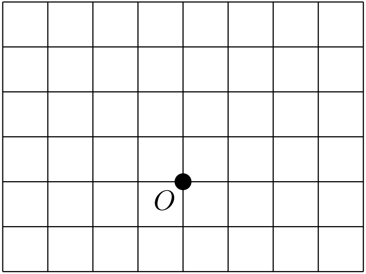
Delocalization of the endpoint of self-avoiding walk

Ioan Manolescu
Joint work with Hugo Duminil-Copin, Alexander Glazman, Alan Hammond

University of Geneva

27 June 2013

What is SAW?



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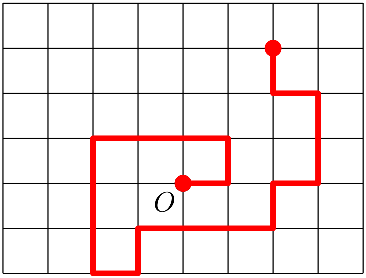

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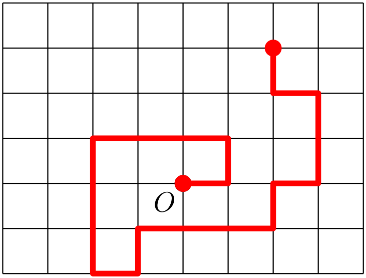

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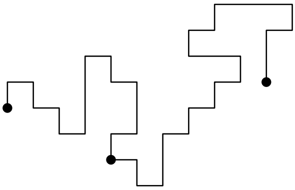
$c_n = |\mathrm{SAW}_n|$ $\mathbb{P}_n$ - uniform measure on SAW_n.

Connective constant

Sub-multiplicativity: $c_{m+n} \leq c_n c_m$.



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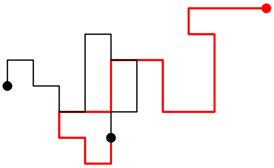

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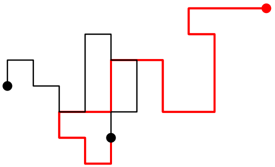

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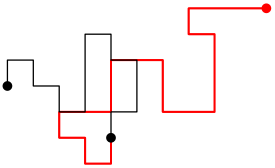

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For the hexagonal lattice $\mu_c(\mathbb{H}) = \sqrt{2+\sqrt{2}}$ (Duminil-Copin, Smirnov '11)

Critical exponent for the number of walks γ

We expect the existence of γ such that

$$c_n \approx A n^{\gamma-1} \mu_c^n.$$

γ only depends on the dimension:

$$\gamma = \begin{cases} 1 & \text{for } d = 1, \\ \frac{43}{32} & \text{for } d = 2, \\ 1.16\ldots & \text{for } d = 3, \\ 1 & \text{for } d = 4, \\ 1 & \text{for } d \geq 5. \end{cases}$$

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For general dimension: $\mu_c^n \leq c_n \leq e^{c\sqrt{n}} \mu_c^n$.

Critical exponent for the mean-square displacement

We expect the existence of ν such that

$$\mathbb{E}_n[||\gamma_n||^2] = n^{2\nu+o(1)}$$

ν only depends on the dimension:

$$\nu = \begin{cases} 1 & \text{for } d = 1, \\ \frac{3}{4} & \text{for } d = 2, \\ 0.59\ldots & \text{for } d = 3, \\ 1/2 & \text{for } d \geq 4, \end{cases}$$

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Known (in general dimension $d \geq 2$):

- Sub-balisticity: $\mathbb{P}_n(||\gamma_n|| \geq vn) \leq e^{-c_v n}$
- Madras lower bound: $\nu \geq \frac{2}{3d}$

Results

Theorem (Duminil-Copin, Glazman, Hammond, M.)

For any $x \in \mathbb{Z}^d$ and $\epsilon > 0$, for n large enough,

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Theorem (Duminil-Copin, Glazman, Hammond, M.)

$$\sup_{x \in \mathbb{Z}^d} \mathbb{P}_n(\gamma_n = x) \to 0.$$

Pattern: a piece of a walk that can occur in the middle of a walk.

Theorem (Kesten '63)

Let P be a pattern. There exist $\delta, c > 0$ such that

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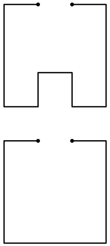

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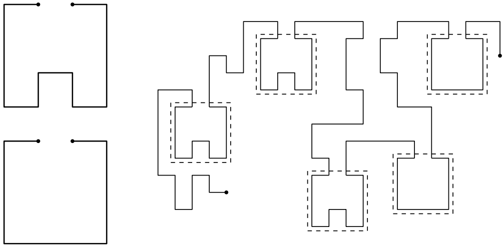

Theorem

$$c_{n+2}/c_n \to \mu_c^2.$$

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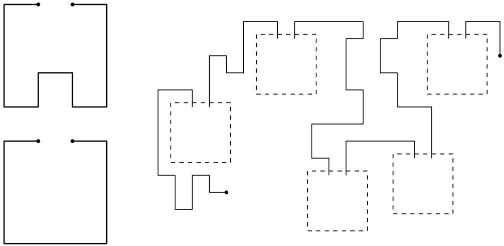

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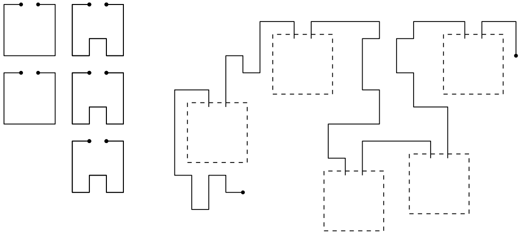

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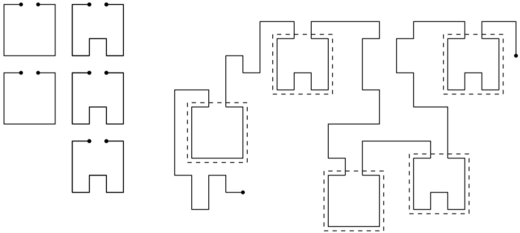

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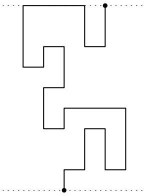
Self-avoiding bridges and the Hammersley-Welsh bound

Bridge: $\gamma \in \mathrm{SAW}_n$ such that

$$\langle \gamma_0 | e_1 \rangle < \langle \gamma_k | e_1 \rangle \leq \langle \gamma_n | e_1 \rangle, \qquad \text{for } 0 < k \leq n.$$

SAB_n is the set of bridges of length n.



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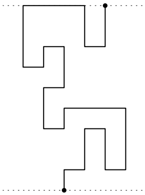

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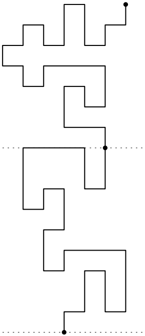

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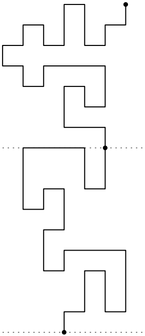
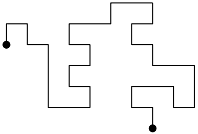


Theorem (Hammersley-Welsh '62)

There exists a constant c_{HW} such that

$$c_n \leq e^{c_{HW}\sqrt{n}}|SAB_n| \leq e^{c_{HW}\sqrt{n}}\mu_c^n.$$

Proof of the Hammersley-Welsh bound – for half space walks



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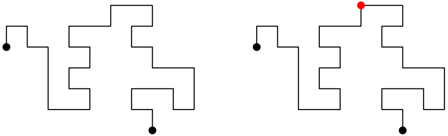

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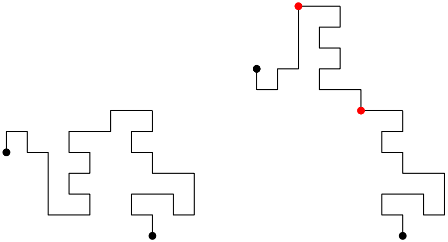

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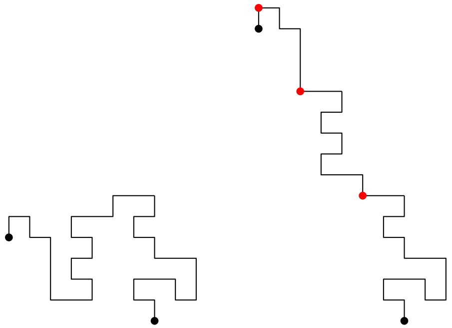

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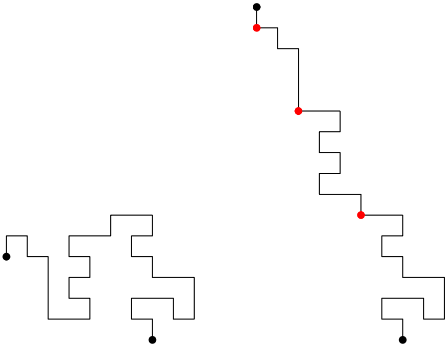

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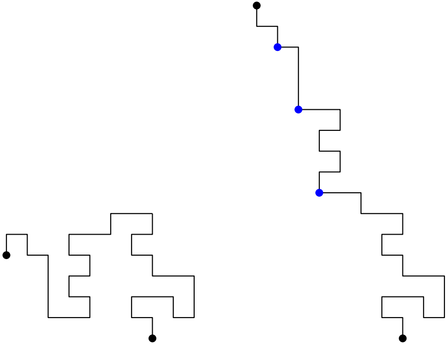

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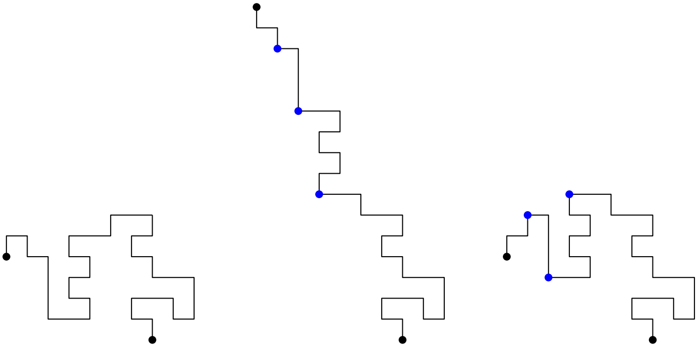

$\Phi : \mathrm{SAW}_n \to \mathrm{SAB}_n$.

$\Phi^{-1}(b) \leq e^{c\sqrt{n}}$. $\Rightarrow$ $|\mathrm{SAW}_n| \leq e^{c\sqrt{n}} |SAB_n|$

Closing walk

We say a walk closes if $||\gamma_n|| = 1$

Theorem (Duminil-Copin, Glazman, Hammond, M.)

For any $\epsilon > 0$ and n large enough,

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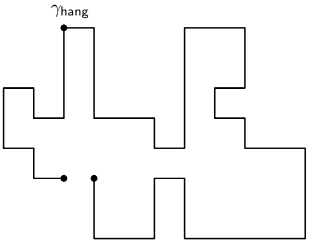
The proof is done by contradiction. Assume n is large and

$$\mathbb{P}_n(\gamma \text{ closes}) > n^{-\frac{1}{4}+\epsilon}.$$

The hanging point

γ_{hang} is the lexicographically maximal point of $\{\gamma_k : k = 0, \dots, n\}$.

$$\gamma^1 = \gamma[0, \text{hang}], \qquad \gamma^2 = \gamma[\text{hang}, n].$$



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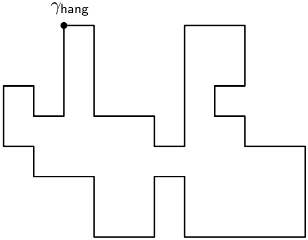

Polygonal invariance: The hanging index is uniform in a closing walk.

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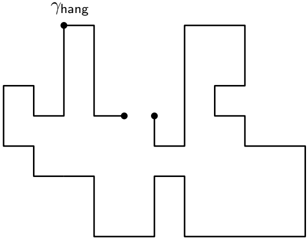

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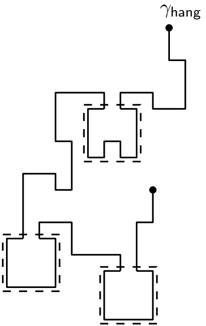
Fix some length $\ell_0 \in [\frac{n}{4}, \frac{3n}{4}]$ and say a walk χ is good if

$$\mathbb{P}_{n-\ell_0+|\chi|}(\gamma \text{ closes} \,|\, \gamma^1 = \chi) \geq n^{-\frac{1}{4}+\epsilon}.$$



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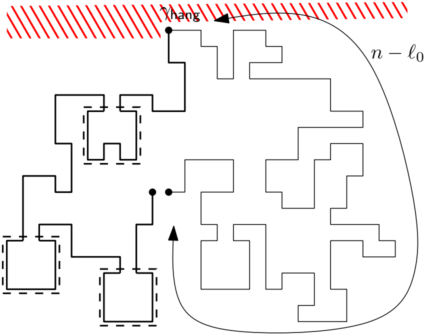

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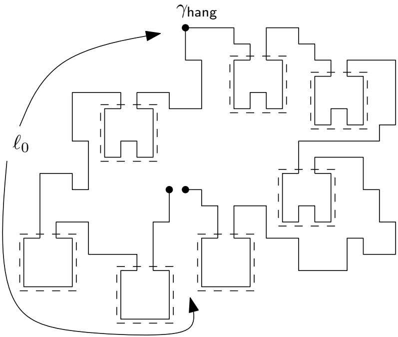

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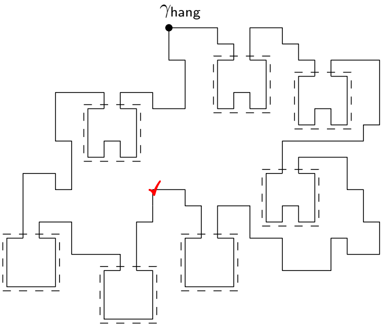

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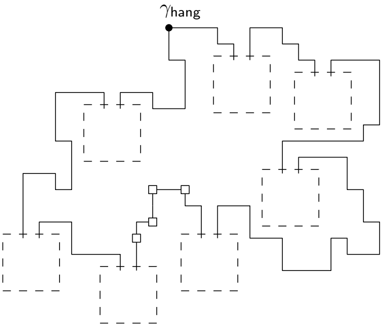

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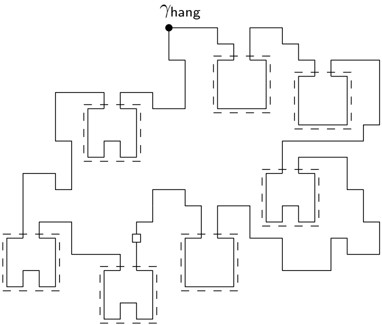

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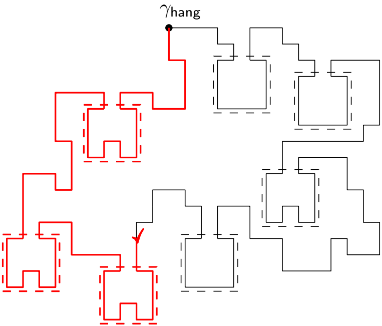
Fix some length $\ell_0 \in [\frac{n}{4}, \frac{3n}{4}]$ and say a walk χ is good if

$$\mathbb{P}_{n-\ell_0+|\chi|}(\gamma \text{ closes} \,|\, \gamma^1 = \chi) \geq n^{-\frac{1}{4}+\epsilon}.$$



Then $$\mathbb{P}_{n-\ell_0}(\gamma^1 \text{ is good} \,|\, \gamma \text{ closes}; |\gamma^1| = \ell_0) \geq n^{-\frac{1}{4}+\epsilon}.$$

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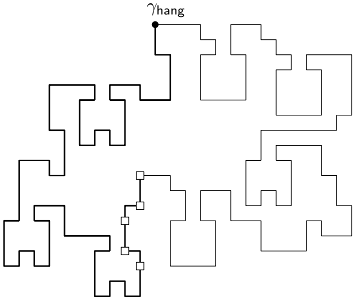

Let Γ be a uniform $n - \ell_0$ walk started at the end of χ^i and with hanging point χ_i.
Conditionally on avoiding χ^i, Γ closes χ^i with prob $n^{-\frac{1}{4}+\epsilon}$.

- The events $\{\Gamma \text{ closes } \chi^i\}$ are mutually exclusive
- The events $\{\Gamma \text{ avoids } \chi^i\}$ are decreasing in i.

Conclusion: with $\mathbb{P}_n(.\,||\gamma^1| = \ell_0; \gamma \text{ closes})$ - probability at least $n^{-\frac{1}{4}+\epsilon}$

$$\mathbb{P}\left(\Gamma \text{ avoids } \gamma^1\right) \leq e^{-n^{\epsilon}}.$$

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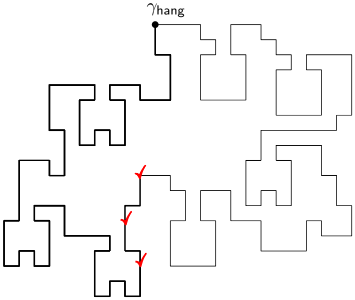

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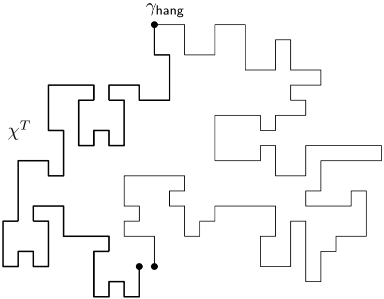

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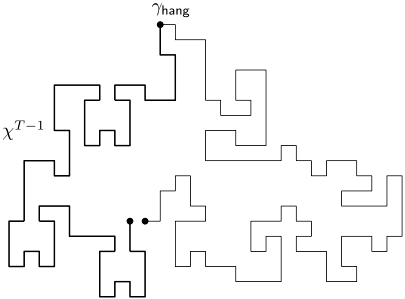

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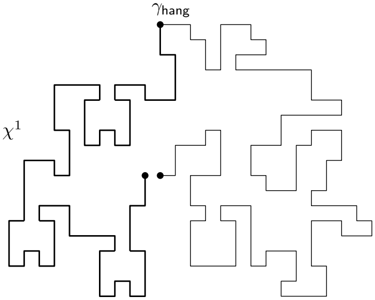

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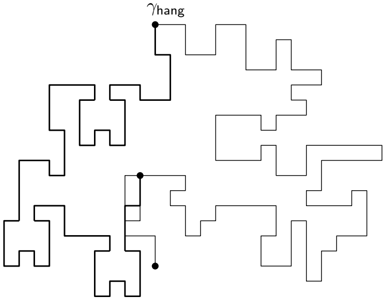

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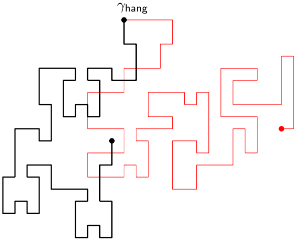
If χ, with $|\chi| = \ell_0$, is such that

$$\mathbb{P}\left(\Gamma \text{ avoids } \chi\right) \leq e^{-n^{\epsilon}},$$

call χ *untouchable*.

For such χ, using unfoldings,

$$\mathbb{P}_n\left(\text{hang}(\gamma) = \ell_0 \,|\, \gamma[0, \ell_0] = \chi\right) \leq e^{-n^{\epsilon}}.$$



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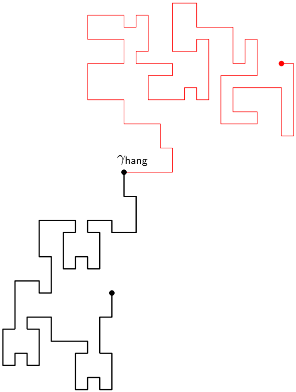

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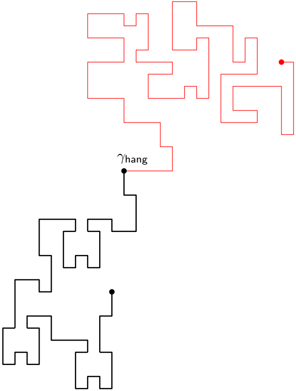

Conclusion:

$$n^{-5/2} \leq \mathbb{P}_n\left(\gamma[0, \ell_0] \text{ untouchable; hang}(\gamma) = \ell_0\right) \leq e^{-n^{\epsilon}}.$$

CONTRADICTION!!!

Thank you!