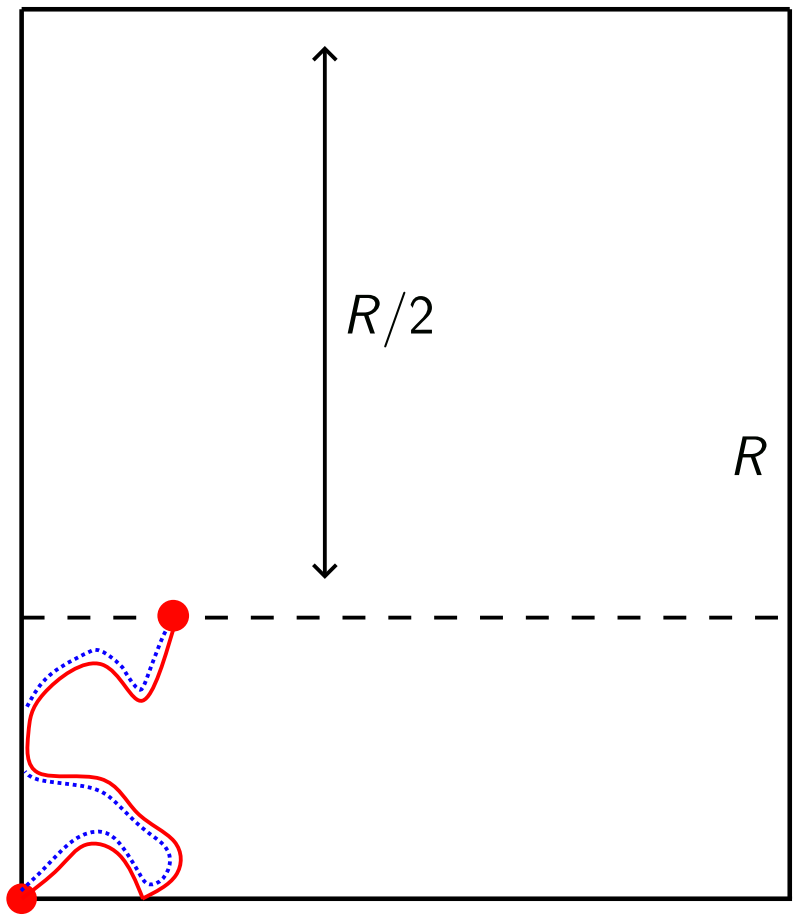

1	0	1	1	1

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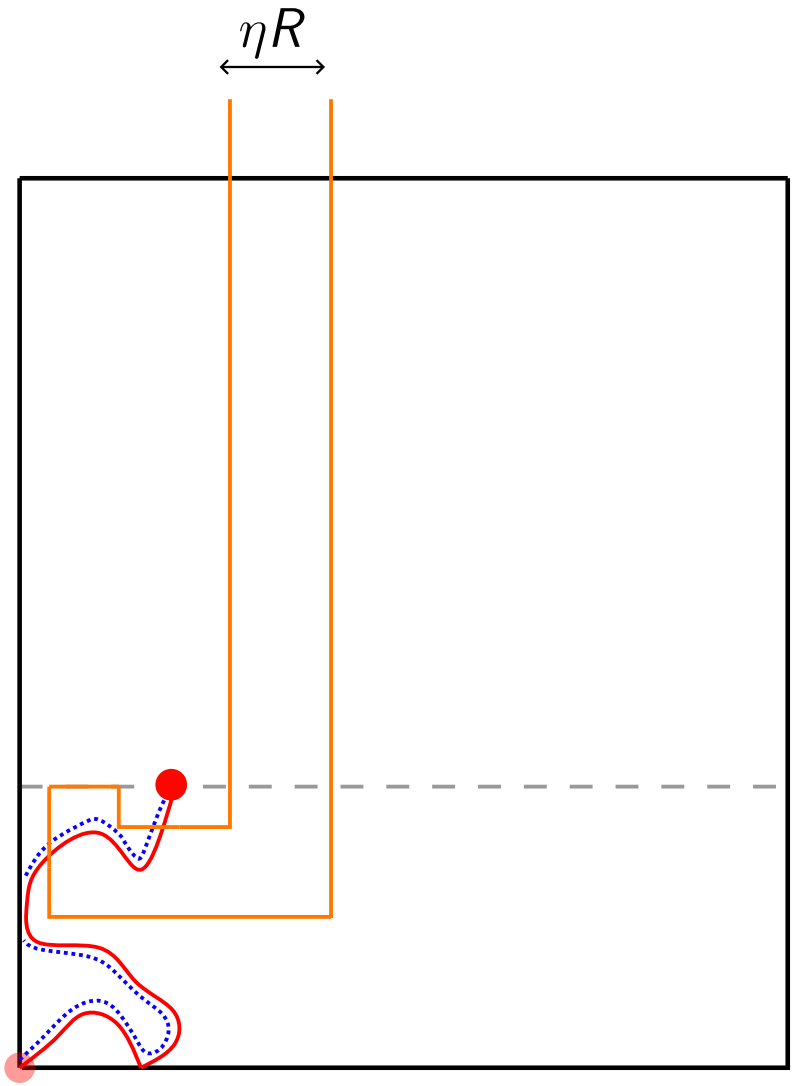

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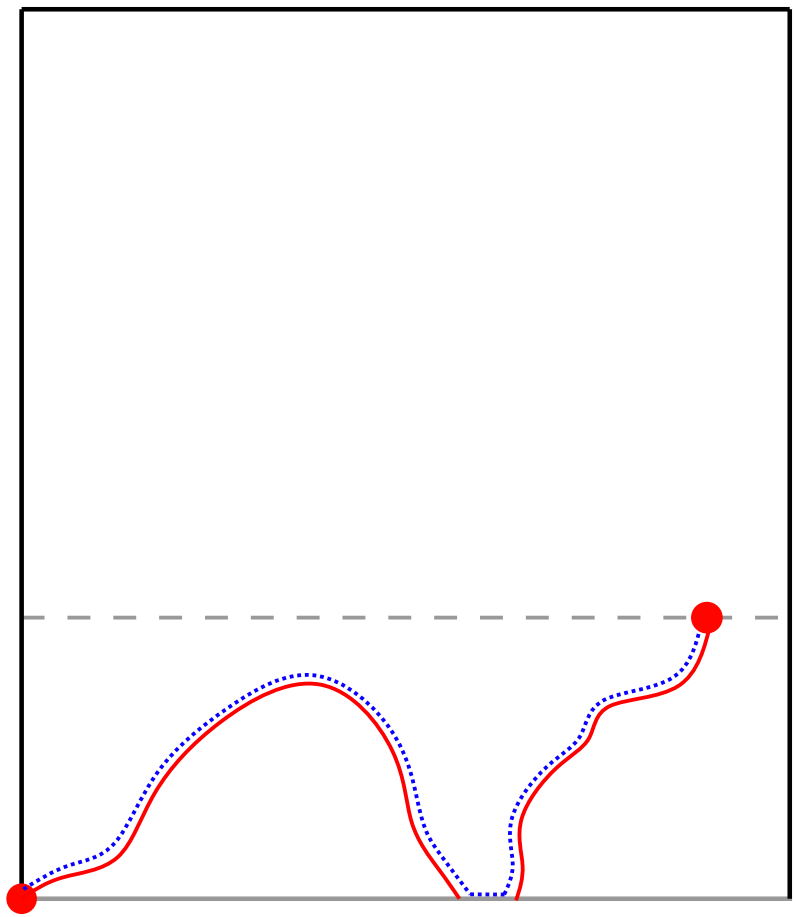

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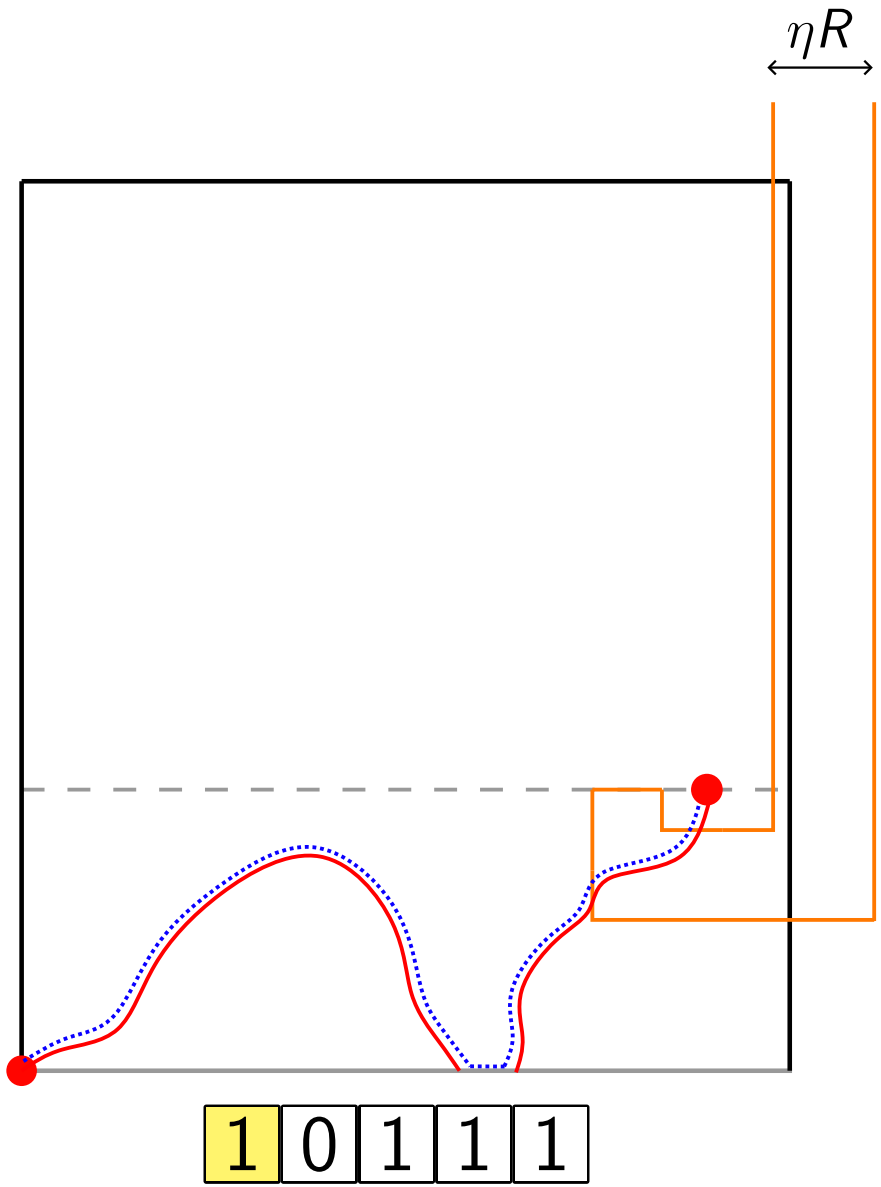

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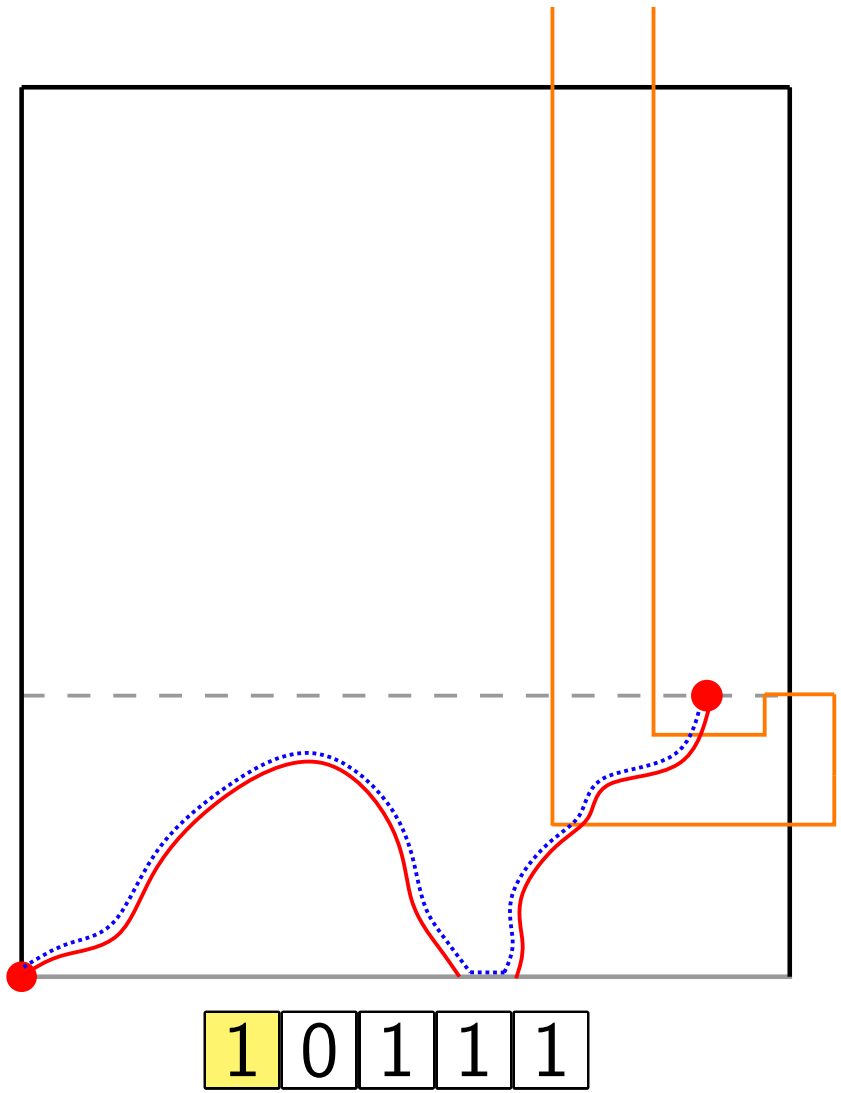

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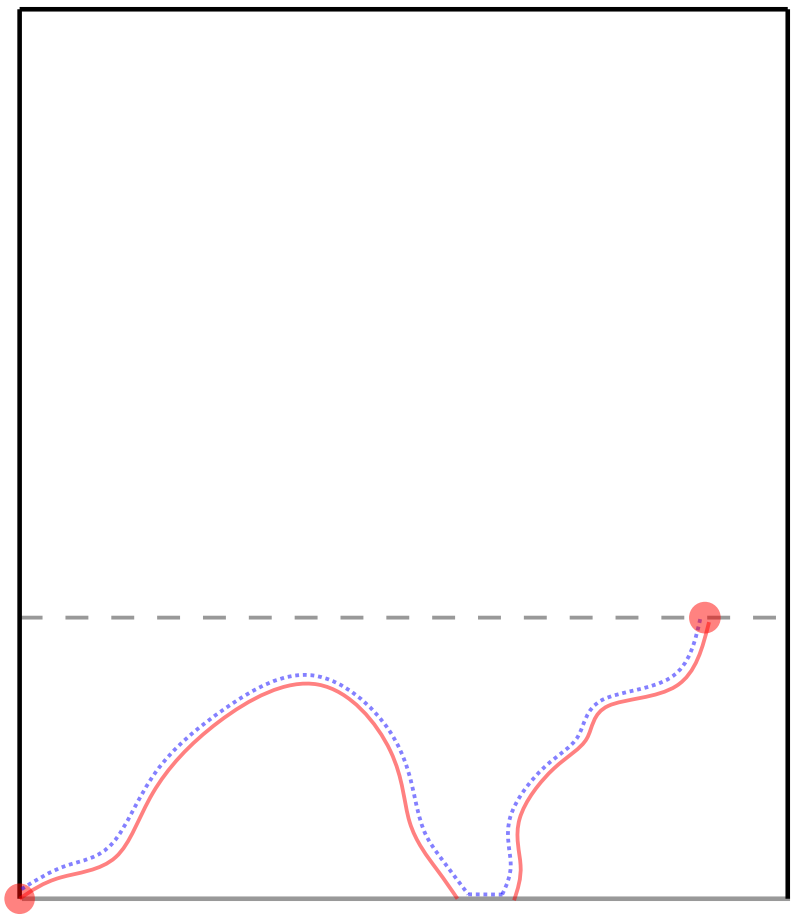

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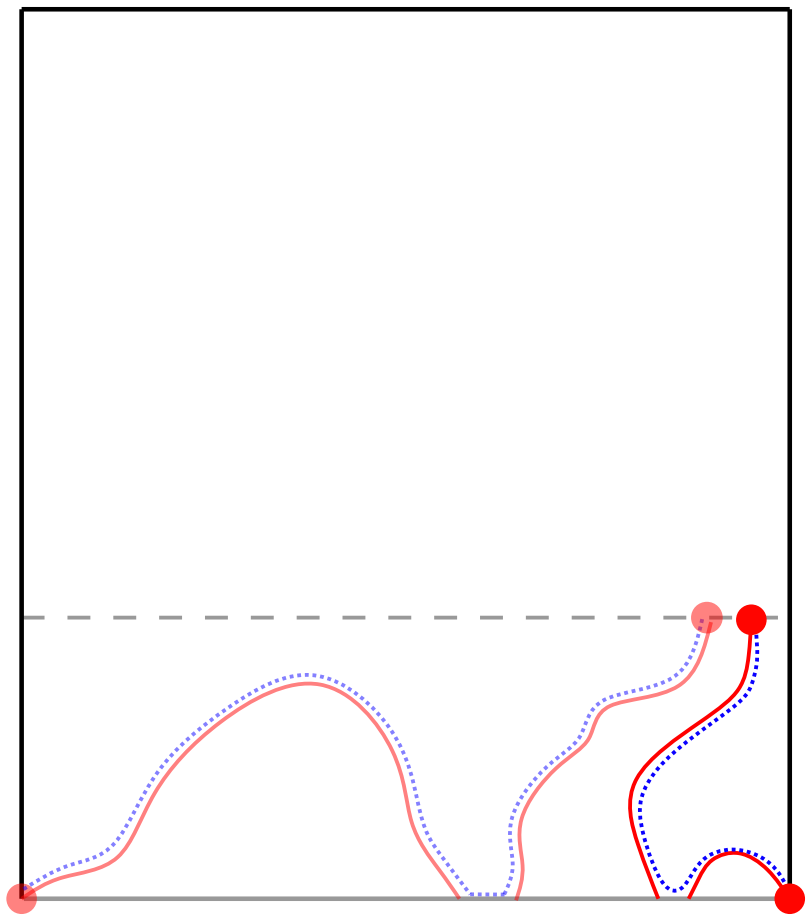

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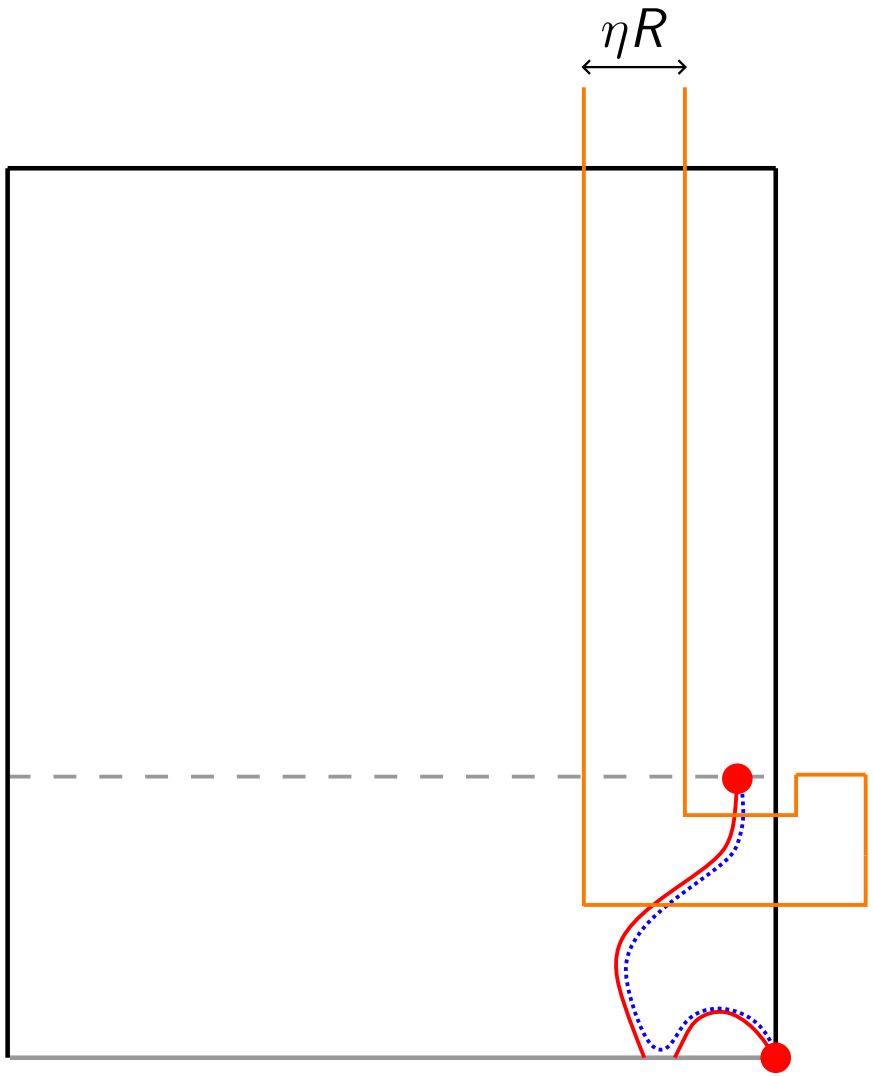

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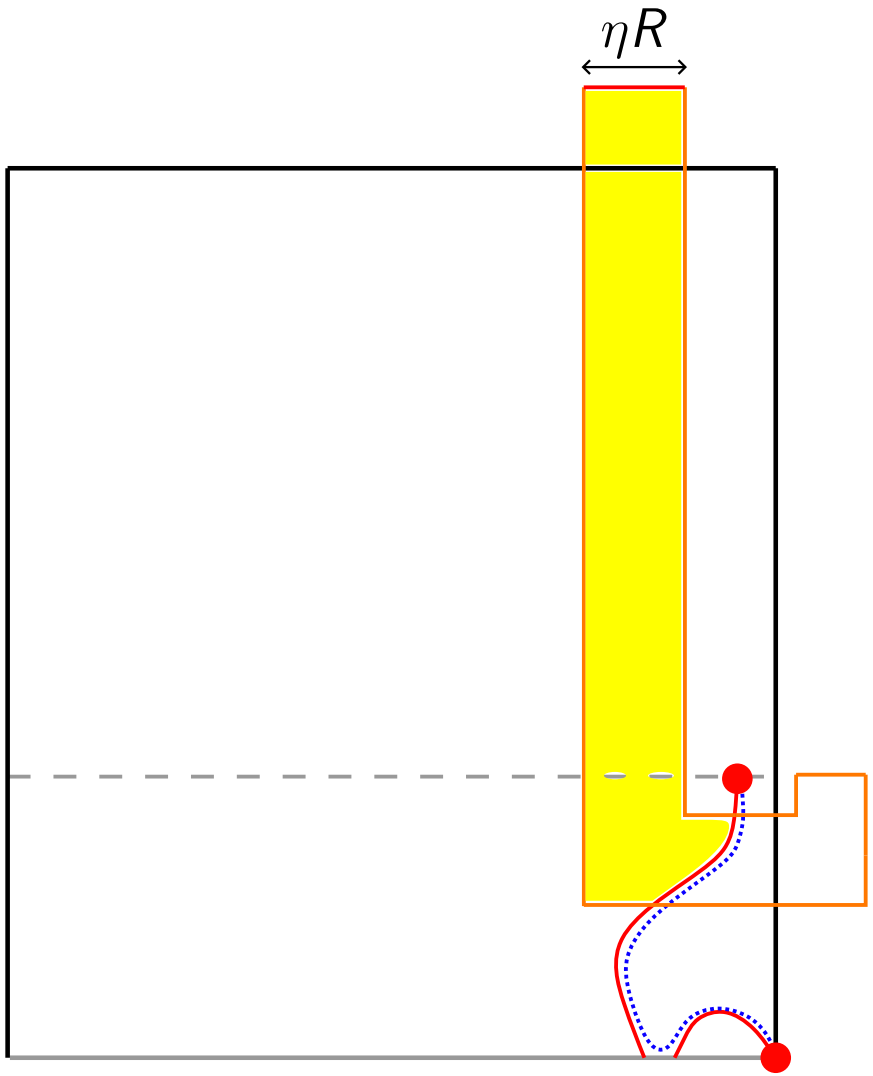

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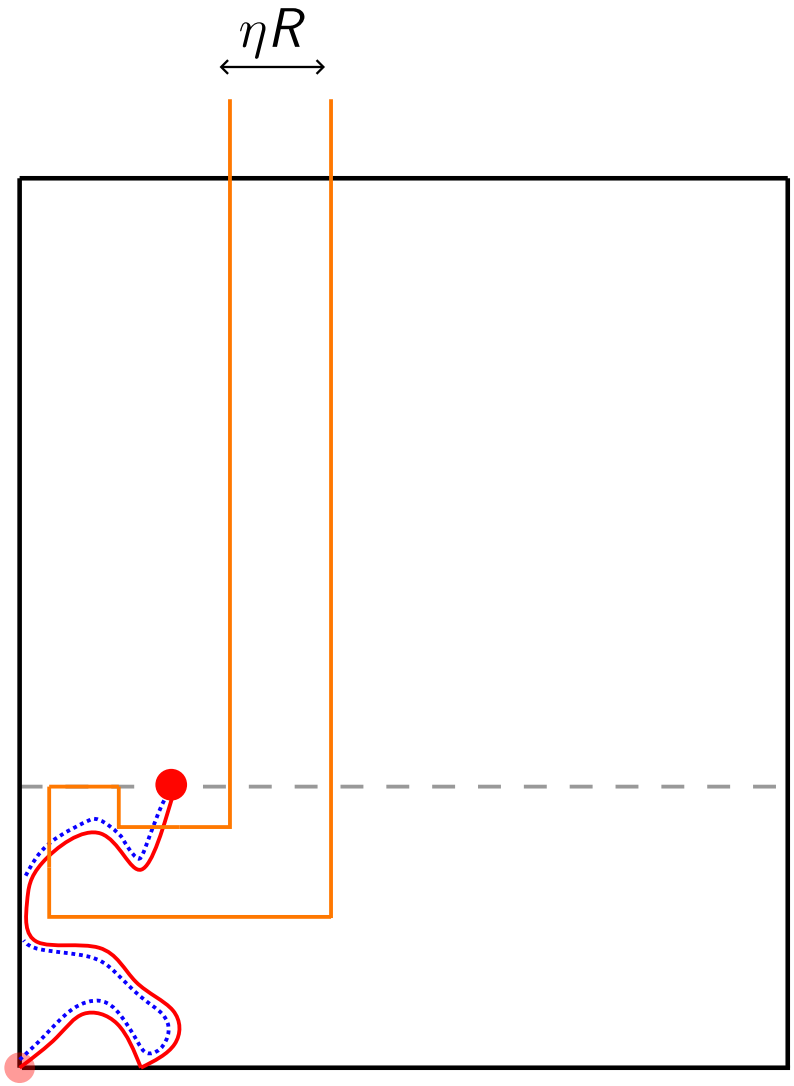

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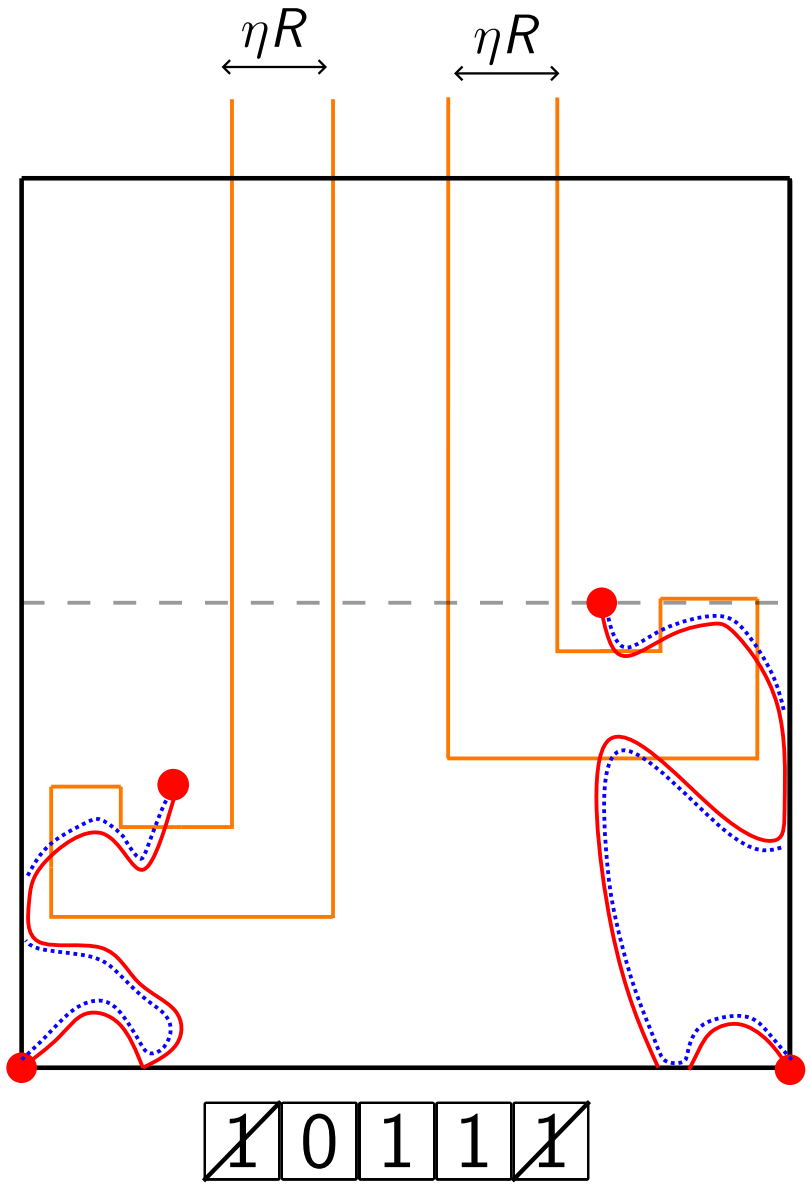

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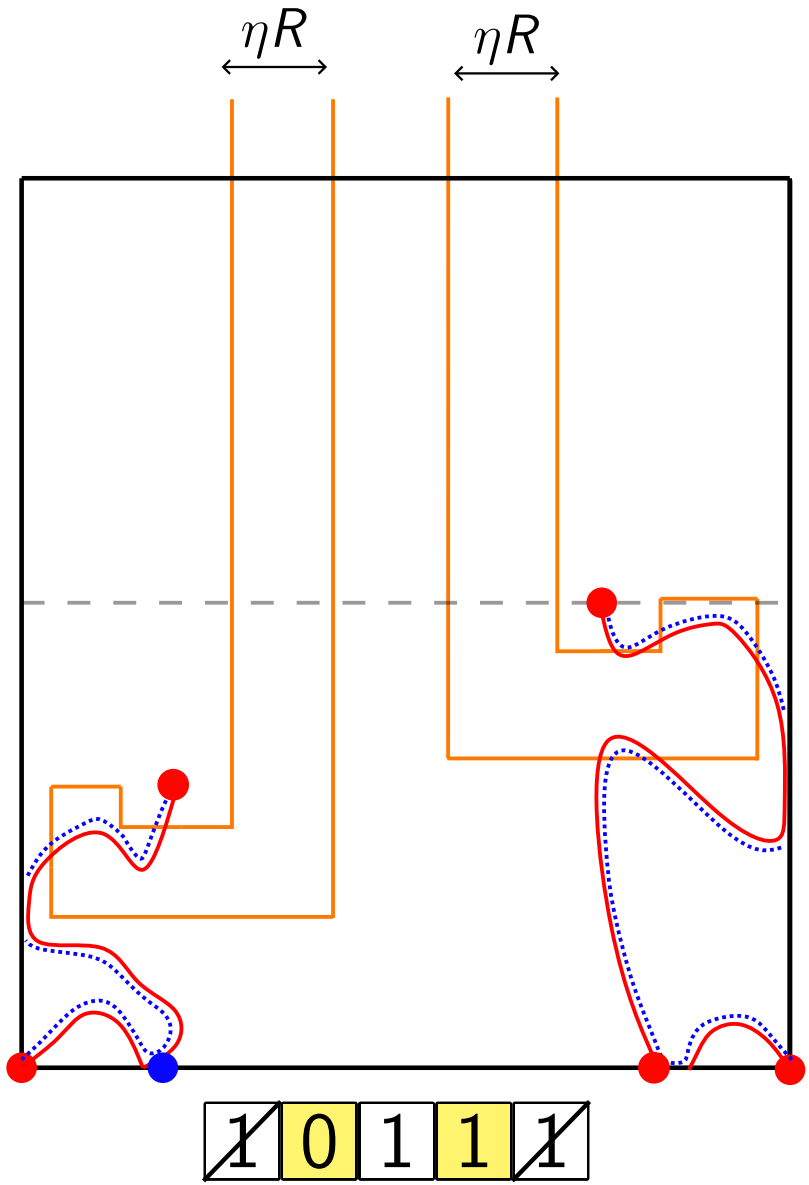

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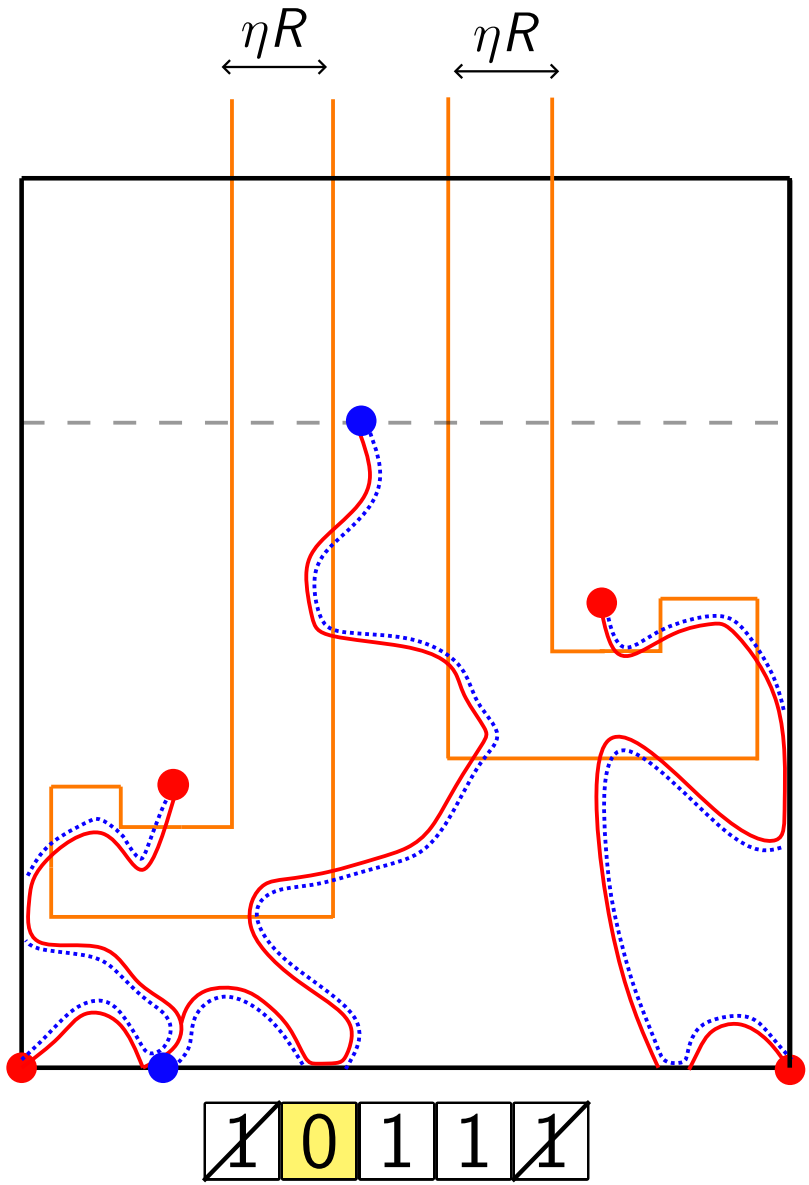

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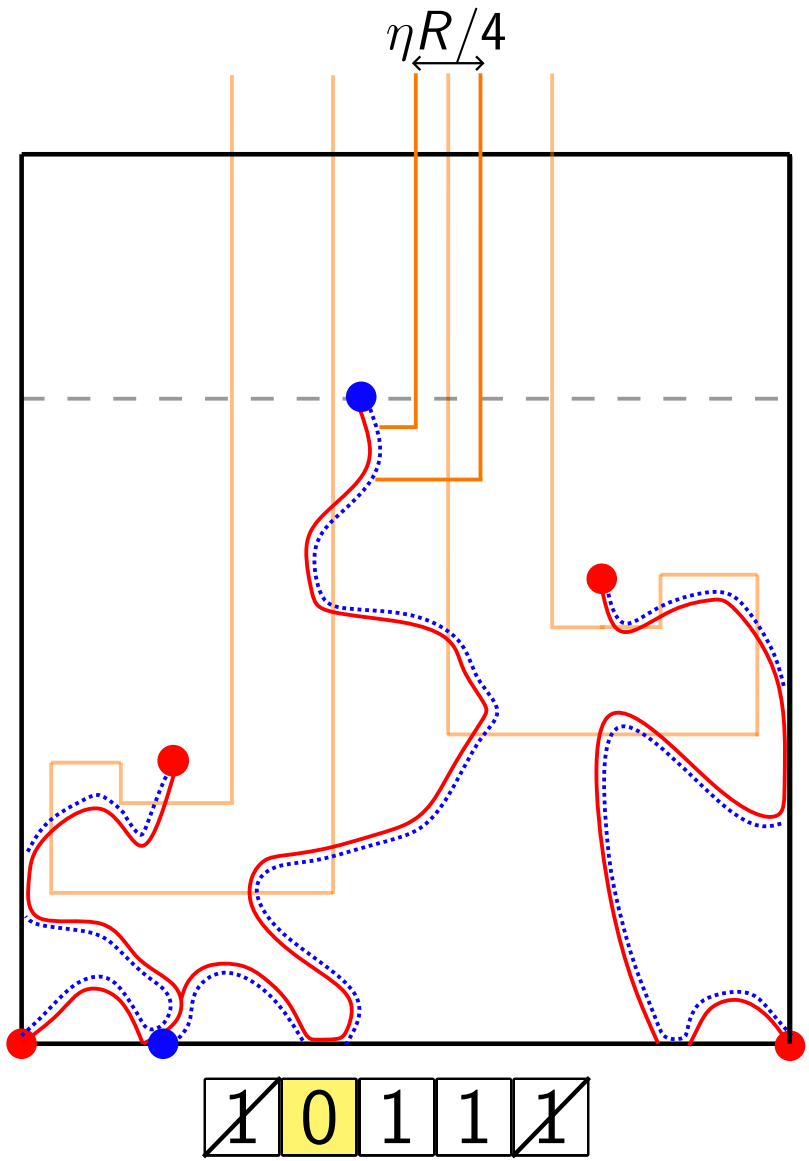

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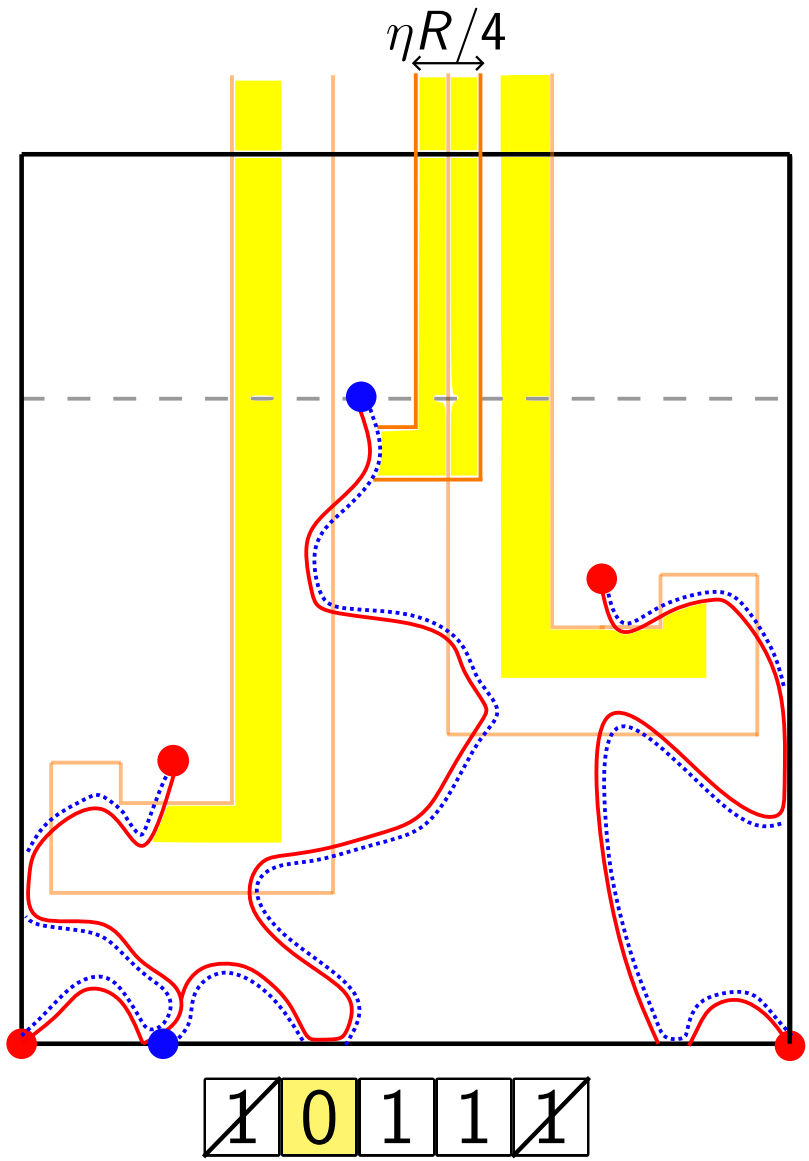

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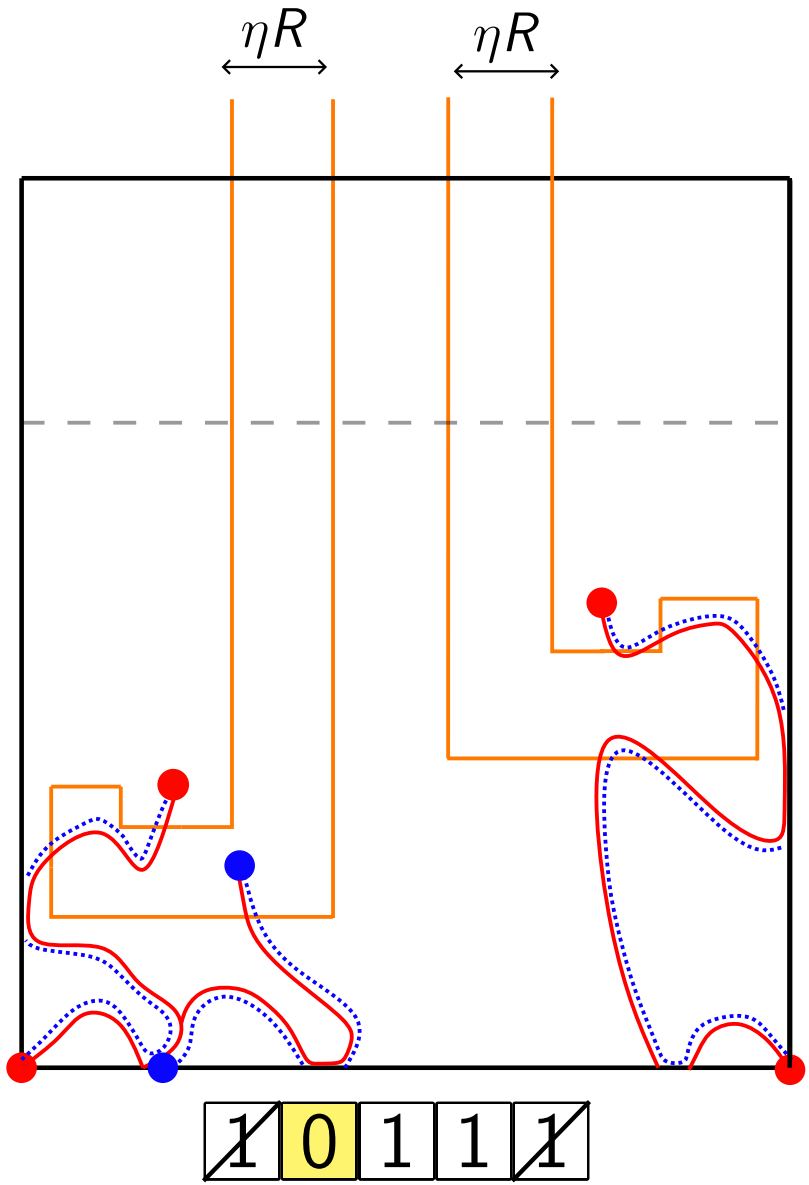

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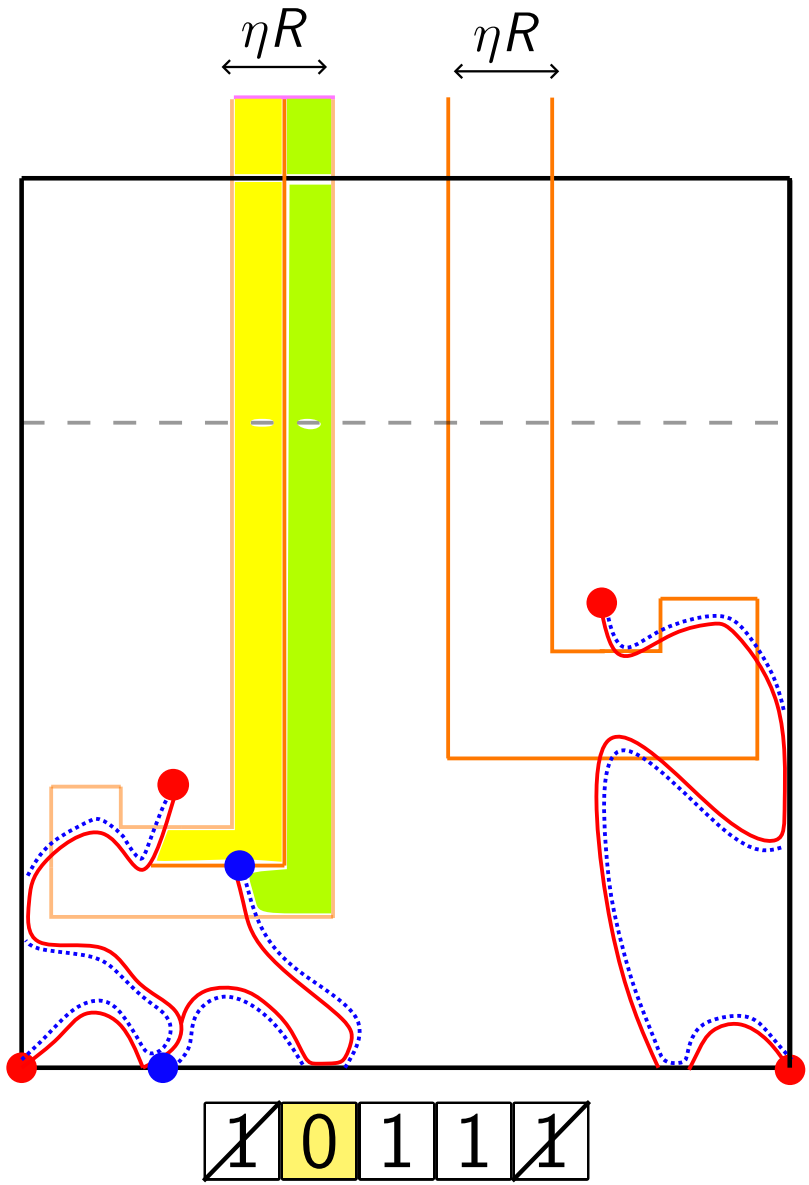

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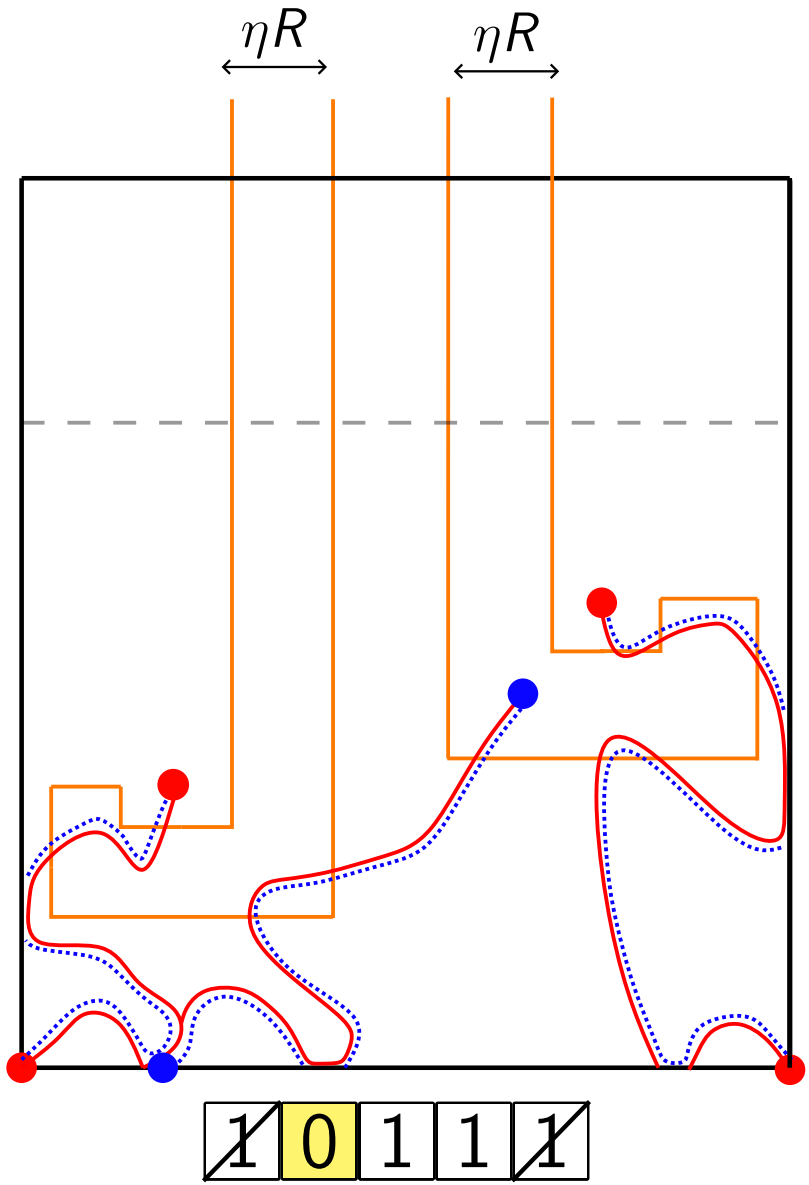

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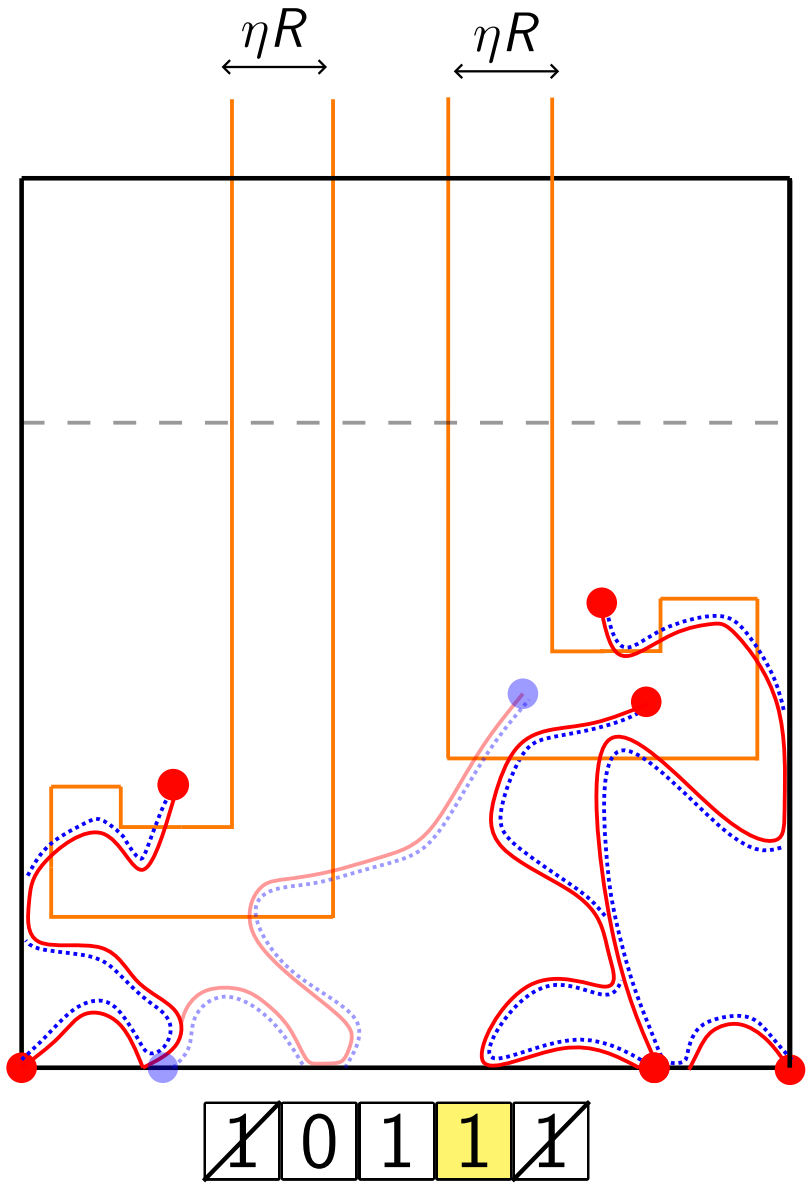

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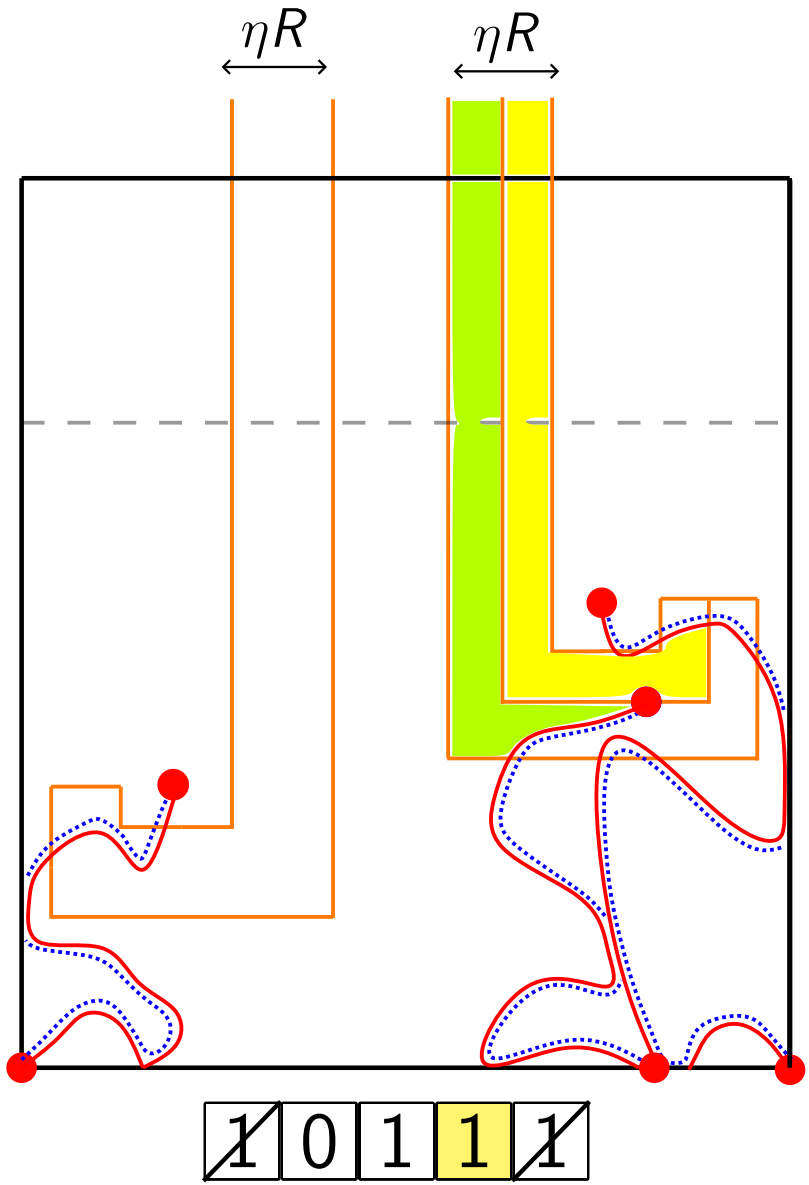

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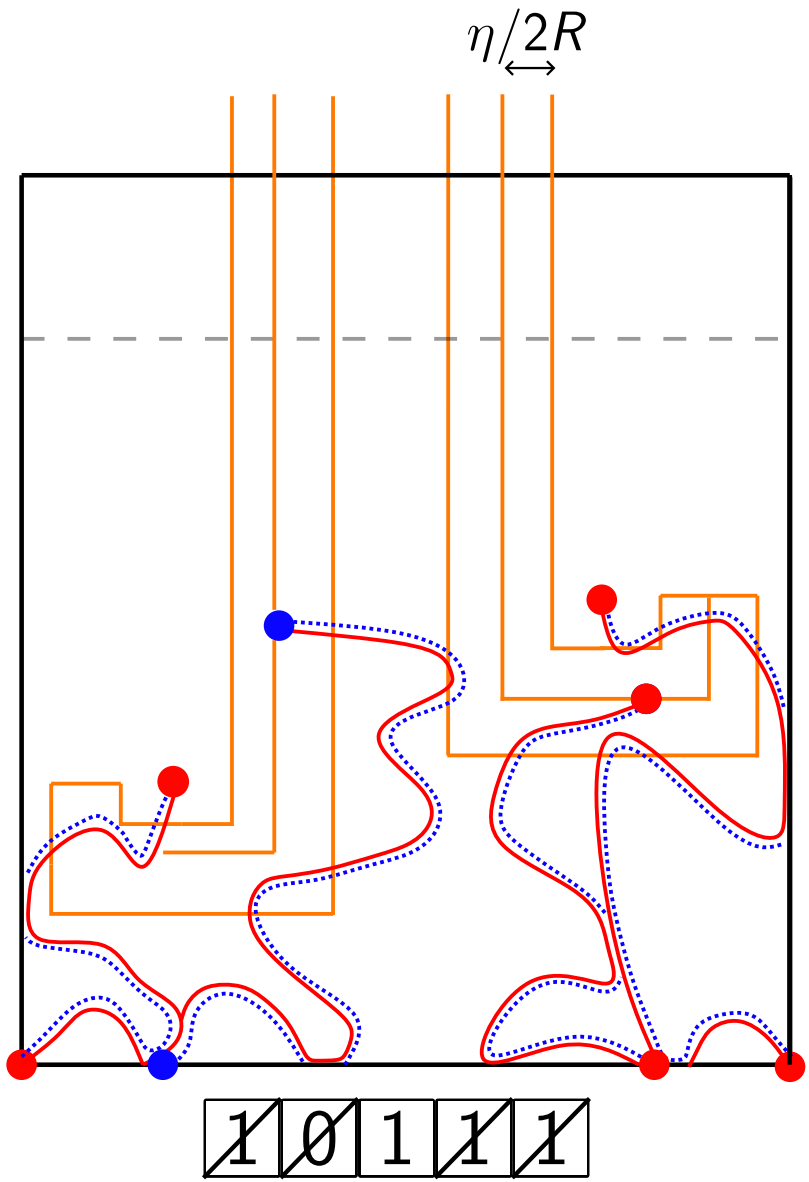

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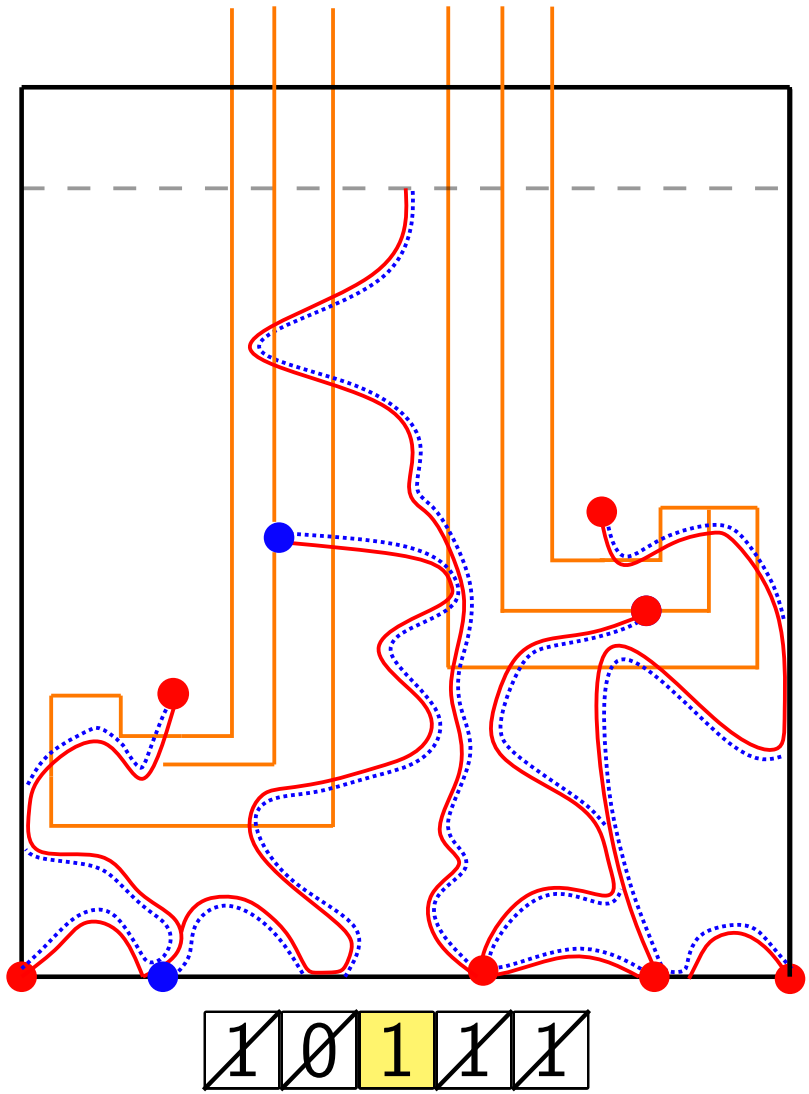

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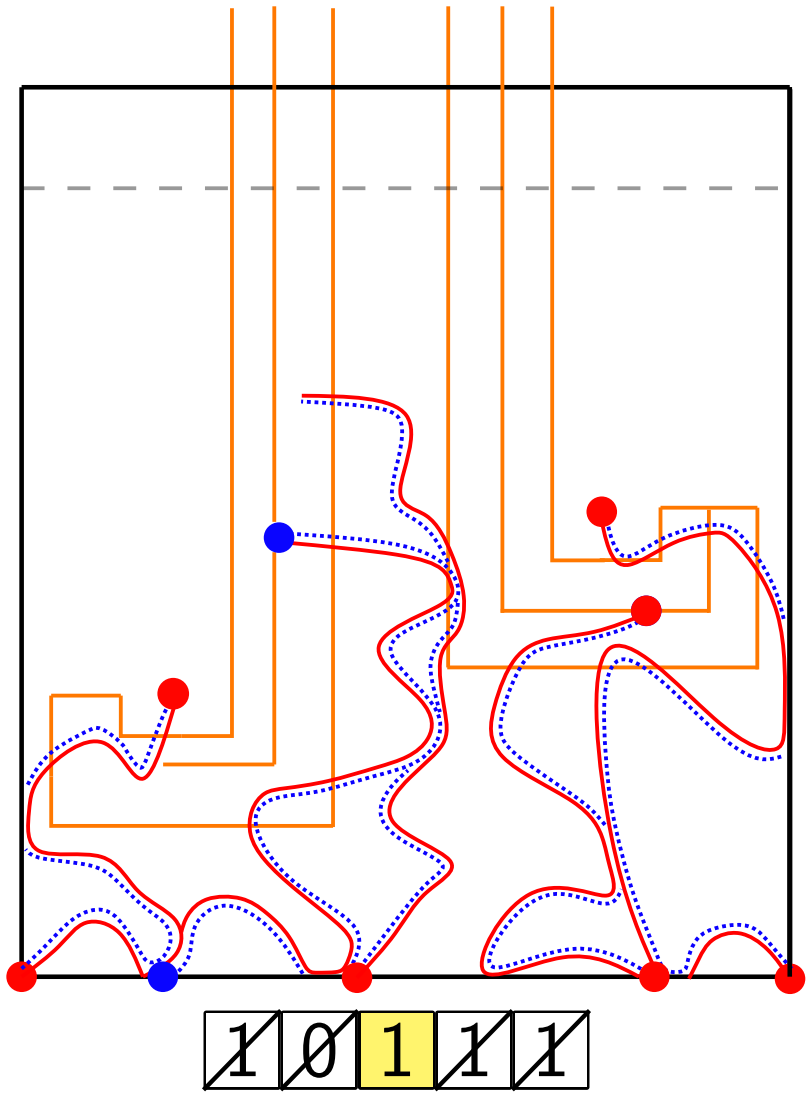

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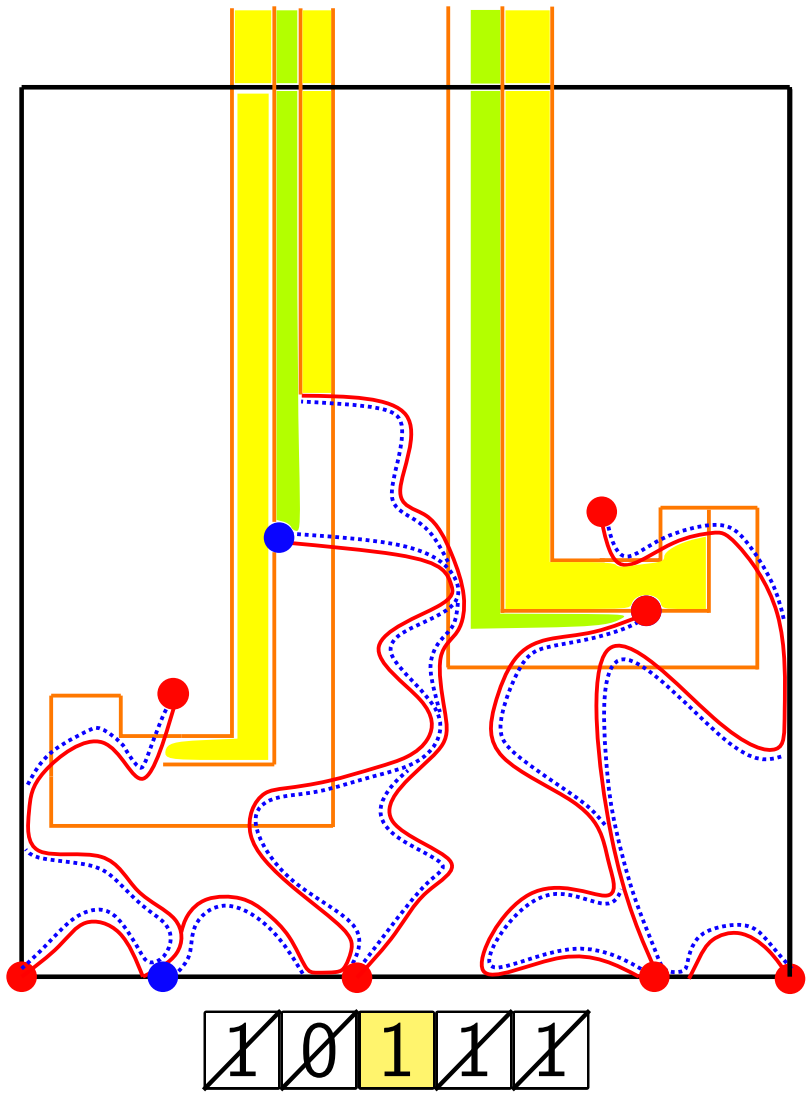

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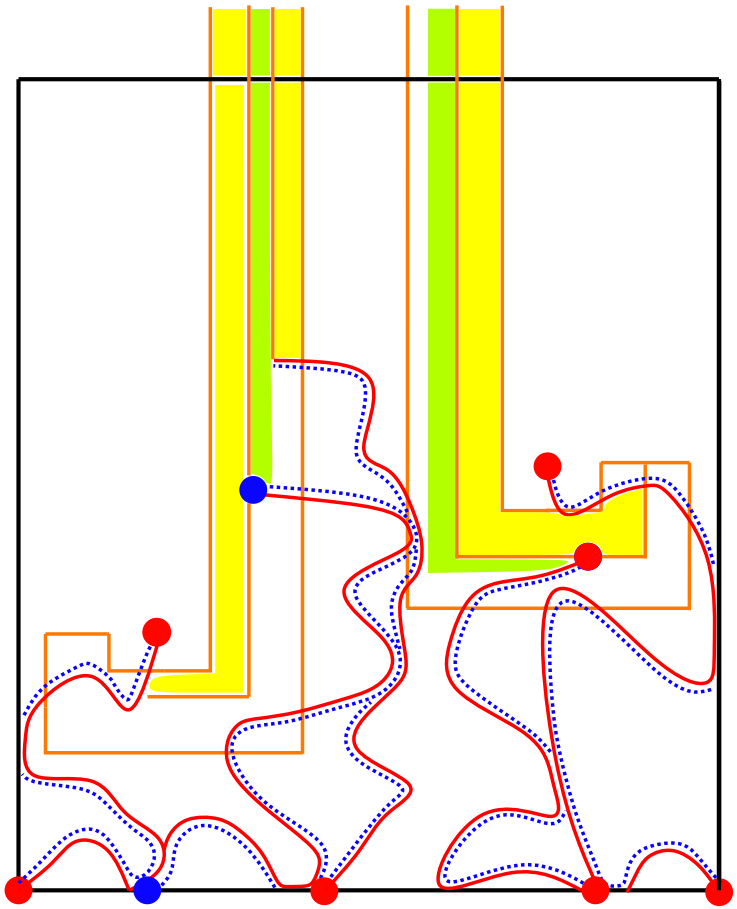
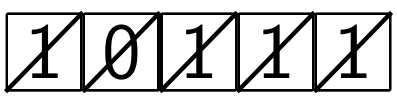




For N arms:

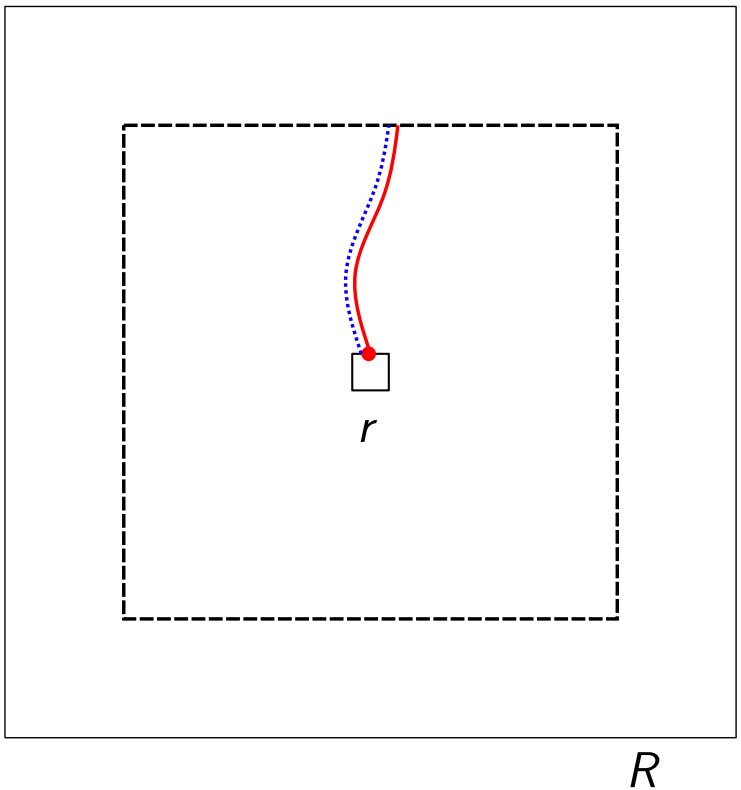
N steps Left/Right

$$\mathbb{P}(\mathcal{A} \text{ succeed}) \geq \frac{1}{2^N}$$

"arms are $\frac{\eta}{2^N}$-exposed"

Non-simply connected domains?

Example with $\sigma = 0101$



Non-simply connected domains?

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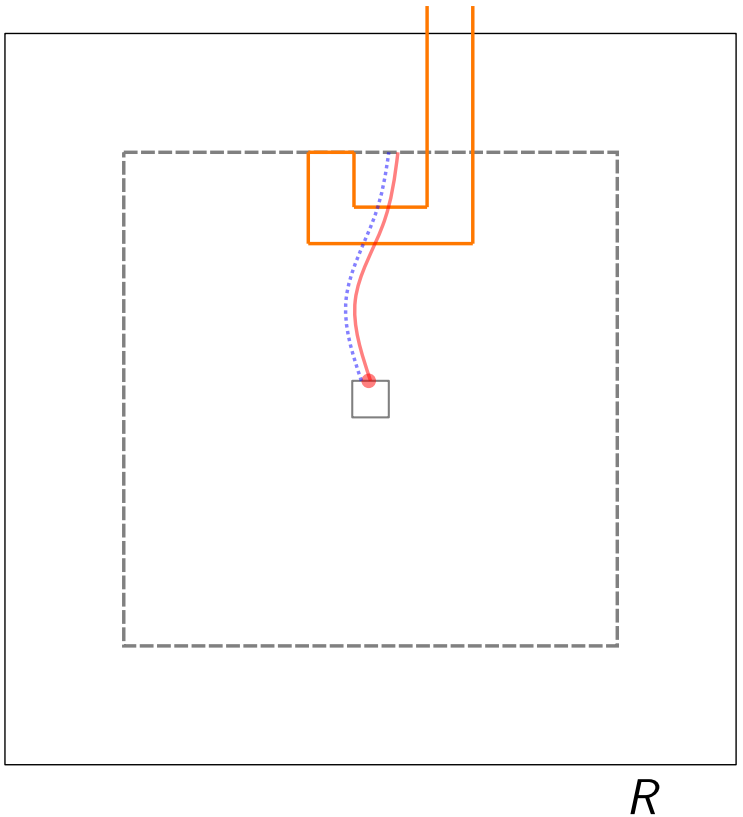

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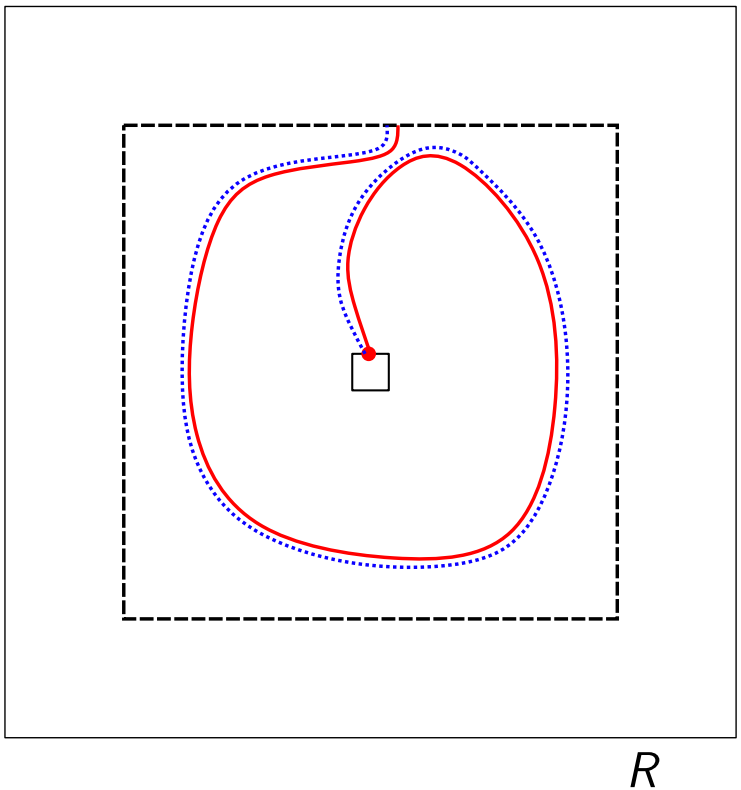
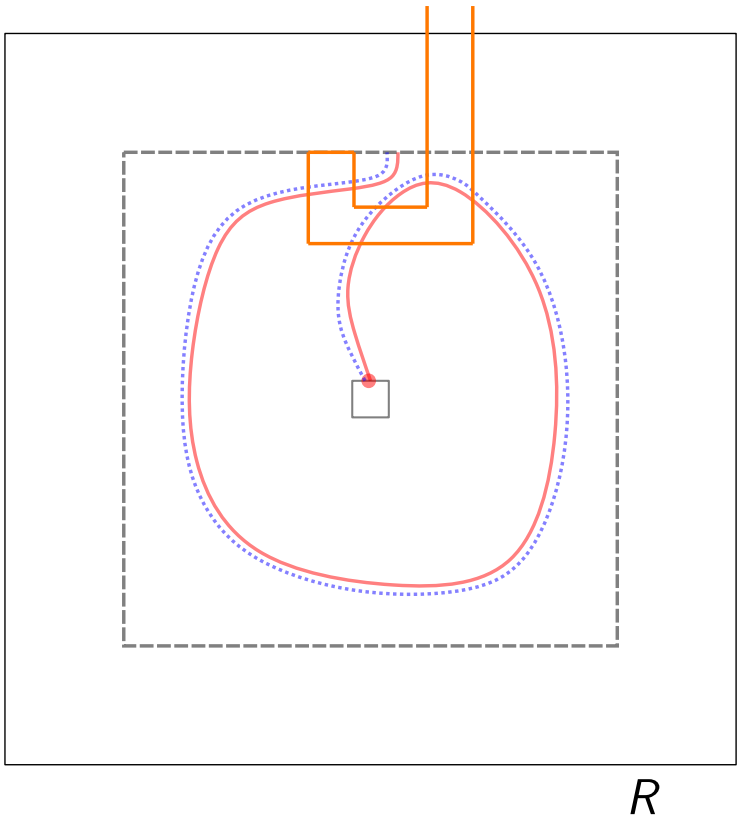


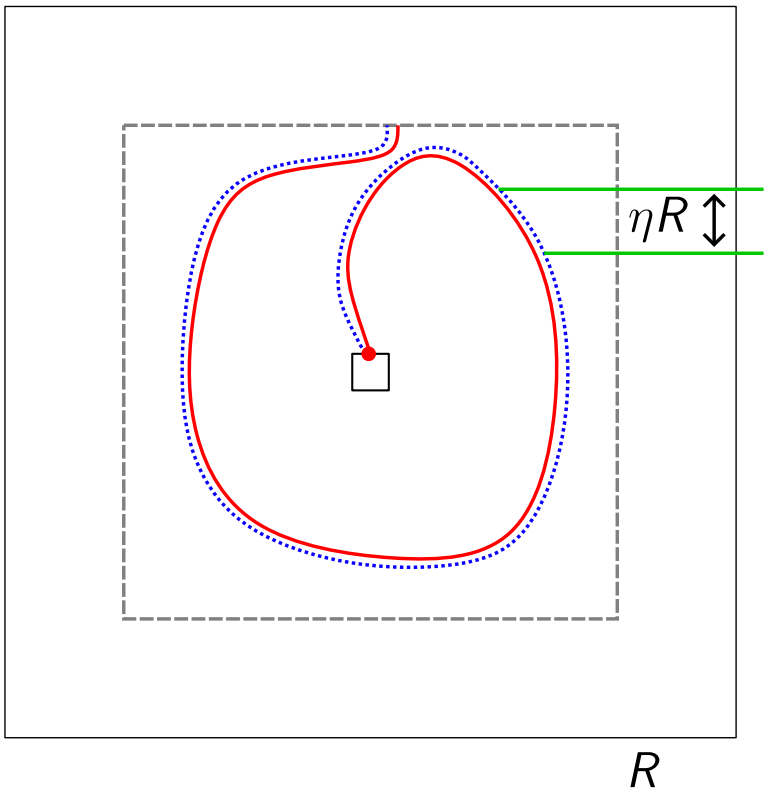
Non-simply connected domains?

Example with $\sigma = 0101$



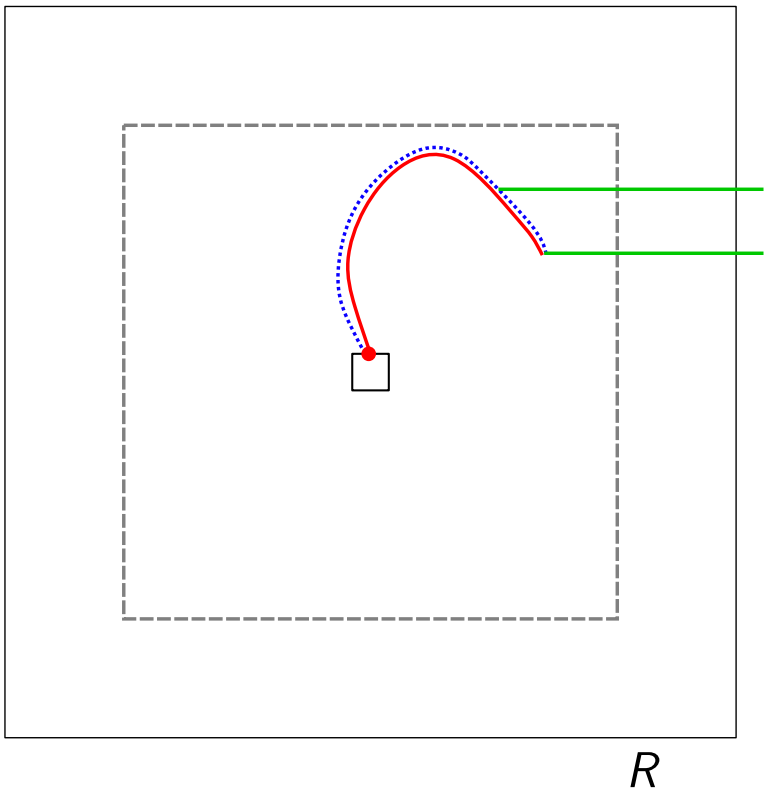
Non-simply connected domains?

Example with $\sigma = 0101$



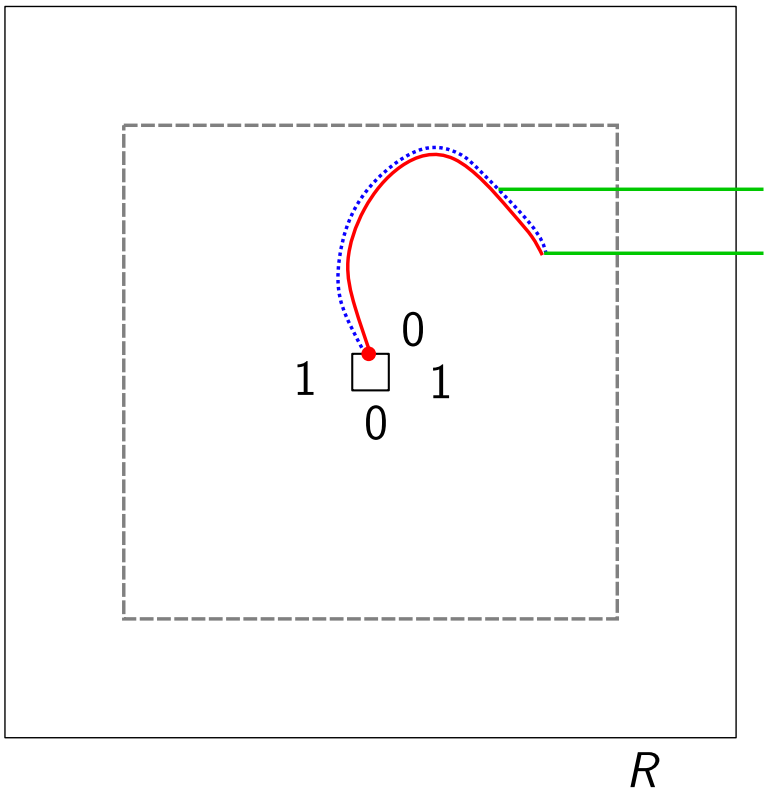
Non-simply connected domains?

Example with $\sigma = 0101$



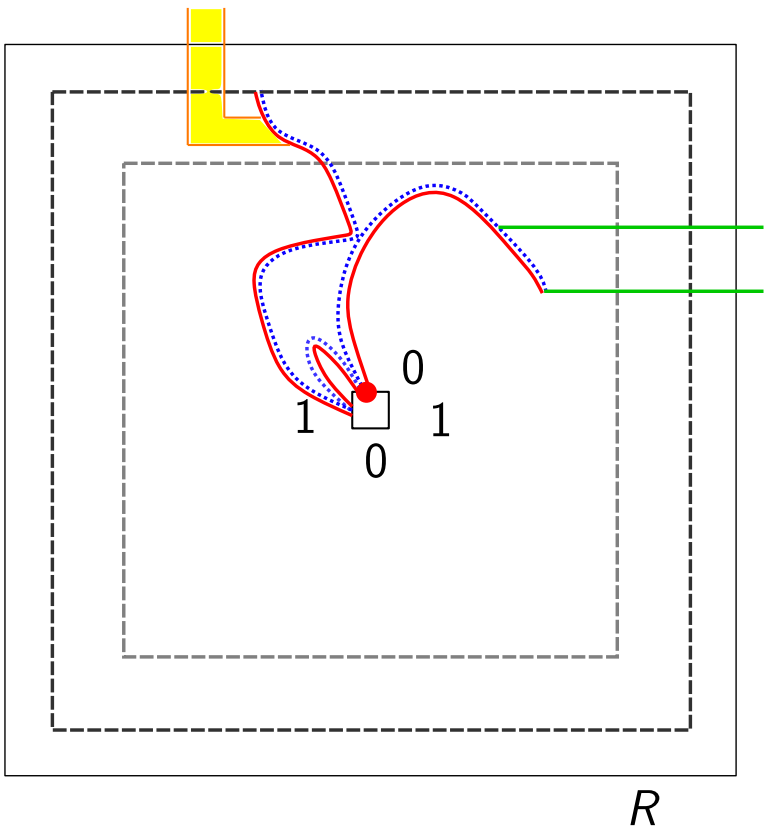
Non-simply connected domains?

Example with $\sigma = 0101$



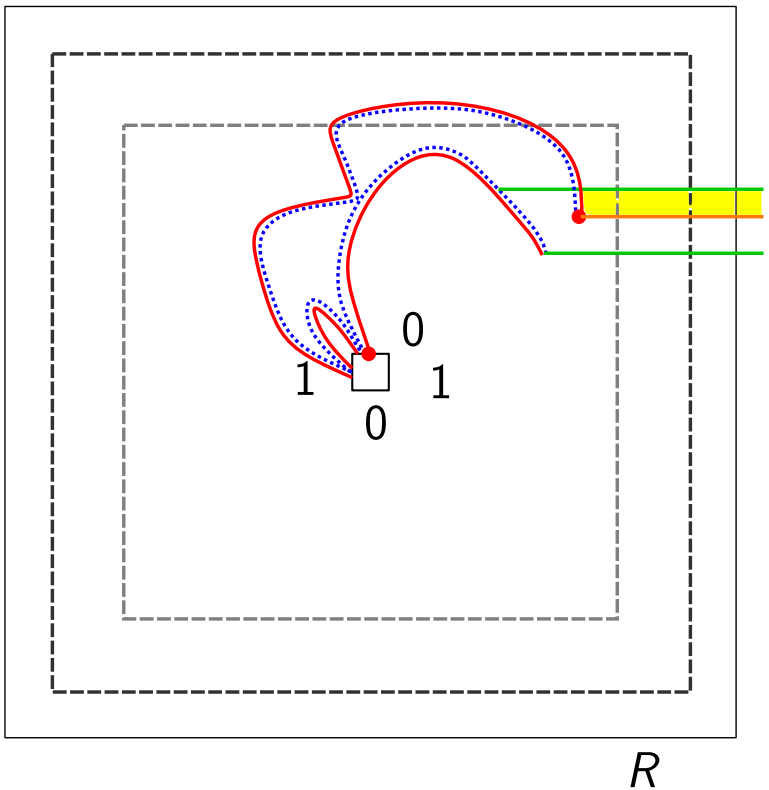
Non-simply connected domains?

Example with $\sigma = 0101$



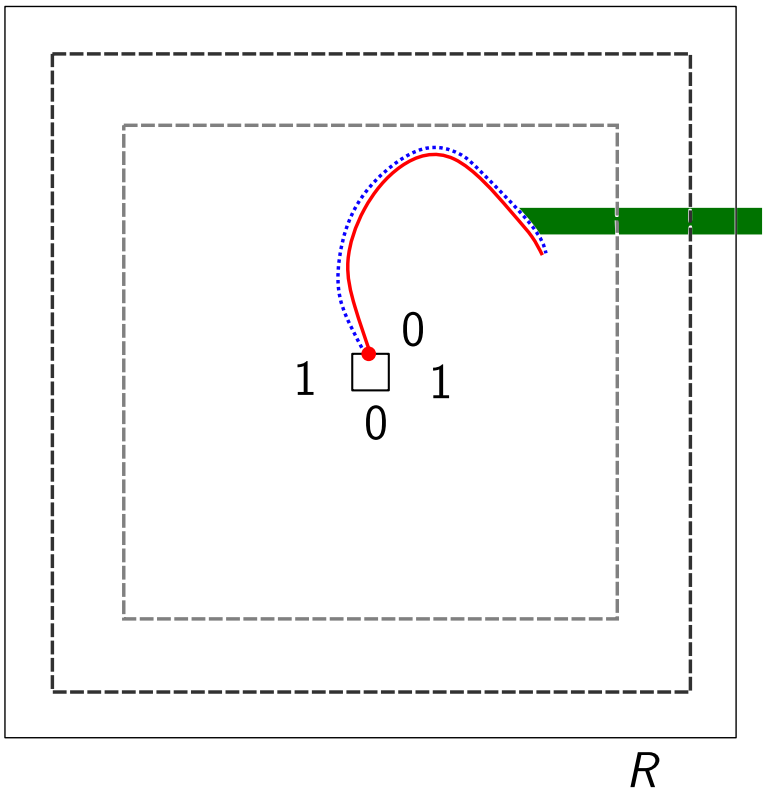
Non-simply connected domains?

Example with $\sigma = 0101$



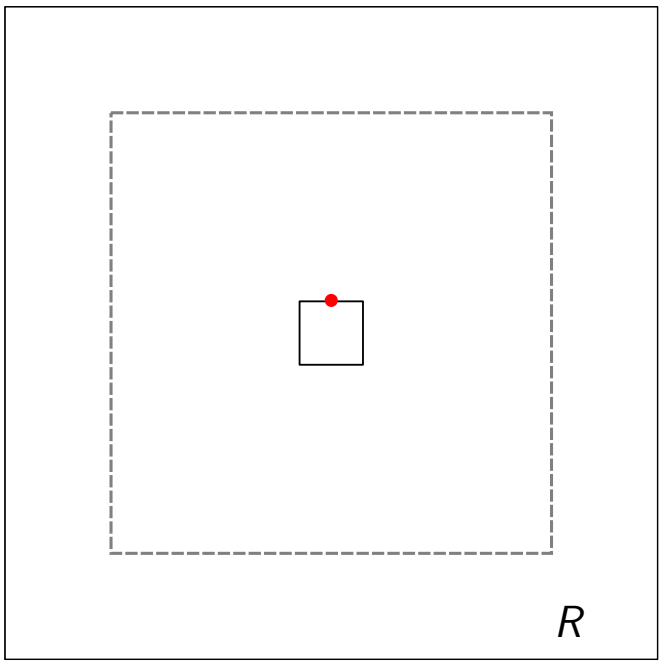
Non-simply connected domains?

Example with $\sigma = 0101$



Non-simply connected domains?

Example with $\sigma = 0101$



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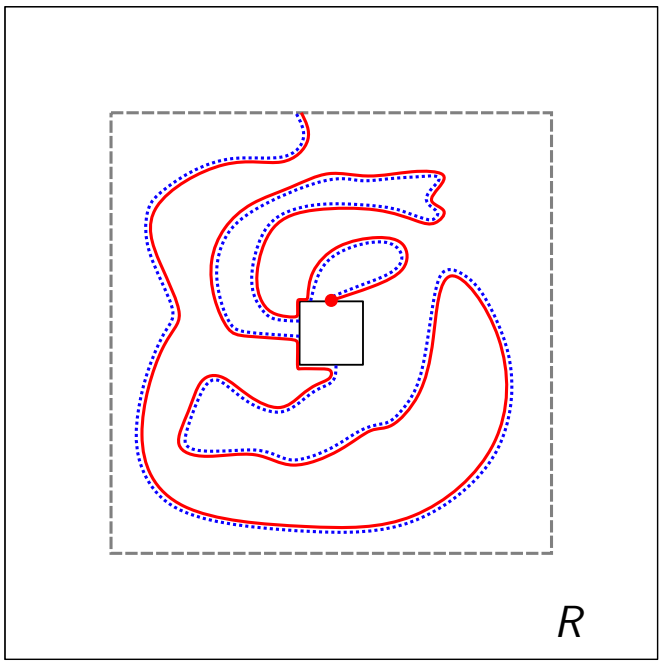

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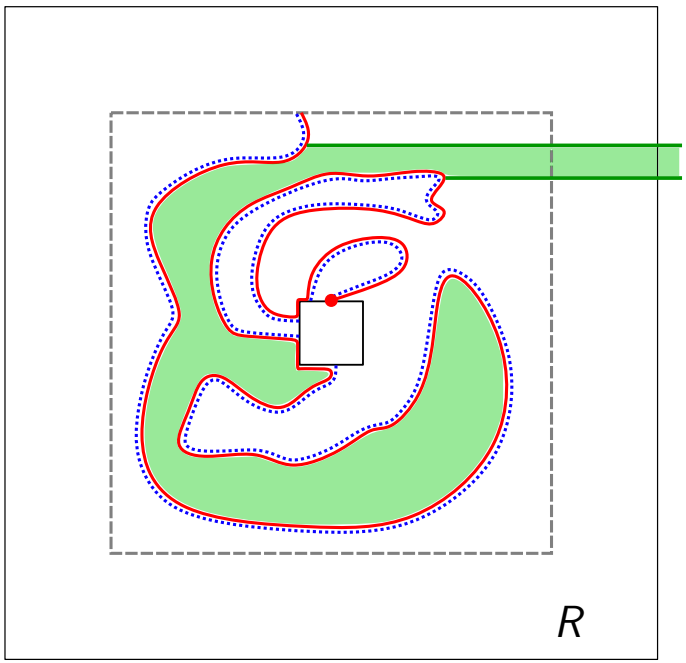

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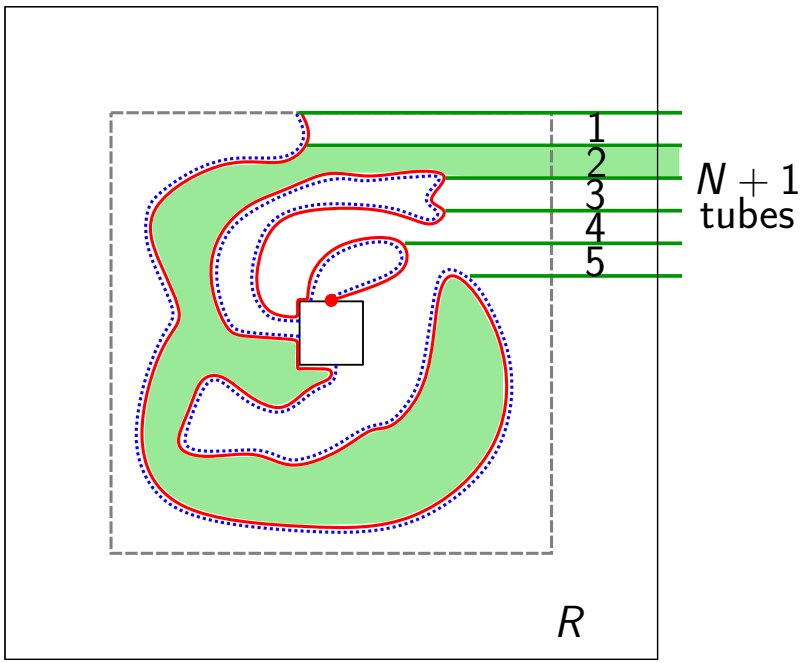

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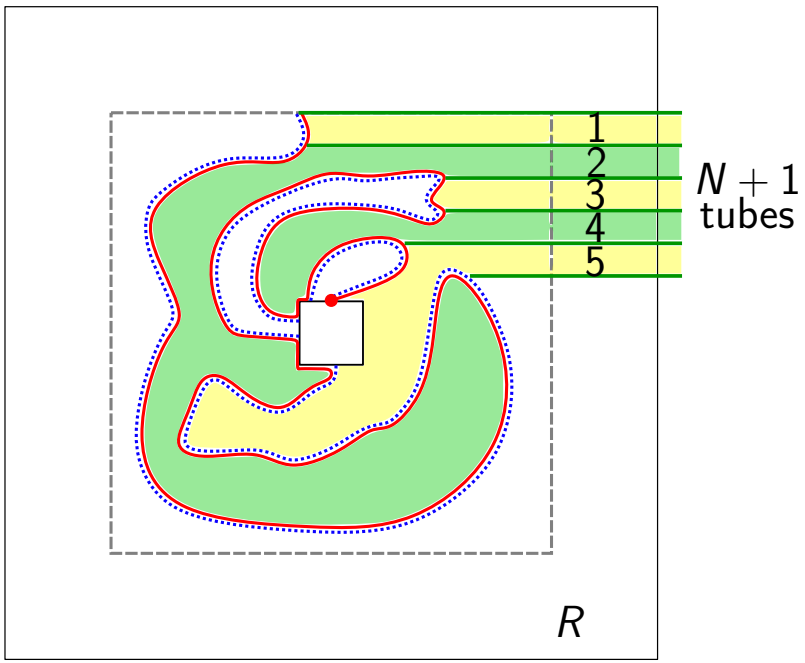

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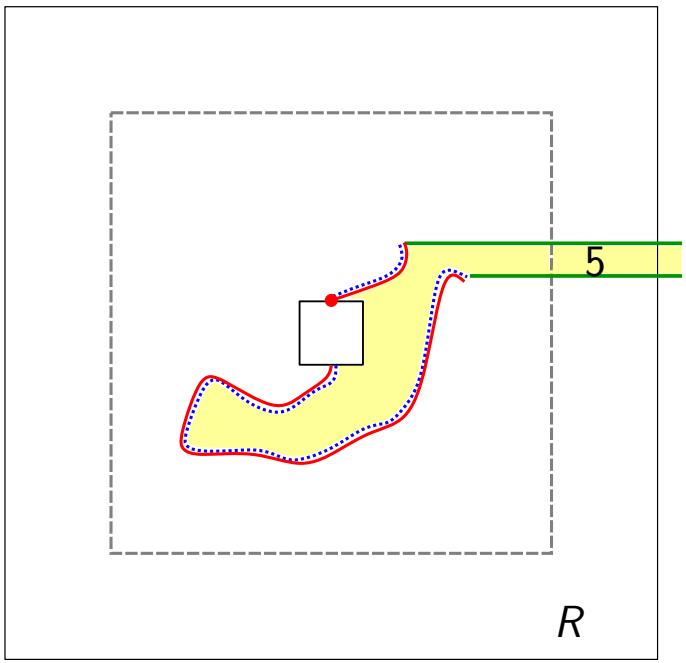
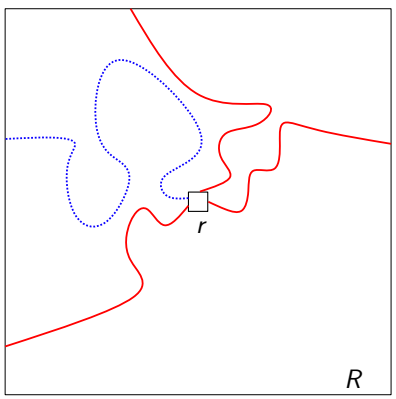
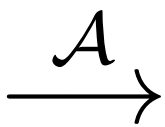


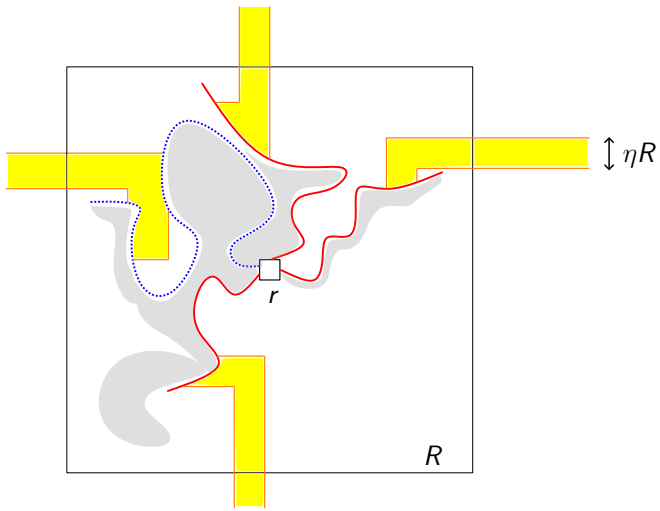
Arm exposition

Conclusion: For any percolation model $\mathbb{P}$ on $\mathbb{Z}^2$



$A_\sigma(r, R)$





Revealement of $\mathcal{A}$

Arm exposition

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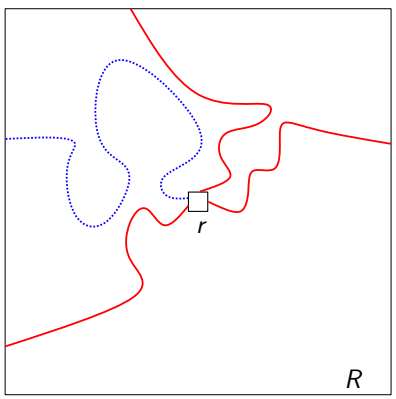

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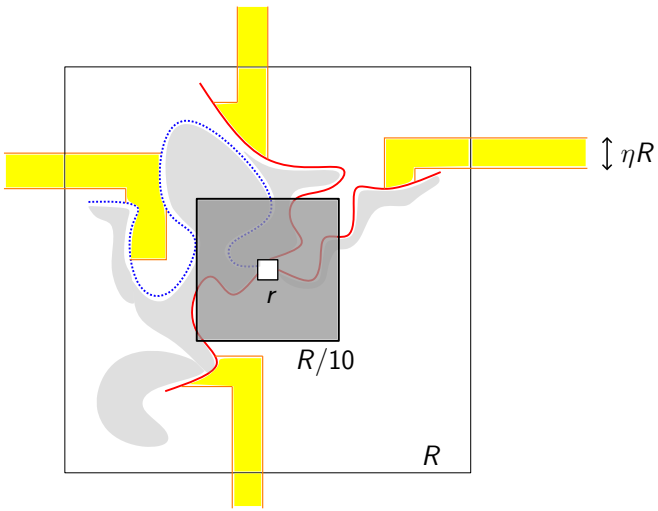


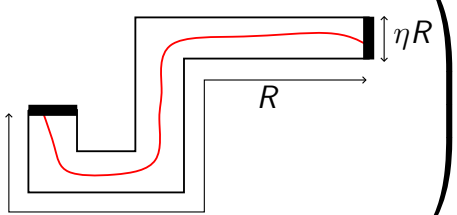
Revealement of $\mathcal{A}$

Arm separation for FK-percolation

Setup: Fix $q \in [1, 4]$. Take ϕ **critical** FK-percolation on $\mathbb{Z}^2$ ($p = p_c(q)$).

- Planarity
- FKG
- Box crossing estimates (RSW) / Spatial Markov Property

$$\forall R > 0, \quad \phi \left(\right) \geq C_\eta$$



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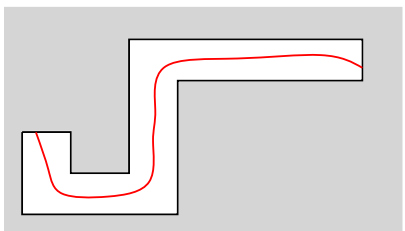

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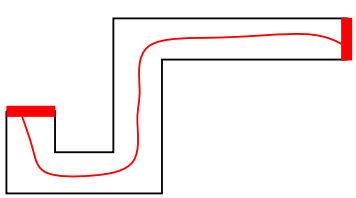

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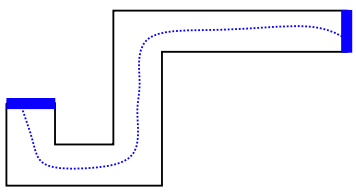
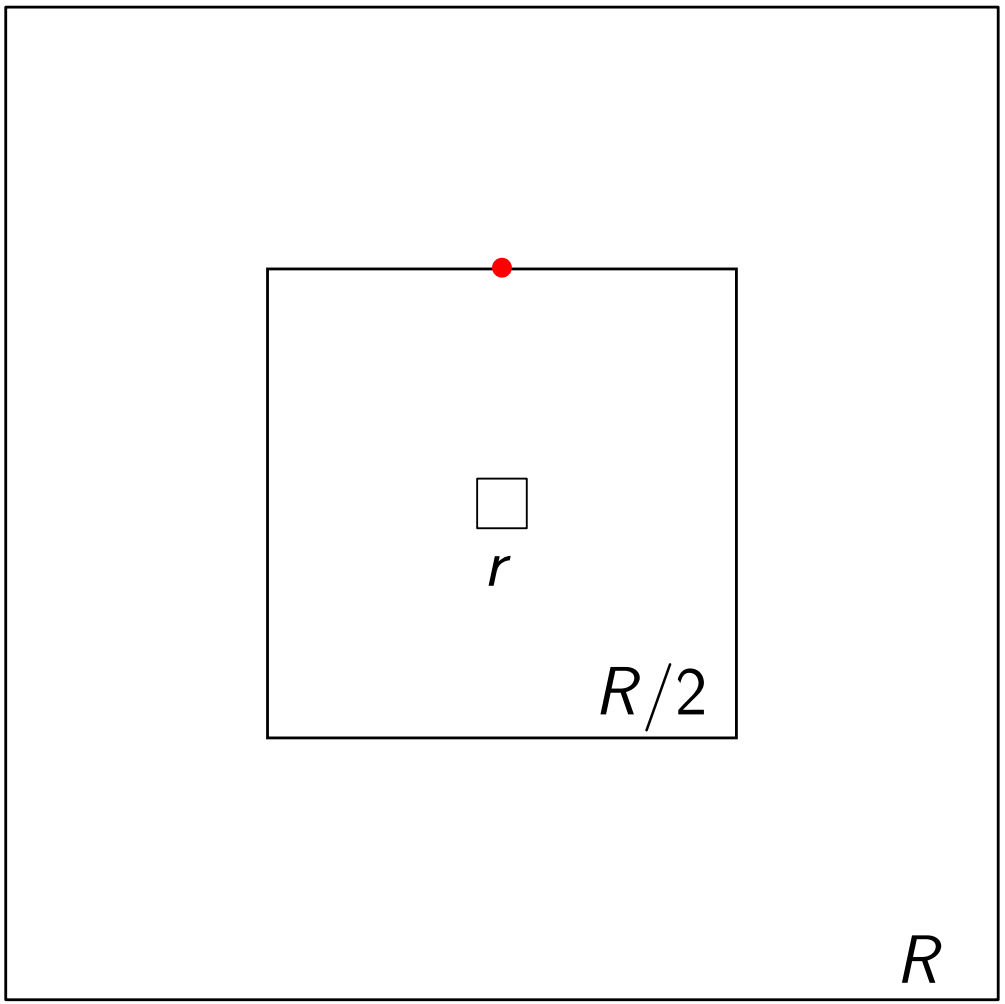


Flower domains



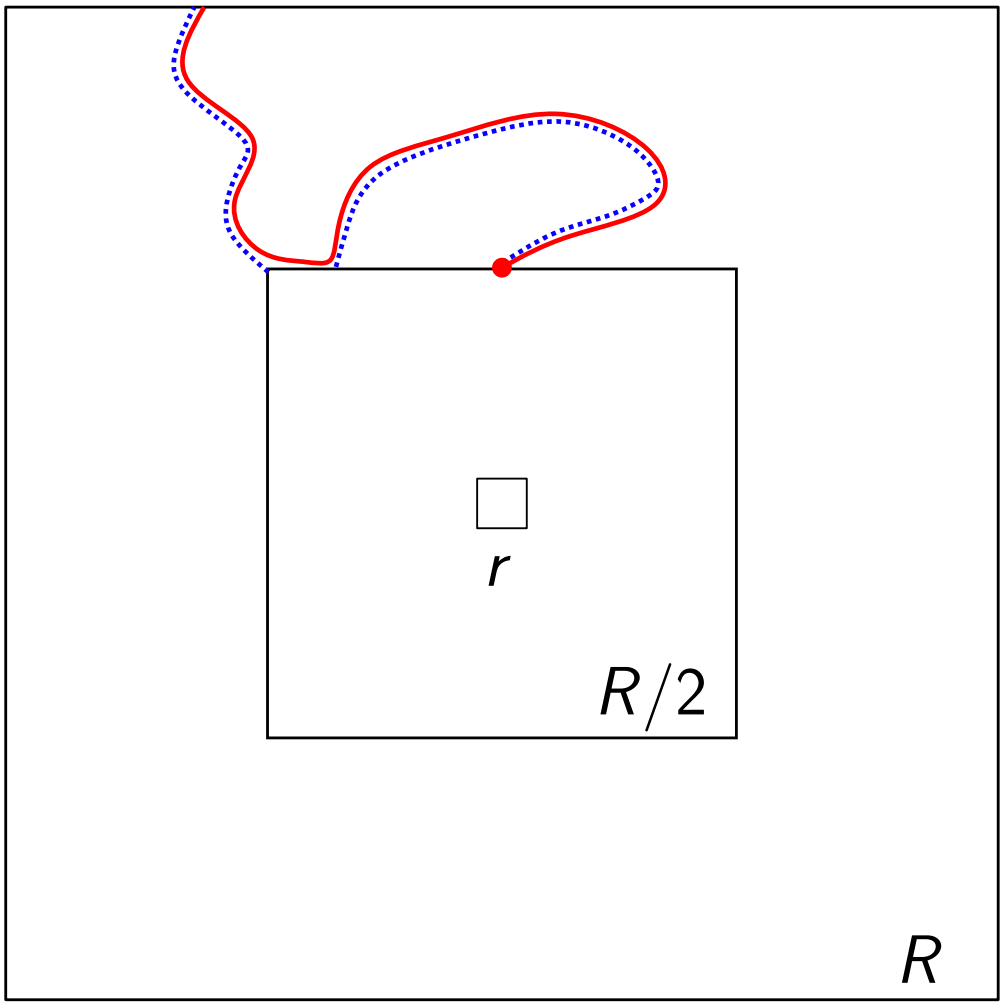
"Scaling relations"
Duminil-Copin,
Manolescu 2020

η-well separated

$\updownarrow$

Petals $\geq \eta R$

Flower domains



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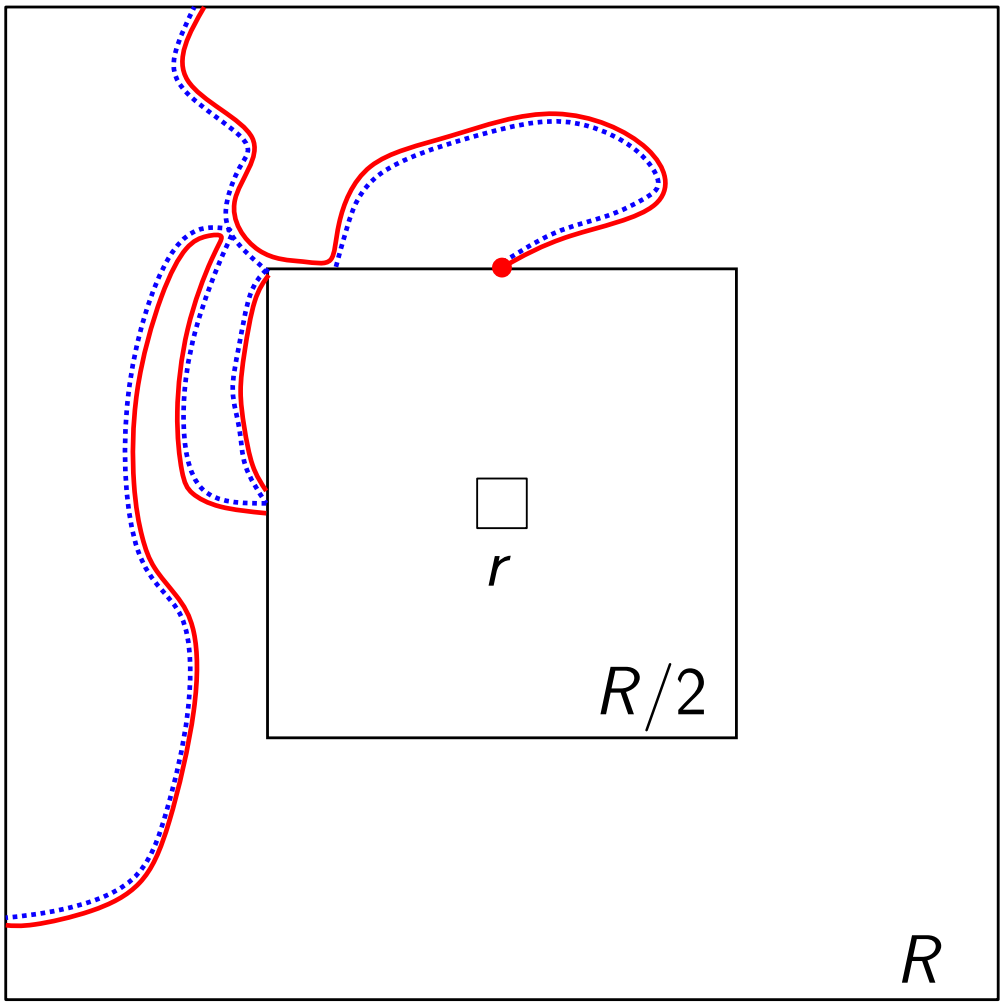

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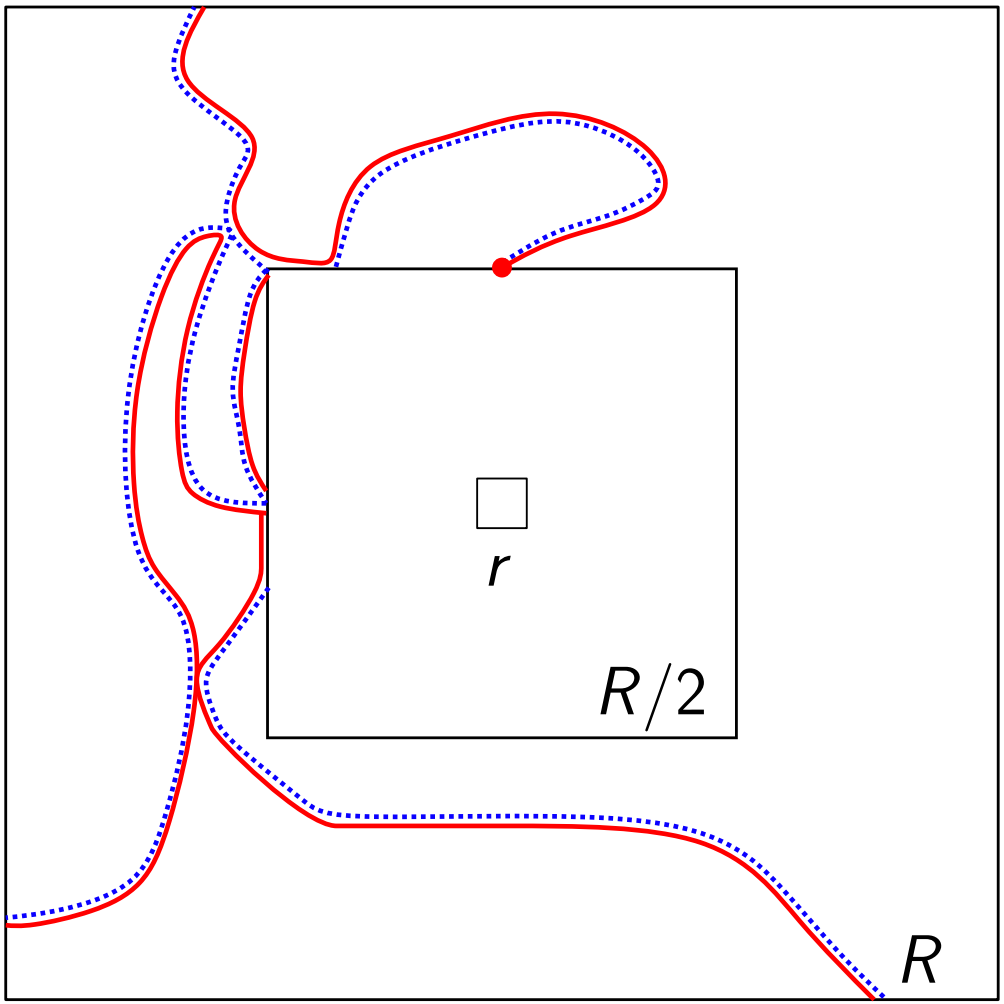

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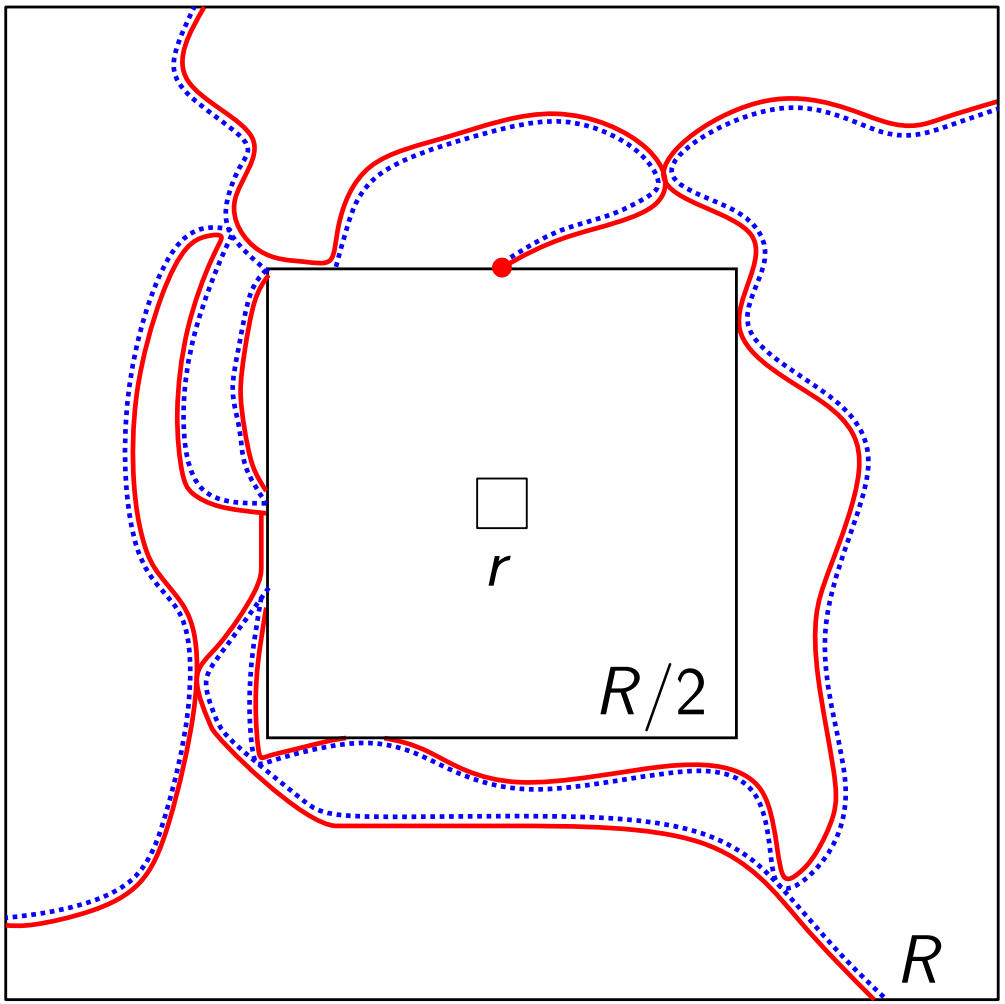

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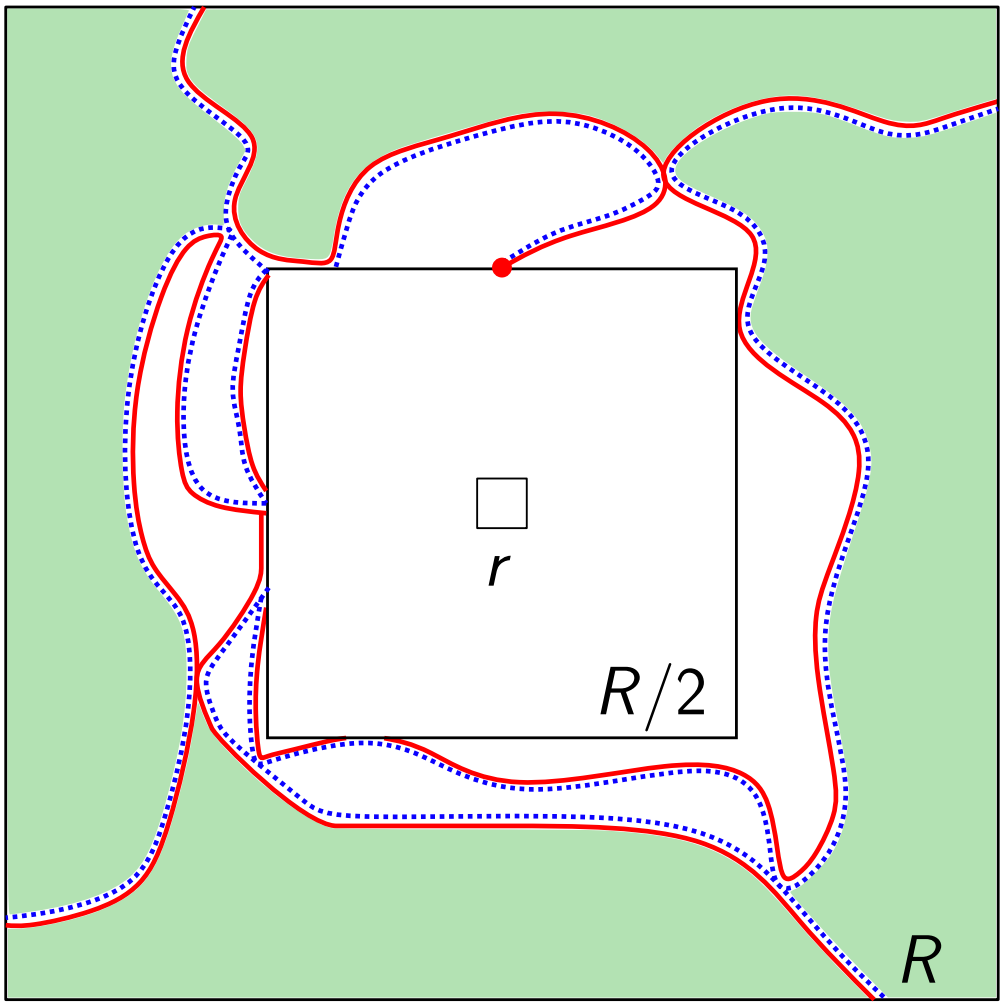

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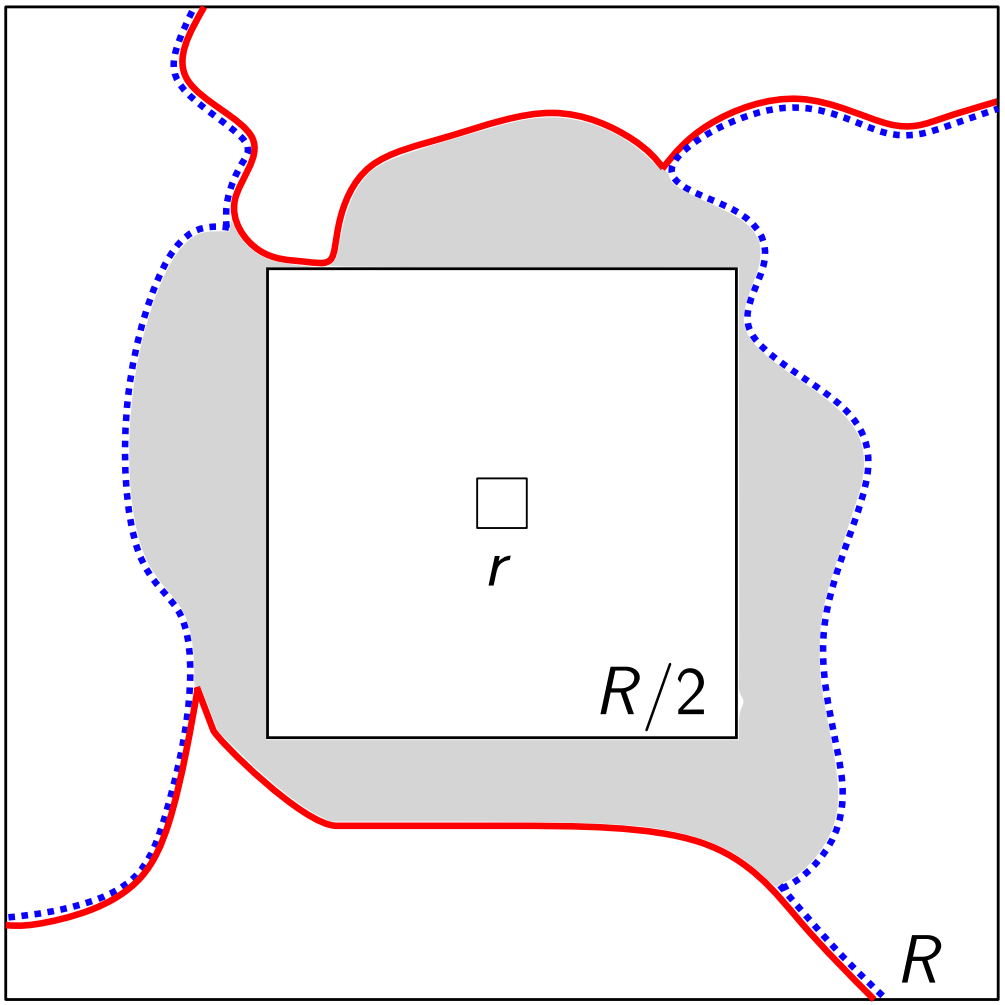

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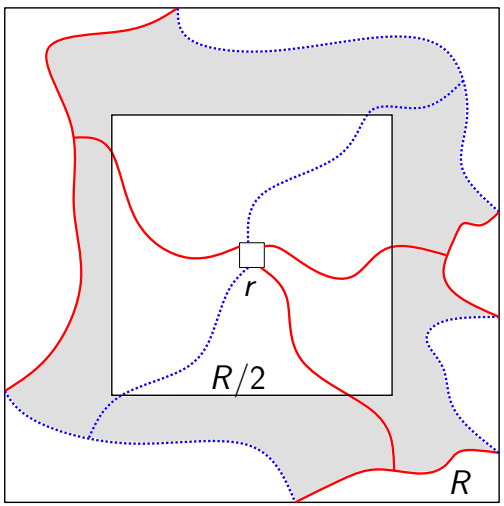
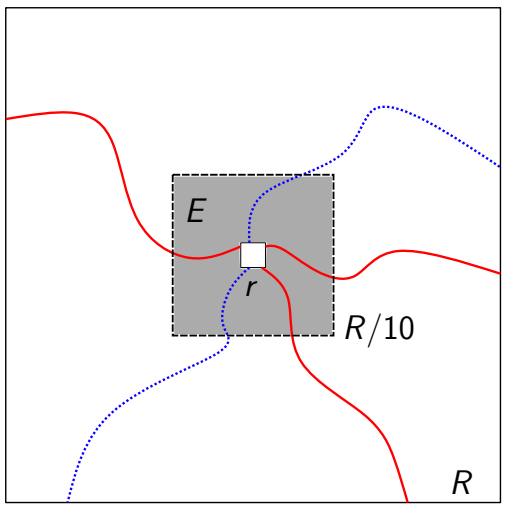
Arm separation for FK-percolation

Theorem (Manolescu, Mohanarangan, G, 26+)

Fix $q \in [1,4]$ and a color sequence $\sigma \in \{0,1\}^N$.
There exists $c > 0$, such that for any $R \geq 2r$

$$\phi[\text{"arms are } \eta\text{-well-separated"} \,|\, \omega = \omega_0 \text{ on } E,\ A_\sigma(r,R)] \geq c.$$

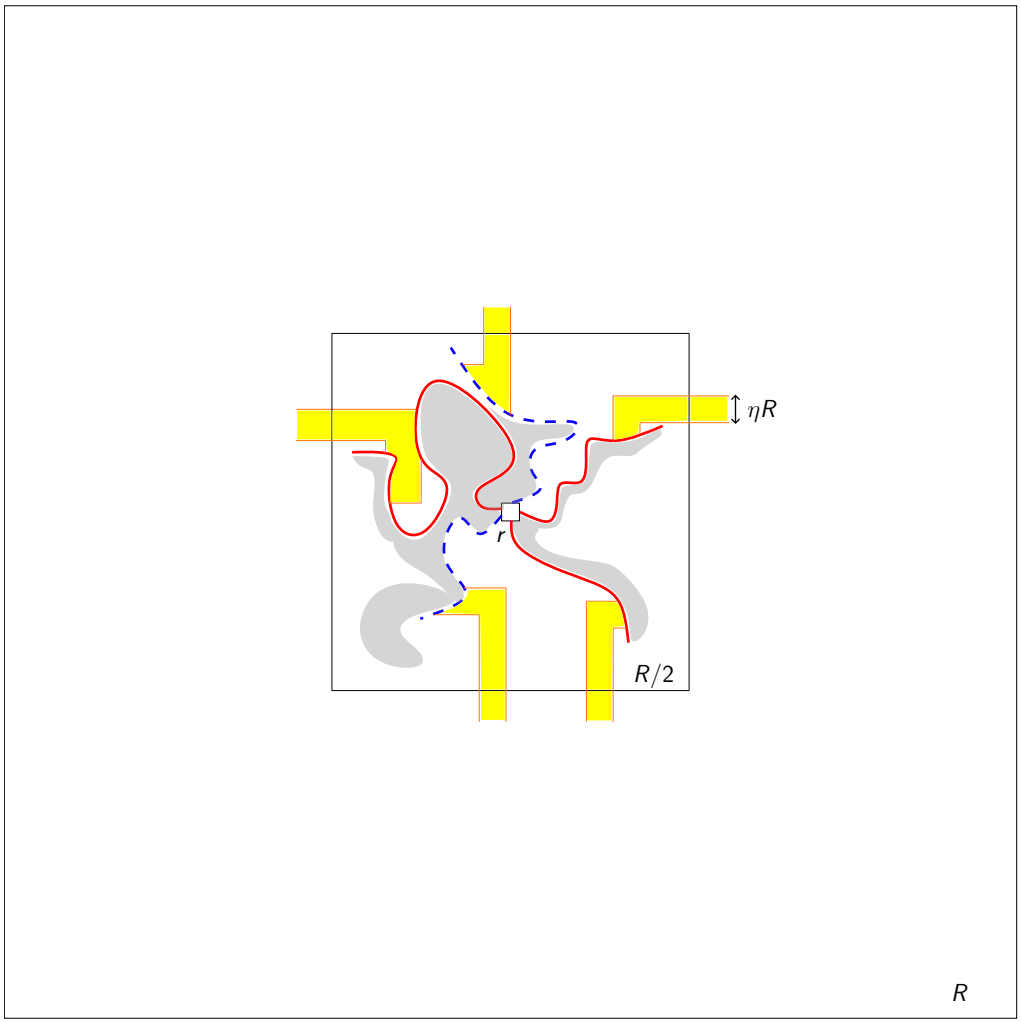




Previous results: Garban, Pete, Schramm 2014 (*Strong arm separation for interfaces*)

Arm separation for FK-percolation

Example with $\sigma = 10110$



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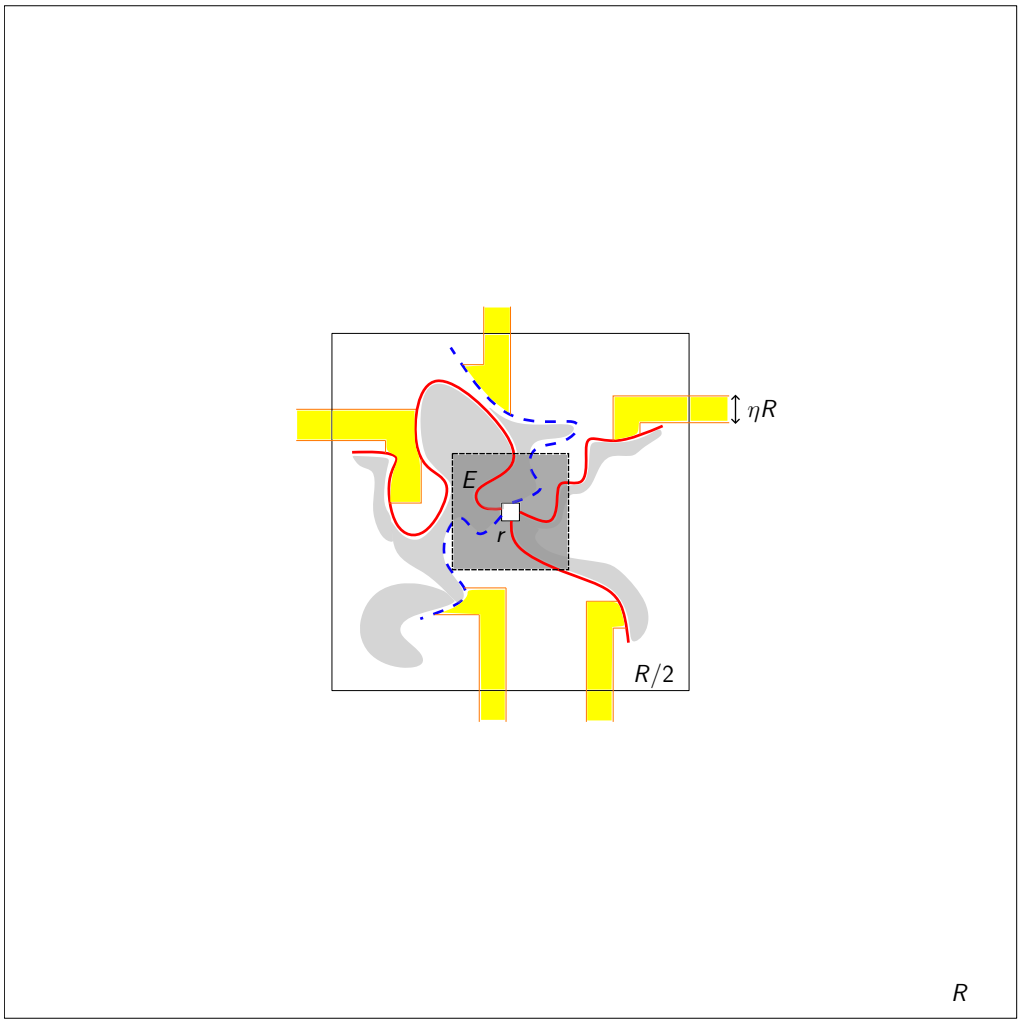

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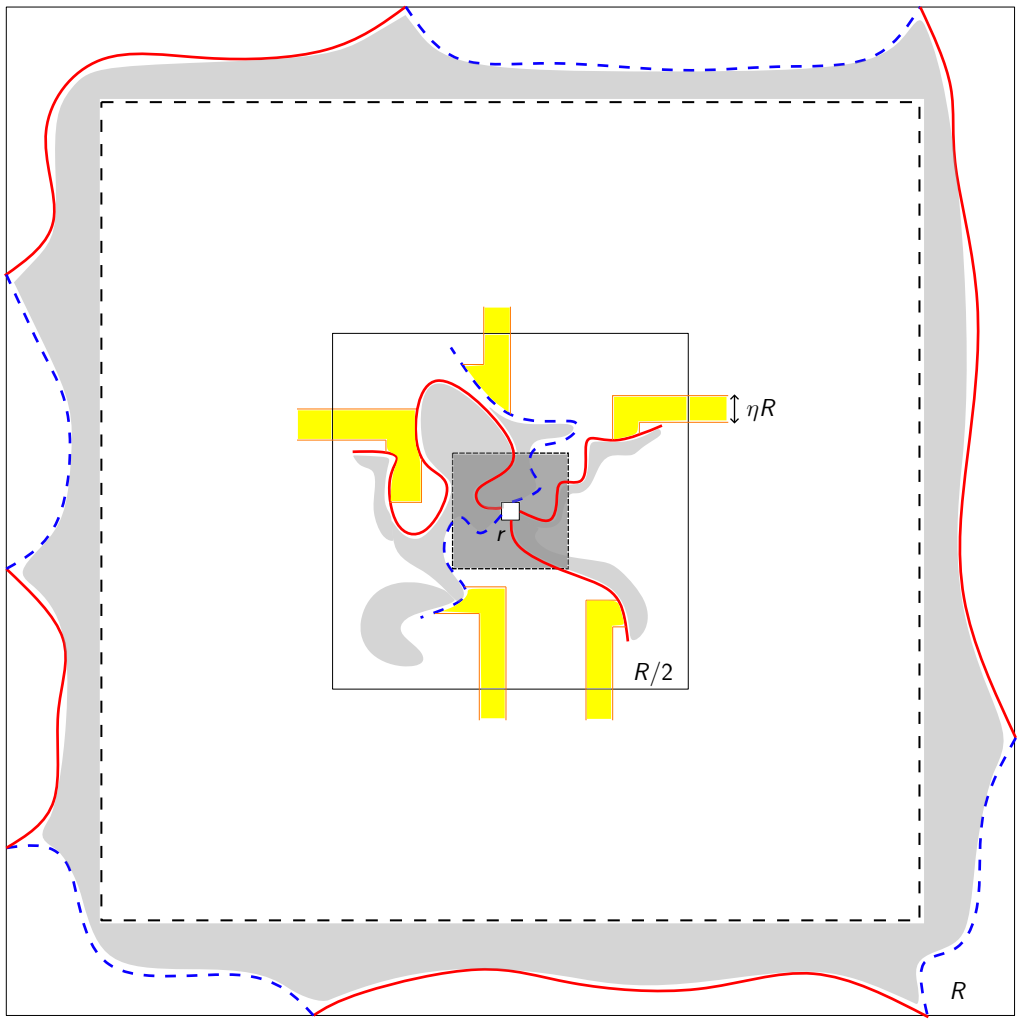

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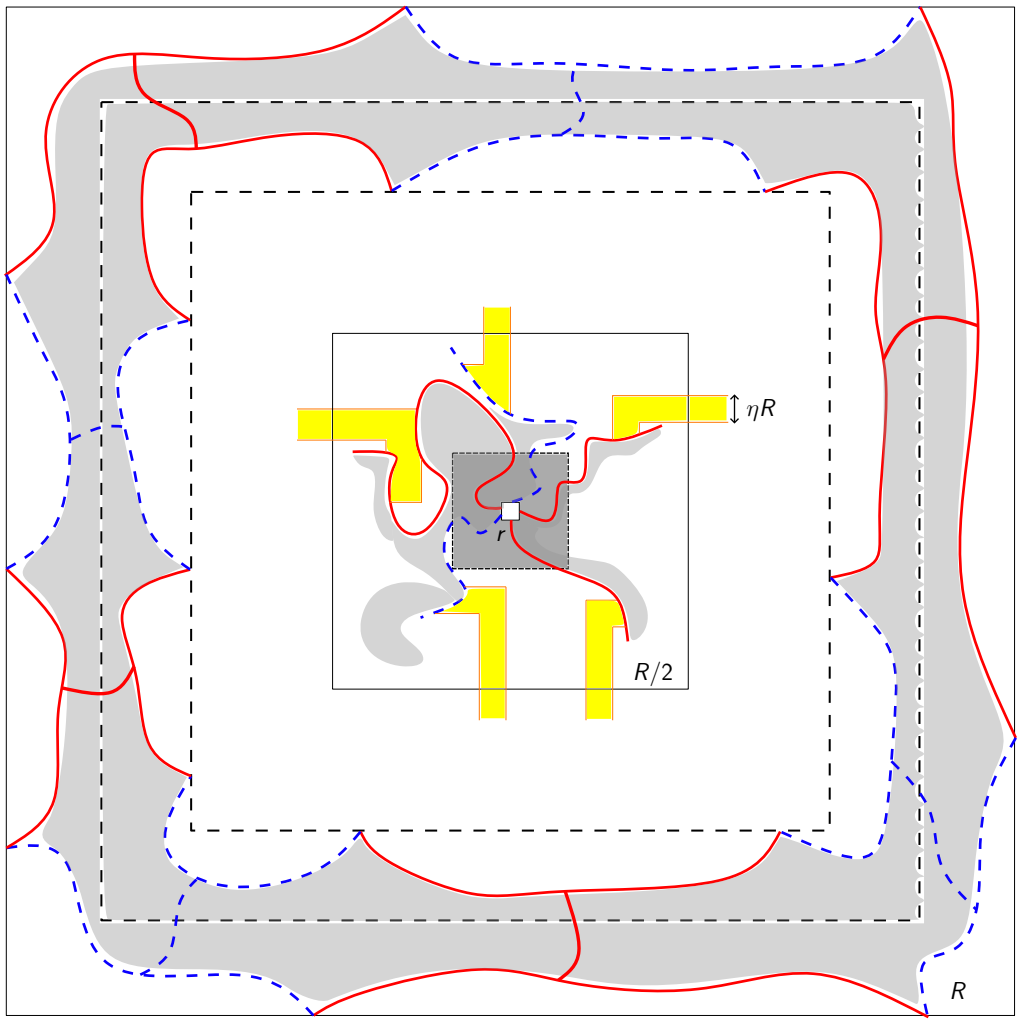

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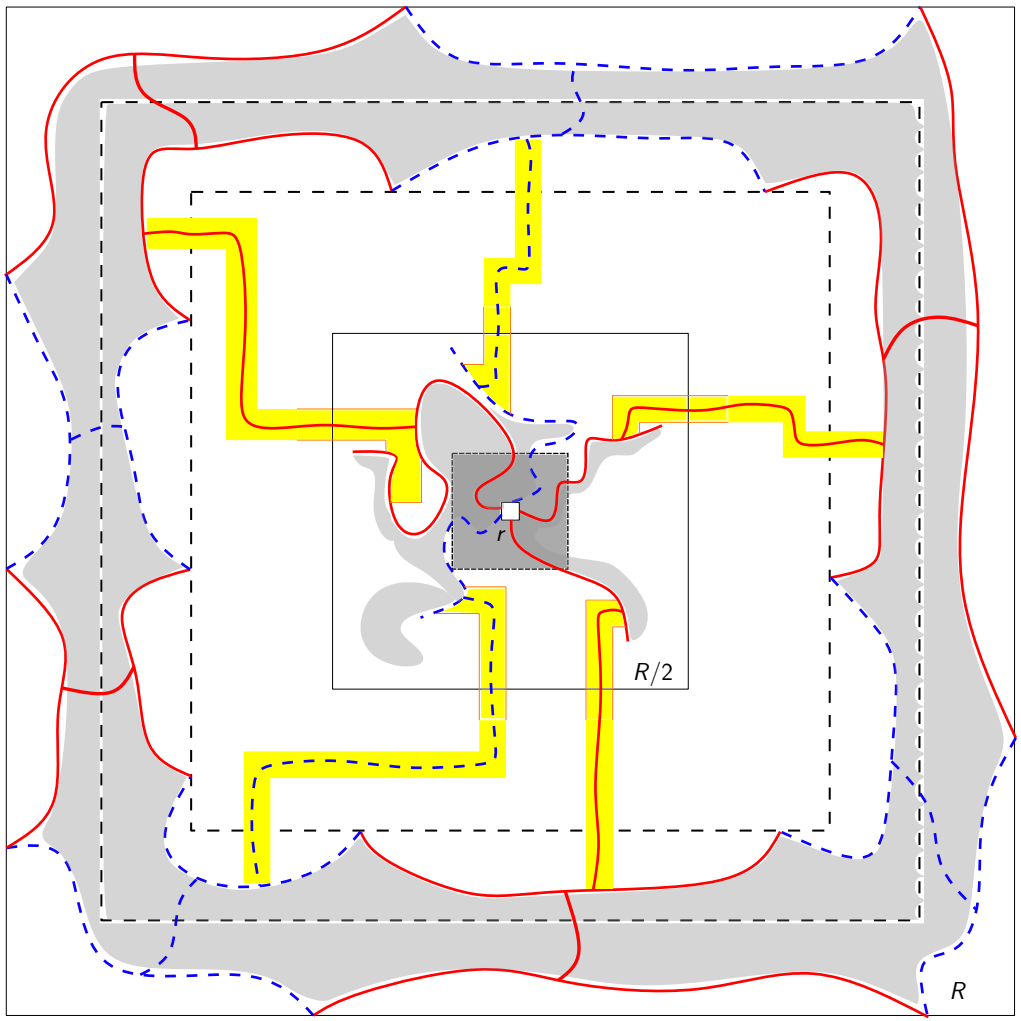

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