Ordering and Convergence of Large Degrees in Random Hyperbolic Graphs

Loïc Gassmann (Université de Fribourg, Suisse)

Project started with Pierre Calka (Université de Rouen, France)

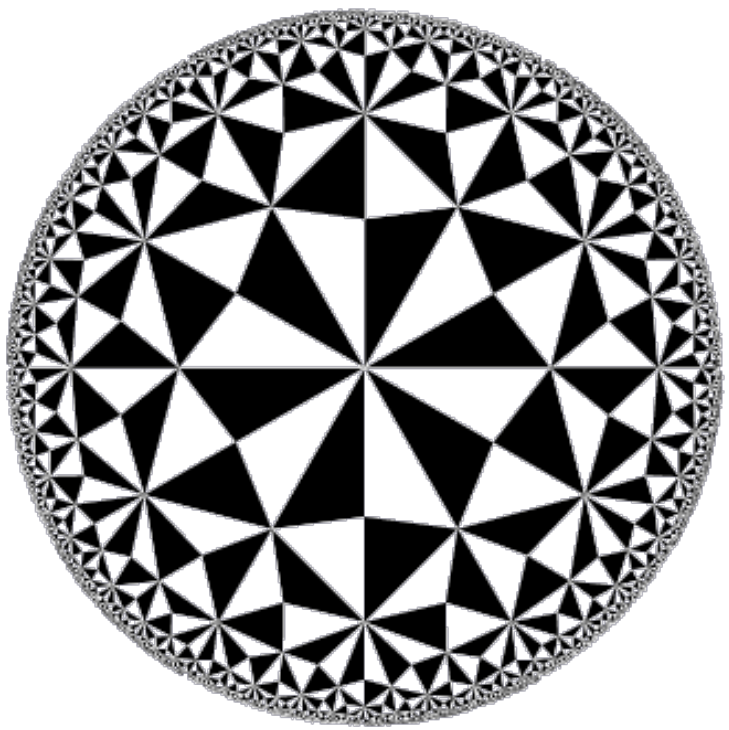
1. The Random Hyperbolic Graph model
2. Convergence of maximum degrees
3. Ordering of large degree nodes

Hyperbolic geometry

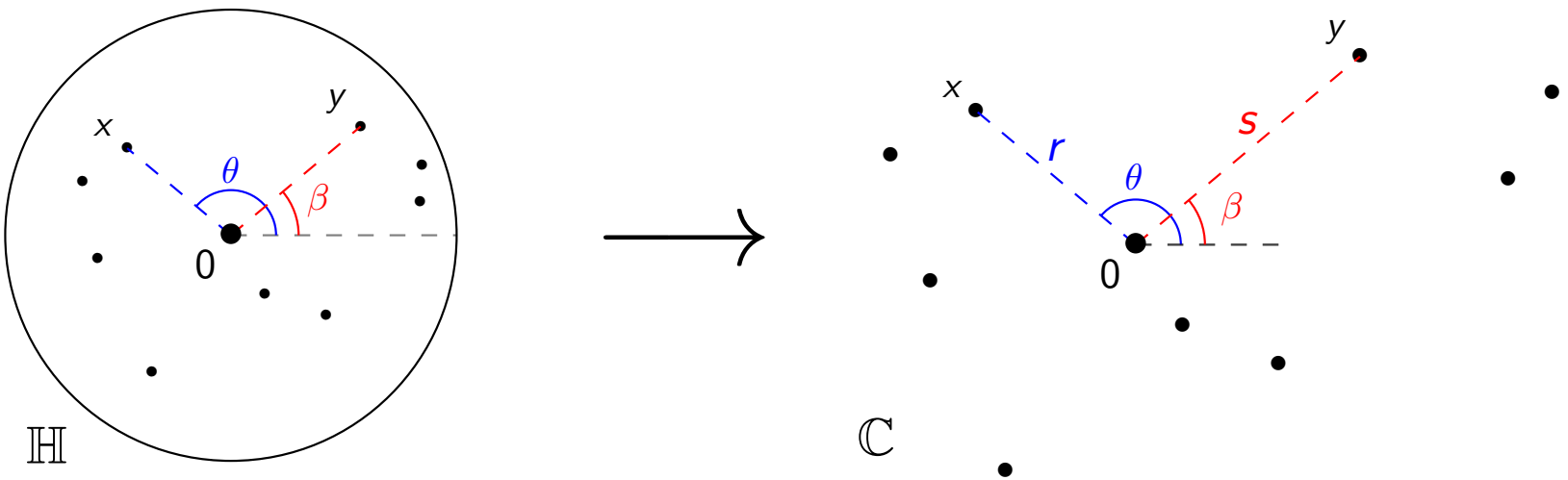
Poincaré disc $\mathbb{H}$: unit disc of $\mathbb{C}$ equipped with

$$\mathbf{g}_{\mathbb{H}} := \frac{4\mathbf{g}_{\mathbb{C}}}{(1-|w|^2)^2}, \quad \text{where } \mathbf{g}_{\mathbb{C}} \text{ is the Euclidean metric on } \mathbb{C}$$

$\rightarrow d_{\mathbb{H}}$ Riemannian distance



Coordinates and native representation



For $x = (r : \theta)$ and $y = (s : \beta)$ two points of $\mathbb{H}$

$$\cosh(d_{\mathbb{H}}(x, y)) = \cosh(r)\cosh(s) - \sinh(r)\sinh(s)\cos(\theta - \beta)\,.$$

$\mathcal{B}_x(s)$ open ball for distance $d_{\mathbb{H}}$

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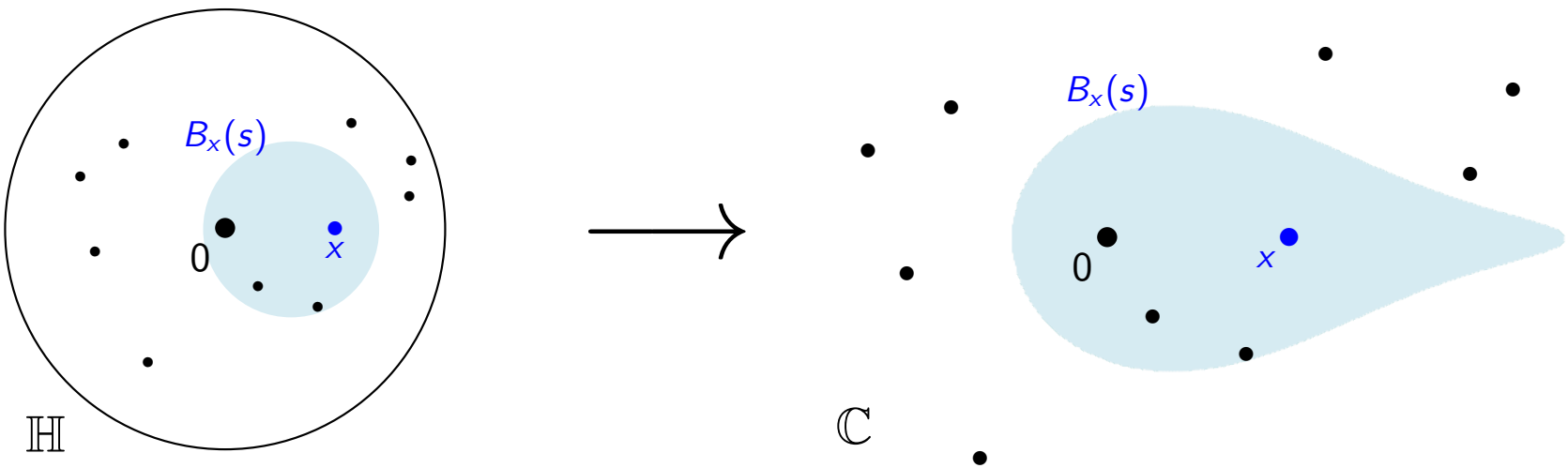

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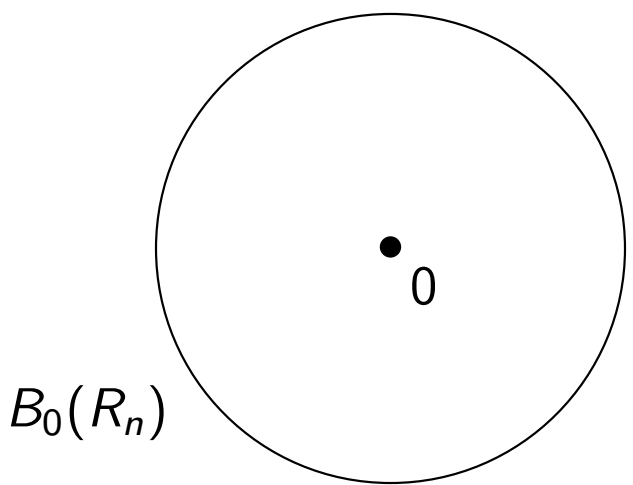

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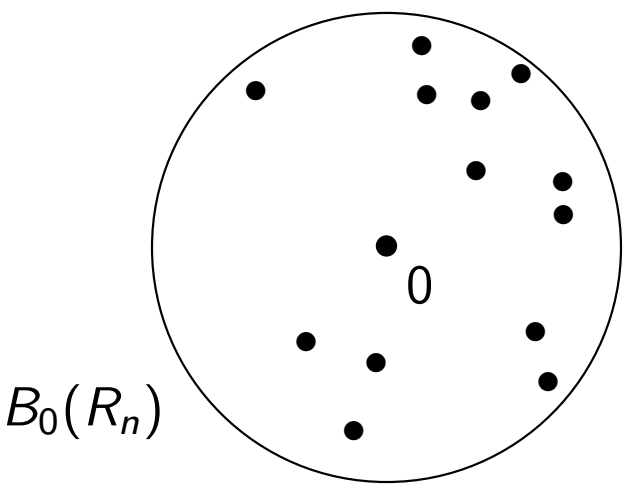
Measure μ_n on $\mathcal{B}_0(R_n)$ with radial density:

$$\rho_n(r) := \frac{\alpha \sinh(\alpha r)}{(\cosh(\alpha R_n) - 1)} \mathbf{1}_{\{r < R_n\}}$$

$(X_1, X_2, \ldots, X_n)$ i.i.d with distribution μ_n



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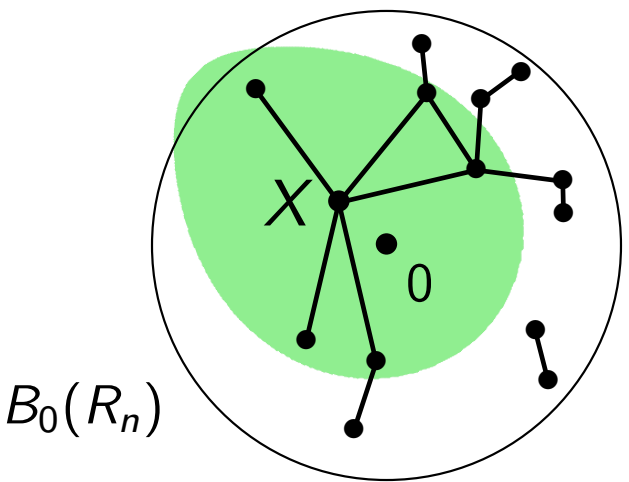

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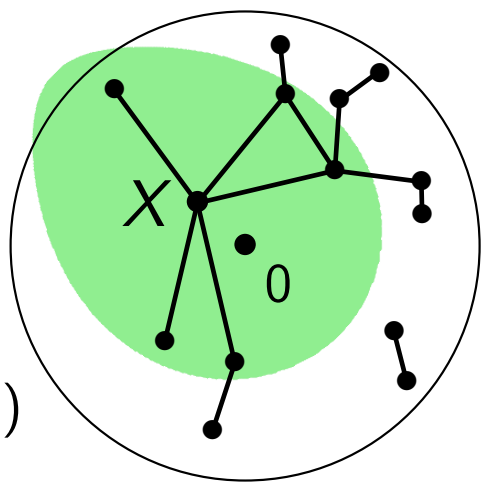
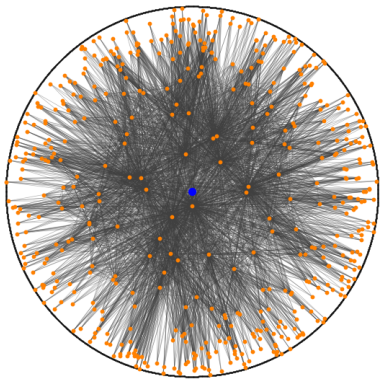


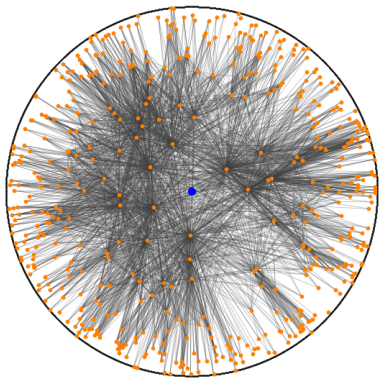
$\deg(X) =$ number of neighbours of X

$\mathbb{E}_X[\deg(X)] = (n-1)\mu_n(\mathcal{B}_X(R_n)) \searrow$ in $r(X)$

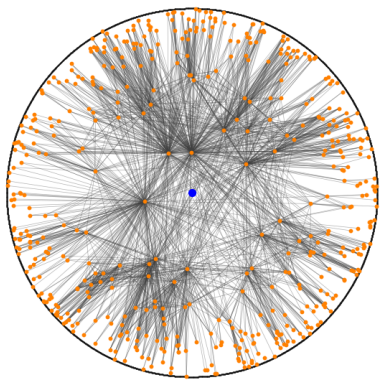
Three different regimes (see [BFM16])



$\alpha < 1/2$, dense regime, connected with high probability (hubs near the centre)



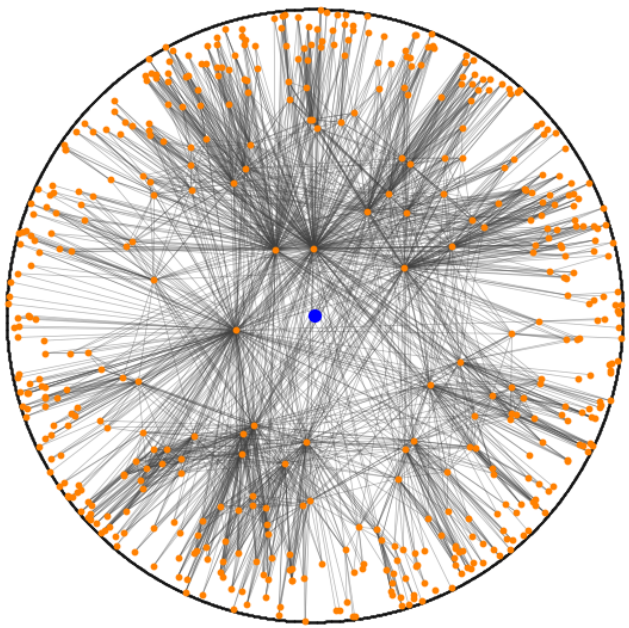
$\alpha = 1/2$, probability of connection converges to a constant



$\alpha > 1/2$, disconnected with high probability

For $\alpha > 1/2$, model for complex networks (Barabási and Albert) [AB02]:

- Sparseness [Pet14]
- Small-world [ABF17]
- High clustering [CF16, FvdHMS21, GPP12]
- **Scale-free degree distribution**
  - [GPP12] Random Hyperbolic Graphs: degree sequence and clustering



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Theorem: [GPP12] Power-law with exponent $2\alpha + 1$

For $\alpha > 1/2$ and $v_n \to \infty$, w.h.p., the maximum degree belongs to

$$[n^{1/(2\alpha)} v_n^{-1}, n^{1/(2\alpha)} v_n]$$

For $n^\delta \leq k \leq \frac{n^{1/(2\alpha)}}{\log(n)}$, w.h.p., the fraction of vertices of degree $\geq k$ is

$$(1 + o(1)) C_{\alpha,\nu} k^{-2\alpha}$$

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Better estimate on maximum/large degrees? For $\alpha \leq 1/2$?

Notations

Point process of the degrees:

$$\mathcal{D}_n := \sum_{i=1}^{n} \delta_{\deg(X_i)}$$

$r(X)$ radius of the node X

$X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ ordering by increasing radius

Ordering and convergence of maximum degrees

Theorem: [LG24] Maximum degree in RHGs

For $\alpha > 0$ and k fixed, with high probability,

$$\deg(X_{(1)}) > \deg(X_{(2)}) > \cdots > \deg(X_{(k)}) > \deg(X_{(i)}), \quad \forall i > k$$

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Moreover, for $\alpha < 1/2$,

$$\mathcal{D}_n(n - n^{\alpha+1/2} \cdot) \xrightarrow[n\to\infty]{(d)} \eta_{m_1}, \quad \text{in } M_p([0,\infty))$$

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$$\forall\, 0 \le a \le b < \infty,\ m_1([a,b]) = (b^2 - a^2)\pi^2\nu^{2\alpha}$$

$$\forall\, 0 \le a \le b \le 1,\ m_2([a,b]) = \nu(\cosh(V^{-1}(a)/2) - \cosh(V^{-1}(b)/2))$$

$$\forall\, 0 < a \le b \le \infty,\ m_3([a,b]) = (C_\alpha\nu)^{2\alpha}(a^{-2\alpha} - b^{-2\alpha})$$

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In particular, for $\alpha < 1/2$,

$$n^{-(\alpha+1/2)}(n - D_n^{\max}) \xrightarrow[n\to\infty]{(d)} \text{Weibull}\left(2, \pi^{-1}\nu^{-\alpha}\right)$$

For $\alpha = 1/2$,

$$n^{-1} D_n^{\max} \xrightarrow[n\to\infty]{(d)} V(2\,\text{arcosh}(\text{Exponential}\,(\nu) + 1))$$

For $\alpha > 1/2$,

$$n^{-\frac{1}{2\alpha}} D_n^{\max} \xrightarrow[n\to\infty]{(d)} \text{Fréchet}\,(2\alpha, C_\alpha \nu)$$

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[BS22] Large degrees in scale-free inhomogeneous random graphs

[GPP12] Random hyperbolic graphs: degree sequence and clustering

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Proof in the case $\alpha > 1/2$

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Point process of node radii, $\mathcal{R}_n := \sum_{i=1}^{n} \delta_{r(X_i)}$

$$\mathcal{R}_n((1 - \tfrac{1}{2\alpha})R_n + \ \cdot\ ) \xrightarrow[n\to\infty]{(d)} \eta_{m_6}, \quad \text{in } M_p(\mathbb{R})$$

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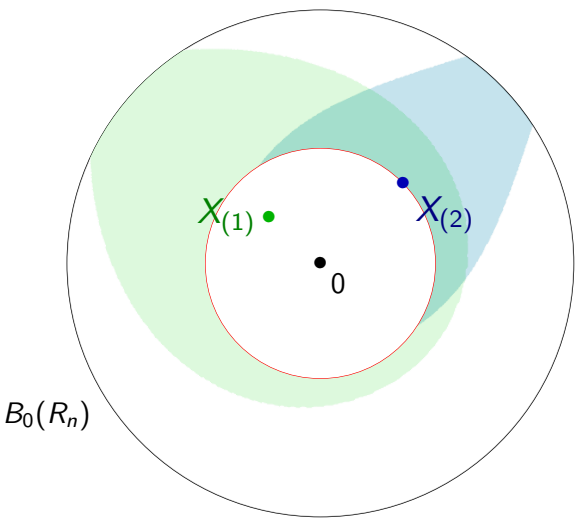

Estimate of [GPP12]

$$\mu_n\left(\mathcal{B}_x\left(R_n\right)\right) = C_\alpha e^{-r(x)/2}\left(1 + O(e^{-(\alpha-1/2)r(x)} + e^{-r(x)})\right)$$

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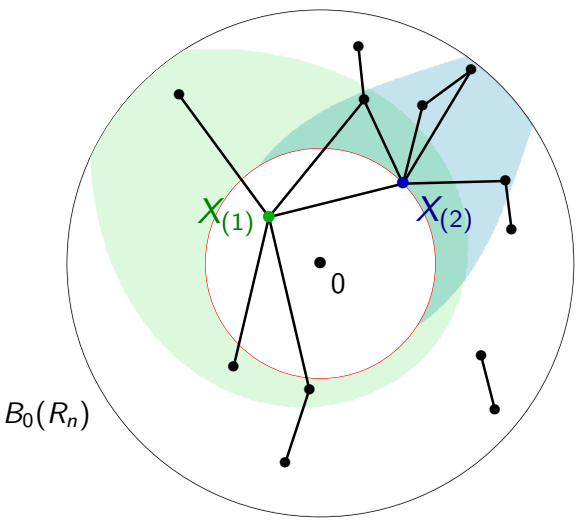

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By Chernoff bounds

$$\mathbb{P}\left[\deg(X_{(1)}) \leq \deg(X_{(2)})\right] \leq \exp\left(-n^{1/(2\alpha)+o(1)}\right)$$

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For $\alpha > 1/2$ and k fixed, with high probability,

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Moreover

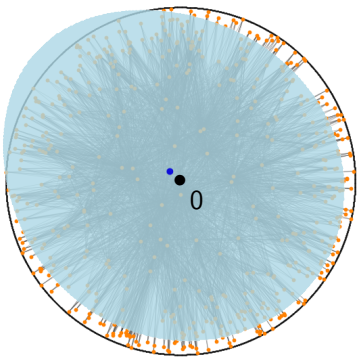
$$\mathcal{D}_n(n^{1/(2\alpha)} \cdot) \xrightarrow[n\to\infty]{(d)} \eta_{m_3}, \quad \text{in } M_p((0,\infty])$$

$(\deg(X_{(1)}), \ldots, \deg(X_{(k)}))$ concentrate on there conditional expectations

$$\mathbb{E}_{X_{(i)}}[\deg(X_{(i)})] \sim n\mu_n(\mathcal{B}_{X_{(i)}}(R_n)) \sim C_\alpha n e^{-r(X_{(i)})/2}$$

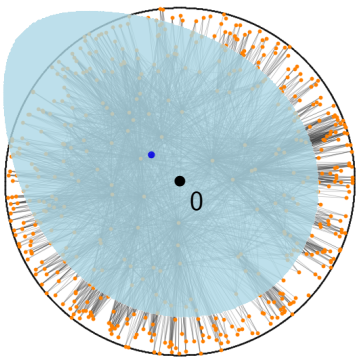
$$\mathcal{R}_n((1-\tfrac{1}{2\alpha})R_n + \cdot) \xrightarrow[n\to\infty]{(d)} \eta_{m_6} \quad \rightsquigarrow \quad \mathcal{D}_n(n^{1/(2\alpha)} \cdot) \xrightarrow[n\to\infty]{(d)} \eta_{m_3}$$

What about the other regimes in α?



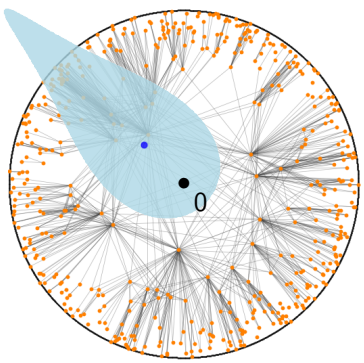
For $\alpha < 1/2$,

$$\mathcal{R}_n(n^{-\frac{1}{2}(1-2\alpha)} \cdot) \xrightarrow[n\to\infty]{(d)} \eta_{m_4}, \text{ in } M_p([0,\infty))$$



For $\alpha = 1/2$,

$$\mathcal{R}_n(\cdot) \xrightarrow[n\to\infty]{(d)} \eta_{m_5}, \text{ in } M_p([0,\infty))$$



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Ordering of the nodes?

$$\deg(X_{(1)}) > \deg(X_{(2)}) > \cdots > \deg(X_{(k)}) > \deg(X_{(i)}), \quad \forall i > k$$



Ordering up to a polynomial rank (scale free regime)

Let us fix $\alpha > 1/2$ and $v_n \to \infty$. Define

$$\beta_c := \frac{1}{1+8\alpha} \quad \text{and} \quad k_n := n^{\beta_c} / \log(n)^{2\alpha}$$

Theorem: [LG24+]

For $\alpha > 1/2$, with high probability,

$$\deg(X_{(1)}) > \deg(X_{(2)}) > \cdots > \deg(X_{(k_n)}) > \deg(X_{(i)}),\ \forall i > k_n$$

and there exists $i \in [n^{\beta_c}, n^{\beta_c} v_n]$ such that

$$\deg(X_{(i)}) < \deg(X_{(i+1)})$$

Proof sketch: ordering up to k_n

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Proof sketch: ordering up to k_n

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$W_k := n\mu_n\left(\mathcal{B}_{X_{(k)}}(R_n)\right) \sim C_\alpha n e^{-r(X_{(k)})/2}$

$\Delta_k := r(X_{(k+1)}) - r(X_{(k)})$

Proof sketch: ordering up to k_n

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Using Chernoff bounds, we get

$$\begin{aligned}\mathbb{P}_{X_{(k)},X_{(k+1)}}\left[\deg(X_{(k)}) \leq \deg(X_{(k+1)})\right] &\leq \exp\left(-C_1 W_k \left(1 - \tfrac{W_{k+1}}{W_k}\right)^2\right)\\ &\leq \exp\left(-C_2 W_k \Delta_k^2\right)\end{aligned}$$

Proof sketch: ordering up to k_n

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$\rightsquigarrow$ choose k_n such that w.h.p., for all $k \leq k_n$,

$$\exp\left(-C_2 W_k \Delta_k^2\right) = o(1/n)$$

By a union bound,

$$\mathbb{P}\left[\deg(X_{(1)}) > \deg(X_{(2)}) > \cdots > \deg(X_{(k_n)})\right] = 1 - o(k_n/n) - o(1)$$

Proof sketch: no ordering after n^{β_c}

Theorem: [LG24+]

Fix $v_n \to \infty$. W.h.p., $\exists k \in [n^{\beta_c}, n^{\beta_c} v_n]$, $\deg(X_{(k)}) < \deg(X_{(k+1)})$

Reminder: $W_k := n\mu_n\left(\mathcal{B}_{X_{(k)}}(R_n)\right)$, $\quad \Delta_k := r(X_{(k+1)}) - r(X_{(k)})$

Proof sketch: no ordering after n^{β_c}

Theorem: [LG24+]

Fix $v_n \to \infty$. W.h.p., $\exists k \in [n^{\beta_c}, n^{\beta_c} v_n]$, $\deg(X_{(k)}) < \deg(X_{(k+1)})$

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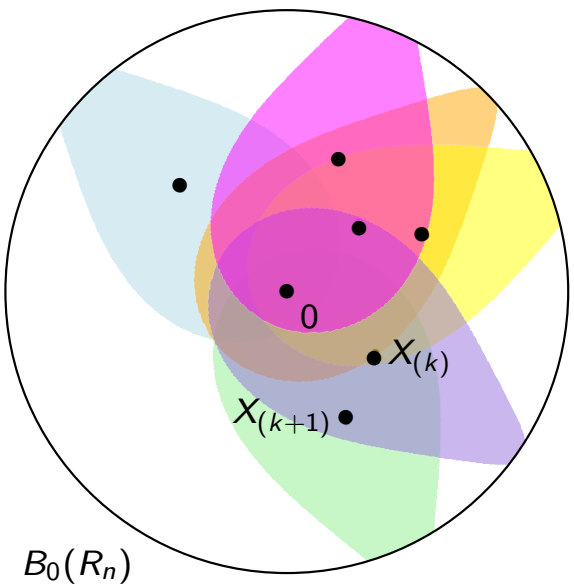

W.h.p, more than $C_1 v_n$ indices $k \in [n^{\beta_c}, n^{\beta_c} v_n]$ satisfy

$$W_k \Delta_k^2 \leq \delta_1.$$

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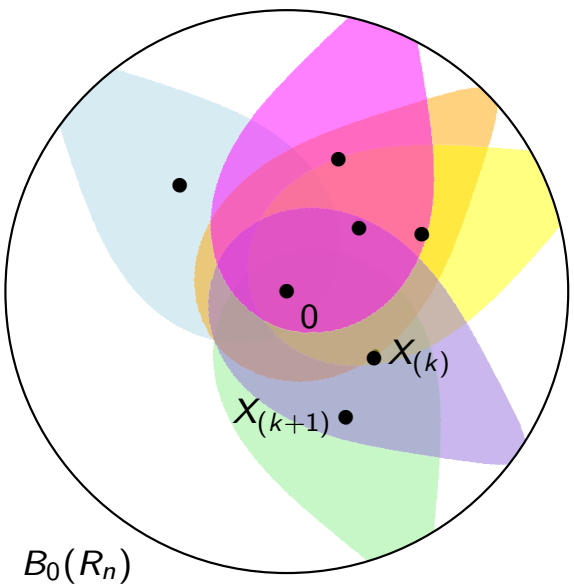

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$$W_k \Delta_k^2 \leq \delta_1.$$

This can be rewritten

$$W_k - W_{k+1} \leq \delta_2 \operatorname{Var}_{X_{(k+1)}}(\deg(X_{(k+1)})),$$

which implies

$$\mathbb{P}_{X_{(k)}, X_{(k+1)}}\left[\deg(X_{(k+1)}) > W_k\right] \geq \delta_3.$$

Proof sketch: no ordering after n^{β_c}

Theorem: [LG24+]

Fix $v_n \to \infty$. W.h.p., $\exists k \in [n^{\beta_c}, n^{\beta_c} v_n]$, $\deg(X_{(k)}) < \deg(X_{(k+1)})$

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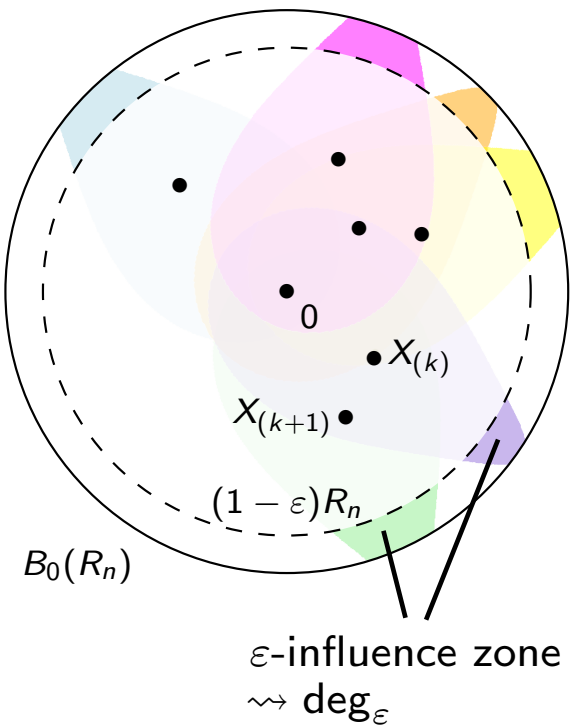

W.h.p, we can find $C_2 v_n$ indices $k \in [n^{\beta_c}, n^{\beta_c} v_n]$ s.t.

$$\mathbb{P}_{X_{(k)}, X_{(k+1)}}\left[\deg_\varepsilon(X_{(k+1)}) > W_k^\varepsilon\right] \geq \delta$$

+ disjoint condition after $(1-\varepsilon)R_n$ (for all points)

Proof sketch: no ordering after n^{β_c}

Theorem: [LG24+]

Fix $v_n \to \infty$. W.h.p., $\exists k \in [n^{\beta_c}, n^{\beta_c} v_n]$, $\deg(X_{(k)}) < \deg(X_{(k+1)})$

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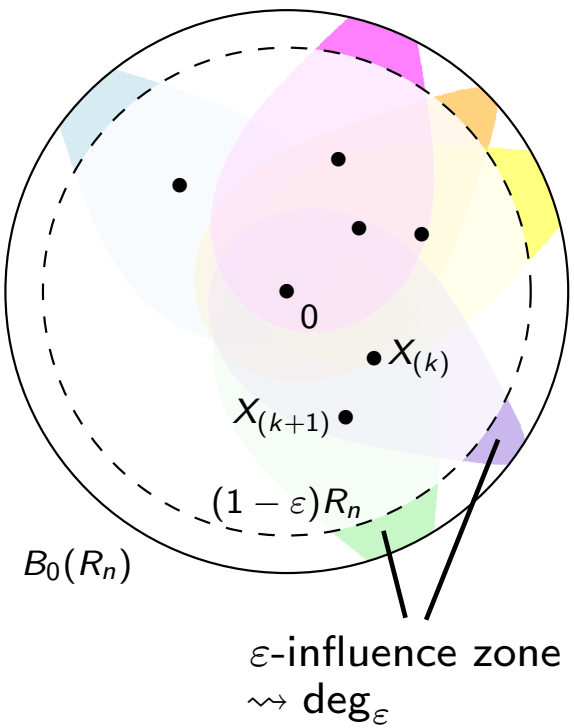

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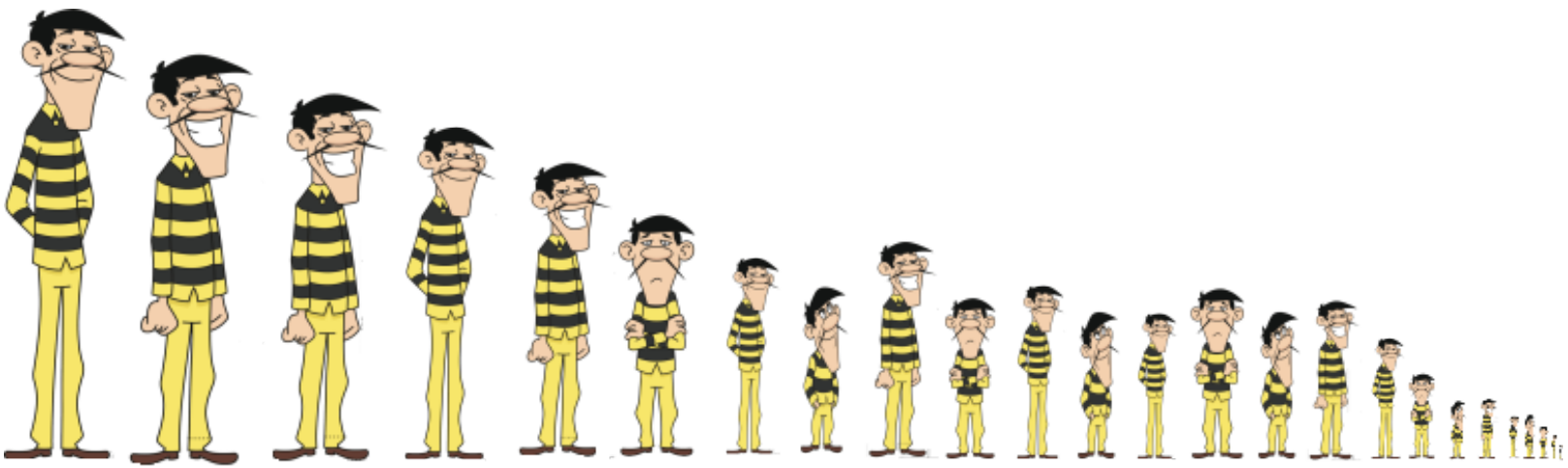
$\rightsquigarrow$ W.h.p, there exists $k \in [n^{\beta_c}, n^{\beta_c} v_n]$, such that

$$\deg_\varepsilon(X_{(k)}) < \deg_\varepsilon(X_{(k+1)})$$

Thank you for your attention!



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