Part 2: Strict inequalities of arm exponents in FK-percolation

Loïc Gassmann (Fribourg)

Joint work with Ioan Manolescu

based on arXiv:2410.23013

Two strict inequalities for critical planar FK-percolation

Goal: State the following inequalities (for $1 \leq q \leq 4$)

Inequality 1: Alternated 2-arms V.S. 1-arm

$$\phi_{p_c}[A_R(01)] \leq R^{-c}\, \phi_{p_c}[A_R(0)]\phi_{p_c}[A_R(1)]$$

Inequality 2: Alternated 4-arms V.S. mixing rate (for $q > 1$)

$$\phi_{p_c}[A_{r,R}(0101)] \leq C(R/r)^{-c}\Delta(r,R)$$

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Inequality 1: Alternated 2-arms V.S. 1-arm

$$\alpha_{01} > \alpha_0 + \alpha_1$$

Inequality 2: Alternated 4-arms V.S. mixing rate (for $q > 1$)

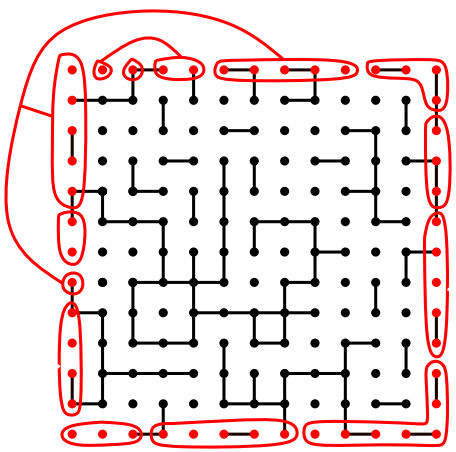
$$\alpha_{0101} > \iota$$

Notation: FK-percolation

FK-percolation on $\mathbb{Z}^2$, parameters $p \in (0,1)$ and $q \geq 1$

For G a finite subgraph of $\mathbb{Z}^2$ and ξ a partition of ∂G

$$\phi^{\xi}_{G,p,q}[\omega] := \frac{p^{o(\omega)}(1-p)^{c(\omega)}q^{k(\omega^{\xi})}}{Z^{\xi}(G,p,q)}$$



$o(\omega)$ = number of open edges
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$k(\omega^{\xi})$ = number of connected components of ω^{ξ}

Monotonicity in boundary conditions:

$$\forall \xi \leq \xi', \quad \phi^{\xi}_{G,p,q} \leq_{st} \phi^{\xi'}_{G,p,q}$$

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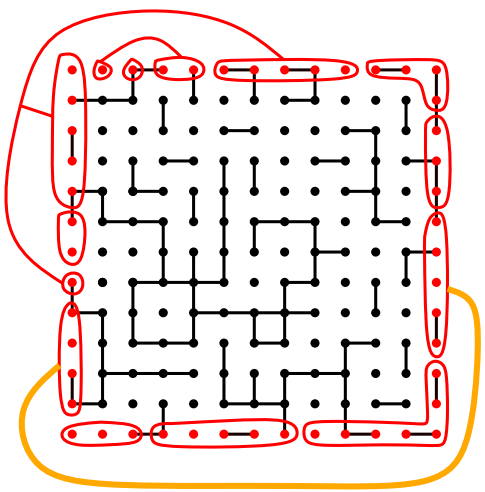

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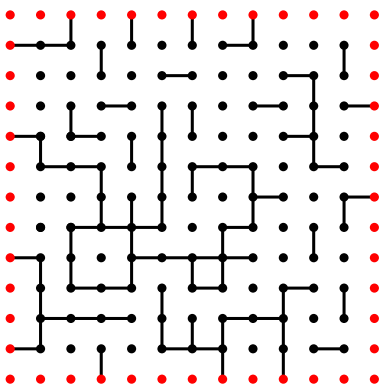

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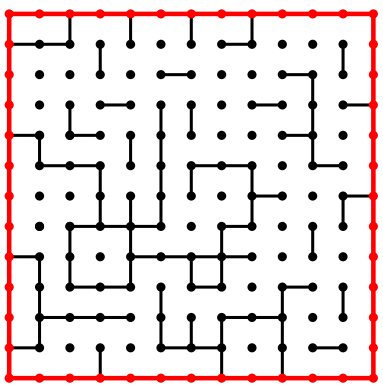

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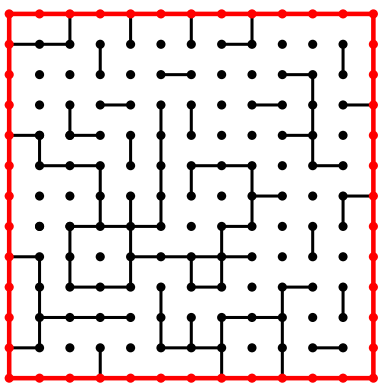

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$q = 1 \rightsquigarrow$ Bernoulli percolation

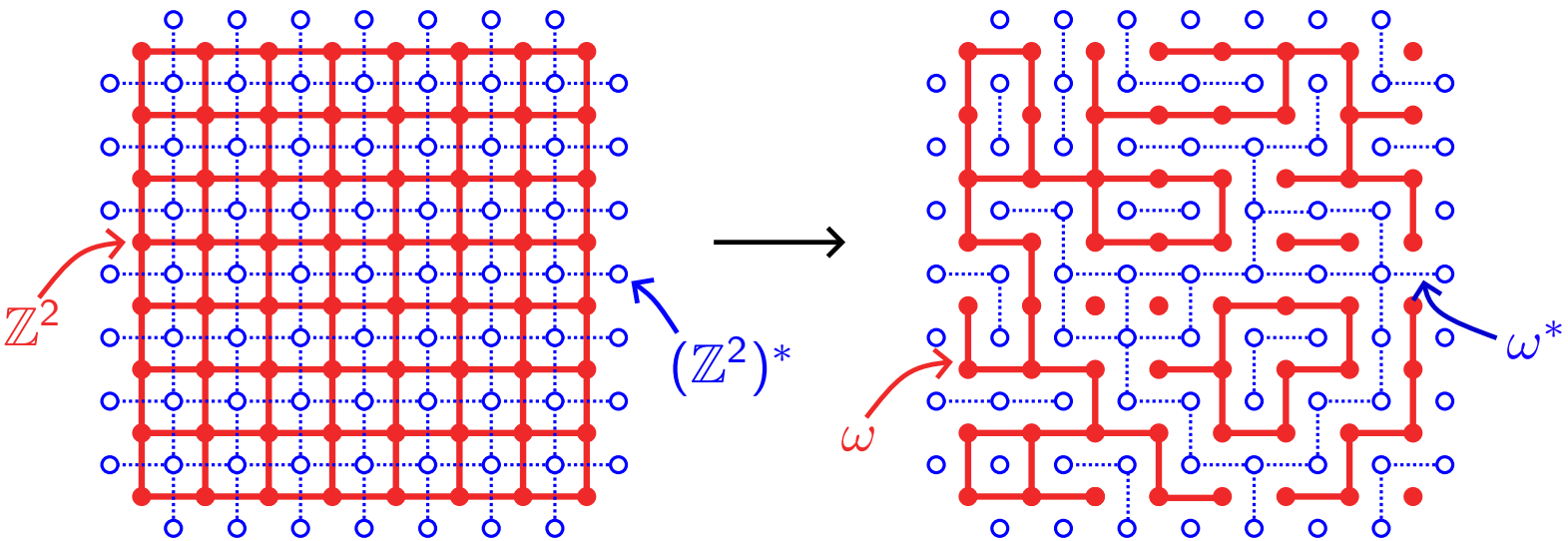
$q > 1 \rightsquigarrow$ positive correlation between edges

Critical FK-percolation

For the rest of the talk we fix $q \in [1, 4]$ and take

$$p = p_c(q) = \frac{\sqrt{q}}{1 + \sqrt{q}}$$

Self-duality:



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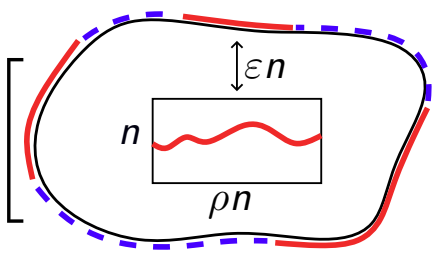
Continuous phase transition:

- Unicity of the measure

$$\phi := \phi^0_{p_c,q} = \phi^1_{p_c,q}$$

- RSW: for any $\rho, \varepsilon > 0$, there exists $0 < c < 1$ such that

$$\forall n \text{ large enough}, \quad c \leq \phi^{\xi}_{G_n,p_c,q}\left[\ \right] \leq 1 - c$$



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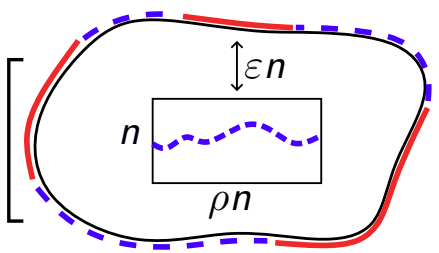
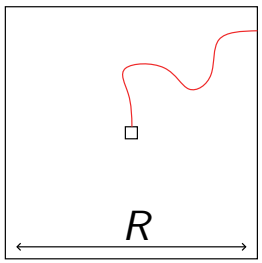
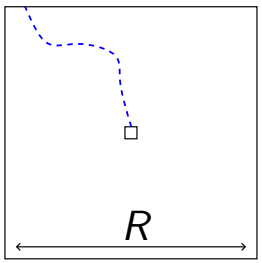
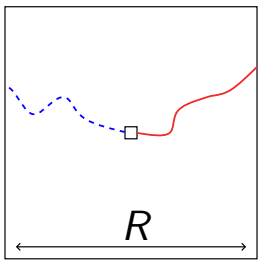


Arm events

For $R > 0$,

$A_R(1) =$ 

$A_R(0) =$ 

$A_R(01) =$ 

More generally: $A_R(\sigma)$ with $\sigma \in \{0,1\}^k$ ($0 \to$ dual, $1 \to$ primal)

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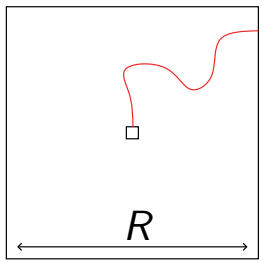
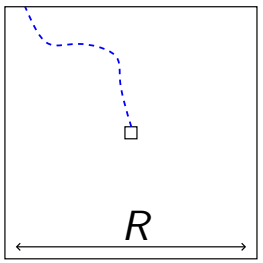
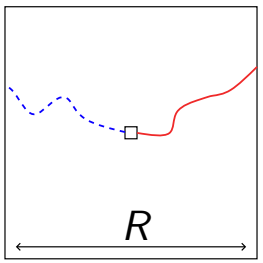
$A_R(1) =$ 

$A_R(0) =$ 

$A_R(01) =$ 

More generally: $A_R(\sigma)$ with $\sigma \in \{0,1\}^k$ ($0 \to$ dual, $1 \to$ primal)

Conjecture:

$$\phi[A_R(\sigma)] = R^{-\alpha_\sigma + o(1)}$$

$$\alpha_0 = \alpha_1 = \frac{(8-\kappa)(3\kappa-8)}{32\kappa}, \quad \alpha_{01} = \frac{16-(4-\kappa)^2}{8\kappa}, \quad \alpha_{0101} = \frac{64-(\kappa-4)^2}{8\kappa}$$

Inequality 1: 2-arm V.S. 2 × 1-arm

Theorem: [Manolescu and G. 2024]

For $1 \leq q \leq 4$, for large enough R,

$$\phi[A_R(01)] \leq R^{-c}\, \phi[A_R(0)]\phi[A_R(1)]$$

In particular, if α_{01}, α_0 and α_1 exist then

$$\alpha_{01} > \alpha_0 + \alpha_1$$

Proof of "$\alpha_{01} > \alpha_0 + \alpha_1$"

$$\phi[A_R(01)] \leq R^{-c}\, \phi[A_R(0)]\phi[A_R(1)]$$

Can be rewritten as

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By the FKG inequality

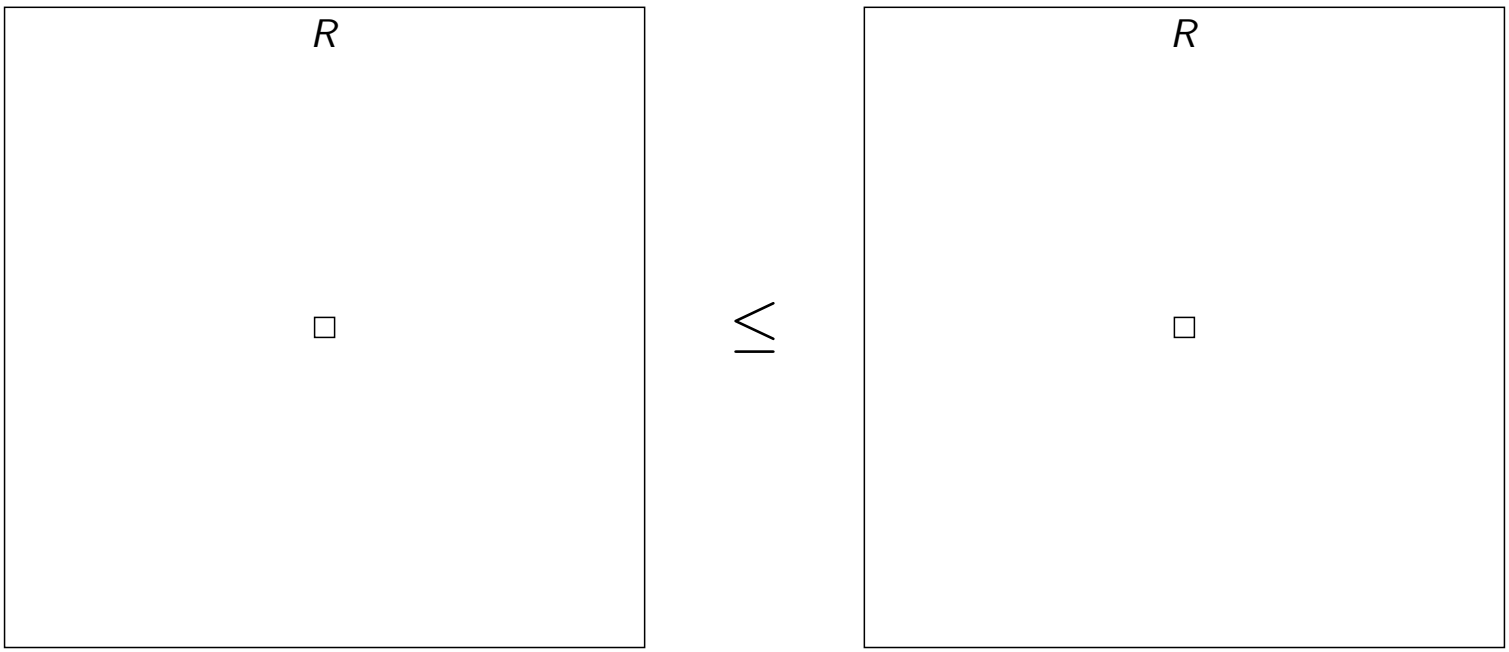
$$\phi[\,\cdot\,|A_R(0)] \leq_{st} \phi[\,\cdot\,]$$

We will construct an **increasing** coupling $\mathbb{P}$ between

$$\omega \sim \phi[\,\cdot\,|A_R(0)] \quad \text{and} \quad \omega' \sim \phi[\,\cdot\,]$$

(**increasing** means that $\omega \leq \omega'$ a.s.)

Proof of "$\alpha_{01} > \alpha_0 + \alpha_1$" (via an increasing coupling $\mathbb{P}$)



$\omega \sim \phi[\cdot | A_R(0)]$ $\omega' \sim \phi[\cdot]$

We want to show that

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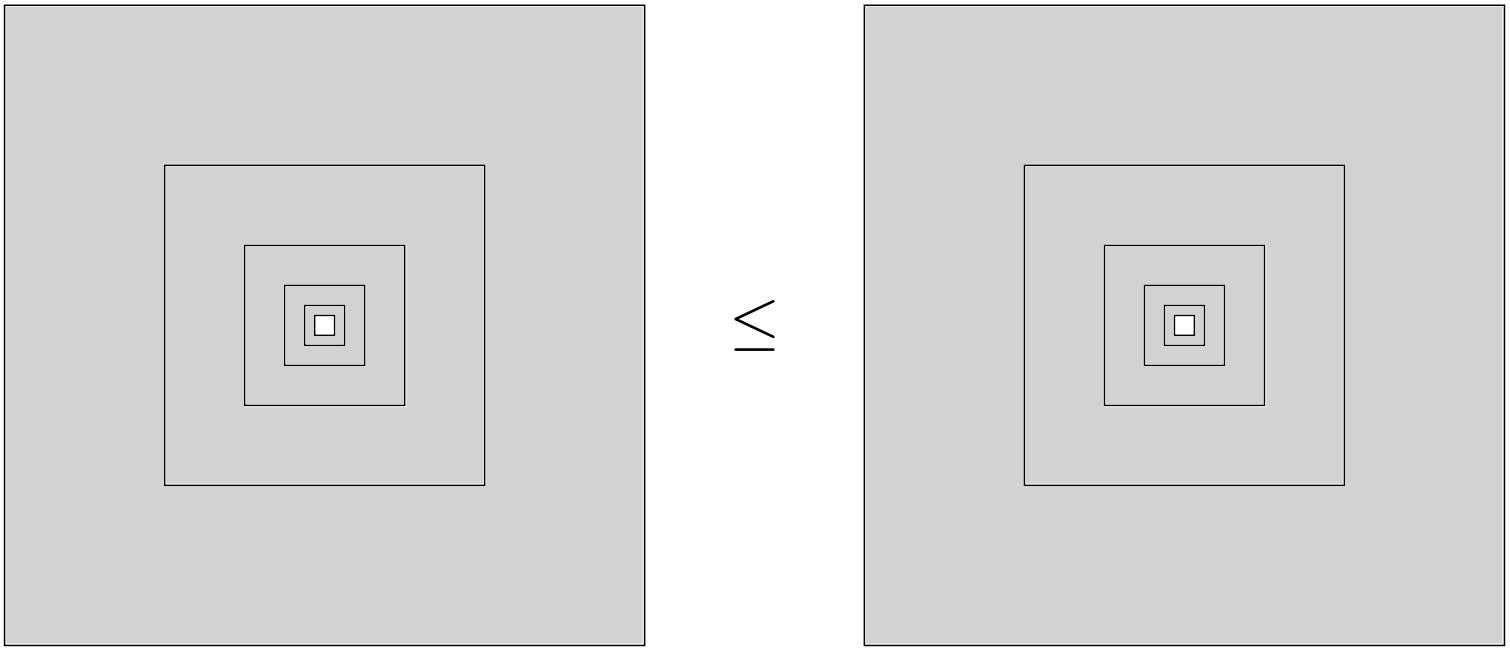

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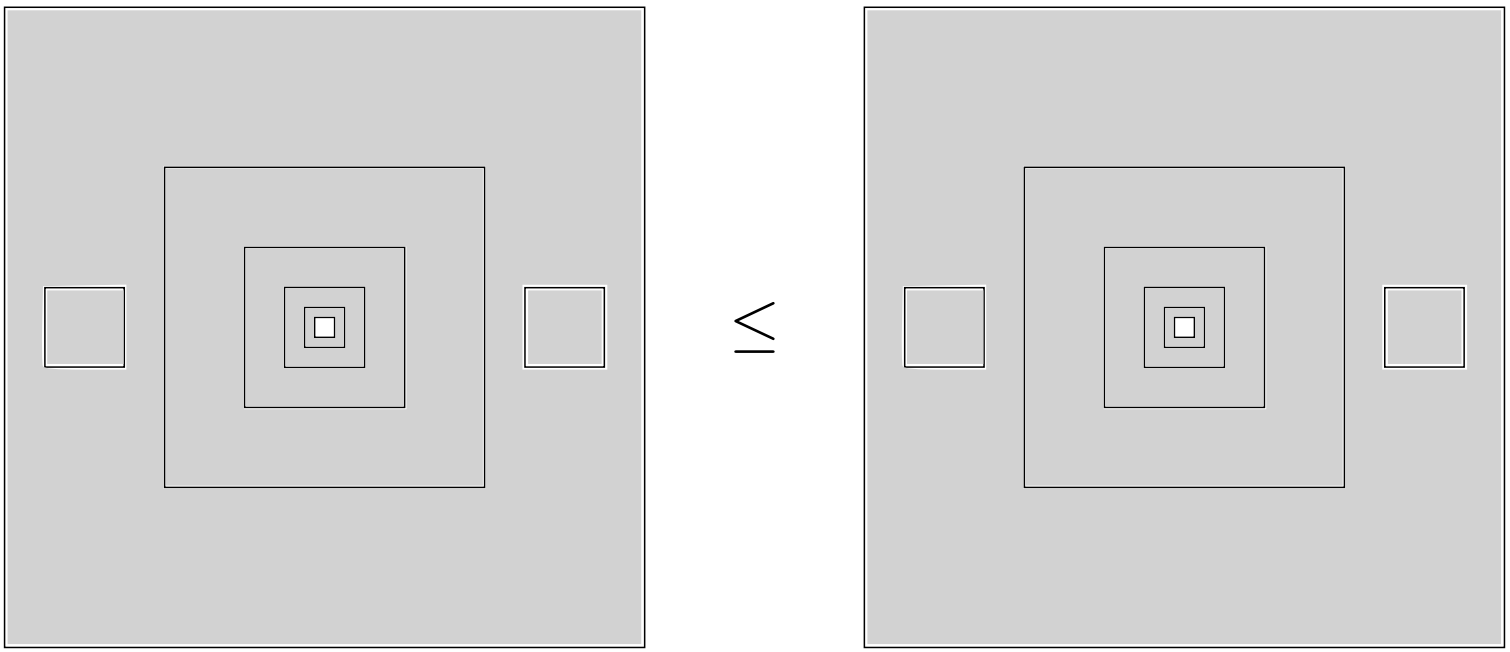

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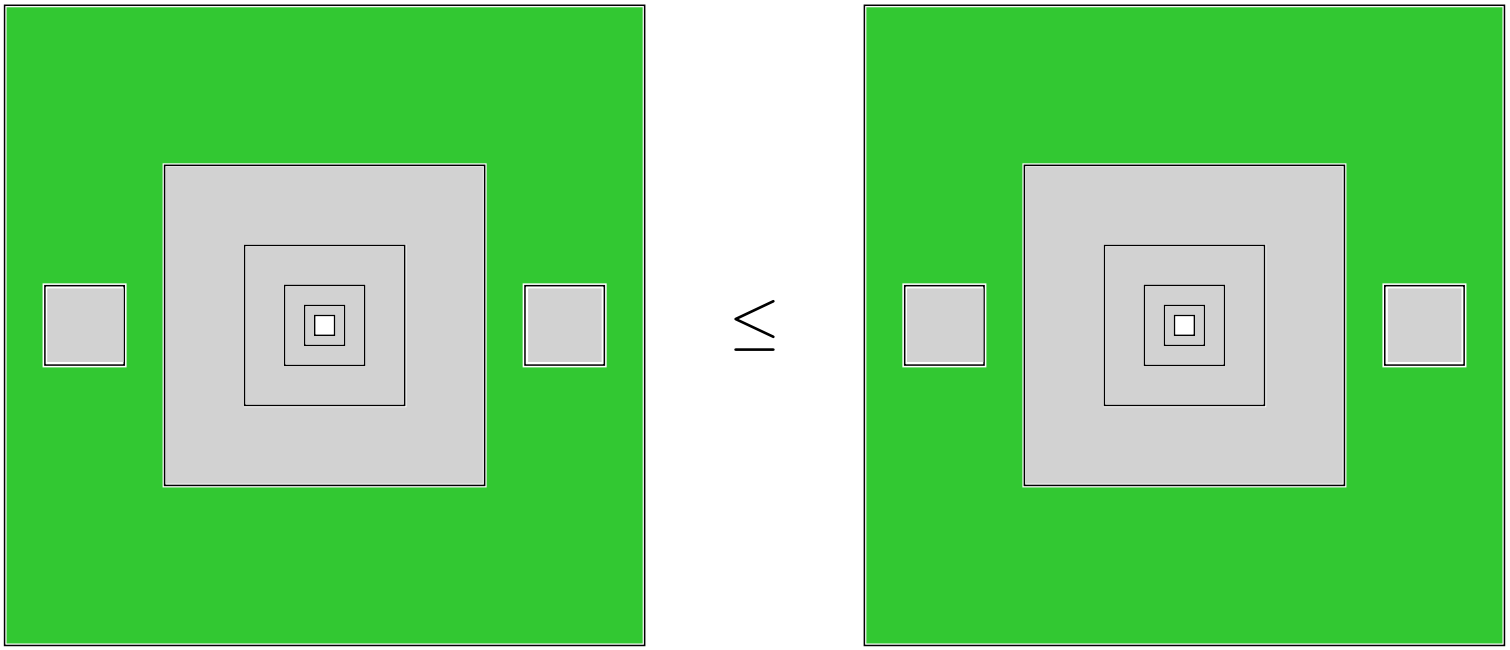

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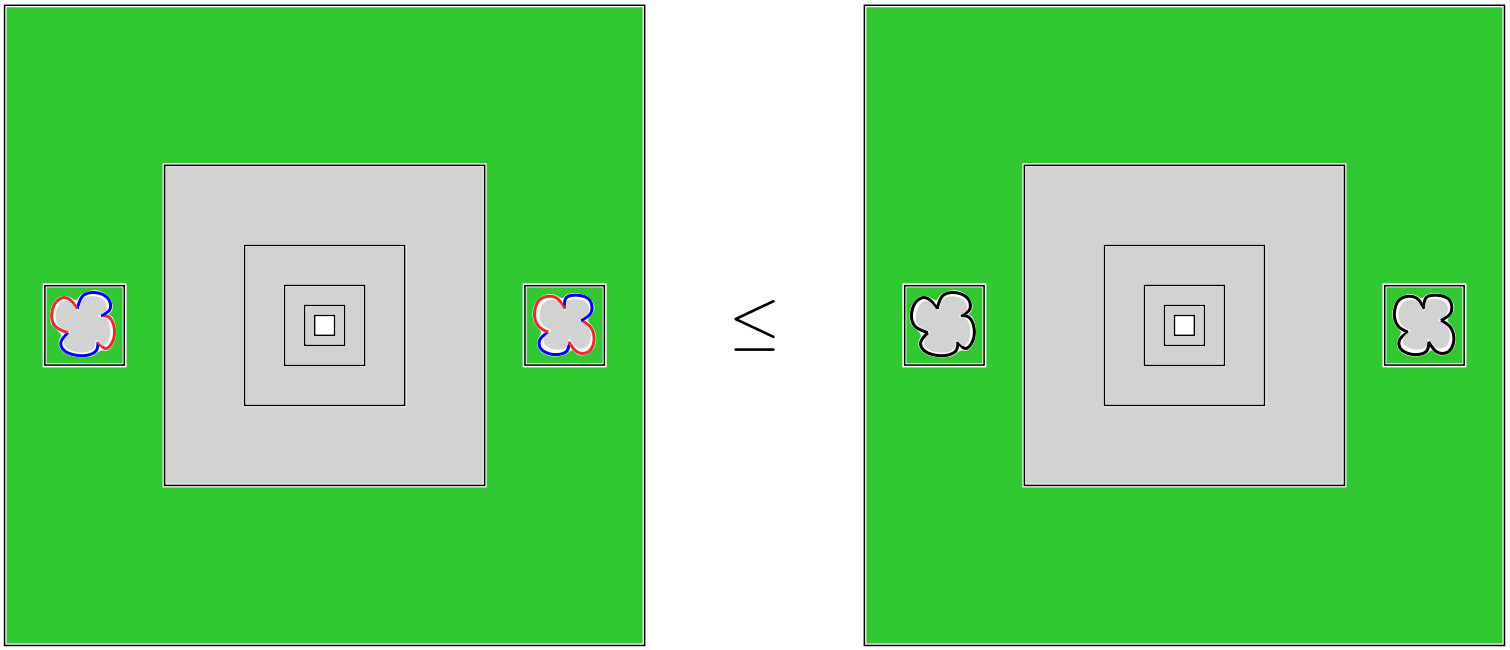

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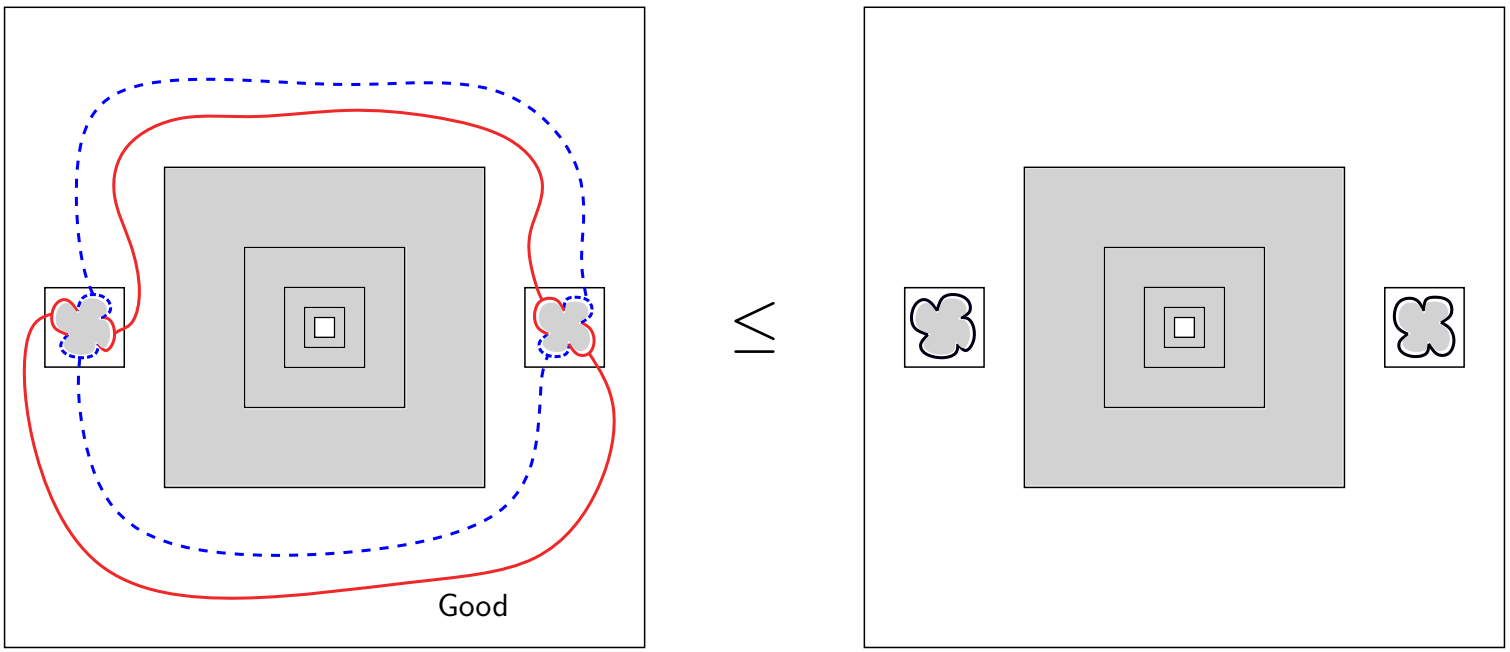

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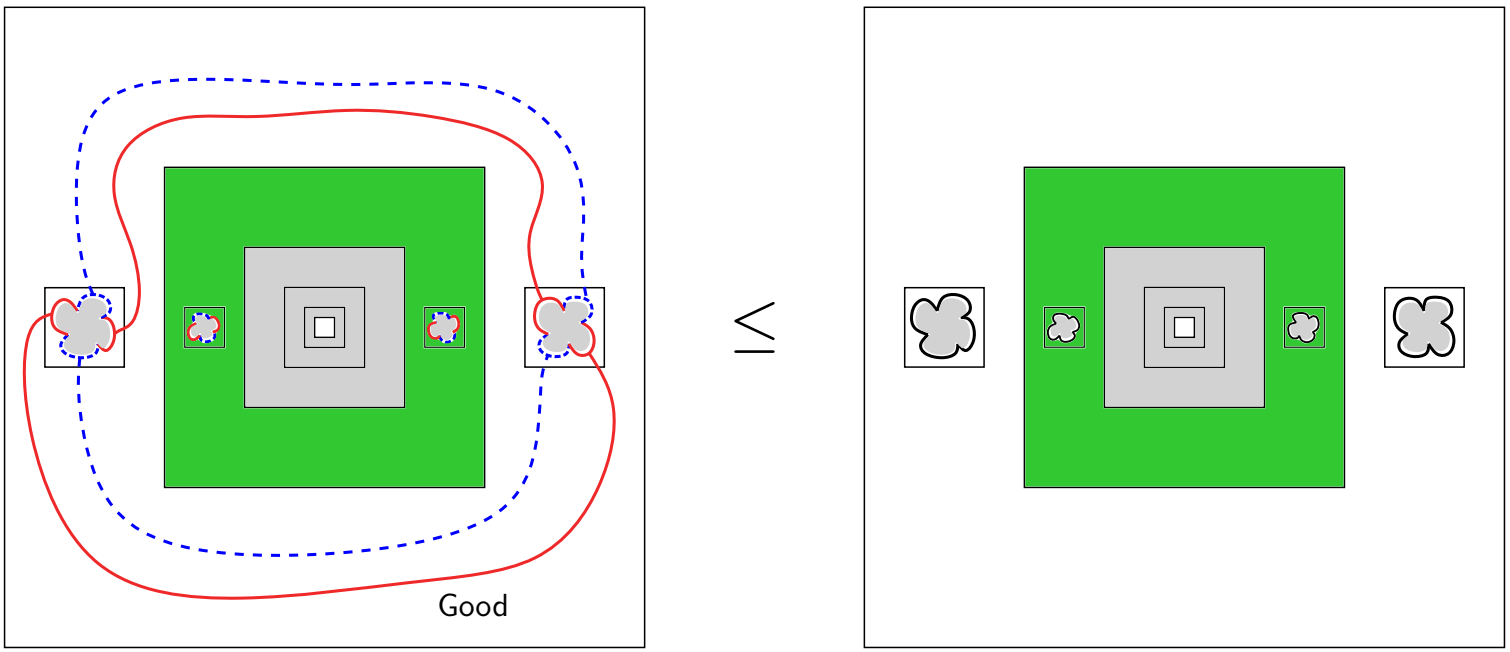

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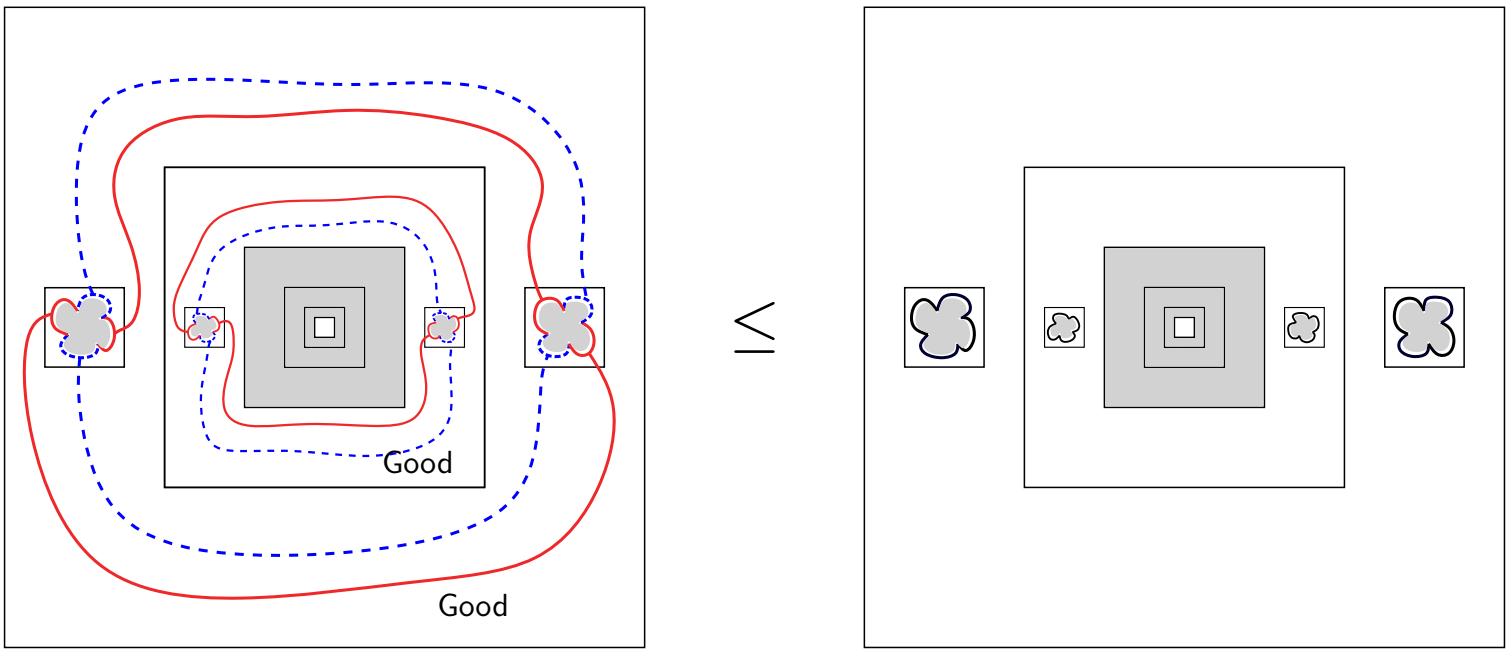

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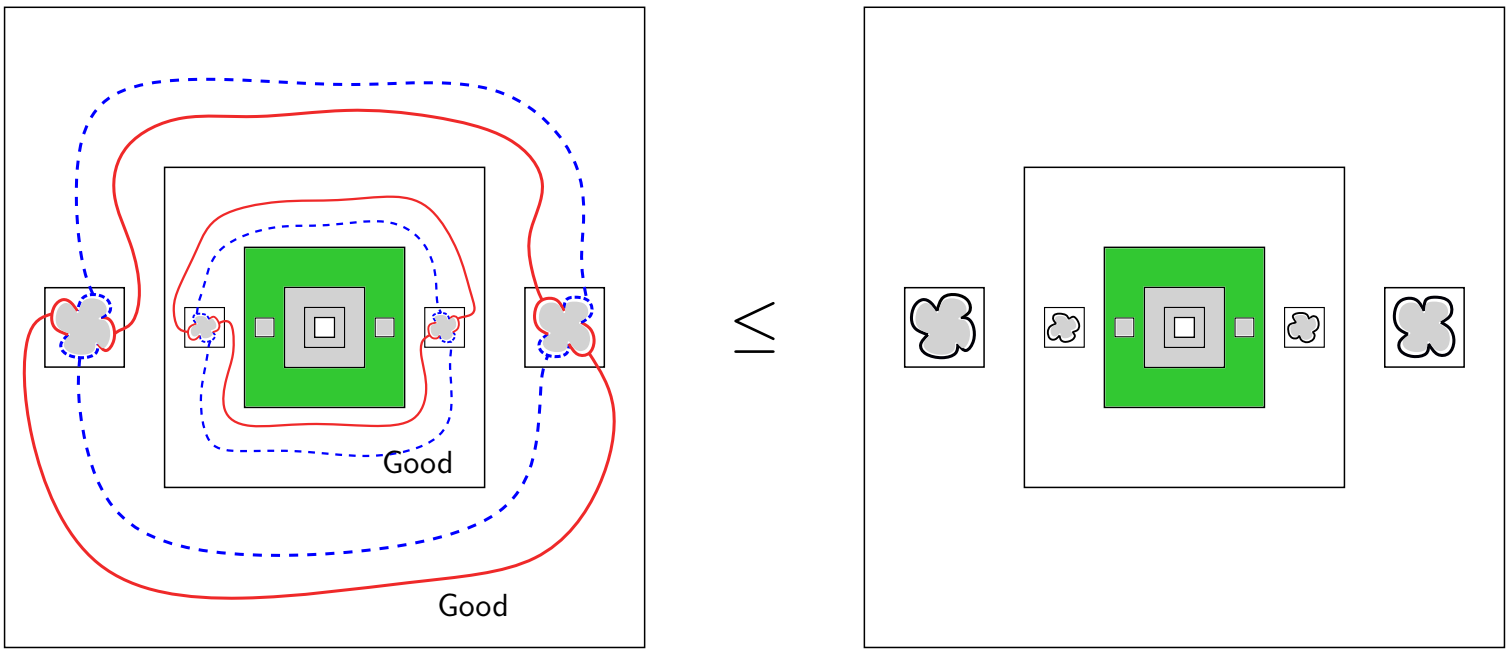

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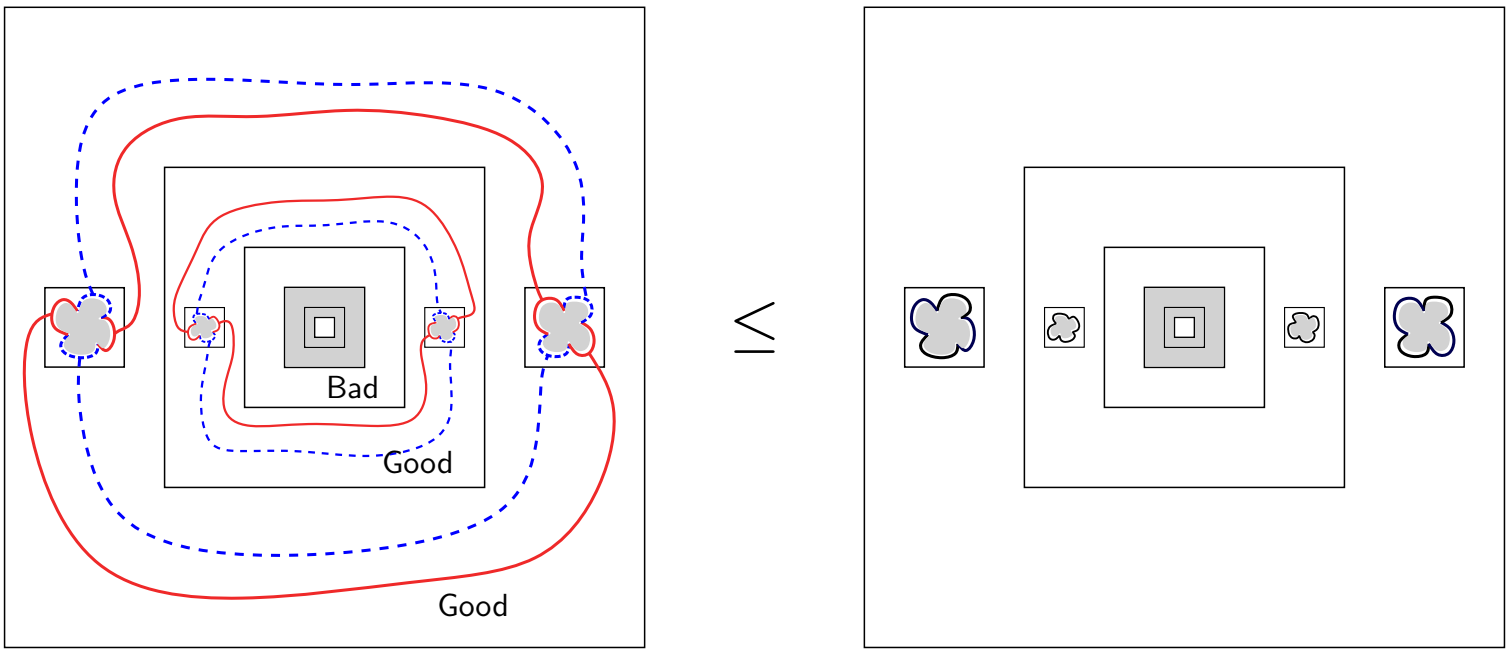

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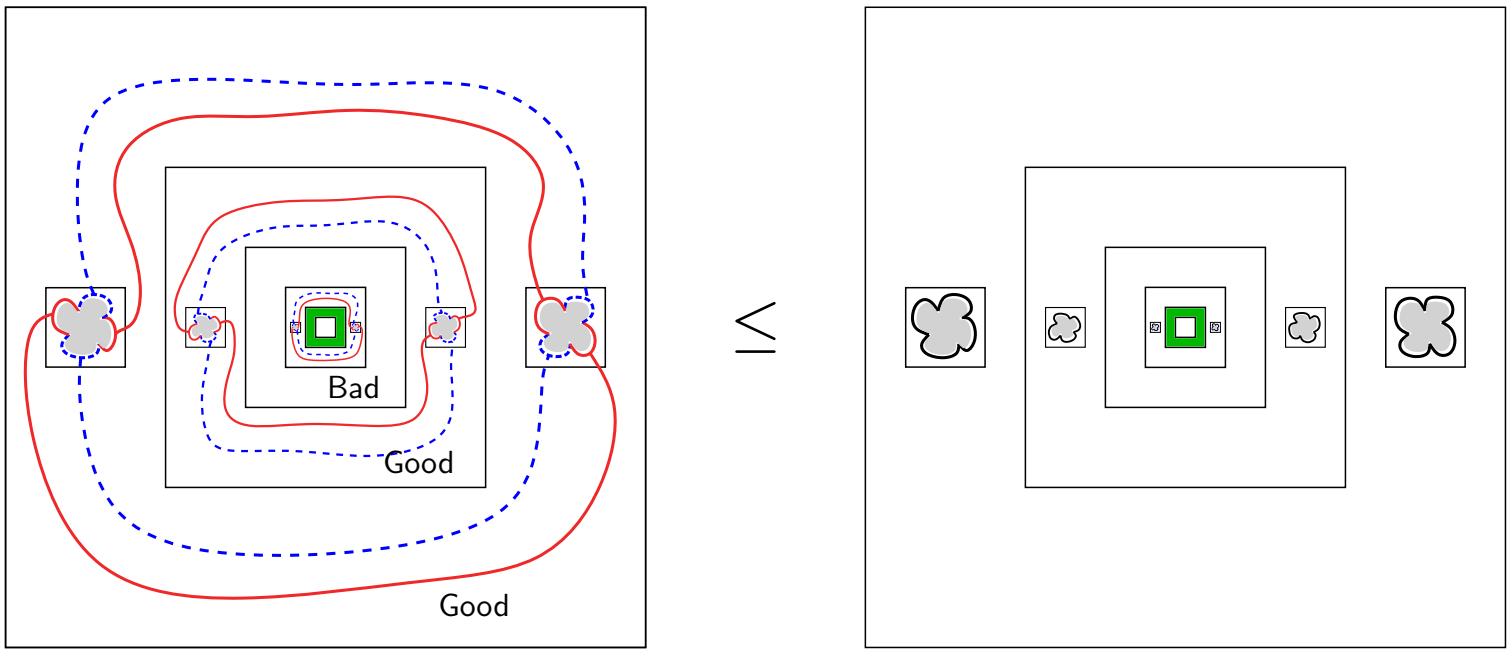

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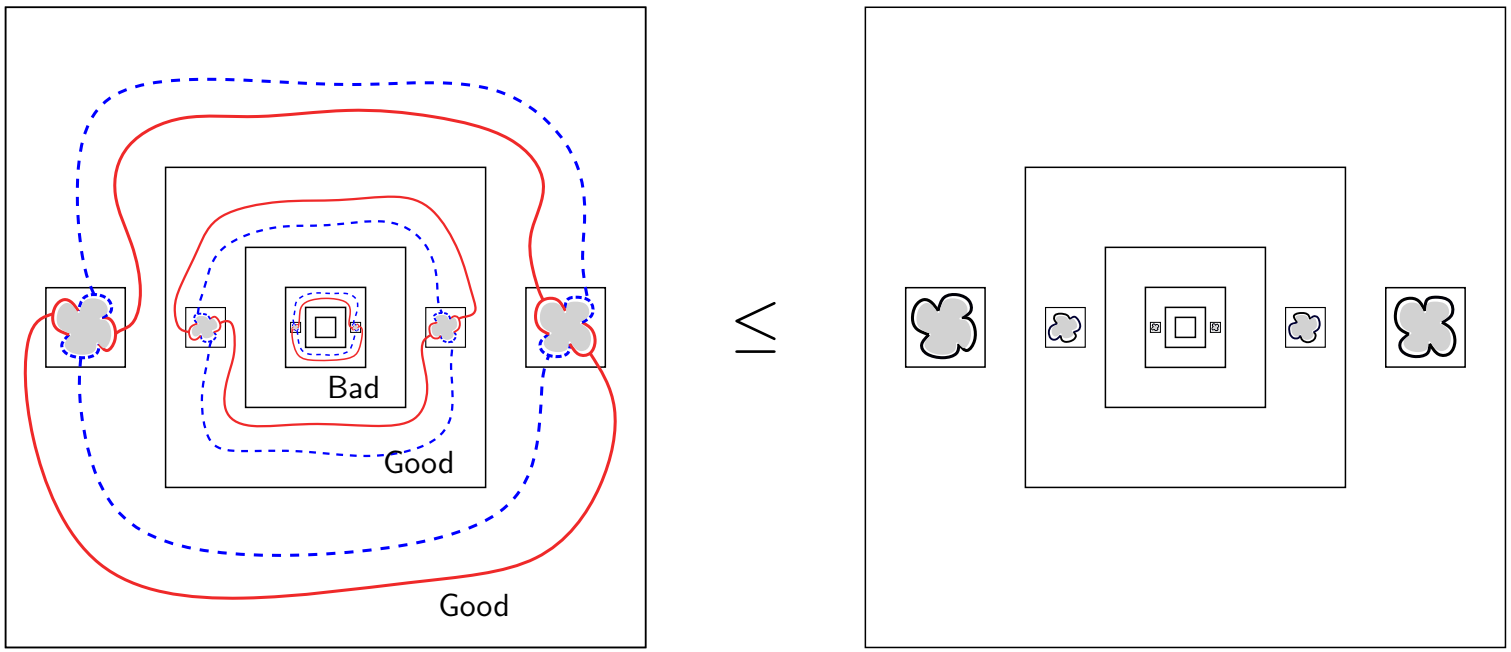

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Good event:

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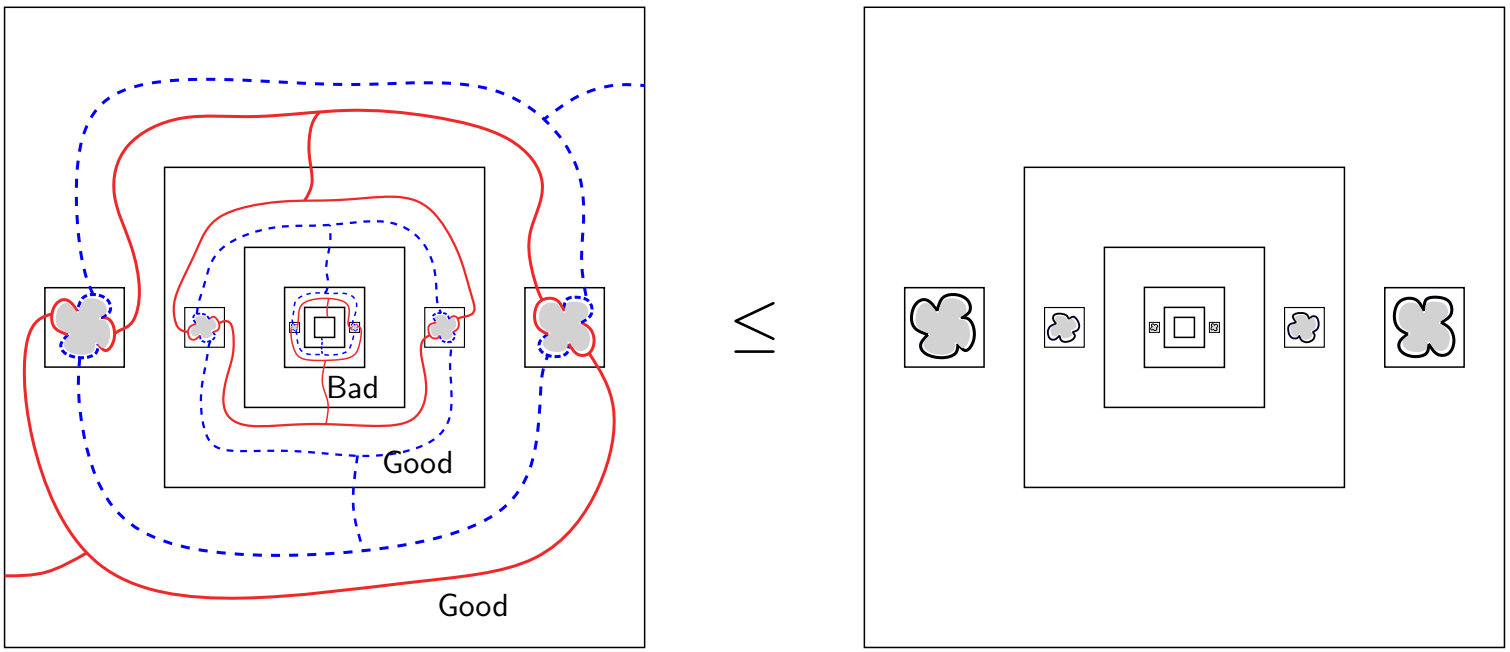

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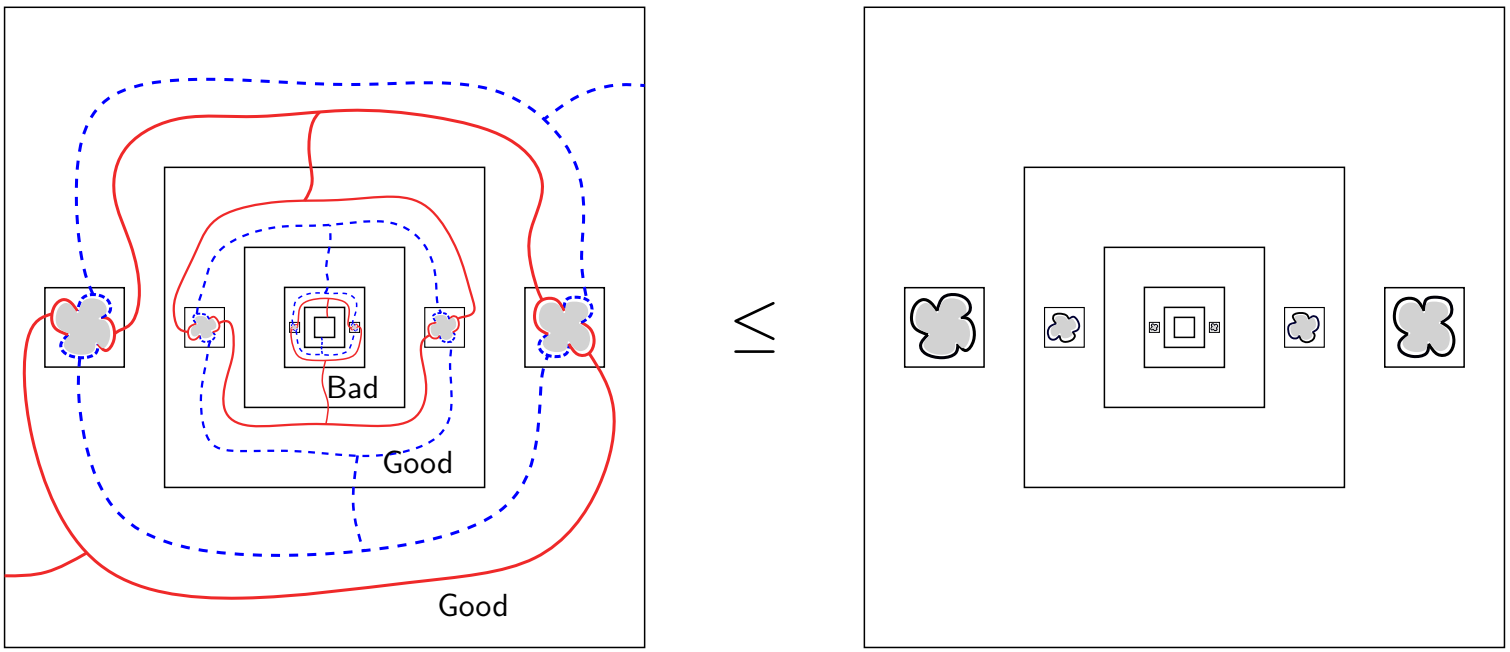

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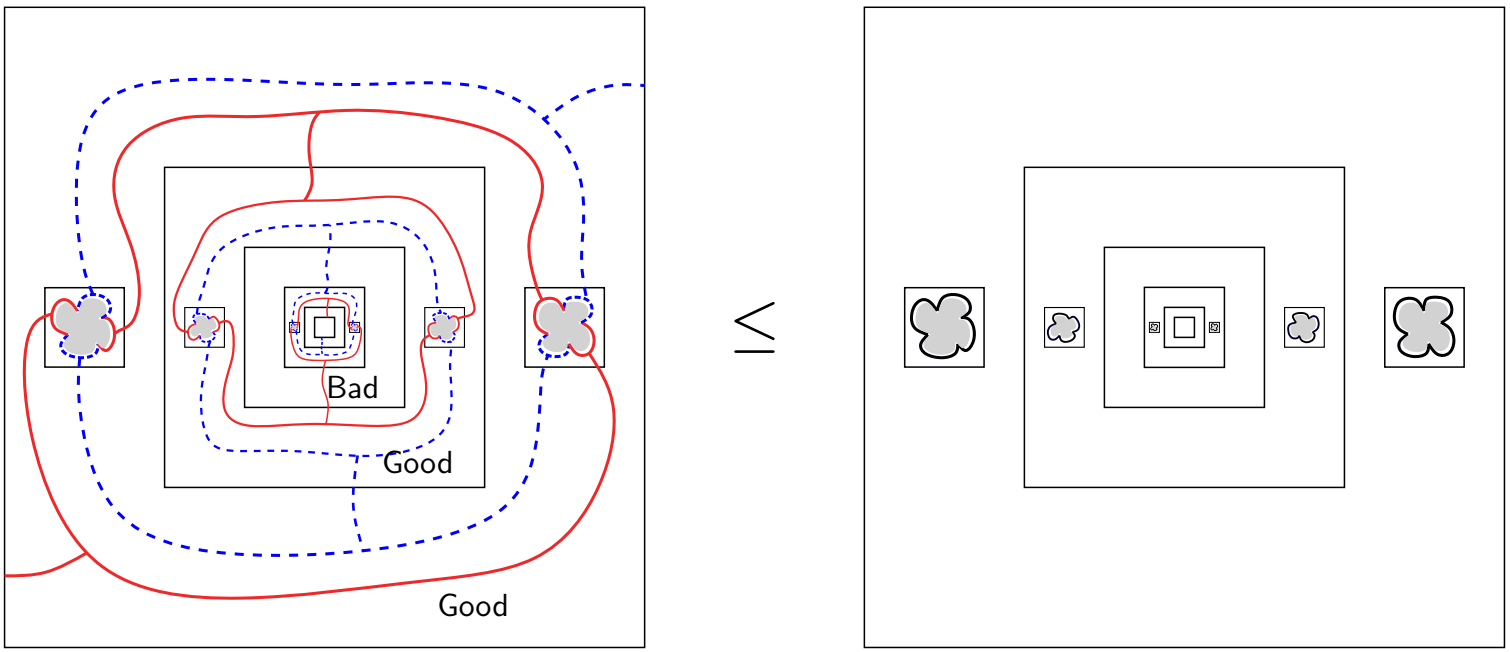

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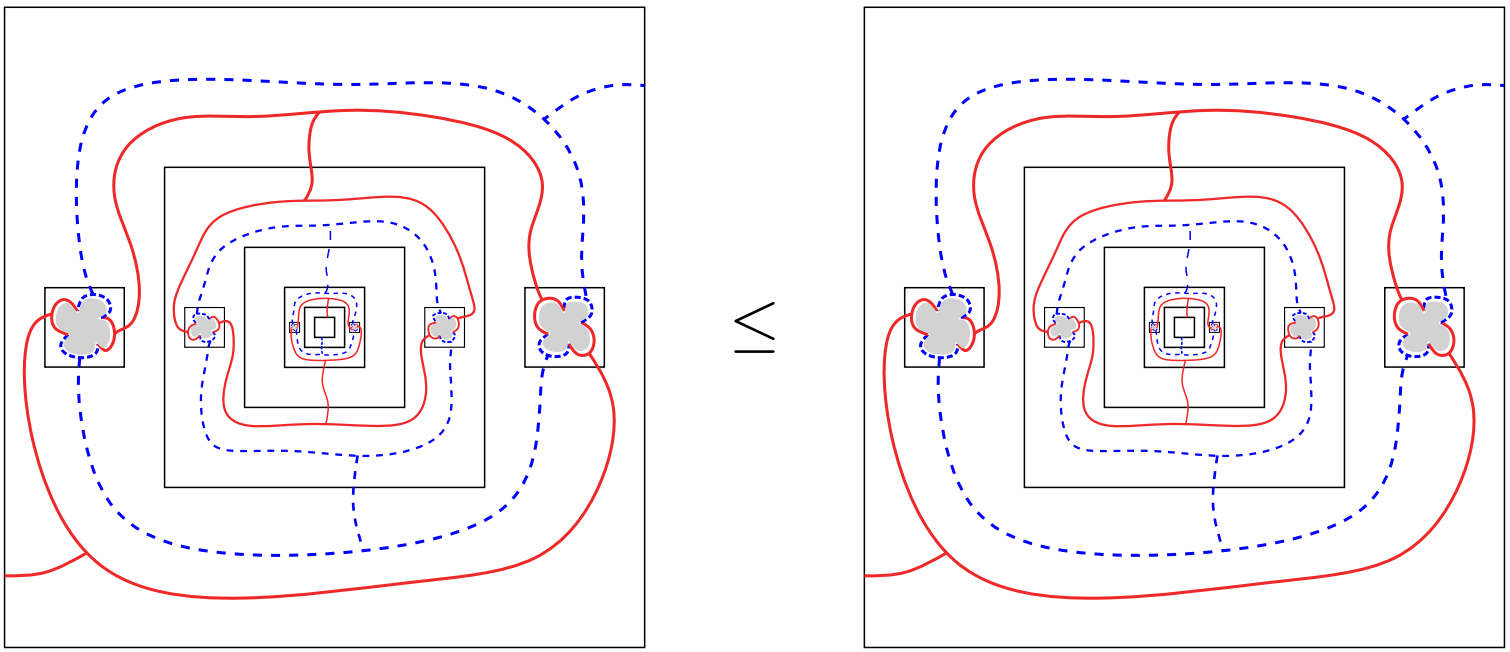

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$$\mathbb{P}[\omega \in A_R(1)\,|\,\omega \in G] \leq R^{-c}\mathbb{P}[\omega' \in A_R(1)\,|\,\omega \in G]$$

Good event:

$$G := \{\text{more than } \beta \log_2(R) \text{ good scales + connections}\}$$

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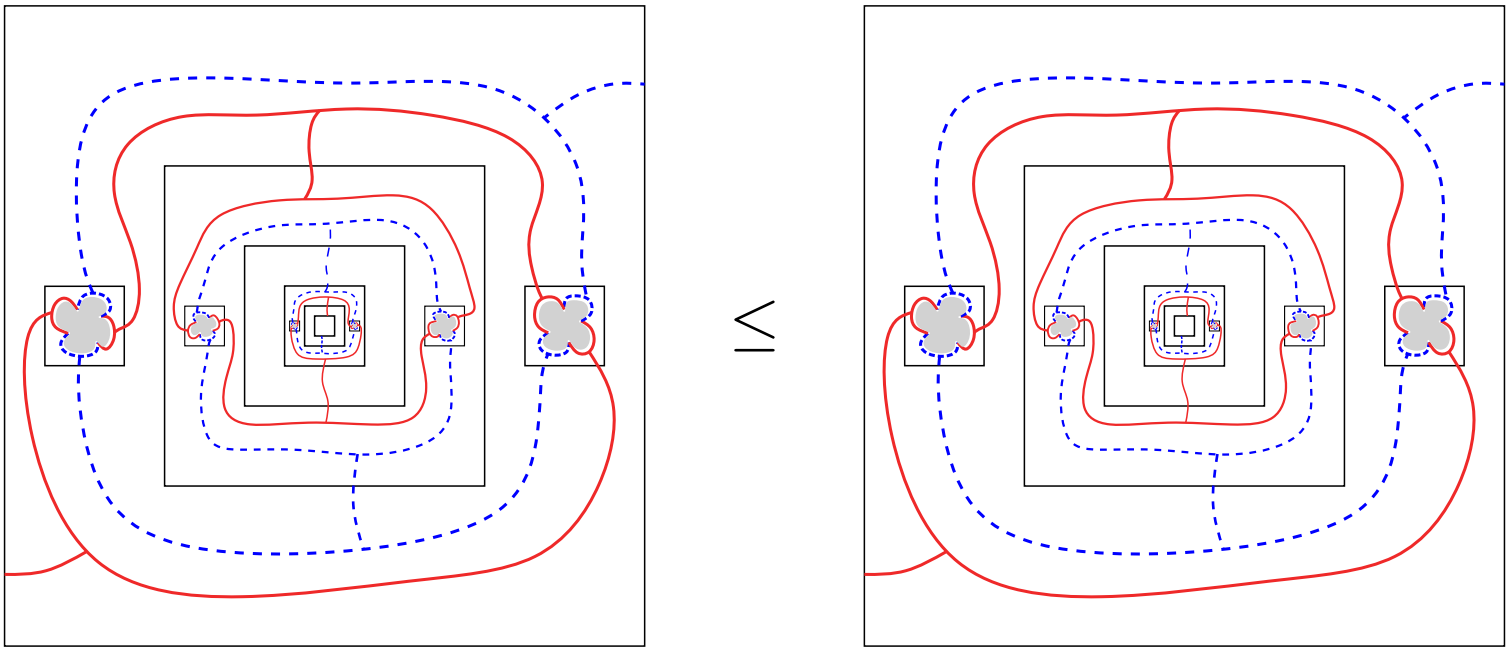

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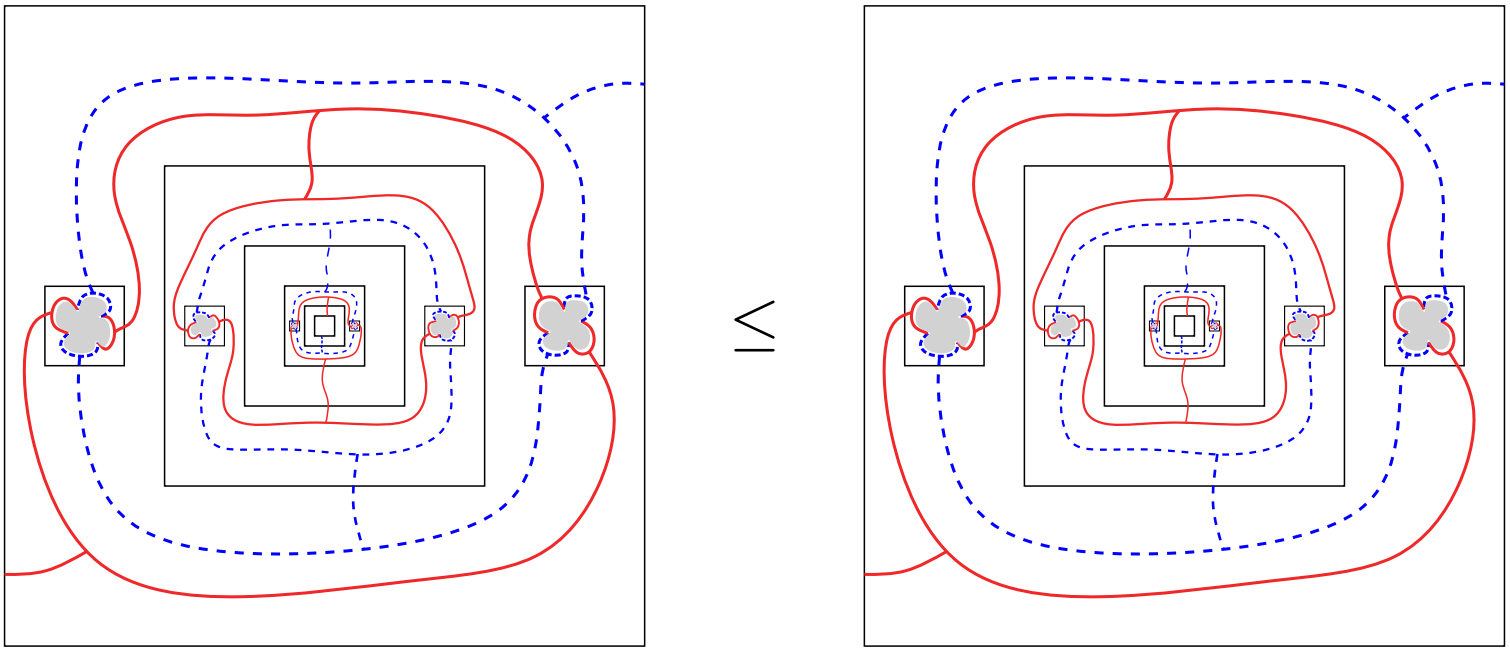

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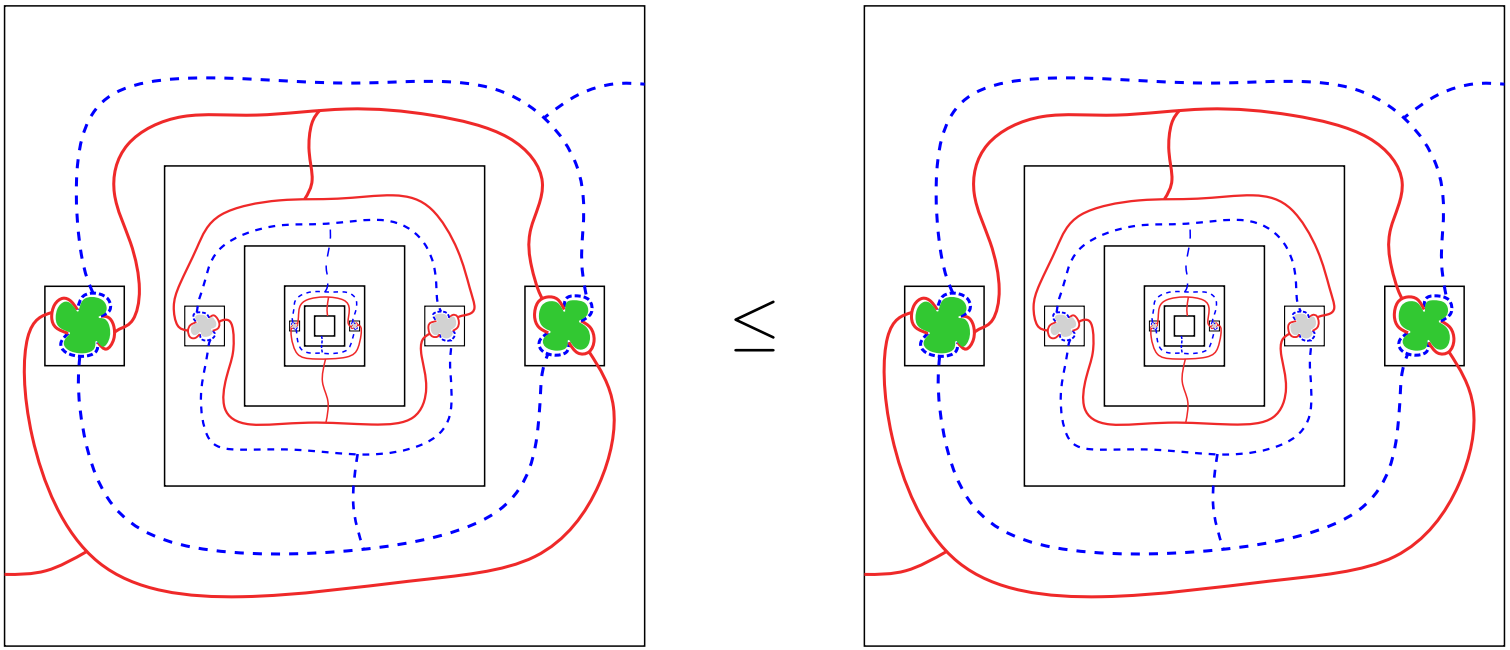

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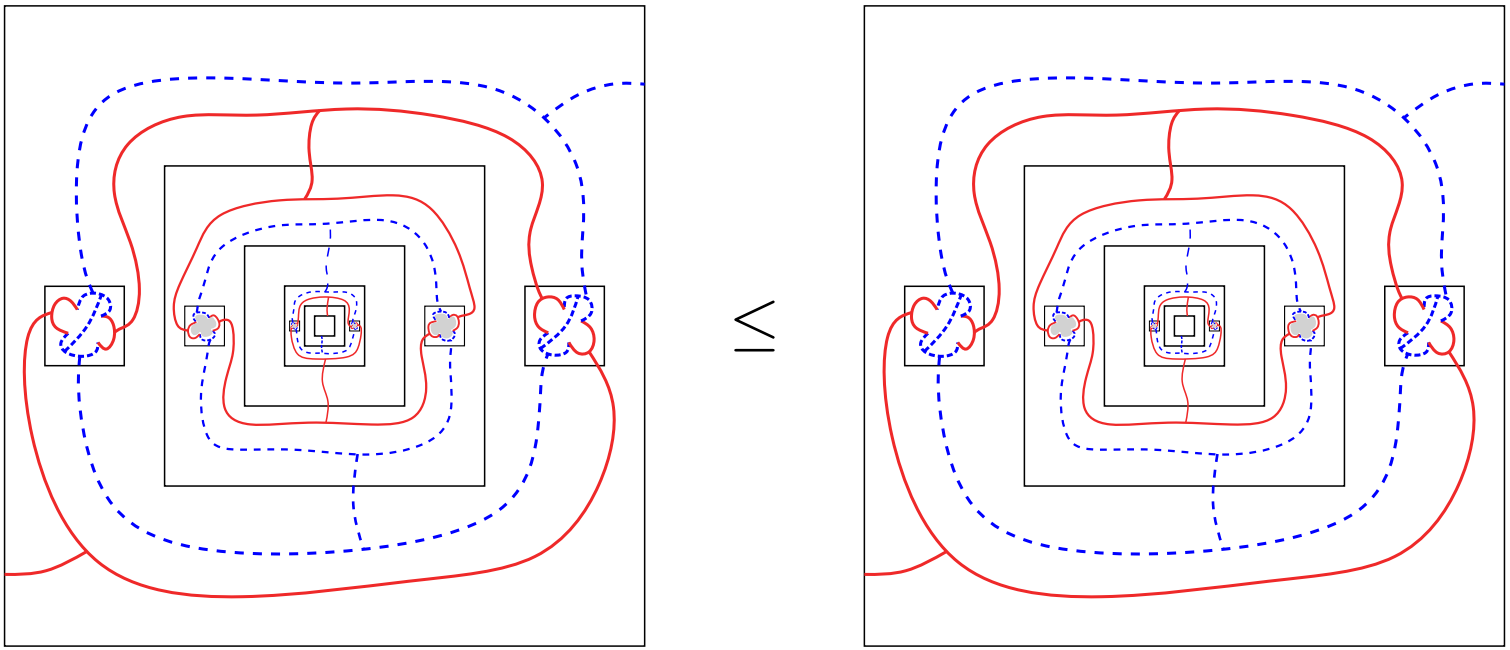

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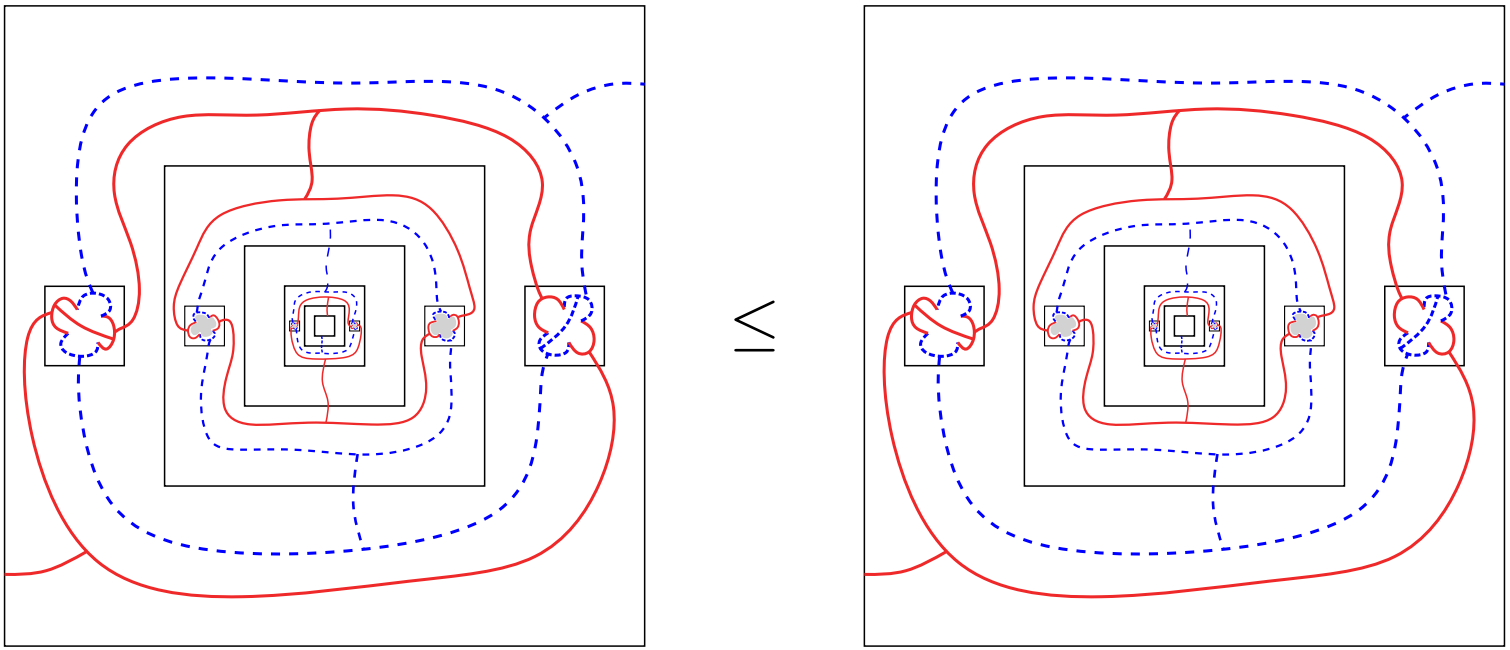

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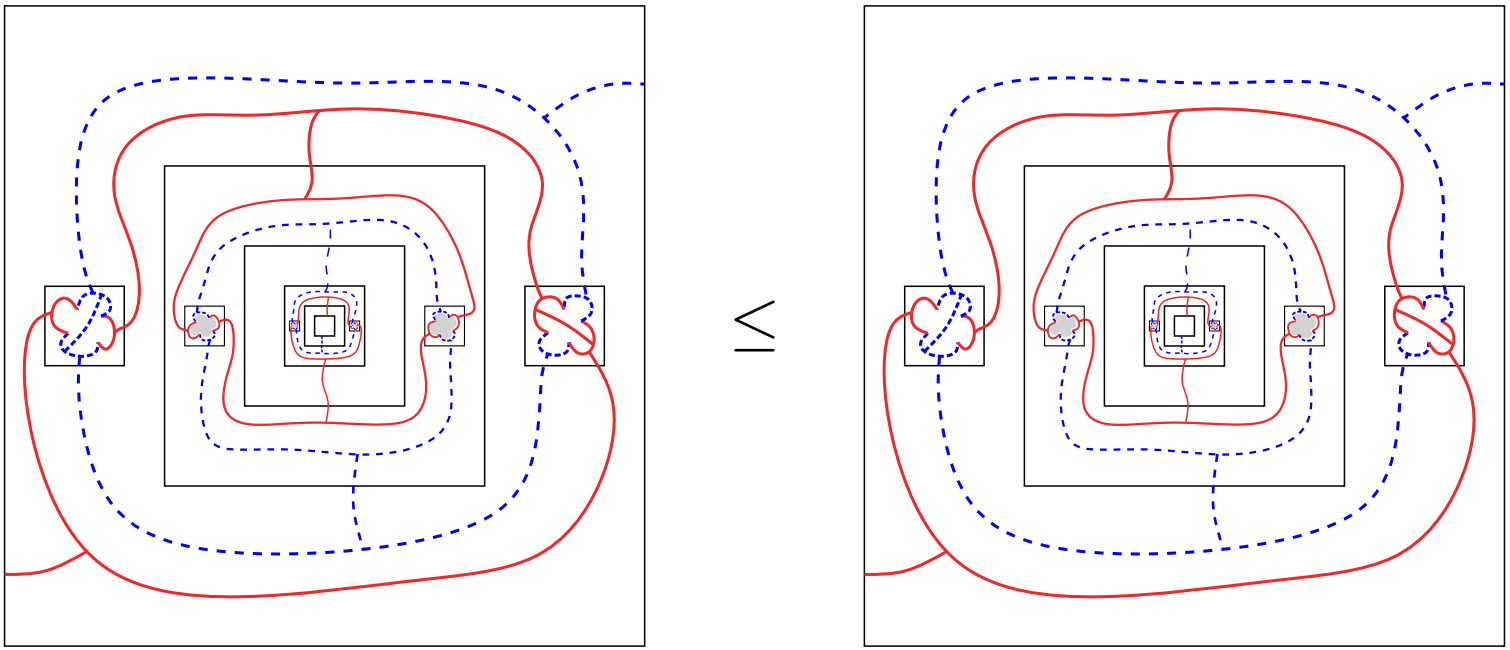

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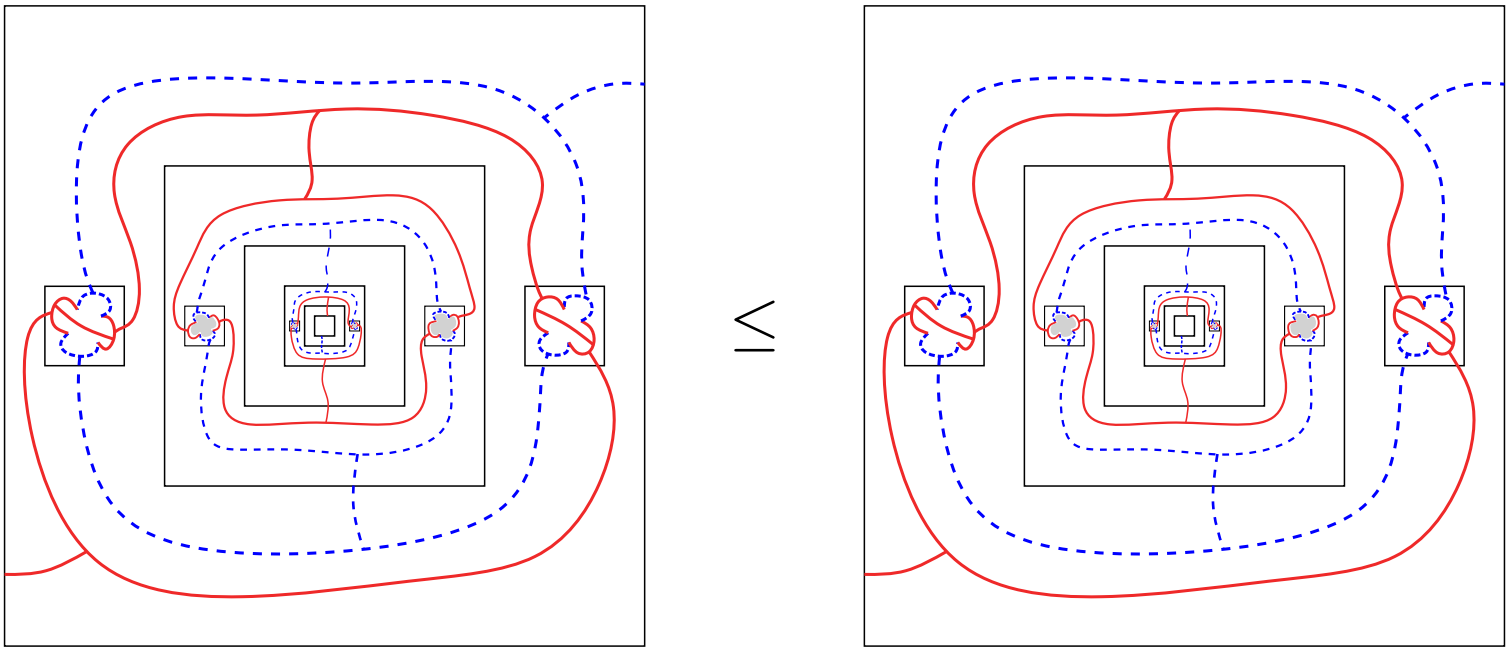

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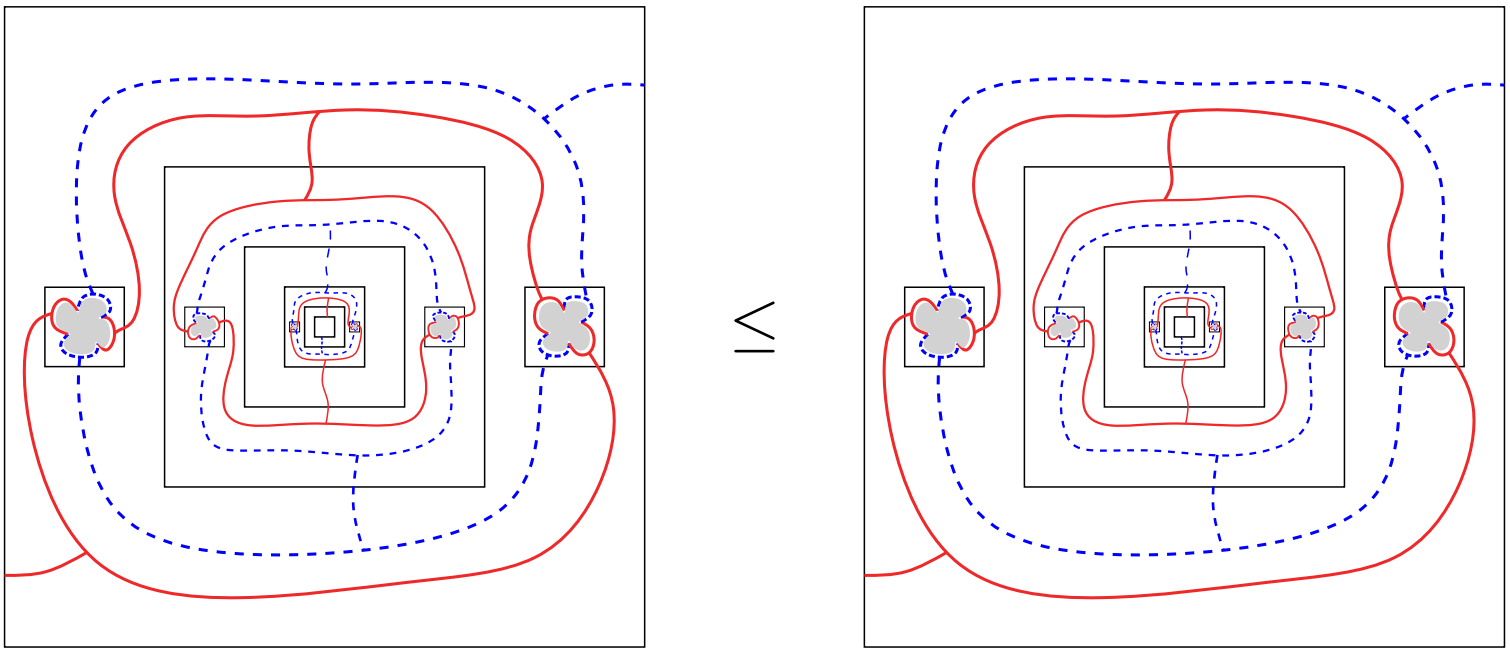

$\omega \sim \phi[\cdot | A_R(0)]$ $\leq$ $\omega' \sim \phi[\cdot]$

It follows that

$$\mathbb{P}[\omega \in A_R(1)] \leq (R/r)^{-c} \mathbb{P}[\omega' \in A_R(1)]$$

We conclude that

$$\phi[A_R(1) | A_R(0)] \leq (R/r)^{-c} \phi[A_R(1)]$$

Remarks about *Inequality 1*

Inequality 1: Alternated 2-arms exponent V.S. 2×1-arm exponent

$$\phi[A_R(01)] \leq R^{-c}\, \phi[A_R(0)]\phi[A_R(1)] \quad \checkmark$$

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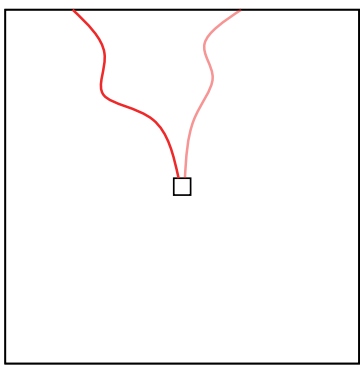

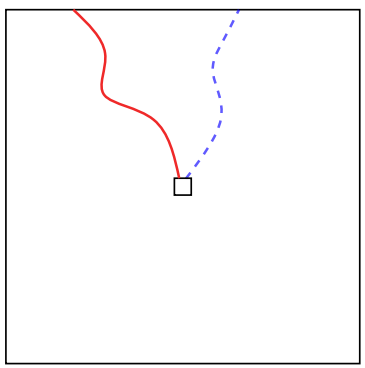

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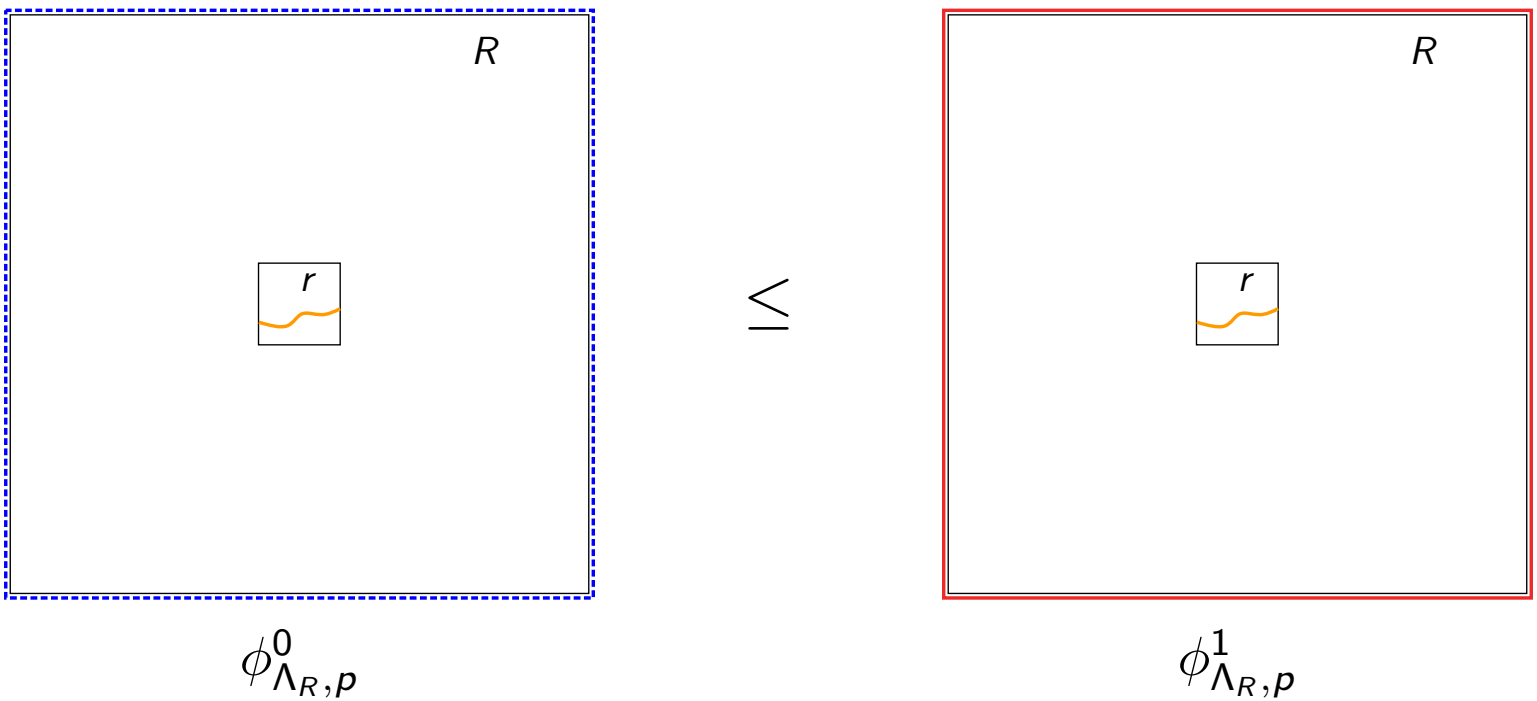
Inequality 2: Alternated 4-arms V.S. mixing rate (only for $q > 1$)

$$\phi_{p_c}[A_{r,R}(0101)] \leq C(R/r)^{-c}\Delta(r,R)$$

Mixing rate

For $r < R$, we define the *mixing rate*

$$\Delta(r, R) := \phi^1_{\Lambda_R, p_c}[\mathcal{C}(\Lambda_r)] - \phi^0_{\Lambda_R, p_c}[\mathcal{C}(\Lambda_r)]$$



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Properties of $\Delta(r, R)$: [Duminil-Copin and Manolescu 2020]

- Controls the mixing between r and R
- Quasi-multiplicativity
- Conjecture: algebraic decay

$$\Delta(r, R) = (R/r)^{-\iota + o(1)}, \quad \iota = \frac{3\kappa}{8} - 1 \text{ [Liu,Wu,Zhuang 2025]}$$

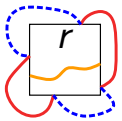
- Determines the critical window $\xi(p)^2 \Delta(1, \xi(p)) \asymp |p - p_c|^{-1}$

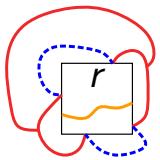
$$q = 1 \rightsquigarrow \xi(p)^2 \pi_4(1, \xi(p)) \asymp |p - p_c|^{-1}$$

Fix $q > 1$.

If the petals are all of size $\geq \eta r$,

$$\phi^{\xi}[\mathcal{C}(\Lambda_r)] + \delta \leq \phi^{\xi'}[\mathcal{C}(\Lambda_r)]$$





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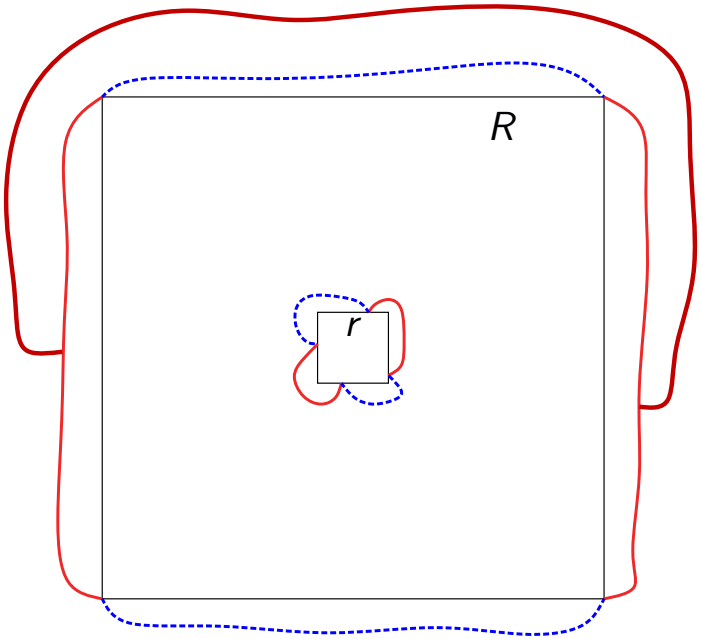

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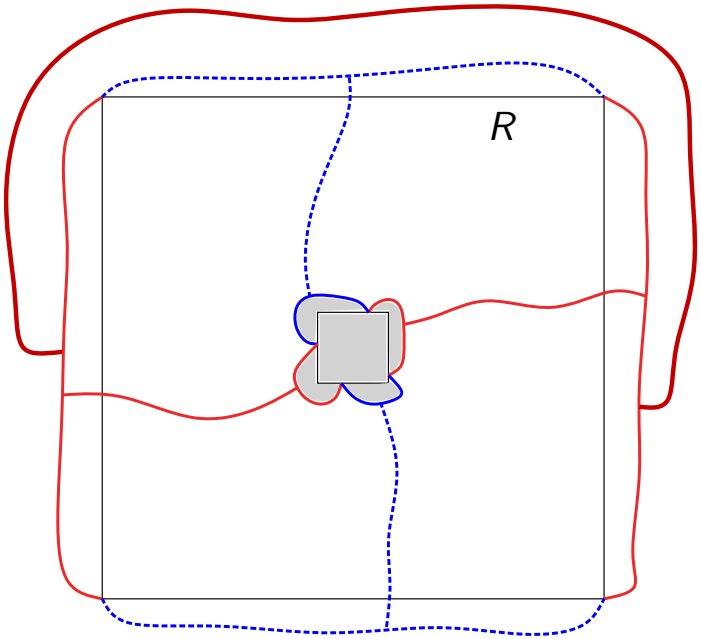

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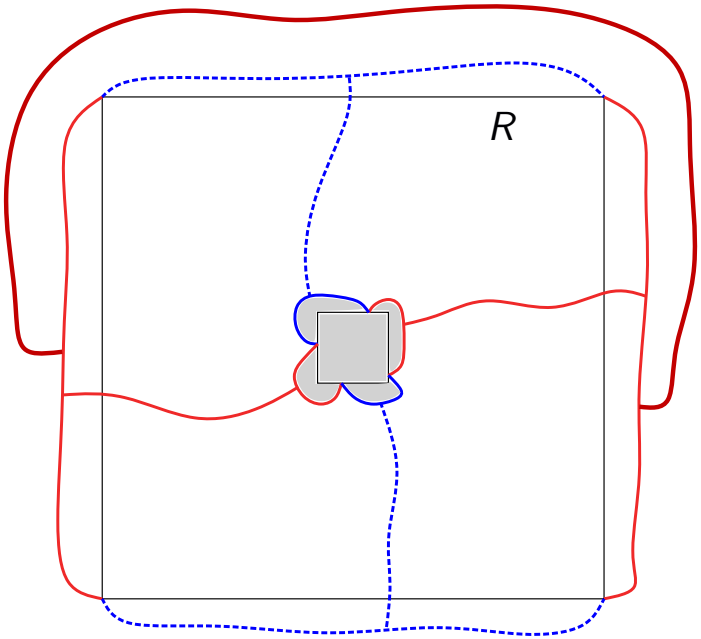
It follows that

$$\phi_{p_c}[A_{r,R}(0101)] \leq C.\Delta(r, R)$$



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Theorem: **[Duminil-Copin and Manolescu 2020]**

For all $1 < q \leq 4$,

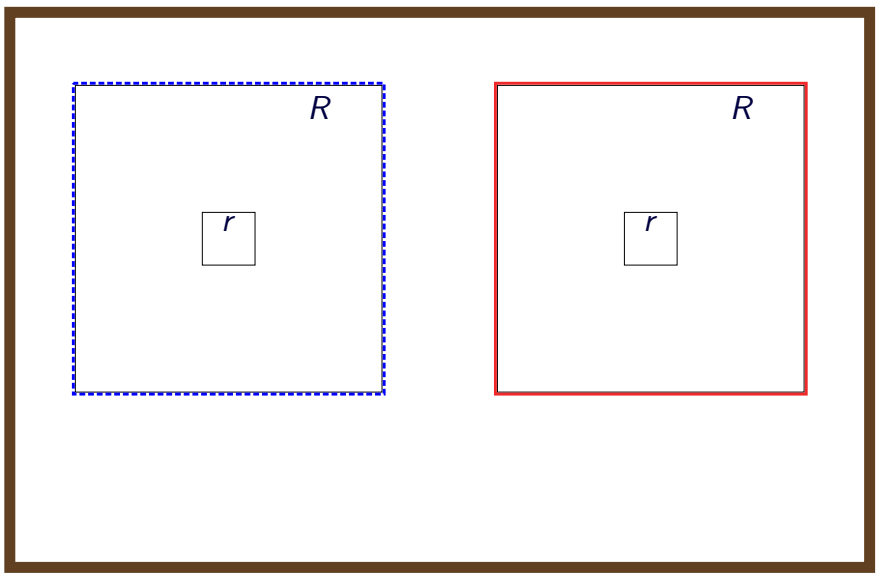
$$\phi_{p_c}[A_{r,R}(0101)] \leq C(R/r)^{-c}\Delta(r,R)$$

So if the exponents α_{0101} and ι exist, then

$$\alpha_{0101} > \iota$$

Proof of $\alpha_{0101} > \iota$ (for $q > 1$)

Goal: $\phi_{p_c}[A_{0101}(r, R)] \leq C(R/r)^{-c}\Delta(r, R)$



$\Delta(r, R)$

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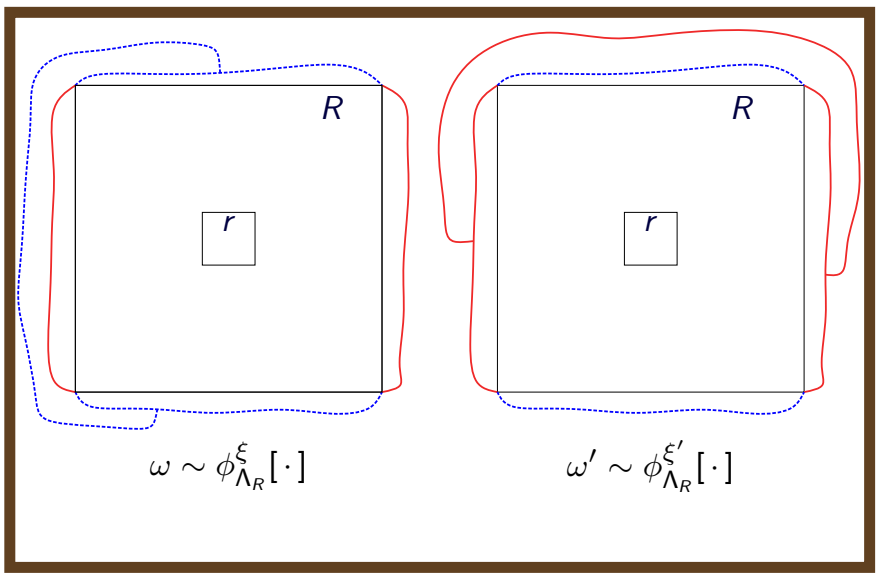

$$\Delta(r,R) \geq \phi^{\xi'}_{\Lambda_R}[\mathcal{C}(\Lambda_r)] - \phi^{\xi}_{\Lambda_R}[\mathcal{C}(\Lambda_r)]$$

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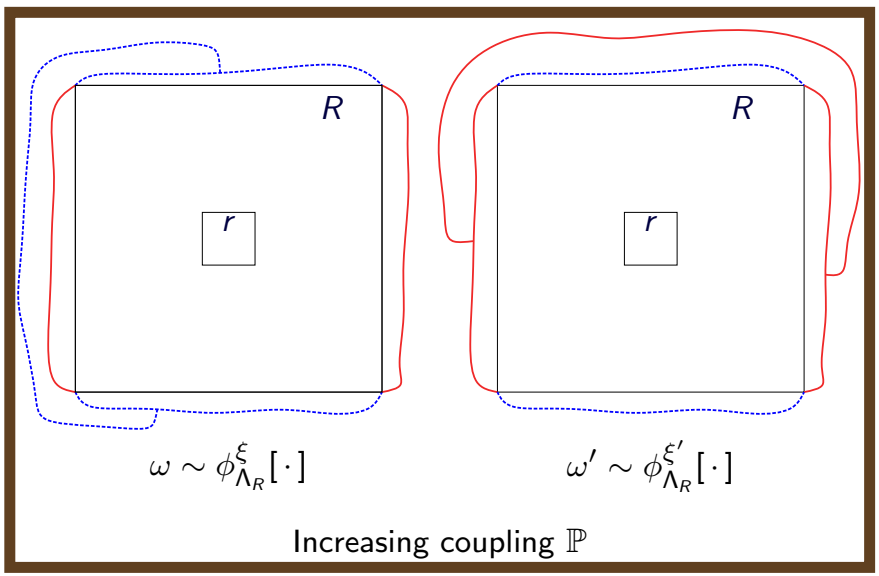

$$\Delta(r,R) \geq \phi^{\xi'}_{\Lambda_R}[\mathcal{C}(\Lambda_r)] - \phi^{\xi}_{\Lambda_R}[\mathcal{C}(\Lambda_r)]$$
$$= \mathbb{P}[\omega' \in \mathcal{C}(\Lambda_r), \omega \notin \mathcal{C}(\Lambda_r)]$$

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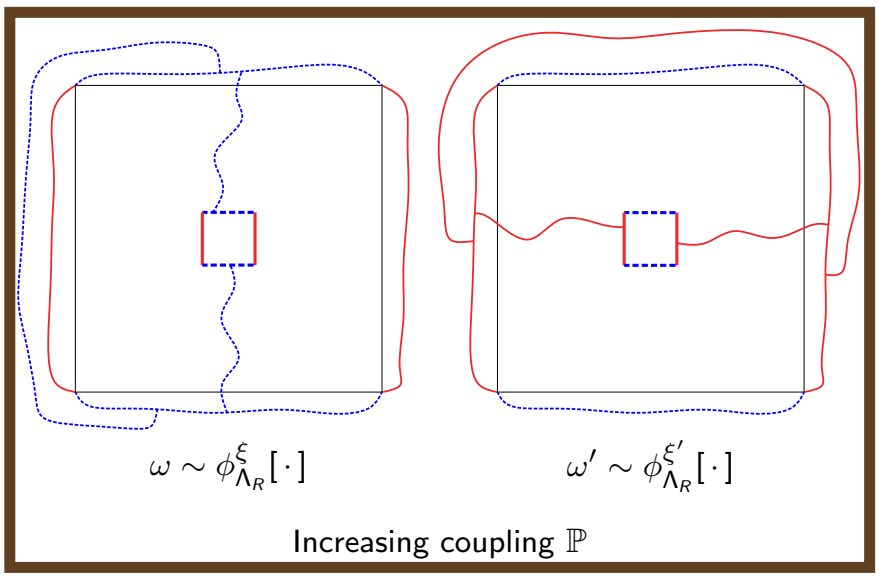

$$\begin{aligned}\Delta(r,R) &\geq \phi^{\xi'}_{\Lambda_R}[\mathcal{C}(\Lambda_r)] - \phi^{\xi}_{\Lambda_R}[\mathcal{C}(\Lambda_r)] \\ &= \mathbb{P}[\omega' \in \mathcal{C}(\Lambda_r), \omega \notin \mathcal{C}(\Lambda_r)] \\ &\geq \tfrac{1}{C}.\mathbb{P}[\widetilde{\mathcal{A}}_4(r,R)]\end{aligned}$$

$$\widetilde{\mathcal{A}}_4(r,R) := \{\text{2 dual arms in } \omega \text{ and 2 primal arms in } \omega'\}$$

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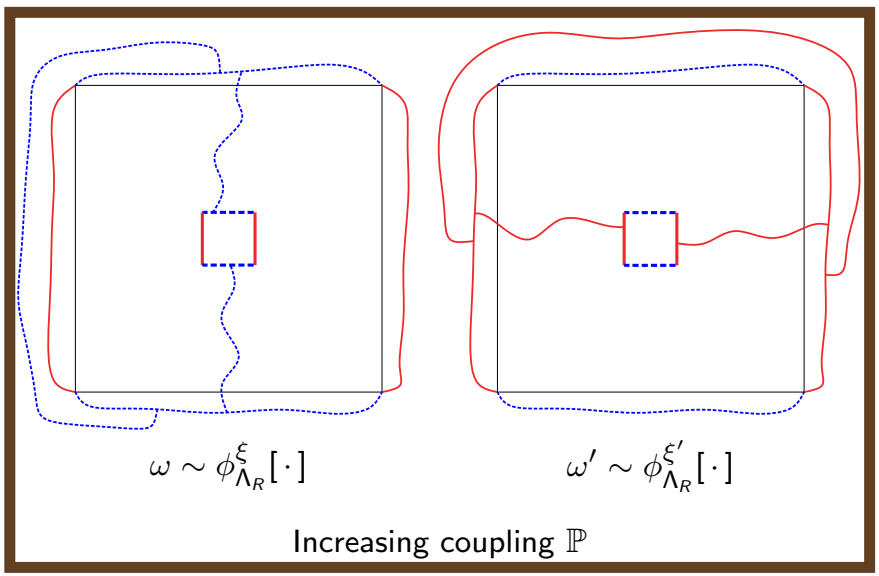

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$\widetilde{\mathcal{A}}_4(r,R) \coloneqq \{\text{2 dual arms in } \omega \text{ and 2 primal arms in } \omega'\}$

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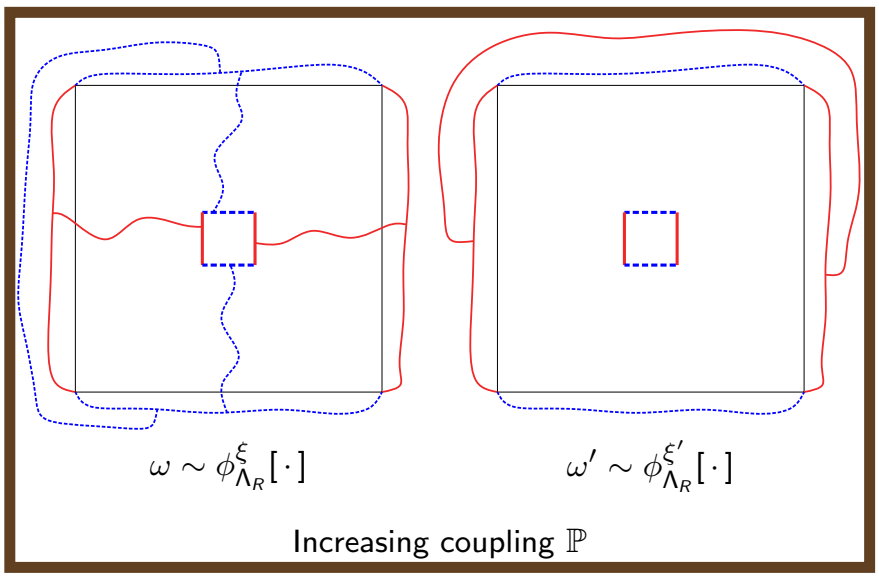

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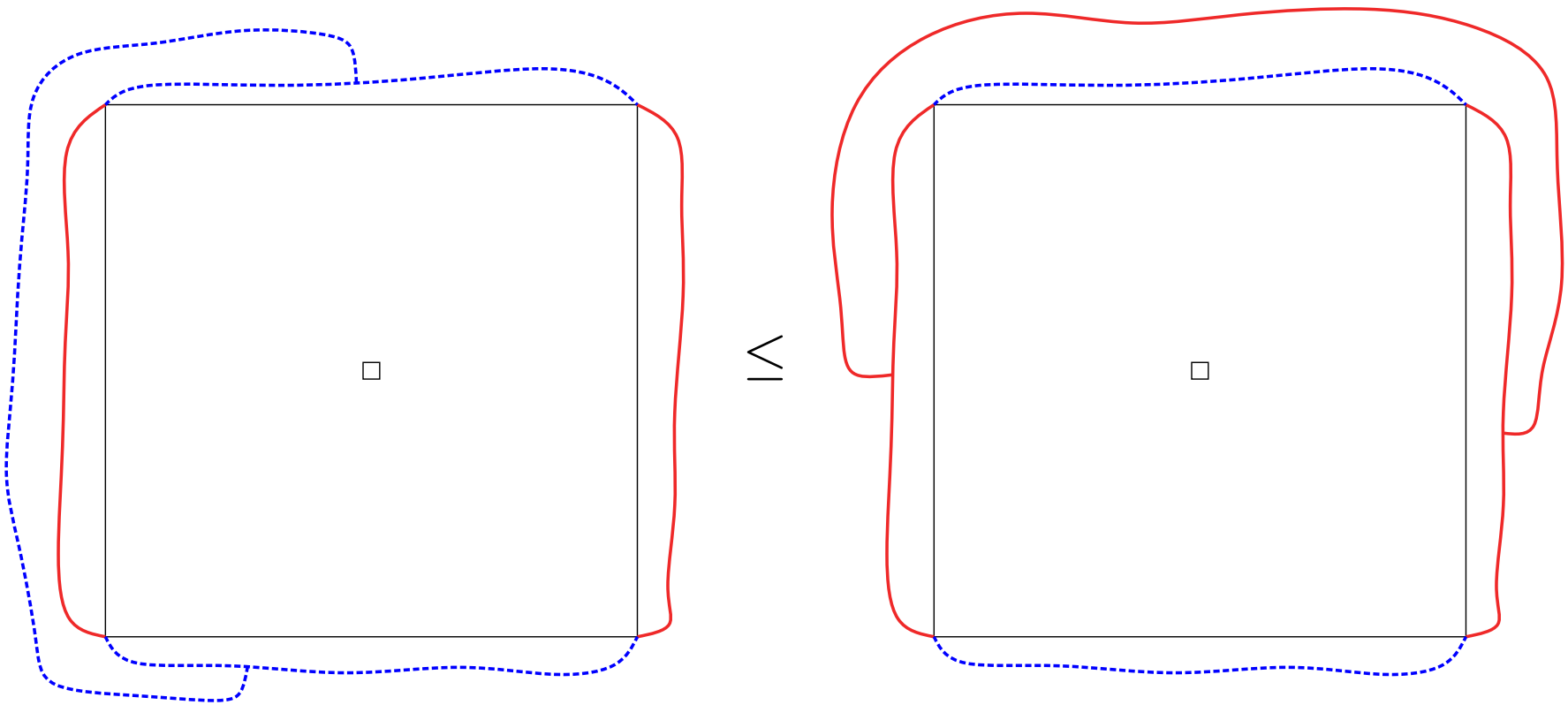

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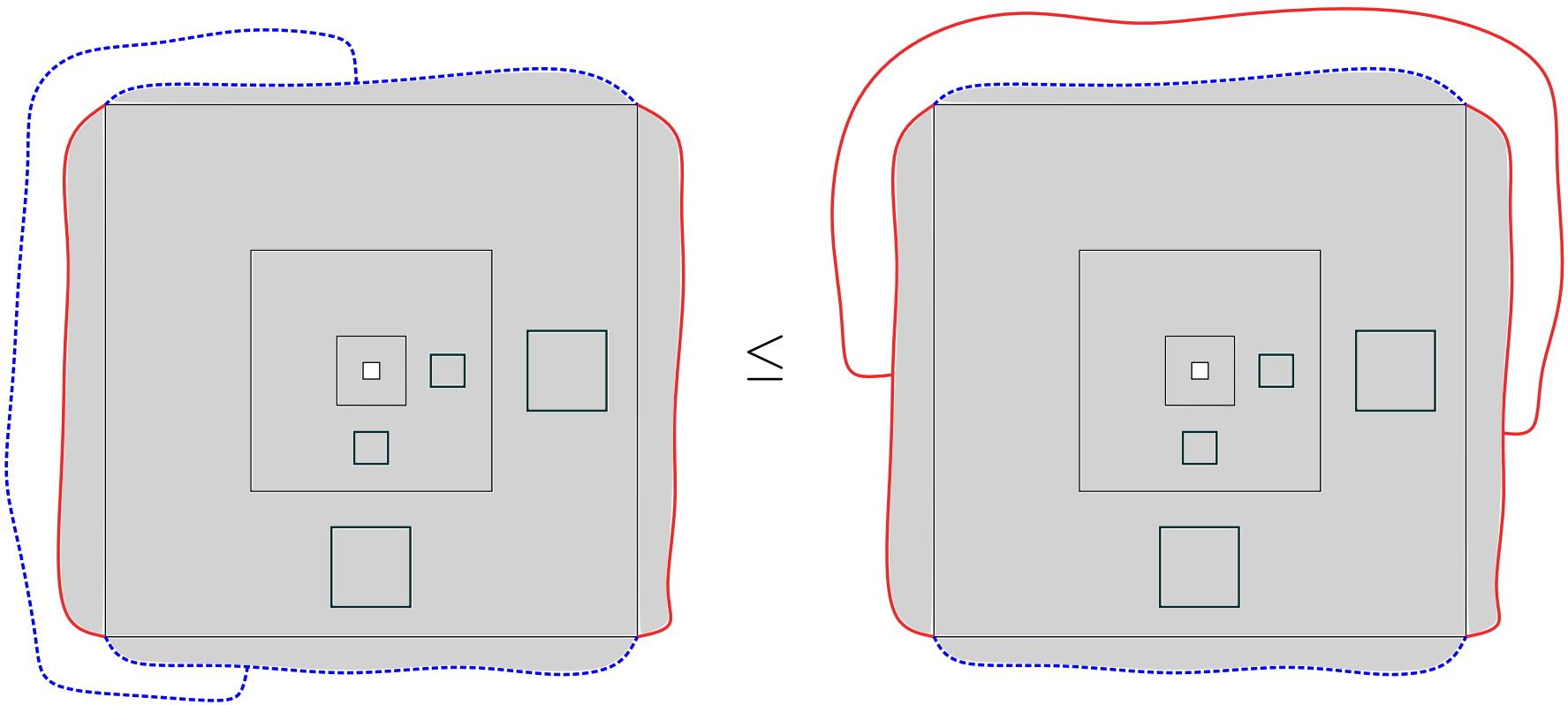

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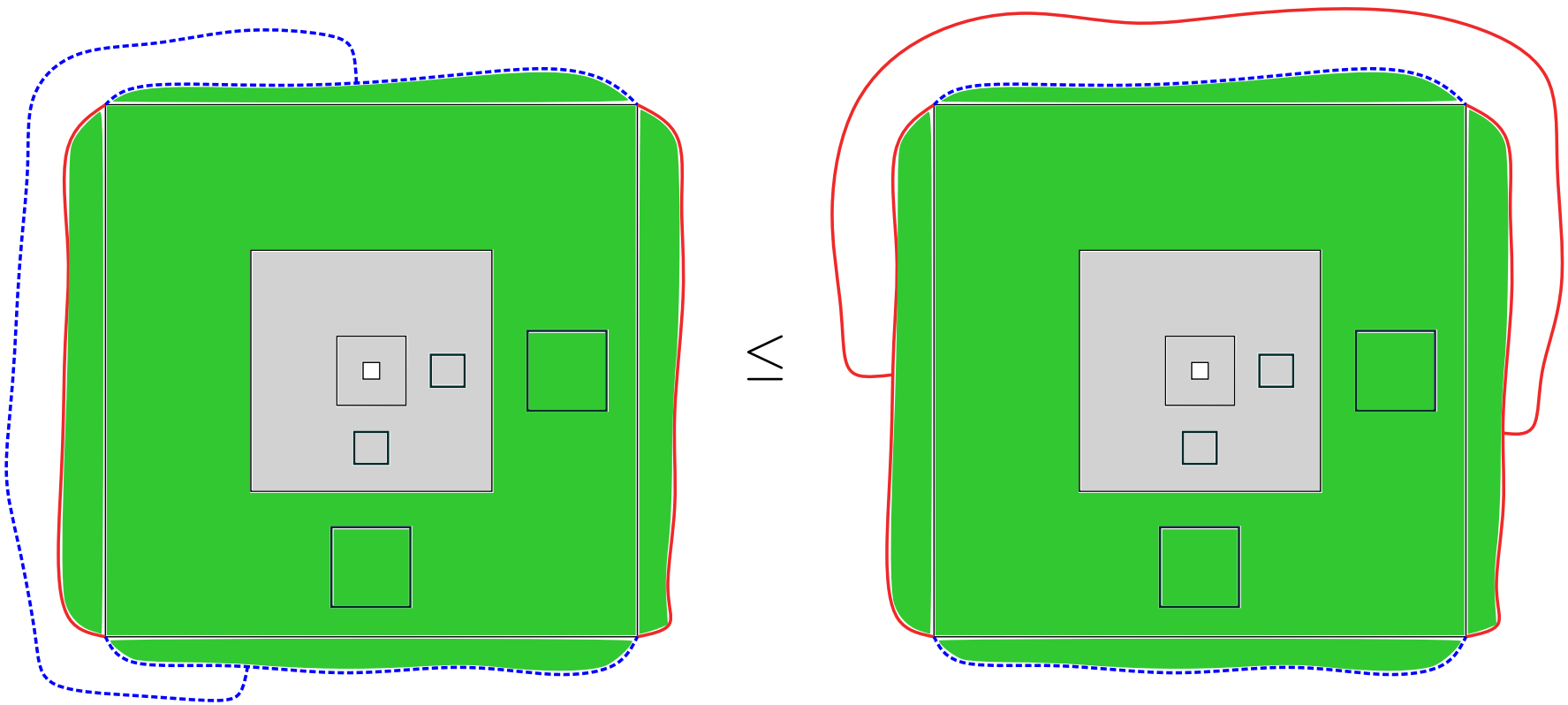

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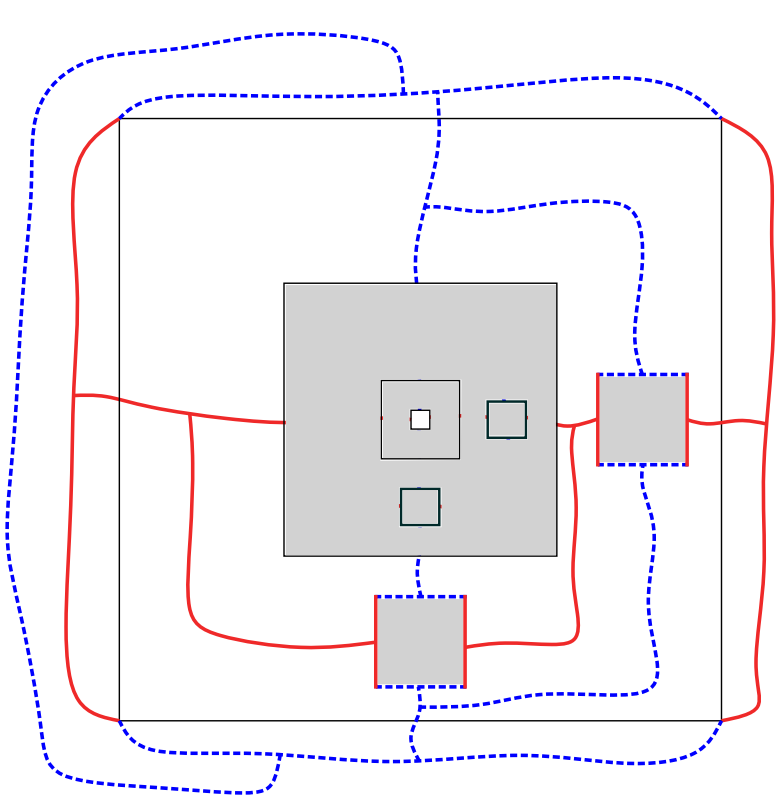
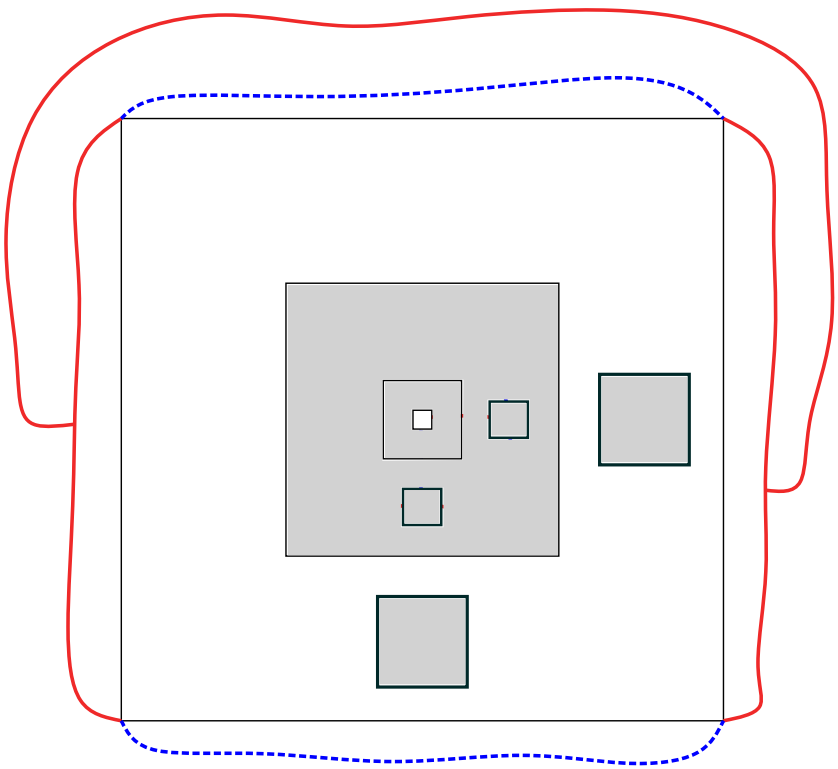


$\omega \sim \phi^{\xi}[\cdot]$

$\leq$



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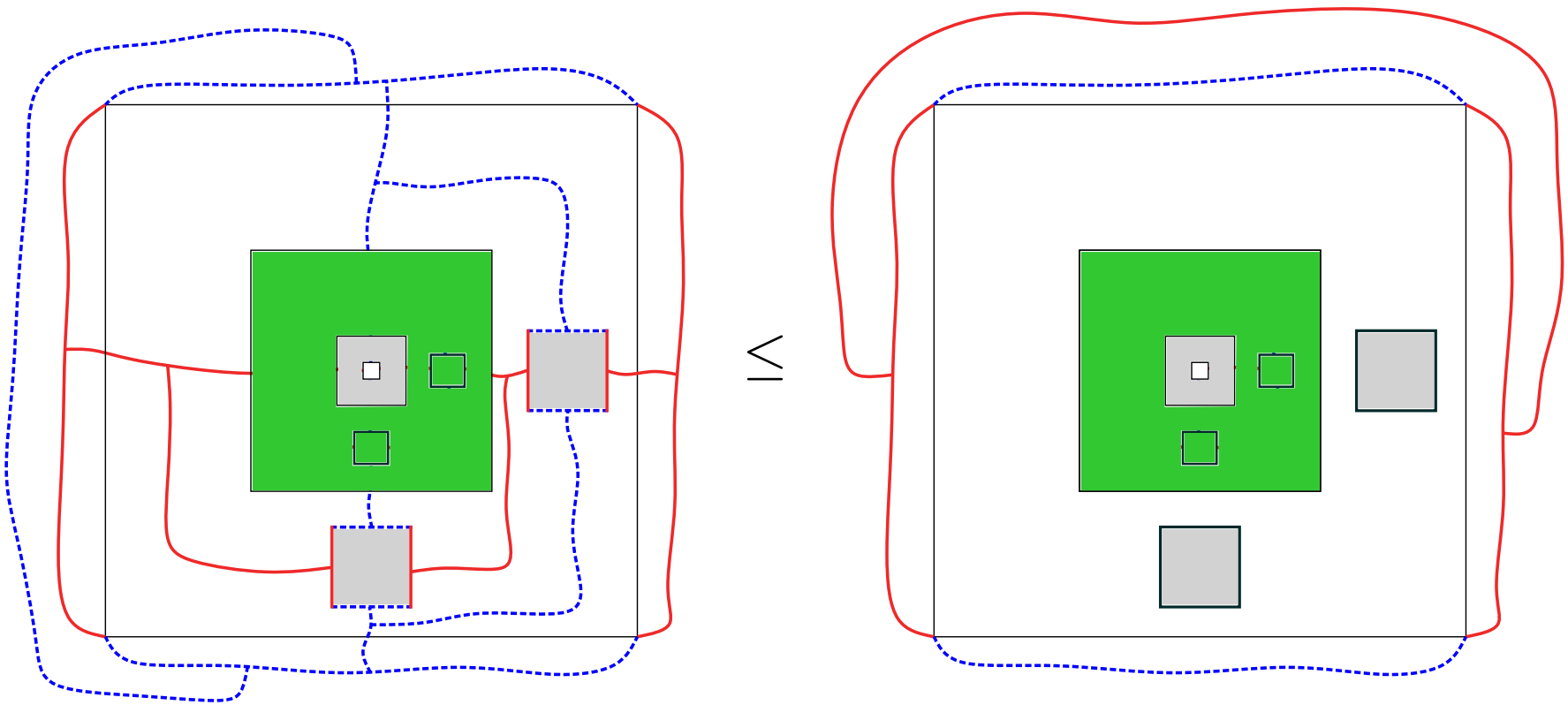

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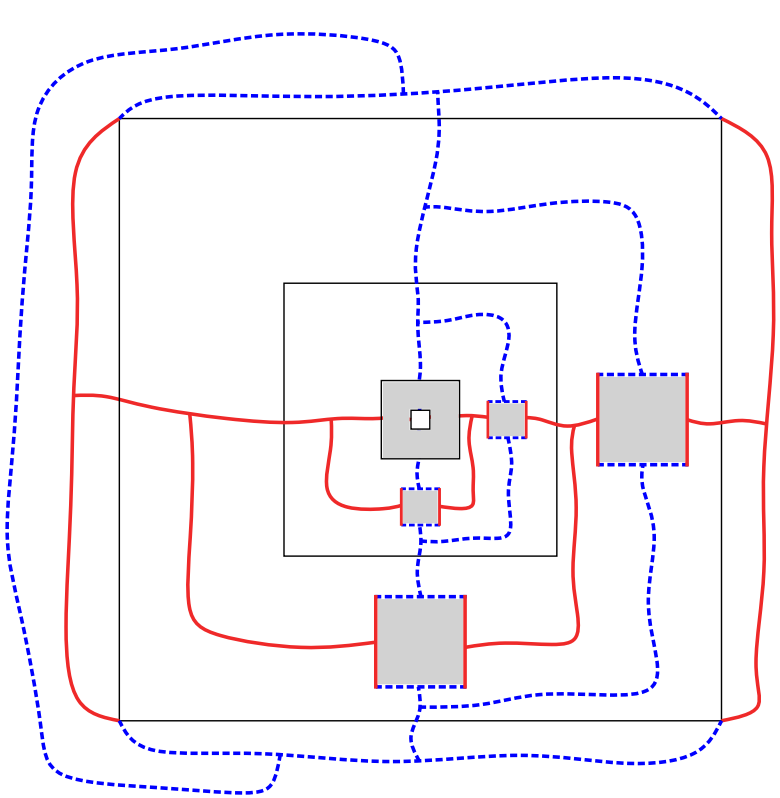

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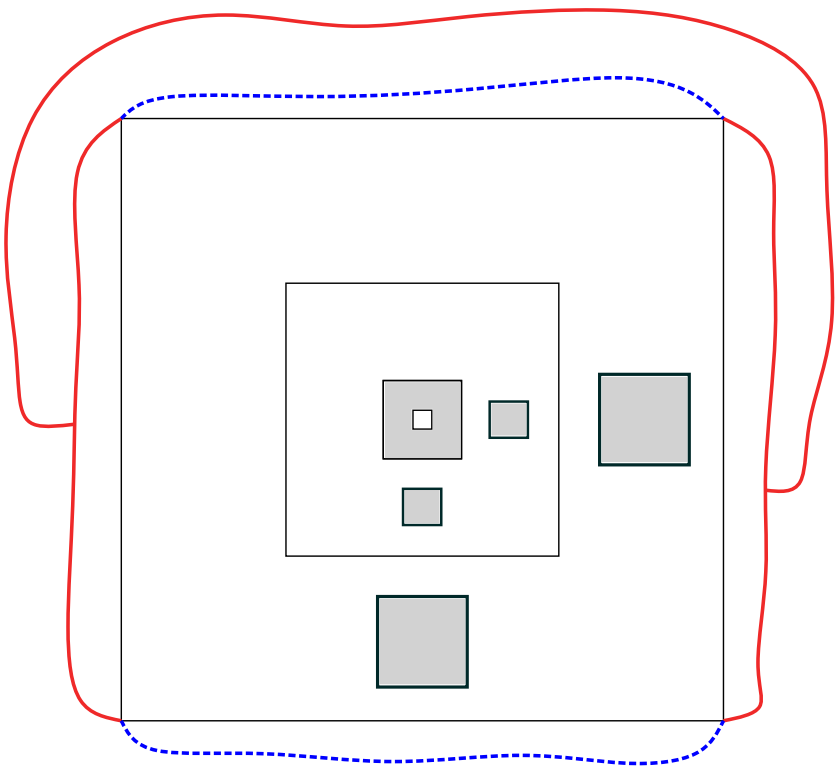

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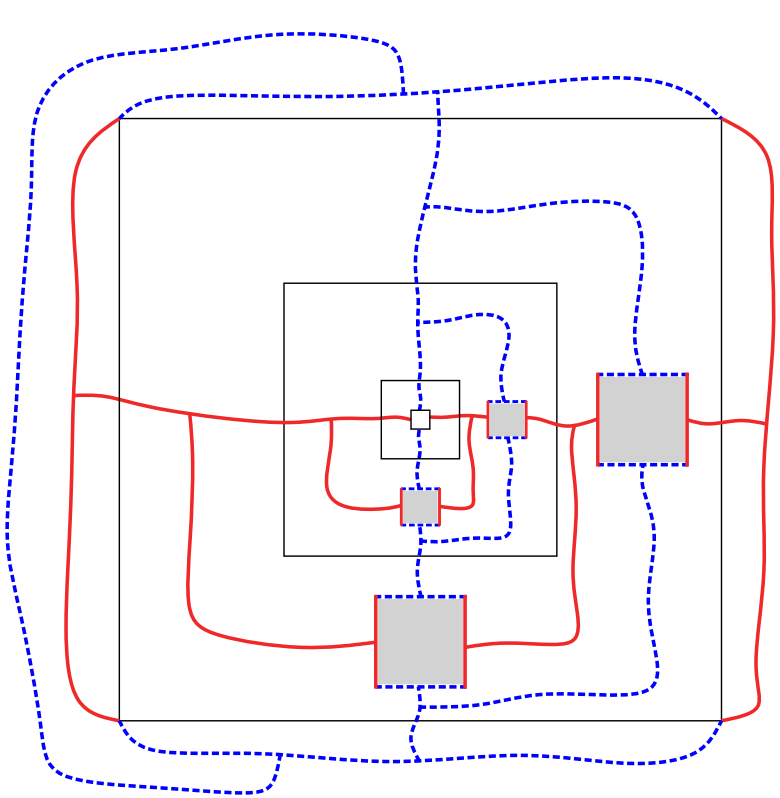

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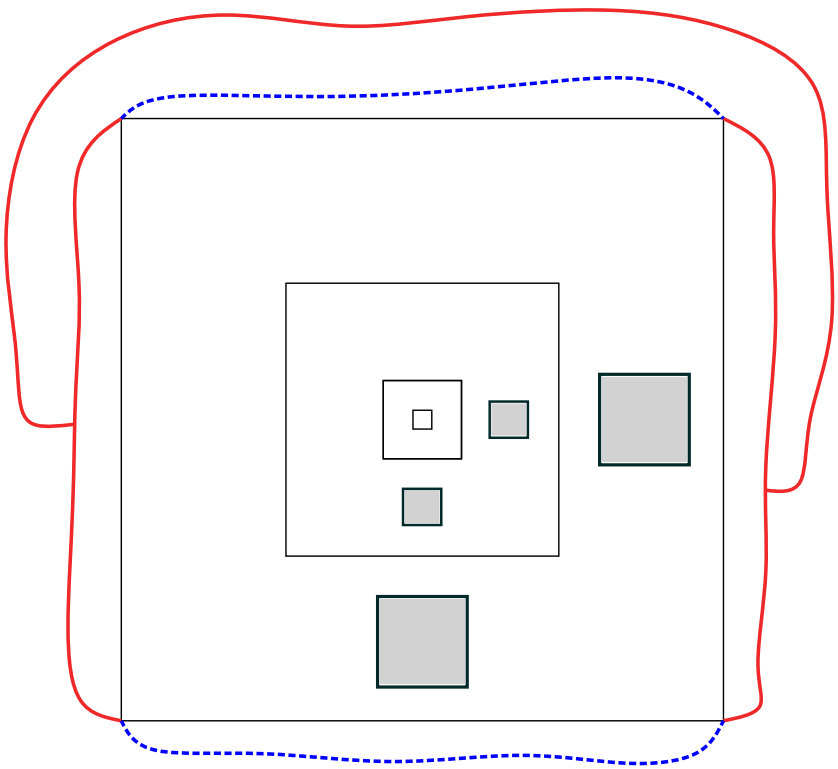

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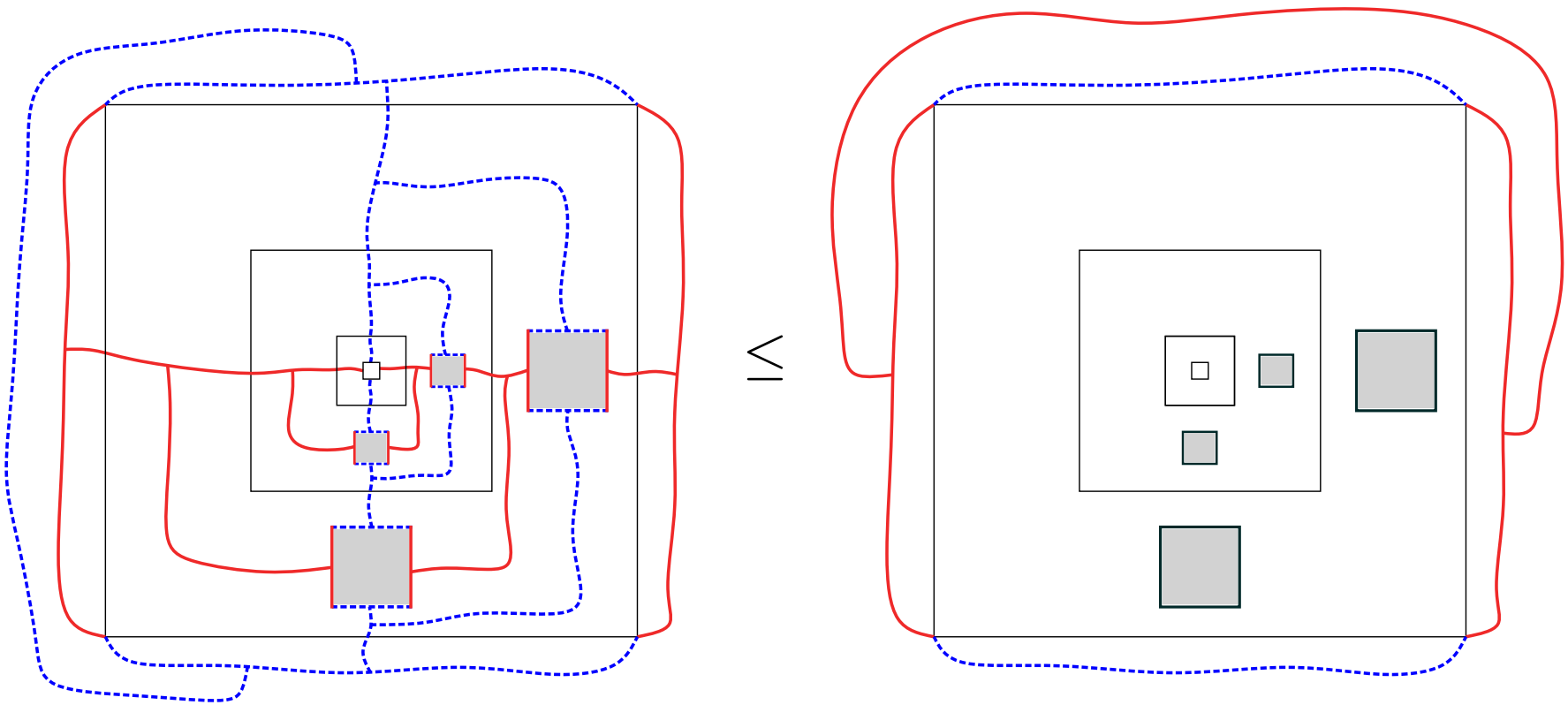

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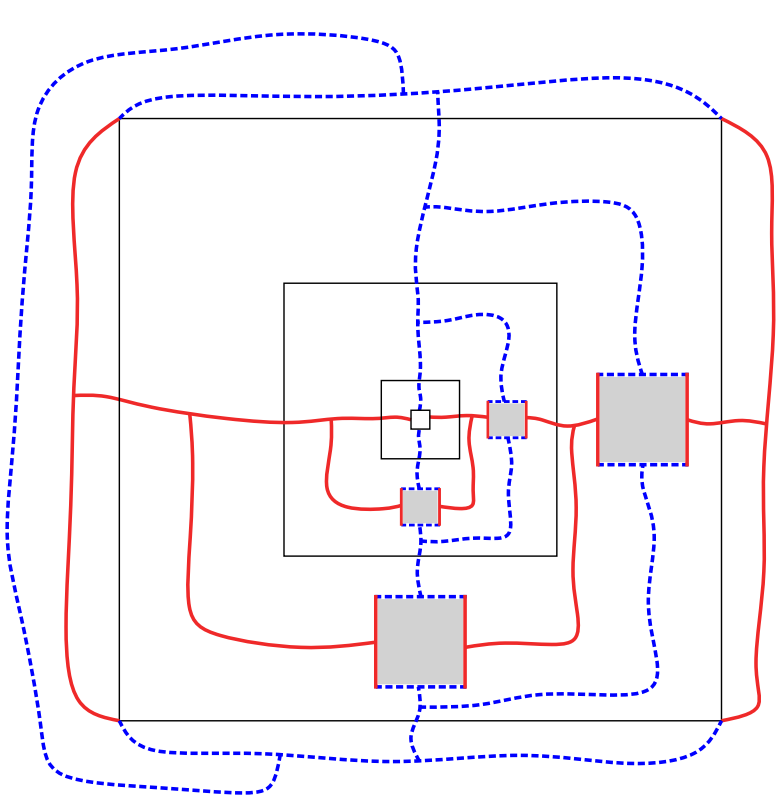

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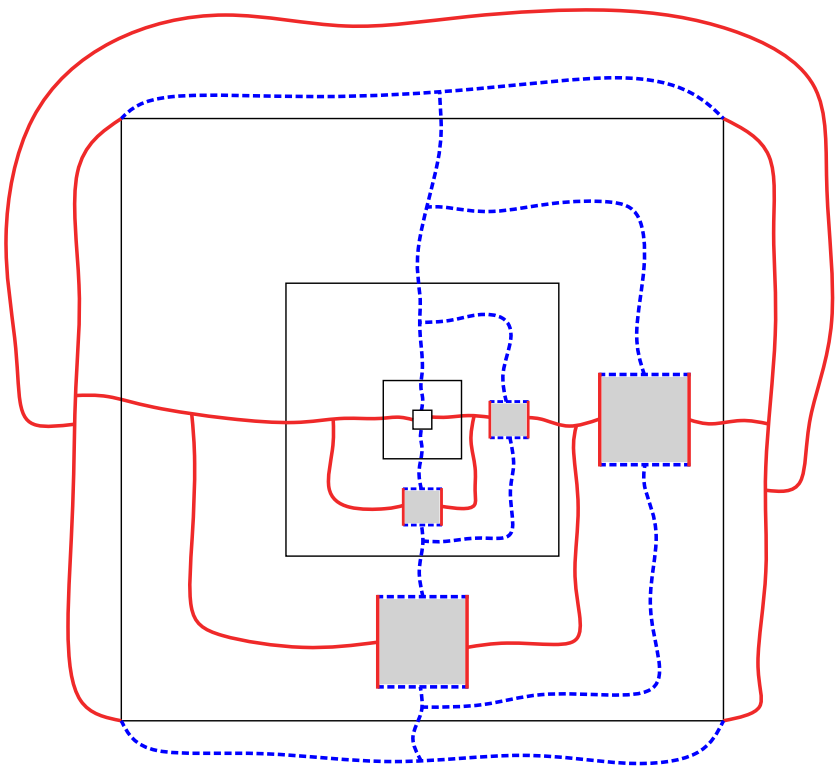

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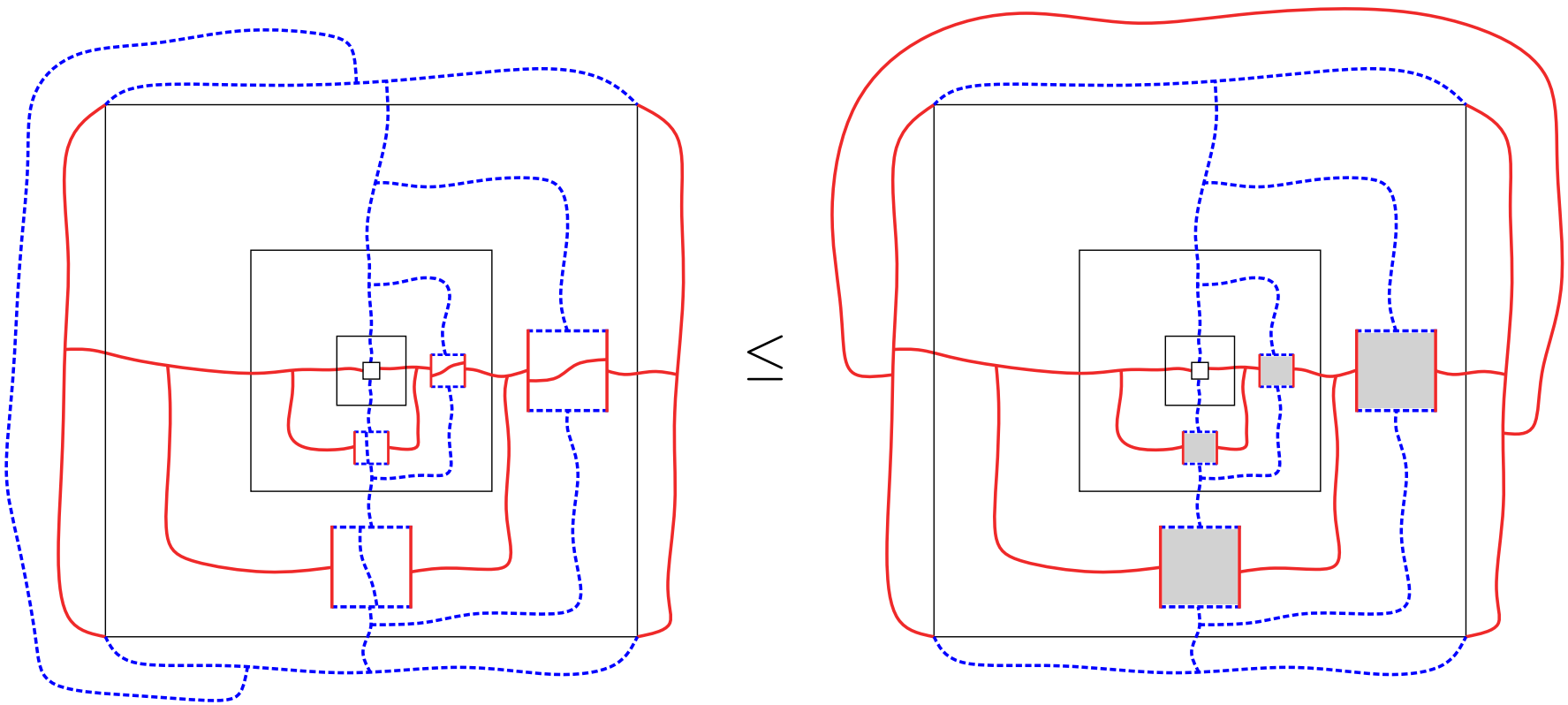

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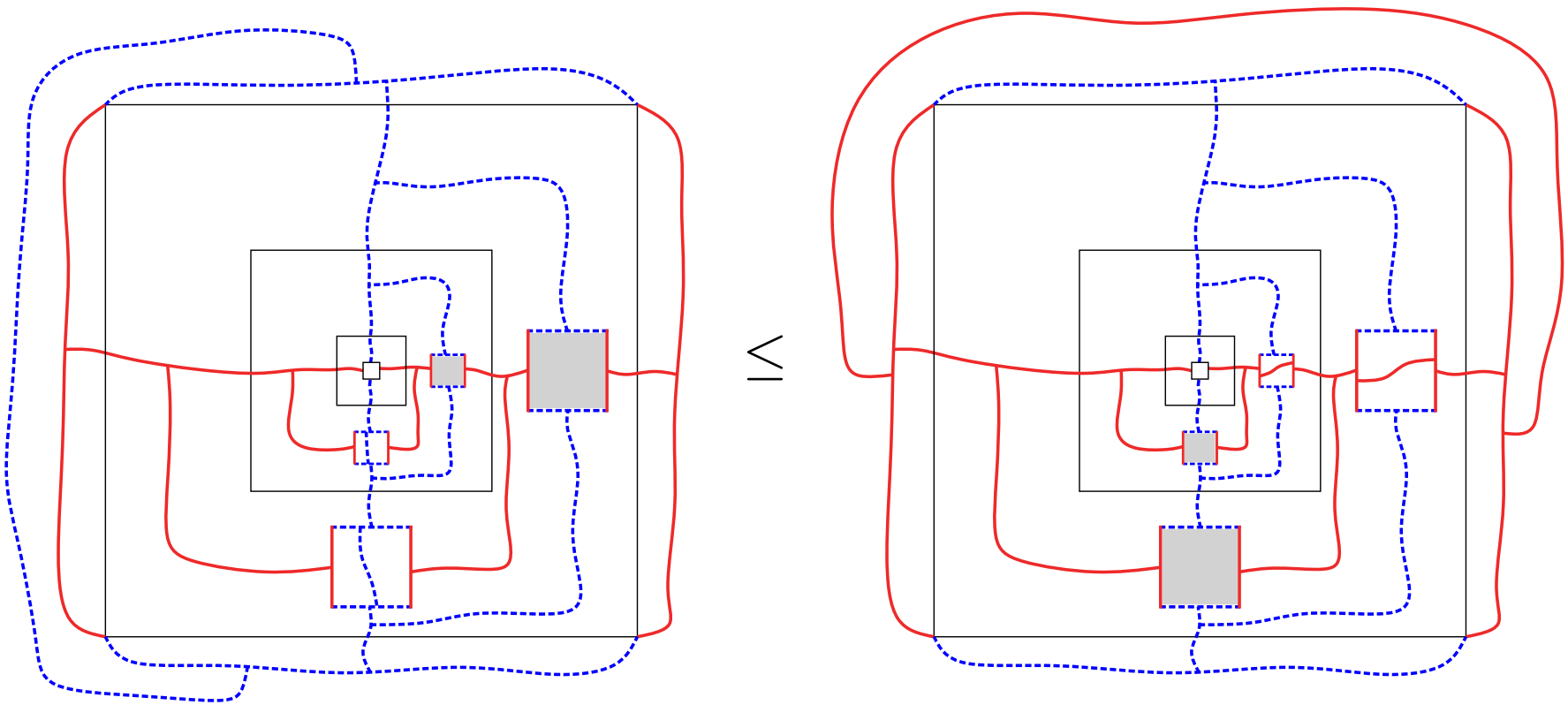

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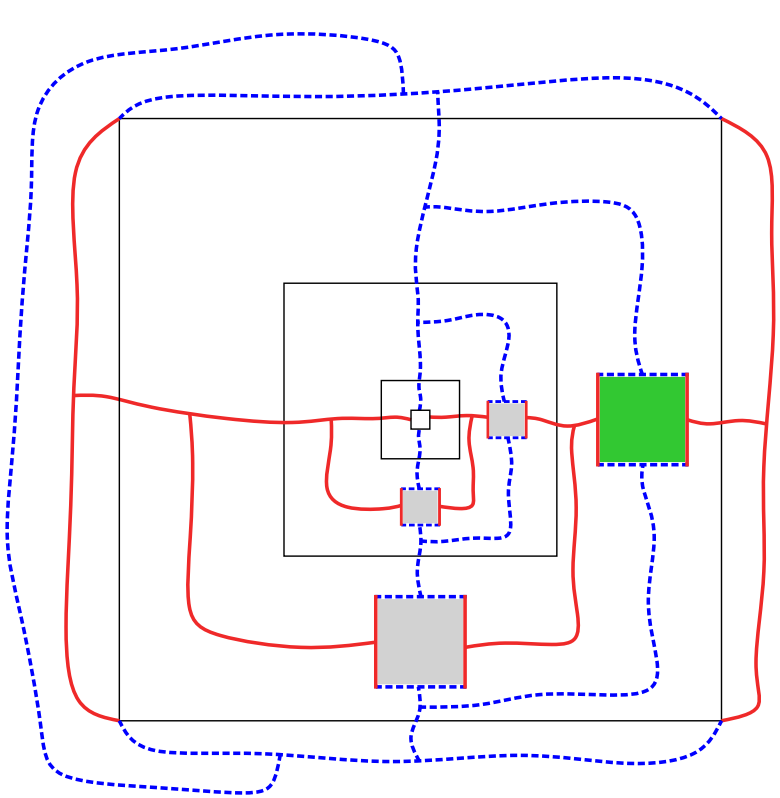

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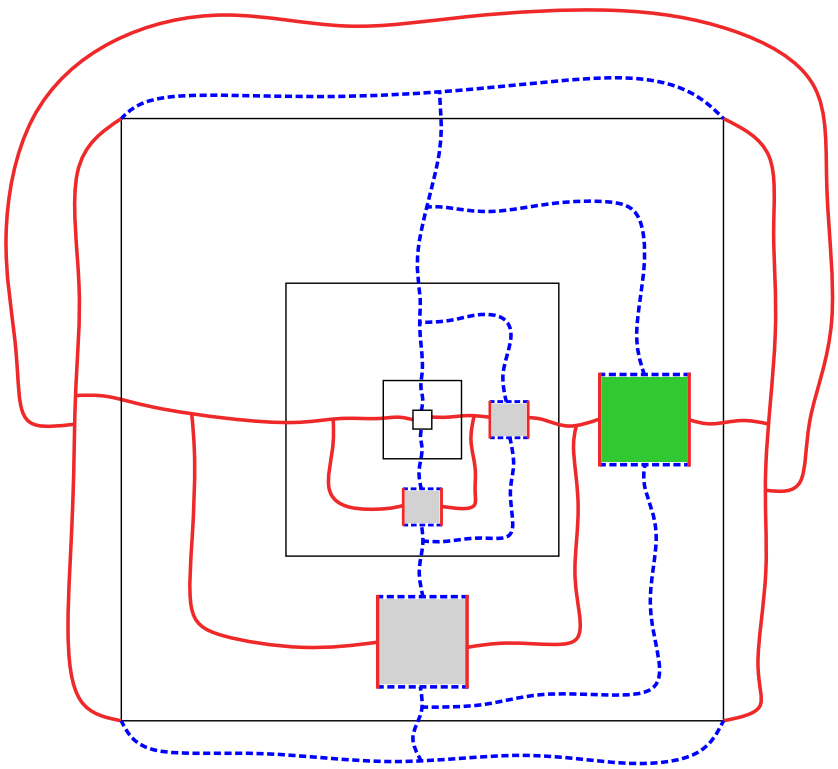

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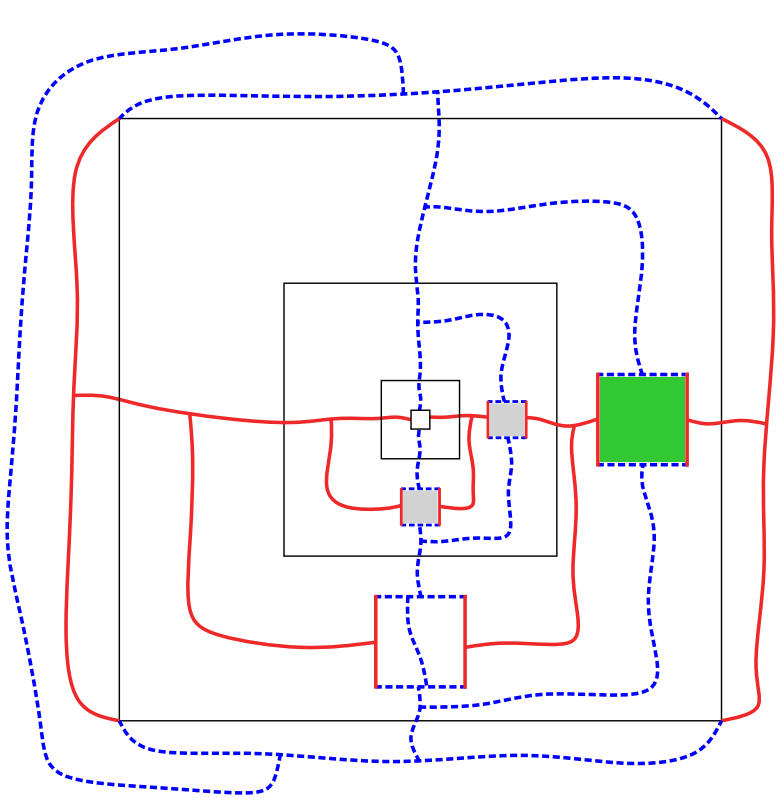

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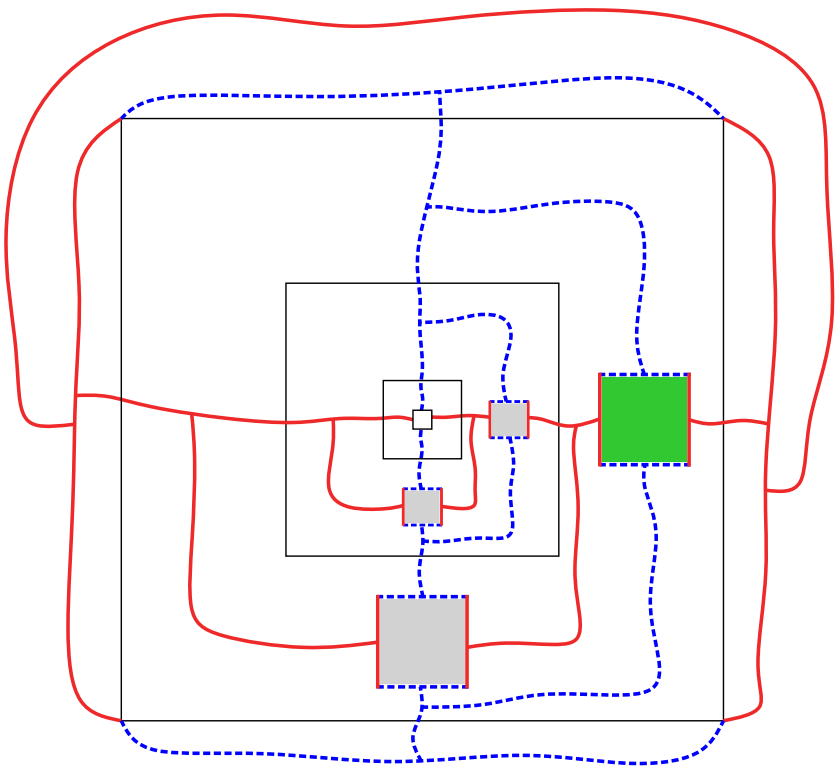

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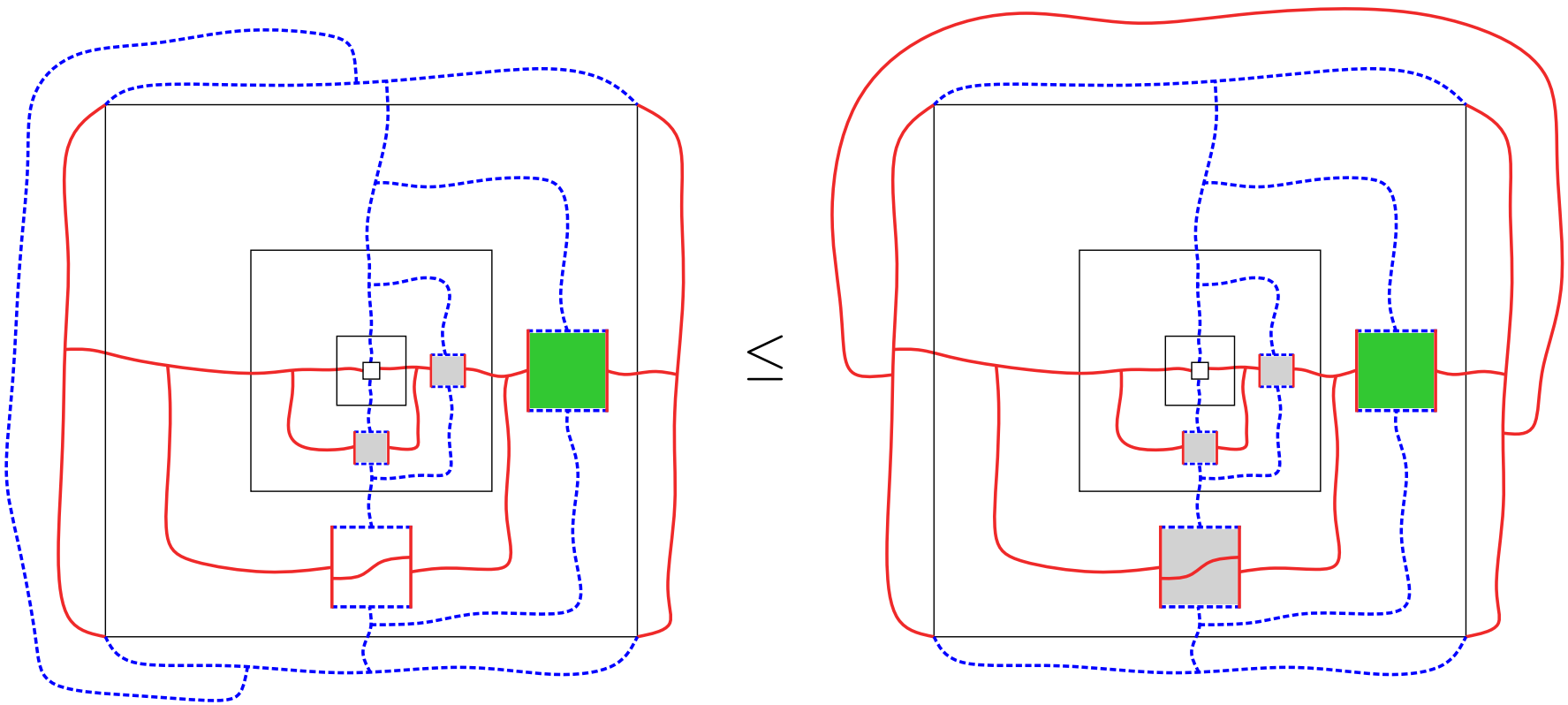

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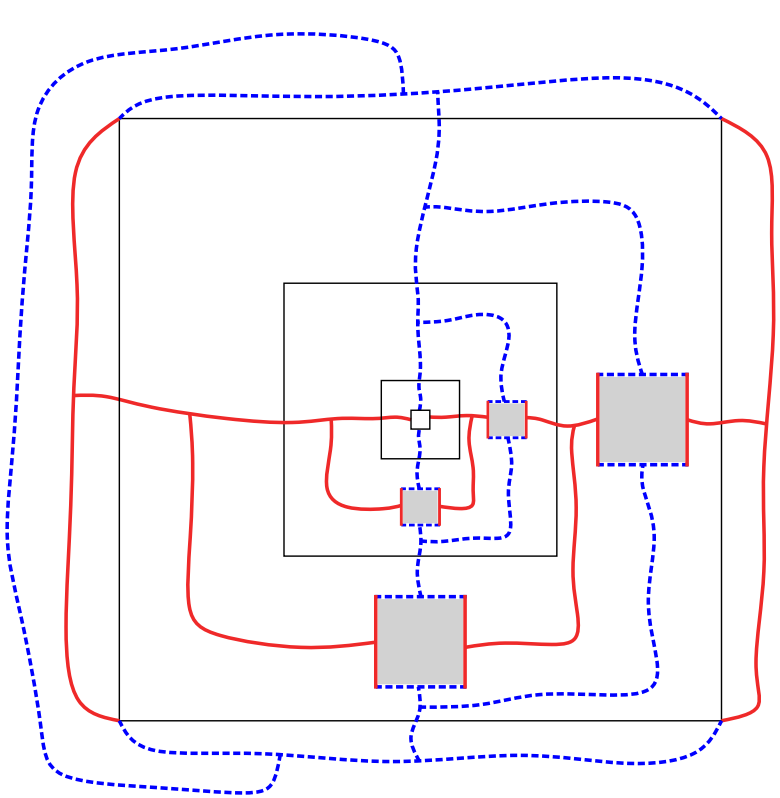

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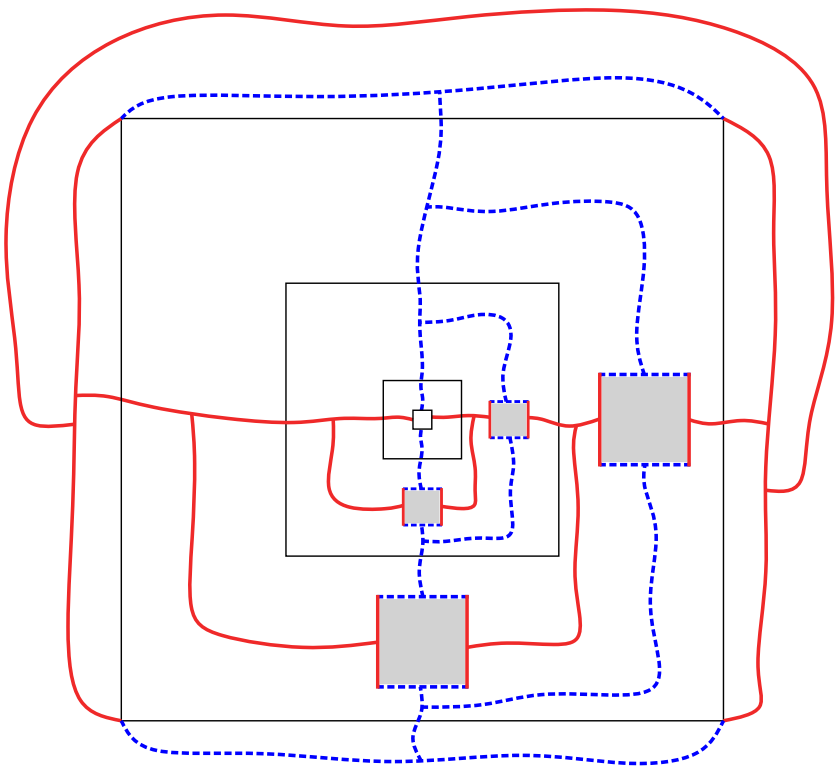

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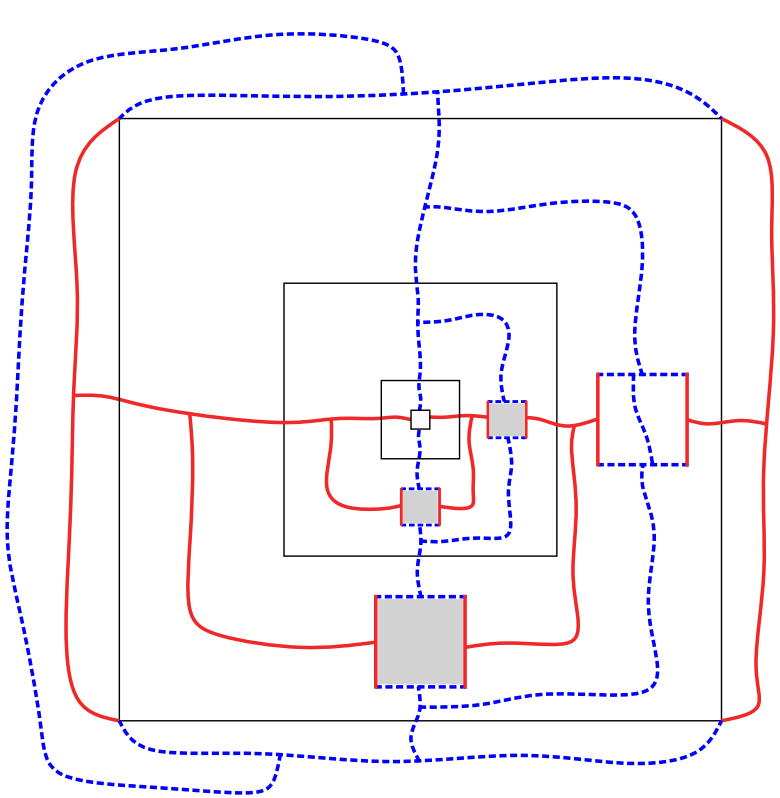

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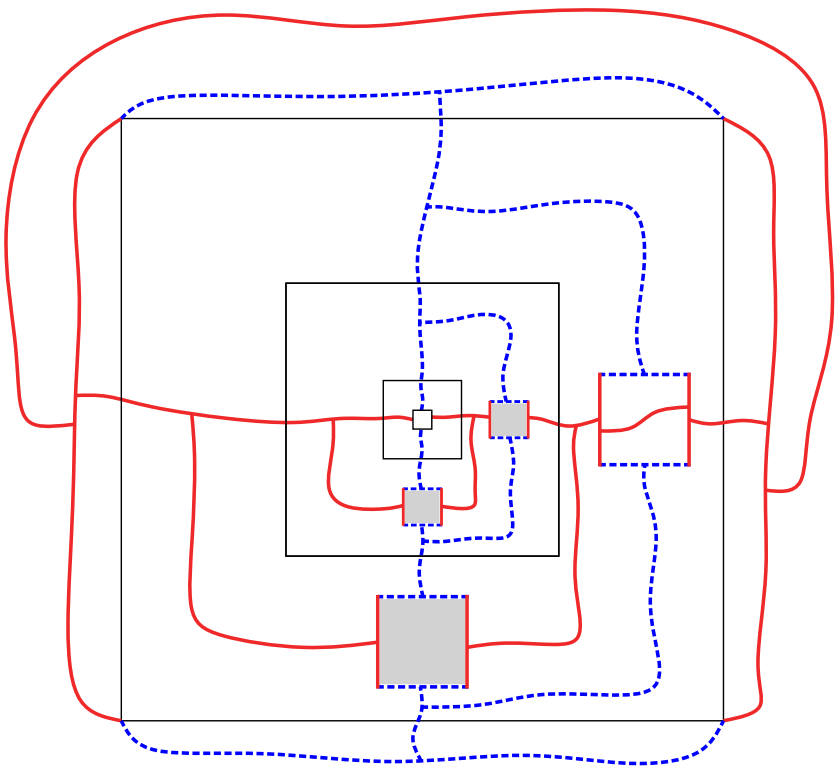

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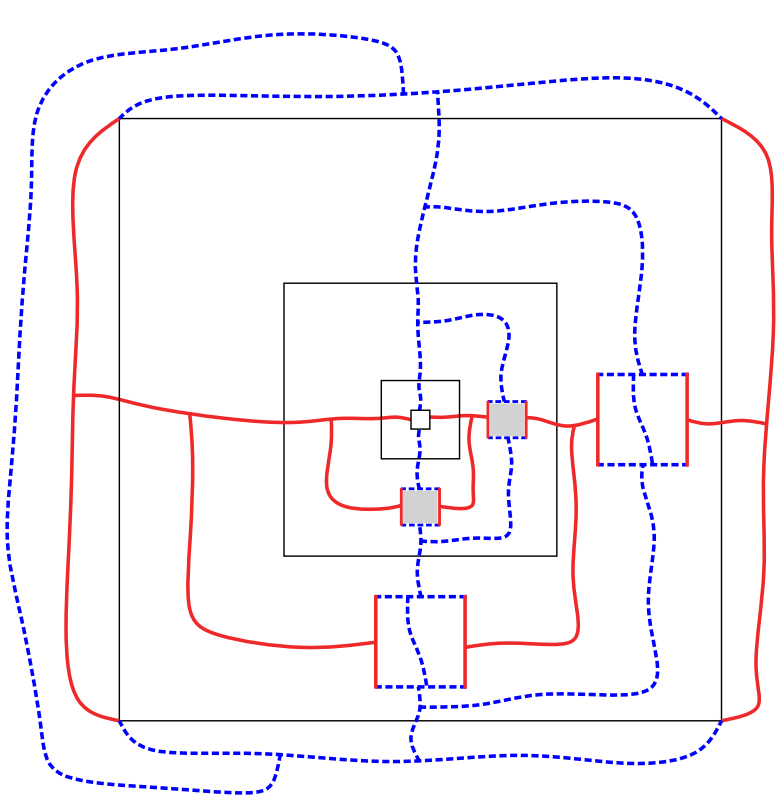

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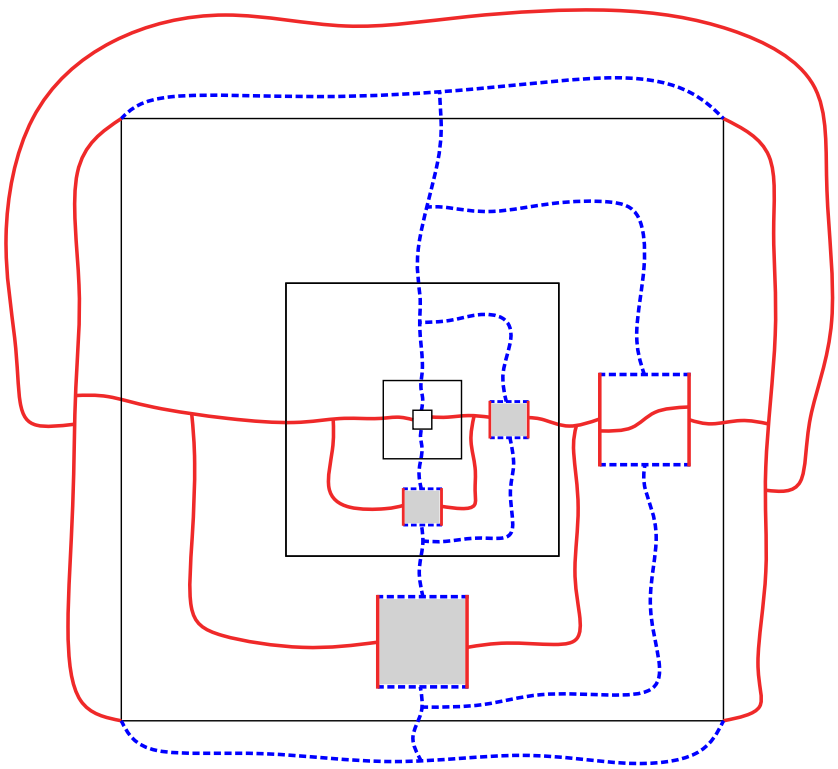
$\leq$



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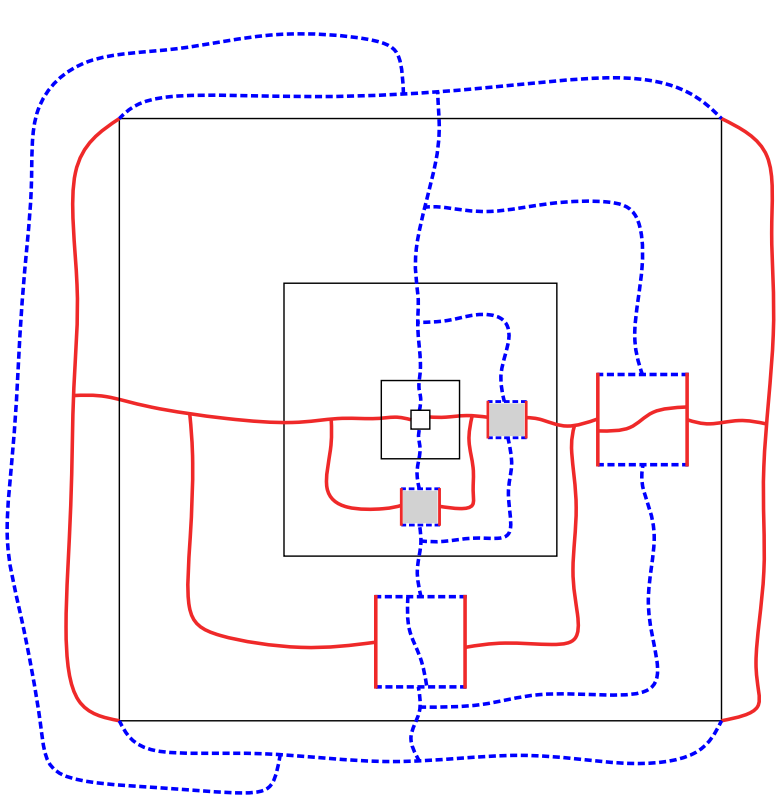

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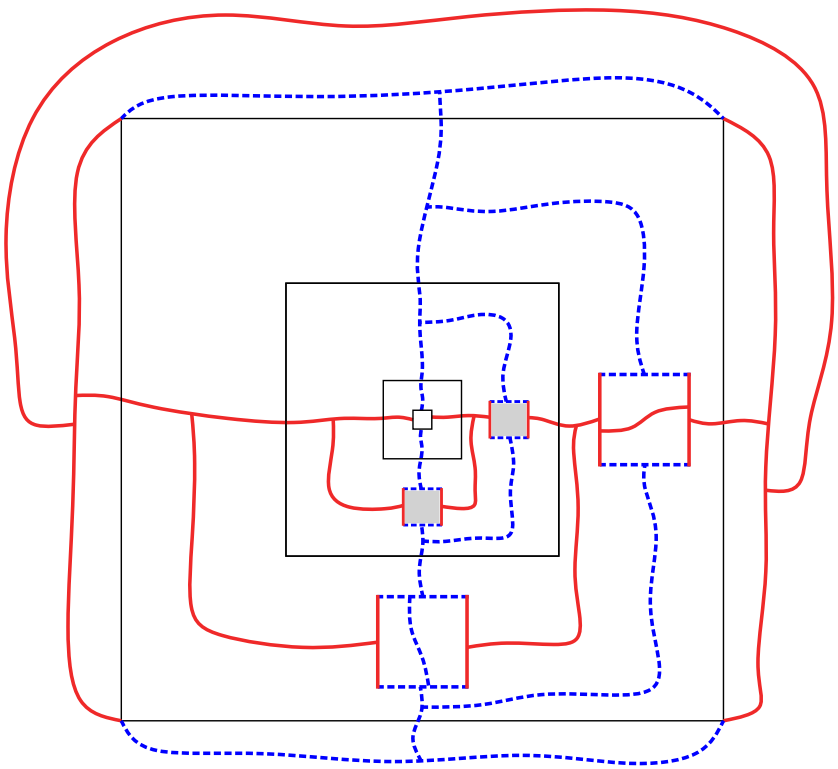


$\omega' \sim \phi^{\xi'}[\,\cdot\,]$

Properties of $\Delta_p(r,R)$: [Duminil-Copin and Manolescu 2020]

- **Controls the mixing:** For every $2r \leq R$,

$$\Delta(r,R) \asymp \max\left\{ \left| \frac{\phi_p[A \cap B]}{\phi_p[A]\phi_p[B]} - 1 \right| \, : \, A \in \mathcal{F}(\Lambda_r) \text{ and } B \in \mathcal{F}(\mathbb{Z}^2 \setminus \Lambda_R) \right\}$$

- **Stability in critical window:** For $p \in (0,1)$ and $1 \leq 2r \leq R \leq \xi(p)$,

$$\Delta_p(r,R) \asymp \Delta(r,R)$$

- **Quasi-multiplicativity:**

$$\forall r \leq R, \Delta(1,r)\Delta(r,R) \asymp \Delta(1,R)$$

- **Determines the critical window:**

$$\xi(p)^2 \Delta(1,\xi(p)) \asymp |p - p_c|^{-1}$$

$$\nu(2-\iota) = 1$$

with ν satisfying $\xi(p) = |p - p_c|^{-\nu + o(1)}$

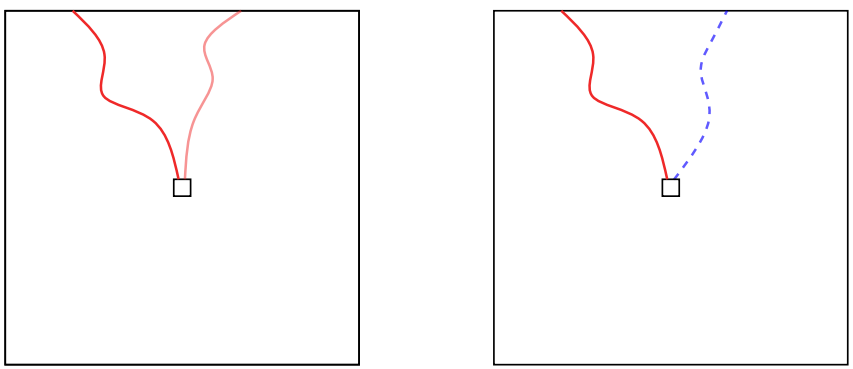
Scaling scaling relation for ν and ι

Suppose that $\xi(p) = |p - p_c|^{-\nu+o(1)}$

$q = 1$	$q \in (1, 4]$
$\frac{d}{dp}\phi_{p,1}[\mathcal{C}(\mathcal{D})] = \sum_{e\in\mathbb{E}} \phi_{p,1}[e \text{ pivotal for } \mathcal{C}(\mathcal{D})]$	$\frac{d}{dp}\phi_p[\mathcal{C}(\mathcal{D})] = \frac{1}{p(1-p)} \sum_{e\in\mathbb{E}} \mathrm{Cov}_p(\omega_e, \mathcal{C}(\mathcal{D}))$
$\xi(p)^2\pi_4(1, \xi(p)) \asymp \lvert p - p_c\rvert^{-1}$	$\xi(p)^2\Delta(1, \xi(p)) \asymp \lvert p - p_c\rvert^{-1}$
$\nu(2 - \alpha_4) = 1$	$\nu(2 - \iota) = 1$

$$\alpha_{01} > \alpha_0 + \alpha_1 \quad \checkmark$$

$$\alpha_{11} > \alpha_{01} \quad \times$$



$$\alpha_{01} > \alpha_0 + \alpha_1 \quad \checkmark$$

$$\alpha_{11} > \alpha_{01} \quad \times$$

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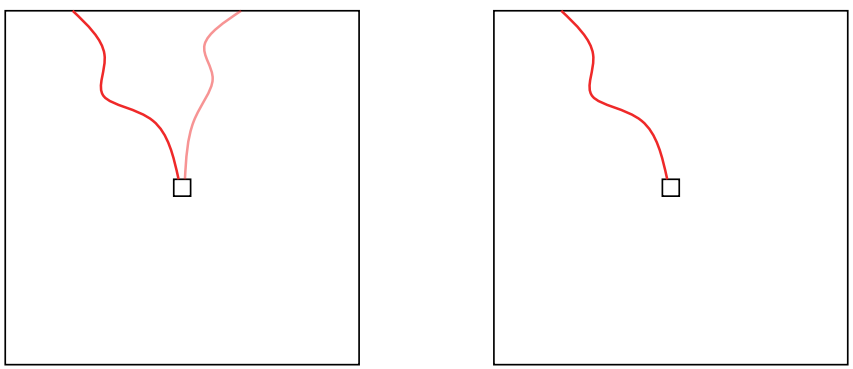
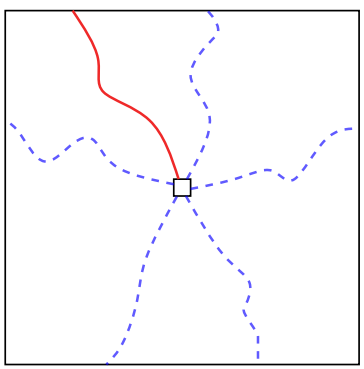


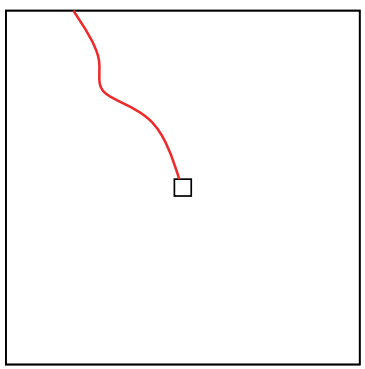
$$\alpha_{01} > \alpha_0 + \alpha_1 \quad \checkmark$$

$$\alpha_{11} > \alpha_{01} \quad \times$$

$$\alpha_{11} > \alpha_1 + \alpha_1 \quad \times$$

$$\alpha_{00\ldots01} > \alpha_{00\ldots0} + \alpha_1 \quad \checkmark$$





$$\alpha_{01} > \alpha_0 + \alpha_1 \quad \checkmark$$

$$\alpha_{11} > \alpha_{01} \quad \times$$

$$\alpha_{11} > \alpha_1 + \alpha_1 \quad \times$$

$$\alpha_{00\ldots01} > \alpha_{00\ldots0} + \alpha_1 \quad \checkmark$$

$$\alpha_{\underbrace{0\ldots0}_{k+1}1} - \alpha_{\underbrace{0\ldots0}_{k+1}} > \alpha_{\underbrace{0\ldots0}_{k}1} - \alpha_{\underbrace{0\ldots0}_{k}} \quad ?$$

