Brouwer's fixed point and the continuous Hex theorem

Loïc Gassmann (Fribourg)



Brouwer's fixed point theorem

Theorem: Brouwer's fixed point

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How many proofs do you know?

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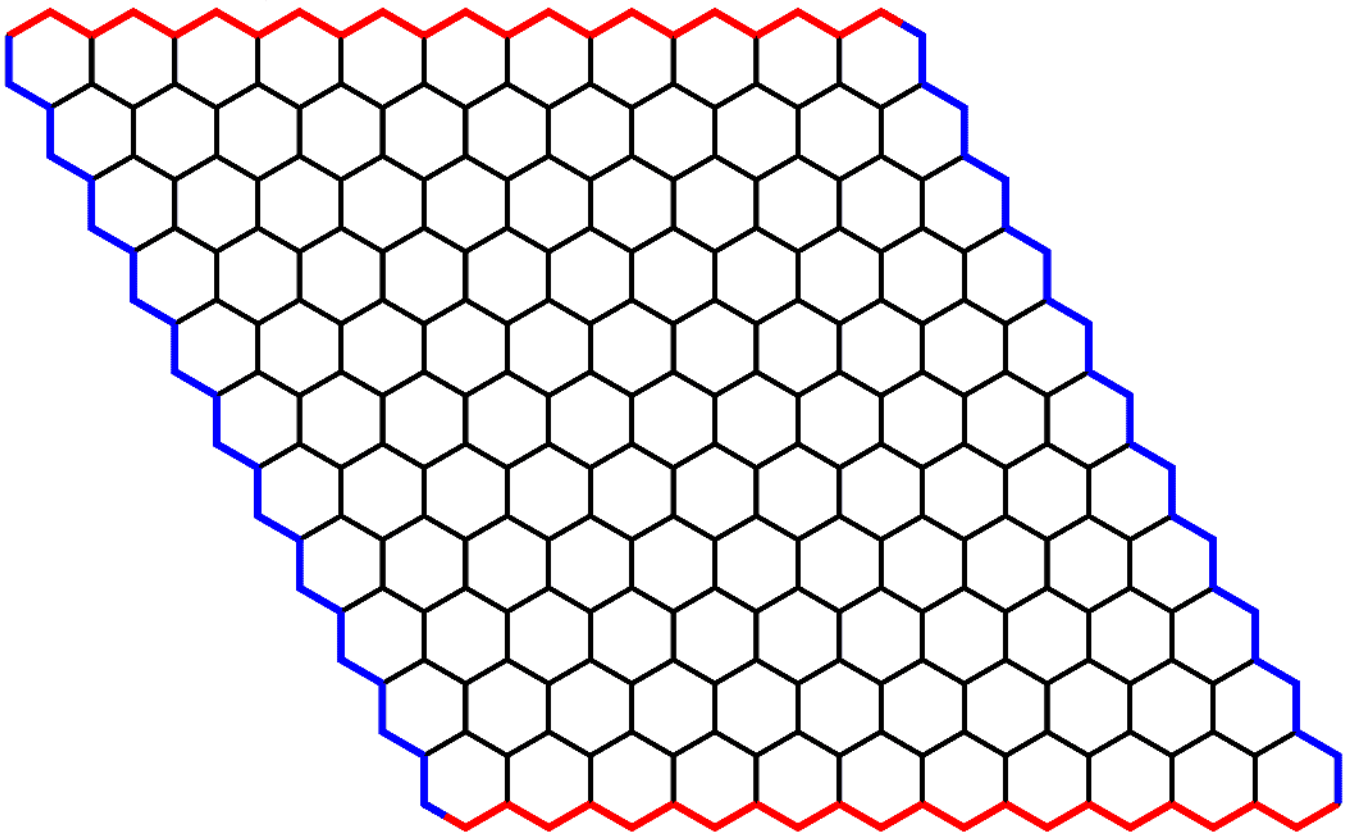
How many proofs do you know?

Wikipedia: 10 proofs

analysis, algebraic topology, combinatorics, ...

Hex game



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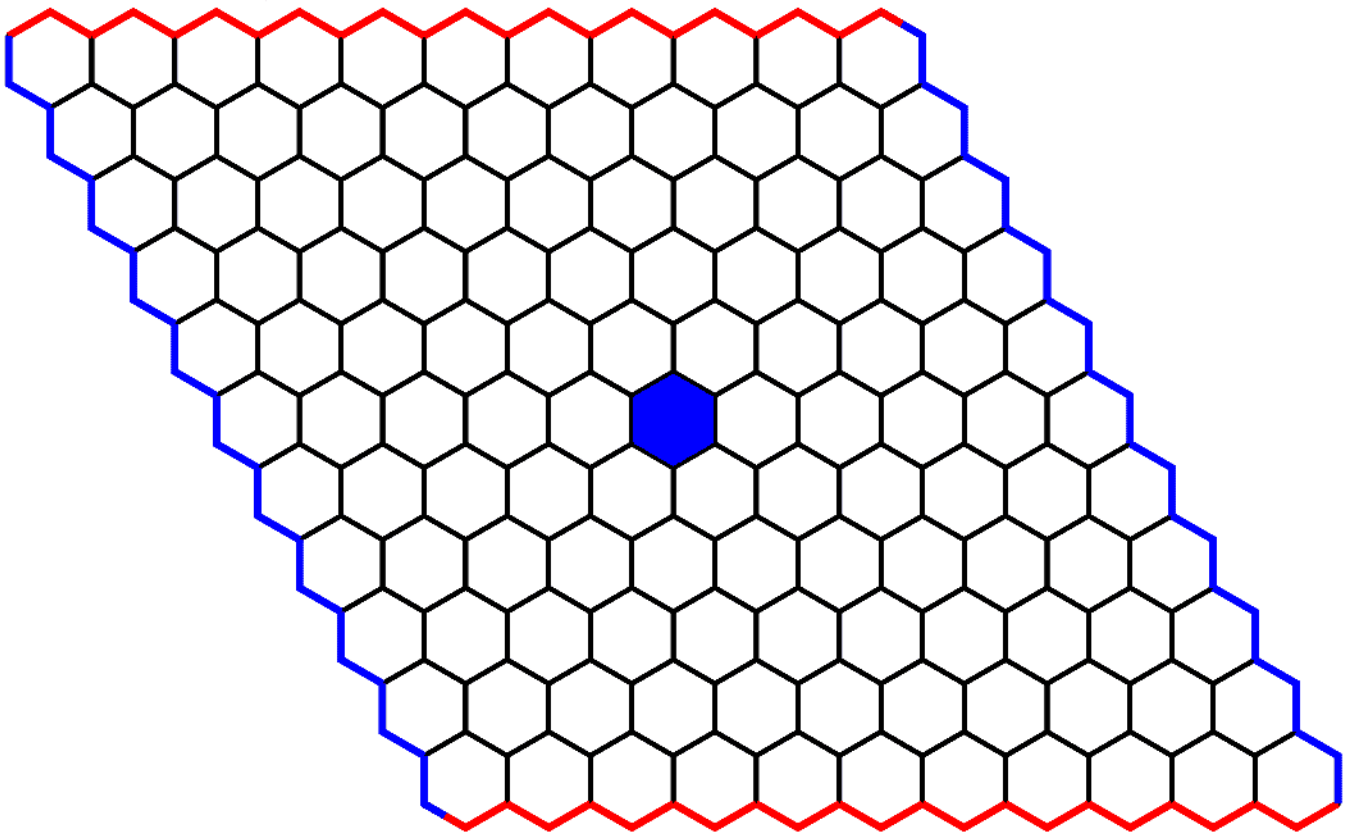

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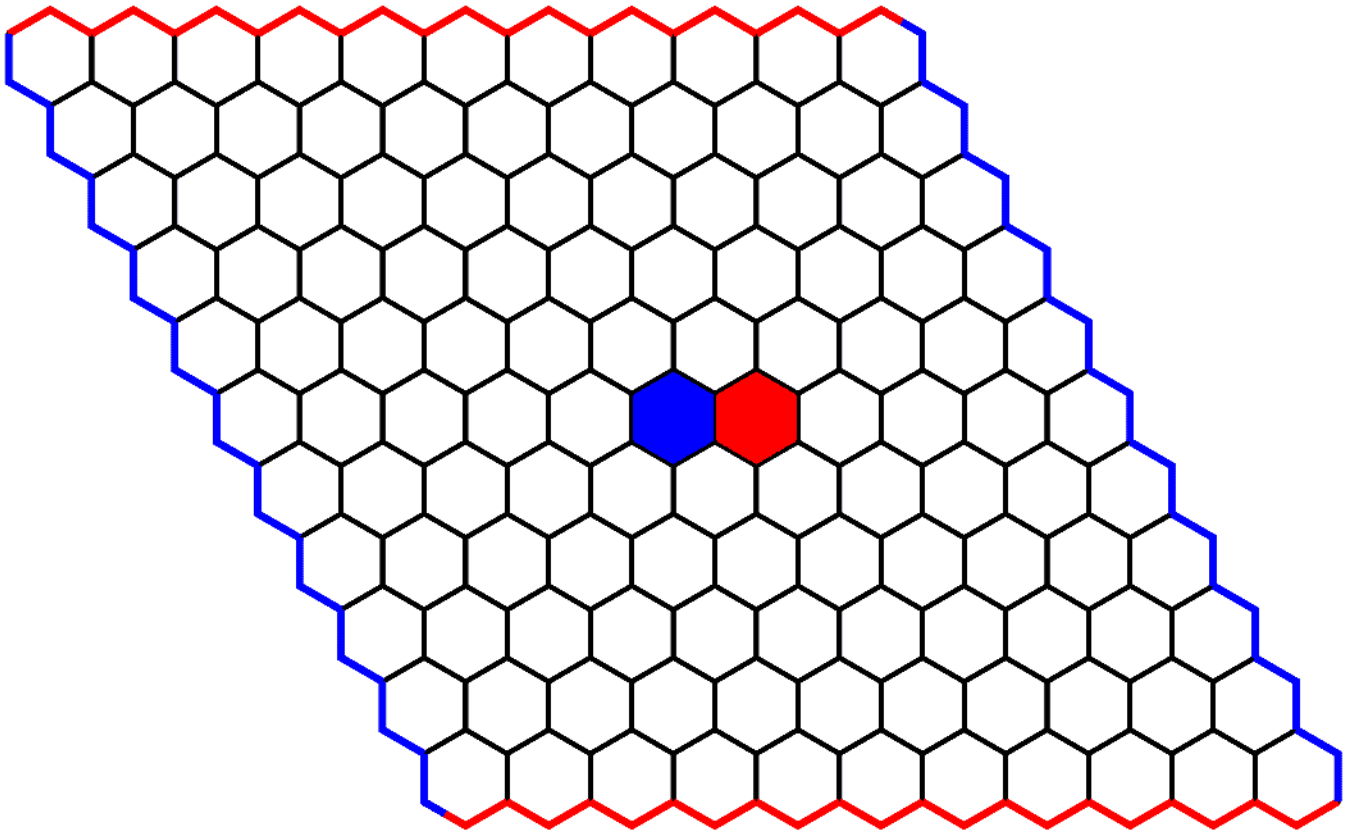

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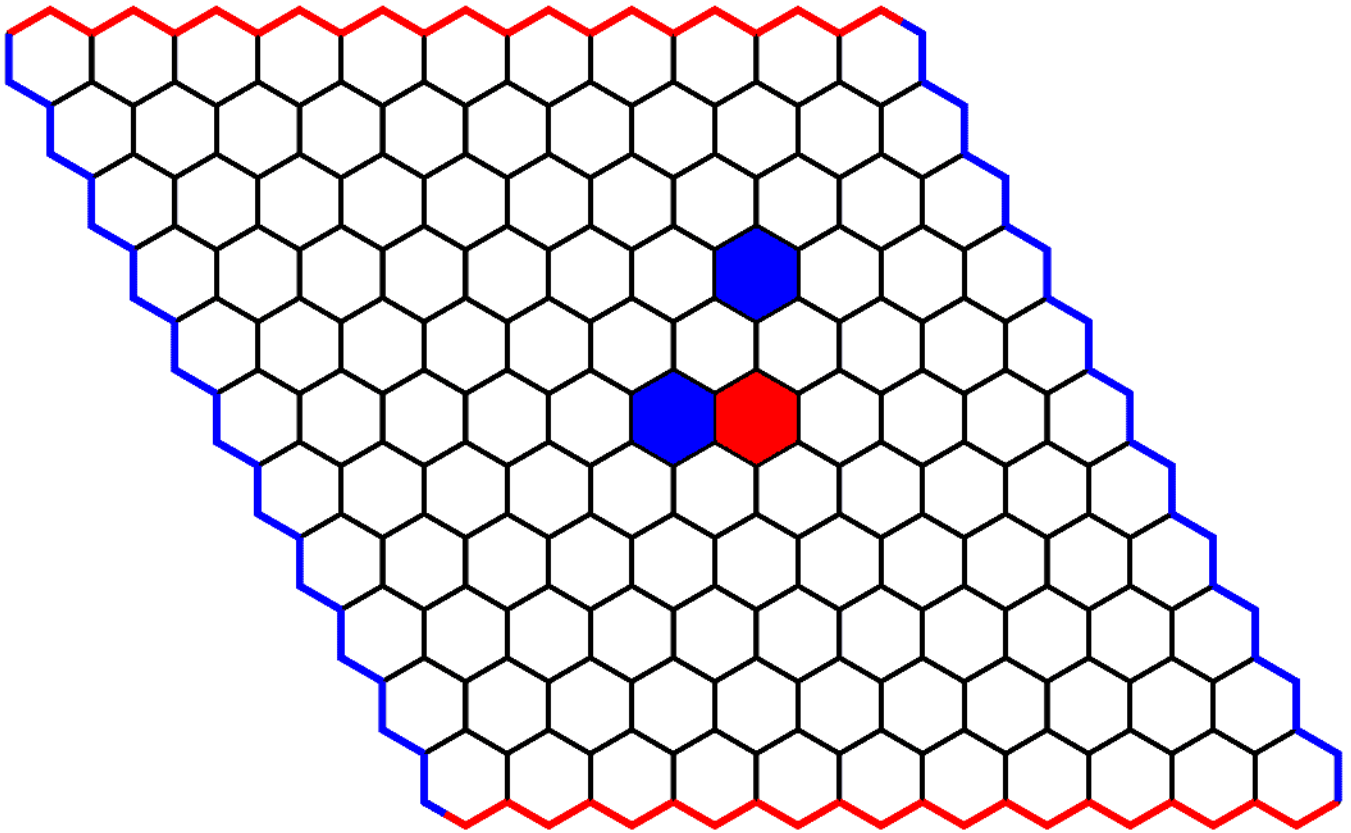

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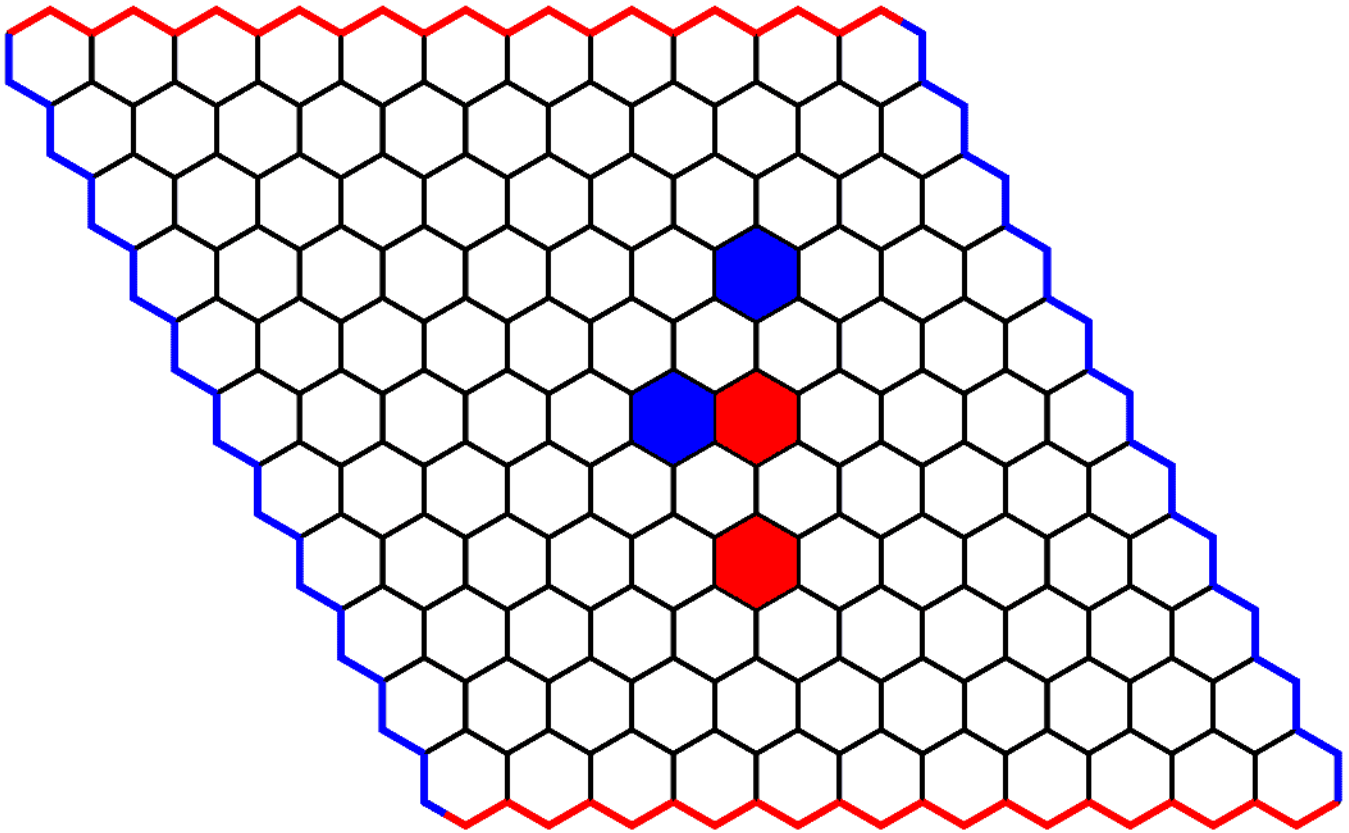

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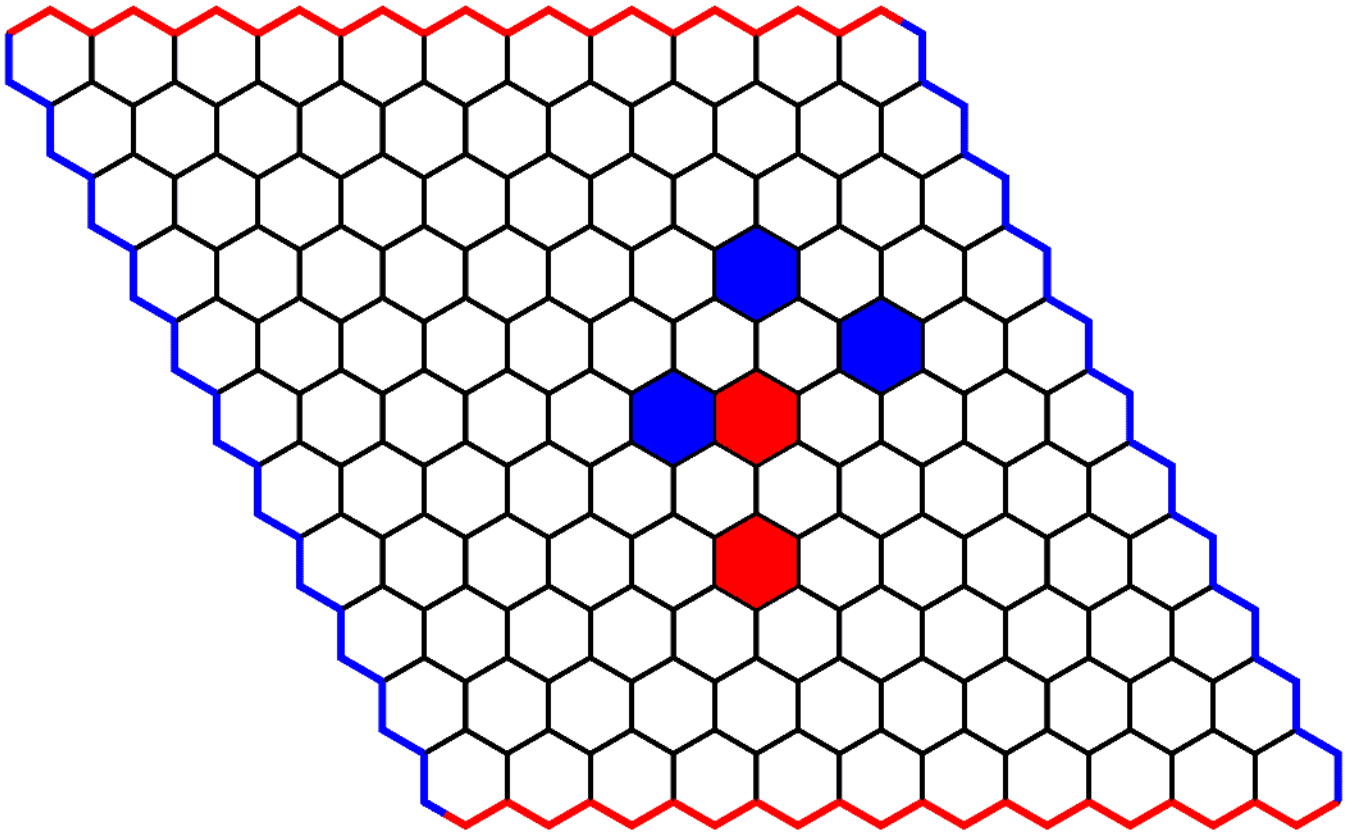

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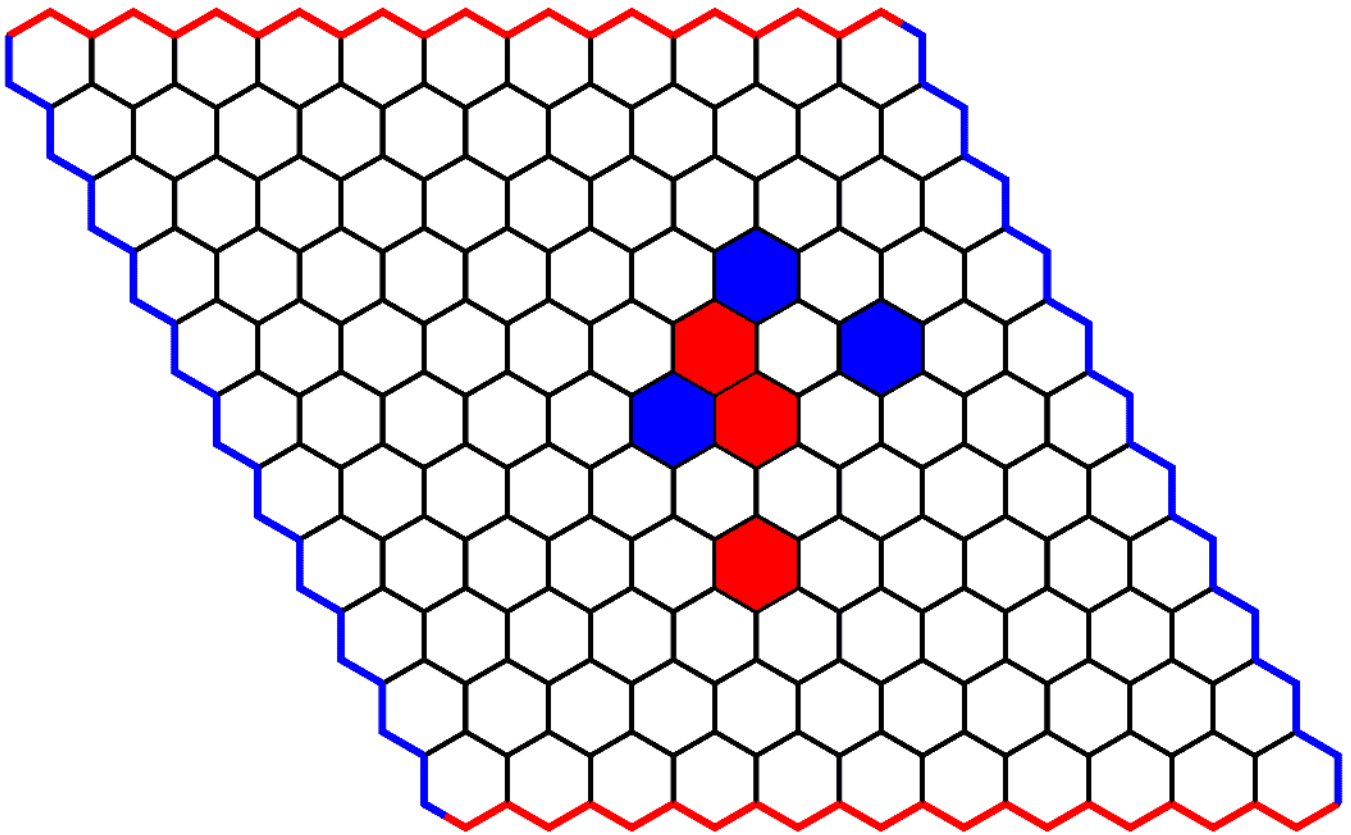

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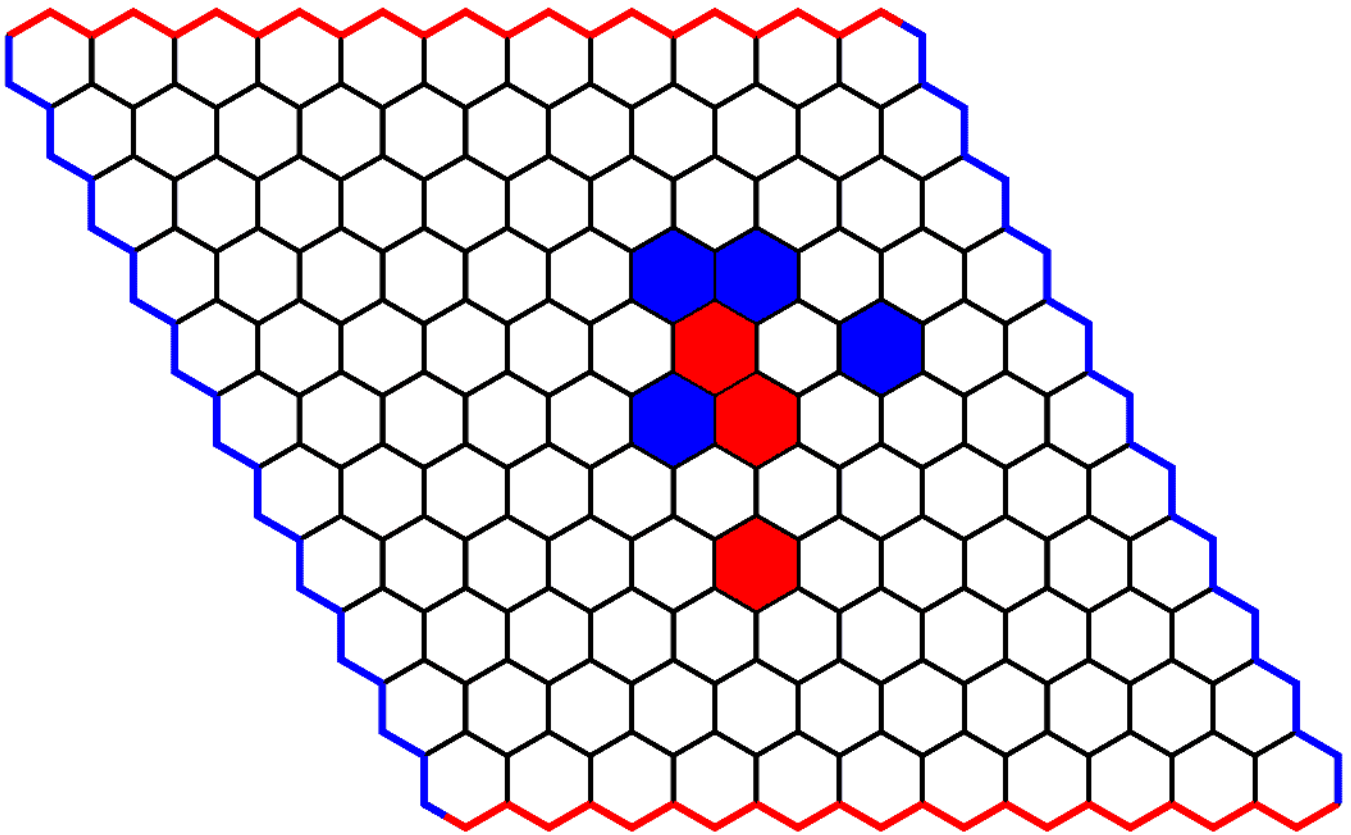

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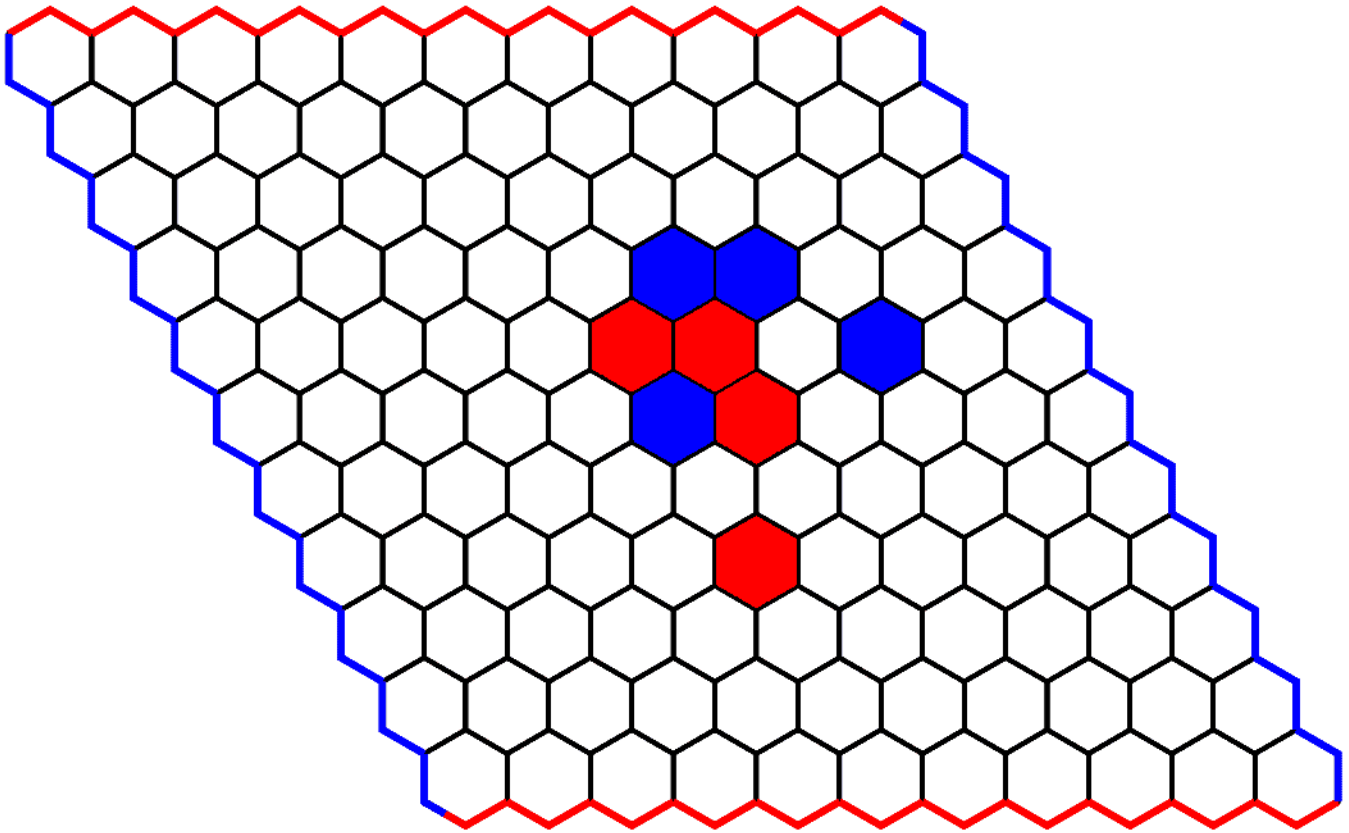

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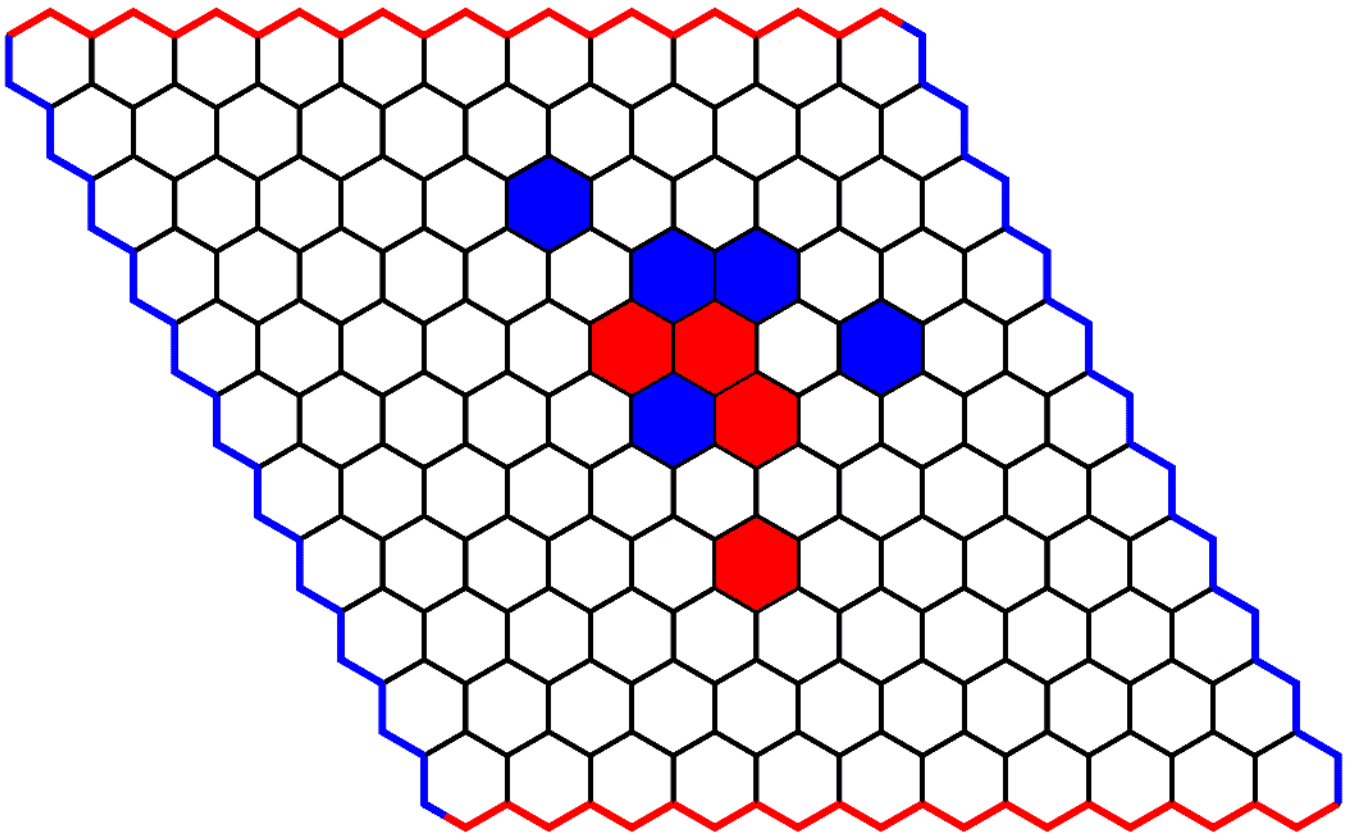

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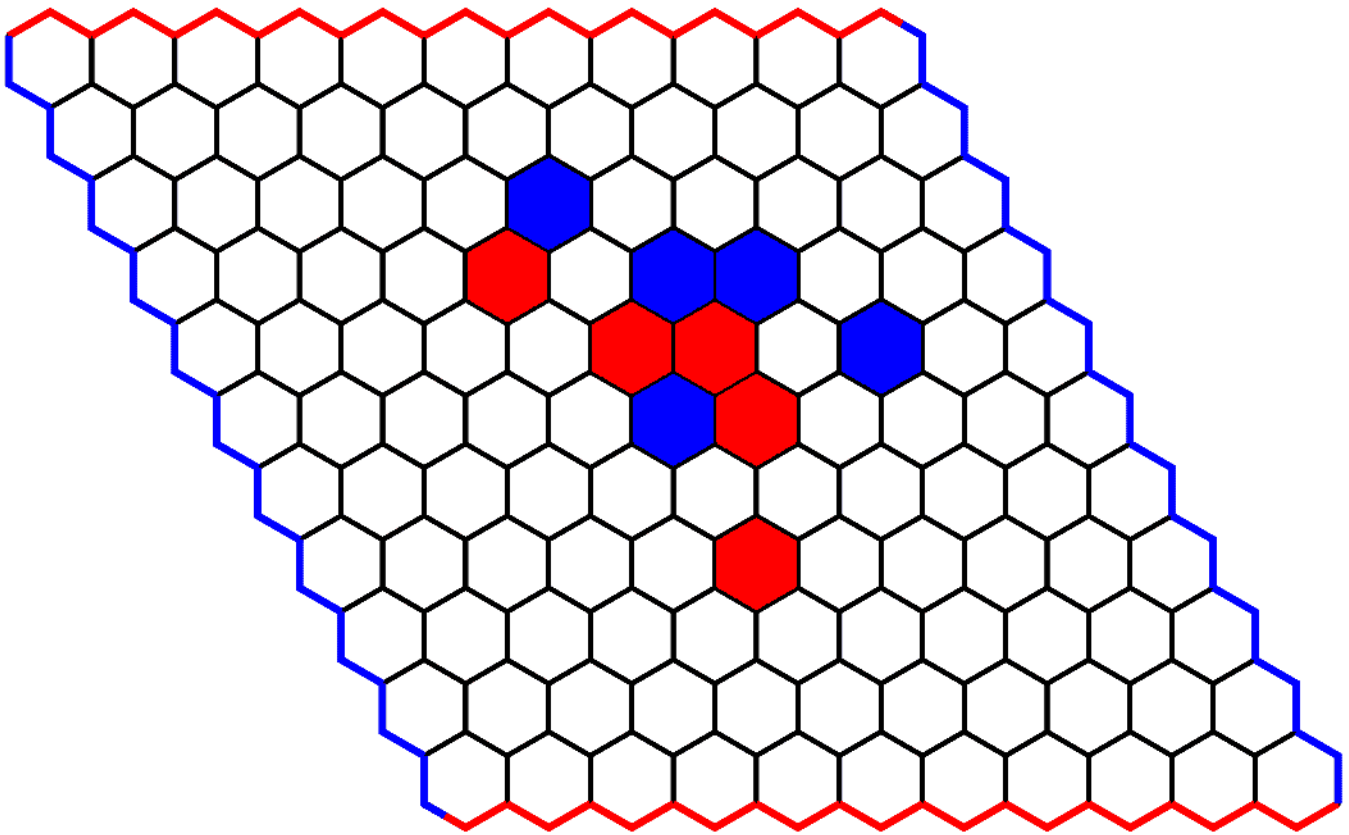

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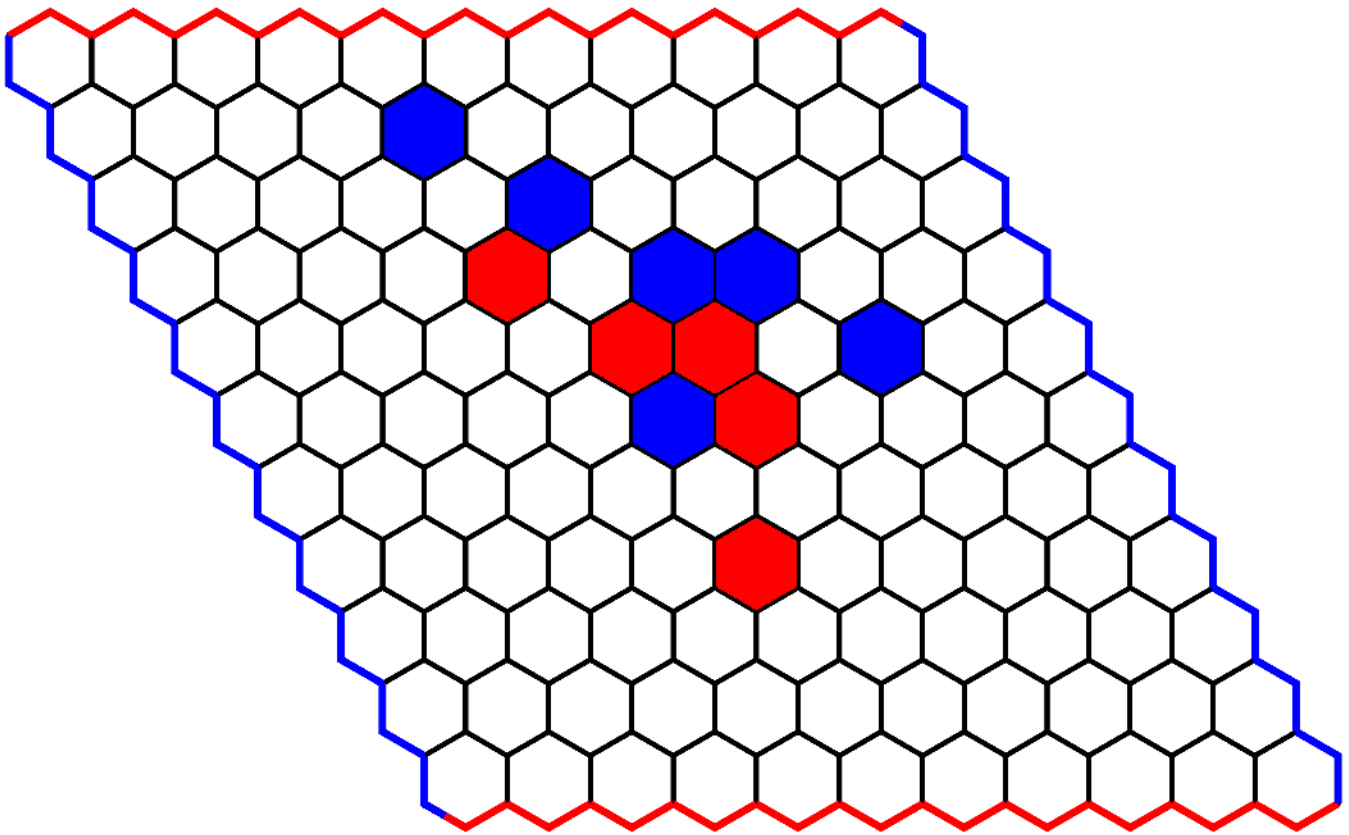

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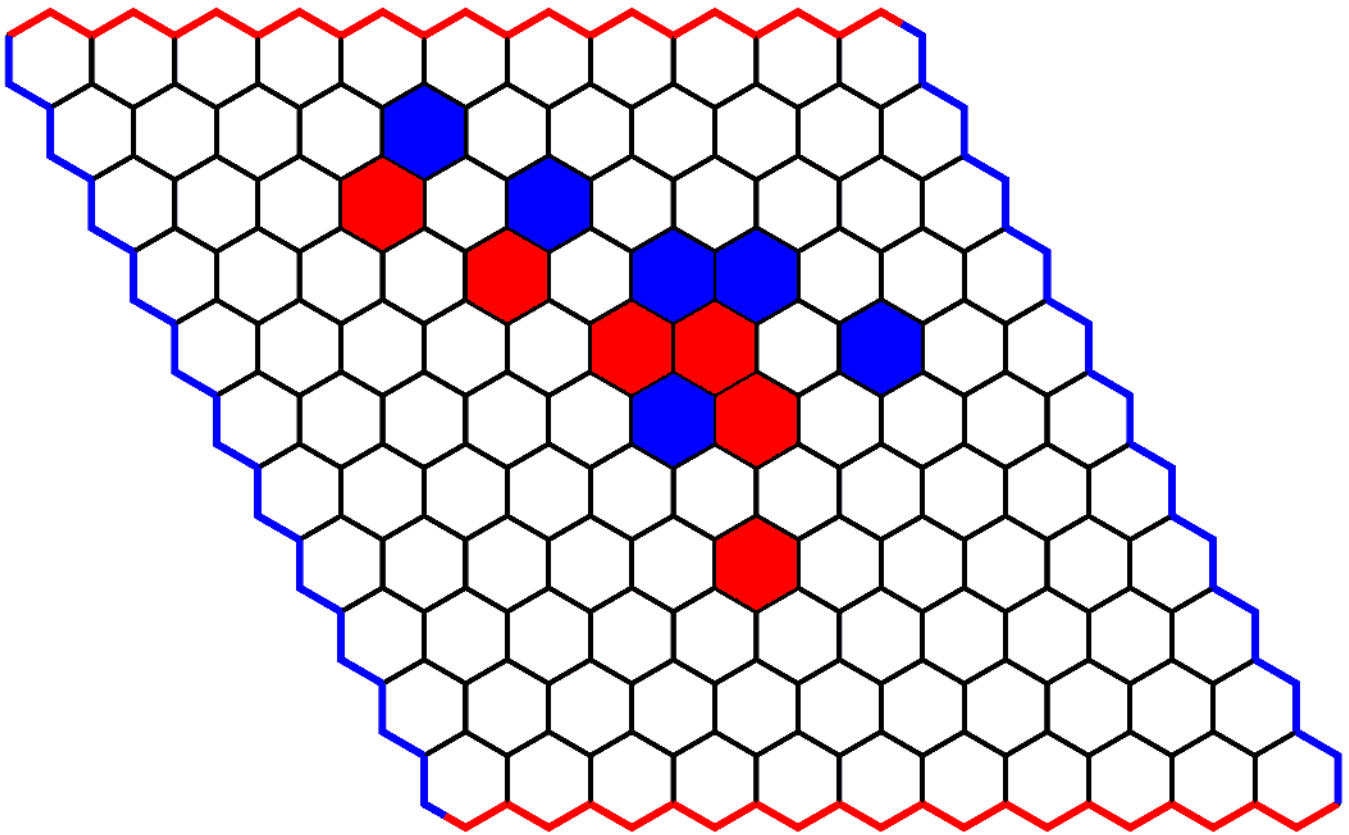

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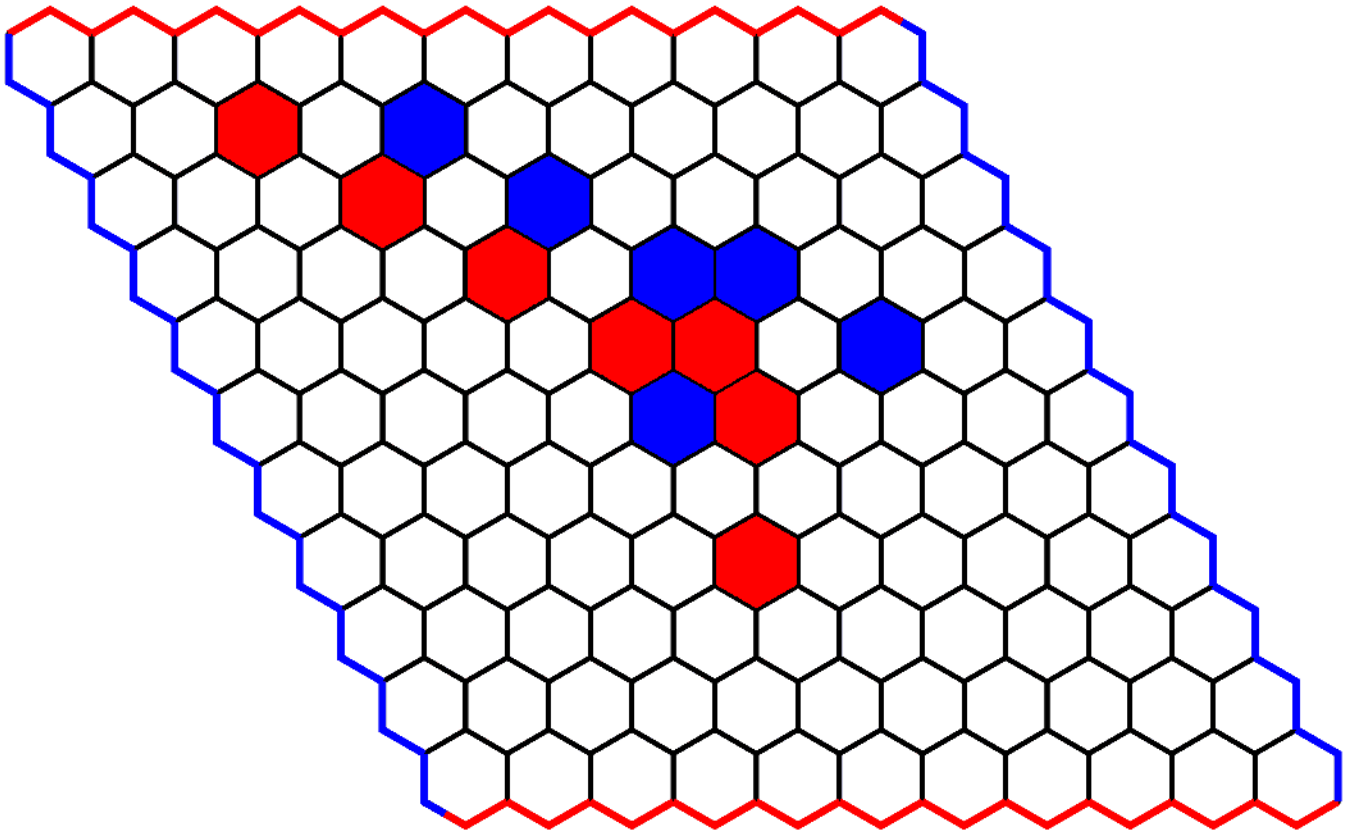

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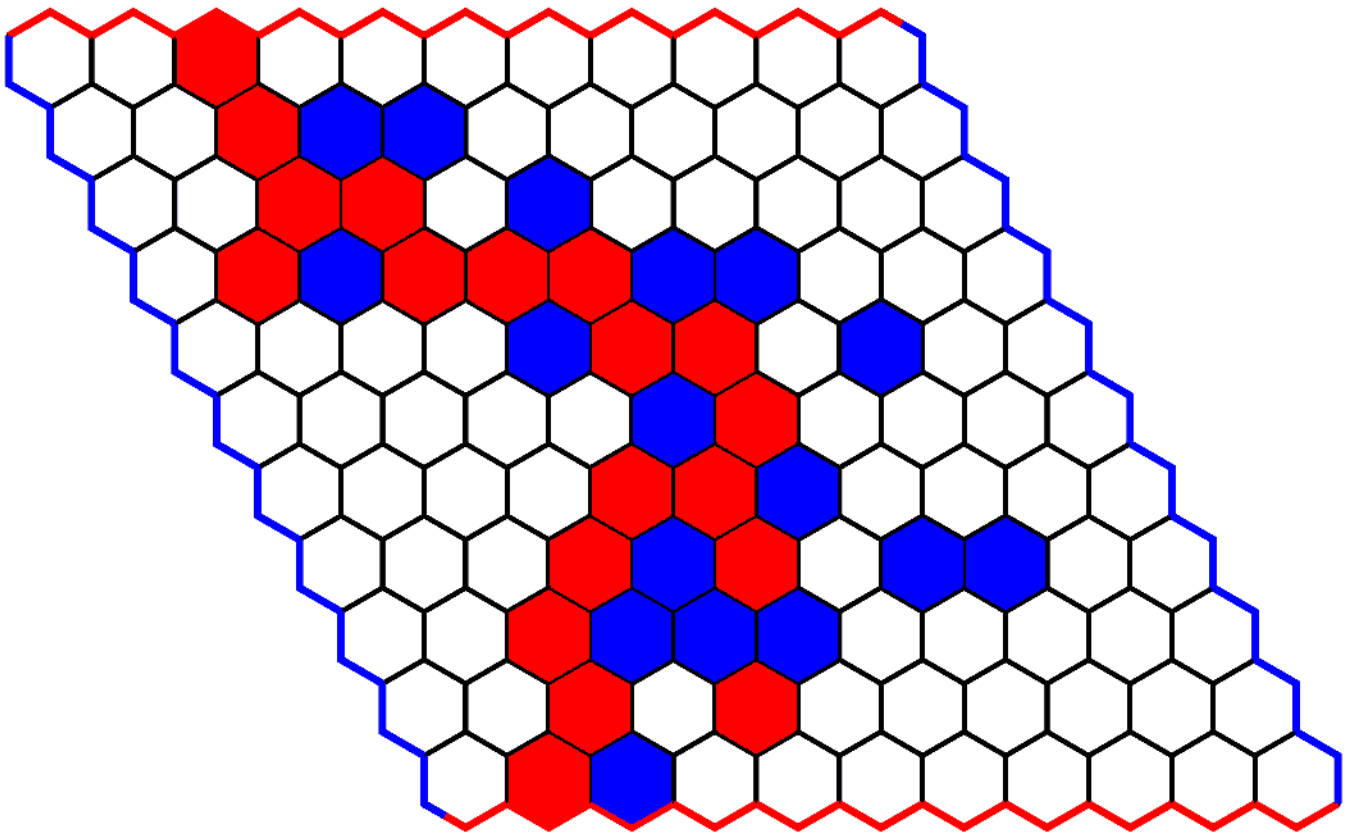

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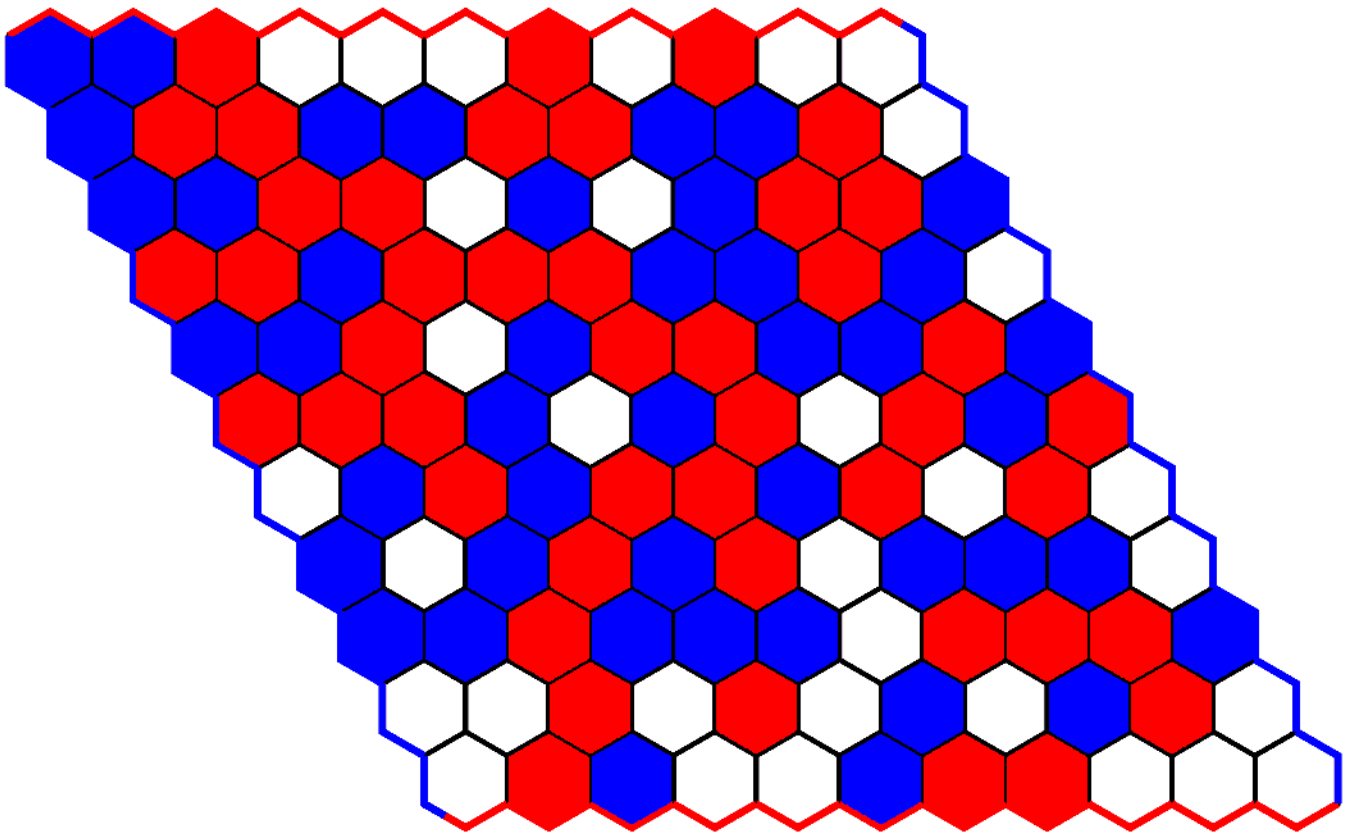

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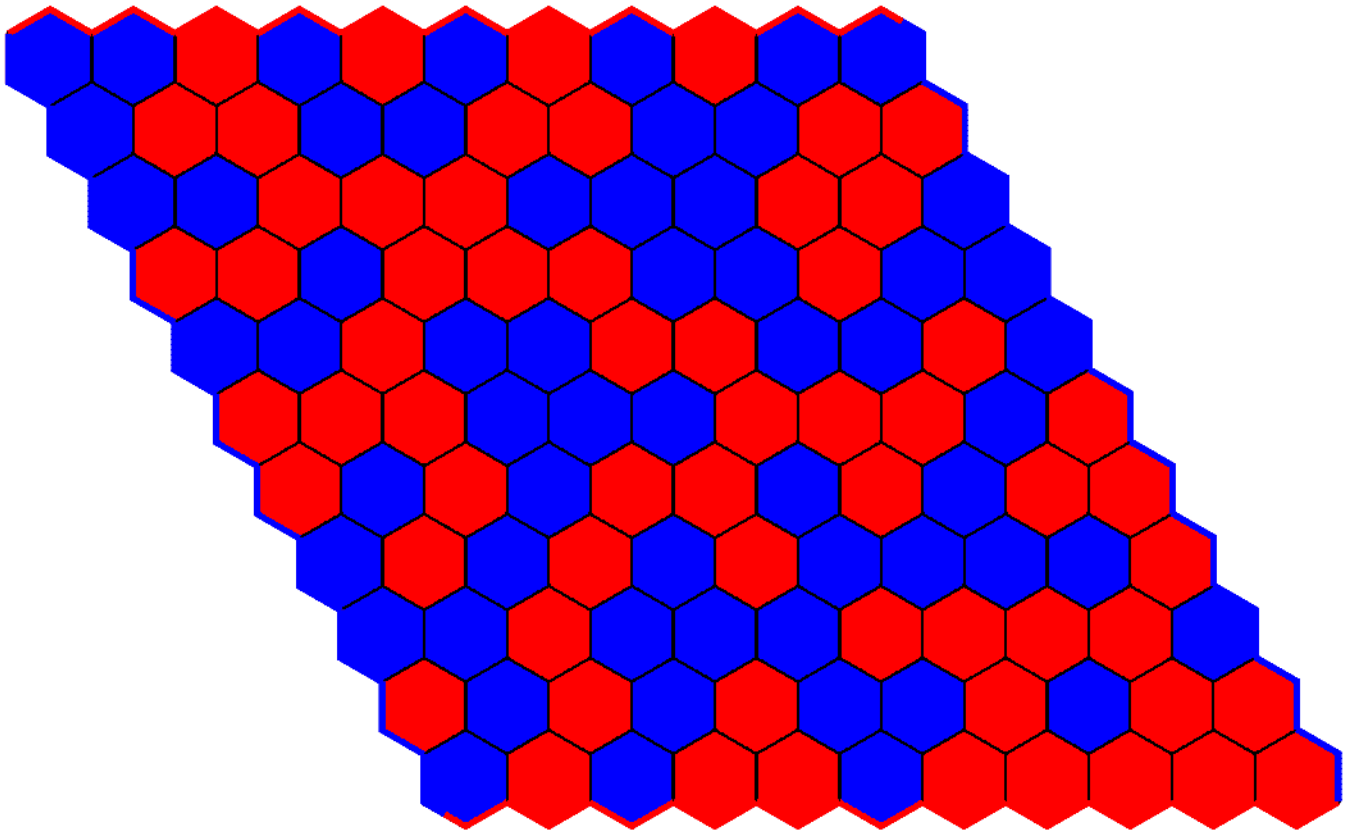

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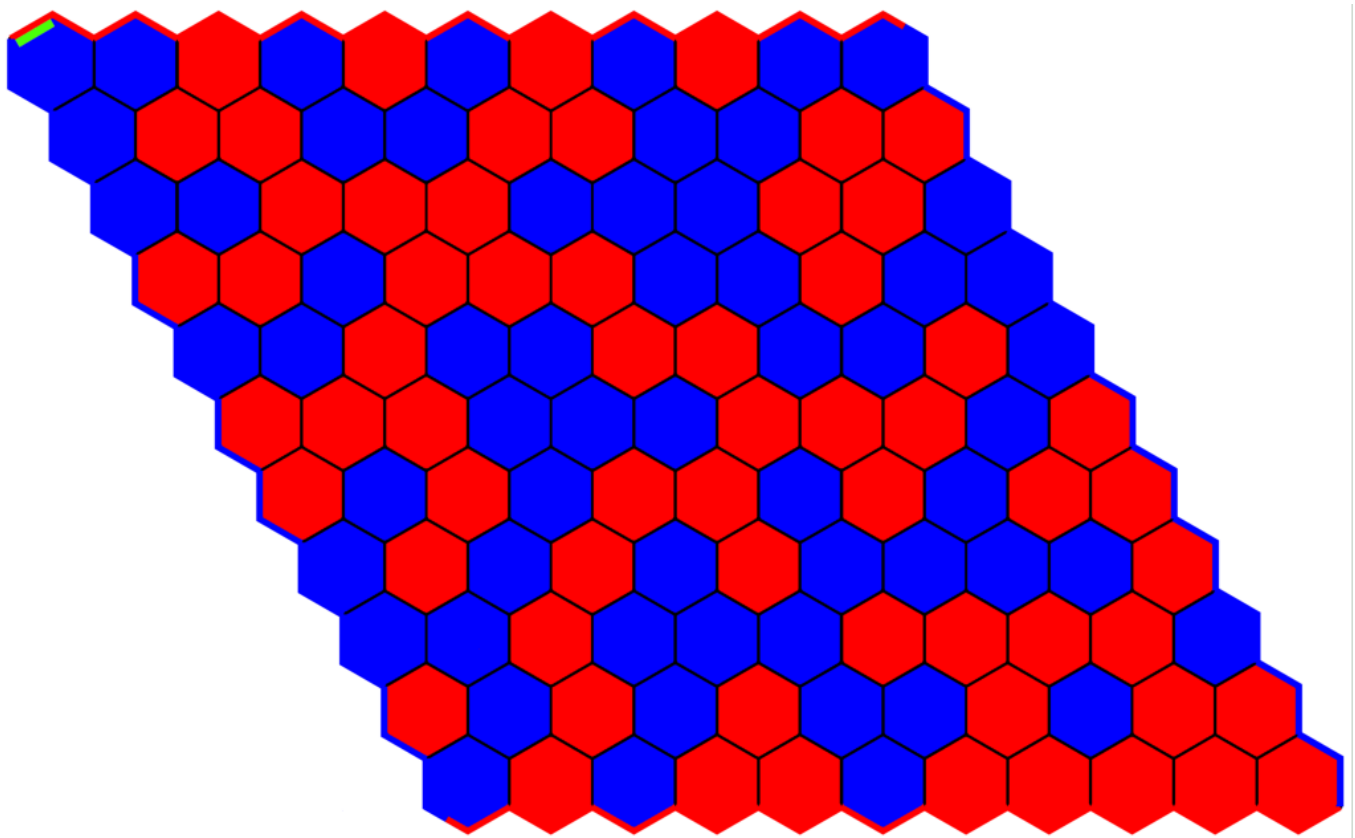

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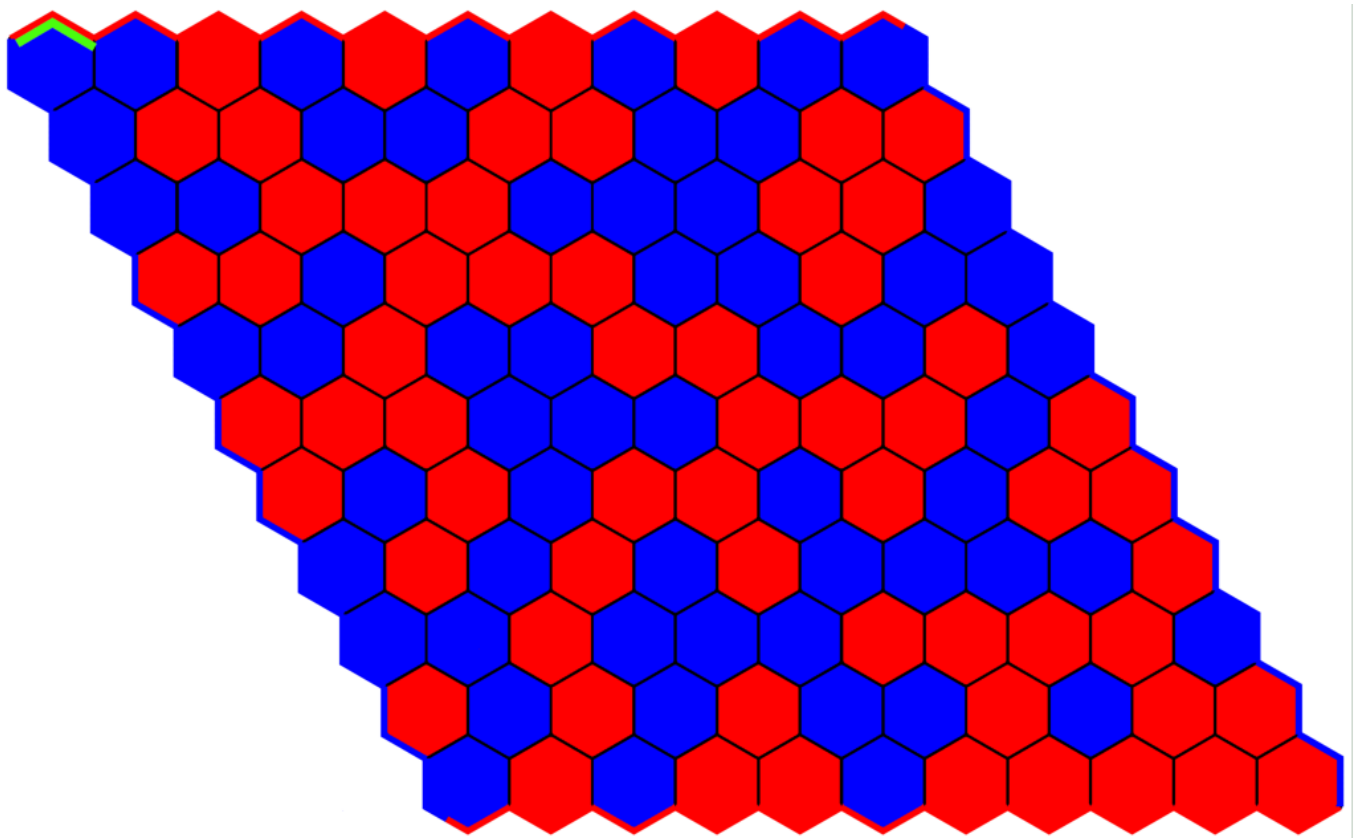

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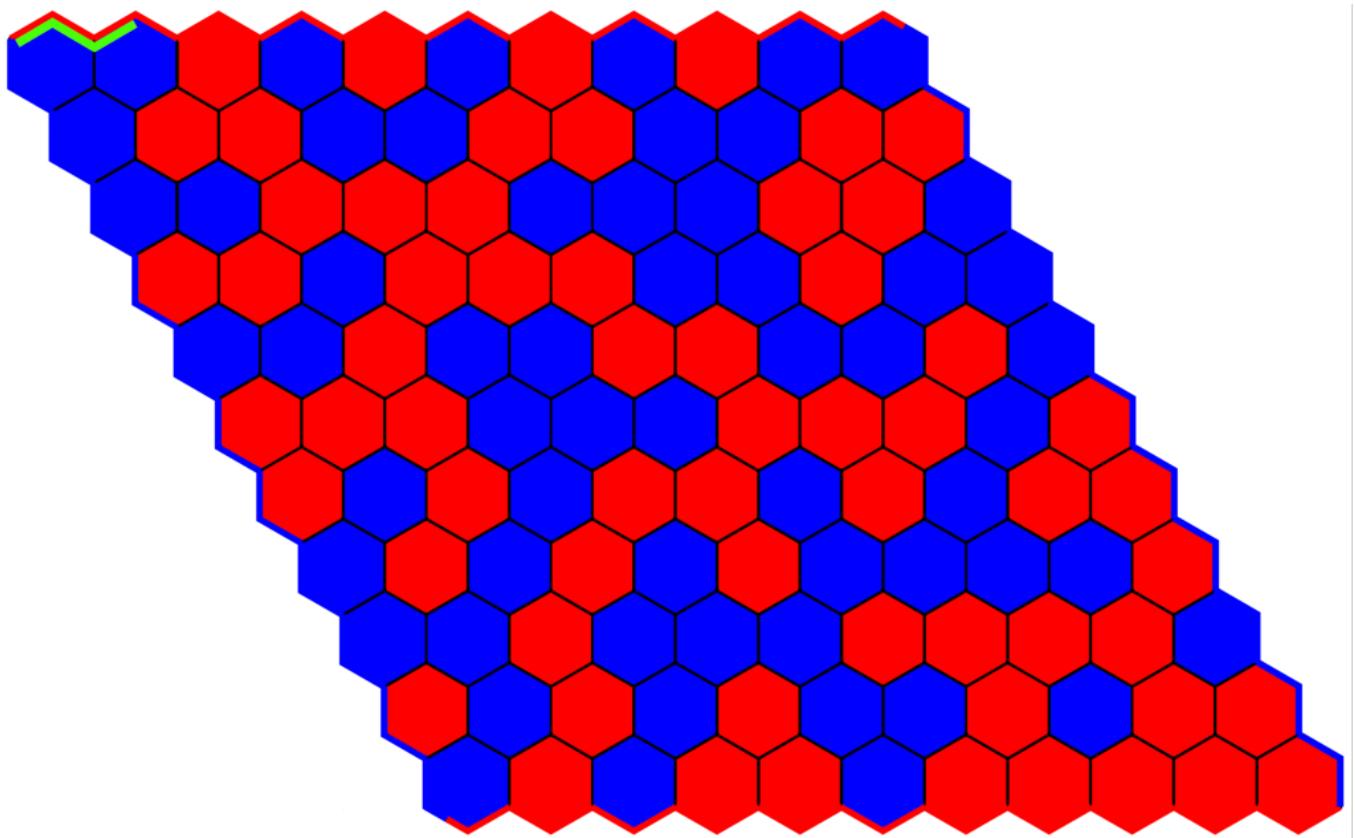

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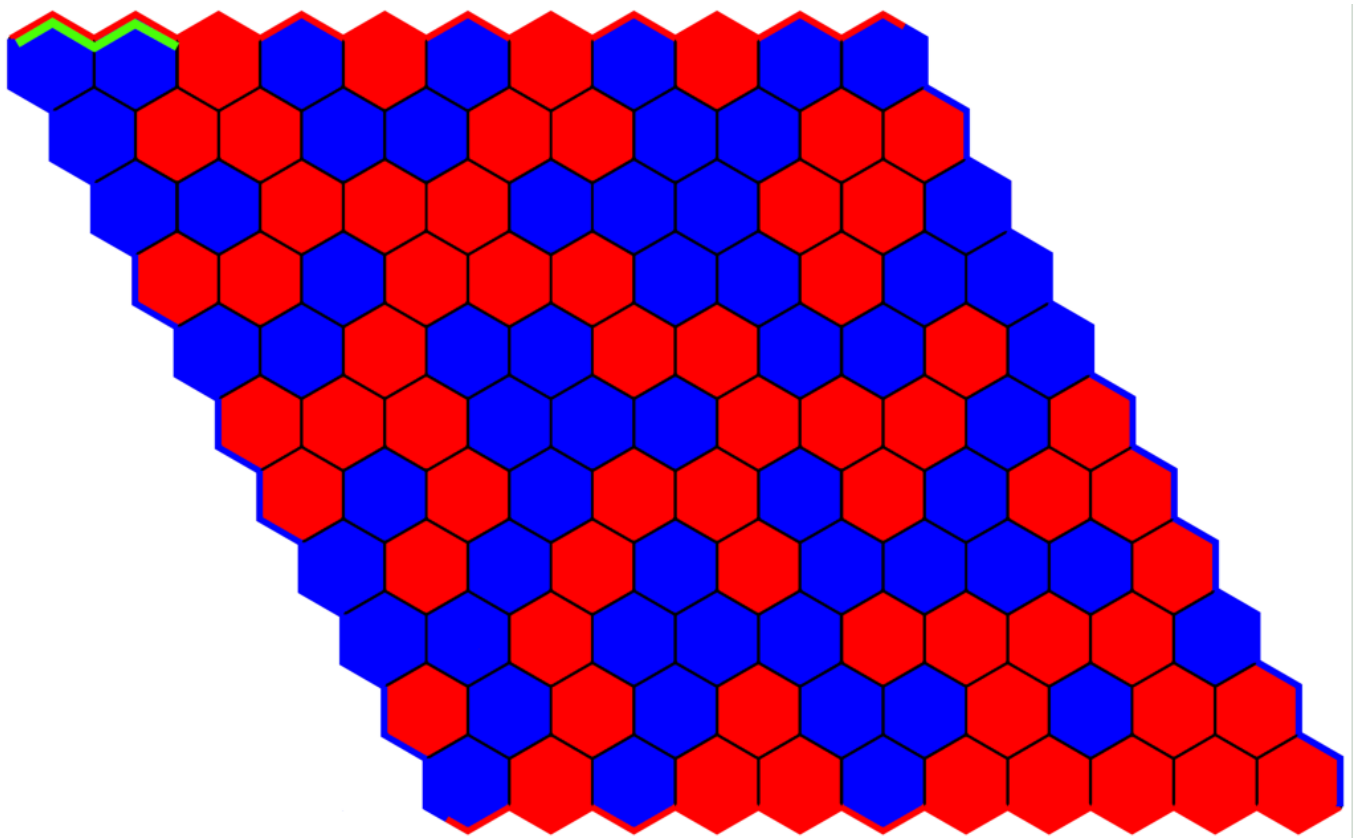

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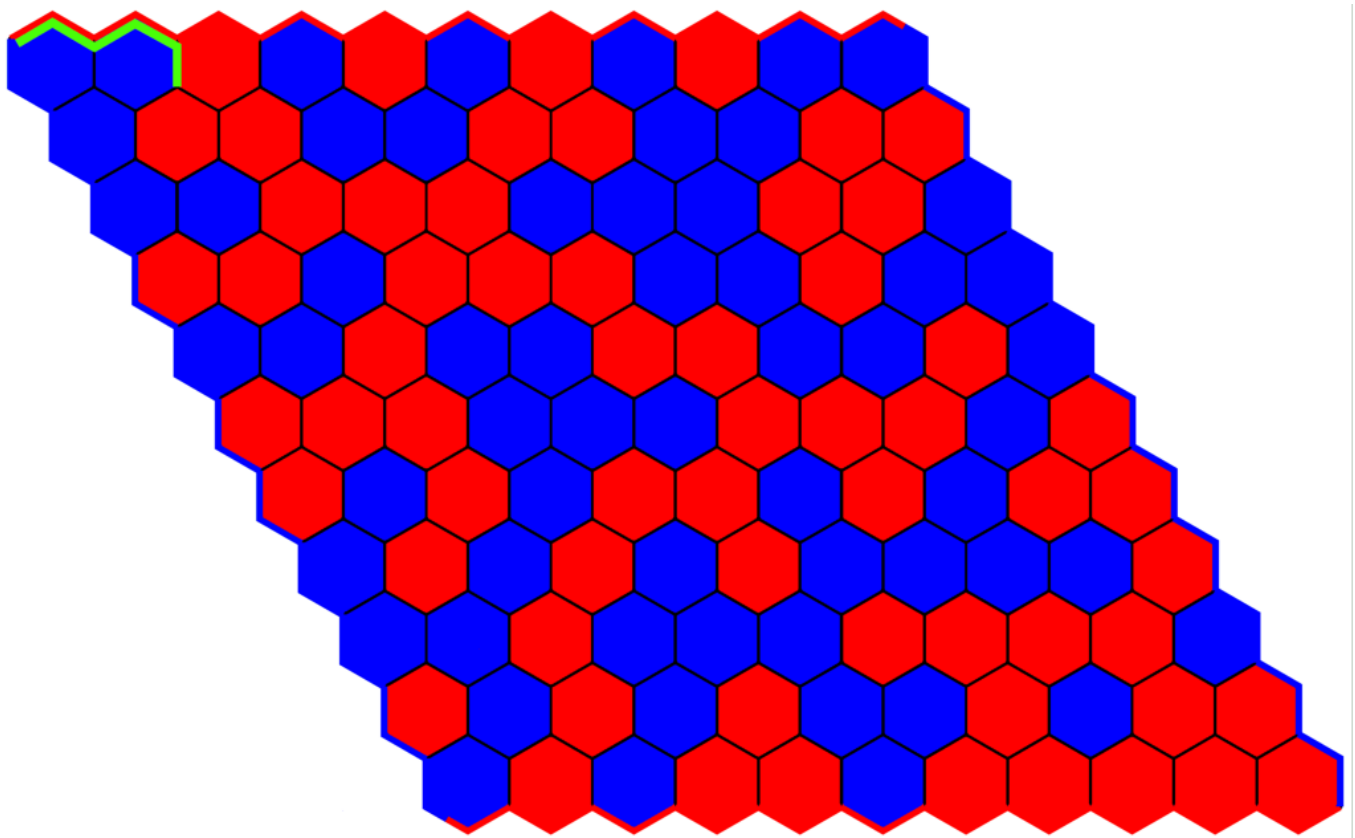

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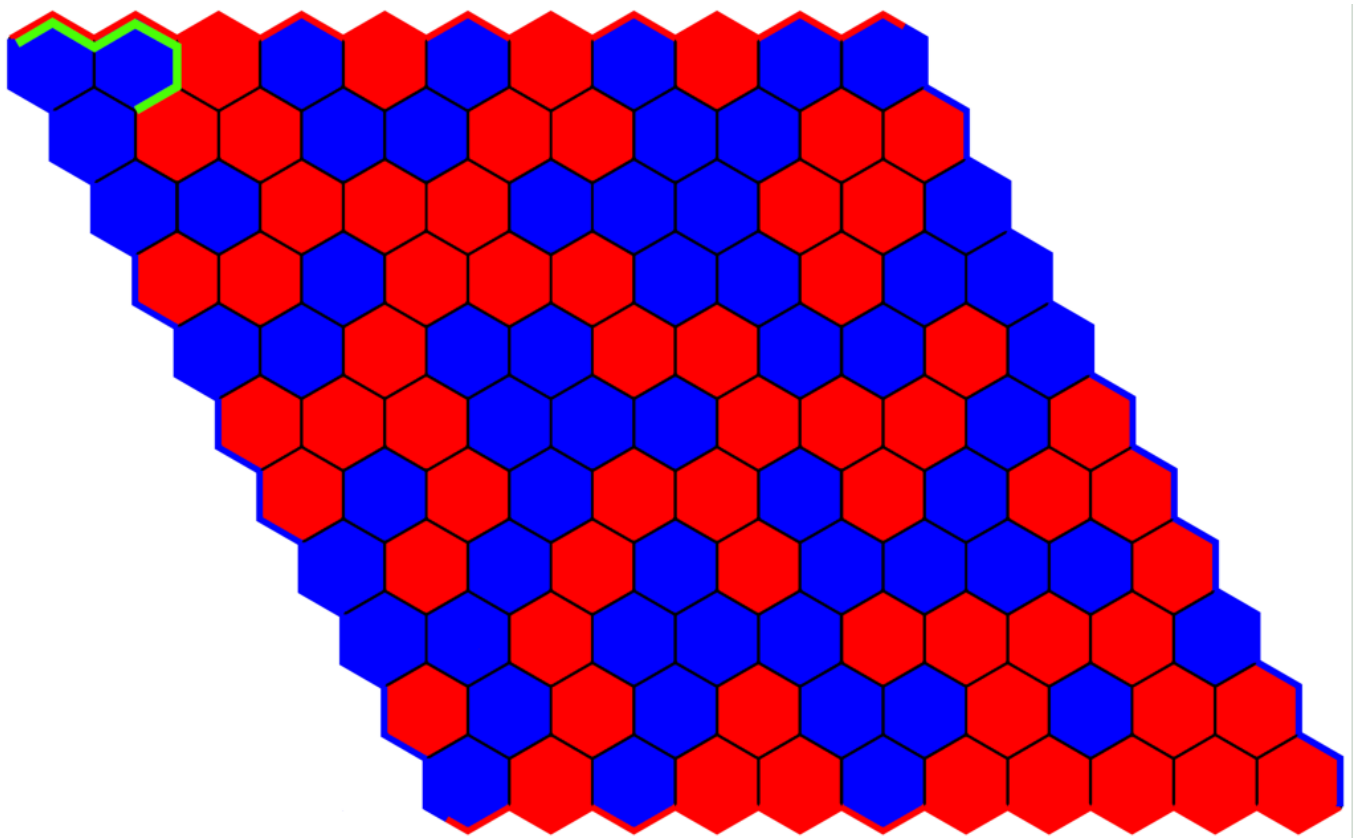

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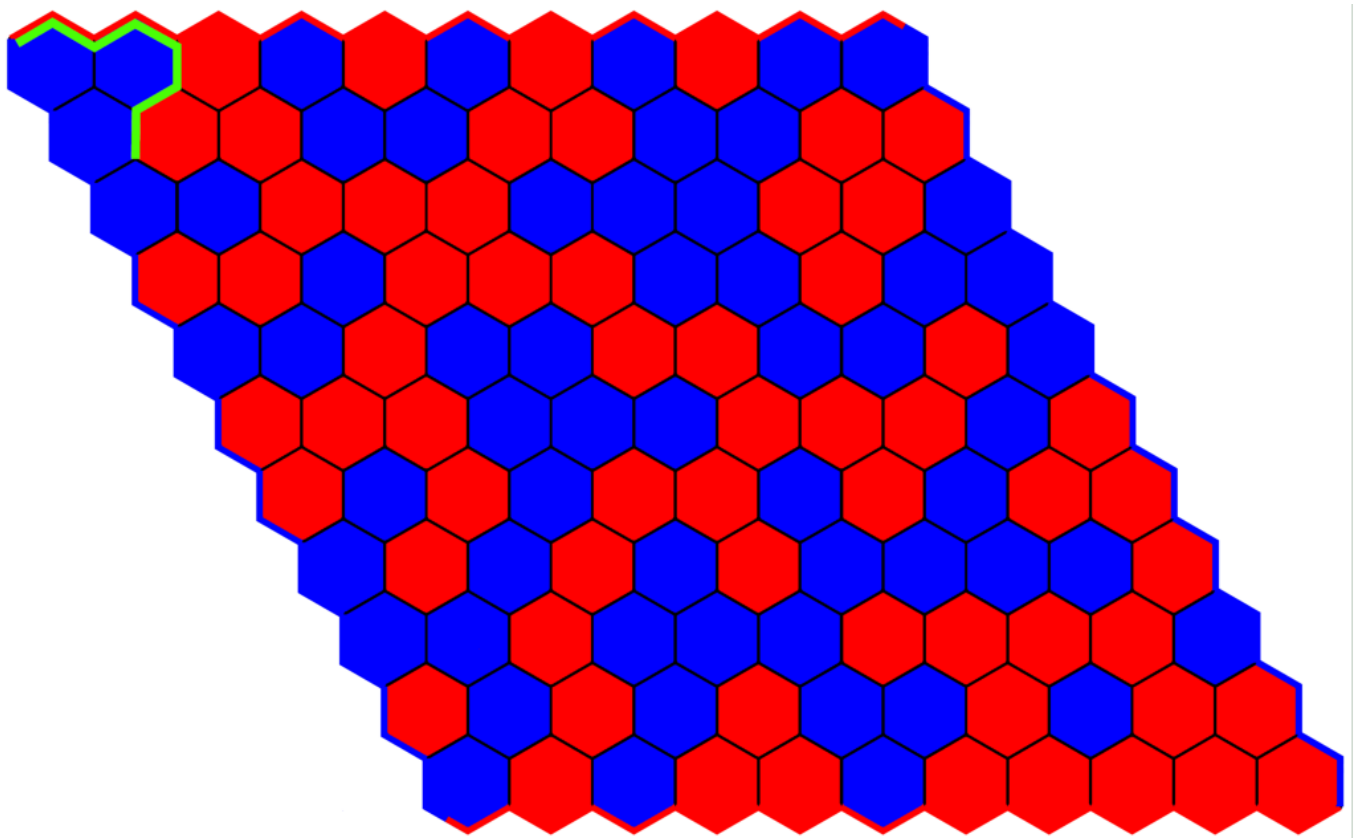

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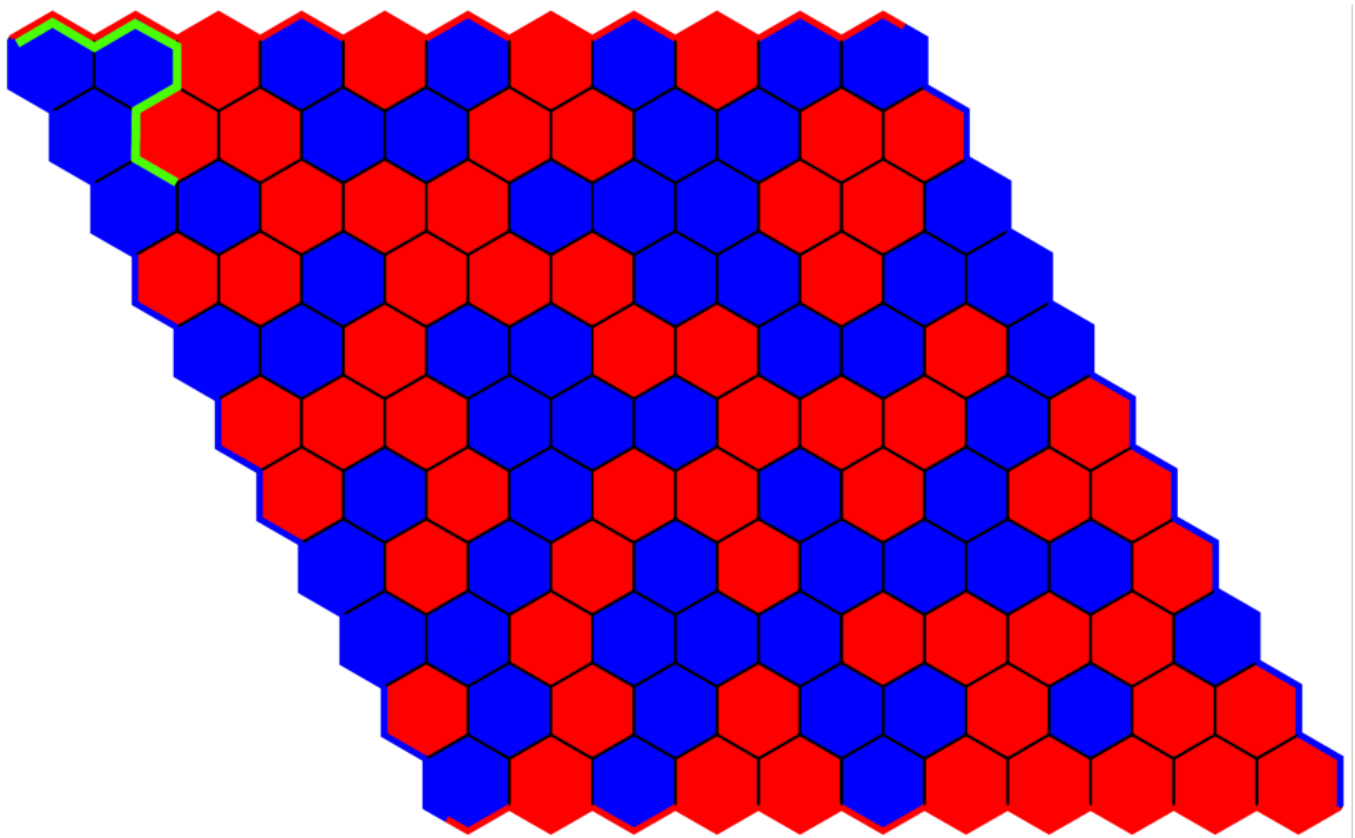

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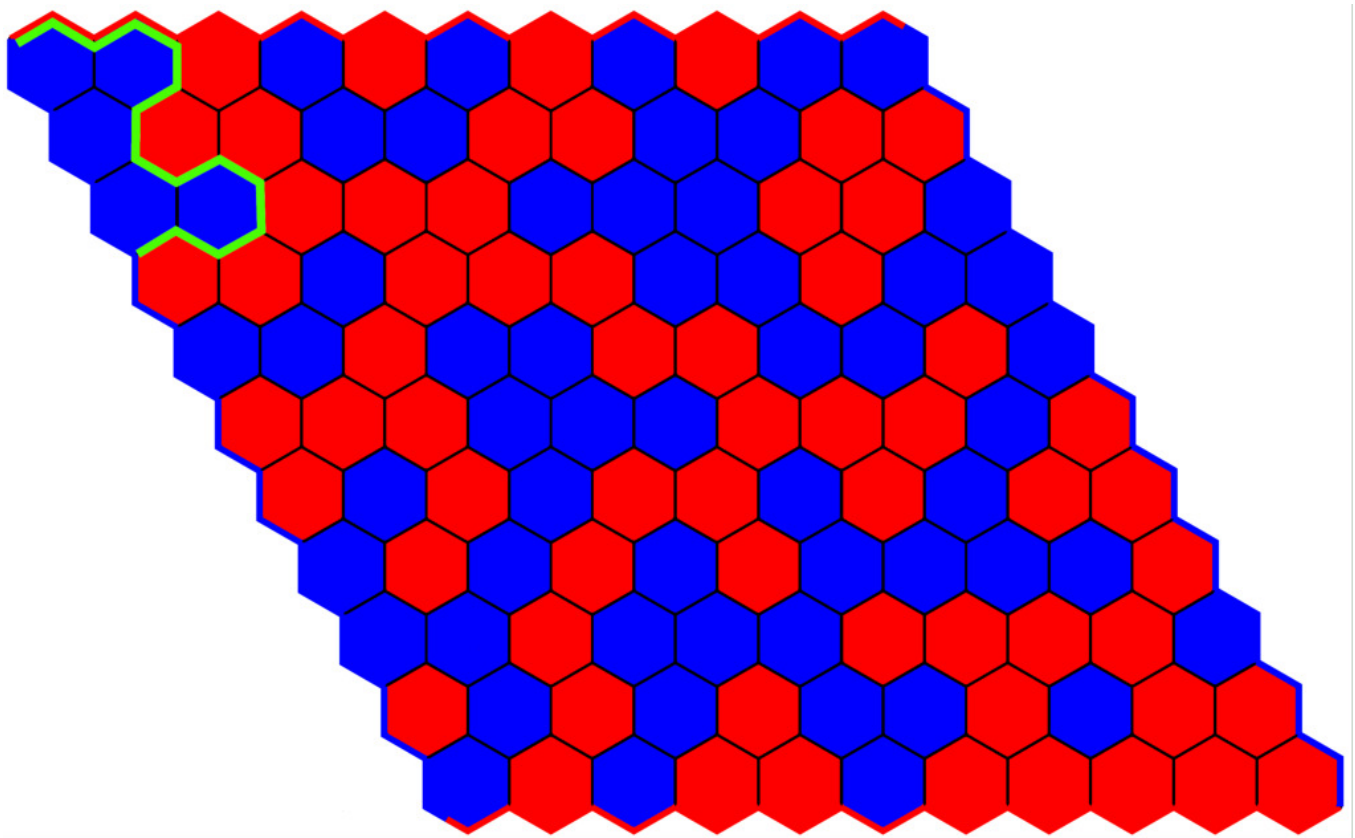

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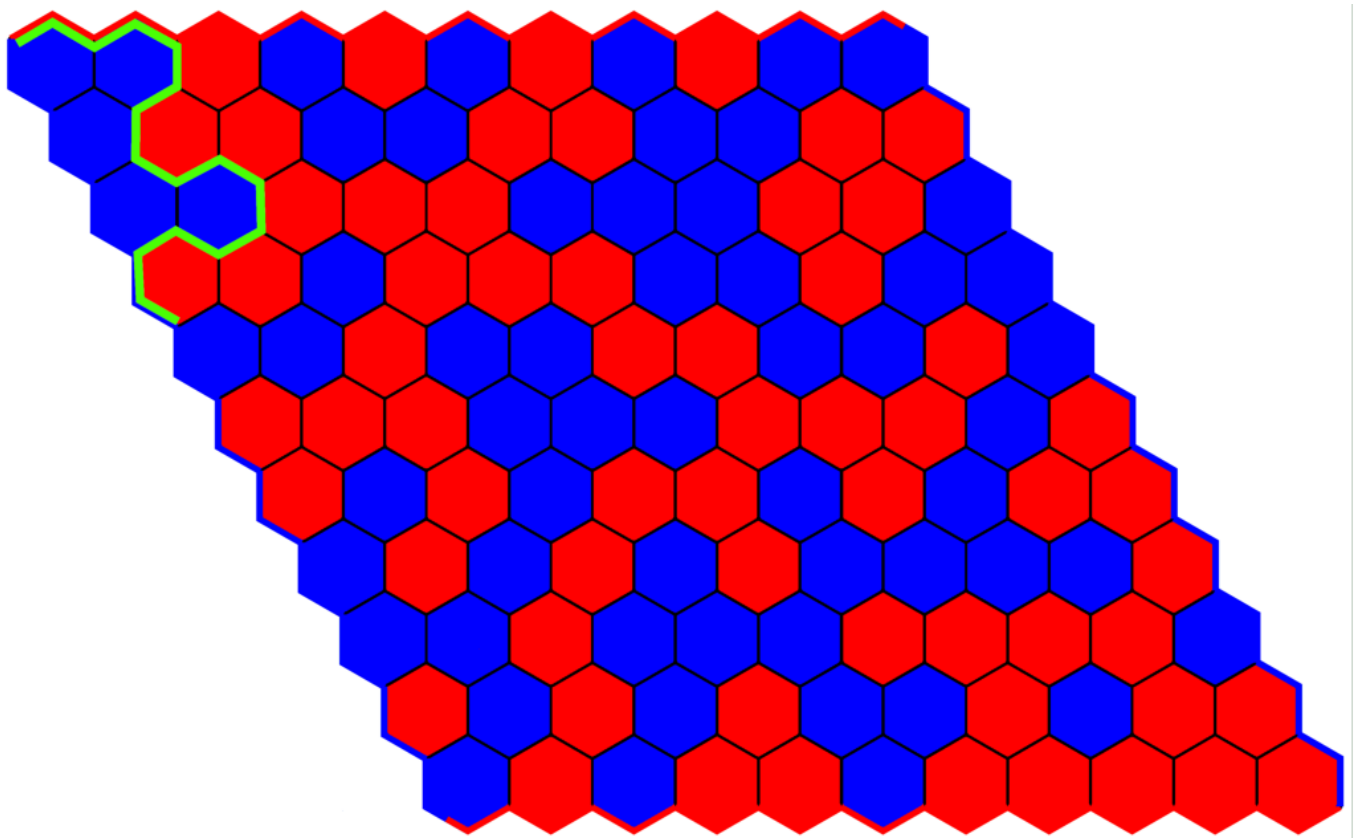

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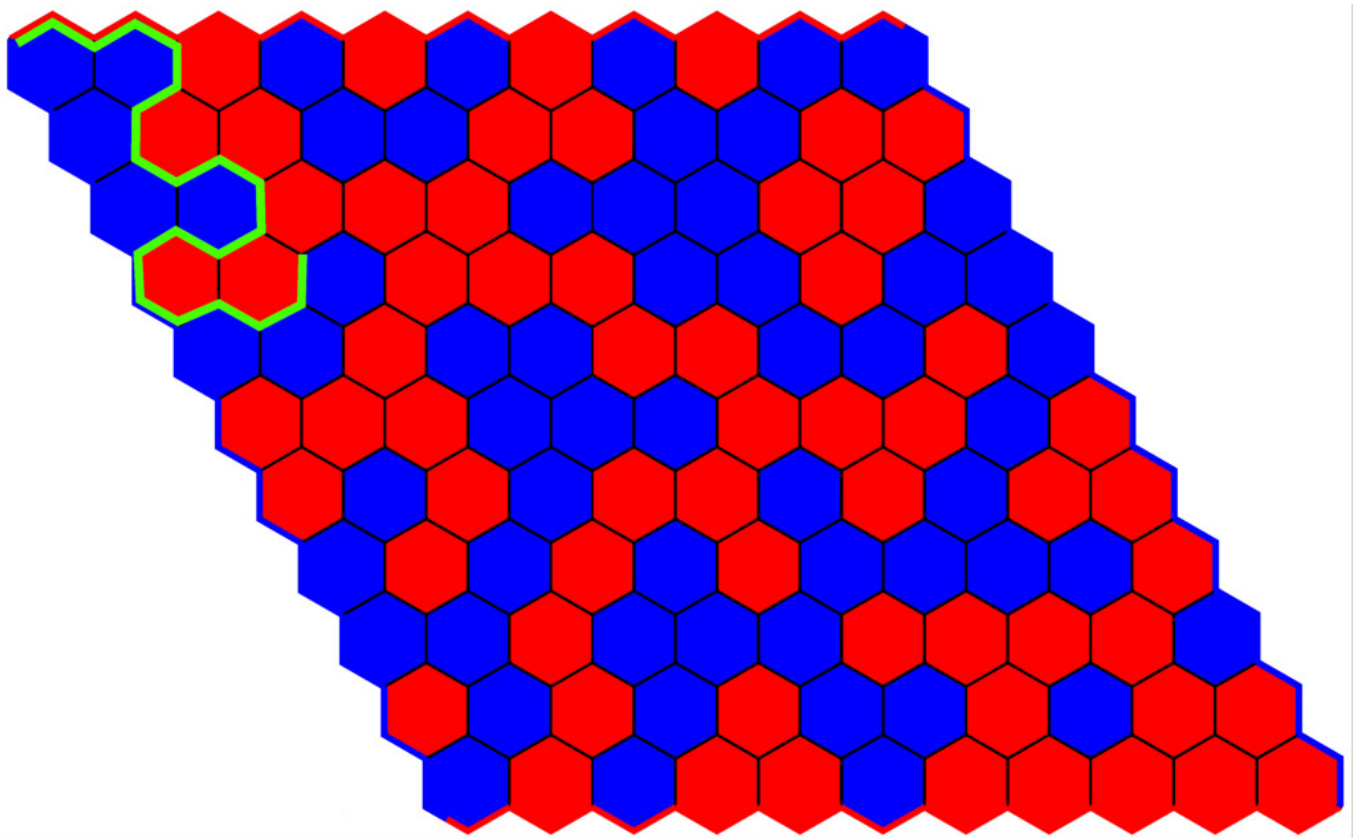

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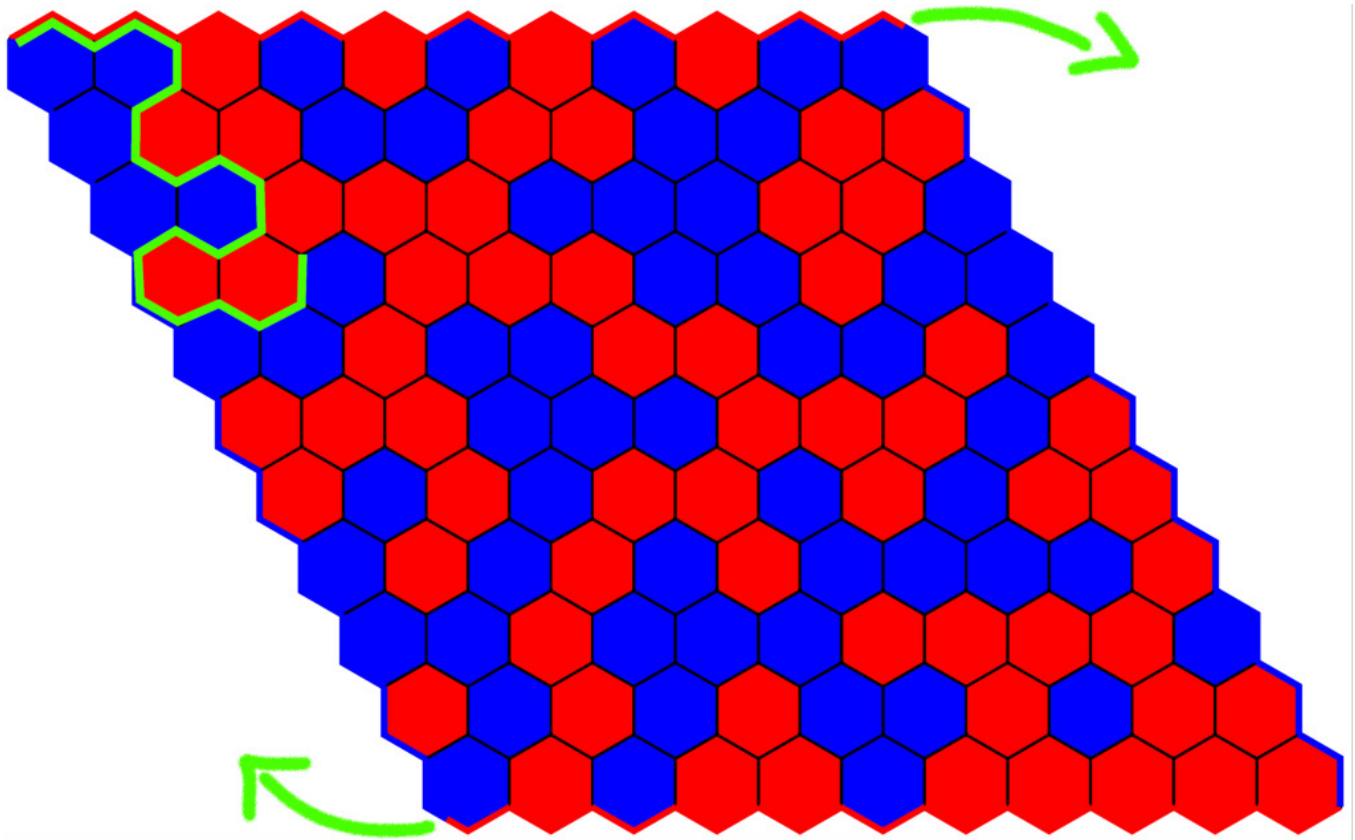

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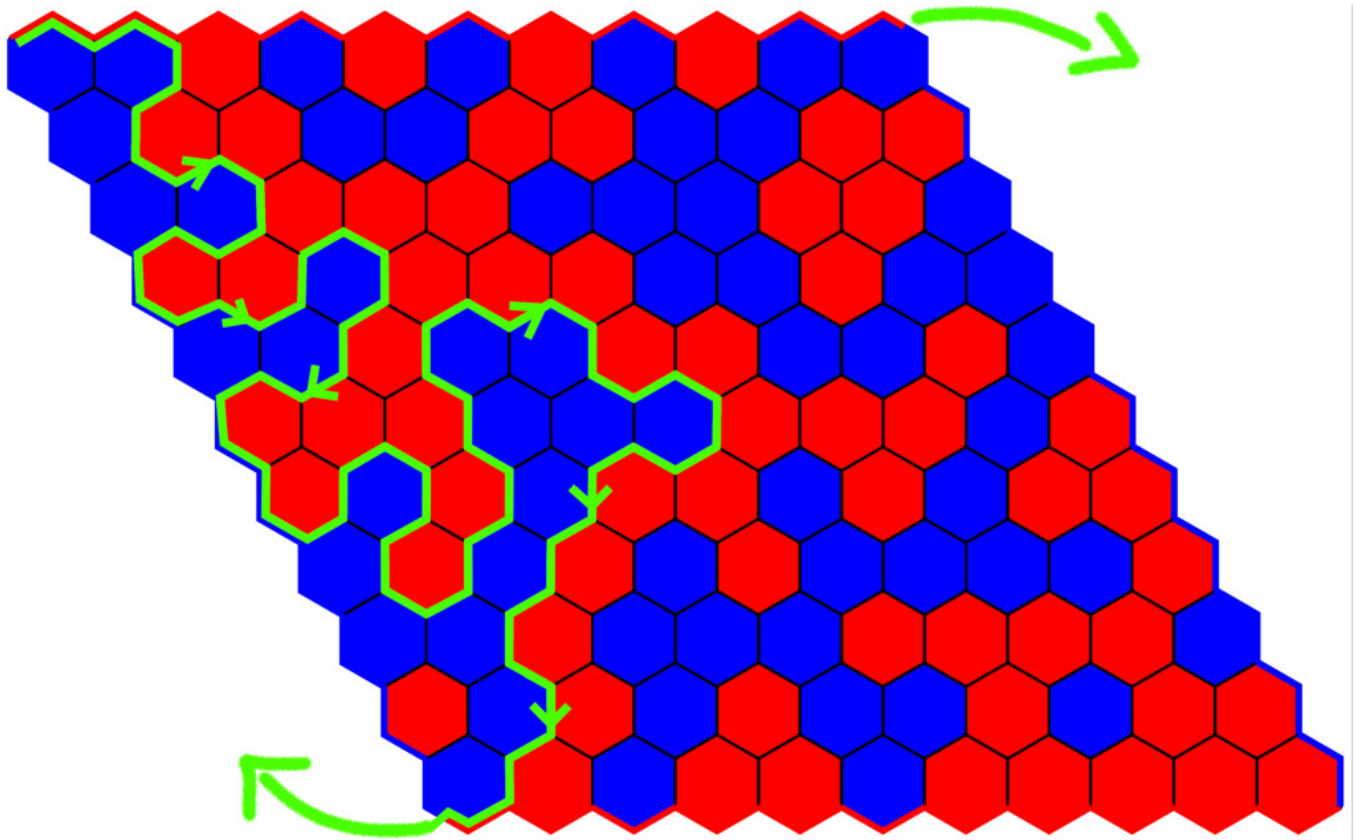

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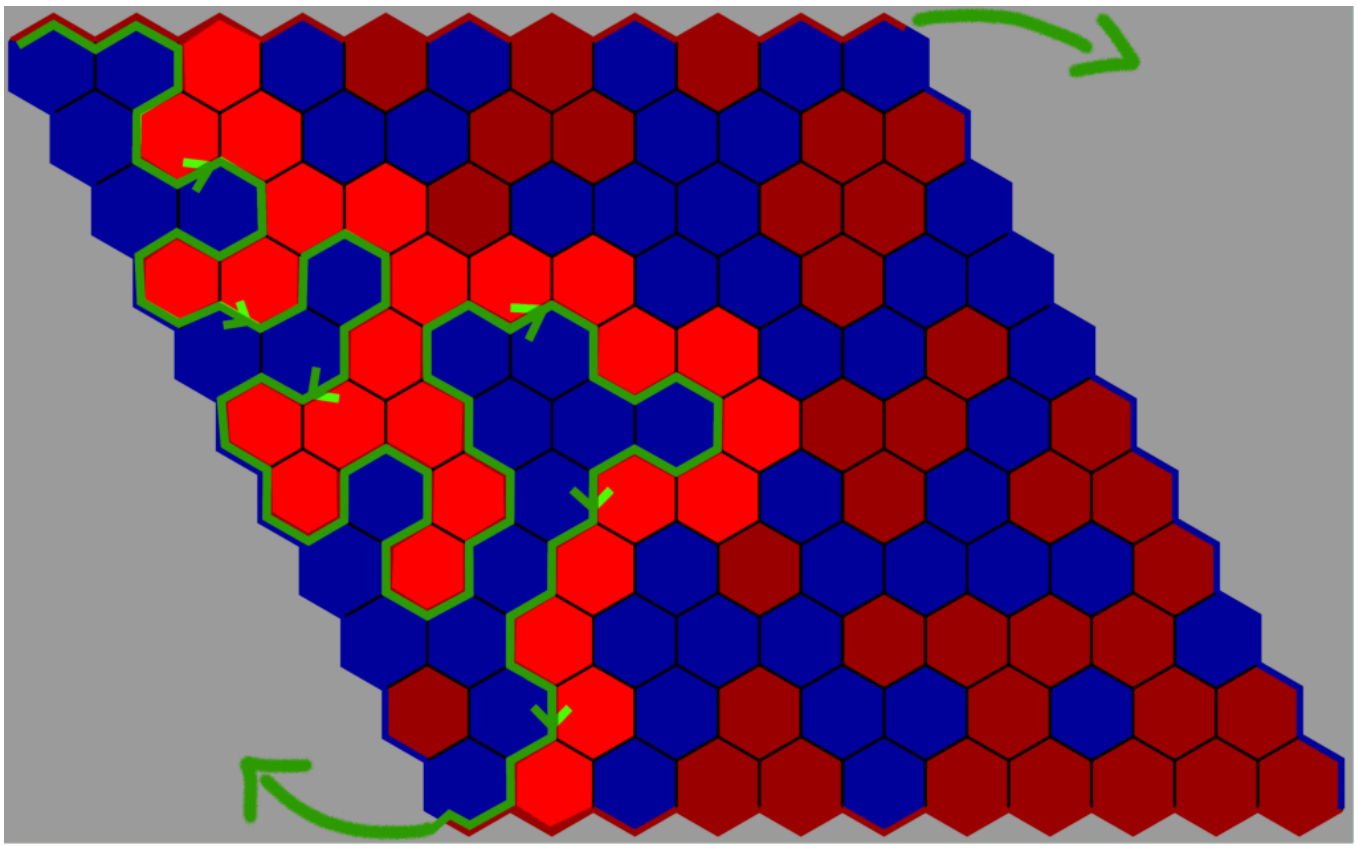

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Hex $\Leftrightarrow$ Brouwer

Hex Theorem

$\Downarrow$

Brouwer's Theorem

$\Downarrow$

Hex Theorem

David Gale 1979

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continuous Hex Theorem

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Proof of Brouwer's Theorem using Hex

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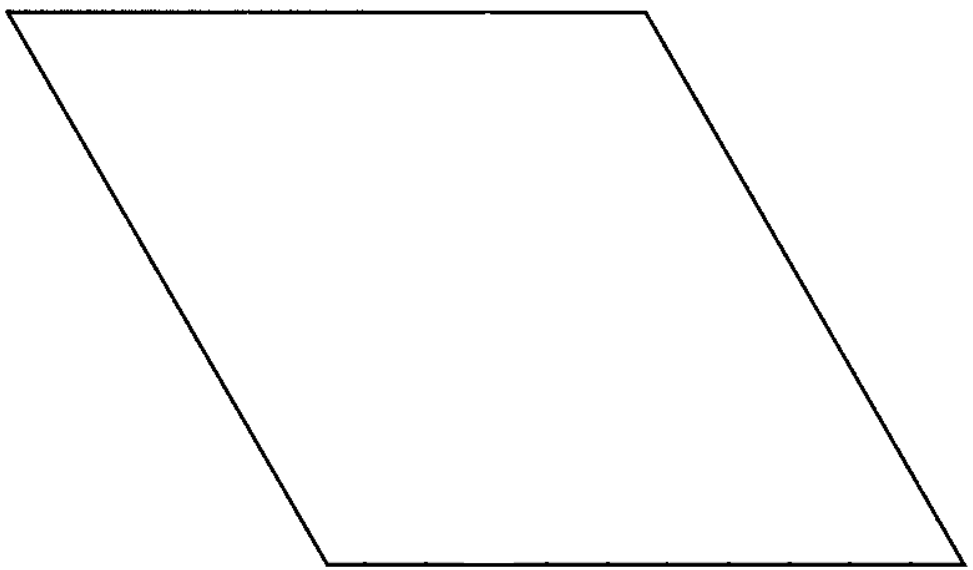

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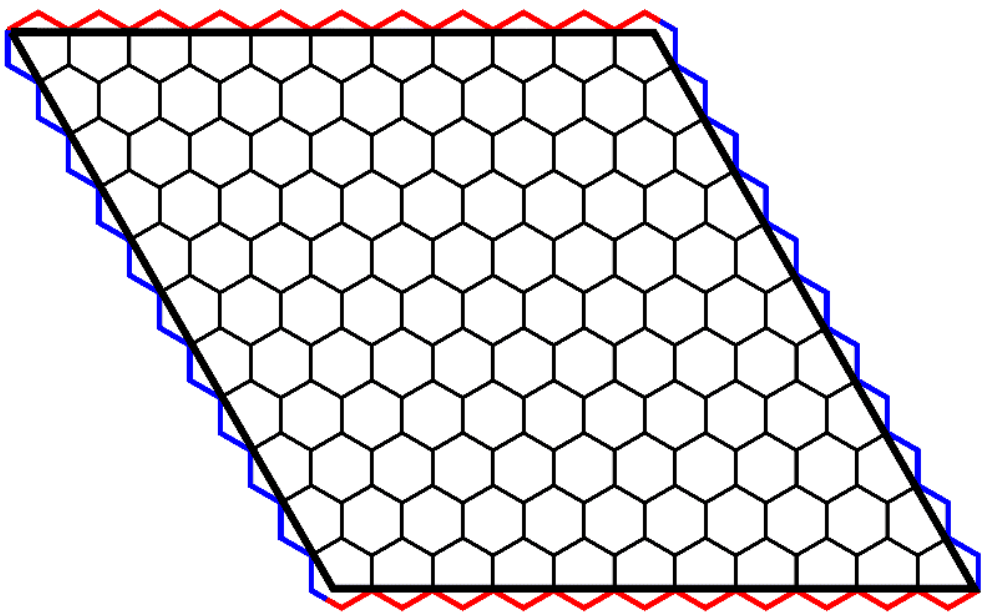

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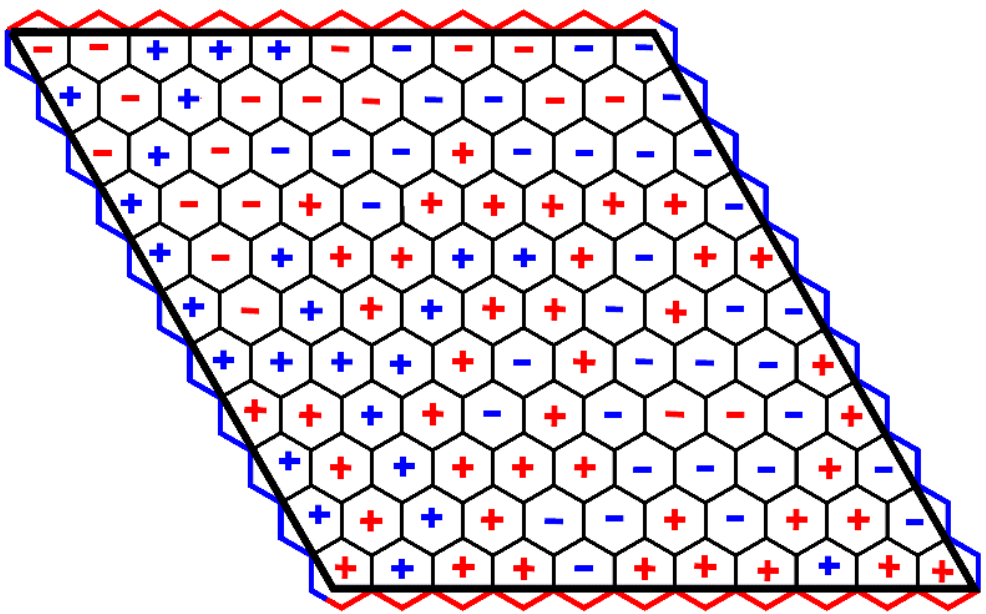

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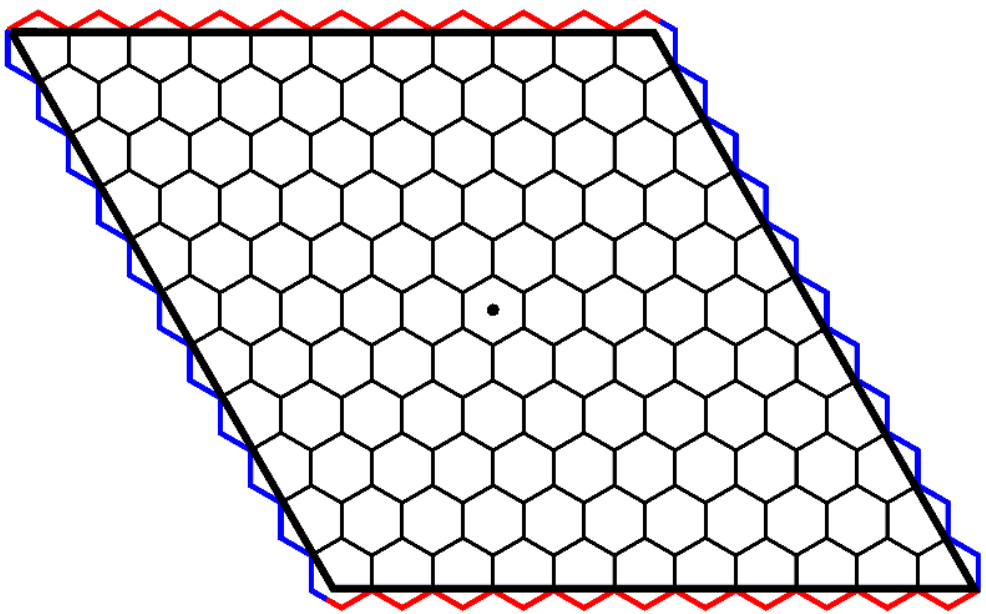

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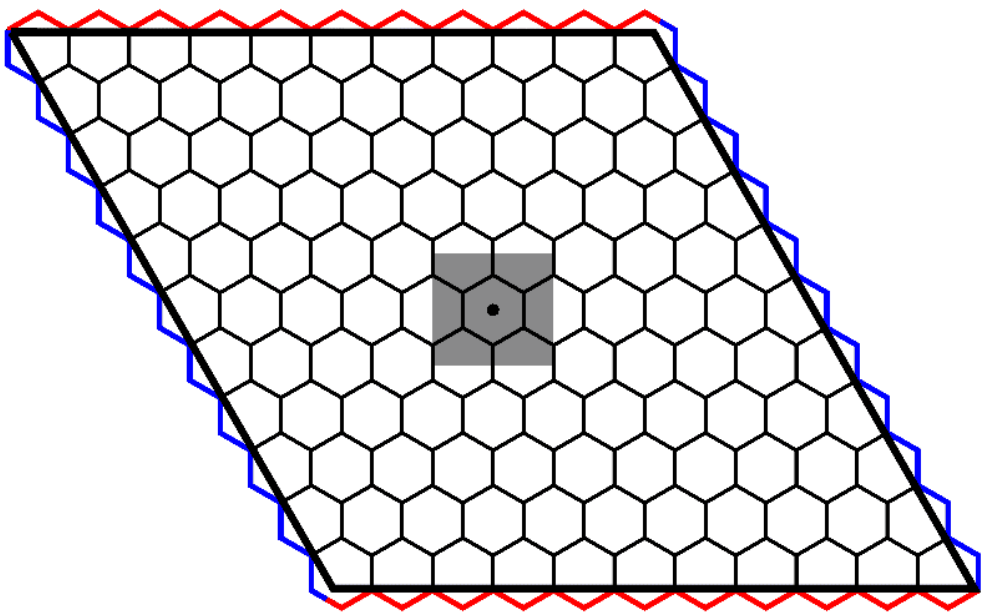

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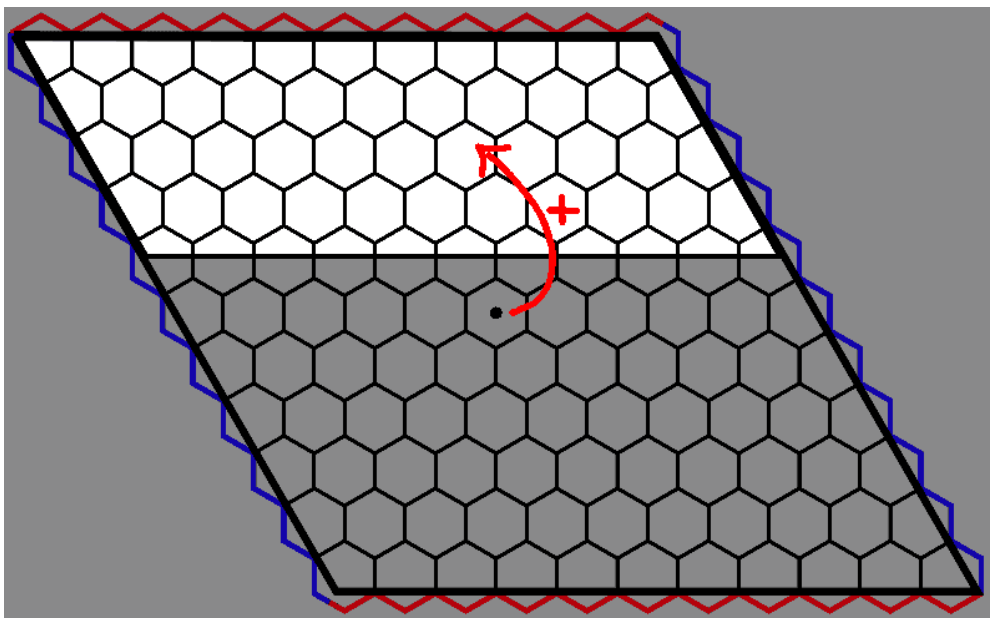

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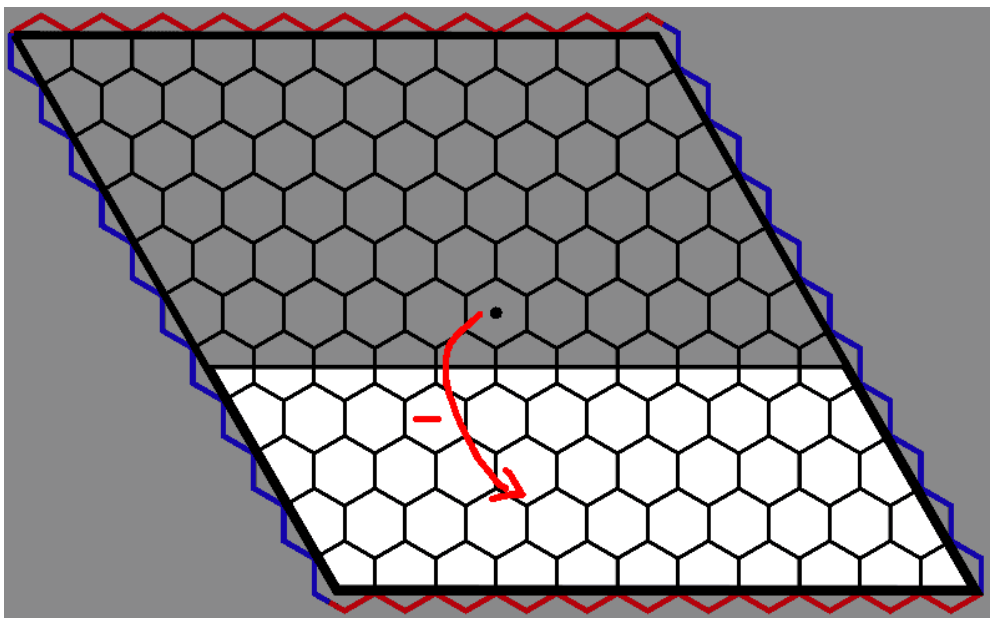

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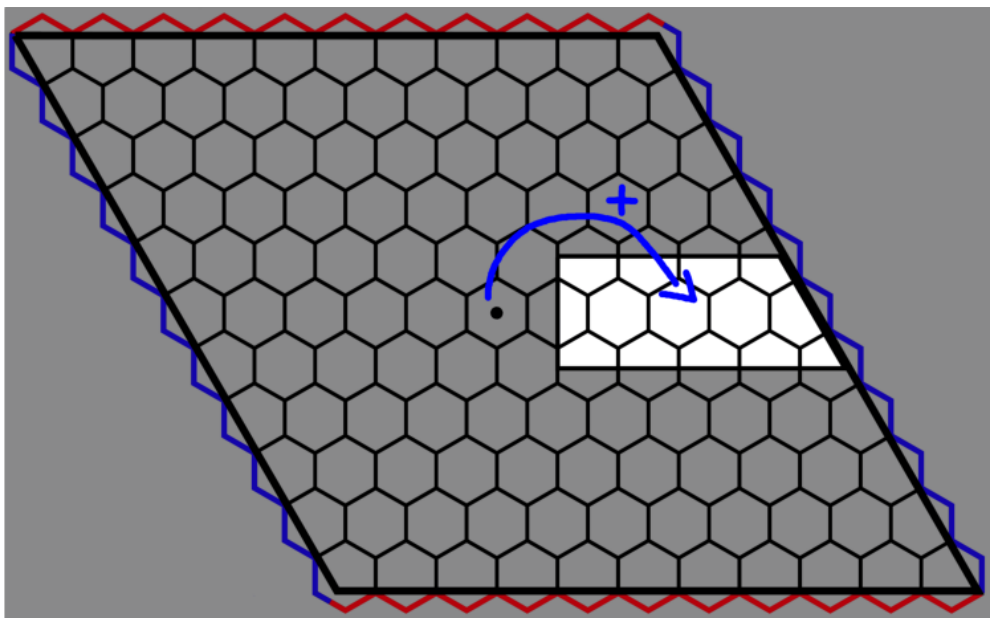

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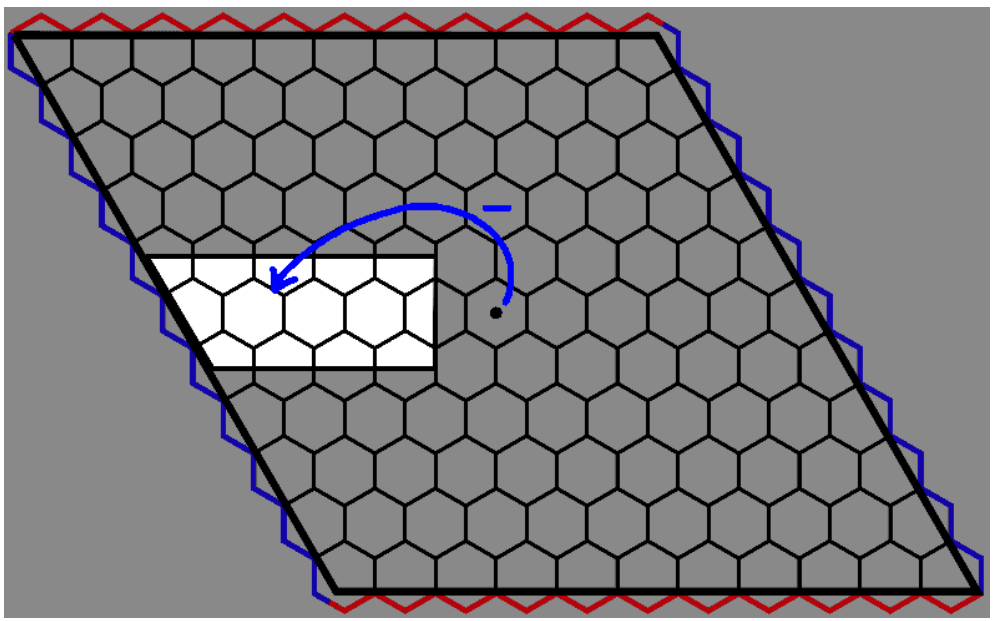

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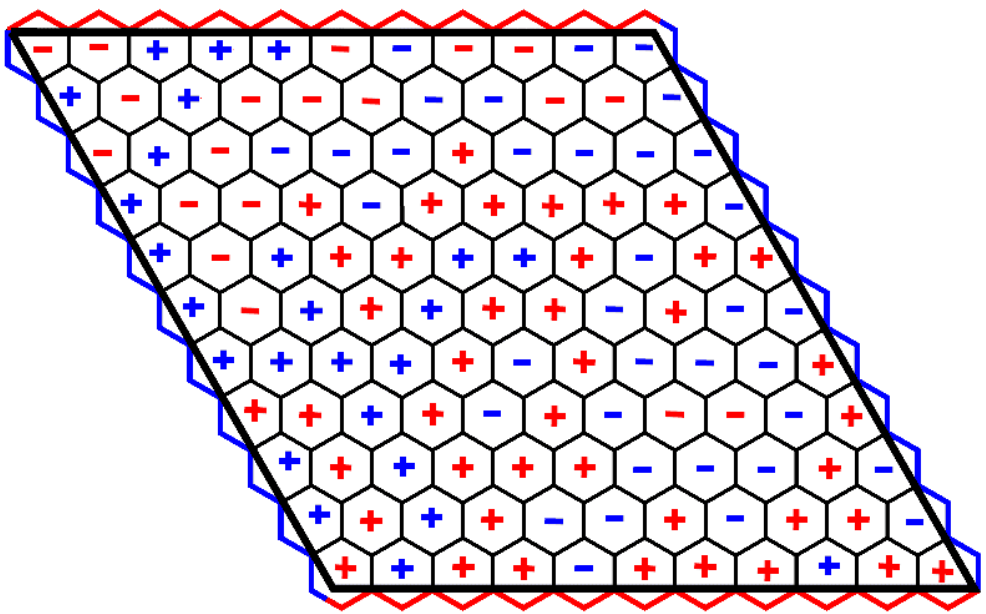

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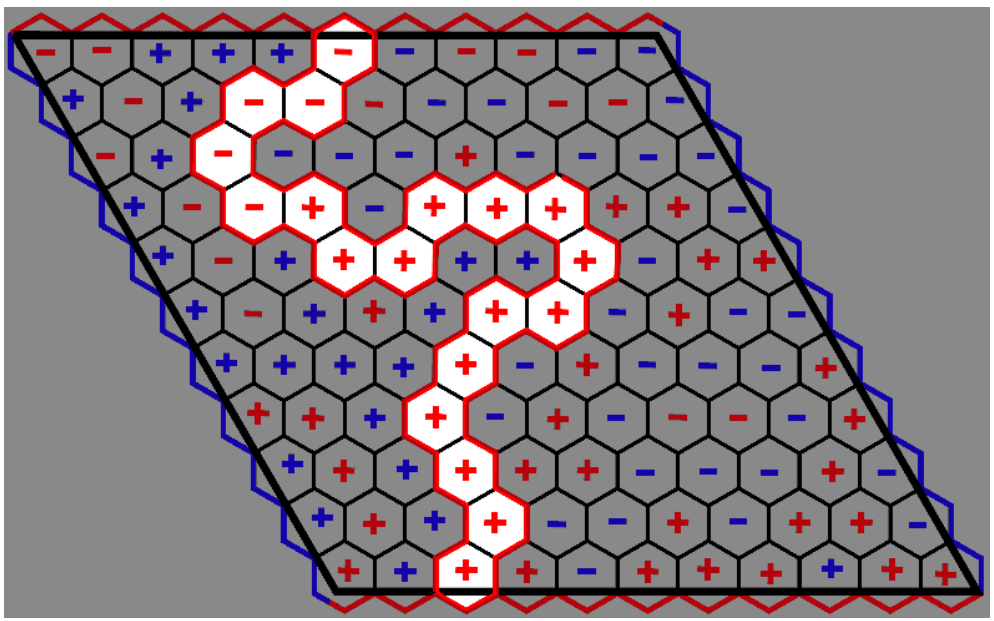

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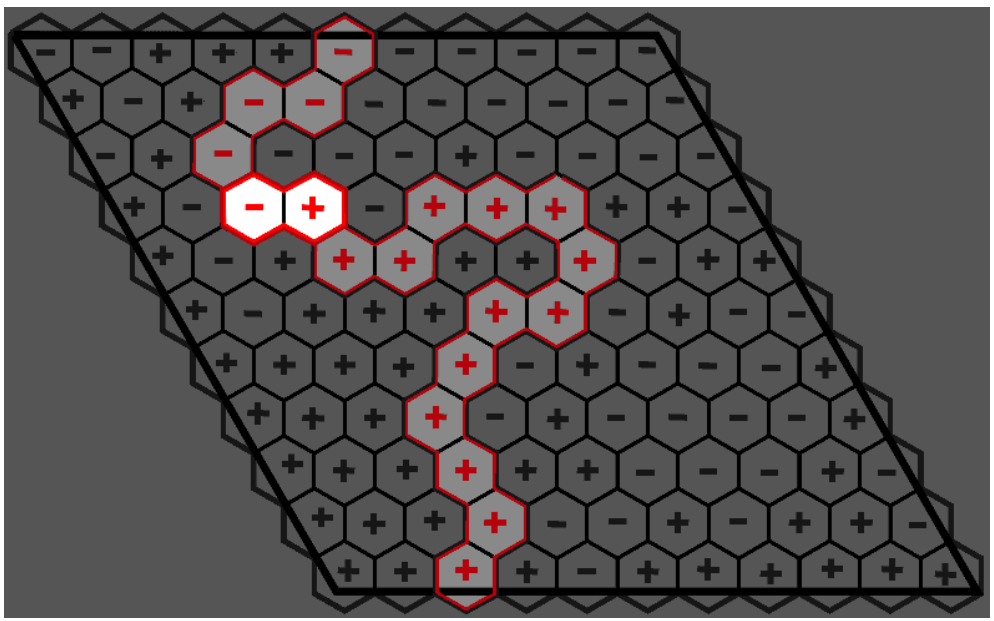
$$|u - v| \leq \delta \implies |f(u) - f(v)| \leq \varepsilon/2$$

Proof of Brouwer's Theorem using Hex

Theorem: Brouwer's fixed point

$\forall f \in \mathcal{C}^0(B^n, B^n)$, $\exists u \in B^n$ such that $f(u) = u$

Proof in 2D, by contradiction (Gale 1979)



$f : R \to R$, $\mathcal{C}^0$ such that

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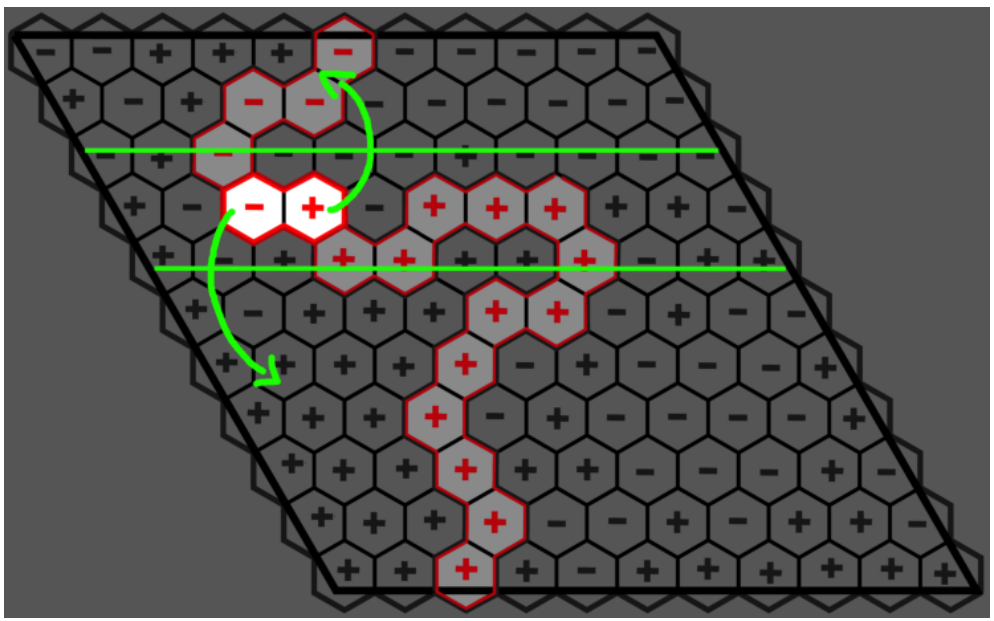

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Hex $\Leftrightarrow$ Brouwer

Hex Theorem

$\Downarrow$

Brouwer's Theorem

David Gale 1979

$\Downarrow$

continuous Hex Theorem

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$\Downarrow$

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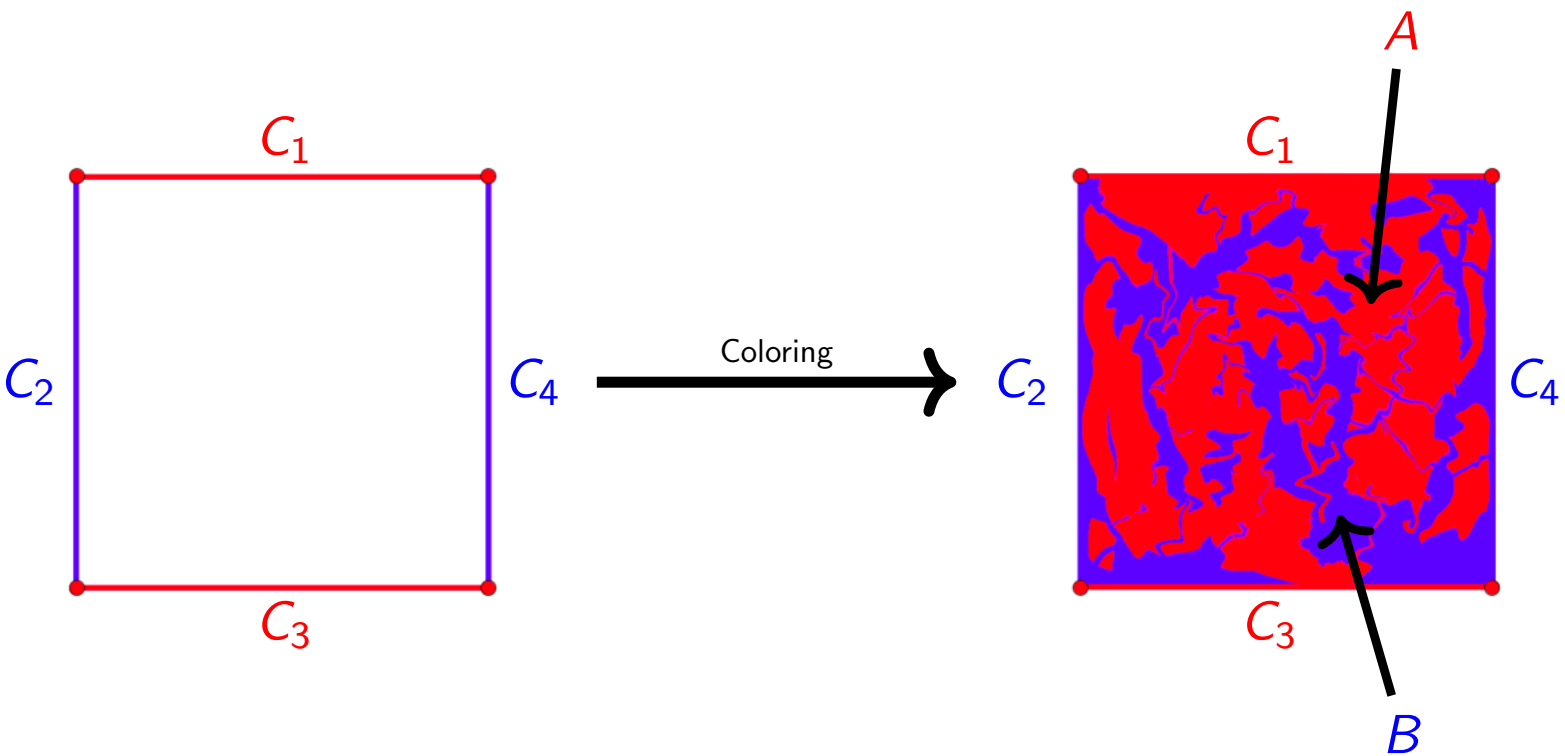
continuous Hex Theorem

Continuous Hex



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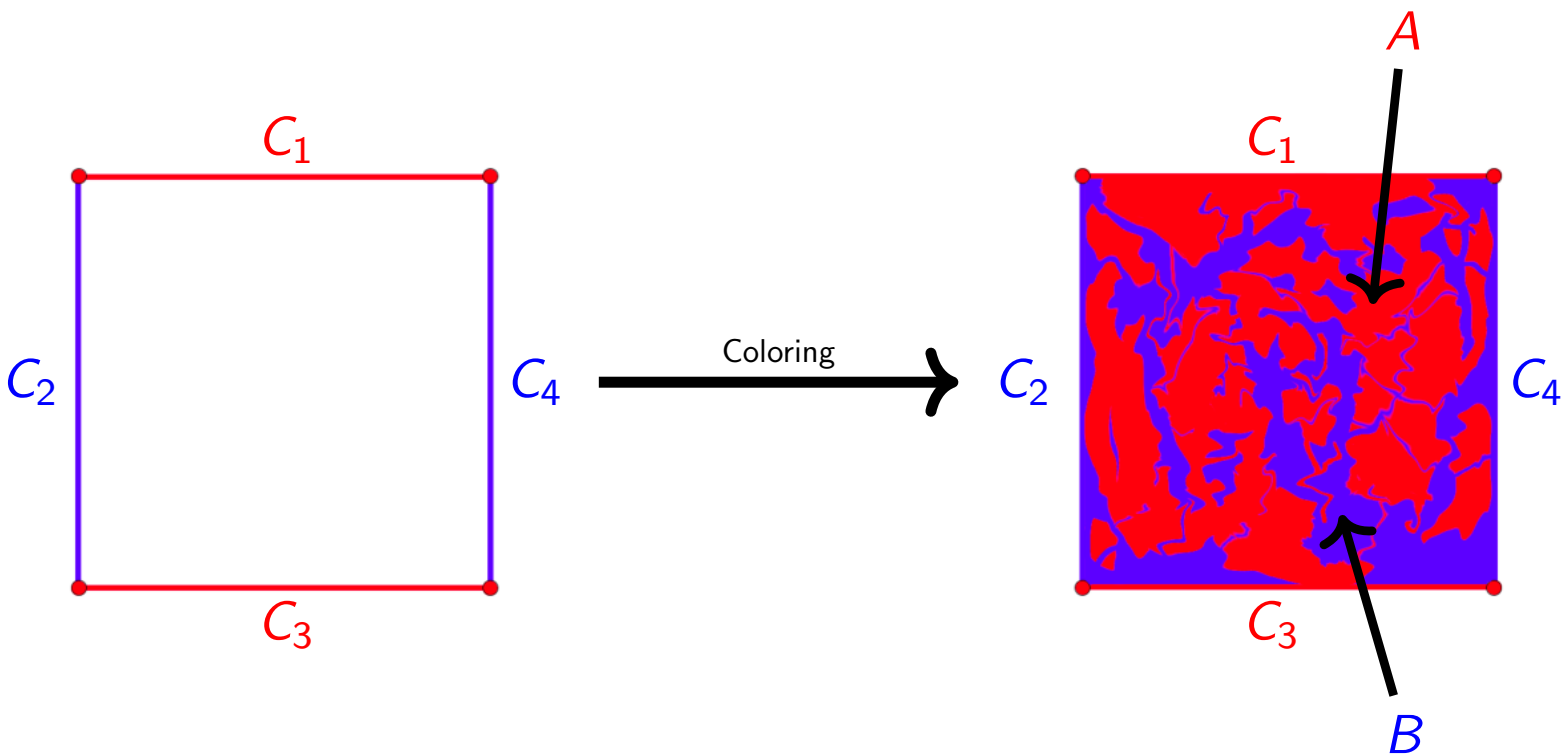

$C_1 \overset{A}{\longleftrightarrow} C_3$, means that $\exists$ a connected component of A containing $C_1 \cap C_3$.

$C_2 \overset{B}{\longleftrightarrow} C_4$, similar definition for B

The continuous Hex Theorem

Theorem: Continuous Hex

Consider $E = [0,1]^2$ (Euclidean). If A and B are such that:

- $E = A \uplus B$
- A is closed and B is open
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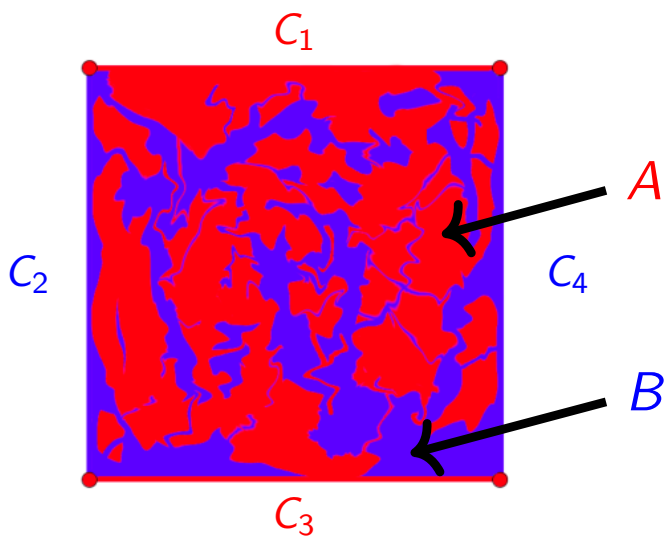

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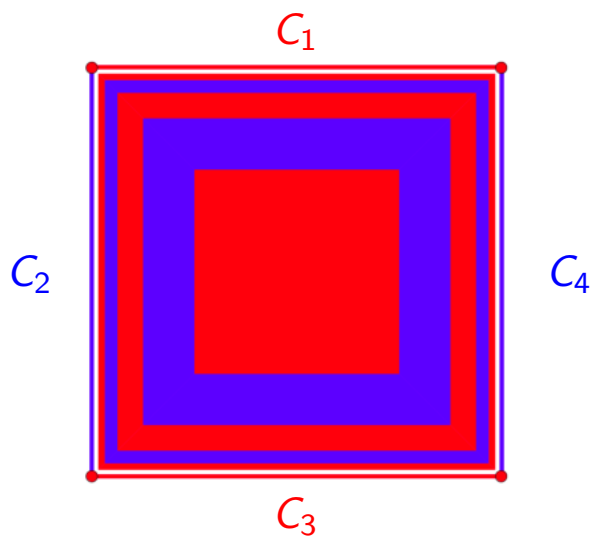
Counterexample 1:



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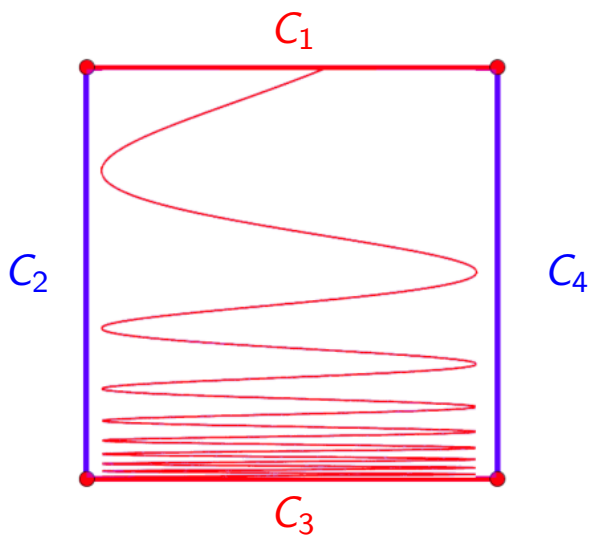

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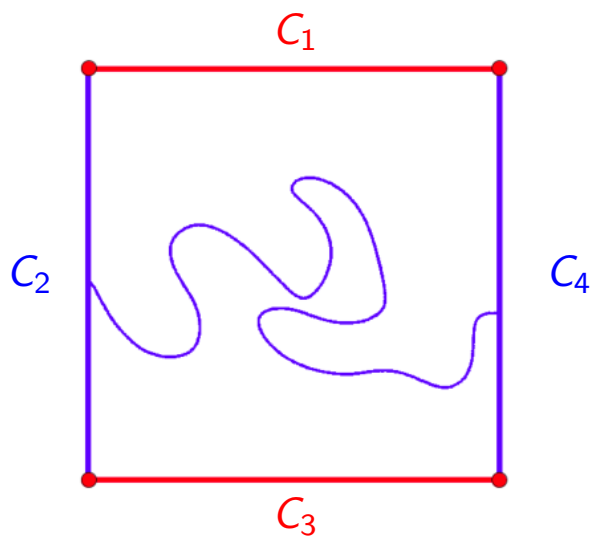
Uniqueness:



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Proof:

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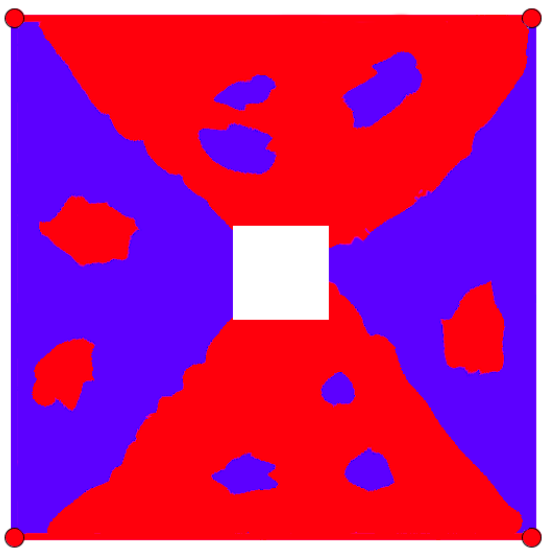

$f : E \to E$ continuous

without fixed point

Proof of the continuous Hex theorem

Theorem: Continuous Hex

- $E = A \uplus B$
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$A_\varepsilon := \varepsilon$-neighbourhood $= \{u \in E : d(A, u) < \varepsilon\}$

Since A is closed

$$\exists \varepsilon > 0, C_1 \overset{A_\varepsilon}{\not\longleftrightarrow} C_3 \quad \text{and} \quad C_2 \overset{B}{\not\longleftrightarrow} C_4$$

Construction of $f \in \mathcal{C}(E, E)$ with no fixed point

Take $\varepsilon > 0$ such that,

$$C_1 \overset{A_\varepsilon}{\longleftrightarrow\!\!\!\!\!\!/} C_3 \quad \text{and} \quad C_2 \overset{B}{\longleftrightarrow\!\!\!\!\!\!/} C_4$$

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Note that

$$E \subset B \cup A_\varepsilon$$

We want to construct

$$f(x, y) = (f_B(x, y), f_{A_\varepsilon}(x, y))$$

such that

$$\forall (x, y) \in B, f_B(x, y) \neq x$$
$$\forall (x, y) \in A_\varepsilon, f_{A_\varepsilon}(x, y) \neq y$$

Let us construct f_B (similar for f_{A_ε} because A_ε also open).

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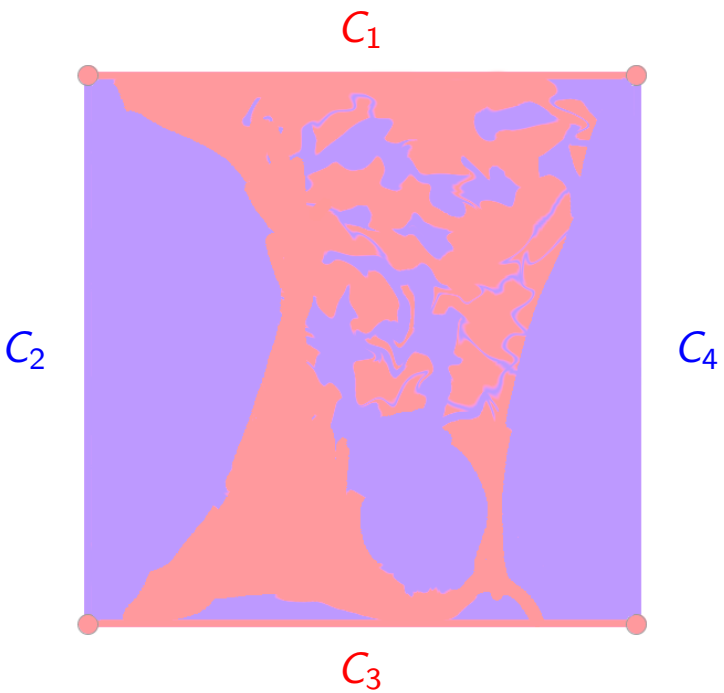

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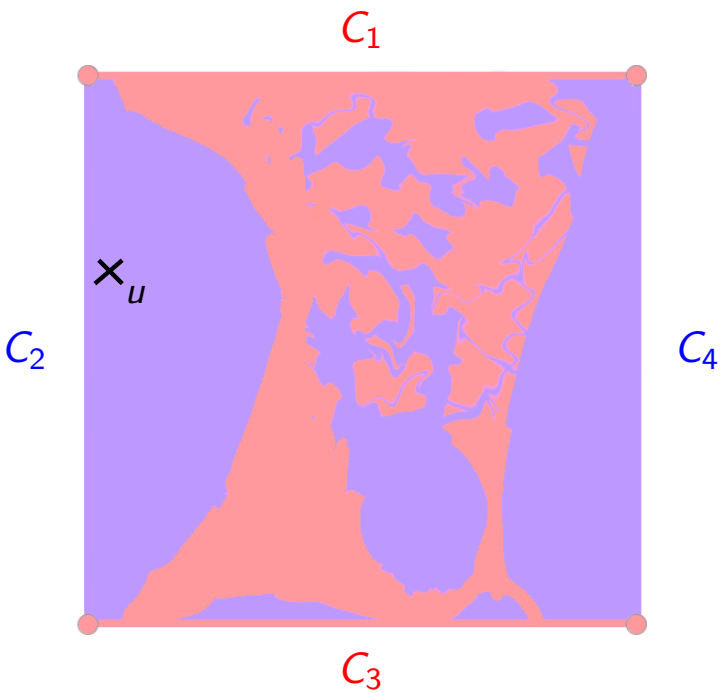

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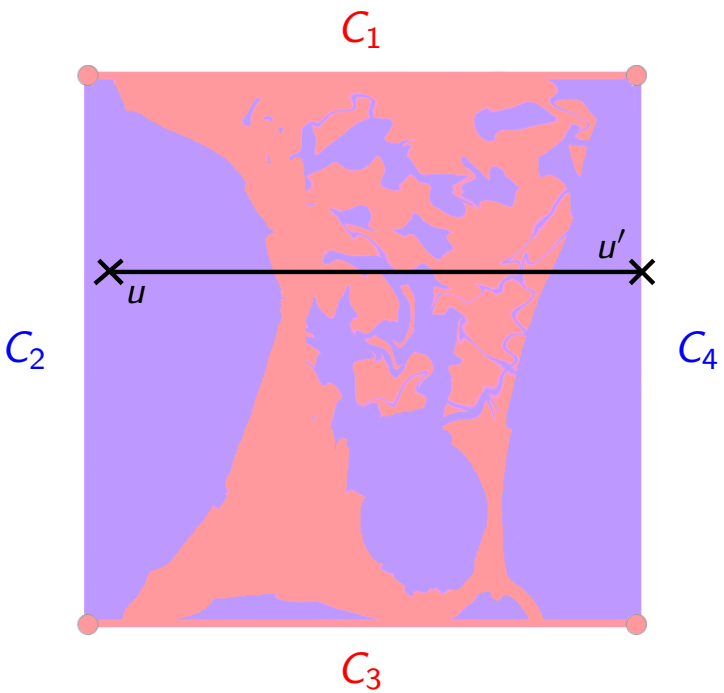

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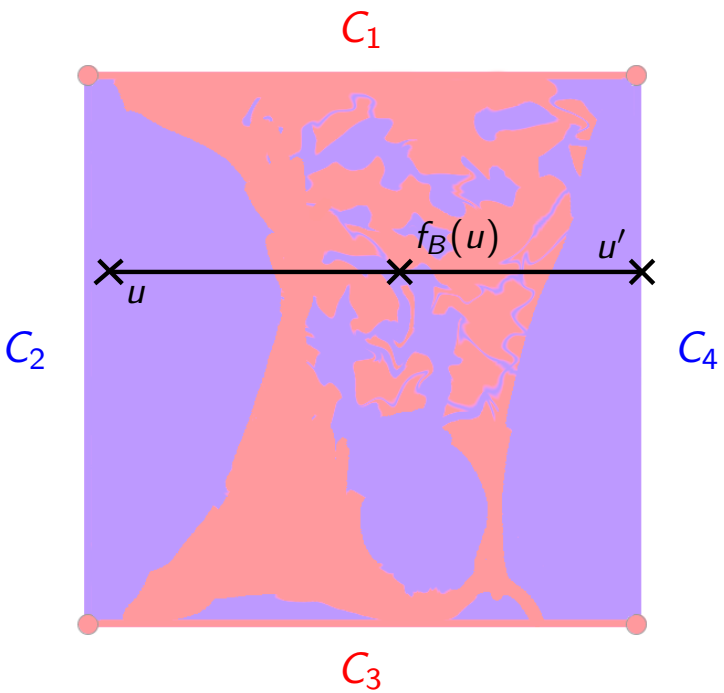

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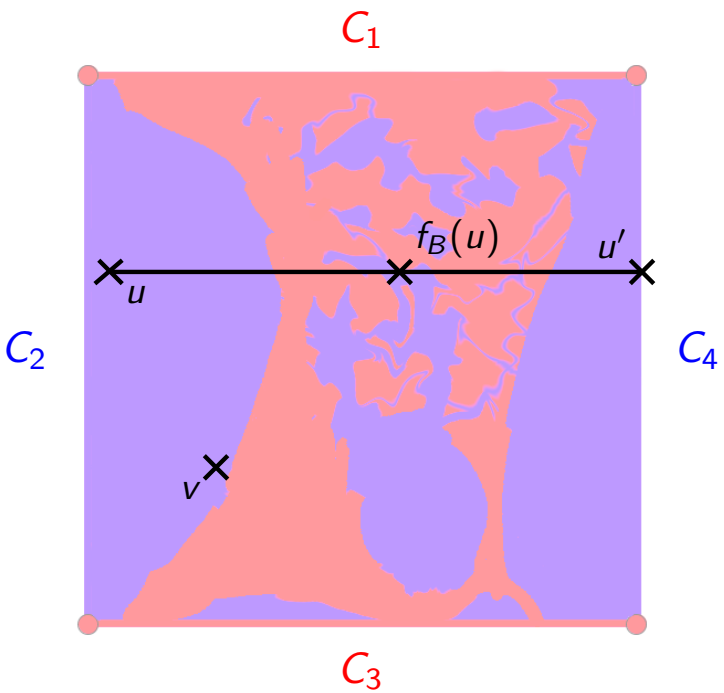

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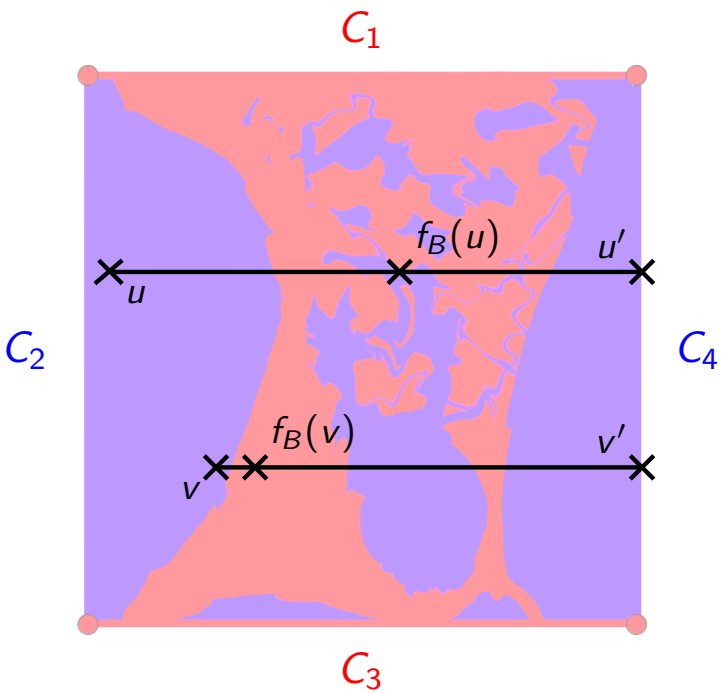

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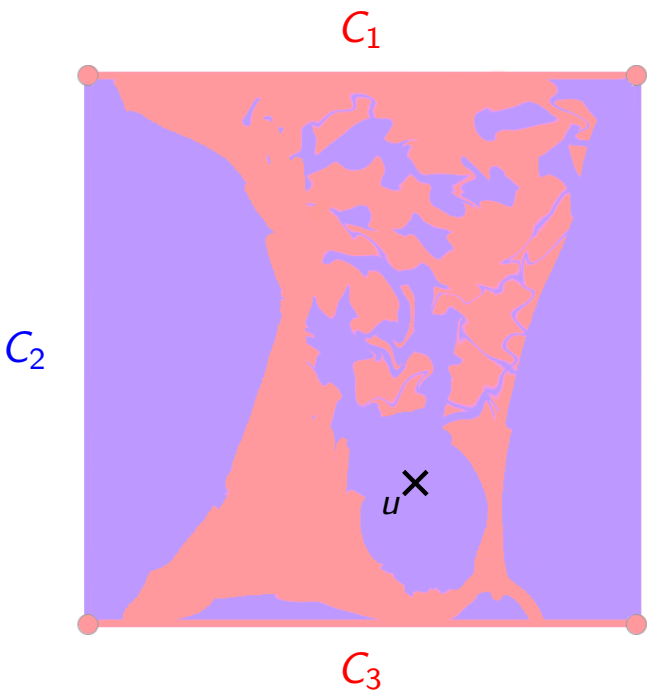

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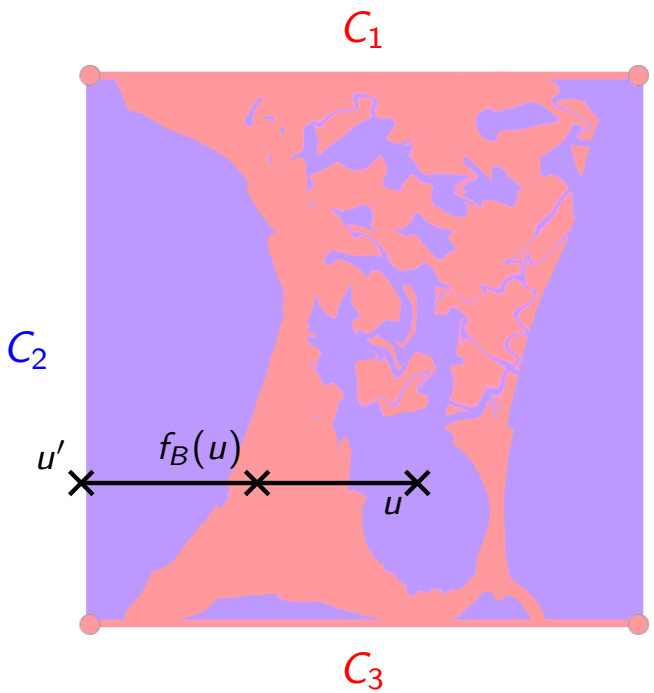

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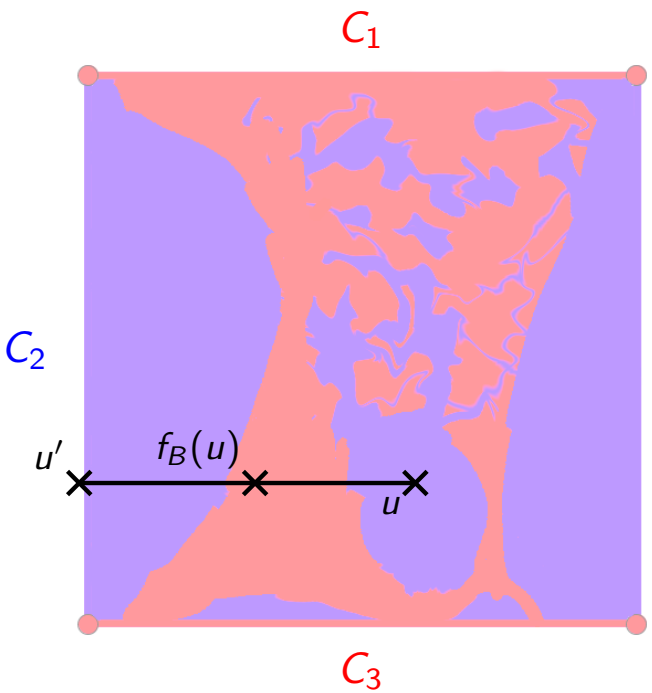

Take $(x, y) \in B$

$$d((x, y), B^c) \neq 0 \quad (B \text{ open})$$

Since $C_2 \overset{B}{\longleftrightarrow} C_4$, the image $f_B(x, y)$ is a convex combination of two **different** coordinates, so

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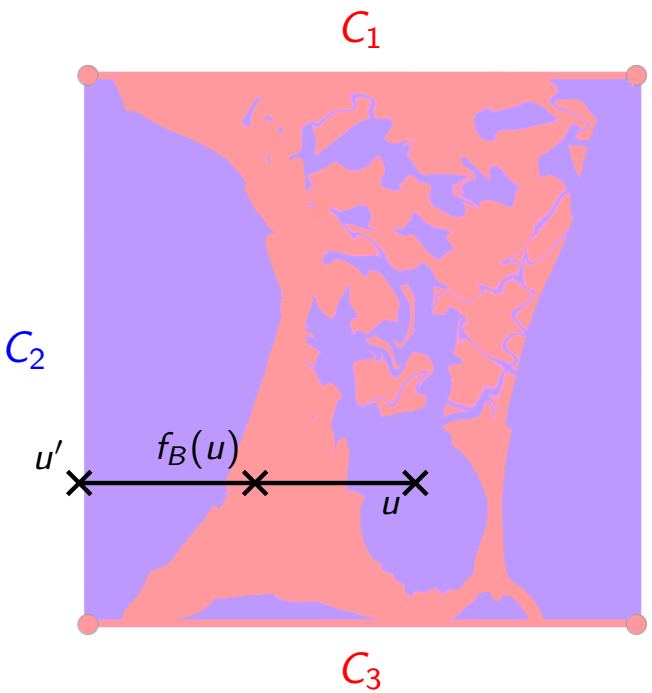

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So $f(x, y) = (f_B(x, y), f_{A_\varepsilon}(x, y))$ is continuous with no fixed point

$\rightsquigarrow$ contradicts Brouwer's theorem.

The continuous Hex theorem

Theorem: Continuous Hex

- $E = A \uplus B$
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Generalisation with quasicomponents

For a topological space X, we define

$$u \sim v \Leftrightarrow \nexists U, V \text{ open s.t. } X = U \uplus V \text{ and } u \in U, v \in V$$

Define

$$\begin{aligned} QC_X(u) &= \text{equivalence class of u for } \sim \\ &= \text{intersection of all the clopen containing } u \end{aligned}$$

We always have:

$$C_X(u) \subset QC_X(u)$$

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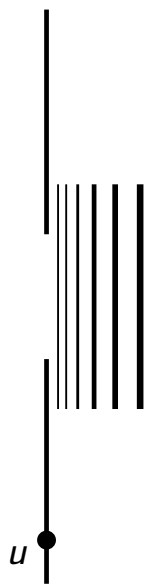

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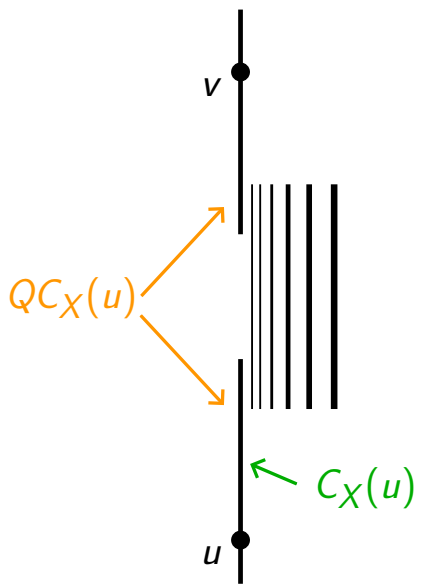

$$C_X(u) \subsetneq QC_X(u)$$

Generalisation with quasicomponents

Theorem: Continuous Hex (generalisation)

Consider $E = [0,1]^2$ (Euclidean). If A and B are such that

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But no uniqueness of the "winner".

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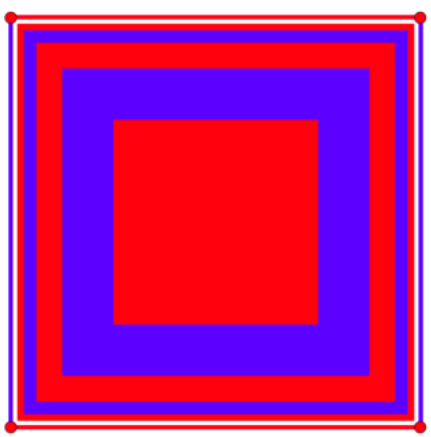

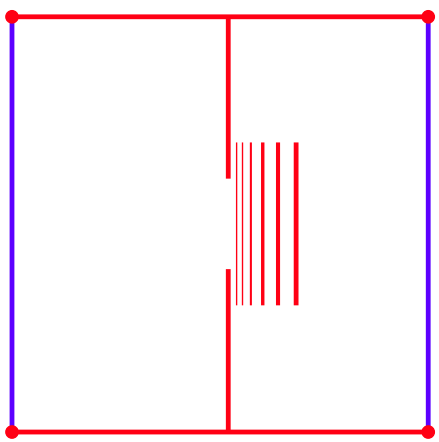

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Remark

If X is a compact Haussdorf space or if X is locally connected then

$$QC_X = C_X$$

Application : Equidistant sets are connected

Theorem

If X and Y are two **connected** subsets of the Euclidean space $\mathbb{R}^n$, then their equidistant set

$$Z = \{z \in \mathbb{R}^n \mid d(z, X) = d(z, Y)\}$$

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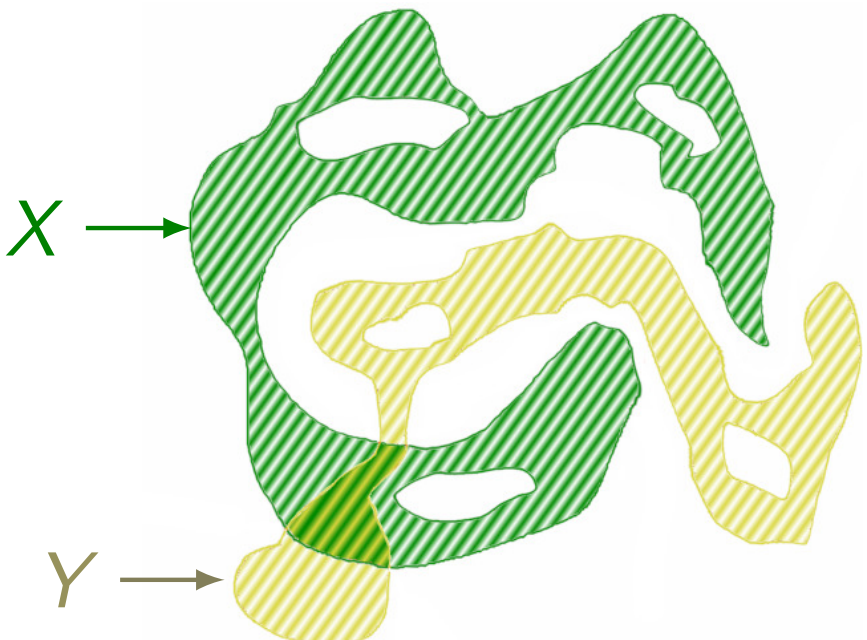

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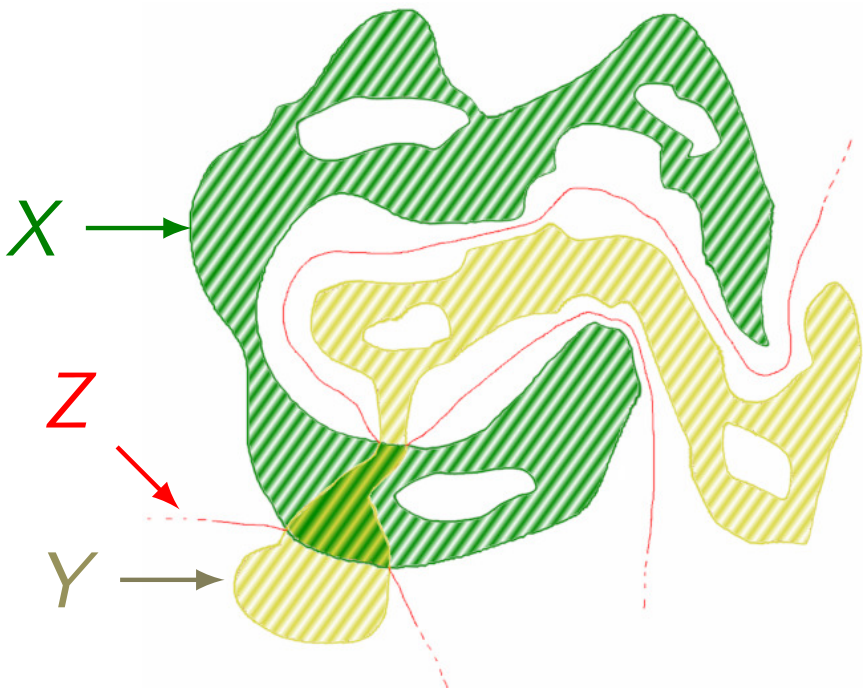

Application : Equidistant sets are connected

Theorem

If X and Y are two **connected** subsets of the Euclidean space $\mathbb{R}^n$, then their equidistant set

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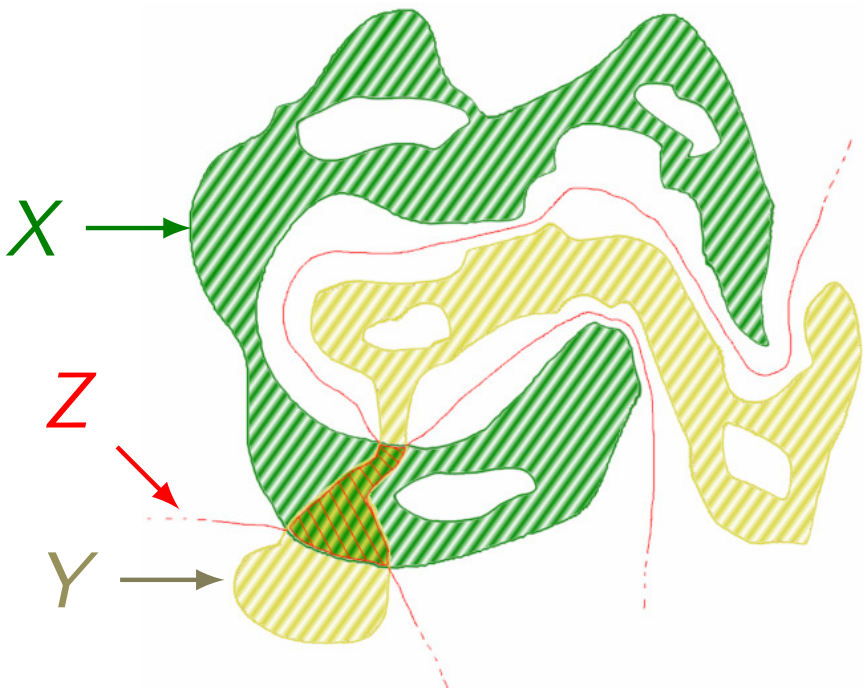

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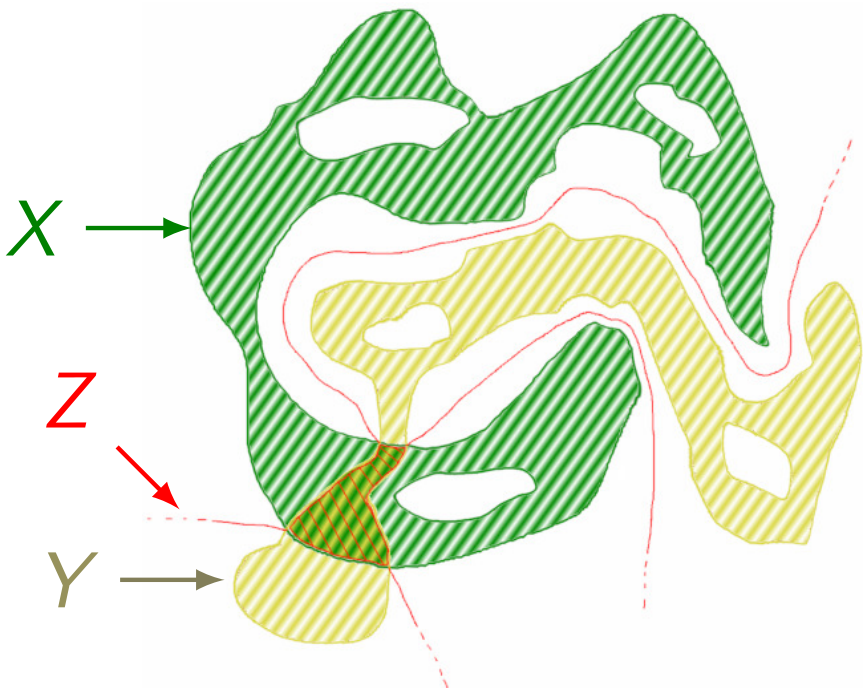
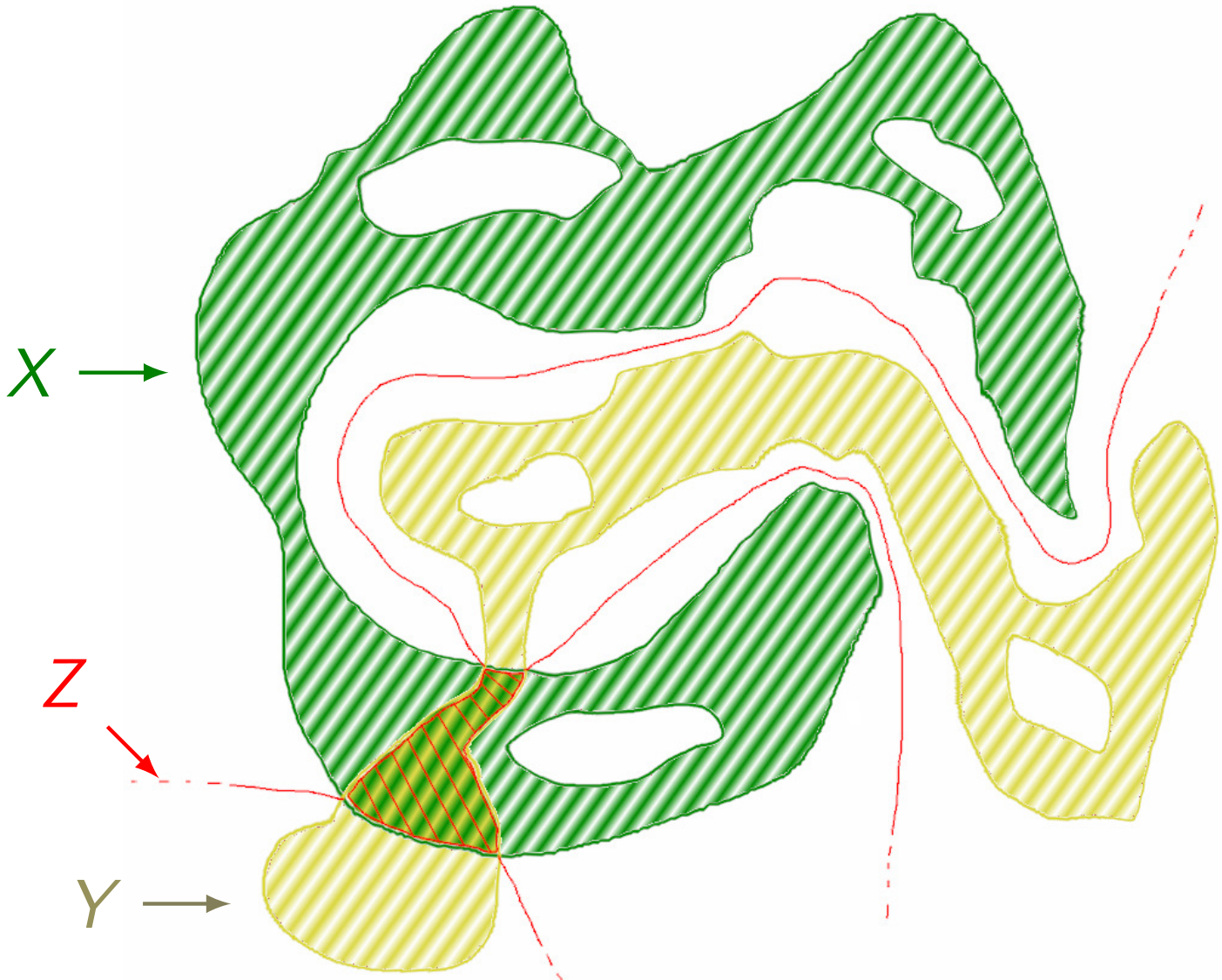


J. B. Wilker 1975,
"Equidistant sets and their connectivity properties"

Euclidean metric

Application: Equidistant sets are connected

Proof Idea (2D):



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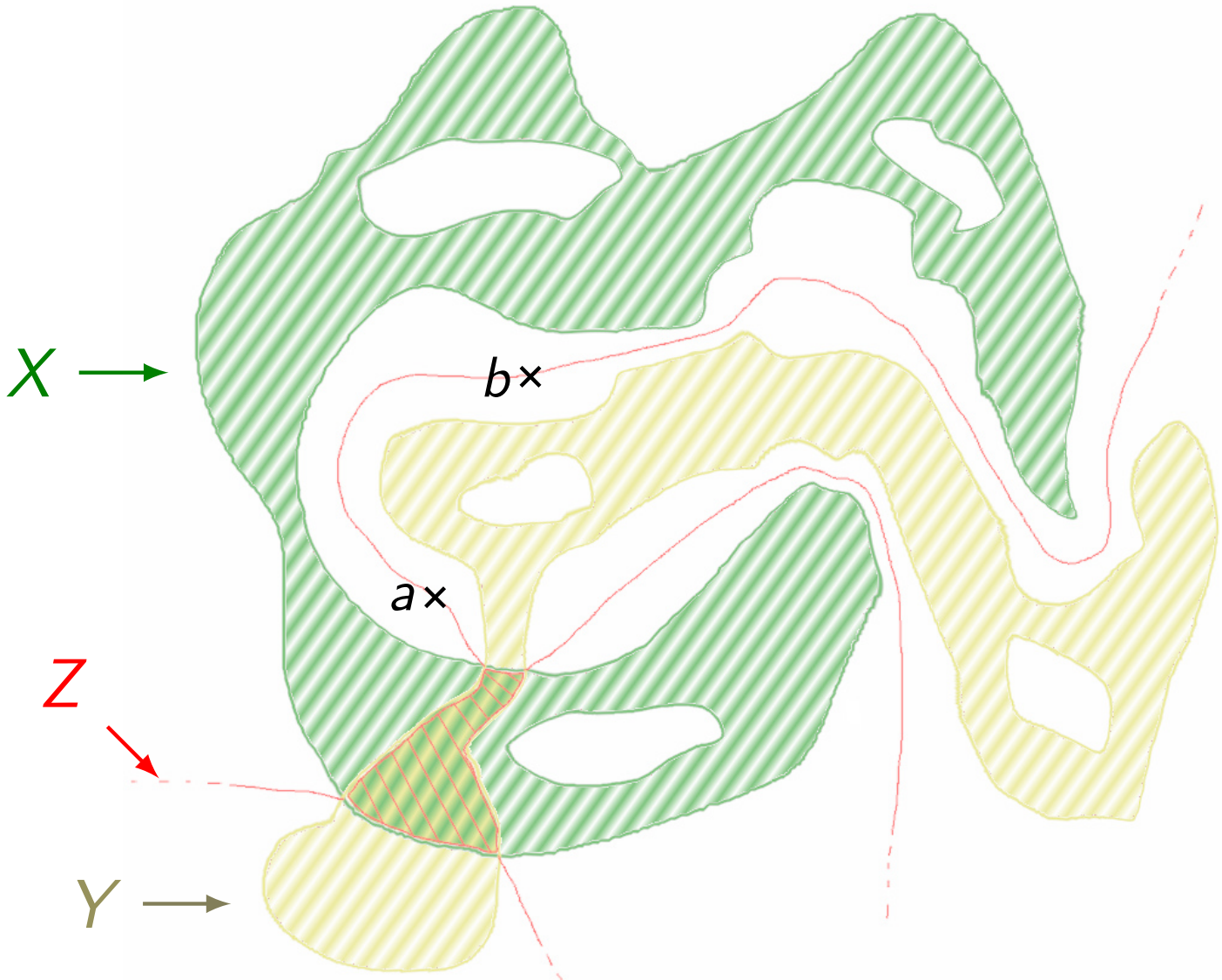

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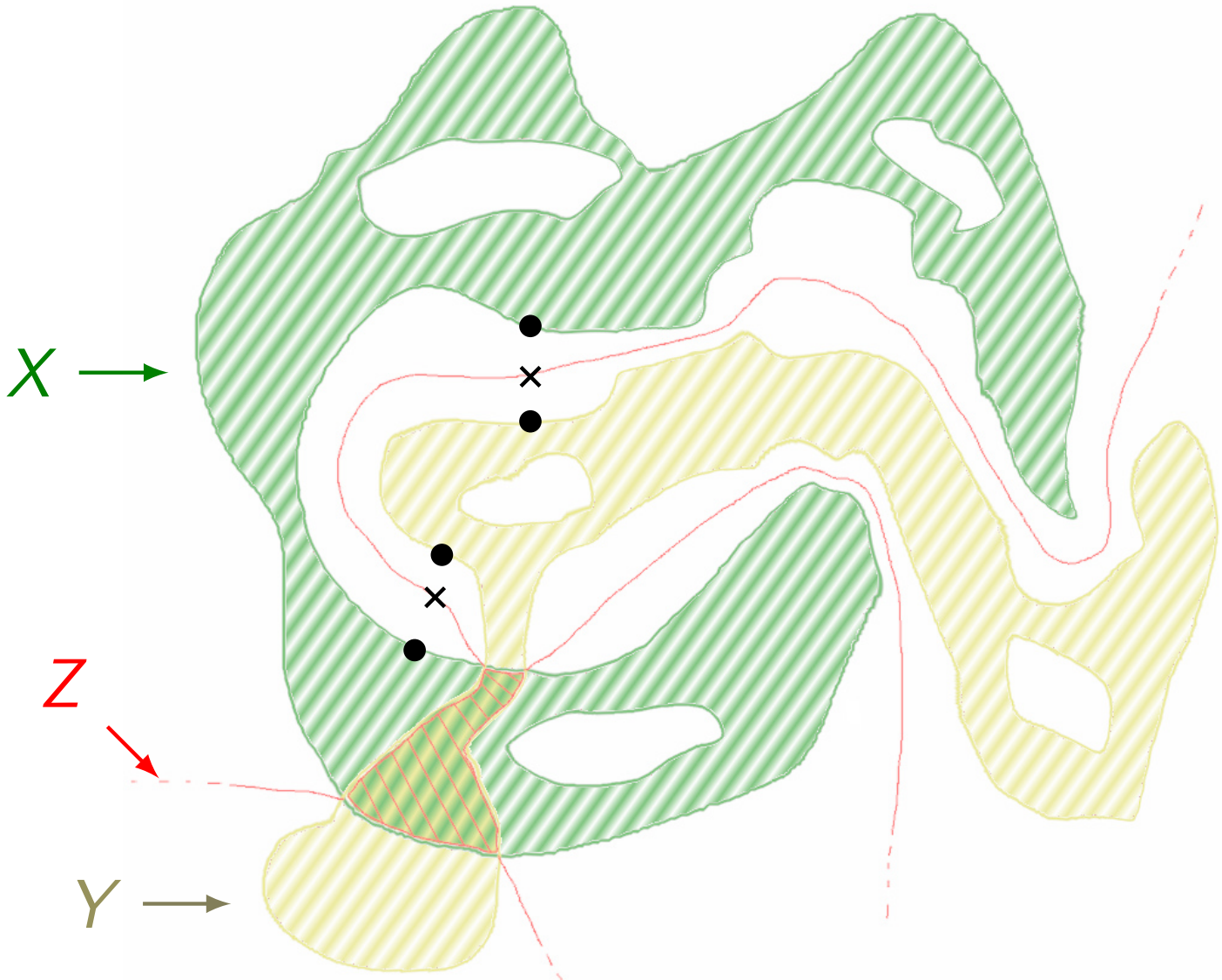

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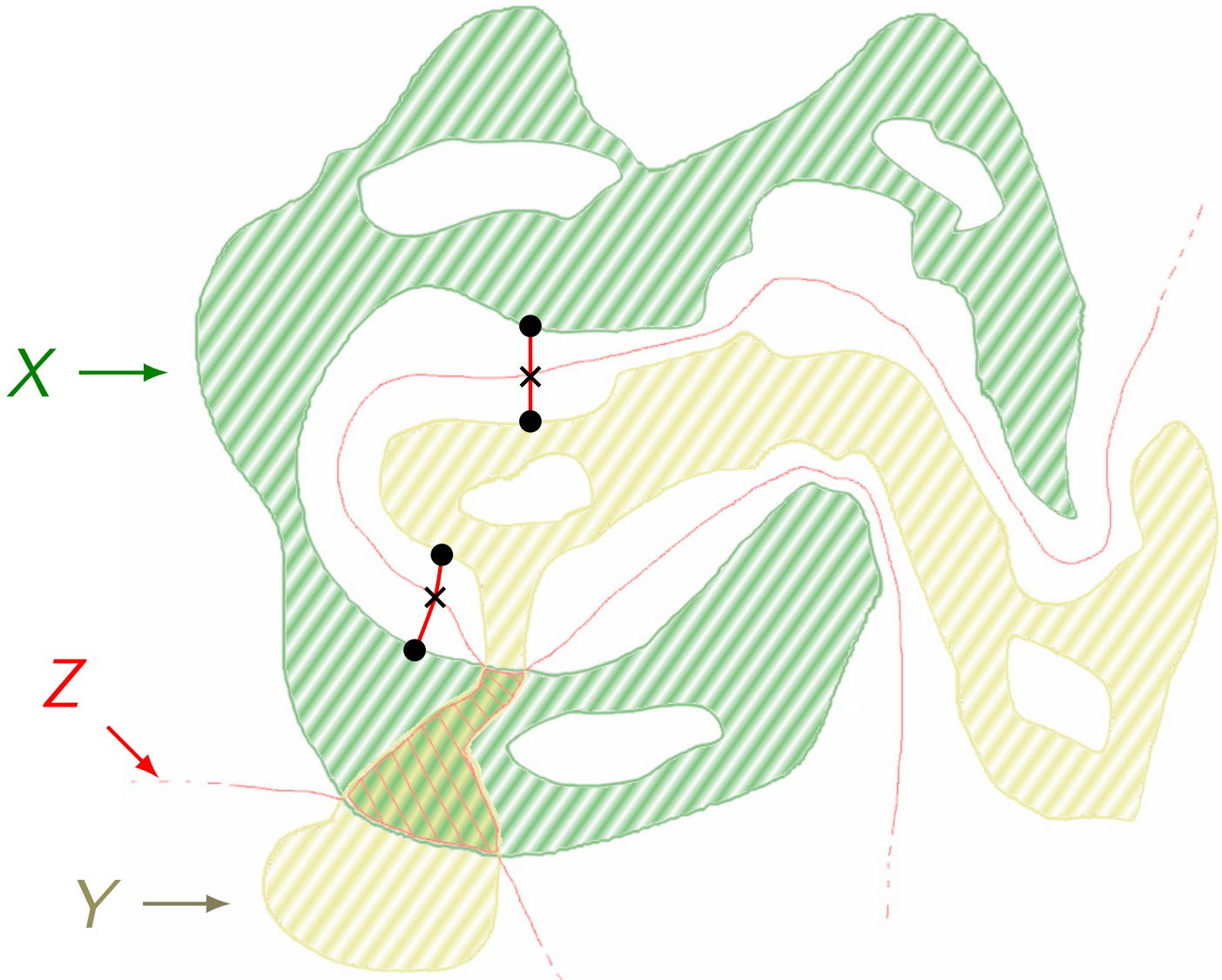

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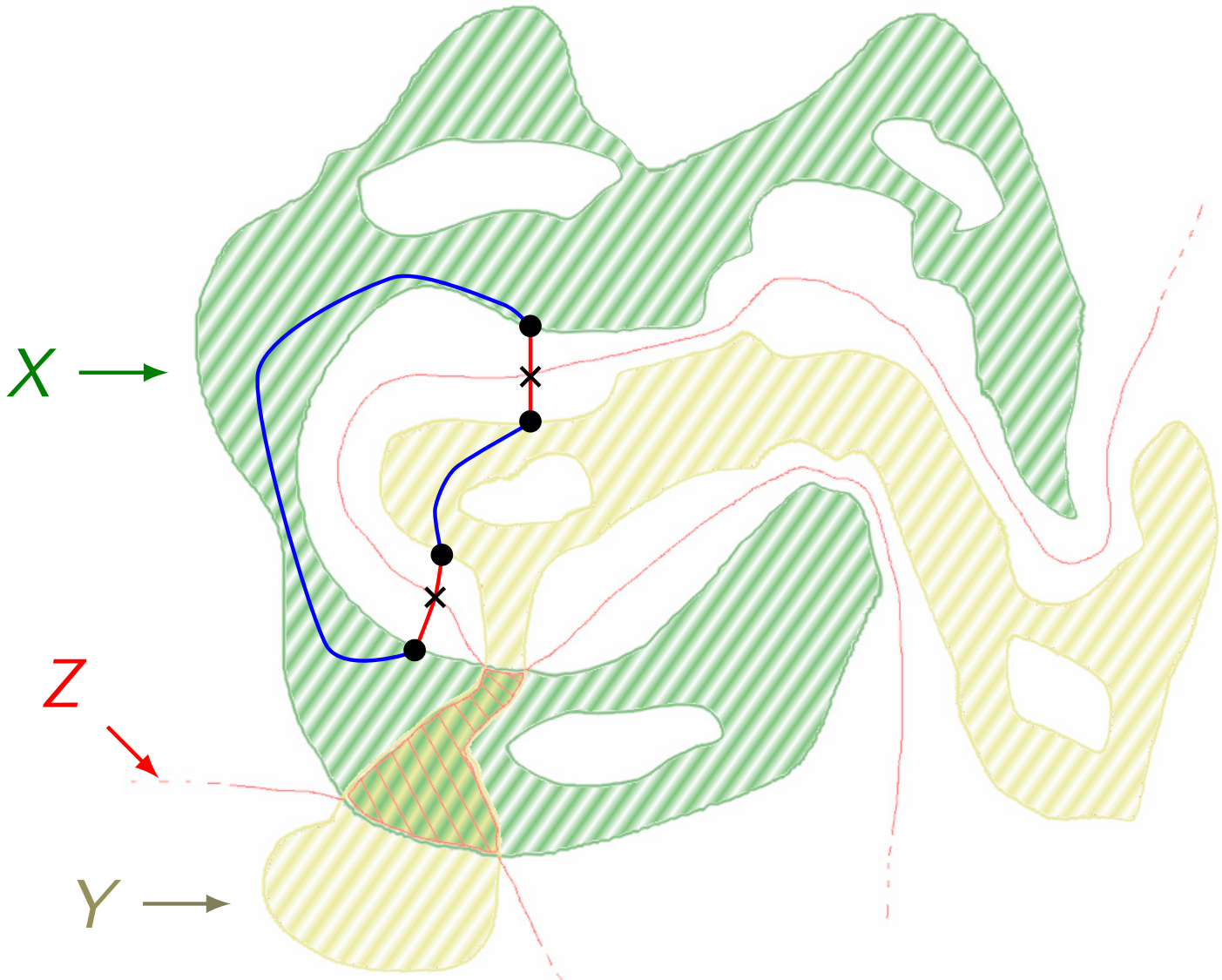

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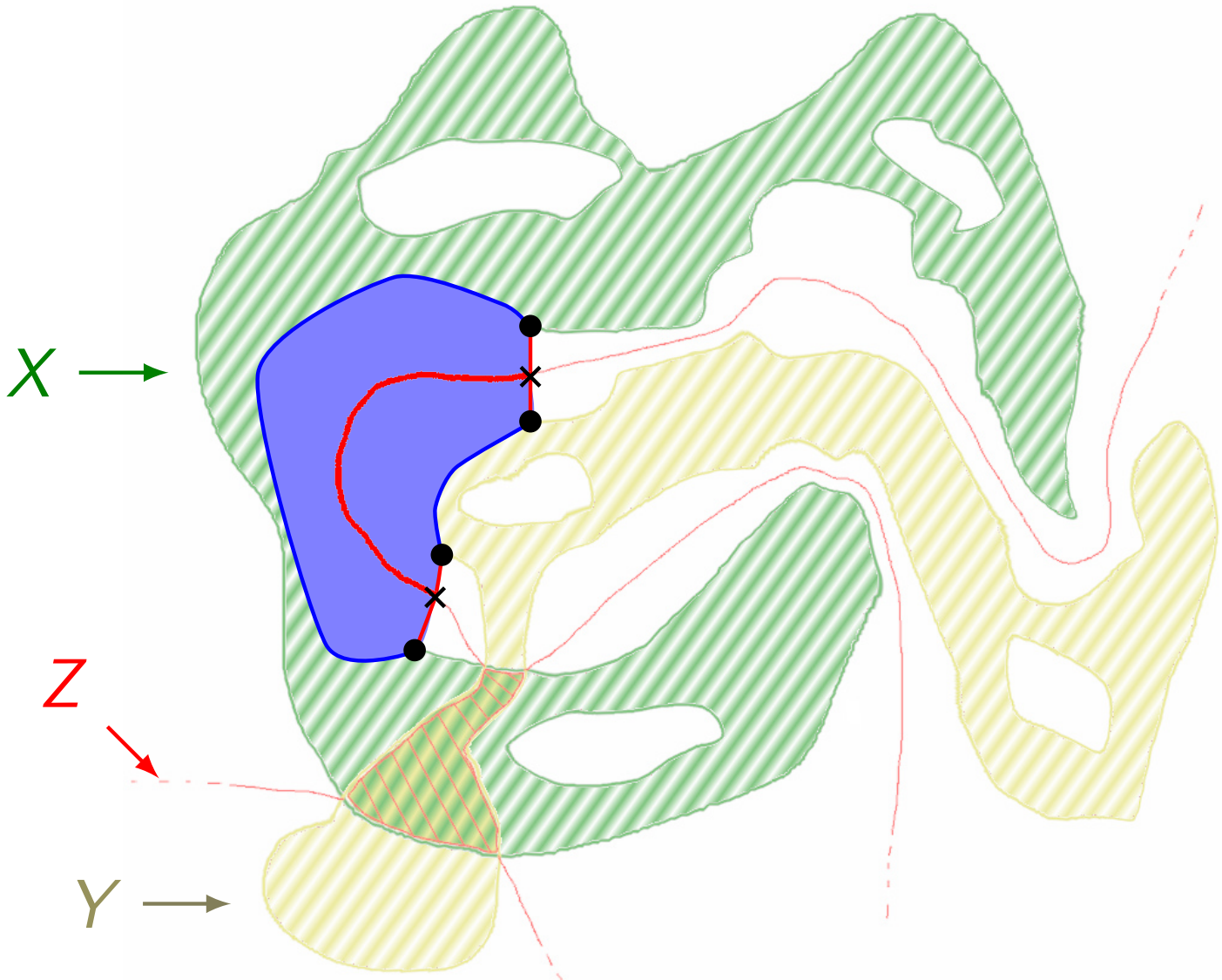

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Take X and Y **connected** subsets of $\mathbb{R}^n$ (Euclidean).
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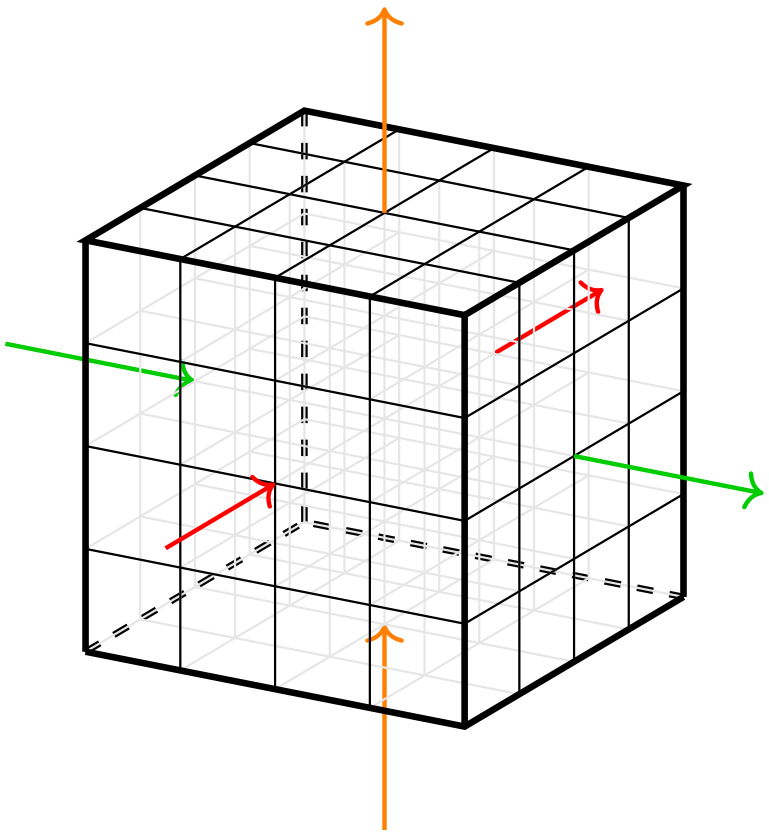
In higher dimensions (discrete)

Fix $d \geq 2$ and consider the cubical complex $T_N^d = \mathbb{Z}^d/(N\mathbb{Z})^d$



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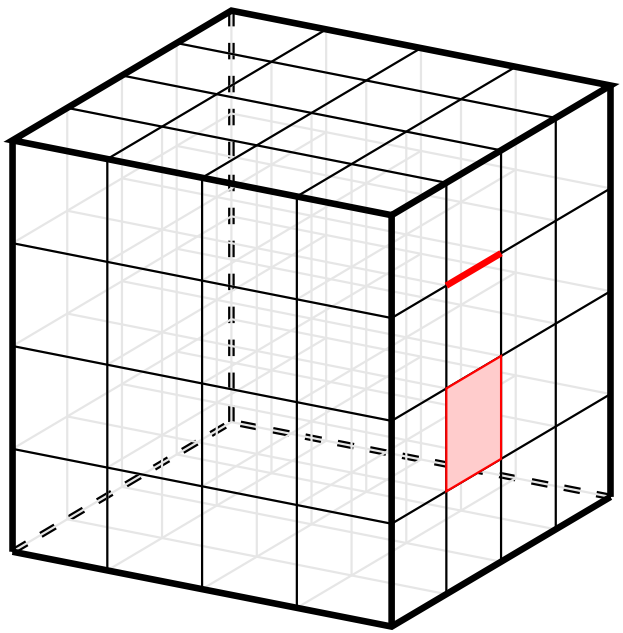

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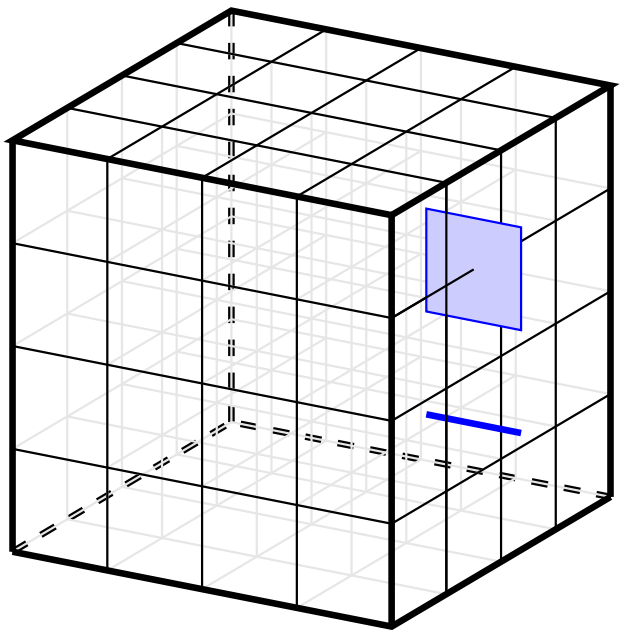

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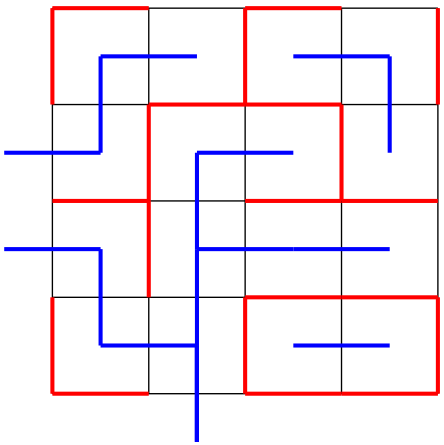

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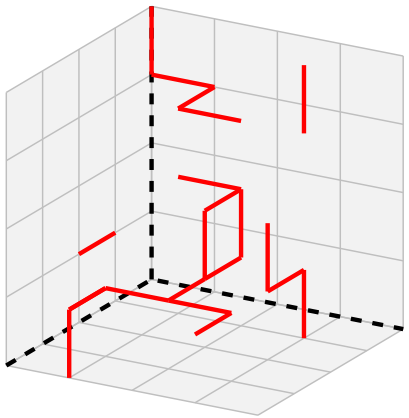

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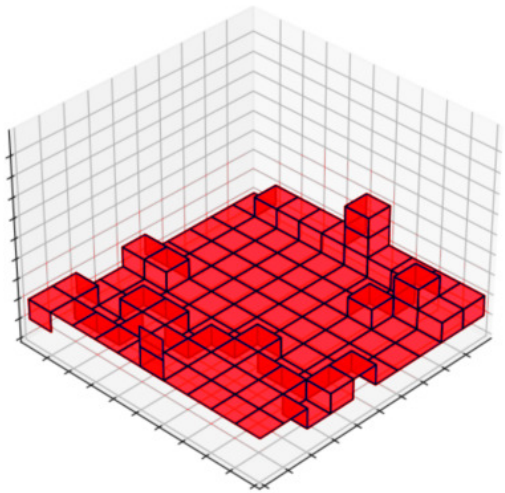

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Duncan et al.

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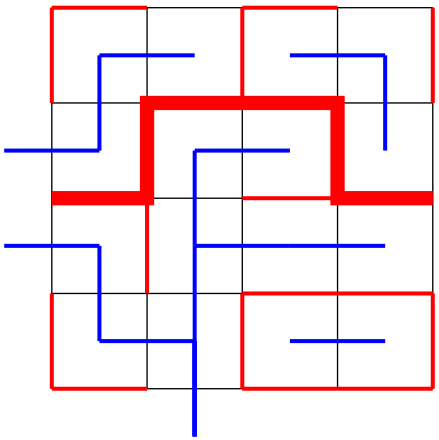

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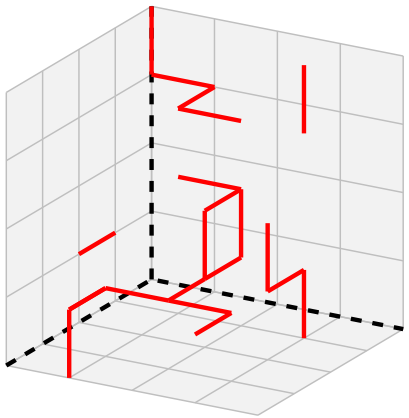

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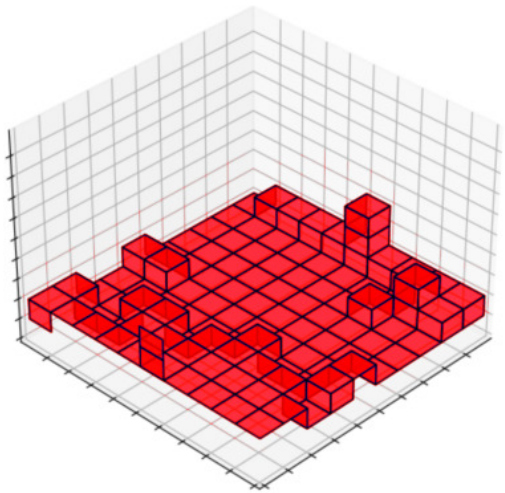

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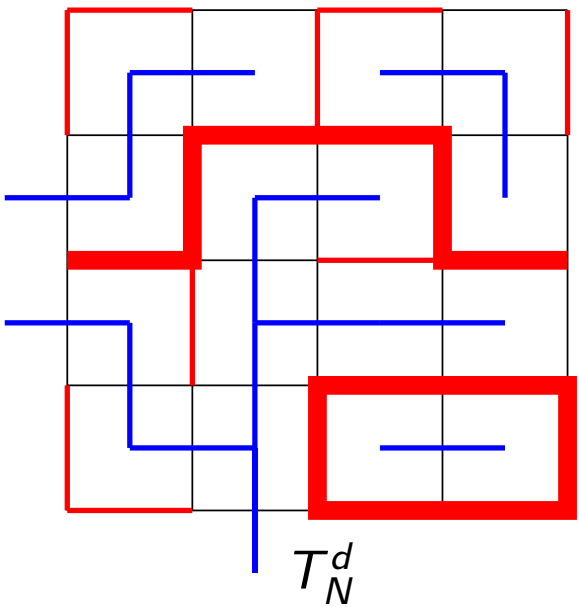
Duncan et al.

How to describe *crossings/cycles* of dimension i (or $d - i$)?

- HOMOLOGICAL PERCOLATION ON A TORUS, 2023,
  Duncan, Kahle, Schweinhart
- HOMOLOGICAL PERCOLATION, 2020
  Bobrowski, Skraba

Homology (discrete)

We are interested in $H_i(A)$ (and the "dual" $H_{d-i}(B)$)



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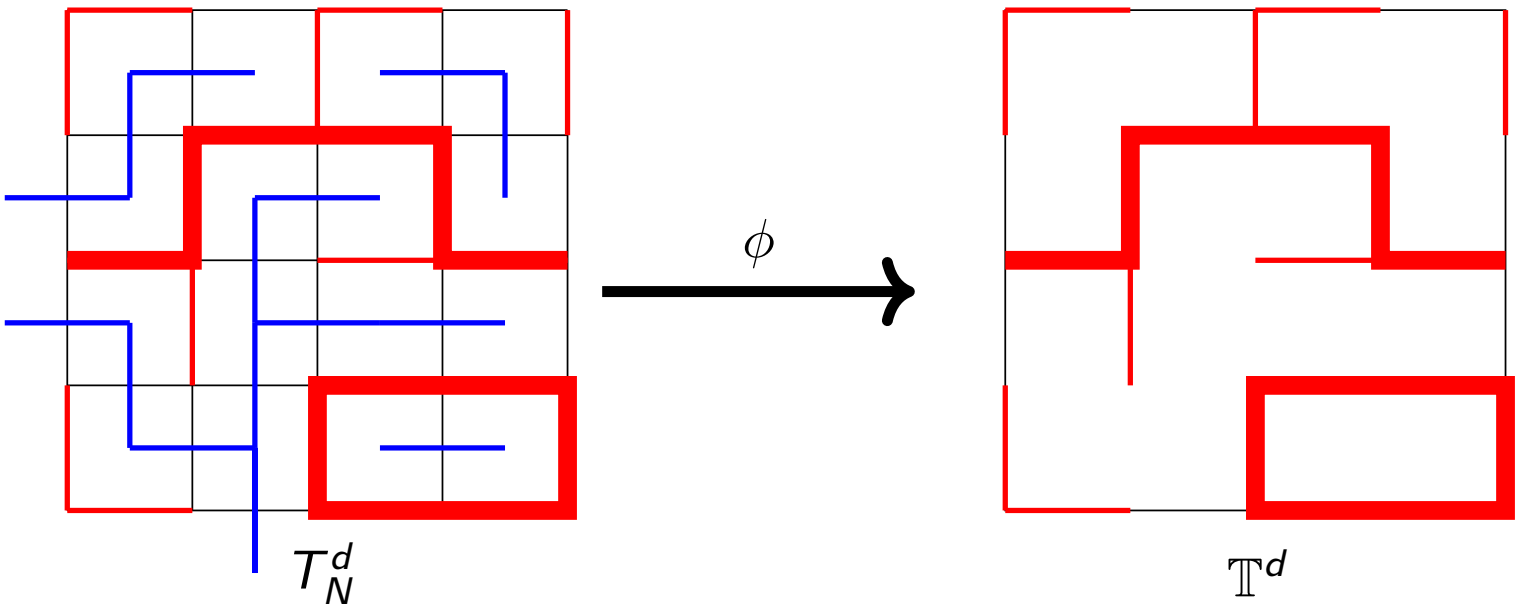

Inclusion maps: $\phi : A \to \mathbb{T}^d$ and $\psi : B \to \mathbb{T}^d$.

$$\phi_\star : H_i(A) \to H_i(\mathbb{T}^d) \quad \text{and} \quad \psi_\star : H_{d-i}(B) \to H_{d-i}(\mathbb{T}^d)$$

"Select the non-trivial i-cycles (or $d - i$ cycles)"

In higher dimensions (discrete)

coef. on a field $\mathbb{F}$

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Duality:

$$\operatorname{rank} \phi_\star + \operatorname{rank} \psi_\star = \operatorname{rank} H_i(\mathbb{T}^d) = \binom{d}{i}$$

In particular

$$\operatorname{rank} \phi_\star > 0 \quad \text{or} \quad \operatorname{rank} \psi_\star > 0$$

see e.g. Duncan, Kahle, Schweinhart 2023

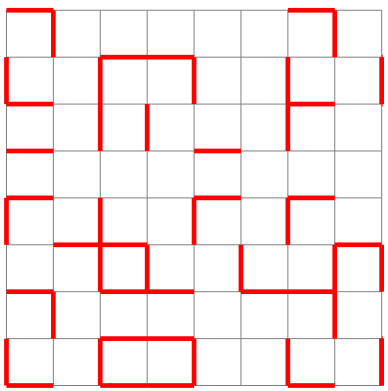
$\frac{d}{2}$-dim percolation in T_N^d with $d = 2k$

Choose A of dimension $\frac{d}{2}$ at random with parameter $p \in (0,1)$

Theorem: Duncan, Kahle, Schweinhart, 2023

With high probability as $N \to \infty$,

$$\operatorname{rank} \phi_\star = \begin{cases} 0 & \text{for } p < 1/2 \\ \operatorname{rank}(H_i(\mathbb{T}^d)) & \text{for } p > 1/2 \end{cases}$$



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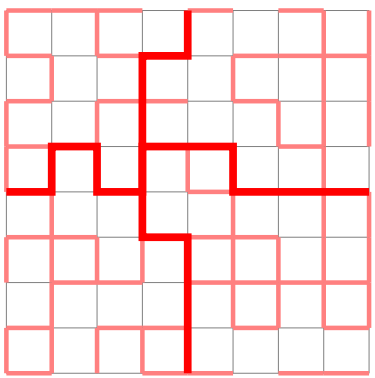

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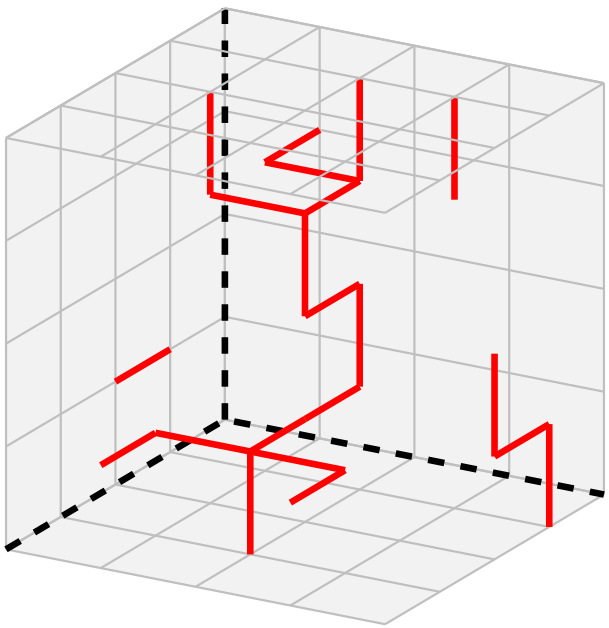
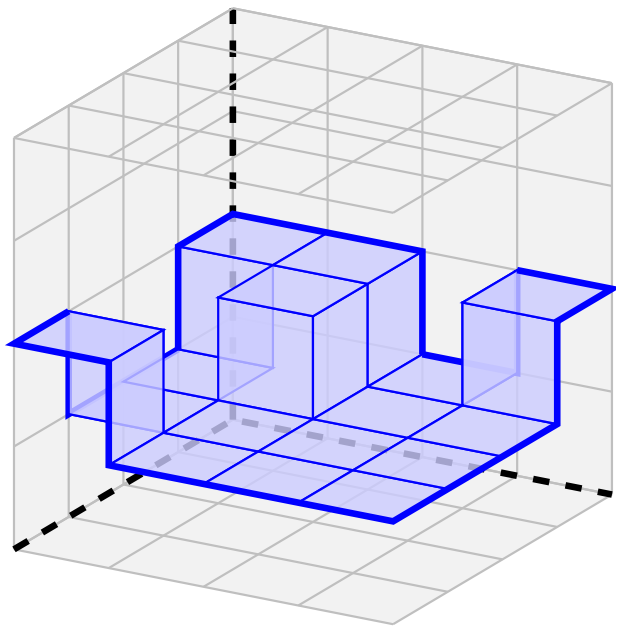
Duality of "crossings" in a box (discrete)

$$[0,N]^3 = \underset{\substack{\uparrow \\ \text{dim 1}}}{A} \uplus \underset{\substack{\uparrow \\ \text{dim 2}}}{B}$$



Top $\overset{A}{\longleftrightarrow}$ Bottom

or



separating surface in B

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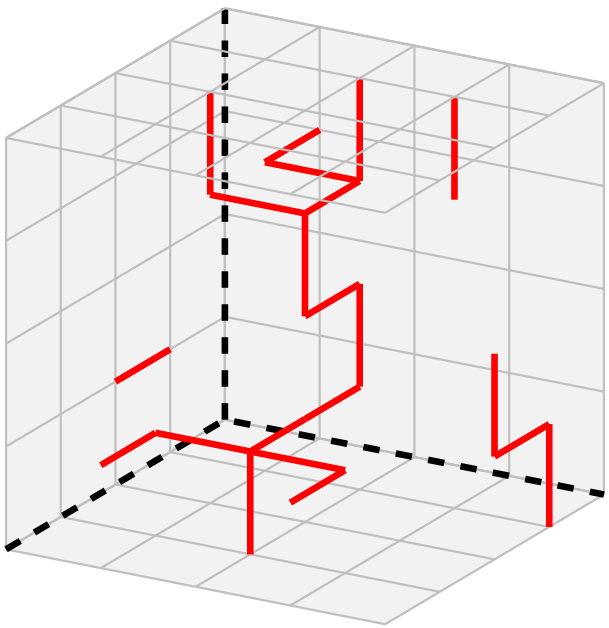

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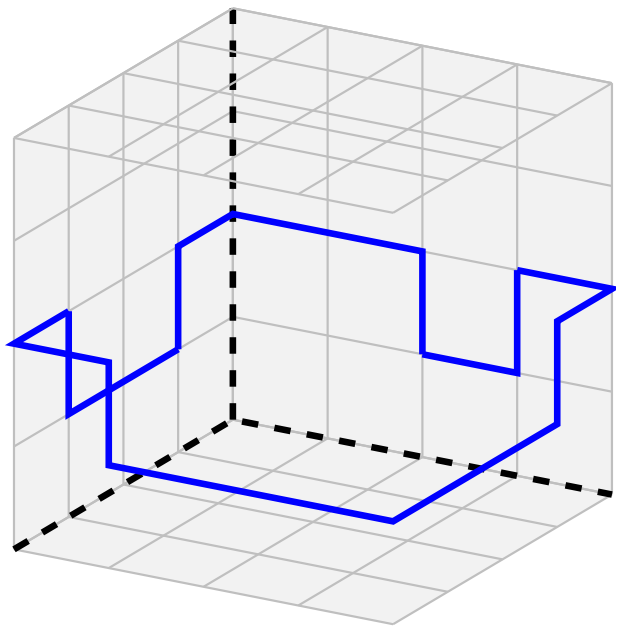

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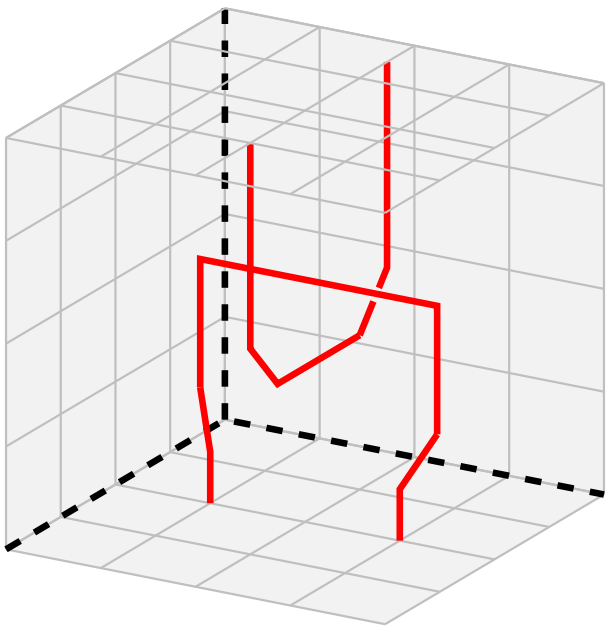

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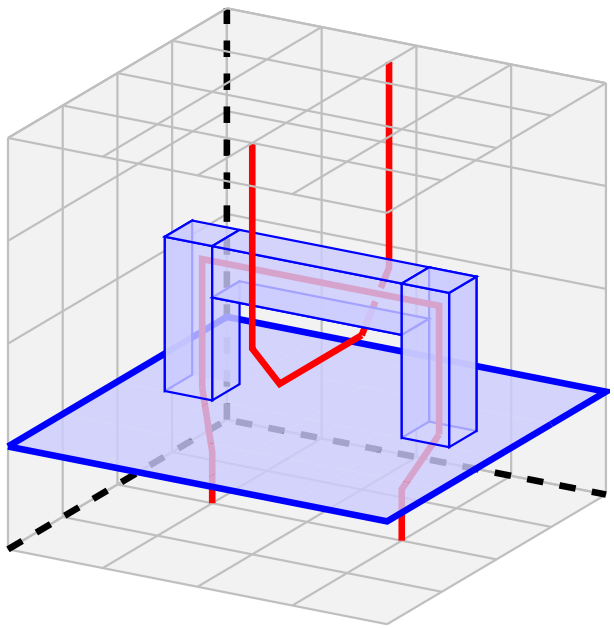

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The End