INTRODUCTION TO GEOMETRIC MEASURE THEORY

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Contents

1. Motivation

Typical problem in Geometric measure Theory (GMT): "Given a k-dimensional submanifold $M \subset \mathbb{R}^n$ with boundary, does there exist a k-dimensional submanifold N of minimal volume with $\partial N = \partial M$?" In other words: does there exist a k-dimensional minimal surface with prescribed $(k-1)$-dimensional boundary?

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Some history:

(1) Already formulated by Lagrange around 1760.
(2) Solution for 2-dimensional surface "of disc type" by Douglas and independently by Radó around 1930. Here, surface of disc type means mappings from $\overline{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ to $\mathbb{R}^n$. Thus, look for $u\colon \overline{D} \to \mathbb{R}^n$ continuous on $\overline{D}$, piecewise C^1 on D such that $u|_{S^1}$ parametrizes Γ and u has minimal area among all such maps.
(3) Federer-Fleming around 1960. Solution for surfaces of general genus and for higher dimensional surfaces in $\mathbb{R}^n$.

Direct method in the calculus of variations:

a) Let (N_m) be "volume minimizing" sequence with $\partial N_m = \partial M$ for all m.
b) Try to extract "convergent subsequence", $N_{m_l} \to N$ and show that $\partial N = \partial M$ and

$$\mathrm{Vol}(N) \leq \liminf_{l\to\infty} \mathrm{Vol}(N_{m_l}).$$

c) Show that N is smooth submanifold up to a small singularity set.

Principal idea of de Rham and Federer-Fleming: Every compact oriented k-dimensional submanifold $N \subset \mathbb{R}^n$ gives rise to a linear functional

$$[N]\colon \mathcal{D}^k(\mathbb{R}^n) \to \mathbb{R}$$

by

$$[N](\omega) := \int_N \omega,$$

where $\mathcal{D}^k(\mathbb{R}^n)$ is the space of compactly supported k-forms in $\mathbb{R}^n$. Convergence of functionals $T_m\colon \mathcal{D}^k(\mathbb{R}^n) \to \mathbb{R}$, $T_m(\omega) \to T(\omega)$ for all $\omega \in \mathcal{D}^k(\mathbb{R}^n)$.

Aim of course: Develop Federer-Fleming theory of currents and solve the generalized Plateau's problem.

2. Measure theoretic background

2.1. **Outer measure.** Let X be a set and

$$2^X := \{A : A \subset X\}.$$

Definition 2.1. A function $\mu\colon 2^X \to [0,\infty]$ is called measure on X if

i) $\mu(\emptyset) = 0$
ii) $\mu(A) \leq \sum_{k=1}^\infty \mu(A_k)$ whenever $A \subset \cup_{k=1}^\infty A_k$.

In the literature, such μ is usually called an outer measure but we will simply call it a measure.

Remark 2.2. If μ is a measure on X, then $\mu(A) \leq \mu(B)$ for all $A \subset B \subset X$.

Definition 2.3. Let μ be a measure on X and $B \subset X$ a subset. The restriction of μ to B is the measure $\mu \llcorner B$ on X defined by

$$(\mu \llcorner B)(A) := \mu(B \cap A)$$

for each $A \in 2^X$.

It follows directly from the definition of measure that $\mu\llcorner B$ is a measure.

Definition 2.4. A subset $A \subset X$ is μ-measurable if for every set $E \subset X$

$$\mu(E) = \mu(E \cap A) + \mu(E \backslash A).$$

We will simply say "measurable" if there is no danger of ambiguity. Note that the inequality "$\leq$" always holds.

Example 2.5. i) If $\mu(A) = 0$, then A is measurable.
ii) $A \subset X$ is measurable if and only if $A^c = X \backslash A$ is measurable.
iii) Let $A, B \subset X$ be measurable sets. If A us μ-measurable, then A is $\mu\llcorner B$-measurable.

Important properties of measurable sets.

Theorem 2.6. Let μ be a measure on X and $\{A_k\}_{k=1}$ a countable collection of measurable sets. Then
i) $\bigcup_{k=1}^{\infty} A_k$ and $\bigcap_{k=1}^{\infty} A_k$ are measurable.
ii) If the sets $\{A_k\}$ are pairwise disjoint, then

$$\mu(\bigcup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} \mu(A_k).$$

iii) If $A_1 \subset A_2 \subset \cdots$, then

$$\mu(\bigcup_{k=1}^{\infty} A_k) = \lim_{k\to\infty} \mu(A_k).$$

iv) If $A_1 \supset A_2 \supset \cdots$ and $\mu(A_1) < \infty$, then

$$\mu(\bigcap_{k=1}^{\infty} A_k) = \lim_{k\to\infty} \mu(A_k).$$

As a special case we obtain: Let $A, B \subset X$ be measurable. Then
i) $A \cup B$ and $A \cap B$ are measurable.
ii) If $A \cap B = \emptyset$, then

$$\mu(A \cup B) = \mu(A) + \mu(B).$$

Sketch of the proof. 1) It is straightforward to check that the previous comments hold and thus i) and ii) holds by induction for finite collections.

2) It thus follows, if $\{A_k\}$ are pairwise disjoint, then

$$\mu(\bigcup_{k=1}^{\infty} A_k) \geq \mu(\bigcup_{k=1}^{n} A_k) = \sum_{k=1}^{n} \mu(A_k)$$

for any finite n and hence

$$\mu(\bigcup_{k=1}^{\infty} A_k) \geq \sum_{k=1}^{n} \mu(A_k)$$

and the equality follows from the σ-subadditivity of the measure.

3) Now iii) and iv) follow easily.

4). Measurability of $A := \bigcup_{k=1}^\infty A_k$. Write $B_n := \bigcup_{k=1}^n A_k$, then $B_1 \subset B_2 \subset \cdots$ and $A = \bigcup_{n=1}^\infty B_n$.

Let $E \subset X$. Without loss of generality, we assume $\mu(E) < \infty$. Then A_k and B_n are μ-measurable and hence $(\mu\llcorner E)$-measurable. It follows

$$\begin{aligned}\mu(E\cap A)+\mu(E\backslash A) &= (\mu\llcorner E)(A)+(\mu\llcorner E)(A^c)\\ &= (\mu\llcorner E)(\bigcup_{n=1}^\infty B_n)+(\mu\llcorner E)(\bigcap B_n^c)\\ &= \lim_{n\to\infty}[(\mu\llcorner E)(B_n)+(\mu\llcorner E)(B_n^c)]\\ &= (\mu\llcorner E)(X)=\mu(E),\end{aligned}$$

which implies that A is measurable.

5) Easy to see that $\bigcap_{k=1}^\infty A_k$ is measurable. □

Definition 2.7. A collection of subsets of $\mathcal{A} \subset 2^X$ is called σ-algebra if

i) $\emptyset, X \in \mathcal{A}$
ii) $A \in \mathcal{A}$ implies $A^c \in \mathcal{A}$
iii) $A_k \in \mathcal{A}$, $k = 1, 2, \ldots$ implies $\bigcup_{k=1}^\infty A_k \in \mathcal{A}$

Theorem 2.6 implies that

Corollary 2.8. If μ is a measure on X, then the collection $\mathcal{A}$ of μ-measurable subsets of X forms a σ-algebra.

Definition 2.9. A measure μ on X is called regular if for all $A \subset X$, there exists a μ-measurable set B with $A \subset B$ and $\mu(B) = \mu(A)$.

Exercise: If μ is a regular measure, then *iii*) of Theorem 2.6 holds even when the sets A_k are not measurable.

2.2. **Borel measures.** Let (X, d) be a metric space.

Definition 2.10. A measure μ on X is called Borel measure on X if every open set in X is μ-measurable.

The smallest σ-algebra containing all open sets in X is called Borel σ-algebra of X and is denoted by $\mathcal{B}(X)$. The elements in $\mathcal{B}(X)$ are called Borel sets.

Remark 2.11. If μ is a Borel measure on X, then every Borel set in X is μ-measurable.

Theorem 2.12. (Carathéodory's criterion) Let μ be a measure on X. Then μ is a Borel measure if and only if

$$\mu(A \cup B) = \mu(A) + \mu(B) \tag{2.1}$$

for all $A, B \subset X$ with $d(A, B) > 0$.

Here, the distance between two sets A, B is defined by

$$d(A, B) := \inf\{d(a, b) : a \in A, b \in B\}.$$

Remark 2.13. In specific examples, (2.1) is usually easy to verify, see below (Hausdorff measure).

Proof. $\Rightarrow$: Suppose $A, B \subset X$ satisfy $d(A, B) > 0$. Then there exists $U \subset X$ open nbhd of A such that $U \cap B = \emptyset$. Since U is measurable,

$$\begin{aligned}\mu(A \cup B) &= \mu((A \cup B) \cap U) + \mu((A \cup B)\backslash U) \\ &= \mu(A) + \mu(B).\end{aligned}$$

$\Leftarrow$: It suffices to show that closed sets are measurable. Let $A \subset X$ be closed and $E \subset X$ with $\mu(E) < \infty$. We want to show that

$$\mu(E) \geq \mu(E \cap A) + \mu(E\backslash A).$$

For $k \in \mathbb{N}$, define

$$A_k := \{x \in X : d(x, A) \leq 1/k\}.$$

Then $d(E \cap A, E\backslash A_k) > 0$ and so

$$\begin{aligned}\mu(E) &\geq \mu\big((E \cap A) \cup (E\backslash A_k)\big) \\ &\geq \mu(E \cap A) + \mu(E\backslash A_k).\end{aligned}$$

It remains to show that $\mu(E\backslash A_k) \to \mu(E\backslash A)$ as $k \to \infty$.

For this, set $R_m := A_m\backslash A_{m+1}$ and notice that

$$E\backslash A = (E\backslash A_k) \cup \bigcup_{m=k}^{\infty} (E \cap R_m)$$

because A is closed. Then

$$\mu(E\backslash A) \leq \mu(E\backslash A_k) + \sum_{m=k}^{\infty} \mu(E \cap R_m).$$

In order to prove our claim, it is enough to show that

$$\sum_{m=1}^{\infty} \mu(E \cap R_m) < \infty.$$

Since $d(R_m, R_k) > 0$ whenever $|m - k| \geq 2$ it follows that for all $N \in \mathbb{N}$,

$$\sum_{m=1}^{N} \mu(E \cap R_{2m-1}) = \mu(E \cap \bigcup_{m=1}^{N} R_{2m-1}) \leq \mu(E)$$

and similarly $\sum_{m=1}^{N} \mu(E \cap R_{2m}) \leq \mu(E)$ from which we deduce

$$\sum_{m=1}^{2N} \mu(E \cap R_m) \leq 2\mu(E) \quad \forall N \in \mathbb{N}$$

and so $\sum_{m=1}^{\infty} \mu(E \cap R_m) \leq 2\mu(E) < \infty$. □

Theorem 2.14. Let μ be a Borel measure on X and $B \subset X$ a Borel set.

i) If $\mu(B) < \infty$, then

$$\mu(B) = \sup\{\mu(C) : C \subset B \text{ and } C \text{ is closed}\}.$$

ii) If $\exists U_i \subset X$ open, $i = 1, 2, \dots$, such that $\mu(U_i) < \infty$ and $B \subset \bigcup_{i=1}^{\infty}(U_i)$, then

$$\mu(B) = \inf\{\mu(V) : B \subset V \text{ and } V \text{ is open}\}.$$

A special case is if $\mu(X) < \infty$.

A sketch of proof. 1) ii) follows from i):

For given $\varepsilon > 0$, by i), choose $C_i \subset U_i \backslash B$ closed and such that

$$\mu\big((U_i \backslash B) \backslash C_i\big) \leq \frac{\varepsilon}{2^i} \quad \forall i \in \mathbb{N}.$$

Then $U := \bigcup_{i=1}^{\infty} U_i \backslash C_i$ is open, contains B and satisfies

$$\mu(U \backslash B) \leq \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} = \varepsilon.$$

Therefore,

$$\mu(U) = \mu(U \backslash B) + \mu(B) \leq \mu(B) + \varepsilon.$$

2) By considering $\mu \llcorner B$ instead of μ, we may assume that $\mu(X) < \infty$. We want to show that for every Borel set $A \subset X$

$$\mu(A) = \sup\{\mu(C) : C \subset A \text{ and } C \text{ is closed}\}. \tag{2.2}$$

Define

$$\mathcal{A} := \{A \subset X : A \text{ satisfies } (2.2)\}.$$

Then A contains all closed sets, in particular, $\emptyset$ and X. Verify that if $A_k \in \mathcal{A}$, $k \in \mathbb{N}$, then

$$\bigcup_{k=1}^{\infty} A_k \in \mathcal{A} \quad \text{and} \quad \bigcap_{k=1}^{\infty} A_k \in \mathcal{A}.$$

It follows that

$$\mathcal{A}' := \{A \in \mathcal{A} : A^c \in \mathcal{A}\}$$

is a σ-algebra. Show that $\mathcal{A}$ contains all open sets, hence $\mathcal{A}'$ contains all closed sets, so $\mathcal{B}(X) \subset \mathcal{A}'$. □

Definition 2.15. A measure μ on X is called Borel regular if μ is a Borel measure and for all $A \subset X$, there exists $B \in \mathcal{B}(X)$ such that $A \subset B$ and $\mu(B) = \mu(A)$.

Lemma 2.16. If μ is a Borel regular measure on X and $A \subset X$ is μ-measurable with $\mu(A) < \infty$, then $\mu \llcorner A$ is Borel regular.

Proof. Let $U \subset X$ be open and $E \subset X$ an arbitrary set. Since U is μ-measurable,

$$\begin{aligned}\mu \llcorner A(E) = \mu(A \cap E) &= \mu(A \cap E \cap U) + \mu(A \cap E \backslash U) \\ &= \mu \llcorner A(E \cap U) + \mu \llcorner A(E \backslash U),\end{aligned}$$

which implies that U is $\mu\llcorner A$ measurable. Consequently, $\mu\llcorner A$ is Borel. Next, let $E \subset X$ be arbitrary. Since μ is Borel regular, there exists a Borel set B such that $A \subset B$ and $\mu(B) = \mu(A)$. Since $\mu(A) < \infty$ and since A and B are μ-measurable, it follows that $\mu(B \setminus A) = 0$. There thus exists a Borel set C such that $B \setminus A \subset C$ and $\mu(C) = 0$. There exists furthermore a Borel set D such that $A \cap E \subset D$ and $\mu(D) = \mu(A \cap E)$.

Set $G := B^c \cup C \cup D$. Then G is a Borel set and $E \subset G$. Since $A \cap B^c = \emptyset$ it follows that

$$\begin{aligned}\mu(A \cap G) &\leq \mu(A \cap B^c) + \mu(A \cap C) + \mu(A \cap D)\\ &\leq \mu(C) + \mu(D) = \mu(A \cap E).\end{aligned}$$

This shows the Borel regularity of $\mu\llcorner A$. □

2.3. **Hausdorff measures.** Let (X, d) be a metric space and $s \in [0, \infty)$. For $A \subset X$ and $\delta > 0$, we set

$$\mathcal{H}^s_\delta(A) := \inf\Big\{\sum_{i=1}^\infty \omega_s\big(\frac{\operatorname{diam} A_i}{2}\big)^s : A \subset \bigcup_i A_i,\ \operatorname{diam} A_i < \delta\Big\},$$

where we use the conventions

- $\inf \emptyset = \infty$
- $(\operatorname{diam} \emptyset)^s = 0$ for all $s \geq 0$
- $0^0 = 1$.

The number ω_s are normalizing constants given by $\omega_s = \frac{\pi^{s/2}}{\Gamma(s/2+1)}$, where Γ is the Gamma-function:

$$\Gamma(t) = \int_0^\infty e^{-x} x^{t-1} dx.$$

Note that ω_n is the Lebesgue measure of the unit ball in $\mathbb{R}^n$.

Definition 2.17. The Hausdorff s-measure of $A \subset X$ is

$$\mathcal{H}^s(A) := \lim_{\delta \to 0} \mathcal{H}^s_\delta(A).$$

Note that $0 < \delta < \delta' \Longrightarrow \mathcal{H}^s_\delta(A) \geq \mathcal{H}^s_{\delta'}(A)$.

Remark 2.18. In Definition 2.17, we may equivalently take the sets A_i to be closed or open.

Simple properties:

i) $\mathcal{H}^0$ is the counting measure on X
ii) If $X = \mathbb{R}^n$, then
  - $\mathcal{H}^s(x + A) = \mathcal{H}^s(A)\ \forall x \in \mathbb{R}^n, \forall A \subset \mathbb{R}^n$
  - $\mathcal{H}^s(\lambda A) = \lambda^s \mathcal{H}^s(A)\ \forall \lambda > 0, \forall A \subset \mathbb{R}^n$
  - For every bounded open set $U \subset \mathbb{R}^n$, $0 < \mathcal{H}^n(U) < \infty$.

By Haar's theorem, there exists a unique (up to a multiplicative constant) translation invariant Borel measure on $\mathbb{R}^n$. The normalizing constants ω_s are chosen in such a way that $\mathcal{H}^n = \mathcal{L}^n$ on $\mathbb{R}^n$.

We shall prove later that for an open set U of an m-dim submanifold $M \subset \mathbb{R}^n$,

$$\mathcal{H}^m(U) = \mathrm{Vol}_M(U).$$

Theorem 2.19. Let X be a metric space. Then

i) $\mathcal{H}^s$ is a Borel regular measure on X for all $s \in [0, \infty)$
ii) For all $0 \leq s < t < \infty$ and $A \subset X$ we have

$$\mathcal{H}^s(A) < \infty \Longrightarrow \mathcal{H}^t(A) = 0$$
$$\mathcal{H}^t(A) > 0 \Longrightarrow \mathcal{H}^s(A) = \infty.$$

Proof. i) Straightforward to check $\mathcal{H}^s$ is a measure.

Borel measure: $A, B \subset X$ with $d(A, B) > 0$. Then $\forall 0 < \delta < d(A, B)$,

$$\mathcal{H}^s_\delta(A \cup B) = \mathcal{H}^s_\delta(A) + \mathcal{H}^s_\delta(B)$$

and so

$$\mathcal{H}^s(A \cup B) = \mathcal{H}^s(A) + \mathcal{H}^s(B).$$

By Theorem 2.12, $\mathcal{H}^s$ is Borel.

Borel regular: $A \subset X$ w.l.o.g. $\mathcal{H}^s(A) < \infty$. Then $\forall m \in \mathbb{N}, \exists A^m_k \subset X$ closed, $k \in \mathbb{N}$, such that $\mathrm{diam}\, A^m_k < \frac{1}{m}$, $A \subset \bigcup_k A^m_k$ and

$$\sum_{k=1}^{\infty} \omega_s \big(\frac{\mathrm{diam}\, A^m_k}{2}\big)^s \leq \mathcal{H}^s_{1/m}(A) + 1/m.$$

Then $B := \bigcap_m \bigcup_k A^m_k$ is Borel and satisfies $A \subset B$ and for each $m > 0$

$$\mathcal{H}^s_{1/m}(B) \leq \mathcal{H}^s_{1/m}(A) + 1/m \leq \mathcal{H}^s(A) + 1/m.$$

Thus $\mathcal{H}^s(B) = \mathcal{H}^s(A)$.

ii) This follows because for $0 \leq s < t$ and $\mathrm{diam}\, A_k < \delta$

$$\big(\frac{\mathrm{diam}\, A_k}{2}\big)^t \leq (\delta/2)^{t-s} \cdot \big(\frac{\mathrm{diam}\, A_k}{2}\big)^s.$$

Left as exercise. □

Definition 2.20. The Hausdorff dimension of a set $\emptyset \neq A \subset X$ is defined by

$$\dim_{\mathcal{H}}(A) := \inf\{s \geq 0 : \mathcal{H}^s(A) = 0\} = \sup\{s \geq 0 : \mathcal{H}^s(A) = \infty\}.$$

Examples 2.21. 1) As mentioned above, $\mathcal{H}^n = \mathcal{L}^n$ and $X = \mathbb{R}^n$.

Claim: $\dim_{\mathcal{H}}(\mathbb{R}^n) = n$.

$\mathcal{H}^n(\mathbb{R}^n) = \mathcal{L}^n(\mathbb{R}^n) = \infty \Longrightarrow \dim_{\mathcal{H}}(\mathbb{R}^n) \geq n$. On the other hand, $\mathbb{R}^n = \bigcup_k B(0, k)$ and $\mathcal{H}^n(B(0, k)) < \infty$ which implies for $s > n$, $\mathcal{H}^s(B(0, k)) = 0, \forall k \in \mathbb{N}$ and thus $\mathcal{H}^s(\mathbb{R}^n) = 0, \forall s > n$. In particular, $\dim_{\mathcal{H}}(\mathbb{R}^n) \leq n$.

2) Similarly, if $M \subset \mathbb{R}^n$ is m-dim submanifold, then $\dim_{\mathcal{H}}(M) = m$.

3) Standard Cantor set $C := \bigcap_k C_k \subset [0, 1]$, where $C_0 = [0, 1]$ and

$$C_k = \frac{1}{3}\big(C_{k-1} \cup (2 + C_{k-1})\big) \quad k \geq 1.$$

Then $C = \frac{1}{3}\big(C \cup (2 + C)\big)$ and so

$$\mathcal{H}^s(C) = 3^{-s}\mathcal{H}^s(C \cup (2 + C)) = 2 \cdot 3^{-s}\mathcal{H}^s(C).$$

If $0 < \mathcal{H}^s(C) < \infty$ for some $s > 0$, then $s = \frac{\log 2}{\log 3}$. One can show that for $s = \frac{\log 2}{\log 3}$, $\mathcal{H}^s(C) = \frac{\omega_s}{2^s}$.

Remark 2.22. One can show that for any metric space X,

$$\dim_{\mathcal{H}}(X) \geq \dim_{top}(X).$$

2.4. **Covering theorems.** Open and closed balls in a metric space (X, d) will be denoted by

$$B(x, r) := \{y \in X : d(y, x) < r\} \quad \text{and} \quad \bar{B}(x, r) := \{y \in X : d(y, x) \leq r\}.$$

When we say a ball B in X, we understand that a center x and a radius $r > 0$ were chosen (note that center and radii are not unique in general). For a ball $B = B(x, r)$ we write $\mathrm{rad}(B) := r$. Given $\lambda > 0$ we write λB for $B(x, \lambda r)$.

Theorem 2.23. ($5r$-covering) Let X be a metric space and $\mathcal{B}$ a family of (open or closed) balls such that

$$\sup\{\mathrm{rad}(B) : B \in \mathcal{B}\} < \infty.$$

Then there exists a disjoint subfamily $\mathcal{B}' \subset \mathcal{B}$ such that

$$\bigcup_{B \in \mathcal{B}} B \subset \bigcup_{B \in \mathcal{B}'} 5B.$$

Remark 2.24. In general, $\mathcal{B}'$ can be uncountable. However, if for example X is proper (i.e. bounded closed sets in X are compact), then $\mathcal{B}'$ is a countable family.

Proof. Set $R := \sup\{\mathrm{rad}(B) : B \in \mathcal{B}\}$ and define for $k = 0, 1, 2, \ldots$

$$\mathcal{B}_k := \{B \in \mathcal{B} : \frac{R}{2^{k+1}} < \mathrm{rad}(B) \leq \frac{R}{2^k}\}.$$

We choose inductively subfamilies $\mathcal{B}'_k \subset \mathcal{B}_k$ as follows:

$k = 0$ By Zorn's lemma, there exists maximal disjoint subfamily $\mathcal{B}'_0 \subset \mathcal{B}_0$. Here maximal means cannot add disjoint balls from $\mathcal{B}_0$.

$k \rightsquigarrow k + 1$ Suppose $\mathcal{B}'_0, \ldots, \mathcal{B}'_k$ have been constructed for some $k \geq 0$. Zorn's lemma implies there exists maximal disjoint subfamily $\mathcal{B}'_{k+1}$ of

$$\Big\{B \in \mathcal{B}_{k+1} : B \cap B' = \emptyset \quad \forall B' \in \bigcup_{j=0}^{k} \mathcal{B}'_j\Big\}.$$

Set $\mathcal{B}' := \bigcup_k \mathcal{B}'_k$. Claim: For every $B \in \mathcal{B}_k$, there exists $B' \in \bigcup_{j=0}^k \mathcal{B}'_j$ such that $B \cap B' \neq \emptyset$. Since otherwise, $\mathcal{B}'_k$ is not maximal.

For such B, B',

$$\mathrm{rad}(B') > \frac{R}{2^{k+1}} \geq \frac{1}{2}\mathrm{rad}(B)$$

and so $B \subset 5B'$.

□

Exercise: Give a constructive proof of a maximal disjoint subfamily in the case the metric space is proper.

Definition 2.25. A family $\mathcal{G}$ of subsets of X is said to be a fine covering of a subset A if $\forall x \in A$, $\forall \varepsilon > 0$, $\exists C \in \mathcal{G}$ such that $x \in C$ and $0 < \operatorname{diam} C < \varepsilon$.

Theorem 2.26 (Vitali covering theorem)**.** Let X be a metric space, $A \subset X$ and $s \geq 0$. Let $\mathcal{G}$ be a fine covering of A by closed sets. Then there exists a (finite or) countable disjoint subfamily $\{C_k\} \subset \mathcal{G}$ such that one of the following holds:

i) $\sum_k (\operatorname{diam} C_k)^s = \infty$
ii) $\mathcal{H}^s(A \backslash \bigcup_k C_k) = 0$.

Remark 2.27. In practice, one has to exclude option i). Often this is possible. For example, one choose $\mathcal{G}$ such that every $C \in \mathcal{G}$ is contained in a fixed set U with $\mathcal{H}^s(U) < \infty$.

Proof. We choose $C_k \in \mathcal{G}$ inductively as follows:

$k = 0$ Define
- $\mathcal{G}_0 = \{C \in \mathcal{G} : \operatorname{diam} C \leq 1\}$
- $d_0 := \sup\{\operatorname{diam} C : C \in \mathcal{G}_0\}$.

$\Longrightarrow$: $\exists C_0 \in \mathcal{G}_0$ such that $\operatorname{diam} C_0 > \frac{1}{2} d_0$.

$k \rightsquigarrow k+1$ Suppose we have chosen disjoint $C_0, \ldots, C_k \in \mathcal{G}_0$ for some $k \geq 0$. Define

$$\mathcal{G}_{k+1} := \Big\{ C \in \mathcal{G}_0 : C \cap \bigcup_{j=1}^{k} C_j = \emptyset \Big\}.$$

We consider two cases:

Case 1: $\mathcal{G}_{k+1} = \emptyset$.

$A \subset \bigcup_{j=0}^{k} C_j$ since C_j closed and by definition of fine covering and thus (ii) holds trivially.

Case 2: $\mathcal{G}_{k+1} \neq \emptyset$.

$\exists C_{k+1} \in \mathcal{G}_{k+1}$ with $\operatorname{diam} C_{k+1} > \frac{1}{2} d_{k+1}$, where $d_{k+1} = \sup\{\operatorname{diam} C : C \in \mathcal{G}_{k+1}\}$.

By above, we may assume the process does not stop. We assume that

$$\sum_k (\operatorname{diam} C_k)^s < \infty. \tag{2.3}$$

We will show that for every $\delta > 0$, $\mathcal{H}^s_\delta(A \backslash \bigcup_k C_k) = 0$.

For each $k \geq 0$. Fix $x_k \in C_k$.

Claim: For all $n \geq 0$, $A \backslash \bigcup_{k=0}^{n} C_k \subset \bigcup_{m \geq n+1} \bar{B}(x_m, 3 \operatorname{diam} C_m)$.

Let $x \in A \backslash \bigcup_{k=0}^{n} C_k$. Let $C \in \mathcal{G}_0$ such that $x \in C$ and $\operatorname{diam} C > 0$ and $C \cap \bigcup_{k=0}^{n} = \emptyset$. By (2.3), we have $d_m \to 0$ as $m \to \infty$. Hence $\exists m \geq n+1$ such that $C \cap C_m \neq \emptyset$. Let m be the smallest such number. Then $\operatorname{diam} C < 2 \operatorname{diam} C_m$ and so $x \in \bar{B}(x_m, 3 \operatorname{diam} C_m)$.

Let $\delta > 0$. For all n large enough, $\operatorname{diam} C_m < \delta/6$, $\forall m \geq n$ and hence

$$\mathcal{H}^s_\delta(A \backslash \bigcup_{k=0}^{n} C_k) \leq \sum_{m=n+1}^{\infty} \omega_s \Big(\frac{6 \operatorname{diam} C_m}{2} \Big)^s \overset{n \to \infty}{\to} 0.$$

Thus $\mathcal{H}^s(A\setminus\bigcup_{k=0}^{\infty} C_k) = 0$.

□

Covering theorems are very useful. For example, Vitali's covering thm can be used to prove

Theorem 2.28 (Lebesgue differentation theorem)**.** Let $f \in L^1(\mathbb{R}^n)$. Then

$$\lim_{r\to 0} \frac{1}{\mathcal{L}^n(B(x,r))} \int_{B(x,r)} |f(y) - f(x)|dy = 0$$

for $\mathcal{L}^n$-a.e. $x \in \mathbb{R}^n$. In particular,

$$\lim_{r\to 0} \frac{1}{\mathcal{L}^n(B(x,r))} \int_{B(x,r)} f(y)dy = f(x)$$

for $\mathcal{L}^n$-a.e. $x \in \mathbb{R}^n$.

Exercise: Try to prove Theorem 2.28.

Proof. We first show that if g is a non-negative, integrable and constant outside a compact set K, then the function

$$\bar{g}(x) := \limsup_{r\to 0} \frac{1}{\mathcal{L}^n(B(x,r))} \int_{B(x,r)} g(y)dy$$

satisfies $\bar{g}(x) \leq g(x)$ for a.e. x. To this end, for each rational q, set $B_q = \{x : g(x) < q < \bar{g}(x)\}$ and select a bounded open set $A \supset B_q$. Consider the family

$$\mathcal{F} = \Big\{\bar{B}(x,r) : x \in B_q, \bar{B}(x,r) \subset A \text{ and } \frac{1}{\mathcal{L}^n(B(x,r))} \int_{B(x,r)} g(y)dy > q\Big\}.$$

Then $\mathcal{F}$ is a fine covering of B_q. By the Vitali Covering Theorem 2.26, we know that there is a countable subfamily $\{\bar{B}(x_i,r_i)\}$ that covers B_q up to a set of measure zero. If $\mathcal{L}^n(B_q) > 0$, then

$$\begin{aligned}\int_A g(y)dy &\geq \sum_i \int_{B(x_i,r_i)} g(y)dy \geq \sum_i q\mathcal{L}^n(B(x_i,r_i)) \\ &= q\mathcal{L}^n(B_q).\end{aligned}$$

Since A is arbitrary, $\int_{B_q} g(y)dy \geq q\mathcal{L}^n(B_q)$. However, this is a contradiction, since $g < q$ on B_q. Thus $\bar{g} \leq g$ a.e.

Now for any rational q, we consider $g_q(y) := |f(y) - q|$ and apply our previous conclusion to infer for a.e. x

$$\begin{aligned}&\limsup_{r\to 0} \frac{1}{\mathcal{L}^n(B(x,r))} \int_{B(x,r)} |f(y) - f(x)|dy \\ &\leq \limsup_{r\to 0} \frac{1}{\mathcal{L}^n(B(x,r))} \int_{B(x,r)} g_q(y) + |q - f(x)|dy \\ &\leq g_q(x) + |q - f(x)| = 2|q - f(x)|.\end{aligned}$$

The last inequality holds for a.e. x and so holds for a.e. x and all q. The claim follows since q can be made arbitrarily close to $f(x)$. □

Another application of Vitali's covering theorem.

Theorem 2.29. Let $f\colon (a,b)\to\mathbb{R}$ be an increasing function. Then f is differentiable $\mathcal{L}^1$-a.e. on (a,b).

Exercise: Try to prove Theorem 2.29.

Proof. See http://people.math.sc.edu/schep/diffmonotone.pdf or http://www.math.uiuc.edu/~mjunge/54004-diffmon.pdf for the complete proof. We give a sketch here. Without loss of generality, we may assume that f is increasing. We need to show that

$$\begin{aligned} A &= \Big\{x\in(a,b): f \text{ not differentiable at } x\Big\}\\ &= \Big\{x\in(a,b): \liminf_{h\to 0}\frac{f(x+h)-f(x)}{h} < \limsup_{h\to 0}\frac{f(x+h)-f(x)}{h}\Big\}\end{aligned}$$

has measure zero (we can write $\liminf_{h\to 0}$ as minimum of $\liminf_{h\to 0^-}$ and $\liminf_{h\to 0^+}$ and similarly for $\limsup_{h\to 0}$). We only show that

$$A' := \Big\{x\in(a,b): F_-(x) < F_+(x)\Big\}$$

has measure zero, where $F_-(x)=\liminf_{h\to 0^+}\frac{f(x+h)-f(x)}{h}$ and $F_+(x)=\limsup_{h\to 0^+}\frac{f(x+h)-f(x)}{h}$. We can write

$$A' = \bigcup_{s<t,s,t\in\mathbb{Q}} A_{s,t} = \bigcup_{s<t,s,t\in\mathbb{Q}}\Big\{x\in(a,b): F_-(x) < s < t < F_+(x)\Big\}$$

and so it is enough to show that $\mathcal{L}^1(A_{s,t})=0$ for fixed rationals $s<t$.

Intuitively, $A''=A_{s,t}$ is the set on which f grows slower than s on certain scales and strictly faster than t on other scales. In order to apply the Vitali covering theorem, we set

$$\mathcal{I}_- = \Big\{[x,x+h]: x\in A'', h>0, \frac{f(x+h)-f(x)}{h} < s\Big\}.$$

If $\mathcal{L}^1(A'')>0$, then we obtain by Vitali covering that for each $\varepsilon>0$, there exists $[x_k,x_k+h_k]\in\mathcal{I}_-$ disjoint such that

- $\sum_{k=1}^n h_k < (1+\varepsilon)\mathcal{L}^1(A'')$
- $\mathcal{L}^1\big(A''\cap\bigcup_{k=1}^n[x_k,x_k+h_k]\big) > \mathcal{L}^1(A'')-\varepsilon$
- $\sum_{k=1}^n\big(f(x_k+h_k)-f(x_k)\big) < s\sum_{k=1}^n h_k < s(1+\varepsilon)\mathcal{L}^1(A'')$

We apply the theorem again to

$$A''' := A''\cap\bigcup_{k=1}^n[x_k,x_k+h_k]$$

$$\mathcal{I}_+ := \Big\{[y,y+r]: y\in A''', r>0, \frac{f(x+r)-f(x)}{r} > t, [y,y+r] \text{ contained in some } [x_k,x_k+h_k]\Big\}$$

For $\varepsilon>0$ sufficiently small and $\mathcal{L}^1(A''')>0$, Vitali covering implies $\exists[y_l,y_l+r_l]\in\mathcal{I}_+$ disjoint such that

- $\mathcal{L}^1\big(A'''\cap\bigcup_{l=1}^m[y_l,y_l+r_l]\big) > \mathcal{L}^1(A''')-\varepsilon$
- $\sum_{l=1}^m\big(f(y_l+r_l)-f(y_l)\big) > t\sum_{l=1}^m r_l$

Consequently,

$$\begin{aligned}\sum_{l=1}^{m} r_l &\geq \mathcal{L}^1\big(A''' \cap \bigcup_{l=1}^{m}[y_l, y_l + r_l]\big)\\ &> \mathcal{L}^1(A''') - \varepsilon > \mathcal{L}^1(A'') - 2\varepsilon.\end{aligned}$$

Since f is increasing and each $[y_l, y_l + r_l]$ is contained in some $[x_k, x_k + h_k]$, it follows that

$$\begin{aligned}t(\mathcal{L}^1(A'') - 2\varepsilon) &< \sum_{l=1}^{m} \big(f(y_l + r_l) - f(y_l)\big)\\ &\leq \sum_{k=1}^{n} \big(f(x_k + h_k) - f(x_k)\big) < s(1+\varepsilon)\mathcal{L}^1(A'').\end{aligned}$$

This is impossible if $\varepsilon > 0$ is small enough and so $\mathcal{L}^1(A'') = 0$.

□

2.5. **Densities.** Often important to compare a given measure μ to Hausdoff measures.

Definition 2.30. Let X be a metric space and μ a Borel measure on X. For $x \in X$ and $s \geq 0$, the upper and lower s-dim density of μ at x is defined by

$$\overline{\Theta}_s(\mu, x) := \limsup_{r \to 0} \frac{\mu(B(x,r))}{\omega_s r^s},$$

$$\underline{\Theta}_s(\mu, x) := \liminf_{r \to 0} \frac{\mu(B(x,r))}{\omega_s r^s}.$$

Note that the open balls $B(x,r)$ can be replaced by closed balls $\bar{B}(x,r)$ without changing value of $\overline{\Theta}_s$ and $\underline{\Theta}_s$.

Notation: Given a set $A \subset X$, we write

$$\overline{\Theta}_s(A, x) := \overline{\Theta}_s(\mathcal{H}^s \llcorner A, x)$$

and similarly for $\underline{\Theta}_s(A, x)$.

Lemma 2.31. Let μ be a Borel measure on X. Fix $s, \varepsilon > 0$. Then the function

$$f(x) := \sup_{0<r<\varepsilon} \frac{\mu(B(x,r))}{\omega_s r^s}$$

is lower semincontinuous. In particular, the function $x \mapsto \overline{\Theta}_s(\mu, x)$ is a Borel function. Analogously, $x \mapsto \underline{\Theta}_s(\mu, x)$ is Borel.

Recall that a map $f \colon X \to Y$, where Y is a topological space, is Borel if $f^{-1}(U)$ is Borel $\forall U \subset Y$ open.

Proof. For each $\lambda > 0$, we need to show that the level set

$$F_\lambda := \{x \in X : f(x) > \lambda\}$$

is open. Fix $x \in F_\lambda$, we shall show that $B(x,\delta) \subset F_\lambda$ for some $\delta > 0$. Since $x \in F_\lambda$, there exists a $r_x \in (0,\varepsilon)$ such that

$$\frac{\mu(B(x,r_x))}{\omega_s r_x^s} > \lambda.$$

Since

$$\lim_{k\to\infty} \mu(B(x, r_x - \frac{1}{k})) = \mu(B(x,r_x)),$$

we know that for some k_0, all $k \geq k_0$,

$$\frac{\mu(B(x,r_x-\frac{1}{k}))}{\omega_s r_x^s} > \lambda.$$

Note that for each $y \in B(x,\frac{1}{k})$, $B(y,r_x) \supset B(x, r_x - \frac{1}{k})$, we thus get

$$\frac{\mu(B(y,r_x))}{\omega_s r_x^s} > \frac{\mu(B(x,r_x-\frac{1}{k}))}{\omega_s r_x^s} > \lambda,$$

which implies $y \in F_\lambda$ as desired. □

Theorem 2.32. Let X be a metric space, μ a Borel measure on X, and $B \subset X$ a Borel set such that $\exists V_i \subset X$ open, $i \in \mathbb{N}$, with $\mu(V_i) < \infty$ and $B \subset \bigcup_{i=1}^\infty V_i$. Let $\lambda, s > 0$.

i) If $\overline{\Theta}_s(\mu,x) \geq \lambda$ for all $x \in B$, then

$$\mu(B) \geq \lambda \mathcal{H}^s(B).$$

ii) If $\overline{\Theta}_s(\mu,x) \leq \lambda$ for all $x \in B$, then

$$\mu(B) \leq 2^s \lambda \mathcal{H}^s(B).$$

Proof. i). Without loss of generality we assume $\mu(B) < \infty$.

Let $0 < \delta < 1$. Then $\exists U \subset X$ open such that $B \subset U$ and $\mu(U) < \mu(B) + \delta$. Define a fine covering of B by

$$\mathcal{G} := \Big\{\bar{B}(x,r) : x \in B, r < \delta/2, \bar{B}(x,r) \subset U, \mu(B(x,r)) \geq (1-\delta)\lambda\omega_s r^s\Big\}.$$

This is a fine covering of B since $\overline{\Theta}_s(\mu,x) \geq \lambda$ for all $x \in B$. Apply Vitali's covering theorem, $\exists$ finite or countable disjoint subfamily $\{C_j\} \subset \mathcal{G}$ such that one of the following holds:

$$\sum_j (\operatorname{diam} C_j)^s = \infty$$

or

$$\mathcal{H}^s(B \backslash \bigcup C_j) = 0.$$

By definition of $\mathcal{G}$, we have

$$(\operatorname{diam} C_j)^s \leq \frac{2^s}{(1-\delta)\lambda\omega_s}\mu(C_j)$$

and hence

$$\sum_j (\operatorname{diam} C_j)^s \leq \frac{2^s}{(1-\delta)\lambda\omega_s}\mu(\bigcup C_j) < \infty.$$

Thus $\mathcal{H}^s(B\backslash\bigcup C_j)=0$. It follows

$$\begin{aligned}\mathcal{H}^s_\delta(B)&\le \mathcal{H}^s_\delta(B\backslash\bigcup C_j)+\mathcal{H}^s_\delta(\bigcup C_j)\\ &\le \sum_j \omega_s\big(\frac{\operatorname{diam} C_j}{2}\big)^s \le \frac{1}{(1-\delta)\lambda}\sum_j \mu(C_j)\\ &\le \frac{1}{(1-\delta)\lambda}\mu(U)\le \frac{1}{(1-\delta)\lambda}(\mu(B)+\delta).\end{aligned}$$

Sending $\delta\to 0$ and we obtain $\lambda\mathcal{H}^s(B)\le\mu(B)$.

ii). Fix $\lambda'>\lambda$. For $k\in\mathbb{N}$ define

$$B_k:=\Big\{x\in B:\frac{\mu(\bar B(x,r))}{\omega_s r^s}\le\lambda',\forall r\in(0,1/k)\Big\}.$$

Note that B_k is Borel by lemma above and that

$$B_1\subset B_2\subset\cdots\quad\text{and}\quad B=\bigcup_k B_k.$$

Thus $\mu(B)=\lim_{k\to\infty}\mu(B_k)$. It then suffices to show that for every $k\in\mathbb{N}$,

$$\mu(B_k)\le 2^s\lambda'(\mathcal{H}^s(B)+1/k). \tag{2.4}$$

Fix $k\in\mathbb{N}$. Then $\exists A_i\subset X$, $i\in\mathbb{N}$ such that $B_k\subset\bigcup_i A_i$, $\operatorname{diam} A_i<1/k$ and

$$\sum_k \omega_s\cdot\Big(\frac{\operatorname{diam} A_i}{2}\Big)^s<\mathcal{H}^s_{1/k}(B_k)+1/k.$$

Without loss of generality, we assume $A_i\cap B_k\ne\emptyset$ for all $i\in\mathbb{N}$. Choose $x_i\in A_i\cap B_k$. Then

$$B_k\subset\bigcup_i A_i\subset\bigcup_i \bar B(x_i,\operatorname{diam} A_i)$$

and so

$$\begin{aligned}\mu(B_k)&\le\sum_i\mu(\bar B(x_i,\operatorname{diam} A_i))\le\sum_i\lambda'\omega_s(\operatorname{diam} A_i)^s\\ &<2^s\lambda'(\mathcal{H}^s(B)+1/k),\end{aligned}$$

from which (2.4) follows. □

Remark 2.33. Part ii) of Theorem 2.32 holds for arbitrary sets $B\subset X$. Indeed, define $B'_k=\{x\in X:\dots\}$ then B'_k Borel and hence $\mu\llcorner B$-measurable, which implies

$$\mu(B)=(\mu\llcorner B)(\bigcup_k B'_k)=\lim(\mu\llcorner B)(B'_k)=\lim_{k\to\infty}\mu(B\cap B'_k)=\lim_{k\to\infty}\mu(B_k).$$

Corollary 2.34. Let X be a metric space, $s>0$, and $A\subset X$ such that $\mathcal{H}^s(A)<\infty$. Then

i) For $\mathcal{H}^s$-a.e. $x\in A$, $2^{-s}\le\overline{\Theta}_s(A,x)\le 1$.
ii) If A is $\mathcal{H}^s$-measurable, then for $\mathcal{H}^s$-a.e. $x\in A^c$, $\overline{\Theta}_s(A,x)=0$.

Proof. i) a. $\overline{\Theta}_s(A,x) \le 1$: Without loss of generality, suppose A is Borel by Borel regularity of $\mathcal{H}^s$. For $\delta > 0$, apply Theorem 2.32 with $\mu = \mathcal{H}^s \llcorner A$ and

$$B = \{x \in A : \overline{\Theta}_s(\mu, x) \ge 1+\delta\}$$

to obtain

$$\infty > \mathcal{H}^s(B) = \mu(B) \ge (1+\delta)\mathcal{H}^s(B).$$

Consequently, $\mathcal{H}^s(B) = 0$.

i) b. $\overline{\Theta}_s(A,x) \ge 2^{-s}$. Apply Theorem 2.32 and Remark 2.33.

ii) $\exists A' \subset X$ Borel such that $A \subset A'$ and $\mathcal{H}^s(A') = \mathcal{H}^s(A)$. Since A', A are $\mathcal{H}^s$-measurable, $\mathcal{H}^s(A' \backslash A) = 0$. Now apply Theorem 2.32 with $\mu = \mathcal{H}^s \llcorner A$ and for $\varepsilon > 0$,

$$B := \{x \in X \backslash A' : \overline{\Theta}_s(\mu, x) \ge \varepsilon\}.$$

Details are left as exercises. □

Definition 2.35. Let μ be a Borel measure on X and $s \ge 0$ and $x \in X$. If

$$\overline{\Theta}_s(\mu, x) = \underline{\Theta}_s(\mu, x),$$

then we write $\Theta_s(\mu, x)$ for this value and call it the s-density of μ at x.

Corollary 2.36 (Lebesgue density theorem)**.** i). Let $A \subset \mathbb{R}^n$ be an arbitrary set. Then for $\mathcal{L}^n$-a.e. $x \in A$,

$$\Theta_n(A, x) = 1.$$

ii). Let $A \subset \mathbb{R}^n$ be $\mathcal{L}^n$-measurable. Then for $\mathcal{L}^n$-a.e. $x \in A^c$,

$$\Theta_n(A, x) = 0.$$

<u>Exercise:</u> Let μ be a measure on X and $A \subset A' \subset X$ such that

$$\mu(A') = \mu(A) < \infty.$$

Then for every μ-measurable set $B \subset X$ we have

$$\mu(A' \cap B) = \mu(A \cap B).$$

By definition of μ-measurability,

$$\begin{aligned}
\mu(A) &= \mu(A \cap B) + \mu(A \backslash B) \\
&\le \mu(A' \cap B) + \mu(A' \backslash B) \\
&= \mu(A') = \mu(A).
\end{aligned}$$

So we have equalitiy everywhere above. In particular, if $A \subset A' \subset X$ with $\mathcal{H}^s(A', x) = \mathcal{H}^s(A) < \infty$, then

$$\overline{\Theta}_s(A', x) = \overline{\Theta}_s(A, x) \quad \forall x \in X$$

and the same for $\underline{\Theta}_s$, Θ_s hold.

Proof of Corollary 2.36. Without loss of generality, we may assume $\mathcal{L}^n(A) < \infty$. ii) follows immediately from Corollary 2.34 ii).

i) $\mathcal{L}^n$ is Borel regular $\Longrightarrow \exists A' \subset \mathbb{R}^n$ Borel such that $A \subset A'$ and $\mathcal{L}^n(A') = \mathcal{L}^n(A)$. By ii), $\Theta_n((A')^c, x) = 0$ for $\mathcal{L}^n$-a.e. $x \in A'$. Since

$$1 = \frac{\mathcal{L}^n(A' \cap B(x,r))}{\omega_n r^n} + \frac{\mathcal{L}^n((A')^c \cap B(x,r))}{\omega_n r^n}$$

and $\frac{\mathcal{L}^n((A')^c \cap B(x,r))}{\omega_n r^n} \to 0$ for $\mathcal{L}^n$-a.e. $x \in A'$, it follows that

$$\Theta_n(A', x) = 1 \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in A'.$$

By the Exercise above,

$$\Theta_n(A, x) = \Theta_n(A', x) = 1 \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in A.$$

□

2.6. **Riesz representation theorem.** For the rest of this section, let (X, d) be a locally compact and separable metric space.

Definition 2.37. A measure μ on X is called Radon measure on X if μ is Borel regular and $\mu(K) < \infty$ for all $K \subset X$ compact.

Exercise: There exists a countable family $\{B_k\}_k$ of open balls $B_k = B(x_k, r_k) \subset X$ such that $\bar{B}(x_k, r_k)$ is compact for each k and $X = \bigcup_k B_k$.

Proposition 2.38. Let μ be a Radon measure on X. Then

i) For every set $A \subset X$,

$$\mu(A) = \inf\{\mu(U) : A \subset U, U \text{ open }\}.$$

ii) For every μ-measurable $A \subset X$,

$$\mu(A) = \sup\{\mu(C) : C \subset A, C \text{ compact }\}.$$

Proof. i) follows directly from Borel regularity, exercise above and Theorem 2.14.

ii) For $n \geq 1$, define $K_n := \bigcup_k \bar{B}(x_k, r_k)$, where $\bar{B}(x_k, r_k)$ is given as in the exercise above. Then K_n is compact and $\mu(A \cap K_n) \to \mu(A)$ as $n \to \infty$. Thus it suffices to prove ii) for $A \subset X$ μ-measurable with $\mu(A) < \infty$ and $A \subset K$ compact.

Define $\nu := \mu \llcorner A$. Then ν is Borel regular by Lemma 2.16. Since $\nu(A^c) = 0$, there exists $A^c \subset B$ Borel with $\nu(B) = 0$. By Theorem 2.14, $\forall \varepsilon > 0$, $\exists U \subset X$ open with $B \subset U$ and $\nu(U) < \varepsilon$. Set $C := U^c$. Then C is closed and $C \subset A \subset K$, which implies C is compact. Since $A \backslash C = U \cap A$, we have

$$\mu(A \backslash C) = \mu(U \cap A) = \nu(U) < \varepsilon,$$

from which we infer that

$$\mu(C) = \mu(A) - \mu(A \backslash C) > \mu(A) - \varepsilon.$$

□

Define

$$C_c(X, \mathbb{R}) := \Big\{ f \colon X \to \mathbb{R} : f \text{ is continuous with } \operatorname{spt}(f) \text{ being compact} \Big\},$$

the support of a function f is

$$\operatorname{spt}(f) := \overline{\{x \in X : f(x) \neq 0\}}.$$

Observation: Every Radon measure μ on X gives rise to a linear functional

$$L\colon C_c(X, \mathbb{R}) \to \mathbb{R}$$
$$L(f) := \int_X f d\mu,$$

which is positive

$$f \geq 0 \Longrightarrow L(f) \geq 0.$$

The following theorem gives an important converse.

Theorem 2.39 (Riesz representation theorem)**.** Let X be a locally compact and separable metric space. Let $L\colon C_c(X, \mathbb{R}) \to \mathbb{R}$ be a positive linear functional. Then there exists a Radon measure μ on X such that

$$L(f) = \int_X f d\mu \quad \forall f \in C_c(X, \mathbb{R}).$$

Thus there is a 1-to-1 correspondence between Radon measures on X and positive linear functionals on $C_0(X, \mathbb{R})$. For the proof, we need the following.

Lemma 2.40. (Partition of unity) Let X be as in Theorem 2.39.

i). Let $K \subset X$ be compact and let $U \subset X$ be open such that $K \subset U$. Then there exists $f \in C_c(X, \mathbb{R})$ such that $0 \leq f \leq 1$ and $\operatorname{spt}(f) \subset U$ and $f = 1$ on a nbhd of K.

ii). Let $K \subset X$ be compact and $U_1, \ldots, U_n \subset X$ be open with $K \subset \bigcup_{k=1}^n U_k$. Then $\exists \lambda_1, \ldots, \lambda_n \in C_c(X, \mathbb{R})$ such that $\lambda_k \geq 0$, $\operatorname{spt}(\lambda_k) \subset U_k$, $\sum_{k=1}^n \lambda_k \leq 1$ and

$$\sum_{k=1}^n \lambda_k(x) = 1 \qquad \text{for all } x \text{ in a nbhd of } K.$$

Proof. Exercise. □

Proof of Theorem 2.39. For $U \subset X$ open, we define

$$\mu(U) := \sup\{L(f) : f \in C_c(X, \mathbb{R}), 0 \leq f \leq 1, \operatorname{spt}(f) \subset U\}$$

and for $A \subset X$ arbitrary,

$$\mu(A) := \inf\{\mu(U) : A \subset U, U \text{ open}\}.$$

We claim that μ is a Radon measure on X.

(a) μ is a measure.

- $\mu(\emptyset) = 0$
- $A \subset \bigcup_{i=1}^\infty A_k$, Without loss of generality, $\mu(A_k) < \infty$ for all k.
  Let $\varepsilon > 0$ and let $U_k \subset X$ be open such that $A_k \subset U_k$ and

$$\mu(U_k) < \mu(A_k) + \varepsilon/2^k.$$

Set $U = \bigcup_{k=1}^{\infty} U_k$. Then U is open and $A \subset U$. Let $f \in C_c(X,\mathbb{R})$ be such that $0 \le f \le 1$ and $\operatorname{spt}(f) \subset U$. Since $K := \operatorname{spt}(f)$ is compact, there exists $n \in \mathbb{N}$ such that $K \subset \bigcup_{k=1}^{n} U_k$.

Let $\lambda_1, \dots, \lambda_n$ be given as in Lemma 2.40 ii) and set $f_k := \lambda_k f$ for $k = 1, \dots, n$. Then $f_k \in C_c(X,\mathbb{R})$ with $0 \le f_k \le 1$ and $\operatorname{spt}(f_k) \subset U_k$ and so $L(f_k) \le \mu(U_k)$. Since

$$\sum_{k=1}^{n} f_k = f \sum_{k=1}^{n} \lambda_k = f,$$

it follows that

$$\begin{aligned} L(f) &= \sum_{k=1}^{n} L(f_k) \\ \Longrightarrow L(f) &\le \sum_{k=1}^{n} \mu(U_k) < \varepsilon + \sum_{k=1}^{\infty} \mu(A_k). \end{aligned}$$

Since f was arbitrary,

$$\mu(A) \le \mu(U) \le \varepsilon + \sum_{k=1}^{\infty} \mu(A_k)$$

and thus μ is a measure.

(b) μ is a Borel measure.
If $U, V \subset X$ open and $U \cap V = \emptyset$, then

$$\mu(U \cap V) = \mu(U) + \mu(V).$$

If $A, B \subset X$ with $d(A,B) > 0$, then $\exists U, V \subset X$ open such that $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$. Let $W \subset X$ be open such that $A \cup B \subset W$. Then

$$\begin{aligned} \mu(W) &\ge \mu(W \cap (U \cup V)) = \mu(W \cap U) + \mu(W \cap V) \\ &\ge \mu(A) + \mu(B). \end{aligned}$$

Since W was arbitrary,

$$\mu(A \cup B) \ge \mu(A) + \mu(B),$$

which implies μ is Borel.

(C) μ is Borel regular.
This follows from the definition of $\mu(A)$.

(d) $\mu(K) < \infty$ for $K \subset X$ compact.

Let $K \subset X$ compact. Then $\exists f \in C_c(X)$ such that $0 \le f \le 1$ and $f = 1$ on an open set V with $K \subset V$. Since $\mu(K) \le \mu(U)$ it is enough to show that $\mu(U) < \infty$. Let $h \in C_c(X)$ with $\operatorname{spt}(h) \subset V$ and $0 \le h \le 1$. Then

$$\begin{aligned} h \le f &\Longrightarrow f - h \ge 0 \\ \Longrightarrow L(f) \ge L(h) &\Longrightarrow \mu(V) \le L(f) < \infty. \end{aligned}$$

(e) For every $f \in C_c(X)$, we have

$$L(f) = \int_X f d\mu.$$

It suffices to show this for $f \geq 0$. Let $\varepsilon > 0$ and $M := \sup\{f(x) : x \in X\}$. Let $V \subset X$ be open with $\operatorname{spt}(f) \subset V$ and $\mu(V) < \infty$ and let

$$0 = y_0 < y < y_1 < \cdots < y_{n-1} < M < y_n$$

be such that $y_i - y_{i-1} < \varepsilon$ and $\mu(\{f = y_i\}) = 0$ for all $i = 1, \ldots, n$. Note that $\mu(\{f = y\}) > 0$ only for countably many $y \in \mathbb{R}$.

Define

$$U_1 := \{f < y_1\} \cap V$$

and, for $i \geq 2$,

$$U_i := \{y_{i-1} < f < y_i\} \subset V.$$

Then U_i is open and pairwise disjoint, with

$$V = \bigcup_{i=1}^n \big(U_i \cap \{f = y_i\}\big)$$

and thus $\exists K_i \subset U_i$ compact such that $\mu(U_i) \leq \mu(K_i) + \frac{\varepsilon}{Mn}$. By Lemma 2.40, $\exists h_i \in C_c(X)$ such that $0 \leq h_i \leq 1$ and $\operatorname{spt}(h_i) \subset U_i$ and $h_i = 1$ on a neighborhood of K_i. Then the function $g := f \cdot \big(1 - \sum_{i=1}^n h_i\big) \in C_c(X)$ and satisfies $0 \leq g \leq M$ and

$$\operatorname{spt}(g) \subset \operatorname{spt}(f) \backslash \bigcup_{i=1}^n K_i \subset \bigcup_{i=1}^n (U_i \backslash K_i) \cup \{f = y_i\}.$$

Thus, by approximating $\{f = y_i\}$ by open sets and using definition of μ, we obtain

$$L(g) \leq M \sum_{i=1}^n \mu(U_i \backslash K_i) \leq \varepsilon$$

and thus $0 \leq L(f) - \sum_{i=1}^n L(f \cdot h_i) \leq \varepsilon$.

Now observe that

- $y_{i-1} \cdot h_i \leq f \cdot h_i \leq y_i \cdot h_i$
- $\mu(U_i) - \frac{\varepsilon}{Mn} \leq \mu(K_i) \overset{\text{as in } (a)}{\leq} L(h_i) \leq \mu(U_i)$ for all i.

Thus we calculate

$$\begin{aligned}\sum_{i=1}^n L(f \cdot h_i) \leq \sum_{i=1}^n L(y_i \cdot h_i) &\leq \sum_{i=1}^n \int_{U_i} f + \varepsilon d\mu \\ &\leq \int_X f d\mu + \varepsilon \mu(U)\end{aligned}$$

and

$$\sum_{i=1}^{n} L(f \cdot h_i) \geq \sum_{i=1}^{n} y_{i-1}(\mu(U_i) - \varepsilon/Mn) \geq \sum_{i=1}^{n} \int_{U_i} f - \varepsilon d\mu - \varepsilon$$

$$\overset{\mu(\{f=y_i\})=0}{\geq} \int_X f d\mu - \varepsilon - \varepsilon\mu(U).$$

Therefore,

$$|L(f) - \int_X f d\mu| \leq \varepsilon + \varepsilon\mu(V).$$

Since ε was arbitrary, $L(f) = \int_X f d\mu$.

(f) μ is unique (exercise).

Let μ, μ' be two Radon measures on X such that

$$\int_X f d\mu = \int_X f d\mu' \quad \forall f \in C_c(X).$$

Enough to show that for all $U \subset X$ open, we have $\mu'(U) = \mu(U)$. Let $U \subset X$ be open.

Case 1: $\mu(U) < \infty$.

Let $\varepsilon > 0$. Then $\exists K \subset U$ compact such that $\mu(U) \leq \mu(K) + \varepsilon$, which implies by Lemma 2.40 that $\exists h \in C_c(X)$ such that $0 \leq h \leq 1$ and $\mathrm{spt}(h) \subset U$ and $h = 1$ on K. Thus

$$\mu(U) \leq \mu(K) + \varepsilon \leq \int_X h d\mu + \varepsilon$$

$$= \int_X h d\mu' + \varepsilon \leq \mu'(U) + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we have $\mu(U) \leq \mu'(U)$.

Case 2: $\mu(U) = \infty$.

Almost the same argument $\implies \mu'(U) = \infty$.

Switching the roles of μ and μ' gives $\mu(U) = \mu'(U)$ for all $U \subset X$ open.

□

We now give a non-trivial generalization of the Riesz representation theorem: Let $(H, \langle \cdot, \cdot \rangle)$ be a finite-dimensional Hilbert space. Define

$$C_c(X, H) = \{f \colon X \to H : f \text{ is continuous and } \mathrm{spt}(f) \text{ is compact}\}.$$

Let μ be a Radon measure on X and $\tau \in L^\infty(X, \mu, H)$. Then μ and τ give rise to a linear functional

$$L \colon C_c(X, H) \to \mathbb{R}$$

$$L(f) := \int_X \langle f(x), \tau(x) \rangle d\mu(x)$$

and L satisfies the finiteness property

$$\sup \Big\{ L(f) : f \in C_c(X, H), |f| \leq 1, \mathrm{spt}(f) \subset K \Big\} < \infty \tag{2.5}$$

for every compact $K \subset X$.

Remark 2.41. The function $x \mapsto \langle f(x), \tau(x)\rangle$ is μ-integrable. (Exercise)

The following theorem gives a converse.

Theorem 2.42 (Riesz representation theorem II)**.** Let X be a locally compact and separable metric space and $\langle \cdot, \cdot \rangle$ a finite dimensional Hilbert space. Let $L\colon C_c(X, H) \to \mathbb{R}$ be a linear functional satisfying the finiteness property (2.5) for all $K \subset X$ compact. Then there exists a unique Radon measure μ on X and a unique μ-measurable map $\tau\colon X \to H$ such that $|\tau(x)| = 1$ for μ-a.e. $x \in X$ and

$$L(f) = \int_X \langle f(x), \tau(x)\rangle d\mu(x) \quad \forall f \in C_c(X, H).$$

Remark 2.43. If $H = \mathbb{R}$ and $L\colon C_c(X) \to \mathbb{R}$ is a positive linear functional, then L satisfies (2.5) (exercise). Thus Riesez represenation theorem I, Theorem 2.39, is a special case of Theorem 2.42 since $\tau = +1$ μ-a.e. by positivity.

We do not prove this theorem here and refer e.g. [4] for a proof. We mention that the Radon measure μ is constructed as in the proof of Theorem 2.39: for each $U \subset X$ open,

$$\mu(U) := \sup\Big\{L(f) : f \in C_c(X, H), |f| \le 1, \operatorname{spt}(f) \subset U\Big\}$$

and for $A \subset X$ arbitrary

$$\mu(A) := \inf\Big\{\mu(U) : A \subset U, U \text{ open }\Big\}.$$

One shows actually as in the previous proof that μ is a Radon measure.

2.7. **Weak compactness of Radon measures and Banach-Alaoglu theorem.** We recall some elements from functional analysis that will play a role later.

Let V be a vector space over $\mathbb{R}$. A norm $\|\cdot\|$ on V induces a metric on V by

$$d(v, w) := \|v - w\| \quad \forall v, w \in V.$$

Proposition 2.44. Let $V = (V, \|\cdot\|)$ be a normed vector space and $T\colon V \to \mathbb{R}$ linear. Then TFAE

i) T is continuous at 0.
ii) T is continuous on V.
iii) T is Lipschitz continuous.
iv) There exists $L \ge 0$ such that
$$\|T(v)\| \le L\|v\|$$
for all $v \in V$.

Proof. Exercise. □

The vector space of continuous linear functional on $V = (V, \|\cdot\|)$ is called the dual space of V and denoted by V^*. It is equipped with the operator norm

$$\|T\|_{V^*} := \sup\{T(v) : \|v\| \le 1\},$$

which defines a norm on V^*. We often write $\|T\|$ instead of $\|T\|_{V^*}$.

Proposition 2.45. The dual space $(V^*, \|\cdot\|_{V^*})$ is a Banach space, i.e., the metric on V^* induced by $\|\cdot\|_{V^*}$ is complete.

Proof. Exercise. □

Example 2.46. For a sequence $\{a_n\} \subset \mathbb{R}$ define

$$\|(a_n)\|_1 := \sum_n |a_n|$$

$$\|(a_n)\|_\infty := \sup_n |a_n|.$$

Define

$$l^1 := \{(a_n) \subset \mathbb{R} : \|(a_n)\|_1 < \infty\}$$
$$l^\infty := \{(a_n) \subset \mathbb{R} : \|(a_n)\|_\infty < \infty\}$$

Then $(l^1, \|\cdot\|_1)$ and $(l^\infty, \|\cdot\|_\infty)$ are Banach spaces and

$$(l^1, \|\cdot\|_1)^* = (l^\infty, \|\cdot\|_\infty).$$

Let $V = (V, \|\cdot\|)$ be a normed space. If $\dim(V) = \infty$, then the closed unit balls in V and in V^* are not compact. In particular, there exists a sequence $(T_n) \subset V^*$ with

$$\|T_n\| \leq 1 \quad \forall n$$

which does not have a converging subsequence in V^*.

Example 2.47. The sequence $(e_n) \subset l^\infty$ with

$$e_n = (0, \cdots, 0, 1, 0, \cdots)$$

does not have a convergent subsequence since

$$\|e_n - e_m\|_\infty = 1 \quad \forall n \neq m.$$

Definition 2.48. A sequence $(T_n) \subset V^*$ is said to converge weakly-$*$ to $T \in V^*$ if $T_n(v) \to T(v)$ for all $v \in V$.

Notation: $T_n \overset{*}{\rightharpoonup} T$.

Proposition 2.49. Let $T, T_n \in V^*$, $n \in \mathbb{N}$. Then

i) If $T_n \to T$, then $T_n \overset{*}{\rightharpoonup} T$.
ii) If $T_n \overset{*}{\rightharpoonup} T$, then $\|T\| \leq \liminf_{n\to\infty} \|T_n\|$.

Proof. Exercise. □

Example 2.50. Let (e_n) be given as in Example 2.47. Then $e_n \overset{*}{\rightharpoonup} 0$.

Theorem 2.51. (Banach-Alaoglu) Let V be a separable normed space and $(T_n) \subset V^*$ a sequence satisfying

$$\sup_n \|T_n\| < \infty.$$

Then there exists a subsequence (T_{n_k}) and $T \in V^*$ such that

$$T_{n_k} \overset{*}{\rightharpoonup} T.$$

Proof. Set

$$M := \sup_n \|T_n\| < \infty.$$

Fix $x \in V$, we have

$$|T_n(x)| \leq \|T_n\| \cdot \|x\| \leq M\|x\| \quad \forall n \in \mathbb{N}$$

and so the sequence $\{T_n(x)\} \subset \mathbb{R}$ is bounded.

Let $A := \{x_m : m \in \mathbb{N}\} \subset V$ be a countable dense subset of V. Then for each m, the sequence $(T_n(x_m))$ has a convergent subsequence and so by the diagonal sequence argument, there exist subsequence (T_{n_k}) such that $(T_{n_k}(x_m))$ converges for every $m \in \mathbb{N}$.

Define $T\colon A \to \mathbb{R}$ by

$$T(x_m) := \lim_{k\to\infty} T_{n_k}(x_m) \quad \forall m \in \mathbb{N}.$$

Then T is M-Lipschitz since

$$\begin{aligned}|T(x_m) - T(x_l)| &= \lim_{k\to\infty} |T_{n_k}(x_m) - T_{n_k}(x_l)| \\ &\leq M\|x_m - x_l\| \quad \forall m, l.\end{aligned}$$

Thus there exists a unique M-Lipschitz extension of T to all of V, which we denote by T again.

Let $x \in V$ and let (x_{m_l}) be a sequence such that $x_{m_l} \to x$. Then

$$\begin{aligned}|T(x) - T_{n_k}(x)| &\leq |T(x) - T(x_{m_l})| + |T(x_{m_l}) - T_{n_k}(x_{m_l})| + |T_{n_k}(x_{m_l}) - T_{n_k}(x)| \\ &\leq 2M\|x - x_{m_l}\| + |T(x_{m_l}) - T_{n_k}(x_{m_l})|.\end{aligned}$$

Thus $T_{n_k}(x) \to T(x)$ as $k \to \infty$, which implies T is linear and $T_n \overset{*}{\rightharpoonup} T$. □

Theorem 2.52. Let X be a compact metric space. Then the vector space $C(X) = \{f\colon X \to \mathbb{R} : f \text{ continuous}\}$, equipped with the sup-norm

$$\|f\|_X := \sup\{|f(x)| : x \in X\}$$

is a separable Banach space.

Note. $(C_c(\mathbb{R}), \|\cdot\|_{\mathbb{R}})$ is not complete.

Proof. It is straightforward to show that $(C(X), \|\cdot\|_X)$ is Banach. We next verify the separability. For each $n \in \mathbb{N}$ choose finitely many points $x_1^n, \cdots, x_{m_n}^n$ such that

$$X = \bigcup_{k=1}^{m_n} B(x_k^n, \frac{1}{n}).$$

Let $\lambda_1^n, \cdots, \lambda_{m_n}^n$ be a partition subordinate to the balls $B(x_k, 1/n)$. Then the set

$$\mathcal{F} := \{\sum_{k=1}^{m_n} q_k \cdot \lambda_k^n : n \in \mathbb{N}, q_k \in \mathbb{Q}\} \subset C(X)$$

is countable.

We show that $\mathcal{F}$ is dense. Let $f \in C(X)$ and $\varepsilon > 0$. Select $n \in \mathbb{N}$ such that for all $x, y \in X$ with $d(x,y) < 1/n$ we have

$$|f(x) - f(y)| < \frac{\varepsilon}{2}.$$

Choose $q_k \mathbb{Q}$ such that $|f(x_k^n) - q_k| < \frac{\varepsilon}{2}$ for all $k = 1, \dots, m_n$. Let $h \in \mathcal{F}$ be the function given by

$$h := \sum_{k=1}^{m_n} q_k \lambda_k^n.$$

Let $x \in X$. If k is such that $d(x, x_k^n) \geq 1/n$, then $\lambda_k^n(x) = 0$. If k is such that $d(x, x_k^n) < 1/n$, then

$$|f(x) - q_k| \leq |f(x) - f(x_k^n)| + |f(x_k^n) - q_k| < \varepsilon.$$

Since $\sum_{k=1}^{m_n} \lambda_k^n(x) = 1$, we obtain

$$|f(x) - h(x)| = |\sum_{k=1}^{m_n} (f(x) - q_k)\lambda_k^n| < \varepsilon.$$

□

Theorem 2.53. Let X be a compact metric space and (μ_n) a sequence of Radon measures on X such that

$$\sup_n \mu_n(X) < \infty.$$

Then there exists a subsequence (μ_{n_k}) and a Radon measure μ on X such that

$$\int_X f d\mu_{n_k} \to \int_X f d\mu$$

for every $f \in C(X)$.

Note. We say that μ_{n_k} converges weakly to μ, we write

$$\mu_{n_k} \rightharpoonup \mu.$$

Proof. For each $n \in \mathbb{N}$, define $T_n \colon C(X) \to \mathbb{R}$ by

$$T_n(f) := \int_X f d\mu_n \quad \forall f \in C(X).$$

Then $T_n \in (C(X), \|\cdot\|_X)^*$ and $\|T_n\| \leq \mu_n(X)$. By the Banach-Alaoglu theorem, there exists a subsequence (T_{n_k}) and $T \in (C(X), \|\cdot\|_X)^*$ such that $T_{n_k} \overset{*}{\rightharpoonup} T$, i.e.,

$$T_{n_k}(f) \to T(f)$$

for every $f \in C(X)$. Since T_n is positive for all $n \in \mathbb{N}$, T is positive as well. By the Riesz representation theorem, there exists a Radon measure μ on X such that

$$T(f) = \int_X f d\mu.$$

Therefore, $\mu_{n_k} \rightharpoonup \mu$. □

3. Lipschitz maps and rectifiable sets

3.1. **Lipschitz extensions.**

Definition 3.1. A map $\varphi\colon X \to Y$ between metric spaces (X, d_X) and (Y, d_Y) is called λ-Lipschitz if

$$d_Y(\varphi(x), \varphi(x')) \leq \lambda d_X(x, x') \quad \forall x, x' \in X.$$

Theorem 3.2. (McShane) Let X be a metric space, $A \subset X$ and $\varphi\colon A \to \mathbb{R}$ λ-Lipschitz. Then there exits a λ-Lipschitz extension $\hat{\varphi}\colon X \to \mathbb{R}$ of φ, i.e., $\hat{\varphi}|_A = \varphi$.

Note. If such $\hat{\varphi}$ exists, then

$$\hat{\varphi}(x) \leq \varphi(a) + \lambda d(x, a) \quad \forall a \in A.$$

Proof. For $x \in X$, define

$$\hat{\varphi}(x) := \inf\{\varphi(a) + \lambda d(x, a) : a \in A\}.$$

Then

- $\hat{\varphi}(x) > -\infty$.
  Fix $a_0 \in A$. Then
  $$\begin{aligned} \varphi(a) + \lambda d(x, a) &\geq \varphi(a_0) - \lambda d(a, a_0) + \lambda\big(d(a, a_0) - d(a_0, x)\big) \\ &= \varphi(a_0) - \lambda d(a_0, x), \end{aligned}$$
  from which we infer that $\hat{\varphi}(x) \geq \varphi(a_0) - \lambda d(a_0, x)$.
- $\hat{\varphi}$ extends φ.
  Let $x \in A$. Then $\varphi(a) + \lambda d(x, a) \geq \varphi(x) = \varphi(x) + \lambda d(x, x)$ for all $a \in A$. Thus $\hat{\varphi}(x) = \varphi(x)$.
- $\hat{\varphi}$ is λ-Lipschitz.
  Let $x, x' \in X$ and without loss of generality we assume $\hat{\varphi}(x) \leq \hat{\varphi}(x')$. Fix $\varepsilon > 0$. Then there exists $a \in A$ such that $\varphi(a) + \lambda d(x, a) \leq \hat{\varphi}(x) + \varepsilon$. It follows
  $$\begin{aligned} \hat{\varphi}(x') - \hat{\varphi}(x) &\leq \varphi(a) + \lambda d(x', a) - \varphi(a) - \lambda d(x, a) + \varepsilon \\ &\leq \lambda d(x, x') + \varepsilon. \end{aligned}$$
  Since ε is arbitrary, $\hat{\varphi}$ is λ-Lipschitz.

□

Exercise: For $A = [0, 1] \cup [2, 3]$, $\lambda = 1$ and $\varphi(a) = 1$ for all $a \in A$, draw the extension given in the proof.

Corollary 3.3. Let X be a metric space, $A \subset X$ and $\varphi\colon A \to \mathbb{R}^n$ λ-Lipschitz. Then φ has $\sqrt{n}\lambda$-Lipschitz extension $\hat{\varphi}\colon X \to \mathbb{R}^n$.

Here, $\mathbb{R}^n$ is equipped with the Euclidean norm

$$|(x_1, \ldots, x_n)| := \sqrt{x_1^2 + \cdots + x_n^2}.$$

Proof. Write $\varphi = (\varphi_1, \ldots, \varphi_n)$ and apply McShane's extension to each φ_i. □

Exercise: Let X be the tripod with the length metric and let $A := \{v_1, v_2, v_3\}$. Find a 1-Lipschitz map $\varphi\colon A \to \mathbb{R}^2$ which does not extend to an L-Lipschitz map $\hat{\varphi}\colon X \to \mathbb{R}^2$ for any $L < \sqrt{2}$.

Question: What happens when the Euclidean norm in the corollary is replaced by l^∞-norm on $\mathbb{R}^n$.

Remark 3.4.
(a) Kirszbraun's theorem: Every λ-Lipschitz map $\varphi\colon A \to \mathbb{R}^n$, $A \subset \mathbb{R}^n$, extends to a λ-Lipschitz map $\hat{\varphi}\colon \mathbb{R}^m \to \mathbb{R}^n$.
(b) Federer: Every λ-Lipschitz map $\varphi\colon A \subset \mathbb{R}^m \to E$, where E is a Banach space extends to an $L\lambda$-Lipschitz map $\hat{\varphi}\colon \mathbb{R}^m \to E$, where L depends only on m.
(c) The result in (b) is not true when $\mathbb{R}^m$ is replaced by an arbitrary metric space.

Proposition 3.5. Let $\varphi\colon X \to Y$ be a λ-Lipschitz map between metric spaces X and Y. If $A \subset X$ and $s \geq 0$, then

$$\mathcal{H}^s(\varphi(A)) \leq \lambda^s \mathcal{H}^s(A).$$

In particular,

$$\operatorname{diam}_{\mathcal{H}}(\varphi(A)) \leq \dim_{\mathcal{H}}(A).$$

Proof. Exercise. □

3.2. **Differentiability of Lipschitz functions.** Recall $U \subset \mathbb{R}^m$ open and quasiconvex, $\varphi\colon U \to \mathbb{R}^n$ C^1-smooth with $\|d\varphi\|$ bounded $\to$ φ Lipschitz.

Let $c\colon [a,b] \to U$ be C^1-smooth curve from x to y with $l(c) \leq L|x-y|$. Then

$$\begin{aligned}|\varphi(y) - \varphi(x)| &= \Big|\int_a^b d\varphi_{c(t)}(c'(t))dt\Big| \\ &\leq \int_a^b \|d\varphi_{c(t)}\||c'(t)|dt \leq ML|y-x|,\end{aligned}$$

where $M = \sup\{\|d\varphi_z\| : z \in U\}$.

Question: φ Lipschitz $\to$ φ differentiable?

Answer: Not everywhere, but almost everywhere.

First, we have the following classical theorem for functions of one variable.

Theorem 3.6. Let $f\colon [a,b] \to \mathbb{R}$ be an absolutely continuous function. Then f is differentiable a.e. with $f' \in L^1([a,b])$ and

$$f(x) = f(a) + \int_a^x f'(y)dy \quad \forall x \in [a,b].$$

Recall that f is absolutely continuous on $[a,b]$ if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that for every finite collection $\{[a_i,b_i]\}_{i=1}^N$ of non-overlapping intervals $[a_i,b_i] \subset [a,b]$ with $\sum_{i=1}^N |b_i - a_i| < \delta$ we have

$$\sum_i |f(b_i) - f(a_i)| < \varepsilon.$$

Exercise: If f is Lipschitz, then f is absolutely continuous.

We now come to the important differentiability theorem for Lipschitz maps in higher dimensions.

Theorem 3.7. (Rademacher's theorem) Every Lipschitz map $\varphi\colon U \to \mathbb{R}^m$ is differentiable $\mathcal{L}^n$-a.e. in U.

Here, differentiability means differentiable in the sense of Frechet: $\exists L\colon \mathbb{R}^n \to \mathbb{R}^m$ linear such that

$$\lim_{v\to 0} \frac{\varphi(x+v)-\varphi(x)-L(v)}{|v|} = 0.$$

If it exists, L is unique and will be denoted by $d\varphi_x$.

An (a priori) weaker notion of differentiability is differentiability in the sense of Gateanx: $\forall v \in \mathbb{R}^n$, the directional derivative

$$D_v\varphi(x) = \lim_{r\to} \frac{\varphi(x+rv)-\varphi(x)}{r}$$

exists and the map $v \mapsto D_v\varphi(x)$ is linear.

Remark 3.8. For a map $\varphi\colon U \to \mathbb{R}^m$ and $x \in U$, we have Frechet differentiability at x implies continuity at x and Gateanx differentiability at x. However, Gateaux differentiability at x does not imply continuity at x.

However, we have

Proposition 3.9. Let $\varphi\colon U \subset \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz map and $x \in U$. If φ is Gateaux differentiable at x, then φ is Frechet differentiable at x and

$$d\varphi_x(v) = D_v\varphi(x) \quad \forall v \in \mathbb{R}^n.$$

Proof. Exercise. Use Gateaux diff. at x in finitely many directions $\{v_1,\dots,v_k\} \subset S^{n-1}$ that are sufficiently dense in S^{n-1}. □

Proof of Theorem 3.7. Without loss of generality $m = 1$ and $U = \mathbb{R}^n$.

Claim 1: For a.e. $x \in \mathbb{R}^n$, the directional derivative $D_v\varphi(x)$ exists for all $v \in \mathbb{R}^n$.

Let $\{v_k\} \subset S^{n-1}$ be countable and dense. Fix k and let $y \in v_k^\perp$. Then by Theorem 3.6, the function $t \mapsto \varphi(y+tv_k)$ is differentiable at a.e. $t \in \mathbb{R}$. By Fubini's theorem, for a.e. $x \in \mathbb{R}^n$, the directional derivative $D_{v_k}\varphi(x)$ exists and so for a.e. $x \in \mathbb{R}^n$, the directional derivative $D_{v_k}\varphi(x)$ exists for all $k \in \mathbb{N}$.

Let λ be the Lipschitz constant of φ. Since

$$\Big|\frac{\varphi(x+tv)-\varphi(x)}{t} - \frac{\varphi(x+tw)-\varphi(x)}{t}\Big| \le \lambda|v-w|,$$

we obtain

$$|D_v\varphi(x) - D_w\varphi(x)| \le \lambda|v-w| \tag{3.1}$$

whenever $D_v\varphi(x)$ and $D_w\varphi(x)$ exist. Thus for a.e. $x \in \mathbb{R}^n$ the directional derivative $D_v\varphi(x)$ exists for all $v \in S^{n-1}$ and thus for all $v \in \mathbb{R}^n$ since $D_{sv}\varphi(x) = sD_v\varphi(x)$.

Claim 2: For a.e. x,

$$D_v\varphi(x) = \langle \nabla\varphi(x), v\rangle \quad \forall v \in \mathbb{R}^n. \tag{3.2}$$

In particular, $v \mapsto D_v\varphi(x)$ is linear.

Due to (3.1), it is enough to show (3.2) for a fixed $v \in \mathbb{R}^n \backslash \{0\}$, we will show that

$$\int_{\mathbb{R}^n} \psi(x) D_v \psi(x) dx = \int_{\mathbb{R}^n} \psi(x) \langle \nabla \varphi(x), v \rangle dx$$

for every $\psi \in C_c^1(\mathbb{R}^n)$. From this it follows with the fundamental lemma of the calculus of variations that

$$D_v \varphi(x) = \langle \nabla \varphi(x), v \rangle \quad \text{a.e. } x \in \mathbb{R}^n.$$

Fix $\psi \in C_c^1(\mathbb{R}^n)$. Note that

$$\varphi_k(x) := \frac{\varphi(x + \frac{1}{k} v) - \varphi(x)}{1/k} \to D_v \varphi(x) \quad \text{a.e. } x \in \mathbb{R}^n$$

and $|\varphi_k(x)| \le \lambda |v|$ for all x, thus

$$\int_{\mathbb{R}^n} D_v \varphi(x) \psi(x) dx = \lim_{k \to \infty} \int_{\mathbb{R}^n} \varphi_k(x) \psi(x) dx.$$

We calculate

$$\begin{aligned}
\int_{\mathbb{R}^n} \varphi_k(x) \psi(x) dx &= k \Big(\int_{\mathbb{R}^n} \varphi(x + \frac{1}{k} v) \psi(x) dx - \int_{\mathbb{R}^n} \varphi(x) \psi(x) dx \Big) \\
&= k \Big(\int_{\mathbb{R}^n} \varphi(y) \psi(y - \frac{1}{k} v) dy - \int_{\mathbb{R}^n} \varphi(y) \psi(y) dy \Big) \\
&= - \int_{\mathbb{R}^n} \varphi(y) \frac{\psi(y - \frac{1}{k} v) - \psi(y)}{-1/k} dy \\
&\overset{k \to \infty}{\to} - \int_{\mathbb{R}^n} \varphi(y) D_v \psi(y) dy \\
&= - \int_{\mathbb{R}^n} \varphi(y) \langle \nabla \psi(y), v \rangle dy.
\end{aligned}$$

So

$$\int_{\mathbb{R}^n} D_v \varphi(x) \psi(x) dx = - \int_{\mathbb{R}^n} \varphi(y) \langle \nabla \psi(y), v \rangle dy.$$

This holds in particular with $v = e_i$ and thus

$$\int_{\mathbb{R}^n} \frac{\partial \varphi}{\partial e_i}(x) \psi(x) dx = - \int_{\mathbb{R}^n} \varphi(x) \frac{\partial \psi}{\partial e_i}(x) dx.$$

For arbitrary $v \in \mathbb{R}^n$, writing $v = (v_1, \dots, v_n)$,

$$\begin{aligned}
\int_{\mathbb{R}^n} D_v \varphi(x) \psi(x) dx &= - \sum_{i=1}^n v_i \int_{\mathbb{R}^n} \varphi(x) \frac{\partial \psi}{\partial e_i}(x) dx \\
&= \sum_{i=1}^n v_i \int_{\mathbb{R}^n} \frac{\partial \varphi}{\partial e_i}(x) \psi(x) dx \\
&= \int_{\mathbb{R}^n} \langle \nabla \varphi(x), v \rangle \psi(x) dx.
\end{aligned}$$

Claims 1 and 2 show that φ is Gateaux diff. a.e. on $\mathbb{R}^n$. The theorem follows from Proposition 3.9.

□

Remark 3.10. The notions of Gateaux and Frechet differentiability make sense for maps $\varphi\colon U \subset \mathbb{R}^n \to E$, where E is a Banach space. However, if $\dim E = \infty$, then Lipschitz maps to E need not be differentiable anywhere.

Example 3.11. Let $\varphi\colon (0,1) \to L^1((0,1))$ be given by

$$\varphi(t) := \chi_{(0,t)}.$$

Then φ is isometric embedding but nowhere differentiable.

Lipschitz maps are even more similar to C^1-smooth maps than one would expect from Rademacher's theorem. Namely, we have

Theorem 3.12. Let $\varphi\colon \mathbb{R}^m \to \mathbb{R}^n$ be Lipschitz and $\varepsilon > 0$. Then there exists a C^1-smooth map $g\colon \mathbb{R}^m \to \mathbb{R}^n$ such that

$$\mathcal{L}^m\Big(\big\{x \in \mathbb{R}^m : \varphi(x) \neq g(x)\big\}\Big) \leq \varepsilon.$$

In other words: outside a set of arbitrarily small measure, φ agrees with a C^1-smooth map. The proof of the theorem relies on Radmarcher's theorem and the so-called Whitney extension theorem.

Theorem 3.13. Let $\varphi\colon \mathbb{R}^m \to \mathbb{R}^n$ be Lipschitz and

$$A := \{x \in \mathbb{R}^m : d\varphi_x \text{ exists and } \operatorname{rank}(d\varphi_x) < m\}.$$

Then $\mathcal{H}^m(\varphi(A)) = 0$.

Notice that the theorem only gives a non-trivial statement when $m \leq n$. The statement of the theorem is a kind of "Sard" type theorem. Recall that the classical theorem of Morse-Sard asserts: If $\varphi\colon U \subset \mathbb{R}^m \to \mathbb{R}^n$ is C^k-smooth for some

$$k \geq \max\{m - n + 1, 1\},$$

then the image of the set $\{x \in U : \operatorname{rank}(d\varphi_x) < n\}$ has Lebesgue n-measure zero. It is important that φ is sufficiently regular. Indeed, there exists a C^1-smooth map $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^2$ which maps the unit cube $[0,1]^3$ surjectively onto $[0,1]^2$.

Example 3.14. A Lipschitz version of Kaufman's example: divide $[0,1]^3$ into cubes of side-length $1/3$ and let $\hat{I}_{1,j}$ be the open cube of side-length $1/4$ with same center. Divide $[0,1]^2 =: J_0$ into 16 squares $J_{1,j}$ of side-length $1/4$. Define a map $\varphi\colon I_0 \backslash \bigcup \hat{I}_{1,j} \to [0,1]^2$ by:

- φ maps $\partial I_{1,j}$ to the center of $[0,1]^2$ $\forall j$
- φ maps $\partial \hat{I}_{1,j}$ to the center of $J_{1,j}$ and $I_{1,j} \backslash \hat{I}_{1,j}$ "linearly" to the segment between the two centers for $j = 1, \ldots, 16$
- For $j = 17, \ldots, 27$, φ maps $I_{1,j}$ to center of $[0,1]^2$

Repeat the procedure above to each $\hat{I}_{1,j}$ and $J_{1,j}$ by scaling and to smaller and smaller cubes. This implies φ Lip. map defined on I_0 minus Cantor set of measure zero. Since φ

is Lipschitz, it has Lip. extension to I_0 and so $\varphi(I_0) = J_0$ and $\operatorname{rank}(d\varphi_x) \leq 1$ for almost every $x \in I_0$.

Proof of Theorem 3.13. By possibly replacing A by $A \cap B(0,R)$ we may assume that A is bounded. Thus there exists an open set $U \subset \mathbb{R}^m$ open such that $A \subset U$ and $\mathcal{L}^m(U) < \infty$. Let $0 < \varepsilon, \delta < 1$, we define a fine covering of A by closed sets as follows: Let $x \in A$, then there exists $v \in \ker(d\varphi_x)$ with $|v| = 1$. Let $W \subset \mathbb{R}^m$ be the subspace orthogonal to v. Set $L := \lambda + \sqrt{2}$, where λ is the Lipschitz constant of φ. For all sufficiently small $r > 0$ with $2L\varepsilon r < \delta$ the closed set

$$C_{x,r} := \{x + w + tv : |t| \leq r, w \in W, |w| \leq \varepsilon r\}$$

satisfies $C_{x,r} \subset U$ and has the property: if $x \neq y \in C_{x,r}$, then

$$\frac{|\varphi(y) - \varphi(x) - d\varphi_x(y - x)|}{|y - x|} \leq \varepsilon.$$

Let $\mathcal{C}$ be the family of all such closed sets $C_{x,r}$ with $x \in A$ and note that $\mathcal{C}$ is a fine covering of A in the sense of Vitali. By Vitali Covering, there exists a countable disjoint subfamily $\{C_i\}$ of sets

$$C_i = C_{x_i,r_i} \in \mathcal{C}$$

such that $\mathcal{L}^m(A \setminus \bigcup C_i) = 0$. Note that $\operatorname{diam} C_i \leq 2\sqrt{1+\varepsilon^2} r_i \leq 2\sqrt{2} r_i$ and thus

$$\begin{aligned}
\sum_i (\operatorname{diam} C_i)^m &\leq \frac{2^{\frac{3m}{2}}}{\omega_{m-1}\varepsilon^{m-1}} \sum_i \mathcal{L}^n(C_i) \\
&\leq \frac{2^{\frac{3m}{2}}}{\omega_{m-1}\varepsilon^{m-1}} \mathcal{L}^m(U) < \infty.
\end{aligned}$$

Hence,

$$\mathcal{H}^m_\delta(\varphi(A)) \leq \mathcal{H}^m_\delta(\varphi(A \setminus \bigcup C_i)) + \mathcal{H}^m_\delta(\varphi(\bigcup C_i)).$$

In order to bound the second term, notice first that if $y \in C_i = C_{x_i,r_i}$, then

$$\begin{aligned}
|\varphi(y) - \varphi(x_i)| &\leq |d\varphi_{x_i}\varphi(y - x_i)| + \varepsilon|y - x_i| \\
&\overset{y=x_i+w+tv}{=} |d\varphi_{x_i}(w)| + \varepsilon|y - x_i| \\
&\leq \varepsilon\lambda r_i + \sqrt{2}\varepsilon r_i
\end{aligned}$$

and thus

$$\varphi(C_i) \subset \bar{B}(\varphi(x_i), L\varepsilon r_i).$$

It follows that

$$\begin{aligned}
\mathcal{H}^m_\delta(\varphi(\bigcup C_i)) &\leq \sum_i \omega_m (L\varepsilon r_i)^m \\
&= \omega_m L^m \frac{1}{2\omega_{m-1}} \varepsilon \sum_i 2r_i\omega_{m-1}(\varepsilon r_i)^{m-1} \\
&\leq \frac{\omega_m L^m}{2\omega_{m-1}} \varepsilon \mathcal{L}^m(U).
\end{aligned}$$

Thus $\mathcal{H}^m_\delta(\varphi(A)) \le \varepsilon\frac{\omega_m L^m}{2\omega_{m-1}}\mathcal{L}^m(U)$. Since ε and δ were arbitrary, $\mathcal{H}^m(\varphi(A)) = 0$.

□

3.3. **Area and coarea formula.** Recall that the usual change of variable formula shows that

$$\int_V f(y)dy = \int_U f\circ\varphi(x)|\det d\varphi_x|dx$$

whenever $\varphi\colon U \to V$ is a diffeomorphism with $U, V \subset \mathbb{R}^m$ open and $f \in C_c(V)$. The area formula (for Lipschitz maps) provides a generalization of this for maps between spaces of different dimensions.

Definition 3.15. Let $L\colon \mathbb{R}^m \to \mathbb{R}^n$ be a linear map with $m \le n$ and let A be the matrix representing L in standard coordinates. The m-Jacobian of L is defined by

$$J_m(L) := \sqrt{\det(A^TA)}.$$

Note. Notice that A^TA is symmetric, positive semi-definite $m \times m$-matrix and so $\det(A^TA) \ge 0$.

Example 3.16. 1) $L\colon \mathbb{R} \to \mathbb{R}^n$ defined by $L(t) = tv$ for some $v \in \mathbb{R}^n$. Then $J_1(v) = |v|$.

2) $L\colon \mathbb{R}^n \to \mathbb{R}^n$ defined by $L(x) = Ax$ for some $n \times n$-matrix A. Then

$$J_n(L) = \sqrt{\det(A^TA)} = |\det(A)|.$$

Theorem 3.17 (Area formula)**.** Let $\varphi\colon \mathbb{R}^m \to \mathbb{R}^n$ be a Lipschitz map with $m \le n$. Then

i) If $A \subset \mathbb{R}^m$ is $\mathcal{L}^m$-measurable, then

$$\int_A J_m(d\varphi_x)d\mathcal{L}^m(x) = \int_{\mathbb{R}^n} N(\varphi|_A, y)d\mathcal{H}^m(y), \tag{3.3}$$

where $N(\varphi|_A, y) = \mathcal{H}^0(A \cap \varphi^{-1}(\{y\}))$.

ii) If $g \in L^1(\mathbb{R}^m)$, then

$$\int_{\mathbb{R}^m} g(x)J_m(d\varphi_x)d\mathcal{L}^m(x) = \int_{\mathbb{R}^n}\Big(\sum_{x\in\varphi^{-1}(\{y\})} g(x)\Big)d\mathcal{H}^m(y).$$

Note. Implicitly contained in the statement is the assertion that the functions in the integrals are measurable, respectively, integrable.

Example 3.18. 1) If $c\colon [a, b] \to \mathbb{R}$ is an injective Lipschitz curve, then

$$\mathcal{H}^1(c([a, b])) = \int_a^b |c'(t)|dt = l(c).$$

2) If $M \subset \mathbb{R}^n$ is m-dim smooth submanifold and (U, ψ) a chart and $V \subset U$ open, bounded with $\overline{V} \subset U$, then

$$\begin{aligned}\mathrm{Vol}_M(V) &\overset{\text{def}}{=} \int_{\psi(V)} J_m(d\psi_x^{-1})d\mathcal{L}^m(x)\\ &\overset{\text{Area formula}}{=} \mathcal{H}^m(V).\end{aligned}$$

Statement (ii) of the theorem follows by approximating g by simple functions. We therefore only prove statement (i). The following lemma shows that (3.3) holds when φ is a linear map.

Lemma 3.19. Let $L\colon \mathbb{R}^m \to \mathbb{R}^n$ be linear with $m \leq n$. Then for every $A \subset \mathbb{R}^m$ $\mathcal{L}^m$-measurable with $\mathcal{L}^m(A) < \infty$, we have

$$\mathcal{H}^m(L(A)) = J_m(L)\mathcal{L}^m(A). \tag{3.4}$$

Proof. If $m = n$, then

$$\mathcal{L}^m(L(A)) = |\det(L)|\mathcal{L}^m(A)$$

by the usual transformation formula for Lebesgue measure and hence (3.4).

Suppose now $m < n$. Let $T \in O(n)$ be such that $T \circ L$ has image in the subspace $\mathbb{R}^m \times \{0\}^{n-m} \subset \mathbb{R}^n$. Notice that

$$\det\big((T \circ L)^T (T \circ L)\big) = \det(L^T T^T T L) = \det(L^T L).$$

We write

$$T \circ L = \begin{bmatrix} C \\ -- \\ 0 \end{bmatrix}$$

for some $m \times m$-matrix C. Then

$$\begin{aligned} (T \circ L)^T (T \circ L) &= C^T C \\ \Rightarrow \det(L^T L) &= \det(C^T C) = (\det(C))^2 \\ \Rightarrow \mathcal{H}^m(L(A)) &= \mathcal{H}^m(T(L(A))) = \mathcal{L}^m(C(A)) \\ &= |\det(C)|\mathcal{L}^m(A) = J_m(L)\mathcal{L}^m(A). \end{aligned}$$

□

For the general case of the area formula we will need:

Lemma 3.20. Let $U \subset \mathbb{R}^m$ be open and $\varphi\colon U \to \mathbb{R}^n$ Lipschitz, where $m \leq n$. Then for every $t > 1$, there exists a countable collection $\{E_k\}_{k \geq 0}$ of Borel sets $E_k \subset U$ such that

i) $U = \bigcup_{k=0}^{\infty} E_k$
ii) If $x \in E_0$, then either φ is not differentiable at x or $J_m(d\varphi_x) = 0$
iii) For every $k \geq 1$, the restriction $\varphi|_{E_k}$ is injective
iv) For every $k \geq 1$, there exists $L_k\colon \mathbb{R}^m \to \mathbb{R}^n$ linear such that

$$\frac{1}{t^m} J_m(L_k) \leq J_m(d\varphi_x) \leq t^m J_m(L_k) \quad \forall x \in E_k$$

and the maps $\varphi \circ (L_k|_{E_k})^{-1}$ and $L_k \circ (\varphi|_{E_k})^{-1}$ are both t-Lipschitz.

Roughly speaking, this asserts that φ is close to a linear map on suitable sets. Moreover, the restriction $\varphi|_{E_k}$ is bi-Lipschitz for all $k \geq 1$. This will also be important later.

Proof. Let E_0 be the set of points $x \in U$ such that either φ is not differentiable at x or $J_m(d\varphi_x) = 0$. It can be shown that E_0 is a Borel set (hard exercise).

Fix $\varepsilon > 0$ such that $\frac{1}{t} + \varepsilon < 1 < t - \varepsilon$, $A \subset U$ countable and dense, G countable and dense in the set of injective linear maps $\mathbb{R}^m \to \mathbb{R}^n$. Let $z \in A$, $L \in G$, $j \geq 1$. Let $E(z, L, j)$ be the set of points $x \in U \cap B(z, 1/j)$ such that

1) φ differentiable at x and $d\varphi_x$ injective
2) For every $y \in U \cap B(x, 2/j)$ we have

$$|\varphi(y) - \varphi(x) - d\varphi_x(y - x)| \leq \varepsilon |L(y - x)| \tag{3.5}$$

3) The map $d\varphi_x \circ L^{-1}$ is $(t - \varepsilon)$-Lipschitz
4) The map $L \circ (d\varphi_x)^{-1}$ is $\frac{1}{\frac{1}{t}+\varepsilon}$-Lipschitz

Claim 1: $E(z, L, j)$ satisfies properties (iii) and (iv) of the Lemma.

If $x, y \in E(z, L, j)$, then $|x - y| < 2/j$ and thus, by our choice, we have

$$\frac{1}{t}|L(y - x)| \leq |\varphi(y) - \varphi(x)| \leq t|y - x|. \tag{3.6}$$

Indeed,

$$\begin{aligned}
|\varphi(y) - \varphi(x)| &\leq |d\varphi_x(y - x)| + \varepsilon|L(y - x)| \\
&= |d\varphi_x \circ L^{-1}(L(y - x))| + \varepsilon|L(y - x)| \\
&\leq (t - \varepsilon)|L(y - x)| + \varepsilon|L(y - x)| \\
&= t|L(y - x)|
\end{aligned}$$

and the other inequality follows similarly.

Since L is injective, it follows from (3.6) that $\varphi|_{E(z,L,j)}$ is injective. By (3.6) again, the maps $\varphi \circ \left(L|_{E(z,L,j)}\right)^{-1}$ and $L \circ (\varphi|_{E(z,L,j)})^{-1}$ are t-Lipschitz, which implies (iii) and the second part of (iv).

It remains to estimate $J_m(d\varphi_x)$. We have

$$\begin{aligned}
J_m(d\varphi_x) &= \mathcal{H}^m(d\varphi_x([0, 1]^m)) \leq (t - \varepsilon)^m \mathcal{H}^m(L([0, 1]^m)) \\
&\leq t^m J_m(L)
\end{aligned}$$

and similarly, $J_m(d\varphi_x) \geq t^{-m} J_m(L)$. This proves claim 1.

Claim 2: the set $E(z, L, j)$ covers $U \backslash E_0$.

Let $x \in U \backslash E_0$. Then φ is differentiable at x and $d\varphi_x$ is injective. Since G is dense in the set of linear maps $\mathbb{R}^m \to \mathbb{R}^n$ there exists $L \in G$ such that 3) and 4) hold. Since L is injective, linear, there exists $\delta > 0$ such that

$$|L(v)| \geq \delta |v| \quad \forall v \in \mathbb{R}^m$$

and $\exists j \geq 1$ such that

$$\begin{aligned}
|\varphi(y) - \varphi(x) - d\varphi_x(y - x)| &\leq \varepsilon\delta|y - x| \\
&\leq \varepsilon|L(y - x)|
\end{aligned}$$

for all $y \in B(x, 2/j) \cap U$. Since $A \subset U$ is dense, $\exists z \in A$ such that $x \in B(z, 1/j)$ and so $x \in E(z, L, j)$. Finally, one can show that the sets $E(z, L, j)$ are Borel sets (exercise).

□

Proof of Theorem 3.17. Let $A \subset \mathbb{R}^m$ be $\mathcal{L}^m$-measurable. One can show that $\varphi(A)$ is $\mathcal{H}^m$-measurable and that the function $y \mapsto N(\varphi|_A, y)$ is $\mathcal{H}^m$-measurable (see Evans-Ganepy).

It is easy to see that the function $x \mapsto J_m(d\varphi_x)$ is a Borel function. Let $t > 1$ and let $\{E_k\}_{k\geq 0}$ be a collection of Borel sets as in Lemma 3.20.

Define $A_0 := A \cap E_0$ and $A_k := (A \cap E_k) \setminus \bigcup_{j=0}^{k-1} E_j$ for $k \geq 1$. Then A_k measurable and pairwise disjoint with $A = \bigcup_{k=0}^{\infty} A_k$. For every $y \in \mathbb{R}^n$, we have

$$N(\varphi|_A, y) = \sum_{k=0}^{\infty} N(\varphi|_{A_k}, y)$$

and

$$\int_{\mathbb{R}^n} N(\varphi|_A, y) d\mathcal{H}^m(y) = \sum_{k=0}^{\infty} \int_{\mathbb{R}^n} N(\varphi|_{A_k}, y) d\mathcal{H}^m(y).$$

By the definition of E_0, Rademacher's theorem and Theorem 3.13 we have $J_m(d\varphi_x) = 0$ for a.e. $x \in A_0$ and $\mathcal{H}^m(\varphi(A_0)) = 0$, hence,

$$\int_{\mathbb{R}^n} N(\varphi|_{A_0}, y) d\mathcal{H}^m(y) = 0 = \int_{A_0} J_m(d\varphi_x) d\mathcal{L}^m(x).$$

Let $k \geq 1$. Since $\varphi|_{A_k}$ is injective, it follows that

$$\int_{\mathbb{R}^n} N(\varphi|_{A_k}, y) d\mathcal{H}^m(y) = \mathcal{H}^m(\varphi(A_k)).$$

Moreover, by (iv) of Lemma 3.20, we have

$$t^{-m}\mathcal{H}^m(L_k(A_k)) \leq \mathcal{H}^m(\varphi(A_k)) \leq t^m \mathcal{H}^m(L_k(A_k))$$

and

$$t^{-m} J_m(L_k)\mathcal{L}^m(A_k) \leq \int_{A_k} J_m(d\varphi_x) d\mathcal{L}^m(x) \leq t^m J_m(L_k)\mathcal{L}^m(A_k).$$

Since

$$\mathcal{H}^m(L_k(A_k)) = J_m(L_k)\mathcal{L}^m(A)$$

by Lemma 3.20, we obtain

$$t^{-2m} \int_{A_k} J_m(d\varphi_x) d\mathcal{L}^m(x) \leq \int_{\mathbb{R}^n} N(\varphi|_{A_k}, y) d\mathcal{H}^m(y) \leq t^{2m} \int_{A_k} J_m(d\varphi_x) d\mathcal{L}^m(x).$$

Summing over k yields

$$t^{-2m} \int_{A} J_m(d\varphi_x) d\mathcal{L}^m(x) \leq \int_{\mathbb{R}^n} N(\varphi|_{A}, y) d\mathcal{H}^m(y) \leq t^{2m} \int_{A} J_m(d\varphi_x) d\mathcal{L}^m(x)$$

and hence the theorem follows by letting $t \to 1$. □

We next come to the coarea formula. Unlike the area formula it is concerned with Lipschitz maps from a higher to a lower dimensional space. We first define the Jacobian in this situation.

Definition 3.21. Let $L \colon \mathbb{R}^m \to \mathbb{R}^n$ be a linear map with $m \geq n$ and let A be the matrix representing L in standard coordinates. The n-Jacobian of L is defined by

$$J_n(L) = \sqrt{\det(AA^T)}.$$

Note. AA^T is an $n \times n$-matrix.

Example 3.22. Let $L\colon \mathbb{R}^m \to \mathbb{R}$ given by $L(v) = \langle w, v\rangle$ for some $w \in \mathbb{R}^n$. Then $J_1(L) = |w|$.

Theorem 3.23 (Coarea formula)**.** Let $f\colon \mathbb{R}^m \to \mathbb{R}^n$ be a Lipschitz map with $m \geq n$.

i) If $A \subset \mathbb{R}^m$ is $\mathcal{L}^m$-measurable, then

$$\int_A J_n(d\varphi_x)d\mathcal{L}^m(x) = \int_{\mathbb{R}^n} \mathcal{H}^{m-n}(A \cap \varphi^{-1}(\{y\}))d\mathcal{L}^n(y).$$

ii) If $g \in L^1(\mathbb{R}^m)$, then

$$\int_{\mathbb{R}^m} g(x)J_n(d\varphi_x)d\mathcal{L}^m(x) = \int_{\mathbb{R}^n} \Big(\int_{\varphi^{-1}(\{y\})} g(x)d\mathcal{H}^{m-n}(x)\Big)d\mathcal{L}^n(y).$$

Remark 3.24. Implicitly contained in the statement is the assertion that the functions in the integrals on the right are measurable resp. integrable.

The coarea formula is a kind of curvelinear version of Fubini's theorem.

Corollary 3.25. Let $f\colon \mathbb{R}^m \to \mathbb{R}$ be a Lipschitz function. Then

$$\int_{\mathbb{R}^m} |\nabla f| d\mathcal{L}^m = \int_{-\infty}^{\infty} \mathcal{H}^{m-1}(\{f = t\})d\mathcal{L}^1(t).$$

Proof. From the coarea formula since $J_1(df_x) = |\nabla f(x)|$. □

Corollary 3.26 (Polar coordinates)**.** Let $g \in L^1(\mathbb{R}^m)$. Then

$$\int_{\mathbb{R}^m} g d\mathcal{L}^m = \int_0^{\infty} \Big(\int_{\partial B(0,r)} g d\mathcal{H}^{m-1}\Big)d\mathcal{L}^1(r).$$

In particular, for a.e. $r > 0$ we have

$$\int_{\partial B(0,r)} g d\mathcal{H}^{m-1} = \frac{d}{dr}\Big(\int_{B(0,r)} g d\mathcal{L}^m\Big)$$

by Lebesgue differentiation theorem.

Proof. The Lipschitz function $\varphi\colon \mathbb{R}^m \to \mathbb{R}$ given by $\varphi(x) = |x|$ is differentiable everywhere on $\mathbb{R}^m \backslash \{0\}$ with $\nabla\varphi(x) = \frac{x}{|x|}$ and thus $J_1(d\varphi_x) = 1$. Now the result follows from the coarea formula. □

We give two other applications of the coarea formula: Let $\varphi\colon \mathbb{R}^m \to \mathbb{R}^n$ be a Lipschitz map with $m \geq n$.

1). If $A \subset \mathbb{R}^m$ has $\mathcal{L}^m(A) = 0$, then

$$\mathcal{H}^{m-n}(A \cap \varphi^{-1}(\{y\})) = 0$$

for $\mathcal{L}^n$-a.e. $y \in \mathbb{R}^n$.

2). For $\mathcal{L}^n$-a.e. $y \in \mathbb{R}^n$ we have

$$\mathcal{H}^{m-n}\Big(\{x \in \mathbb{R}^m : \operatorname{rank}(d\varphi_x) < n\} \cap \varphi^{-1}(\{y\})\Big) = 0.$$

This follows from the coarea formula and the fact that for a linear map $L\colon \mathbb{R}^m \to \mathbb{R}^n$ we have

$$\operatorname{rank}(L) < n \Rightarrow J_n(L) = 0.$$

Notice that if φ is C^k-smooth with $k \geq m - n + 1$, then the Morse-Sard theorem shows that even

$$\{x \in \mathbb{R}^m : \operatorname{rank}(d\varphi_x) < n\} \cap \varphi^{-1}(\{y\}) = \emptyset$$

for $\mathcal{L}^n$-a.e. $y \in \mathbb{R}^n$.

We now turn to the proof of the coarea formula. The proof uses an approximation which is somewhat similar to that used in the proof of the area formula. We do not provide the whole proof but only sketch it.

Lemma 3.27. Let $L\colon \mathbb{R}^m \to \mathbb{R}^n$ be linear and $A \subset \mathbb{R}^m$ $\mathcal{L}^m$-measurable. Then

$$J_n(L)\mathcal{L}^m(A) = \int_{\mathbb{R}^m} \mathcal{H}^{m-n}(A \cap L^{-1}(\{y\}))d\mathcal{L}^m(y).$$

Proof. We may assume $m > n$. There exists an orthogonal transformation $P\colon \mathbb{R}^m \to \mathbb{R}^m$ such that

$$\{0\}^n \times \mathbb{R}^{m-n} \subset P(\ker(L)). \tag{3.7}$$

Write $T := L \circ P$ and notice that

$$J_n(T) = \sqrt{\det(TT^T)} = \sqrt{\det(LPP^TL^T)} = J_n(L).$$

Now by (3.7), the linear map T can be written as

$$T = (B \mid 0)$$

for some $n \times n$-matrix B. Clearly, we have $TT^T = BB^T$ and hence $J_n(T) = |\det(B)|$. Since the Lemma clear holds if $\det(B) = 0$ and thus we may assume $\det(B) \neq 0$. Observe that

$$L^{-1}(\{y\}) = P(T^{-1}(\{y\}))$$

and hence

$$P^{-1}(A \cap L^{-1}(\{y\})) = P^{-1}(A) \cap T^{-1}(\{y\}) = P^{-1}(A) \cap \Big(\{B^{-1}(y)\} \times \mathbb{R}^{m-n}\Big)$$

for all $y \in \mathbb{R}^n$. It follows therefore from Fubini's theorem that the function

$$y \mapsto \mathcal{H}^{m-n}\Big(A \cap L^{-1}(\{y\})\Big) = \mathcal{H}^{m-n}\Big(P^{-1}(A \cap L^{-1}(\{y\}))\Big)$$

is $\mathcal{L}^n$-measurable and together with the transformation formula

$$\begin{aligned}\int_{\mathbb{R}^n} \mathcal{H}^{m-n}\Big(A \cap L^{-1}(\{y\})\Big)d\mathcal{L}^n(y) &= \int_{\mathbb{R}^n} |\det(B)|\mathcal{H}^{m-n}\Big(A \cap L^{-1}(\{B(y)\})\Big)d\mathcal{L}^n(y)\\ &= J_n(L)\int_{\mathbb{R}^n} \mathcal{H}^{m-n}\Big(P^{-1}(A) \cap (\{y\} \times \mathbb{R}^{n-m})\Big)d\mathcal{L}^n(y)\\ &= J_n(L)\mathcal{L}^m(P^{-1}(A)) = J_n(L)\mathcal{L}^m(A).\end{aligned}$$

□

The next lemma is an analog of the linear approximation lemma 3.20. It will be used in the (sketch of) proof of the coarea formula that will also be important later.

Lemma 3.28. Let $U \subset \mathbb{R}^m$ be open and $\varphi\colon U \to \mathbb{R}^n$ Lipschitz, where $m > n$. Then there exists a countable collection $\{B_k\}_{k\geq 0}$ of Borel sets $B_k \in U$ such that

i) $U = \bigcup_{k=0}^{\infty} B_k$
ii) If $x \in B_0$, then either φ is not differentiable at x or $\operatorname{rank}(d\varphi_x) < n$
iii) For every $k \geq 1$ there exists a projection $p_k\colon \mathbb{R}^m \to \mathbb{R}^{m-n}$ and Lipschitz maps $u_k\colon \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^{m-n}$ and $v_k\colon \mathbb{R}^n \times \mathbb{R}^{m-n} \to \mathbb{R}^m$ such that $u_k(x) = (\varphi(x), p_k(x))$ and $v_k(u_k(x)) = x$ for all $x \in B_k$.

For $1 \leq l \leq m$ we write

$$\Lambda(m,l) = \Big\{\alpha : \{1,\dots,l\} \to \{1,\dots,m\} \text{ strictly increasing}\Big\}$$

and for $\alpha \in \Lambda(m,l)$ we denote by $p_\alpha\colon \mathbb{R}^m \to \mathbb{R}^l$ the projection

$$p_\alpha(x_1,\dots,x_m) = (x_{\alpha(1)},\dots,x_{\alpha(l)}).$$

The projections in the lemma are of the form $p_k = p_{\alpha_k}$ for some $\alpha_k \in \Lambda(m,m-n)$.

Proof. We define B_0 as the set of $x \in U$ such that either φ is not differentiable at x or $\operatorname{rank}(d\varphi_x) < n$. Observe that if $V \subset \mathbb{R}^m$ is a linear subspace of dimension $m-n$, then there exists some $\alpha \in \Lambda(m,m-n)$ such that $p_\alpha(V) = \mathbb{R}^{m-n}$. Thus for every $x \in U\backslash B_0$, there is $\alpha \in \Lambda(m,m-n)$ such that the map $u_\alpha\colon \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^{m-n}$ given by $u_\alpha(y) = (\varphi(y), p_\alpha(y))$ satisfies $\operatorname{rank}(d(u_\alpha)_x) = m$. Thus we have

$$U\backslash B_0 = \bigcup_{\alpha\in\Lambda(m,m-n)} A_\alpha,$$

where $A_\alpha = \{x \in U\backslash B_0 : \operatorname{rank}(d(u_\alpha)_x) = m\}$. By Lemma 3.20, we can cover each A_k by countably many Borel sets $E_{\alpha,k}$ such that u_α is bi-Lipschitz on $A_\alpha \cap E_{\alpha,k}$. Then the map $v_{\alpha,k} := (u_\alpha|_{A_\alpha\cap E_{\alpha,k}})^{-1}$ is Lipschitz and hence has a Lipschitz extension $\bar{v}_{\alpha,k}\colon \mathbb{R}^n \times \mathbb{R}^{m-n} \to \mathbb{R}^m$. Clearly, $\bar{v}_{\alpha,k}(u_\alpha(x)) = x\ \forall x \in A_\alpha \cap E_{\alpha,k}$. Relabeling yields the lemma. □

We give a short sketch of the proof of (i) of the coarea formula. We only consider the case that φ is differentiable at every $x \in A$ and $\operatorname{rank}(d\varphi_x) = n$. Let B_k, u_k, p_k, v_k be as in Lemma 3.28. Fix $k \geq 1$ and set $B := B_k$, $u := u_k$, $p := p_k$ and $v := v_k$. We will show that $A' := A \cap B$ satisfies

$$\int_{A'} J_n(d\varphi_x)d\mathcal{L}^m(x) = \int_{\mathbb{R}^n} \mathcal{H}^{m-n}(A' \cap \varphi^{-1}(\{y\}))d\mathcal{L}^n(y). \tag{3.8}$$

For $y \in \mathbb{R}^n$, define a Lipschitz map $v_y\colon \mathbb{R}^{m-n} \to \mathbb{R}^m$ by $v_y(z) := v(y,z)$ and notice that the restriction

$$v_y\colon u(A') \cap \big(\{y\} \times \mathbb{R}^{m-n}\big) \to A' \cap \varphi^{-1}(y)$$

is bijective. Recall here that $v \circ u(x) = x$ for all $x \in A'$ and $u(x) = (\varphi(x), p(x))$. Hence, we obtain by the area formula that

$$\mathcal{H}^{m-n}\big(A' \cap \varphi^{-1}(y)\big) = \int_{u(A')\cap\big(\{y\}\times\mathbb{R}^{m-n}\big)} J_{m-n}(d(v_y)_z)d\mathcal{H}^{m-n}(z)$$

and by Fubini and change of variable formula, we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} \mathcal{H}^{m-n}\big(A' \cap \varphi^{-1}(y)\big) d\mathcal{L}^n(y) &= \int_{\mathbb{R}^n} \int_{u(A')\cap\big(\{y\}\times\mathbb{R}^{m-n}\big)} J_{m-n}((dv_y)_z) d\mathcal{H}^{m-n}(z) d\mathcal{L}^n(y) \\ &= \int_{u(A')} J_{m-n}((dv_y)_z) d\mathcal{L}^m(y,z) \\ &= \int_{A'} J_{m-n}(d(v_{\varphi(x)})_{p(x)}) |\det(du_x)| d\mathcal{L}^m(x).\end{aligned}$$

It remains to show that

$$J_n(d\varphi_x) = J_{m-n}(d(v_{\varphi(x)})_{p(x)}) |\det(du_x)| \tag{3.9}$$

for a.e. $x \in A'$.

For a.e. $x \in A'$, we have $dv_{u(x)} \circ du_x = 1$. Fix such x and set $W := \ker(d\varphi_x)$. Since $du_x = (d\varphi_x, p)$, we have $du_x|_W = (0, p|_W)$ and hence

$$d(v_{\varphi(x)})_{p(x)} = dv_{u(x)}(0, \cdot) = (p|_W)^{-1}$$

and thus

$$J_{m-n}(d(v_{\varphi(x)})_{p(x)}) = J_{m-n}((p|_W)^{-1}).$$

Let $W^\perp$ be the subspace orthogonal to W. Let $I, I^\perp$ be unit cubes in W and $W^\perp$, respectively. Fubini implies

$$\begin{aligned}|\det(du_x)| &= \mathcal{L}^m\big(du_x(I \times I^\perp)\big) \\ &= \mathcal{L}^n\big(d\varphi_x(I^\perp)\big)\mathcal{L}^{m-n}(p(I)).\end{aligned}$$

Since

$$J_n(d\varphi_x) = \int_{\mathbb{R}^n} \mathcal{H}^{m-n}(I \times I^\perp \cap d\varphi_x^{-1}(y)) d\mathcal{L}^n(y) = \mathcal{L}^n(d\varphi_x(I^\perp))$$

and

$$J_{m-n}((p|_W)^{-1})\mathcal{H}^{m-n}(p(I)) = \mathcal{H}^{m-n}(I) = 1,$$

we get

$$|\det(du_x)| = J_n(d\varphi_x) \cdot \frac{1}{J_{m-n}(d(v_{\varphi(x)})_{p(x)})},$$

proving (3.9).

3.4. **Rectifiable sets.** Rectifiable sets are generalized submanifolds.

Definition 3.29. A set $E \subset \mathbb{R}^n$ is called m-rectifiable if there exist Lipschitz maps $\varphi_k \colon \mathbb{R}^m \to \mathbb{R}^n$, $k \in \mathbb{N}$, such that

$$\mathcal{H}^m\big(E \setminus \bigcup_{k=1}^{\infty} \varphi_k(\mathbb{R}^m)\big) = 0.$$

Example 3.30. 1) Write $\mathbb{Q}^2 = \{q_1, q_2, \cdots\} \subset \mathbb{R}^2$. Then the set

$$E = \bigcup_{k=1}^{\infty} \partial B(q_k, 2^{-k}) \subset \mathbb{R}^2$$

is 1-rectifiable, $\mathcal{H}^1$-measurable, $\mathcal{H}^1(E) < \infty$, and dense in $\mathbb{R}^2$.

2) We construct a compact subset $C \subset [0,1]^2$ as follows

- $C_0 = Q_1^0 := [0,1]^2$
- $C_1 = \bigcup_{j=1}^4 Q_j^1$, where the Q_j^1 are the four squares of side-length $\frac{1}{4}$ in the corners of Q_1^0
- Replace each Q_j^1 by the four square of side-length $\frac{1}{4^2}$ in its corners to obtain

$$C_2 = \bigcup_{j=1}^{4^2} Q_j^2$$

and set

$$C := \bigcup_{k=1}^{\infty} C_k.$$

Then C is compact. Since C_k consists of 4^k squares of diameter $\sqrt{2}4^{-k}$, it follows that

$$\mathcal{H}^1_\delta(C) \leq 2 \cdot 4^k \cdot \frac{\sqrt{2}4^{-k}}{2} = \sqrt{2}$$

for all $\delta > 0$, and thus $\mathcal{H}^1(C) < \infty$. We can show that $\mathcal{H}^1(C) > 0$. For this, let $L \subset \mathbb{R}^2$ be a line of slope -2 (as in the course). The orthogonal projection $P(C_k)$ of C_k to L is always the same interval. Thus

$$\mathcal{H}^1(C) \geq \mathcal{H}^1(P(C)) > 0.$$

As we will see later, the set C is not 1-rectifiable even though ∂C_k is 1-rectifiable.

Theorem 3.31. Let $E \subset \mathbb{R}^n$ be m-rectifiable, $\mathcal{H}^m$-measurable, with $\mathcal{H}^m(E) < \infty$. Then for every $\lambda > 1$ there exist compact sets $K_i \subset \mathbb{R}^m$, $i \in \mathbb{N}$, and λ-bi-Lipschitz maps

$$\varphi_i \colon K_i \to E$$

such that the images $\varphi_i(K_i)$ are pairwise disjoint and satisfy

$$\mathcal{H}^m\big(E \setminus \bigcup_{i=1}^{\infty} \varphi_i(K_i)\big) = 0.$$

This is a bit like an atlas for a smooth submanifold.

Proof. Let $\varphi \colon \mathbb{R}^m \to \mathbb{R}^n$ be one of the countably many Lipschitz maps whose images cover $\mathcal{H}^m$-a.e. of E. By Lemma 3.20, there exist Borel sets $E_0, E_1, \dots \subset \mathbb{R}^m$ such that

$$\mathbb{R}^m = \bigcup_{k=0}^{\infty} E_k$$

with $\mathcal{H}^m(\varphi(E_0)) = 0$ and for $k \geq 1$ the restriction $\varphi|_{E_k}$ is injective and there exists an injective linear map $L_k \colon \mathbb{R}^m \to \mathbb{R}^n$ such that

$$\varphi \circ (L_k|_{E_k})^{-1} \quad \text{and} \quad L_k \circ (\varphi|_{E_k})^{-1}$$

are both λ-Lipschitz.

Fix $k \geq 1$ and identify the subspace $L_k(\mathbb{R}^m)$ with $\mathbb{R}^m$ via a linear isometric map $T\colon \mathbb{R}^m \to L_k(\mathbb{R}^m)$. Set $A := T^{-1}(L_k(E_k))$, which is Borel. Then the map $\tilde{\varphi}_k\colon A \to \mathbb{R}^n$,

$$\tilde{\varphi}_k := \varphi \circ L^{-1} \circ T|_A$$

is λ-bi-Lipschitz. Moreover, the subset $A_k = \tilde{\varphi}_k^{-1}(E)$ is $\mathcal{L}^m$-measurable.

Doing this for all $k \geq 1$ and every φ we obtain the existence of countably many λ-bi-Lipschitz maps

$$\varphi_k\colon A_k \subset \mathbb{R}^m \to \mathbb{R}^n$$

with $\varphi_k(A_k) \subset E$ and $\mathcal{H}^m\big(E \setminus \bigcup \varphi_k(A_k)\big) = 0$. Now the proof of the theorem can be completed using the inner regularity of the Lebesgue measure.

□

A source of examples is given by the following result.

Theorem 3.32. Let $\varphi\colon \mathbb{R}^m \to \mathbb{R}^n$ be a Lipschitz map with $m > n$. Then for almost every $y \in \mathbb{R}^n$ the fiber $\varphi^{-1}(\{y\})$ is $(m-n)$-rectifiable.

Proof. Let B_k, u_k, v_k be as in Lemma 3.28. By the coarea formula,

$$0 = \int_{B_0} J_n(d\varphi_x)dx = \int_{\mathbb{R}^n} \mathcal{H}^{m-n}\big(B_0 \cap \varphi^{-1}(\{y\})\big)dy.$$

Thus, for almost every $y \in \mathbb{R}^n$,

$$\mathcal{H}^{m-n}\big(B_0 \cap \varphi^{-1}(\{y\})\big) = 0.$$

Fix such y. For $k \geq 1$, the map

$$\psi_k\colon u_k(B_k) \cap \{y\} \times \mathbb{R}^{m-n} \to B_k \cap \varphi^{-1}(\{y\})$$

given by

$$\psi_k(z) = v_k(y, z)$$

is Lipschitz and bijective since $v_k \circ u_k(x) = x$ for all $x \in B_k$ and $u_k(x) = (\varphi(x), p_k(x))$. By McShane's lemma there exists a Lipschitz extension $\bar{\psi}_k\colon \mathbb{R}^{m-n} \to \mathbb{R}^n$ of ψ_k. Clearly,

$$\begin{aligned} \varphi^{-1}(\{y\}) &= \varphi^{-1}(\{y\}) \cap \bigcup_{k=0}^{\infty} B_k \\ &= \Big(B_0 \cup \varphi^{-1}(\{y\})\Big) \cup \bigcup_{k\geq 1} \bar{\psi}_k(\mathbb{R}^{m-n}). \end{aligned}$$

Since $B_0 \cup \varphi^{-1}(\{y\})$ has $\mathcal{H}^{m-n}$-measure zero, $\varphi^{-1}(\{y\})$ is $(m-n)$-rectifiable.

□

By combining the previous two theorems we obtain

Theorem 3.33. Let $E \subset \mathbb{R}^N$ be n-rectifiable, $\mathcal{H}^n$-measurable and with $\mathcal{H}^n(E) < \infty$. Let $\varphi\colon \mathbb{R}^N \to \mathbb{R}^m$ be Lipschitz with $m < n$. Then for almost every $y \in \mathbb{R}^m$ the set

$$E \cap \varphi^{-1}(\{y\})$$

is $(m-n)$-rectifiable and $\mathcal{H}^{m-n}$-measurable.

We now show that m-rectifiable sets have approximate tangent planes almost everywhere. For $1 \le m < n$, let $G(n,m)$ denote the space of all m-dimensional linear subspace of $\mathbb{R}^n$. For $V \in G(n,m)$ and $x \in \mathbb{R}^n$ and $0 < s < 1$ we define a cone centered at x by

$$C(x,V,s) := \Big\{ y \in \mathbb{R}^n : d(y-x,V) < s \cdot |y-x| \Big\}.$$

Definition 3.34. Let $E \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$. An m-plane $V \in G(n,m)$ is said to be an approximate tangent m-plane of E at x if $\overline{\Theta}_m(E,x) > 0$ and

$$\lim_{r\to 0} \frac{1}{r^m} \mathcal{H}^m\big(E \cap B(x,r) \backslash C(x,V,s)\big) = 0$$

for every $0 < s < 1$.

The set of all approximate tangent m-planes of E at x is denoted by $\operatorname{apTan}^m(E,x)$. If there is only one, then we write $V = \operatorname{apTan}^m(E,x)$.

Theorem 3.35. Let $E \subset \mathbb{R}^n$ be $\mathcal{H}^m$-measurable and $\mathcal{H}^m(E) < \infty$. Then E is m-rectifiable if and only if for $\mathcal{H}^m$-a.e. $x \in E$ there exists a unique approximate tangent m-plane of E at x. In this case, $\Theta_m(E,x) = 1$ for $\mathcal{H}^m$-a.e. $x \in E$.

Proof. We only prove the implication $\Rightarrow$ and the second statement. For the reverse implication, we refer to Mattila's book [7].

Let $E \subset \mathbb{R}^n$ be m-rectifiable, $\mathcal{H}^m$-measurable and $\mathcal{H}^m(E) < \infty$. By Corollary 2.34,

$$\overline{\Theta}_m(E,z) \le 1$$

for $\mathcal{H}^m$-a.e. $z \in E$.

Let $\lambda > 1$. By Theorem 3.31, there exist compact sets $K_i \subset \mathbb{R}^m$, $i \in \mathbb{N}$ and λ-bi-Lipschitz maps $\varphi_i \colon K_i \to E$ such that

$$\mathcal{H}^m\big(E \backslash \bigcup_{i=1}^{\infty} \varphi_i(K_i)\big) = 0.$$

Fix i and write $\varphi = \varphi_i$ and $K = K_i$. Let $x \in K$ be a Lebesgue density point of K and such that (a Lipschitz extension of) φ is differentiable at x. Note that a.e. $x \in K$ has this property and that $d\varphi_x$ is injective.

Since φ is λ-bi-Lipschitz, it follows that

$$\varphi(K \cap B(x,r/\lambda)) \subset B(\varphi(x),r) \cap E$$

for all $r > 0$ and hence

$$\begin{aligned}\mathcal{H}^m\big(E \cap B(\varphi(x),r)\big) &\ge \mathcal{H}^m\big(\varphi(K \cap B(x,r/\lambda))\big)\\ &\ge \lambda^{-m}\mathcal{L}^m(K \cap B(x,r/\lambda)).\end{aligned}$$

Therefore,

$$\underline{\Theta}_m(E,\varphi(x)) \ge \lambda^{-2m}.$$

Now, let $0 < s < 1$ and set $V := d\varphi_x(\mathbb{R}^m)$. Then $\exists r_0 > 0$ such that

$$|\varphi(y) - \varphi(x) - d\varphi_x(y-x)| \le \frac{s}{2\lambda}|y-x|$$

for all $y \in K \cap B(x, \lambda r_0)$. Thus if $w = \varphi(y) \in \varphi(K) \cap B(\varphi(x), r)$, then $y \in K \cap B(x, \lambda r)$ and thus

$$|w - \varphi(x) - d\varphi_x(y - x)| \leq \frac{s}{2}|w - \varphi(x)|,$$

which implies $w \in C(\varphi(x), V, s) \cup \{\varphi(x)\}$. This shows that

$$\varphi(K) \cap B(\varphi(x), r) \subset C(\varphi(x), V, s) \cup \{\varphi(x)\}$$

for $r < r_0$. By Corollary 2.34, we have

$$\overline{\Theta}_m\big(E \backslash \varphi(K), \varphi(x)\big) = 0$$

for a.e. $x \in K$ and hence, since

$$\begin{aligned} &E \cap B(\varphi(x), r) \backslash C(\varphi(x), V, s) \\ &\quad \subset \big(E \backslash \varphi(K)\big) \cap B(\varphi(x), r) \cup \varphi(K) \cap B(\varphi(x), r) \backslash C(\varphi(x), V, s) \\ &\quad \subset \big(E \backslash \varphi(K)\big) \cap B(\varphi(x), r) \cup \{\varphi(x)\} \end{aligned}$$

for $0 < r < r_0$, we obtain

$$\lim_{r \to 0} \frac{1}{r^m} \mathcal{H}^m\big(E \cap B(\varphi(x), r) \backslash C(\varphi(x), V, s)\big) = 0.$$

Since $\lambda > 1$ and $i \in \mathbb{N}$ were arbitrary, it follows that at $\mathcal{H}^m$-a.e. $z \in E$, the set E has an approximate tangent m-plane and $\Theta_m(E, z) = 1$.

We leave it as an exercise to show that for a.e. x as above, the m-plane V is the unique approximate tangent m-plane of E at $\varphi(x)$.

□

It can be shown that the set $C \subset \mathbb{R}^2$ in Example 3.30 does not have an approximate tangent 1-plane a.e.

4. Review of differential forms

4.1. m-vectors and m-covectors. Let V be an n-dimensional vector space over $\mathbb{R}$. The space of 0-vectors and of 1-vectors are defined by $\Lambda_0 V := \mathbb{R}$ and $\Lambda_1 V := V$.

For $2 \leq m \leq n$, let $F_m(V)$ be the free vector space over the m-fold product $V \times \cdots \times V$, thus $F_m(V)$ consists of formal linear combinations

$$F_m(V) = \Big\{ \sum_{\text{finite}} \lambda_i (v_1^i, \ldots, v_m^i) \Big\}$$

with the obvious addition and scalar multiplication. Let $I_m(V)$ be the subspace of $F_m(V)$ generated by elements of the form

- $(v_1, \ldots, v_i + v_i', \ldots, v_m) - (v_1, \ldots, v_i, \ldots, v_m) - (v_1, \ldots, v_i', \ldots, v_m)$
- $(v_1, \ldots, \lambda v_i, \ldots, v_m) - \lambda(v_1, \ldots, v_i, \ldots, v_m)$
- $(v_1, \ldots, v_i, \ldots, v_i, \ldots, v_m)$

for $\lambda \in \mathbb{R}$ and $1 \leq i \leq m$.

The space of m-vectors in V is defined to be the quotient space

$$\Lambda_m V := F_m(V)/I_m(V)$$

and the equivalence class of $(v_1, \dots, v_m)$ is denoted by

$$v_1 \wedge \cdots \wedge v_m.$$

Elements of this form are called simple m-vectors.

Example 4.1. 1) If $1 \le i < j \le m$, then

$$v_1 \wedge \cdots \wedge v_i \cdots \wedge v_j \cdots \wedge v_m = -v_1 \wedge \cdots \wedge v_j \cdots \wedge v_i \cdots \wedge v_m.$$

2) $v_1, \dots, v_m$ are linearly dependent if and only if

$$v_1 \wedge \cdots \wedge v_m = 0$$

3) Let $\{e_1, e_2, e_3, e_4\} \subset \mathbb{R}^4$ be the standard basis. Then

$$e_1 \wedge e_2 + e_3 \wedge e_4$$

is not a simple 2-vector.

The space $\Lambda_m V$ has dimension

$$\dim \Lambda_m V = \binom{n}{m}.$$

If $\{e_1, \dots, e_n\}$ is a basis of V, then a basis of $\Lambda_m V$ is given by

$$\{e_\alpha = e_{\alpha(1)} \wedge \cdots \wedge e_{\alpha(m)} : \alpha \in \Lambda(n,m)\}.$$

Thus, every element in $\Lambda_m V$ can be written uniquely in the form

$$\sum_{\alpha \in \Lambda(n,m)} \lambda_\alpha e_\alpha$$

with $\lambda_\alpha \in \mathbb{R}$ and we can identify $\Lambda_m V$ with $\mathbb{R}^{\binom{n}{m}}$. If V carries an inner product and $\{e_1, \dots, e_n\}$ is an orthonormal basis of V, then we define an inner product on $\Lambda_m V$ in such a way that $\{e_\alpha : \alpha \in \Lambda(n,m)\}$ is an orthonormal basis.

If $1 \le m, l < n$ are such that $m + l \le n$, then the wedge product is the bilinear map

$$\big(\Lambda_m V\big) \times \big(\Lambda_l V\big) \to \Lambda_{m+l} V$$

given for simple m-vectors and l-vectors by

$$(v_1 \wedge \cdots \wedge v_m, w_1 \wedge \cdots \wedge w_l) = v_1 \wedge \cdots \wedge v_m \wedge w_1 \wedge \cdots \wedge w_l$$

and linear extension. If $u \in \Lambda_m V$ and $w \in \Lambda_l V$, then

$$u \wedge w = (-1)^{ml} w \wedge u.$$

Any linear map $T\colon V \to W$, where W is a finite dimensional vector space, extends to a linear map

$$\Lambda_m T\colon \Lambda_m V \to \Lambda_m W$$

by

$$\Lambda_m T(v_1 \wedge \cdots \wedge v_m) = T(v_1) \wedge \cdots \wedge T(v_m)$$

and linear extension.

We now define the space $\Lambda^m V$ of m-covectors in V. This space can be defined as the dual space of $\Lambda_m V$. However, we prefer to give an alternative but equivalent definition. Let V be an n-dimensional vector space over $\mathbb{R}$ and let

$$V^* := \{T\colon V \to R \text{ linear }\}$$

be its dual space.

Let $m \in \mathbb{N}$. A function $T\colon V \times \cdots \times V \to R$ is called

i) multilinear (or m-linear) if T is linear in each argument:

$$T(v_1, \cdots, v_i + \lambda v_i', \cdots, v_m) = T(v_1, \cdots, v_i, \cdots, v_m) + \lambda T(v_1, \cdots, v_i', \cdots, v_m)$$

ii) alternating if

$$T(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_m) = -T(v_1, \cdots, v_j, \cdots, v_i, \cdots, v_m)$$

for all $1 \le i < j \le m$.

Definition 4.2. The space $\Lambda^m V$ of m-covectors in V is the vector space

$$\Lambda^m V := \{T\colon V \times \cdots \times V \to \mathbb{R} \ m\text{-linear, alternating}\}$$

The determinant is the prime example of an n-covector.

Example 4.3. The map $\mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$ given by

$$(v_1, \cdots, v_n) \mapsto \det(v_1, \cdots, v_m)$$

is n-linear and alternating and this defines an element of $\Lambda^n \mathbb{R}^n$.

Note. $\Lambda^1 V = V^*$.

We now define the wedge product (or exterior product) of m-covectors: for $k \ge 2$ denote by S_k the group of permutations of $\{1, \dots, k\}$. Thus, an element of S_k is just a bijection $\{1, \dots, k\} \to \{1, \dots, k\}$. A permuation $\sigma \in S_k$ is called a transposition if there exist $1 \le i < j \le k$ such that $\sigma(i) = j$, $\sigma(j) = i$ and $\sigma(l) = l$ for all other l. Every permutation σ can be written as a finite number M of transpositions. The sign of σ is defined by

$$\text{Sign}(\sigma) = (-1)^M$$

and is well-defined.

Lemma 4.4. If $T \in \Lambda^m V$ and $\sigma \in S_m$, then for all $v_1, \dots, v_m \in V$,

$$T(v_{\sigma(1)}, \cdots, v_{\sigma(m)}) = \text{Sign}(\sigma) T(v_1, \cdots, v_m).$$

Proof. Exercise. □

Let $m, l \ge 1$. A permutation $\sigma \in S_{m \times l}$ is called an (m, l)-shuffe if

$$\sigma(1) < \sigma(2) < \cdots < \sigma(m) \quad \text{and} \quad \sigma(m+1) < \cdots < \sigma(m+l).$$

Definition 4.5. If $\alpha \in \Lambda^m V$ and $\beta \in \Lambda^l V$, then the wedge product (or exterior product) of α and β is defined by

$$\alpha \wedge \beta(v_1, \cdots, v_{m+l}) := \sum_{\sigma \ (m,l)\text{-shuffe}} \text{Sign}(\sigma) \alpha(v_{\sigma(1)}, \cdots, v_{\sigma(m)}) \beta(v_{\sigma(m+1)}, \cdots, v_{\sigma(m+l)}).$$

Example 4.6. $\xi_1, \xi_2 \in V^* = \Lambda^1 V \Rightarrow$

$$\begin{aligned}\xi_1 \wedge \xi_2(v_1, v_2) &= \xi_1(v_1)\xi_2(v_2) - \xi_1(v_2)\xi_2(v_1) \\ &= \det\begin{pmatrix}\xi_1(v_1) & \xi_1(v_2) \\ \xi_2(v_1) & \xi_2(v_2)\end{pmatrix}.\end{aligned}$$

The most important properties of the wedge product are

Proposition 4.7. Let $\alpha \in \Lambda^m V$, $\beta \in \Lambda^l V$, and $\gamma \in \Lambda^k V$. Then

i) $\alpha \wedge \beta \in \Lambda^{m+l} V$ and

$$\alpha \wedge \beta = (-1)^{ml} \beta \wedge \alpha$$

ii) The wedge product is bilinear

iii) The wedge product is associative:

$$(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \alpha).$$

Proof. See e.g. [6]. □

Proposition 4.8. If $\xi_1, \cdots, \xi_m \in V^*$, then for all $v_1, \cdots, v_m \in V$

$$(\xi_1 \wedge \cdots \wedge \xi_m)(v_1, \cdots, v_m) = \det(\xi_i(v_j)).$$

In particular, from the definition of the determinant we obtain

$$(\xi_1 \wedge \cdots \wedge \xi_m)(v_1, \cdots, v_m) = \sum_{\sigma \in S_m} \mathrm{Sign}(\sigma) \xi_1(v_{\sigma(1)}) \cdots \xi_m(v_{\sigma(m)}).$$

Proof. By induction on m; see e.g. [6]. □

Let $\{e_1, \cdots, e_n\} \subset V$ be a basis of V and let $\{e_1^*, \cdots, e_n^*\} \subset V^*$ be the dual basis, thus

$$e_i^*(e_j) = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{otherwise} \end{cases}$$

Proposition 4.9. The space $\Lambda^m V$ has dimension $\binom{n}{m}$ and a basis is given by

$$\{e_\alpha^* := e_{\alpha(1)}^* \wedge \cdots \wedge e_{\alpha(m)}^* : \alpha \in \Lambda(n, m)\}.$$

Proof. See e.g. [6]. □

Remark 4.10. Any $\omega \in \Lambda^m V$ can be written uniquely as

$$\omega = \sum_{\alpha \in \Lambda(n,m)} \omega_\alpha e_\alpha^*$$

and the ω_α's are given by

$$\omega_\alpha = \omega(e_{\alpha(1)}, \cdots, e_{\alpha(m)}).$$

If V contains an inner product and $\{e_1, \cdots, e_n\}$ is an orthonormal basis, then we can endow $\Lambda^m V$ with an inner product such that the e_α^* are orthonormal.

There exists a natural isomorphism

$$\Lambda^m V \to (\Lambda_m V)^*$$

defined as follows. Given $\omega \in \Lambda^m V$ define an element $\hat\omega$ of $(\Lambda_m V)^*$ by

$$\hat\omega(v_1 \wedge \cdots \wedge v_m) := \omega(v_1, \cdots, v_m)$$

and linear extension. It can easily be checked that $\hat\omega$ is well-defined and that the map $\omega \mapsto \hat\omega$ is an isomorphism. We will write

$$\langle \omega, \tau \rangle := \hat\omega(\tau)$$

for $\tau \in \Lambda_m V$.

Suppose now that V carries an inner product $\langle \cdot, \cdot \rangle$. Then the map

$$R \colon V \to V^*$$

$$v \mapsto \langle v, \cdot \rangle$$

is an isomorphism. If $\{e_1, \cdots, e_n\} \subset V$ is an orthonormal basis of V and $\{e_1^*, \cdots, e_n^*\} \subset V^*$ is the dual basis, then

$$R(e_i) = e_i^* \quad \forall i = 1, \cdots, n.$$

The map R induces an isomorphism

$$R_m \colon \Lambda_m V \to \Lambda^m V$$

by

$$R_m\Big(\sum_{\alpha \in \Lambda(n,m)} v_\alpha e_\alpha\Big) := \sum_{\alpha \in \Lambda(n,m)} v_\alpha e_\alpha^*$$

which satisfies

$$R_m(v_1 \wedge \cdots \wedge v_m) = R(v_1) \wedge \cdots \wedge R(v_m)$$

for all $v_1, \cdots, v_m \in V$.

The inner product on $\Lambda_m V$ for which $\{e_\alpha : \alpha \in \Lambda(n,m)\}$ is an orthonormal basis satisfies

$$\langle v_1 \wedge \cdots \wedge v_m, \omega_1 \wedge \cdots \wedge \omega_m \rangle = \langle R_m(v_1 \wedge \cdots \wedge v_m), \omega_1 \wedge \cdots \wedge \omega_m \rangle, \tag{4.1}$$

where on the right-hand side, $\langle \cdot, \cdot \rangle$ denotes the dual pairing.

Exercise: prove (4.1).

Similarly, the inner product on $\Lambda^m V$ for which $\{e_\alpha^* : \alpha \in \Lambda(n,m)\}$ is an orthonormal basis satisfies

$$\langle \xi \wedge \cdots \wedge \xi_m, \eta_1 \wedge \cdots \wedge \eta_m \rangle = \langle \xi_1 \wedge \cdots \wedge \xi_m, R_m^{-1}(\eta_1 \wedge \cdots \wedge \eta_m) \rangle.$$

We now specialize to $V = \mathbb{R}^n$ and let $\langle \cdot, \cdot \rangle$ be the standard Euclidean inner product and $\{e_1, \cdots, e_n\}$ the standard basis. We denote by $|\cdot|$ the norms on $\Lambda_m \mathbb{R}^n$ and $\Lambda^m \mathbb{R}^n$ coming from the inner products induced as above.

Lemma 4.11. If $T \colon \mathbb{R}^m \to \mathbb{R}^n$ is linear, then

$$|T(e_1) \wedge \cdots \wedge T(e_m)| = J_m(T).$$

In particular, $|T(e_1) \wedge \cdots \wedge T(e_m)|$ is just the Hausdorff m-measure of the parallelepiped spanned by the vectors $T(e_1), \cdots, T(e_m)$.

Proof. By the formula (4.1) and Proposition 4.8, we have

$$\begin{aligned}|T(e_1)\wedge\cdots\wedge T(e_m)|^2 &= \langle T(e_1)\wedge\cdots\wedge T(e_m), T(e_1)\wedge\cdots\wedge T(e_m)\rangle \\ &= \langle R\big(T(e_1)\wedge\cdots\wedge T(e_m)\big), T(e_1)\wedge\cdots\wedge T(e_m)\rangle \\ &= \det\big(\langle T(e_i), T(e_j)\rangle\big) = \det(T^T T) = J_m(T)^2.\end{aligned}$$

□

4.2. **Differential forms and Stokes' theorem.** Let $U\subset\mathbb{R}^n$ be open and $m\geq 0$. The space of smooth differential m-forms on U is denoted by

$$\mathcal{E}^m(U) := C^\infty(U, \Lambda^m\mathbb{R}^n).$$

Every element $\omega\in\mathcal{E}^m(U)$ can be uniquely written as

$$\omega = \sum_{\alpha\in\Lambda(n,m)} \omega_\alpha dx^\alpha$$

for some functions $\omega_\alpha\in C^\infty(U)$. Here, $\{dx^1,\cdots,dx^n\}$ denotes the dual basis of $\{e_1,\cdots,e_n\}$ and

$$dx^\alpha := dx^{\alpha(1)}\wedge\cdots\wedge dx^{\alpha(m)}.$$

Notice that

- $\mathcal{E}^0(U) = C^\infty(U)$.
- $\mathcal{E}^1(U) = \Big\{\sum_{i=1}^n f_i dx^i : f_i\in C^\infty(U)\Big\}$
- $\mathcal{E}^n(U) = \Big\{f dx^1\wedge\cdots\wedge dx^n : f\in C^\infty(U)\Big\}$.

The space of compactly supported smooth differential m-forms in U is denoted by

$$\mathcal{D}^m(U) = C_c^\infty(U, \Lambda^m\mathbb{R}^n)\subset\mathcal{E}^m(U).$$

By definition, the support of $\omega\in\mathcal{E}^m(U)$ is

$$\operatorname{spt}(\omega) := \overline{\{x\in U : \omega(x)\neq 0\}}\cap U.$$

Definition 4.12. The exterior derivative of $\omega\in\mathcal{E}^m(U)$ is the form $d\omega\in\mathcal{E}^{m+1}(U)$ defined by

$$d\omega := \sum_{\alpha\in\Lambda(n,m)}\sum_{i=1}^n \frac{\partial\omega_\alpha}{\partial x_i^\alpha} dx^i\wedge dx^\alpha,$$

where $\omega = \sum_{\alpha\in\Lambda(n,m)}\omega_\alpha dx^\alpha$.

For example, if $f\in C^\infty(U) = \mathcal{E}^0(U)$, then

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx^i.$$

The following properties are obtained by a direct calculation.

Proposition 4.13. If $\omega\in\mathcal{E}^m(U)$, $\nu\in\mathcal{E}^k(U)$, and $f\in\mathcal{E}^0(U)$, then

(i) $d(d\omega) = 0$
(ii) $d(f\omega) = df\wedge\omega + f d\omega$

(iii) $d(\omega \wedge \nu) = (d\omega) \wedge \nu + (-1)^m \omega \wedge (d\nu)$

Proof. Exercise. □

Notice that the wedge product $\omega \wedge \nu$ is defined pointwise:

$$(\omega \wedge \nu)(x) := \omega(x) \wedge \nu(x).$$

Notice also that

$$\operatorname{spt}(d\omega) \subset \operatorname{spt}(\omega)$$

and that the inclusion can be strict.

Definition 4.14. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^N$ be open and $\varphi\colon U \to V$ smooth. The pull-back of $\omega \in \mathcal{E}^m(V)$ under φ is the differential form $\varphi^*\omega \in \mathcal{E}^m(U)$ defined by

$$\big(\varphi^*\omega\big)(x)(v_1, \cdots, v_m) := \omega(\varphi(x))(d\varphi_x(v_1), \cdots, d\varphi_x(v_m))$$

for all $x \in U$ and $v_1, \cdots, v_m \in \mathbb{R}^n$.

If we denote

- $\{dx^1, \cdots, dx^n\}$ the dual basis of the standard basis in $\mathbb{R}^n$
- $\{dy^1, \cdots, dy^N\}$ the dual basis of the standard basis in $\mathbb{R}^N$
- $\varphi = (\varphi_1, \cdots, \varphi_N)$
- $\omega = \sum_{\alpha \in \Lambda(N,m)} \omega_\alpha dy^\alpha$

then

$$\varphi^*\omega = \sum_{\beta \in \Lambda(n,m)} \sum_{\alpha \in \Lambda(N,m)} \omega_\alpha \circ \varphi \det\big(\frac{\partial \varphi_{\alpha(i)}}{\partial x_{\beta(j)}}\big) dx^\beta.$$

Indeed, for $x \in U$ and $\beta \in \Lambda(n,m)$, we have

$$\begin{aligned}
(\varphi^*\omega)(x)(e_{\beta(1)}, \cdots, e_{\beta(m)}) &= \omega(\varphi(x))\big(\frac{\partial \varphi}{\partial x_{\beta(1)}}(x), \cdots, \frac{\partial \varphi}{\partial x_{\beta(m)}}(x)\big) \\
&= \sum_{\alpha \in \Lambda(N,m)} \omega_\alpha(\varphi(x)) dx^\alpha\big(\frac{\partial \varphi}{\partial x_{\beta(1)}}(x), \cdots, \frac{\partial \varphi}{\partial x_{\beta(m)}}(x)\big) \\
&= \sum_\alpha \sum_{\sigma \in S_m} \omega_\alpha(\varphi(x)) \operatorname{Sign}(\sigma) \frac{\partial \varphi_{\alpha(\sigma(1))}}{\partial x_{\beta(1)}}(x) \cdots \frac{\partial \varphi_{\alpha(\sigma(m))}}{\partial x_{\beta(m)}}(x) \\
&= \sum_\alpha \omega_\alpha \circ \varphi(x) \det\big(\frac{\partial \varphi_{\alpha(i)}}{\partial x_{\beta(j)}}(x)\big).
\end{aligned}$$

Notice that $\varphi^*\omega$ need not have compactly supported, even if ω has compact support.

A direct calculation shows.

Proposition 4.15. If $\omega \in \mathcal{E}^m(V)$ and $\nu \in \mathcal{E}^k(V)$, then

(i) $\varphi^*(\omega \wedge \nu) = (\varphi^*\omega) \wedge (\varphi^*\nu)$
(ii) $d(\varphi^*\omega) = \varphi^*(d\omega)$

Proof. Exercise. □

We briefly recall Stokes' theorem. For this, let $M \subset U \subset \mathbb{R}^n$ be a smooth oriented m-dimensional submanifold and $\{(V_\lambda, \psi_\lambda)\}$ an oriented atlas, that is,

$$\det\big(d(\varphi_{\lambda'} \circ \psi_\lambda^{-1})\big) > 0 \quad \forall x, \lambda, \lambda'.$$

Lemma 4.16. If $z \in V_\lambda \cap V_{\lambda'}$, then with $x := \psi_\lambda(z)$ and $x' := \psi_{\lambda'}(z)$, we have

$$\frac{\frac{\partial \psi_\lambda^{-1}}{\partial x_1}(x) \wedge \cdots \wedge \frac{\partial \psi_\lambda^{-1}}{\partial x_m}(x)}{\left|\frac{\partial \psi_\lambda^{-1}}{\partial x_1}(x) \wedge \cdots \wedge \frac{\partial \psi_\lambda^{-1}}{\partial x_m}(x)\right|} = \frac{\frac{\partial \psi_{\lambda'}^{-1}}{\partial x_1}(x') \wedge \cdots \wedge \frac{\partial \psi_{\lambda'}^{-1}}{\partial x_m}(x')}{\left|\frac{\partial \psi_{\lambda'}^{-1}}{\partial x_1}(x') \wedge \cdots \wedge \frac{\partial \psi_{\lambda'}^{-1}}{\partial x_m}(x')\right|}.$$

Proof. Since $\psi_\lambda^{-1} = \psi_{\lambda'}^{-1} \circ (\psi_{\lambda'} \circ \psi_\lambda^{-1})$, we have, with $\rho := \psi_{\lambda'} \circ \psi_\lambda^{-1}$,

$$\frac{\partial \psi_\lambda^{-1}}{\partial x_i}(x) = \sum_{j=1}^m \frac{\partial \psi_{\lambda'}^{-1}}{\partial x_j}(x') \frac{\partial \rho_j}{\partial x_i}(x).$$

It follows that

$$\frac{\partial \psi_\lambda^{-1}}{\partial x_1}(x) \wedge \cdots \wedge \frac{\partial \psi_\lambda^{-1}}{\partial x_m}(x) = \det(d\rho_x) \frac{\partial \psi_{\lambda'}^{-1}}{\partial x_1}(x') \wedge \cdots \wedge \frac{\partial \psi_{\lambda'}^{-1}}{\partial x_m}(x')$$

and hence the lemma. □

It follows from the lemma that the m-vector field

$$\tau_M \colon M \to \Lambda_m \mathbb{R}^n$$

given by

$$\tau_M(z) := \frac{\frac{\partial \psi_\lambda^{-1}}{\partial x_1}(x) \wedge \cdots \wedge \frac{\partial \psi_\lambda^{-1}}{\partial x_m}(x)}{\left|\frac{\partial \psi_\lambda^{-1}}{\partial x_1}(x) \wedge \cdots \wedge \frac{\partial \psi_\lambda^{-1}}{\partial x_m}(x)\right|},$$

where $x = \psi_\lambda(z)$ is well-defined. We call τ_M the orientation of M.

We can now rewrite the classically defined integral

$$\int_M \omega$$

for $\omega \in \mathcal{D}^m(U)$ as follows.

Proposition 4.17. For every $\omega \in \mathcal{D}^m(U)$, we have

$$\int_M \omega = \int_M \langle \omega(x), \tau_M(x) \rangle d\mathcal{H}^m(x).$$

For the definition of $\int_M \omega$, see [3, 6] or the proof below.

Proof. If $\operatorname{spt}(\omega) \cap M \subset V_\lambda$ for some λ, then, by definition of $\int_M \omega$, we have with $\varphi := \psi_\lambda^{-1}$ and $W := \psi_\lambda(V_\lambda)$,

$$\begin{aligned}\int_M \omega = \int_W \varphi^*\omega &\overset{\text{definition}}{=} \int_W (\varphi^*\omega)(x)(e_1, \cdots, e_m)dx \\ &= \int_W \omega(\varphi(x))\big(\frac{\partial \varphi}{\partial x_1}(x), \cdots, \frac{\partial \varphi}{\partial x_m}(x)\big)dx \\ &\overset{\text{Lemma 4.16}}{=} \int_W J_m(d\varphi_x)\langle \omega(\varphi(x)), \tau_M \circ \varphi^{-1}(x)\rangle dx \\ &\overset{\text{Theorem 3.17}}{=} \int_{V_\lambda} \langle \omega(x), \tau_M(x)\rangle d\mathcal{H}^m(x).\end{aligned}$$

Now the proof follows with a partition of unity argument. □

If M is a smooth m-dimensional submanifold with boundary ∂M, then ∂M is a smooth $(m-1)$-dimensional submanifold without boundary. An orientation on ∂M can be given.

Theorem 4.18 (Stokes' theorem)**.** If $M \subset U$ is smooth oriented m-dimensional submanifold with boundary and $\omega \in \mathcal{D}^{m-1}(U)$, then

$$\int_M d\omega = \int_{\partial M} \omega.$$

We will need a topology on $\mathcal{D}^m(U)$. For this, we first define a topology on $\mathcal{E}^m(U)$. For each $K \subset U$ compact and each $i \geq 0$ define a semi-norm on $\mathcal{E}^m(U)$ by

$$v_K^i(\omega) := \sup\Big\{\big|\frac{\partial^j \omega_\alpha}{\partial x_{k_1} \cdots \partial x_{k_j}}(x)\big| : x \in K, \alpha \in \Lambda(n,m), 0 \leq j \leq i, k_1, \cdots, k_j \in \{1, \cdots, n\}\Big\},$$

where we have written $\omega = \sum \omega_\alpha dx^\alpha$.

By definition, a subset of $\mathcal{E}^m(U)$ is open if it is the union of finite intersections of sets of the form

$$V(\eta, K, i, r) = \Big\{\eta + \omega : \omega \in \mathcal{E}^m(U), v_K^i(\omega) < r\Big\}$$

for $\eta \in \mathcal{E}^m(U)$ and $K \subset U$ compact, $i \geq 0$, $r > 0$. This defines a topology on $\mathcal{E}^m(U)$.

Now we define a topology on $\mathcal{D}^m(U)$ as follows: a subset $\Gamma \subset \mathcal{D}^m(U)$ is called open if for every $K \subset U$ compact the set

$$\Gamma \cap \Big\{\omega \in \mathcal{E}^m(U) : \operatorname{spt}(\omega) \subset K\Big\}$$

is open in $\{\omega \in \mathcal{E}^m(U) : \operatorname{spt}(\omega) \subset K\}$ in the relative topology in $\mathcal{E}^m(U)$.

Example 4.19. Recall that $\mathcal{D}^0(\mathbb{R}) = C_c^\infty(\mathbb{R})$. The subset

$$\Gamma := \Big\{f \in C_c^\infty(\mathbb{R}) : |f(x)| < \frac{1}{k} \quad \forall x \in [k, k+1], \forall k \in \mathbb{N}\Big\}$$

is open with respect to the topology on $\mathcal{D}^0(\mathbb{R})$.

5. The theory of currents in Euclidean space

5.1. **Definitions and examples.** Let $U \subset \mathbb{R}^n$ be open and $m \geq 0$.

Definition 5.1. An m-current on U is a continuous linear functional on $\mathcal{D}^m(U)$. The space of m-currents on U is denoted by $\mathcal{D}_m(U)$.

A 0-current is also called a distribution.

It follows from the definition of the topology on $\mathcal{D}^m(U)$ that a linear functional $T\colon \mathcal{D}^m(U) \to \mathbb{R}$ is an m-current on U if and only if for every $K \subset U$ compact there exist $M \geq 0$ and $i \geq 0$ such that

$$|T(\omega)| \leq M v_K^i(\omega)$$

for all $\omega \in \mathcal{D}^m(U) \cap \{\eta : \operatorname{spt}(\eta) \subset K\}$.

We first give some examples which will appear later again.

Example 5.2. (1) Let $x \in \mathbb{R}^n$. Then the functional

$$[x](f) := f(x)$$

defines a 0-current on $\mathbb{R}^n$.

(2) Let $a, b \in \mathbb{R}$ with $a < b$. Then the functional

$$[a, b](f dx) := \int_a^b f(x) dx$$

defines a 1-current on $\mathbb{R}$.

(3) Let $M \subset \mathbb{R}^n$ be a smooth, oriented m-dimensional submanifold. Then the functional

$$[M](\omega) := \int_M \omega = \int_M \langle \omega(x), \tau_M(x) \rangle d\mathcal{H}^m(x)$$

defines an m-current on $\mathbb{R}^n$.

(4) The functional on $\mathcal{D}^1(\mathbb{R}^2)$ given by

$$T(\omega) := \int_0^1 \omega(s, 0)(e_2) ds$$

defines a 1-current on $\mathbb{R}^2$.

(5) If $\Theta \in L^1(U)$, then the functional

$$[\Theta](f dx^1 \wedge \cdots \wedge dx^n) := \int_U f \Theta d\mathcal{L}^n$$

defines an n-current on $U \subset \mathbb{R}^n$.

(6) If $T \in \mathcal{D}_m(U)$ and $v \in \mathcal{D}^k(U)$ with $k \leq m$, then the functional on $\mathcal{D}^{m-k}(U)$ defined by

$$(T \llcorner v)(\omega) := T(v \wedge \omega)$$

is an $(m-k)$-current on U, called the restriction on T to v.

Inspired by Stokes' theorem, we define the boundary of a current as follows.

Definition 5.3. The boundary of a current $T \in \mathcal{D}^m(U)$ is the current $\partial T \in \mathcal{D}_{m-1}(U)$ defined by

$$\partial T(\omega) := T(d\omega)$$

for $\omega \in \mathcal{D}^{m-1}(U)$. If $m = 0$, then $\partial T = 0$ by definition.

It follows from the characterization of continuity given after the definition of a current and the fact that

$$v_k^i(d\omega) \leq v_K^{i+1}(\omega)$$

the functional ∂T is continuous and hence an $(m-1)$-current. If $m \geq 1$, then $\partial(\partial T) = 0$ because $d(d\omega) = 0$. We compute the boundary of some of the examples given above:

Example 5.4. (1) If $a, b \in \mathbb{R}$ with $a < b$, then

$$\partial[a, b] = [b] - [a]$$

because

$$\begin{aligned}\partial[a, b](f) = [a, b](f' dx) &= \int_a^b f'(x) dx \\ &= f(b) - f(a) = [b](f) - [a](f).\end{aligned}$$

(2) If $M \subset \mathbb{R}^n$ is a smooth, oriented m-dimensional submanifold with (possibly empty) boundary, then

$$\partial[M] = [\partial M].$$

Indeed, by Stokes' theorem, we have for $\omega \in \mathcal{D}^{m-1}(\mathbb{R}^n)$,

$$\partial[M](\omega) = [M](d\omega) = \int_M d\omega = \int_{\partial M} \omega = [\partial M](\omega).$$

(3) If $T \in \mathcal{D}_1(\mathbb{R}^2)$ is the current given by

$$T(\omega) = \int_0^1 \omega(s, 0)(e_2) ds,$$

then

$$\partial T(f) = T\big(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy\big) = \int_0^1 \frac{\partial f}{\partial y}(s, 0) ds$$

for all $f \in \mathcal{D}^0(\mathbb{R}^2)$.

Definition 5.5. The support of a current $T \in \mathcal{D}_m(U)$ is defined by

$$\mathrm{spt}(T) := U \backslash \Big\{ V \subset \mathbb{R}^n : V \text{ open such that } T(\omega) = 0 \ \forall \omega \in \mathcal{D}^m(U) \ \text{ with } \mathrm{spt}(\omega) \subset V \Big\}.$$

In particular, if $\omega \in \mathcal{D}^m(U)$ satisfies $\mathrm{spt}(\omega) \cap \mathrm{spt}(K) = \emptyset$, then $T(\omega) = 0$.

We now define the mass of a current, which can be thought of as its "volume". We first define a norm on $\mathcal{D}^m(U)$.

Definition 5.6. The comass of $\omega \in \Lambda^m \mathbb{R}^n$ is defined by

$$\|\omega\| := \sup \Big\{ \langle \omega, \tau \rangle : \tau \in \Lambda_m \mathbb{R}^n \text{ simple }, |\tau| \leq 1 \Big\}$$

and the comass of $\omega \in \mathcal{D}^m(U)$ is defined by

$$\|\omega\| := \sup \Big\{ \|\omega(x)\| : x \in U \Big\}.$$

Notice that for $\omega \in \Lambda^m \mathbb{R}^n$ we have

$$\|\omega\| = \sup \Big\{ \omega(v_1, \cdots, v_m) : |v_i| \leq 1 \Big\}$$

and $\|\omega\| \le |\omega|$.

Definition 5.7. The mass of a linear functional $T\colon \mathcal{D}^m(U) \to \mathbb{R}$ is

$$M(T) := \sup\Big\{T(\omega) : \omega \in \mathcal{D}^m(U), \|\omega\| \le 1\Big\}.$$

If a linear functional $T\colon \mathcal{D}^m(U) \to \mathbb{R}$ satisfies $M(T) < \infty$, then we have

$$|T(\omega)| \le M(T)\|\omega\|$$

for all $\omega \in \mathcal{D}^m(U)$ and hence $T \in \mathcal{D}_m(U)$ by the characterization of continuity given after the definition of a current.

Remark 5.8. A somewhat different mass is defined by

$$\underline{M}(T) := \sup\Big\{T(\omega)\colon \omega \in \mathcal{D}^m(U), |w| \le 1\Big\},$$

where $|\omega| = \sup\{|\omega(x)| : x \in U\}$. This is sometimes called the Euclidean mass of T. Since $\|\omega\| \le |\omega|$, we obtain

$$\underline{M}(T) \le M(T).$$

It is not difficult to see that there exists $c = c(n)$ such that

$$cM(T) \le \underline{M}(T) \le M(T).$$

The space

$$M_m(U) = \Big\{T\colon \mathcal{D}^m(U) \to \mathbb{R} : T \text{ linear }, M(T) < \infty\Big\}$$

is called the space of m-currents on U with finite mass.

Clearly, the mass defines a norm on $M_m(U)$ and $M_m(U)$ with this norm is exactly the dual space of the normed space $(\mathcal{D}^m(U), \|\cdot\|)$. In particular, $M_m(U)$ is a Banach space.

Example 5.9. If $M \subset \mathbb{R}^n$ is a smooth, compact, oriented m-dimensional submanifold, then $[M] \in M_m(\mathbb{R}^n)$ and

$$M([M]) = \mathrm{Vol}(M) = \mathcal{H}^m(M).$$

Indeed, let $\tau_M\colon M \to \Lambda_m\mathbb{R}^n$ be the orienting m-vector field of M and let $\omega \in \mathcal{D}^m(\mathbb{R}^n)$ with $\|\omega\| \le 1$. Then

$$|[M](\omega)| \le \int_M |\langle\omega(x), \tau_M(x)\rangle| d\mathcal{H}^m(x) \le \mathcal{H}^m(M)\|\omega\| = \mathcal{H}^m(M)$$

and so $M([M]) \le \mathcal{H}^m(M)$. It is not difficult to see that equality holds and this will also follow from the theorem below.

The following example shows that if $T \in M_m(U)$, then, in general, we need not have $\partial T \in M_{m-1}(U)$.

Define $T \in M_1(\mathbb{R}^2)$ by

$$T(\omega) := \int_0^1 \omega(s,0)(e_2)ds$$

and notice that $M(T) = 1$. However, since

$$\partial T(f) = \int_0^1 \frac{\partial f}{\partial y}(s,0)ds,$$

we clearly have $M(\partial T) = \infty$. Notice, however, that if in the definition of T above, the vector e_2 is replaced by e_1, then

$$\partial T = [(1,0)] - [(0,0)],$$

and so $M(\partial T) = 2$ in this case.

We call elements of the space

$$N_m(U) := \Big\{ T \in M_m(U) : \partial T \in M_{m-1}(U) \Big\}$$

normal m-currents in U. We define $N_0(U) := M_0(U)$. We have the inclusions

$$N_m(U) \subset M_m(U) \subset \mathcal{D}_m(U)$$

and these are strict by the example above if $m > 0$.

We now use the Riesz representation theorem to show that currents of finite mass can be represented by integration. We first define a norm on $\Lambda_m \mathbb{R}^n$ by

$$\|\tau\| := \sup\{\langle \omega, \tau \rangle : \omega \in \Lambda^m \mathbb{R}^n, \|\omega\| \leq 1\},$$

which is called the mass of τ. Notice that if $\tau \in \Lambda_m \mathbb{R}^n$, then

$$|\tau| = \sup\{\langle \omega, \tau \rangle : \omega \in \Lambda^m \mathbb{R}^n, |\omega| \leq 1\} \leq \|\tau\|$$

and $|\tau| = \|\tau\|$ if τ is simple.

Theorem 5.10 (Representation theorem)**.** If $T \in M_m(U)$, then there exist a unique finite Radon measure $\|T\|$ on U and a $\|T\|$-measurable m-vector field $\overrightarrow{T} : U \to \Lambda_m \mathbb{R}^n$ with $\|\overrightarrow{T}(x)\| = 1$ for $\|T\|$-almost every $x \in U$ and such that

$$T(\omega) = \int_U \langle \omega(x), \overrightarrow{T}(x) \rangle d\|T\|(x)$$

for every $\omega \in \mathcal{D}^m(U)$. Moreover, for every $W \subset U$ open, we have

$$\|T\|(W) = \sup\{T(\omega) : \omega \in \mathcal{D}^m(U), \operatorname{spt}(\omega) \subset W, \|\omega\| \leq 1\}$$

and, in particular, $M(T) = \|T\|(U)$. Notice that

$$\operatorname{spt}(T) = U \cap \operatorname{spt}(\|T\|),$$

where the support of a measure μ is defined by

$$\operatorname{spt}(\mu) = \{x \in \mathbb{R}^n : \mu(B(x,r)) > 0 \quad \forall r > 0\}.$$

Conversely, every finite Radon measure μ on U and every μ-measurable $\tau : U \to \Lambda_m \mathbb{R}^n$ with $\|\tau(x)\| = 1$ μ-a.e. $x \in U$ give rise to an element $T \in M_m(U)$ by

$$T(\omega) = \int_U \langle \omega(x), \tau(x) \rangle d\mu(x)$$

for all $\omega \in \mathcal{D}^m(U)$. One often writes $\mu \wedge \tau := T$.

Proof. Since $\mathcal{D}^m(U)$ is dense in $(C_c(U, \Lambda^m(\mathbb{R}^n), \|\cdot\|))$ and since T is continuous with respect to $\|\cdot\|$, it follows that T has a unique extension to $C_c(U, \Lambda^m \mathbb{R}^n)$ which satisfies

$$|T(\omega)| \leq M(T)\|\omega\|$$

for all $\omega \in C_c(U, \Lambda^m(\mathbb{R}^n))$.

We now apply the Riesz representation theorem with $X = U$ and $H = (\Lambda^m\mathbb{R}^n, \langle\cdot,\cdot\rangle)$ to find a Radon measure μ_T on U and $\tau\colon U \to \Lambda^m\mathbb{R}^n$ μ_T-measurable with $|\tau(x)| = 1$ μ_T-a.e. $x \in U$ such that

$$T(\omega) = \int_U \langle\omega(x), \tau(x)\rangle d\mu_T(x)$$

for all $\omega \in C_c(U, \Lambda^m\mathbb{R}^n)$. We define

$$\overrightarrow{T}(x) := \frac{1}{|\tau(x)|} R_m^{-1}(\tau(x))$$

and

$$d\|T\| = \|\tau(\cdot)\| d\mu_T.$$

□

Using the representation theorem above, we can define the restriction of a current of finite mass to a Borel set.

Definition 5.11. Let $T \in M_m(U)$ and let $f\colon U \to \mathbb{R}$ be a bounded Borel function. The restriction of T to f is the current $T \llcorner f$ defined by

$$(T \llcorner f)(\omega) := \int_U f(x)\langle\omega(x), \overrightarrow{T}(x)\rangle d\|T\|(x)$$

for all $\omega \in \mathcal{D}^m(U)$. We write $T \llcorner B := T \llcorner \chi_B$ whenever $B \subset U$ is a Borel set.

A sequence $(T_k) \subset M_m(U)$ is said to converge in mass to $T \in M_m(U)$ if $M(T - T_k) \to 0$. This is just norm convergence, which is often too strong. For example, the sequence (T_k) given by $T_k := [[0,1] \times \{\frac{1}{k}\}] \in M_1(\mathbb{R}^2)$ does not converge in mass, however, it converges weakly to $T = [[0,1] \times \{0\}]$ in the following sense.

Definition 5.12. A sequence $(T_k) \subset \mathcal{D}_m(U)$ is said to converge weakly to $T \in \mathcal{D}_m(U)$ if $T_k(\omega) \to T(\omega)$ for every $\omega \in \mathcal{D}^m(U)$. We write $T_m \rightharpoonup T$.

Notice that if $T, T_k \in M_m(U)$, then this is just the weak-* convergence in the dual space $(M_m(U), M(\cdot))$. A simple but important property.

Theorem 5.13 (Lower semicontinuity of mass)**.** If a sequence $(T_k) \subset \mathcal{D}_m(U)$ converges weakly to $T \in \mathcal{D}_m(U)$, then

$$M(T) \leq \liminf_{k\to\infty} M(T_k).$$

Proof. For every $\omega \in \mathcal{D}^m(U)$ with $\|\omega\| \leq 1$ we have

$$T(\omega) = \lim_{k\to\infty} T_k(\omega) \leq \liminf_{k\to\infty} M(T_k)$$

and hence $M(T) \leq \liminf_{k\to\infty} M(T_k)$. □

We can now solve Plateau's problem in a very weak sense.

Theorem 5.14. Let $S \in N_m(U)$. Then there exists $T \in N_m(U)$ such that $\partial T = \partial S$ and

$$M(T) = \inf\{M(S') : S' \in N_m(U), \partial S' = \partial S\}.$$

Thus T is a mass minimizing normal current with boundary equal to ∂S. This theorem is not satisfying because normal m-currents are in general very far from m-dimensional submanifolds as the following example shows.

Example 5.15. The 1-current on $\mathbb{R}^2$ given by

$$T(\omega) := \int_{[0,1]^2} \omega(x)(e_1) d\mathcal{L}^2(x)$$

is a normal current, that is, $T \in N_1(\mathbb{R}^2)$. Clearly, $M(T) = 1$ and since

$$\begin{aligned}\partial T(f) = T(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy) &= \int_0^1 \int_0^1 \frac{\partial f}{\partial x}(x,y)dxdy \\ &= \int_0^1 f(1,y) - f(0,y)dy,\end{aligned}$$

we have $M(\partial T) = 2 < \infty$.

Proof. Let $(T_k) \subset N_m(U)$ be a mass minimizing sequence with $\partial T_k = \partial S$ for all k. Thus

$$M(T_k) \to L := \inf\big\{M(S') : S' \in N_m(U), \partial S' = \partial S\big\}.$$

By the Banach-Alaoglu Theorem 2.51, there exists a subsequence (T_{k_j}) and $T \in N_m(U)$ such that

$$T_{k_j} \rightharpoonup T.$$

Since $T_{k_j} \rightharpoonup T$, it follows that $\partial T_{k_j} \rightharpoonup \partial T$ and hence $\partial T = \partial S$, in particular, $T \in N_m(U)$. By the lower semi-continuity of mass, we have

$$M(T) \leq \liminf_{j\to\infty} M(T_{k_j}) = L.$$

□

We will soon introduce so-called integral m-currents. These are special normal currents, which are much "closer" to oriented m-dimensional submanifolds. The aim will then be to prove that Theorem 5.14 holds with normal currents replaced by integral currents. The main difficulty will be to show that the weak limit of a bounded sequence of integral currents is again an integral current.

As indicated before, the metric induced by the mass norm is often not suitable. A norm which is of more geometric significance is the flat norm, defined for $T \in \mathcal{D}_m(U)$ by

$$\mathbb{F}(T) := \inf\Big\{M(S) + M(R) : S \in \mathcal{D}_{m+1}(U), R \in \mathcal{D}_m(U), T = \partial S + R\Big\}.$$

We have $\mathbb{F}(T) \leq M(T)$ and for $\omega \in \mathcal{D}^m(U)$

$$|T(\omega)| \leq \mathbb{F}(T) \max\{\|\omega\|, \|d\omega\|\}.$$

It follows that $\mathbb{F}$ is a norm on $M_m(U)$ and that convergence with respect to the flat norm implies weak convergence.

Illustration:

1). $T_2 - T_1 = \partial S + R$. T_1 and T_2 are "geometrically close". Their flat distance

$$d_{\mathbb{F}}(T_1, T_2) = \mathbb{F}(T_2 - T_1)$$

is small but $M(T_2 - T_1)$ is big.

2). $T = \partial S + R$. $M(T) = 4r + 2R$ and $\mathbb{F}(T) \le 4r + R \cdot \varepsilon$.

5.2. **Homotopy formula and push-forward.** The product of the 1-current $[[0,1]] \in \mathcal{D}_1(\mathbb{R})$ with an m-current $T \in \mathcal{D}_m(U)$ is defined as follows: an element $\omega \in \mathcal{D}^{m+1}(\mathbb{R} \times U)$ can be written as

$$\omega(t,x) = \sum_{\alpha \in \Lambda(n,m)} \omega_\alpha(t,x) dx^0 \wedge dx^\alpha + \sum_{\beta \in \Lambda(n,m+1)} \hat{\omega}_\beta(t,x) dx^\beta.$$

We use the notation $\{e_0, e_1, \cdots, e_n\}$ for the standard basis in $\mathbb{R}\times\mathbb{R}^n$ and $\{dx^0, dx^1, \cdots, dx^n\}$ for its dual basis. The linear functional $[[0,1]] \times T \colon \mathcal{D}^{m+1}(\mathbb{R} \times U) \to \mathbb{R}$ given by

$$([[0,1]] \times T)(\omega) = \int_0^1 \sum_{\alpha \in \Lambda(n,m)} T(\omega_\alpha(t,\cdot) dx^\alpha) dt$$

is called the product of $[[0,1]]$ with T and satisfies

Theorem 5.16. We have $[[0,1]] \times T \in \mathcal{D}_{m+1}(\mathbb{R} \times U)$ and

$$\partial([[0,1]] \times T) = [[1]] \times T - [[0]] \times T - [[0,1]] \times \partial T.$$

Moreover, if $T \in M_m(U)$, then

$$[[0,1]] \times T(\omega) = \int_0^1 \int_U \langle \omega(t,x), e_0 \wedge \overrightarrow{T}(x) \rangle d\|T\|(x) dt.$$

We used the notation

$$[[t_0]] \times T(\omega) := T(\omega(t_0,\cdot)) := T(\sum_{\beta \in \Lambda(n,m)} \hat{\omega}(t_0,\cdot) dx^\beta)$$

for $\omega \in \mathcal{D}^m(\mathbb{R} \times U)$ given by

$$\omega(t,x) = \sum_{\alpha \in \Lambda(n,m-1)} \omega_\alpha dx^0 \wedge dx^\alpha + \sum_{\beta \in \Lambda(n,m)} \hat{\omega}_\beta(t,x) dx^\beta$$

and $\partial T = 0$ if $m = 0$.

Proof. Exercise. □

Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^N$ be open and $T \in \mathcal{D}_m(U)$. Let $\varphi \colon U \to V$ be a smooth map such that the restriction $\varphi|_{\mathrm{spt}(T)}$ is proper, that is,

$$\left(\varphi|_{\mathrm{spt}(T)}\right)^{-1}(K) = \varphi^{-1}(K) \cap \mathrm{spt}(T)$$

is compact whenever $K \subset V$ is compact. For $\omega \in \mathcal{D}^m(V)$ we define

$$(\varphi_\sharp T)(\omega) = T(\rho \cdot \varphi^* \omega),$$

where $\rho \in C_c^\infty(U)$ is any function which equals 1 on a neighborhood of the compact set $\mathrm{spt}(T) \cap \varphi^{-1}(\mathrm{spt}(\omega))$. The definition of $(\varphi_\sharp T)(\omega)$ is independent of the choice of ρ. Notice that ρ is needed because $\varphi^*\omega$ need not have compact support. One easily checks that $\varphi_\sharp T \in \mathcal{D}_m(U)$, that

$$\partial(\varphi_\sharp T) = \varphi_\sharp(\partial T)$$

and $\operatorname{spt}(\varphi_\sharp T) \subset \varphi(\operatorname{spt}(T))$.

Lemma 5.17. Let $T \in M_m(U)$ and let φ be as above so that

$$\lambda := \sup\{\|d\varphi_x\| : x \in \operatorname{spt}(T)\} < \infty.$$

Then $\varphi_\sharp T \in M_m(V)$ and $M(\varphi_\sharp T) \leq \lambda^m M(T)$.

In the above lemma, $\|d\varphi_x\|$ stands the operator norm of $d\varphi_x$.

Proof. If $\omega \in \mathcal{D}^m(V)$ satisfies $\|\omega\| \leq 1$, then $\|(\varphi^*\omega)(x)\| \leq \lambda^m$ for every $x \in \operatorname{spt}(T)$ and hence by Theorem 5.10

$$\begin{aligned} |\varphi_\sharp T(\omega)| &\leq \int_U \left|\langle(\varphi^*\omega)(x), \overrightarrow{T}(x)\rangle\right| d\|T\|(x) \\ &\leq \int_U \lambda^m \|\overrightarrow{T}(x)\| d\|T\|(x) = \lambda^m M(T). \end{aligned}$$

It follows that $M(\varphi_\sharp T) \leq \lambda^m M(T)$. □

Theorem 5.18 (Homotopy formula)**.** Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^N$ be open sets and $\varphi, \psi \colon U \to V$ smooth maps. Suppose $h \colon [0,1] \times U \to V$ is a smooth homotopy from φ to ψ. If $T \in \mathcal{D}_m(U)$ and if $h|_{[0,1]\times\operatorname{spt}(T)}$ is proper, then

$$\psi_\sharp T - \varphi_\sharp T = \partial h_\sharp([[0,1]] \times T) + h_\sharp([[0,1]] \times \partial T).$$

Proof. Since $h|_{[0,1]\times\operatorname{spt}(T)}$ is proper, it follows that $\varphi|_{\operatorname{spt}(T)}$ and $\psi|_{\operatorname{spt}(T)}$ are proper. Hence, $\varphi_\sharp T$ and $\psi_\sharp T$ are defined and elements of $\mathcal{D}_m(V)$. Since

$$\operatorname{spt}([[0,1]] \times T) = [0,1] \times \operatorname{spt}(T),$$

it follows that $h_\sharp([[0,1]] \times T)$ and $h_\sharp([[0,1]] \times \partial T)$ are defined and elements of $\mathcal{D}_{m+1}(V)$ and $\mathcal{D}_m(V)$, respectively.

We now calculate

$$\begin{aligned} \partial h_\sharp([[0,1]] \times T) &= h_\sharp(\partial([[0,1]] \times T)) \\ &= h_\sharp\big([[1]] \times T - [[0]] \times T - [[0,1]] \times \partial T\big) \\ &= h_\sharp\big([[1]] \times T\big) - h_\sharp\big([[0]] \times T\big) - h_\sharp\big([[0,1]] \times \partial T\big) \\ &= \psi_\sharp T - \varphi_\sharp T - h_\sharp\big([[0,1]] \times \partial T\big), \end{aligned}$$

where the last equality follows from a direct calculation. □

We now specialize to the straight-line homotopy from φ to ψ:

$$h(t,x) = (1-t)\varphi(x) + t\psi(x)$$

for $t \in [0,1]$ and $x \in U$.

Lemma 5.19. Suppose h is the straight-line homotopy from φ to ψ and $h|_{[0,1]\times\operatorname{spt}(T)}$ is proper. Then

$$M(h_\sharp([[0,1]] \times T)) \leq L \cdot \lambda^m \cdot M(T),$$

where we have set

$$L := \sup\Big\{|\psi(x) - \varphi(x)| : x \in \operatorname{spt}(T)\Big\}$$

and

$$\lambda := \sup\Big\{\max\{\|d\varphi_x\|, \|d\psi_x\|\} : x \in \mathrm{spt}(T)\Big\}.$$

Proof. We may assume that $M(T) < \infty$ and $\lambda, L < \infty$. Let $\omega \in \mathcal{D}^{m+1}(\mathbb{R}^N)$ be such that $\|\omega\| \le 1$. We have

$$\begin{aligned} h_\sharp([[0,1]] \times T)(\omega) &= \int_0^1 \int_U \langle (h^*\omega)(t,x), e_0 \wedge \overrightarrow{T}(x)\rangle d\|T\|(x)dt \\ &= \int_0^1 \int_U \langle \omega(h(t,x)), \Lambda_{m+1} dh_{(t,x)}(e_0 \wedge \overrightarrow{T}(x)) d\|T\|(x)dt. \end{aligned}$$

We set $h_t(x) := h(t,x)$ and note that

$$\Lambda_{m+1} dh_{(t,x)}(e_0 \wedge \overrightarrow{T}(x)) = (\psi(x) - \varphi(x)) \wedge \Lambda_m (dh_t)_x(\overrightarrow{T}(x))$$

and thus

$$\begin{aligned} \|\Lambda_{m+1} dh_{(t,x)}(e_0 \wedge \overrightarrow{T}(x))\| &\le |\psi(x) - \varphi(x)| \cdot \|\Lambda_m (dh_t)_x(\overrightarrow{T}(x))\| \\ &\le L \cdot \lambda^m \cdot \|\overrightarrow{T}(x)\| \end{aligned}$$

for every $x \in \mathrm{spt}(T)$. From this it follows with the above that

$$|h_\sharp([[0,1]] \times T)(\omega)| \le L \cdot \lambda^m \cdot M(T),$$

which concludes the proof. □

As a consequence of the homotopy formula, Lemma 5.19, and the definition of the flat norm, we obtain

Corollary 5.20. If $T \in N_m(U)$, then, with the notation above,

$$\mathbb{F}(\psi_\sharp T - \varphi_\sharp T) \le L \cdot \lambda^{m-1}(1+\lambda) \cdot (M(T) + M(\partial T)).$$

We can specialize even more, namely, to the case that φ is a constant map and ψ the identity map.

Let $T \in \mathcal{D}_m(U)$ with $\mathrm{spt}(T)$ compact. Let $z \in U$ and suppose U is star-like with respect to z. We define the cone over T with vertex z by

$$z \mathbin{⨳} T := h_\sharp([[0,1]] \times T),$$

where we have set

$$h(t,x) := (1-t)z + tx = z + t(x-z).$$

If $\partial T = 0$ and $m \ge 1$, then $\partial(z \mathbin{⨳} T) = T$. If $M(T) < \infty$ and $z \in \mathrm{spt}(T)$, then

$$M(z \mathbin{⨳} T) \le \frac{1}{m+1} \cdot \mathrm{diam}\big(\mathrm{spt}(T)\big) \cdot M(T).$$

This follows from the proof of Lemma 5.19 together with the observation that, since $h_t(x) = z + t(x-z)$, we have $\|(dh_t)_x\| = t$.

Consequence: If $T \in N_m(\mathbb{R}^n)$, $m \ge 1$, with $\mathrm{spt}(T)$ compact and $\partial T = 0$, then there exists $S \in N_{m+1}(\mathbb{R}^n)$ such that $\partial S = T$.

We now use the homotopy formula and approximation to define the push-forward of an element $T \in N_m(\mathbb{R}^n)$ under a Lipschitz map $\varphi\colon \mathbb{R}^n \to \mathbb{R}^N$ such that $\varphi|_{\mathrm{spt}(T)}$ is proper.

For this, we first approximate φ by smooth maps using standard mollifers. Thus, let $\eta \in C^\infty(\mathbb{R}^n)$ be given by

$$\eta(x) = \begin{cases} c\exp(\frac{1}{|x|^2-1}) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1, \end{cases}$$

where $c > 0$ is such that $\int_{\mathbb{R}^n} \eta(x)dx = 1$. For $\varepsilon > 0$, define the standard mollifer $\eta_\varepsilon(x) := \varepsilon^{-n}\eta(\varepsilon^{-1}x)$. We "smoothen" φ using η_ε as follows:

$$\varphi_\varepsilon(x) := (\eta_\varepsilon \ast \varphi)(x) = \int_{\mathbb{R}^n} \eta_\varepsilon(x-y)\varphi(y)dy.$$

One can show, see e.g. [4] that

(i) $\varphi_\varepsilon \in C^\infty(\mathbb{R}^n, \mathbb{R}^N)$
(ii) $\varphi_\varepsilon \to \varphi$ uniformly as $\varepsilon \to 0$
(iii) If φ is λ-Lipschitz, then φ_ε is λ-Lipschitz for all $\varepsilon > 0$.

Let (ε_i) be a sequence of positive real numbers converging to 0 and set $\varphi_i := \varphi_{\varepsilon_i}$. Since $\|\varphi_i - \varphi\|_\infty < \infty$ and since $\varphi|_{\mathrm{spt}(T)}$ is proper, it follows that $\varphi_i|_{\mathrm{spt}(T)}$ is proper. Thus, by Lemma 5.17, we have $\varphi_{i\sharp}T \in M_m(\mathbb{R}^N)$ and

$$M(\varphi_{i\sharp}T) \leq \lambda^m M(T)$$

for all i.

Similarly, for fixed i and j, the straight-line homotopy h from φ_i to φ_j is proper on $[0,1] \times \mathrm{spt}(T)$ and hence, by Corollary 5.20

$$\mathbb{F}(\varphi_{j\sharp}T - \varphi_{i\sharp}T) \leq \|\varphi_j - \varphi_i\|_\infty(\lambda^m + \lambda^{m-1})(M(T) + M(\partial T)).$$

It follows that, for every $\omega \in \mathcal{D}^m(\mathbb{R}^N)$, we have

$$\begin{aligned} |(\varphi_{j\sharp}T)(\omega) - (\varphi_{i\sharp}T)(\omega)| &\leq \mathbb{F}(\varphi_{j\sharp}T - \varphi_{i\sharp}T) \cdot \max\{\|\omega\|, \|d\omega\|\} \\ &\to 0 \end{aligned}$$

as $i, j \to \infty$. Hence, $\big((\varphi_{j\sharp}T)(\omega)\big)$ is a Cauchy sequence and

$$(\varphi_\sharp T)(\omega) := \lim_{j\to\infty} (\varphi_{j\sharp}T)(\omega)$$

exists. It is clear that $\varphi_\sharp T$ is linear. Moreover, by the lower semi-continuity of mass we have

$$M(\varphi_\sharp T) \leq \liminf_{j\to\infty} M(\varphi_{j\sharp}T) \leq \lambda^m M(T),$$

in particular, $\varphi_\sharp T \in M_m(\mathbb{R}^N)$.

Theorem 5.21. Let T and φ be as above. Then

(i) $\varphi_\sharp T \in N_m(\mathbb{R}^N)$ and
$$M(\varphi_\sharp T) \leq \lambda^m M(T).$$
(ii) $\partial(\varphi_\sharp T) = \varphi_\sharp(\partial T)$.

Proof. For $\omega \in \mathcal{D}^{m-1}(\mathbb{R}^N)$, we have

$$\begin{aligned}\partial(\varphi_\sharp T)(\omega) &= (\varphi_\sharp T)(d\omega) = \lim_{j\to\infty}(\varphi_{j\sharp} T)(d\omega)\\ &= \lim_{j\to\infty}\partial(\varphi_{j\sharp}T)(\omega) = \lim_{j\to\infty}\big(\varphi_{j\sharp}(\partial T)\big)(\omega)\\ &= \big(\varphi_\sharp(\partial T)\big)(\omega).\end{aligned}$$

This proves (ii), from which it follows that

$$\begin{aligned}M(\partial(\varphi_\sharp T)) &= M(\varphi_\sharp(\partial T))\\ &\leq \lambda^{m-1}M(\partial T) < \infty.\end{aligned}$$

Hence $\varphi_\sharp T \in N_m(\mathbb{R}^N)$. This proves the remaining statement in (i). □

In the proof that the limit

$$(\varphi_\sharp T)(\omega) := \lim_{j\to\infty}(\varphi_{j\sharp}T)(\omega)$$

exists and defines an element of $M_m(\mathbb{R}^N)$ the only properties of (φ_j) which we used were that $\varphi_j \to \varphi$ uniformly and that the Lipschitz constant of φ_j is independent of j. Any such sequence provides the same limit. In particular, if φ is moreover smooth, then $\varphi_\sharp T$ coincides with the previous definition.

The following example shows that the condition that $T \in N_m(\mathbb{R}^n)$ cannot be relaxed to $T \in M_m(\mathbb{R}^n)$ in general.

Example 5.22. Let $T \in M_1(\mathbb{R}^2)$ be given by

$$T(\omega) = \int_0^1 \omega(s,0)(e_2)ds$$

and $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $\varphi(s,t) = (s,|t|)$. One easily constructs sequences $(\varphi_j),(\bar\varphi_j) \subset C^\infty(\mathbb{R}^2,\mathbb{R}^2)$ such that

- $\varphi_j \to \varphi$ and $\bar\varphi_j \to \varphi$ uniformly
- φ_j and $\bar\varphi_j$ have uniformly bounded Lipschitz constants
- $\frac{\partial\varphi_j}{\partial x_2}(s,0) = 0$ and $\frac{\partial\bar\varphi_j}{\partial x_2}(s,0) = e_2$ for all $s \in \mathbb{R}$

It follows that

$$(\varphi_{j\sharp}T)(\omega) = \int_0^1 \omega(\varphi_j(s,0))\big(\frac{\partial\varphi_j}{\partial x_2}(s,0)\big)ds = 0$$

and

$$((\bar\varphi_j)_\sharp T)(\omega) = \int_0^1 \omega(\bar\varphi_j(s,0))(e_2)ds \overset{j\to\infty}{\to} T(\omega).$$

As for the construction of φ_j and $\bar\varphi_j$, let η_ε be the standard mollifier on $\mathbb{R}$ and define

$$\varphi_j(s,t) = (s, (\eta_{\frac{1}{j}} \ast \rho)(t)),$$

where $\rho(t) = |t|$. Then $\varphi_j \to \varphi$ uniformly and φ_j has uniformly bounded Lipschitz constant. Moreover,

$$\frac{d}{dt}(\eta_{\frac{1}{j}} \ast \rho)(0) = (\eta'_{\frac{1}{j}} \ast \rho)(0) = 0$$

by a direct calculation. We define $\bar{\varphi}_j$ by $\bar{\varphi}_j(s,t) = \varphi_j(s, t+\frac{2}{j})$ and one checks easily that the desired properties hold.

In some sense, the reason for not being able to define the push-forward of T in the example above is that the 1-vector field e_2 is transversal to the set $[0,1]\times\{0\}$ over which we integrate. Had we chosen e_1 instead of e_2, then we could have defined the push-forward un-ambiguously. More generally, Let $\Omega \subset \mathbb{R}^m$ be open and bounded and $\Theta \in L^1(\Omega)$. Let $\varphi\colon \mathbb{R}^m \to \mathbb{R}^n$ be a Lipschitz map. For $\omega \in \mathcal{D}^m(\mathbb{R}^n)$ define

$$\varphi_\sharp[[\Theta]](\omega) := \int_\Omega \Theta(x)\cdot\omega(\varphi(x))(d\varphi_x(e_1),\cdots,d\varphi_x(e_m))dx.$$

Then $\varphi_\sharp[[\Theta]]$ is well-defined and is an element of $M_m(\mathbb{R}^n)$. We use the following lemma to show that $\varphi_\sharp[[\Theta]]$ can be viewed as a limit of push-forward as above.

Lemma 5.23. Let $\lambda \geq 0$ and let $\rho, \rho_j\colon \Omega \to \mathbb{R}^m$ be λ-Lipschitz maps. If $\rho_j \to \rho$ pointwise on Ω, then

$$\int_\Omega \Theta(x)\det(d(\rho_j)_x)dx \to \int_\Omega \Theta(x)\det(d\rho_x)dx.$$

Proof. See e.g. [1, Theorem 2.16]. □

Writing $\omega = \sum_\alpha \omega_\alpha dx^\alpha$, we have

$$\varphi_\sharp[[\Theta]](\omega) = \sum_\alpha \int_\Omega \Theta(x)\omega_\alpha(\varphi(x))\det\big(\frac{\partial\varphi_{\alpha(i)}}{\partial x_j}(x)\big)dx,$$

where we have written $\varphi = (\varphi_1,\cdots,\varphi_n)$. From this and Lemma 5.23 it follows that if $\varphi^j\colon \mathbb{R}^m \to \mathbb{R}^n$ are Lipschitz with uniformly bounded Lipschitz constants and $\varphi^j \to \varphi$ pointwise, then

$$\varphi^j_\sharp[[\Theta]](\omega) \to \varphi_\sharp[[\Theta]](\omega).$$

Clearly, if φ^j is smooth, then $\varphi^j_\sharp[[\Theta]]$ coincides with the previously defined push-forward under smooth mappings.

5.3. **Integer rectifiable and integral currents.** As already observed, general currents of finite mass have very little in common with oriented submanifolds. In this section, we introduce a subclass of currents which are much closer to submanifolds while at the same time being closed under taking limits.

Let $U \subset \mathbb{R}^n$ be open and $m \geq 0$.

Definition 5.24. An element $T \in \mathcal{D}_m(U)$ is called integer rectifiable current if there exist

(i) an m-rectifiable and $\mathcal{H}^m$-measurable set $E \subset U$ with $\mathcal{H}^m(E) < \infty$,
(ii) an $\mathcal{H}^m$-measurable map $\tau\colon E \to \Lambda_m\mathbb{R}^n$ such that $\tau(x)$ is simple and $\|\tau(x)\| = 1$ and $\tau(x)$ spans the approximate tangent m-plane of E at x for $\mathcal{H}^m$-almost every $x \in E$,
(iii) a function $\Theta \in L^1(E, \mathbb{Z}, \mathcal{H}^m)$ with $\Theta(x) > 0$ such that

$$T(\omega) = \int_E \Theta(x)\langle\omega(x),\tau(x)\rangle d\mathcal{H}^m(x)$$

for every $\omega \in \mathcal{D}^m(U)$.

The map τ is called an orientation of E, and Θ is called the multiplicity function.

Remark 5.25. The Condition (iii) in Definition 5.24 does not imply $\operatorname{spt}(T)=E$. Indeed, let $\{x_k\}_{k\in\mathbb{N}}\subset\mathbb{R}^2$ be dense and define

$$T=\sum_{k=1}^{\infty}\Big[\Big[\partial B(x_k,2^{-k})\Big]\Big]\in I_1(\mathbb{R}^2)$$

with $\operatorname{spt}(T)=\mathbb{R}^2$.

If $T\in\mathcal{D}_m(U)$ is integer rectifiable, then $T\in M_m(U)$ and

$$M(T)=\int_E\Theta(x)d\mathcal{H}^m(x)$$

and $\|T\|=\Theta\mathcal{H}^m\llcorner E$.

We introduce the notation

$$\mathcal{I}_m(U):=\Big\{T\in\mathcal{D}_m(U):T\text{ integer rectifiable}\Big\}$$

and

$$I_m(U):=\mathcal{I}_m(U)\cap N_m(U).$$

We have

$$I_m(U)\subset\mathcal{I}_m(U)\subset M_m(U)$$

and

$$I_m(U)\subset N_m(U)\subset M_m(U).$$

Elements of $I_m(U)$ are called integral m-currents.

Example 5.26. 1). A function $T\colon\mathcal{D}^0(U)\to\mathbb{R}$ belongs to $\mathcal{I}_0(U)$ if and only if there exist finitely many points $x_1,\cdots,x_k\in U$ and $\Theta_1,\cdots,\Theta_k\in\mathbb{Z}\backslash\{0\}$ such that

$$T(f)=\sum_{i=1}^{k}\Theta_i f(x_i)$$

for all $f\in\mathcal{D}^0(U)$.

2). If $M\subset U$ is a smooth, compact, oriented m-dimensional submanifold, then $[[M]]\in I_m(U)$.

3). If $S,T\in\mathcal{I}_m(U)$ and $a,b\in\mathbb{Z}$, then $aS+bT\in\mathcal{I}_m(U)$.

4). Let $M:=[0,1]\times\{0\}\subset\mathbb{R}^2$ and $M_j:=[0,1]\times\{1/j\}\subset\mathbb{R}^2$. The sequence $(T_j)\subset I_1(\mathbb{R}^2)$ given by $T_j:=[[M]]+[[M_j]]$ converges weakly to $2[[M]]$.

5). The current $T\in M_1(\mathbb{R}^2)$ given by

$$T(fdx+gdy):=\int_0^1 g(s,0)ds$$

is not integer rectifiable even thought $\|T\|$ is concentrated on the 1-rectifiable set $[0,1]\times\{0\}=:E$.

6). $\mathcal{I}_m(U)$ is a closed subset of $M_m(U)$ (exercise).

7). If $\Theta_i \in L^1(\mathbb{R}^m, \mathbb{Z})$ and $\psi_i\colon \mathbb{R}^m \to \mathbb{R}^n$ are Lipschitz such that

$$\sum_{i=1}^{\infty} M\big(\psi_{i\sharp}[[\Theta_i]]\big) < \infty,$$

then $T := \sum_{i=1}^{\infty} \psi_{i\sharp}[[\Theta_i]] \in \mathcal{I}_m(\mathbb{R}^n)$.

Conversely, we have

Theorem 5.27. Let $U \subset \mathbb{R}^n$ be open and $T \in \mathcal{M}_m(U)$. Then $T \in \mathcal{I}_m(U)$ if and only if there exist $K_i \subset \mathbb{R}^m$ compact, $\psi_i\colon K_i \to U$ bi-Lipschitz, and $\Theta_i \in L^1(K_i, \mathbb{Z})$ such that the images $\psi_i(K_i)$ are pairwise disjoint and

$$T = \sum_{i=1}^{\infty} \psi_{i\sharp}[[\Theta_i]]$$

and

$$M(T) = \sum_{i=1}^{\infty} M\big(\psi_{i\sharp}[[\Theta_i]]\big).$$

Proof. "$\Leftarrow$": From Example 5.26 7).

"$\Rightarrow$": Since $T \in \mathcal{I}_m(U)$, there exist E, τ, Θ such that

$$T(\omega) = \int_E \Theta(x)\langle \omega(x), \tau(x)\rangle d\mathcal{H}^m(x)$$

for all $\omega \in \mathcal{D}^m(U)$.

By Theorem 3.31, there exist $K_i \subset \mathbb{R}^m$ compact and $\psi_i\colon K_i \to E$ bi-Lipschitz such that the images $\psi_i(K_i)$ are pairwise disjoint and

$$\mathcal{H}^m\big(E \setminus \bigcup_{i=1}^{\infty} \psi_i(K_i)\big) = 0.$$

By the proof of Theorem 3.35, we know that $d(\psi_i)_x(\mathbb{R}^m)$ is the approximate tangent plane of E at $\psi_i(x)$ for almost everywhere $x \in K_i$. Define for such $x \in K_i$,

$$\Theta_i(x) = \pm\Theta\big(\psi_i(x)\big)$$

with the sign depending on

$$\frac{\frac{\partial \psi_i}{\partial x_1}(x) \wedge \cdots \wedge \frac{\partial \psi_i}{\partial x_m}(x)}{\left|\frac{\partial \psi_i}{\partial x_1}(x) \wedge \cdots \wedge \frac{\partial \psi_i}{\partial x_m}(x)\right|} = \pm\tau\big(\psi_i(x)\big).$$

We then have

$$\begin{aligned}\psi_{i\sharp}[[\Theta_i]](\omega) &= \int_{K_i} \Theta_i\Big\langle \omega(\psi_i(x)), \frac{\partial \psi_i}{\partial x_1}(x) \wedge \cdots \wedge \frac{\partial \psi_i}{\partial x_m}(x)\Big\rangle dx \\ &\overset{\text{Theorem 3.17}}{=} \int_{\psi_i(K_i)} \pm\Theta(y)\langle \omega(y), \pm\tau(y)\rangle d\mathcal{H}^m(y),\end{aligned}$$

from which the claim easily follows. $\square$

Using the above parametrization result, we define the push-forward of an integer rectifiable current under a Lipschitz map as follows. Let $U \subset \mathbb{R}^n$ be open and $T \in \mathcal{I}_m(U)$. Let $\varphi\colon U \to V \subset \mathbb{R}^N$ be Lipschitz such that $\varphi|_{\mathrm{spt}(T)}$ is proper. We define $\varphi_\sharp T$ by

$$\varphi_\sharp T := \sum_{i=1}^{\infty} (\varphi \circ \psi_i)_\sharp [[\Theta_i]],$$

where $T = \sum_{i=1}^{\infty} \psi_{i\sharp}[[\Theta_i]]$ is the representation from Theorem 5.27.

By Example 5.26 7), we have $\varphi_\sharp T \in \mathcal{I}_m(\mathbb{R}^N)$ and one can easily check that $\varphi_\sharp T$ is independent of the representation of T, and that $\varphi_\sharp T$ agrees with the previously defined push-forward in case φ is smooth or $T \in I_m(U)$. In the latter case, we have

$$\partial\big(\varphi_\sharp T\big) = \varphi_\sharp\big(\partial T\big).$$

If $T \in \mathcal{I}_m(\mathbb{R}^n)$, then $[[0,1]] \times T \in \mathcal{I}_{m+1}(\mathbb{R}^{n+1})$. Indeed, writing

$$T(\omega) = \int_E \Theta(x)\langle \omega(x), \tau(x)\rangle d\mathcal{H}^m(x)$$

for $\omega \in \mathcal{D}^m(\mathbb{R}^n)$, we obtain

$$\begin{aligned}
[[0,1]] \times T(\omega) &= \int_{[0,1]\times E} \Theta(x)\langle \omega(t,x), e_0 \wedge \tau(x)\rangle d\big(\mathcal{L}^1 \times \mathcal{H}^m\big)(t,x) \\
&= \int_{[0,1]\times E} \Theta(x)\langle \omega(t,x), e_0 \wedge \tau(x)\rangle d\mathcal{H}^{m+1}(t,x)
\end{aligned}$$

for all $\omega \in \mathcal{D}^{m+1}(\mathbb{R} \times \mathbb{R}^n)$. The second equality follows easily from the area formula (e.g. Theorem 3.17) and the fact that one can write an m-rectifiable set as the union of bi-Lipschitz pieces. Indeed, let $\psi\colon K \subset \mathbb{R}^m \to E$ be a bi-Lipschitz map and set $\hat{\psi}(t,x) := (t, \psi(x))$. Then a direct calculation shows that $J_{m+1}(d\hat{\psi}_{(t,x)}) = J_m(d\psi_x)$ and thus, by the area formula,

$$\begin{aligned}
\int_0^1 \int_{\psi(K)} f(t,y) dt d\mathcal{H}^m(y) &= \int_0^1 \int_K f(t, \psi(x)) J_m(d\psi_x) dt dx \\
&= \int_{[0,1]\times K} f(\hat{\psi}(t,x)) J_{m+1}(d\hat{\psi}_{(t,x)}) d\mathcal{L}^{m+1}(t,x) \\
&= \int_{\hat{\psi}\left([0,1]\times K\right)} f(t,y) d\mathcal{H}^{m+1}(y)
\end{aligned}$$

for any integrable function f.

Remark 5.28. The m-rectifiability is essential. Indeed, Freilich [2] constructed a set $A \subset \mathbb{R}^2$ such that $\mathcal{H}^1(A) < \infty$ and

$$\mathcal{H}^2\big([0,1] \times A\big) \neq \mathcal{H}^1(A) = \big(\mathcal{L}^1 \times \mathcal{H}^1\big)\big([0,1] \times A\big).$$

From the above we obtain that if $T \in \mathcal{I}_m(\mathbb{R}^n)$ with $\mathrm{spt}(T)$ compact and $z \in \mathbb{R}^n$, then $z \divideontimes T \in \mathcal{I}_{m+1}(\mathbb{R}^n)$. In particular, if $T \in I_m(\mathbb{R}^n)$ with $\partial T = 0$ and $\mathrm{spt}(T)$ compact, then there exists $s \in I_{m+1}(\mathbb{R}^n)$ with $\partial S = T$.

We now come to two of the central theorems about integral currents.

Theorem 5.29 (Closure theorem)**.** Let $(T_k) \subset N_m(\mathbb{R}^n)$ be a sequence which converges weakly to some $T \in N_m(\mathbb{R}^n)$ and such that

$$\sup_k \big[M(T_k) + M(\partial T_k)\big] < \infty.$$

If $T_k \in I_m(\mathbb{R}^n)$ for every $k \in \mathbb{N}$, then also $T \in I_m(\mathbb{R}^n)$.

The following examples illustrate what can go wrong when the boundedness condition is not satisfied.

Example 5.30. 1). The sequence $(T_k) \subset I_1(\mathbb{R})$ given by $I_k := k \cdot [[0, 1/k]]$ converges weakly to the current $T \in M_1(\mathbb{R})$ given by

$$T(\omega) := \omega(0)(e_1),$$

which is not an integer rectifiable 1-current.

2). The sequence $(T_k) \subset I_1(\mathbb{R}^2)$ given by

$$T_k := \sum_{i=0}^{k-1} \sum_{j=1}^{k-1} \Big[\Big[[\frac{i}{k}, \frac{i}{k} + \frac{1}{k^2}] \times \{j/k\}\Big]\Big]$$

converges weakly to the current $T \in N_1(\mathbb{R}^2)$ given by

$$T(\omega) = \int_0^1 \int_0^1 \omega(x, y)(e_1) dx dy,$$

but T is not an integer rectifiable 1-current.

3). The sequence $(T_k) \subset I_1(\mathbb{R}^2)$ given by

$$T_k = \sum_{i=1}^{k} \Big[\Big[\{i/k\} \times [0, 1/k]\Big]\Big]$$

converges weakly to the current $T \in M_1(\mathbb{R}^2)$ given by

$$T(\omega) := \int_0^1 \omega(x, 0)(e_2) dx = \int_{[0,1]\times\{0\}} \langle \omega, e_2 \rangle d\mathcal{H}^1,$$

but T is not integer 1-rectifiable.

Theorem 5.31 (Boundary rectifiable theorem)**.** If $T \in I_m(\mathbb{R}^n)$ with $m \geq 1$, then $\partial T \in I_{m-1}(\mathbb{R}^n)$.

The content of the theorem is that if $T \in M_m(\mathbb{R}^n)$ is integer rectifiable and $M(\partial T) < \infty$, then ∂T is also integer rectifiable.

For the proofs of the closure and boundary rectifiablility theorems, we will need new tools which will be developed in the next sections. We thus postpone their proofs to later sections and give here a first application to the existence of area-minimizing currents.

Generalized Plateau Problem: Given $T \in I_m(\mathbb{R}^n)$ with $m \geq 1$ and $\partial T = 0$, find $S \in I_{m+1}(\mathbb{R}^n)$ with $\partial S = T$ and such that

$$M(S) \leq M(S')$$

for all $S' \in I_{m+1}(\mathbb{R}^n)$ with $\partial S' = T$.

Remark 5.32. The classical problem of Plateau is the analogous problem for $m = 1$ when T is replaced by a closed curve $c\colon S^1 \to \mathbb{R}^n$ and S by a "disc-type surface" with boundary c, that is, a (smooth) map $u\colon \overline{D} \to \mathbb{R}^n$ with $u|_{S^1} = c$.

The theorems above allow us to solve the generalized Plateau problem.

Theorem 5.33. Let $T \in I_m(\mathbb{R}^n)$ with $m \geq 1$ and $\partial T = 0$ and such that $\operatorname{spt}(T)$ is compact. Then there exists $S \in I_{m+1}(\mathbb{R}^n)$ with $\partial S = T$ and such that

$$M(S) \leq M(S')$$

for all $S' \in I_{m+1}(\mathbb{R}^n)$ with $\partial S' = T$.

Thus, S is a mass-minimizing integral current with boundary T.

Proof. Since $\operatorname{spt}(T)$ is compact and $m \geq 1$, the cone $S' := O \times\!\!\!\!\times T$ satisfies $S' \in I_{m+1}(\mathbb{R}^n)$ and $\partial S' = T$ and thus $S' \in I_{m+1}(\mathbb{R}^n)$.

Let $(S_k) \subset I_{m+1}(\mathbb{R}^n)$ be a mass-minimizing sequence with $\partial S_k = T$ for all $k \in \mathbb{N}$, thus

$$M(S_k) \to L := \inf\big\{M(S') : S' \in I_{m+1}(\mathbb{R}^n), \partial S' = T\big\} < \infty.$$

By the Banach-Alaoglu Theorem 2.51, there exist a subsequence (S_{k_j}) and $S \in M_{m+1}(\mathbb{R}^n)$ such that $S_{k_j} \rightharpoonup S$. Thus also $\partial S_{k_j} \rightharpoonup \partial S$ and hence $\partial S = T$. In particular, we have $S \in N_{m+1}(\mathbb{R}^n)$. By the lower semi-continuity of mass we have

$$M(S) \leq \liminf_{j\to\infty} M(S_{k_j}) \leq L.$$

Finally, since

$$\sup_j \Big[M(S_{k_j}) + M(\partial S_{k_j})\Big] < \infty,$$

it follows from the Closure Theorem 5.29 that $S \in I_{m+1}(\mathbb{R}^n)$ and hence also that $M(S) = L$.

□

It is natural to ask whether it makes a difference if one minimizers over the mass of all currents of finite mass with boundary T in the generalized Plateau problem. The following theorem shows that it indeed makes a difference.

Theorem 5.34. There exists $T \in I_1(\mathbb{R}^4)$ with $\partial T = 0$ which is induced by a closed embedded Lipschitz curve and such that

$$\inf\big\{M(S) : S \in M_2(\mathbb{R}^4), \partial S = T\big\} < \inf\big\{M(S) : S \in I_2(\mathbb{R}^4), \partial S = T\big\}.$$

Thus, filling with currents of finite mass is, in general, more efficient than filling with integral currents. For a proof, we refer to Youngs.

We sketch a related construction:

Let $K \subset \mathbb{R}^4$ be an embedded Klein bottle and let $T \in I_1(\mathbb{R}^4)$ be given by integration over an odd number of sufficiently dense, equally spaced closed curves on K as in the figure below:

One can show that if the curves are sufficiently dense, then

$$\inf\left\{M(S) : S \in I_2(\mathbb{R}^4), \partial S = T\right\} \approx \frac{1}{2}\text{Area}(K) + \text{Area}(\text{disc})$$

because T consists of an odd number of curves.

Now, let $R \in I_2(\mathbb{R}^4)$ be the current obtained by integration with changing orientation on each strip as in the figure below:

Then $\partial R = 2T$ and $M(R) = \text{Area}(K)$. It follows that $S_0 := \frac{1}{2}R$ satisfies $S_0 \in M_2(\mathbb{R}^4)$ and $\partial S_0 = T$ and $M(S_0) = \frac{1}{2}\text{Area}(K)$, which implies with above that

$$\begin{aligned}\inf\left\{M(S) : S \in M_2(\mathbb{R}^4), \partial S = T\right\} &\leq \frac{1}{2}\text{Area}(K)\\ &< \frac{1}{2}\text{Area}(K) + \text{Area}(\text{disc})\\ &\approx \inf\left\{M(S) : S \in I_2(\mathbb{R}^4), \partial S = T\right\}.\end{aligned}$$

We end this section with some results about the interior regularity of mass-minimizing currents. We call an element $S \in I_m(\mathbb{R}^n)$ a mass-minimizing m-current if

$$M(S) \leq M(S')$$

for all $S' \in I_m(\mathbb{R}^n)$ with $\partial S' = \partial S$. Thus, the current S in theorem 5.33 is a mass-minimizing $(m+1)$-current.

The regular/singular points of a current $S \in I_m(\mathbb{R}^n)$ are defined as follows.

Definition 5.35. A point $x \in \text{spt}(S)\setminus\text{spt}(\partial S)$ is called an interior regular point of S if there exist $r > 0$, $Q \in \mathbb{N}$, and a smooth embedded m-dimensional submanifold $M \subset \mathbb{R}^n$ such that

$$S \llcorner B(x,r) = Q \cdot [[M]] \llcorner B(x,r).$$

The set of interior regular points of S is denoted $\text{reg}(S)$. The interior singular set of S is

$$\text{Sing}(S) := \text{spt}(S)\setminus\big(\text{spt}(\partial S) \cup \text{reg}(S)\big).$$

The first theorem yields interior regularity for currents in codimension 1:

Theorem 5.36. Let $S \in I_{n-1}(\mathbb{R}^n)$ be a mass-minimizing $(n-1)$-current. Then

(i). If $n \leq 7$, then $\text{Sing}(S)$ is empty.
(ii). If $n = 8$, then $\text{Sing}(S)$ consists of isolated points.
(iii) If $n \geq 9$, then $\text{Sing}(S)$ has Hausdorff dimension at most $n - 8$.

Statement (i) is due to

- Fleming and De Giorgi for $n = 3$
- Almgren for $n = 4$
- Simons for $5 \leq n \leq 7$

Statements (ii) and (iii) are due to Federer. Statement (i) shows in particular that if $T \in I_2(\mathbb{R}^3)$ is mass-minimizing, then $\text{spt}(T)\setminus\text{spt}(\partial T)$ is a smooth embedded submanifold of $\mathbb{R}^3$. In contrast, area minimizing discs are only immersed. The following theorem shows that singularities can occur, that is, Statement (ii) is optimal.

Theorem 5.37. Let $S \in I_7(\mathbb{R}^8)$ be the cone over $S^3\big(0, 1/\sqrt{2}\big) \times S^3\big(0, 1/\sqrt{2}\big) \subset \mathbb{R}^8$, where $S^3(0, r)$ is the sphere of radius r around 0 in $\mathbb{R}^4$. Then S is mass-minimizing and has an interior singularity at 0.

This example is due to Bombien-De Giorgi-Giusi.

For mass-minimizing currents in codimension ≥ 2, we have the following important result:

Theorem 5.38 (Almgren)**.** Let $S \in I_m(\mathbb{R}^n)$ be a mass-minimizing current. Then $\mathrm{Sing}(S)$ has Hausdorff dimension at most $m - 2$.

The original proof is about 1000 pages long. A partially new approach has been developed by De Lellis-Spadaro and has led to a shorter and simpler proof.

5.4. **Slices.** Recall that if $M \subset \mathbb{R}^n$ is a smooth m-dimensional submanifold and $f\colon \mathbb{R}^n \to \mathbb{R}$ a smooth function, then for almost every $s \in \mathbb{R}$ intersection $M \cap \{f = s\}$ is a smooth $(m-1)$-dimensional submanifold, see e.g. [8]. There exists a similar construction for normal currents and Lipschitz functions.

Let $T \in N_m(\mathbb{R}^n)$ with $m \geq 1$ and $f\colon \mathbb{R}^n \to \mathbb{R}$ Lipschitz. For $s \in \mathbb{R}$, we define an $(m-1)$-current $\langle T, f, s\rangle \in \mathcal{D}_{m-1}(\mathbb{R}^n)$, called the slice of T, by

$$\langle T, f, s\rangle := \partial\big(T\llcorner\{f \leq s\}\big) - (\partial T)\llcorner\{f \leq s\}.$$

Example 5.39. Consider the figure below.

If $m \geq 2$, then clearly

$$\partial\langle T, f, s\rangle = -\langle \partial T, f, s\rangle$$

for all $s \in \mathbb{R}$. We can write $\langle T, f, s\rangle$ equivalently as

$$\langle T, f, s\rangle = (\partial T)\llcorner\{f > s\} - \partial\big(T\llcorner\{f > s\}\big). \tag{5.1}$$

Indeed, we write

$$T = T\llcorner\{f \leq s\} + T\llcorner\{f > s\}$$

and thus

$$\partial\big(T\llcorner\{f \leq s\}\big) + \partial\big(T\llcorner\{f > s\}\big) = \partial T = (\partial T)\llcorner\{f \leq s\} + (\partial T)\llcorner\{f > s\},$$

from which (5.1) follows.

We have the following important properties of the slices.

Theorem 5.40. Let $T \in N_m(\mathbb{R}^n)$ with $m \geq 1$ and let $f\colon \mathbb{R}^n \to \mathbb{R}$ be λ-Lipschitz. Then

(i) For almost every $s \in \mathbb{R}$ we have $\langle T, f, s\rangle \in N_{m-1}(\mathbb{R}^n)$ with $\mathrm{spt}\big(\langle T, f, s\rangle\big) \subset \mathrm{spt}(T) \cap \{f = s\}$ and

$$M\big(\langle T, f, s\rangle\big) \leq \lambda \frac{d}{dr}\Big|_{r=s} \|T\|\big(\{f \leq r\}\big).$$

(ii) The function $s \mapsto M\big(\langle T, f, s\rangle\big)$ is measurable and hence

$$\int_{s_0}^{s_1} M\big(\langle T, f, s\rangle\big) ds \leq \lambda \cdot \|T\|\big(\{s_0 < f \leq s_1\}\big)$$

for all $s_0 < s_1$.

Proof. Fix $s \in \mathbb{R}$ and let, for $h > 0$ sufficiently small, $g_h \colon \mathbb{R}^n \to \mathbb{R}$ be a smooth function such that

- $g_h = 0$ on $\{f \leq s\}$
- $g_h = 1$ on $\{f \geq s + h\}$
- g_h is $\frac{\lambda}{h} \cdot (1 + \varepsilon(h))$-Lipschitz,

where $\varepsilon(h) \to 0$ as $h \to 0$. Such a function g_h can for example be obtained as follows. Define $g := \rho \circ f$, where $\rho \colon \mathbb{R} \to \mathbb{R}$ is the piece-wise affine function such that $\rho|_{(-\infty, s+h^2])} = 0$, $\rho|_{[s+h-h^2]} = 1$ and ρ is linear on $[s + h^2, s + h - h^2]$. Let $g_h := \eta_{h^2/\lambda} * g$, where $\eta_{h^2/\lambda}$ is the standard mollifier defined earlier. It is not difficult to check that g_h satisfies the properties listed above.

We define $T_h \in \mathcal{D}_{m-1}(\mathbb{R}^n)$ by

$$T_h(\nu) := T(dg_h \wedge \nu)$$

for all $\nu \in \mathcal{D}^{m-1}(\mathbb{R}^n)$. Since $dg_h \wedge \nu = d(g_h \nu) - g_h d\nu$, we obtain that

$$\begin{aligned}
T_h(\nu) &= (\partial T)(g_h \nu) - T(g_h d\nu) \\
&= \int_{\mathbb{R}^n} g_h \langle \nu, \partial \overrightarrow{T} \rangle d\|\partial T\| - \int_{\mathbb{R}^n} g_h \langle d\nu, \overrightarrow{T} \rangle d\|T\| \\
&\overset{h \to 0}{\to} \int_{\{f>s\}} \langle \nu, \partial \overrightarrow{T} \rangle d\|\partial T\| - \int_{\{f>s\}} \langle d\nu, \overrightarrow{T} \rangle d\|T\| \\
&= \Big((\partial T) \llcorner \{f > s\} \Big)(\nu) - \Big(T \llcorner \{f > s\} \Big)(d\nu) \\
&= \langle T, f, s \rangle (\nu).
\end{aligned}$$

Thus, $T_h \rightharpoonup \langle T, f, s \rangle$. Now observe that $\operatorname{spt}(dg_h) \subset \{s \leq f \leq s + h\}$ and

$$\|dg_h \wedge \nu\| \leq \|dg_h\| \cdot \|\nu\| \leq \frac{\lambda}{h} \cdot \big(1 + \varepsilon(h)\big) \|\nu\|.$$

From this, it follows that

$$\operatorname{spt}(T_h) \subset \{s \leq f \leq s + h\} \cap \operatorname{spt}(T)$$

and for $\nu \in \mathcal{D}^{m-1}(\mathbb{R}^n)$ with $\|\nu\| = 1$ that

$$\begin{aligned}
|T_h(\nu)| = |T(dg_h \wedge \nu)| &\leq \int_{\mathbb{R}^n} \big| \langle dg_h \wedge \nu, \overrightarrow{T} \rangle \big| d\|T\| \\
&\leq \lambda \cdot (1 + \varepsilon(h)) \cdot \frac{1}{h} \cdot \|T\| \big(\{s \leq f \leq s + h\} \big).
\end{aligned}$$

Thus also

$$M(T_h) \leq \lambda \cdot (1 + \varepsilon(h)) \cdot \frac{1}{h} \cdot \|T\| \big(\{s \leq f \leq s + h\} \big).$$

Since $T_h \rightharpoonup \langle T, f, s \rangle$, it follows that

$$\operatorname{spt} \big(\langle T, f, s \rangle \big) \subset \operatorname{spt}(T) \cap \{f = s\}.$$

Moreover, the lower semi-continuity of mass implies

$$M\big(\langle T,f,s\rangle\big)\leq \lambda\cdot\liminf_{h\to 0}\frac{1}{h}\cdot\|T\|\big(\{s\leq f\leq s+h\}\big).$$

Set $F(s):=\|T\|\big(\{f\leq s\}\big)$ and notice that F is non-decreasing on $\mathbb{R}$ and hence differentiable at almost every point and

$$\int_{s_0}^{s_1}F'(s)ds\leq F(s_1)-F(s_0)$$

for all $s_0<s_1$. In particular, $F'(s)<\infty$ for almost all $s\in\mathbb{R}$. Since $\|T\|\big(\{f=s\}\big)=0$ for every point of continuity of F, we obtain

$$\begin{aligned}\frac{1}{h}\cdot\|T\|\big(\{s\leq f\leq s+h\}\big)&=\frac{1}{h}\cdot\|T\|\big(\{s<f\leq s+h\}\big)\\&=\frac{F(s+h)-F(s)}{h}\to F'(s)<\infty\end{aligned}$$

for almost every $s\in\mathbb{R}$. This shows that

$$M\big(\langle T,f,s\rangle\big)\leq\lambda\cdot\frac{d}{dr}\Big|_{r=s}\|T\|\big(\{f\leq r\}\big)<\infty$$

for almost every $s\in\mathbb{R}$. Since $\partial\langle T,f,s\rangle=-\langle\partial T,f,s\rangle$, it follows that $\langle T,f,s\rangle\in N_{m-1}(\mathbb{R}^n)$ for almost every $s\in\mathbb{R}$. This shows (i).

We leave it as an exercise to check that the function

$$s\mapsto M\big(\langle T,f,s\rangle\big)$$

is measurable. It then follows that

$$\begin{aligned}\int_{s_0}^{s_1}M\big(\langle T,f,s\rangle\big)ds&\leq\lambda\int_{s_0}^{s_1}F'(s)ds\\&\leq\lambda\Big(F(s_1)-F(s_0)\Big)\\&=\lambda\cdot\|T\|\big(\{s_0<f\leq s_1\}\big).\end{aligned}$$

□

Theorem 5.41. Let $T\in I_m(\mathbb{R}^n)$ with $m\geq 1$ and let $f\colon\mathbb{R}^n\to\mathbb{R}$ be Lipschitz. Then $\langle T,f,s\rangle\in I_{m-1}(\mathbb{R}^n)$ for almost every $s\in\mathbb{R}$.

It follows, in particular, that $\partial\big(T\llcorner\overline{B}(x,s)\big)\in I_{m-1}(\mathbb{R}^n)$ for every $x\in\mathbb{R}^n$ and almost every $0<s<\operatorname{dist}(x,\operatorname{spt}(\partial T))$. The proof of the theorem relies on the coarea formula. We omit it here and refer to the book of Simon [9].

The method of slicing is often very useful. For example, (iterated) slices will play an important role in the proof of the closure theorem. Here, we give two different applications which use slices.

Theorem 5.42 (Monotonicity)**.** Let $S\in I_m(\mathbb{R}^n)$ be a mass-minimizing current with $m\geq 1$ and let $x\in\operatorname{spt}(S)$. Then the function

$$r\mapsto\frac{\|S\|(\overline{B}(x,r))}{\omega_m r^m}$$

is non-decreasing on the interval $\big(0, \operatorname{dist}(x, \operatorname{spt}(\partial S))\big)$.

As a consequence, we obtain

Corollary 5.43. Let $S \in I_m(\mathbb{R}^n)$ be a mass-minimizing current with $m \geq 1$. Then for every $x \in \operatorname{spt}(S) \setminus \operatorname{spt}(\partial S)$, the density $\Theta_m(\|S\|, x)$ exists and satisfies $\Theta_m(\|S\|, x) \geq 1$. Moreover,

$$\|S\|(\overline{B}(x, r)) \geq \Theta_m(\|S\|, x)\omega_m r^m$$

for all $0 \leq r \leq \operatorname{dist}(x, \operatorname{spt}(\partial S))$.

Proof. The only statement which is not a direct consequence of the theorem is the fact that

$$\Theta_m(\|S\|, x) \geq 1 \quad \text{for all } x \in \operatorname{spt}(S) \setminus \operatorname{spt}(\partial S).$$

Let E, Θ, τ be such that

$$S(\omega) = \int_E \Theta \langle \omega, \tau \rangle d\mathcal{H}^m$$

for all $\omega \in \mathcal{D}^m(\mathbb{R}^n)$, hence $\|S\| = \Theta \cdot \mathcal{H}^m \llcorner E$. It follows that

$$\Theta_m(\|S\|, x) = \Theta(x) \cdot \Theta_m(E, x) = \Theta(x) \in \mathbb{N}$$

for $\mathcal{H}^m$-almost every $x \in E$. Since the set of all such x is dense in $\operatorname{spt}(S)$ and since, by Theorem 5.42, the function $x \mapsto \Theta_m(\|S\|, x)$ is upper semi-continuous on $\operatorname{spt}(S)$, it follows that $\Theta_m(\|S\|, x) \geq 1$ for all $x \in \operatorname{spt}(S)$.

□

Proof of Theorem 5.42. We may assume that $r_0 := \operatorname{dist}(x, \operatorname{spt}(\partial S)) > 0$. Define a non-decreasing function by $F(r) := \|S\|(\overline{B}(x, r))$. For almost every $r \in (0, r_0)$ we have that $S_r := \partial\big(S \llcorner \overline{B}(x, r)\big)$ satisfies $S_r \in I_{m-1}(\mathbb{R}^n)$ and

$$M(S_r) \leq F'(r).$$

For every such r we have $x \divideontimes S_r \in I_m(\mathbb{R}^n)$ with

$$M(x \divideontimes S_r) \leq \frac{r}{m} \cdot M(S_r) \leq \frac{r}{m} F'(r).$$

Since $\partial(x \divideontimes S_r) = S_r$, it follows from the mass-minimizing property of S that

$$M(S \llcorner \overline{B}(x, r)) \leq M(x \divideontimes S_r)$$

and hence

$$F(r) = M(S \llcorner \overline{B}(x, r)) \leq M(x \divideontimes S_r) \leq \frac{r}{m} F'(r)$$

for almost every $r \in (0, r_0)$. In other words,

$$\frac{d}{dr} \log \circ F(r) \geq \frac{m}{r}$$

for almost every $r \in (0, r_0)$ and thus, by integration,

$$\log\left(\frac{F(t)}{F(s)}\right) \geq m \log\left(\frac{t}{s}\right) = \log\left(\frac{t^m}{s^m}\right)$$

and hence

$$\frac{F(t)}{t^m} \geq \frac{F(s)}{s^m}$$

for all $0 < s < t < r_0$. □

As a second application of Theorem 5.41, we prove the following isoperimetric inequality.

Theorem 5.44. Given $m \geq 1$ there exists $C_m > 0$ with the following property. For every $T \in I_m(\mathbb{R}^n)$ with $\partial T = 0$, there exists $S \in I_{m+1}(\mathbb{R}^n)$ such that $\partial S = T$ and

$$M(S) \leq C_m M(T)^{\frac{m+1}{m}}. \tag{5.2}$$

The constant C_m which we will obtain does not depend on the ambient dimension n. however, our constant C_m is not optimal. The optimal constant was obtained by Almgren and is such that equality holds exactly when T is an m-sphere and S is an $(m+1)$-ball.

The proof of Theorem 5.44 will be by induction on m. The idea is as follows. If T is "roundish" in the sense that

$$\dim\big(\operatorname{spt}(T)\big) \leq CM(T)^{\frac{1}{m}},$$

then the cone $S := x \divideontimes T$ with $x \in \operatorname{spt}(T)$ satisfies $\partial S = T$ and

$$\begin{aligned} M(S) &\leq \frac{1}{m+1} \cdot \operatorname{diam}\big(\operatorname{spt}(T)\big) \cdot M(T) \\ &\leq \frac{C}{m+1} M(T)^{\frac{m+1}{m}}, \end{aligned}$$

thus S is the desired filling. In the general case, we will decompose T into a sum of "roundish" cycles and a small rest. Each of the roundish cycles will be filled by a cone and the result will again be decomposed into the sum of roundish cycles plus an even smaller rest.

We first show

Proposition 5.45. Let $m \geq 1$. If $m \geq 2$, then assume that Theorem 5.44 holds with $m-1$. There exist constants $E > 0$, $0 < \delta, \lambda < 1$ depending only on m such that the following holds. Every $T \in I_m(\mathbb{R}^n)$ with $\partial T = 0$ can be written as a finite sum

$$T = T_1 + \cdots + T_N + R,$$

where $T_i, R \in I_m(\mathbb{R}^n)$ satisfy $\partial T_i = 0 = \partial R$ and

(i) $\operatorname{diam}\big(\operatorname{spt}(T_i)\big) \leq E \cdot M(T_i)^{\frac{1}{m}}$
(ii) $M(R) \leq (1-\delta)M(T)$
(iii) $M(T_1) + \cdots + M(T_N) \leq (1+\lambda)M(T)$.

This means that after splitting off roundish cycles we obtain a new cycle R which has essentially smaller mass. It is not difficult to see that Proposition 5.45 implies Theorem 5.44.

Proof of Theorem 5.44. Applying Proposition 5.45 repeatedly, we obtain sequences $(T_k), (R_k) \subset I_m(\mathbb{R}^n)$ with $\partial T_k = 0 = \partial R_k$ for all k and a strictly increasing sequence of integers $(N_k) \subset \mathbb{N}$ such that for all k

(i) $T = T_1 + \dots + T_{N_k} + R_k$

(ii) $\operatorname{diam}\big(\operatorname{spt}(T_k)\big) \leq E \cdot M(T_k)^{\frac{1}{m}}$

(iii) $\sum_{i=1}^{\infty} M(T_i) \leq (1+\lambda) \cdot \Big(\sum_{i=0}^{\infty}(1-\delta)^i\Big) M(T) = \frac{1+\lambda}{\delta} M(T)$.

We fill each T_k by a cone: let $x_k \in \operatorname{spt}(T_k)$ and let $S_k := x_k \divideontimes T_k$. Then $S_k \in I_{m+1}(\mathbb{R}^n)$ with $\partial S_k = T_k$ and

$$M(S_k) \leq \frac{E}{m+1} \cdot M(T_k)^{\frac{m+1}{m}}.$$

Set $S^k := \sum_{i=1}^{N_k} S_i$ and notice that $S^k \in I_m(\mathbb{R}^n) \subset \mathcal{I}_m(\mathbb{R}^n)$ and

$$\begin{aligned} M(S^l - S^k) &\leq \sum_{i=k+1}^{l} M(S_i) \leq \frac{E}{m+1} \cdot \sum_{i=k+1}^{l} M(T_i)^{\frac{m+1}{m}} \\ &\leq \frac{E}{m+1}\Big[\sum_{i=k+1}^{l} M(T_i)\Big]^{\frac{m+1}{m}} \end{aligned}$$

for all $l > k \geq 1$. Since $\sum_{i=1}^{\infty} M(T_i) < \infty$, it follows that (S^k) is a Cauchy sequence with respect to the mass norm and there converges to some $S \in M_{m+1}(\mathbb{R}^n)$. Since $\mathcal{I}_{m+1}(\mathbb{R}^n)$ is a closed subset of $M_{m+1}(\mathbb{R}^n)$ it follows that $S \in \mathcal{I}_{m+1}(\mathbb{R}^n)$. Moreover,

$$\partial S^k = T_1 + \dots + T_{N_k} = T - R_k \to T$$

and hence $\partial S = T$ and $S \in I_{m+1}(\mathbb{R}^n)$. Finally,

$$\begin{aligned} M(S) &\leq \sum_{i=1}^{\infty} M(S_i) \leq \frac{E}{m+1} \sum_{i=1}^{\infty} M(T_i)^{\frac{m+1}{m}} \\ &\leq \frac{E}{m+1}\Big[\sum_{i=1}^{\infty} M(T_i)\Big]^{\frac{m+1}{m}} \leq \frac{E}{m+1}\Big(\frac{1+\delta}{\lambda}\Big)^{\frac{m+1}{m}} M(T)^{\frac{m+1}{m}}. \end{aligned}$$

□

We now prove Proposition 5.45 and let $T \in I_m(\mathbb{R}^n)$ with $\partial T = 0$. For $y \in \operatorname{spt}(T)$, we define

$$F_y(r) := \|T\|(\overline{B}(y,r))$$

and

$$r_0(y) := \max\Big\{r \geq 0 : F_y(r) \geq A \cdot r^m\Big\},$$

where $A > 0$ is a small constant, precisely

$$A := \frac{1}{2} \cdot \min\Big\{1,\ \omega_m,\ \frac{1}{2^{m^2} m^m C_{m-1}^{m-1}}\Big\}.$$

Here, C_{m-1} is the isoperimetric constant in dimension $m-1$ if $m \geq 2$ and $C_0 := 1$ if $m = 1$.

Notice that $F_y(r_0(y)) = A \cdot r_0(y)^m$ and $F_y(r) < Ar^m$ for all $r > r_0(y)$.

Lemma 5.46. There exist $y_1, \cdots, y_N \in \operatorname{spt}(T)$ such that

(i) $r_0(y_i) > 0$
(ii) the balls $\overline{B}(y_i, 2r_0(y_i))$ are pairwise disjoint
(iii) $\sum_{i=1}^N \|T\|\big(\overline{B}(y_i, r_0(y_i))\big) \geq \alpha \cdot M(T)$, where $\alpha > 0$ is a constant only depending on m.

Proof. Define $Y_1 := \operatorname{spt}(T)$ and

$$r_1 := \sup\Big\{r_0(y) : y \in Y_1\Big\} > 0$$

and choose $y_1 \in Y_1$ with $r_0(y_1) > \frac{2}{3}r_1$.

Suppose $y_1, \cdots, y_k$ have been chosen for some $k \geq 1$. Define

$$Y_{k+1} := Y_1 \backslash \bigcup_{i=1}^k \overline{B}(y_i, 5r_0(y_i))$$

and

$$r_{k+1} := \sup\Big\{r_0(y) : y \in Y_{k+1}\Big\}.$$

Notice that $r_1 \geq r_2 \geq \cdots \geq 0$ and for $i < j$

$$|y_j - y_i| > 5r_0(y_i) > 2r_0(y_i) + 2r_j \geq 2r_0(y_i) + 2r_0(y_j),$$

hence the balls $\overline{B}(y_i, 2r_0(y_i))$ are pairwise disjoint. We first assume that $r_k > 0$ for all $k \geq 1$. Then $r_k \to 0$ and

$$\|T\|\Big(\operatorname{spt}(T) \backslash \cup \overline{B}(y_i, 5r_0(y_i))\Big) = 0.$$

Hence

$$\begin{aligned} M(T) &\leq \sum_{i=1}^\infty \|T\|\big(\overline{B}(y_i, 5r_0(y_i))\big) \leq \sum_{i=1}^\infty A \cdot \big(5r_0(y_i)\big)^m \\ &= 5^m \sum_{i=1}^\infty \|T\|\big(\overline{B}(y_i, r_0(y_i))\big). \end{aligned}$$

Thus, if $0 < \alpha < 5^{-m}$, then there exists $N \geq 1$ such that

$$\sum_{i=1}^N \|T\|\big(\overline{B}(y_i, r_0(y_i))\big) \geq \alpha M(T).$$

The case that $r_k = 0$ for some $k \geq 1$ is similar. □

Let $y_1, \cdots, y_N \in \operatorname{spt}(T)$ be given as in Lemma 5.46 and fix $i \in \{1, \cdots, N\}$.

Lemma 5.47. There exists a set of positive measure of $r \in (r_0(y_i), 2r_0(y_i))$ for which $F'_{y_i}(r) \leq A \cdot m \cdot r^{m-1}$.

Proof. We argue by contradiction and assume that

$$F'_{y_i}(r) > Amr^{m-1}$$

for almost every $r \in (r_0(y_i), 2r_0(y_i))$. Then

$$F_{y_i}(2r_0(y_i)) \geq F_{y_i}(r_0(y_i)) + \int_{r_0(y_i)}^{2r_0(y_i)} F'_{y_i}(r)dr$$
$$> Ar_0(y_i)^m + \int_{r_0(y_i)}^{2r_0(y_i)} Amr^{m-1}dr = A \cdot \big(2r_0(y_i)\big)^m,$$

which contradicts the definition of $r_0(y_i)$. □

Proof of Proposition 5.45. By Lemma 5.47 and Theorem 5.41, there exists $r_i \in (r_0(y_i), 2r_0(y_i))$ such that $\partial\big(T\llcorner\overline{B}(y_i, r_i)\big) \in I_{m-1}(\mathbb{R}^n)$ and

$$M\Big(\partial\big(T\llcorner\overline{B}(y_i, r_i)\big)\Big) \leq F'_{y_i}(r_i) \leq Amr_i^{m-1}.$$

We distinguish two cases.

Case 1: $m = 1$.

We have $M\Big(\partial\big(T\llcorner\overline{B}(y_i, r_i)\big)\Big) \leq A \leq \frac{1}{2}$. Since $\partial\big(T\llcorner\overline{B}(y_i, r_i)\big) \in I_0(\mathbb{R}^n)$, it follows that $\partial\big(T\llcorner\overline{B}(y_i, r_i)\big) = 0$.

Set $T_i := T\llcorner\overline{B}(y_i, r_i)$ for $i = 1, \cdots, N$ and $R := T - T_1 - \cdots - T_N$. Then $\partial T_i = 0 = \partial R$ and so $T_i, R_i \in I_1(\mathbb{R}^n)$ and

$$T = T_1 + \cdots + T_N + R.$$

Moreover,

$$M(T_1) + \cdots + M(T_N) \leq M(T)$$

and $M(R) \leq (1 - \alpha) \cdot M(T)$. Finally,

$$\operatorname{diam}\big(\operatorname{spt}(T_i)\big) \leq 2r_i \leq 4r_0(y_i) \leq \frac{4}{A} \cdot M(T_i).$$

This proves the proposition in the case $m = 1$.

Case 2: $m \geq 2$.

By assumption, $\mathbb{R}^n$ admits an isoperimetric inequality for $(m-1)$-cycles. Thus there exists $S_i \in I_m(\mathbb{R}^n)$ such that $\partial S_i = \partial\big(T\llcorner\overline{B}(y_i, r_i)\big)$ and

$$M(S_i) \leq C_{m-1} M\Big(\partial\big(T\llcorner\overline{B}(y_i, r_i)\big)\Big)^{\frac{m}{m-1}} \leq C_{m-1}\big(Am\big)^{\frac{m}{m-1}} r_i^m$$
$$< C_{m-1} m^{\frac{m}{m-1}} A^{\frac{1}{m-1}} 2^m \cdot Ar_0(y_i)^m < \frac{1}{2} Ar_0(y_i)^m.$$

After possibly projecting S_i onto the ball $\overline{B}(y_i, r_i)$, we may assume that $\operatorname{spt}(S_i) \subset \overline{B}(y_i, r_i)$.

Set $T_i := T\llcorner\overline{B}(y_i, r_i) - S_i$ for $i = 1, \cdots, N$ and notice that $\partial T_k = 0$ and $T_i \in I_m(\mathbb{R}^n)$. Since

$$M(T_i) \geq \|T\|\big(\overline{B}(y_i, r_i)\big) - M(S_i) > \frac{1}{2} Ar_0(y_i)^m.$$

It follows that

$$\operatorname{diam}\big(\operatorname{spt}(T_i)\big) \leq 2r_i < 4r_0(y_i) < \frac{4 \cdot 2^{1/m}}{A^{1/m}} \cdot M(T_i)^{1/m}.$$

Finally,

$$\begin{aligned}\sum_{i=1}^{N} M(T_i) &< \sum_{i=1}^{N}\Big(\|T\|\big(\overline{B}(y_i,r_i)\big)+\frac{1}{2}Ar_0(y_i)^m\Big)\\ &\leq (1+\frac{1}{2})M(T)\end{aligned}$$

and hence the cycles $R := T - T_1 - \cdots - T_N \in I_m(\mathbb{R}^n)$ satisfies

$$\begin{aligned}M(R) &\leq \|T\|\Big(\mathbb{R}^n\setminus\bigcup_{i=1}^{N}\overline{B}(y_i,r_i)\Big)+\sum_{i=1}^{N}M(S_i)\\ &= M(T)-\sum_{i=1}^{N}\|T\|\Big(\overline{B}(y_i,r_i)\Big)+\sum_{i=1}^{N}M(S_i)\\ &\leq M(T)-\frac{1}{2}\sum_{i=1}^{N}\|T\|\Big(\overline{B}(y_i,r_0(y_i))\Big)\\ &\leq (1-\frac{\alpha}{2})\cdot M(T).\end{aligned}$$

This proves the proposition in the case $m \geq 2$.

□

5.5. **Proofs of Closure and Boundary Rectifiability Theorems.** The aim of this section is to prove the Closure and Boundary Rectifiability theorems. The proofs are by induction on the dimension of the currents, and the main ingredient, which will be proved in the next section, is a characterization of integral currents via iterated slices.

Proposition 5.48 (Closure theorem for 0-current)**.** Let $(T_k) \subset M_0(\mathbb{R}^n)$ be a sequence such that $\sup_k M(T_k) < \infty$ and $T_k \rightharpoonup T$ for some $T \in M_0(\mathbb{R}^n)$. If $T_k \in I_0(\mathbb{R}^n)$ for all k, then $T \in I_0(\mathbb{R}^n)$.

Proof. Setting $N_k = M(T_k)$, we can write T_k as

$$T_k = \sum_{i=1}^{N_k} m_k^i[[x_k^i]]$$

for some $x_k^i \in \mathbb{R}^n$ and $m_k^i \in \{-1,1\}$. After possibly passing to a subsequence we may assume that $N_k = N$ and $m_k^i = m^i$ for some $N \in \mathbb{N}$ and $m^i \in \{-1,1\}$ for all k and that there exists $0 \leq M \leq N$ such that $x_k^i \to x^i$ for $i \leq M$ and $|x_k^i| \to \infty$ for $i > M$. It follows that for every $f \in C_c^\infty(\mathbb{R}^n)$ we have

$$I_k(f) = \sum_{i=1}^{N} m^i f(x_k^i) \to \sum_{i=1}^{M} m^i f(x^i)$$

as $k \to \infty$ and hence $T = \sum_{i=1}^{N} m^i[[x^i]]$. This shows that $T \in I_0(\mathbb{R}^n)$. □

Lemma 5.49. Let $m \geq 1$ and let $(T_k) \subset N_m(\mathbb{R}^n)$ be a sequence such that

$$\sup_k \Big(M(T_k)+M(\partial T_k)\Big) < \infty$$

and such that $T_k \rightharpoonup T$ for some $T \in N_m(\mathbb{R}^n)$. Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz function. Then for almost every $s \in \mathbb{R}$ there exists a subsequence (T_{k_j}) such that

$$\langle T_{k_j}, f, s\rangle \rightharpoonup \langle T, f, s\rangle$$

and

$$\sup_j \Big(M\big(\langle T_{k_j}, f, s\rangle\big) + M\big(\partial\langle T_{k_j}, f, s\rangle\big)\Big) < \infty.$$

Proof. Set $\mu_k := \|T_k\|$ and $\nu_k := \|\partial T_k\|$. There exists a subsequence (k_j) and Radon measures μ, ν such that (μ_{k_j}) converges weakly to μ and (ν_{k_j}) converges weakly to ν.

Let $s \in \mathbb{R}$ be such that

$$\mu\big(\{f = s\}\big) + \nu\big(\{f = s\}\big) = 0$$

and notice that all but at most countably many $s \in \mathbb{R}$ have this property. We want to show that

$$\langle T_{k_j}, f, s\rangle \rightharpoonup \langle T, f, s\rangle.$$

For this, it is enough to show that

$$T_{k_j} \llcorner \{f \le s\} \rightharpoonup T \llcorner \{f \le s\}$$

and

$$(\partial T_{k_j}) \llcorner \{f \le s\} \rightharpoonup (\partial T) \llcorner \{f \le s\}.$$

Let $\varepsilon > 0$. There exists $h > 0$ such that $\mu\big(\{s - h \le f \le s + 2h\}\big) < \varepsilon$. Let $\omega \in \mathcal{D}^m(\mathbb{R}^n)$. Then

$$\mu_{k_j}\big(\{s \le f \le s + h\} \cap \operatorname{spt}(\omega)\big) < \varepsilon \tag{5.3}$$

for all j sufficiently large. Indeed, let $\eta \in C_c(\mathbb{R}^n)$ be such that

$$\chi_{\{s \le f \le s+h\} \cap \operatorname{spt}(\omega)} \le \eta \le \chi_{\{s-h \le f \le s+2h\}}$$

and notice that

$$\begin{aligned} \mu_{k_j}\big(\{s \le f \le s + h\} \cap \operatorname{spt}(\omega)\big) &\le \int_{\mathbb{R}^n} \eta \, d\mu_{k_j} \overset{j\to\infty}{\to} \int_{\mathbb{R}^n} \eta \, d\mu \\ &\le \mu\big(\{s - h \le f \le s + 2h\}\big) < \varepsilon, \end{aligned}$$

proving (5.3).

Now, let $g \in C^\infty(\mathbb{R}^n)$ be such that

$$|g - \chi_{\{f \le s\}}| \le \chi_{\{s \le f \le s+h\}}.$$

Then

$$\begin{aligned} &|T_{k_j} \llcorner \{f \le s\}(\omega) - T \llcorner \{f \le s\}(\omega)| \\ &\quad \le |T_{k_j} \llcorner \{f \le s\}(\omega) - T_{k_j}(g\omega)| + |T_{k_j}(g\omega) - T(g\omega)| + |T(g\omega) - T \llcorner \{f \le s\}(\omega)| \\ &\quad \le \|\omega\| \cdot \mu_{k_j}\big(\{s \le f \le s + h\} \cap \operatorname{spt}(\omega)\big) + |T_{k_j}(g\omega) - T(g\omega)| \\ &\qquad\qquad + \|\omega\| \cdot \mu\big(\{s \le f \le s + h\} \cap \operatorname{spt}(\omega)\big) \\ &\quad \le \varepsilon \cdot \big(2\|\omega\| + 1\big) \end{aligned}$$

for all sufficiently large j. This shows that $T_{k_j}\llcorner\{f \le s\} \rightharpoonup T\llcorner\{f \le s\}$ and one shows analogously that $(\partial T_{k_j})\llcorner\{f \le s\} \rightharpoonup (\partial T)\llcorner\{f \le s\}$. This proves that

$$\langle T_{k_j}, f, s\rangle \rightharpoonup \langle T, f, s\rangle$$

for all but countably many $s \in \mathbb{R}$.

As for the second statement, let λ denote the Lipschitz constant of f. By Theorem 5.40 and Fatou's lemma, we have

$$\begin{aligned}\int_{\mathbb{R}} \liminf_{j\to\infty} M\big(\langle T_{k_j}, f, s\rangle\big) ds &\le \liminf_{j\to\infty} \int_{\mathbb{R}} M\big(\langle T_{k_j}, f, s\rangle\big) ds \\ &\le \lambda \cdot \liminf_{j\to\infty} M(T_{k_j}) < \infty.\end{aligned}$$

Hence, for almost every $s \in \mathbb{R}$, there exists a subsequence $(T_{k_{j_l}})$ such that

$$\sup_l M\big(\langle T_{k_{j_l}}, f, s\rangle\big) < \infty.$$

Similarly for the boundary.

□

We now define iterated slices. Let $T \in N_m(\mathbb{R}^n)$ and let $\pi = (\pi_1, \cdots, \pi_k)\colon \mathbb{R}^n \to \mathbb{R}^k$ be a Lipschitz map for some $1 \le k \le m$. For almost every $x = (x_1, \cdots, x_k) \in \mathbb{R}^k$, the iterated slice

$$\langle T, \pi, x\rangle := \langle \cdots \langle\langle T, \pi_1, x_1\rangle, \pi_2, x_2\rangle, \cdots, \pi_k, x_k\rangle$$

is well-defined and defines an element in $N_{m-k}(\mathbb{R}^n)$. If $T \in I_m(\mathbb{R}^n)$, then $\langle T, \pi, x\rangle \in I_{m-k}(\mathbb{R}^n)$ for almost every $x \in \mathbb{R}^k$.

The following theorem, whose proof will be given in the next section, is the main ingredient in the proofs of the Closure and Boundary Rectifiability theorems.

Theorem 5.50 (Rectifiable slices)**.** Let $T \in N_m(\mathbb{R}^n)$ with $m \ge 1$. If $\langle T, \pi, x\rangle \in I_0(\mathbb{R}^n)$ for every Lipschitz map $\pi\colon \mathbb{R}^n \to \mathbb{R}^m$ and almost every $x \in \mathbb{R}^m$, then $T \in I_m(\mathbb{R}^n)$.

The converse is also true by the above. We use Theorem 5.50 to prove the Closure and Boundary Rectifiability theorems.

Proof of Theorem 5.29. We argue by induction on m. The case $m = 0$ follows from Proposition 5.48. Assume therefore that $m \ge 1$ and that Theorem 5.29 holds for $m - 1$.

Let $\pi = (\pi_1, \cdots, \pi_m)\colon \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz map. By Theorem 5.41 and Lemma 5.49, we have for almost every $s \in \mathbb{R}$ that $\langle T_k, \pi, s\rangle \in I_{m-1}(\mathbb{R}^n)$ for all $k \in \mathbb{N}$ and there exists a subsequence (T_{k_j}) such that

$$\langle T_{k_j}, \pi_1, s\rangle \rightharpoonup \langle T, \pi_1, s\rangle$$

and

$$\sup_j \Big(M\big(\langle T_{k_j}, \pi_1, s\rangle\big) + M\big(\partial\langle T_{k_j}, \pi_1, s\rangle\big)\Big) < \infty.$$

Since Theorem 5.29 is assumed to hold for $m - 1$ it follows that $\langle T, \pi_1, s\rangle \in I_{m-1}(\mathbb{R}^n)$ for almost every $s \in \mathbb{R}$ and hence

$$\langle T, \pi, x\rangle = \langle\langle T, \pi_1, x_1\rangle, (\pi_2, \cdots, \pi_m), (x_2, \cdots, x_m)\rangle \in I_0(\mathbb{R}^n)$$

for almost every $x = (x_1, \cdots, x_m) \in \mathbb{R}^m$. It follows from Theorem 5.50 that $T \in I_m(\mathbb{R}^n)$. This concludes the proof.

□

For the proof of the Boundary Rectifiability theorem, we will need the following characterization of integral 0-currents.

Lemma 5.51. Let $S \in M_0(\mathbb{R}^n)$. Then $S \in I_0(\mathbb{R}^n)$ if and only if $S(\chi_K) \in \mathbb{Z}$ for every compact set $K \subset \mathbb{R}^n$.

Recall that, by the Representation Theorem 5.10, S is of the form

$$S(f) = \int_{\mathbb{R}^n} f \cdot \overrightarrow{S}\, d\|S\|$$

for some finite Radon measure $\|S\|$ on $\mathbb{R}^n$ and some $\|S\|$-measurable function $\overrightarrow{S} : \mathbb{R}^n \to \mathbb{R}$ with $|\overrightarrow{S}(x)| = 1$ for $\|S\|$-almost every $x \in \mathbb{R}^n$. Therefore, the quantity

$$S(\chi_K) := \int_K \overrightarrow{S}\, d\|S\|$$

makes sense.

Proof. "$\Longrightarrow$": Clear.

"$\Longleftarrow$": Define

$$A := \Big\{ x \in \mathbb{R}^n : \|S\|(\{x\}) \geq 1 \Big\}$$

and notice that A is a finite set. Let $x \in \mathbb{R}^n \backslash A$. Then, there exists $r > 0$ such that $\|S\|(B(x,r)) < 1$. We claim that

$$\|S\|(B(x,r)) = 0.$$

For this, define

$$F_\pm := \Big\{ y \in B(x,r) : \overrightarrow{S}(y) = \pm 1 \Big\}.$$

For every compact subset $K \subset F_+$, we have $S(\chi_K) \in \mathbb{Z}$ and

$$S(\chi_K) = \|S\|(K) \leq \|S\|(B(x,r)) < 1$$

and hence $\|S\|(K) = 0$. Since

$$\|S\|(F_+) = \sup \Big\{ \|S\|(K) : K \subset F_+ \text{ compact } \Big\},$$

it follows that $\|S\|(F_+) = 0$. One shows analogously that $\|S\|(F_-) = 0$ and hence $\|S\|(B(x,r)) = 0$. It follows that $\|S\|(\mathbb{R}^n \backslash A) = 0$. Since

$$\|S\|(\{x\}) = |S(\chi_{\{x\}})| \in \mathbb{Z},$$

we finally obtain that $\|S\|$ is a finite sum of Dirac measure and thus $S \in I_0(\mathbb{R}^n)$. □

Proof of Theorem 5.31. The proof is by induction on m. We first prove the case $m = 1$. Let $T \in I_1(\mathbb{R}^n)$ and let $K \subset \mathbb{R}^n$ be compact. By Lemma 5.51, it suffices to show that $\partial T(\chi_K) \in \mathbb{Z}$. For this, define a Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ by

$$f(x) := \operatorname{dist}(x, K).$$

Fix $R > 0$ such that $K \subset B(0, R)$ and let $g \in C_c^\infty(\mathbb{R}^n)$ be such that $g = 1$ on $\overline{B}(0, 2R)$. For every $0 < r < R$, we have

$$\begin{aligned}(\partial T)(\chi_{\{f\leq r\}}) &= \big((\partial T)\llcorner\{f \leq r\}\big)(g)\\ &= \partial\big(T\llcorner\{f \leq r\}\big)(g) - \langle T, f, r\rangle(g)\\ &= T\big(\chi_{\{f\leq r\}}dg\big) - \langle T, f, r\rangle\big(\chi_{\overline{B}(0,2R)}\big)\\ &= -\langle T, f, r\rangle\big(\chi_{\overline{B}(0,2R)}\big).\end{aligned}$$

Since $\langle T, f, r\rangle \in I_0(\mathbb{R}^n)$, $\langle T, f, r\rangle\big(\chi_{\overline{B}(0,2R)}\big) \in \mathbb{Z}$ for almost every $0 < r < R$, and thus we see that

$$(\partial T)(\chi_{\{f\leq r\}}) \in \mathbb{Z}$$

for almost every $0 < r < R$. Since $\chi_{\{f\leq r\}} \to \chi_K$ as $r \to 0$, it follows that $(\partial T)(\chi_{\{f\leq r\}}) \in \mathbb{Z}$. Since K was arbitrary, it follows from Lemma 5.51 that $\partial T \in I_0(\mathbb{R}^n)$. This proves the case $m = 1$.

Suppose now that Theorem 5.31 holds for some $m \geq 1$. Let $T \in I_{m+1}(\mathbb{R}^n)$. Let $\pi = (\pi_1, \cdots, \pi_m)\colon \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz map. Since

$$\langle \partial T, \pi, x_1\rangle = -\langle T, \pi_1, x_1\rangle \in I_{m-1}(\mathbb{R}^n)$$

for almost every $x_1 \in \mathbb{R}$, it follows that

$$\langle \partial T, \pi, x\rangle = \langle\langle \partial T, \pi, x_1\rangle, (\pi_2, \cdots, \pi_m), (x_2, \cdots, x_m)\rangle \in I_0(\mathbb{R}^n)$$

for almost every $x = (x_1, \cdots, x_m) \in \mathbb{R}^m$. Theorem 5.50 implies that $\partial T \in I_m(\mathbb{R}^n)$. This completes the proof.

□

5.6. **MBV functions and the proof of the Slice-rectifiability Theorem.**

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