First order phase transition for the Random Cluster model with $q > 4$

Ioan Manolescu

joint work with:
Hugo Duminil-Copin, Maxime Gagnebin, Matan Harel, Vincent Tassion

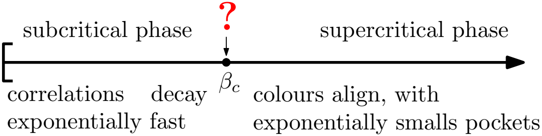
University of Fribourg

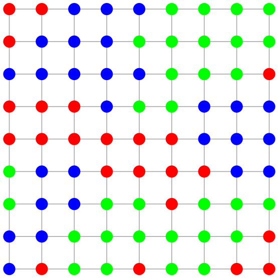
14th February 2017
Diablerets

Setting: G is a finite subgraph of $\mathbb{Z}^2$.

Potts model with $q \geq 2$ states on $G = (V, E)$: $\sigma \in \{1, \dots, q\}^V$ with probability

$$\mu_{\beta,G,q}(\sigma) = \frac{1}{Z} \exp\Big(\beta \sum_{(u,v)\in E} 1_{\sigma(\mathrm{u})=\sigma(\mathrm{v})}\Big) \qquad \beta > 0$$

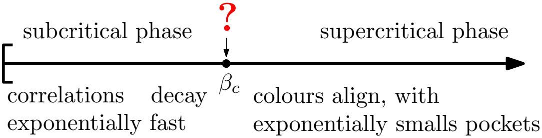


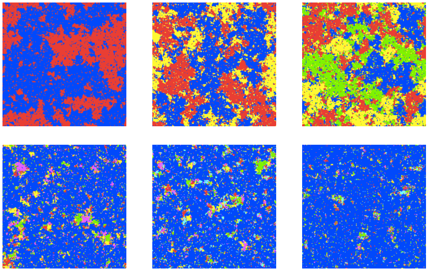


Setting: G is a finite subgraph of $\mathbb{Z}^2$.

Potts model with $q \geq 2$ states on $G = (V, E)$: $\sigma \in \{1, \ldots, q\}^V$ with probability

$$\mu_{\beta,G,q}(\sigma) = \frac{1}{Z} \exp\Big(\beta \sum_{(u,v)\in E} 1_{\sigma(\mathrm{u})=\sigma(\mathrm{v})}\Big) \qquad \beta > 0$$

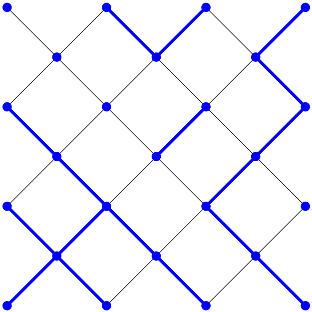




Setting: G is a finite subgraph of $\mathbb{Z}^2$.

Random Cluster model: parameters $q \geq 1$ and $p \in [0,1]$ on $G = (V, E)$:

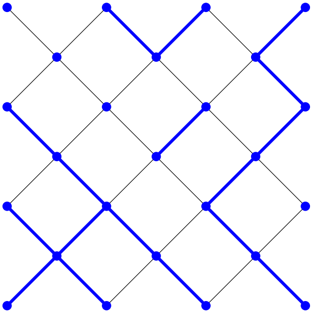
$\omega \in \{0,1\}^E$ with probability $\quad \Phi_{p,G,q}(\omega) = \frac{1}{Z_{p,G,q}} p^{o(\omega)} (1-p)^{c(\omega)} q^{k(\omega)}.$



Setting: G is a finite subgraph of $\mathbb{Z}^2$.

Random Cluster model: parameters $q \geq 1$ and $p \in [0,1]$ on $G = (V, E)$:

$\omega \in \{0,1\}^E$ with probability $\Phi^{0/1}_{p,G,q}(\omega) = \frac{1}{Z_{p,G,q}} p^{o(\omega)} (1-p)^{c(\omega)} q^{k^{0/1}(\omega)}$.



Setting: G is a finite subgraph of $\mathbb{Z}^2$.

Random Cluster model: parameters $q \geq 1$ and $p \in [0,1]$ on $G = (V, E)$:

$\omega \in \{0,1\}^E$ with probability $$\Phi^{0/1}_{p,G,q}(\omega) = \frac{1}{Z_{p,G,q}} p^{o(\omega)}(1-p)^{c(\omega)} q^{k^{0/1}(\omega)}.$$

Infinite volume measures on $\mathbb{Z}^2$ may be defined by taking limits:

$$\Phi^0_{p,G,q} \xrightarrow[G \to \mathbb{Z}^2]{} \Phi^0_{p,q} \quad \text{and} \quad \Phi^1_{p,G,q} \xrightarrow[G \to \mathbb{Z}^2]{} \Phi^1_{p,q}. \qquad \Phi^0_{p,q} \leq \Phi^1_{p,q}.$$

Setting: G is a finite subgraph of $\mathbb{Z}^2$.

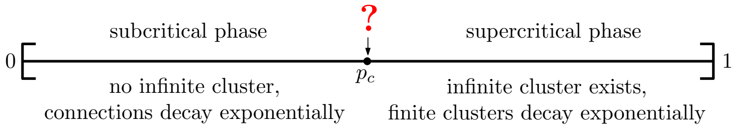
Random Cluster model: parameters $q \geq 1$ and $p \in [0,1]$ on $G = (V, E)$:

$\omega \in \{0,1\}^E$ with probability
$$\Phi^{0/1}_{p,G,q}(\omega) = \frac{1}{Z_{p,G,q}} p^{o(\omega)}(1-p)^{c(\omega)} q^{k^{0/1}(\omega)}.$$

Infinite volume measures on $\mathbb{Z}^2$ may be defined by taking limits:

$$\Phi^0_{p,G,q} \xrightarrow[G\to\mathbb{Z}^2]{} \Phi^0_{p,q} \quad \text{and} \quad \Phi^1_{p,G,q} \xrightarrow[G\to\mathbb{Z}^2]{} \Phi^1_{p,q}. \qquad \Phi^0_{p,q} \leq \Phi^1_{p,q}.$$

Phase transition in terms of infinite cluster ($\Phi_{p,q}$ increasing in p)



Setting: G is a finite subgraph of $\mathbb{Z}^2$.

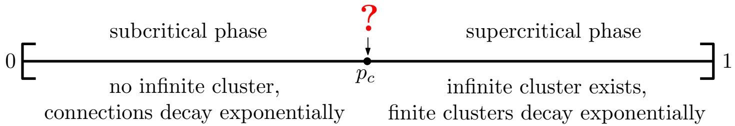
Random Cluster model: parameters $q \geq 1$ and $p \in [0,1]$ on $G = (V, E)$:

$\omega \in \{0,1\}^E$ with probability $$\Phi^{0/1}_{p,G,q}(\omega) = \frac{1}{Z_{p,G,q}} p^{o(\omega)}(1-p)^{c(\omega)} q^{k^{0/1}(\omega)}.$$

Infinite volume measures on $\mathbb{Z}^2$ may be defined by taking limits:

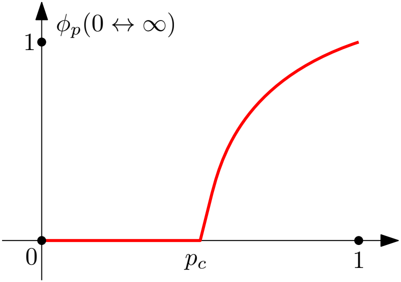
$$\Phi^0_{p,G,q} \xrightarrow[G\to\mathbb{Z}^2]{} \Phi^0_{p,q} \quad \text{and} \quad \Phi^1_{p,G,q} \xrightarrow[G\to\mathbb{Z}^2]{} \Phi^1_{p,q}. \qquad \Phi^0_{p,q} \leq \Phi^1_{p,q}.$$

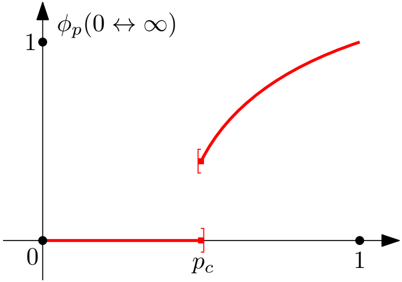
Phase transition in terms of infinite cluster ($\Phi_{p,q}$ increasing in p)



Theorem (Beffara, Duminil-Copin 2012)

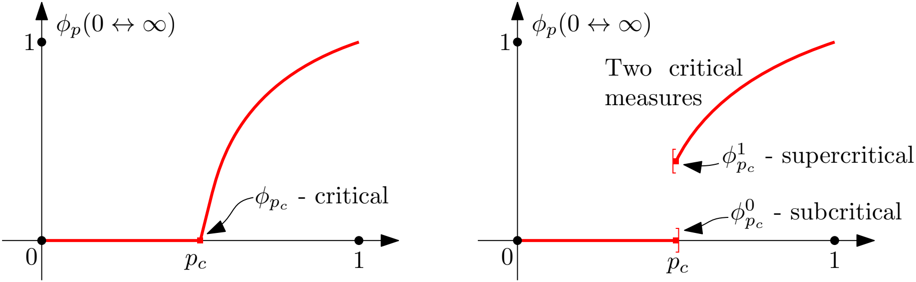
On $\mathbb{Z}^2$, $p_c = \frac{\sqrt{q}}{1+\sqrt{q}}$ (in other words $p_c = p_{sd}$, the self-dual parameter).





Theorem (Duminil-Copin, Sidoravicius, Tassion 2015)

Phase transition: either one or the other.



Theorem (Duminil-Copin, Sidoravicius, Tassion 2015)

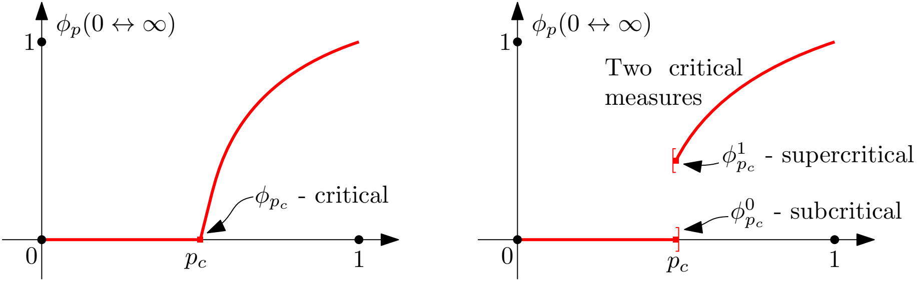
Phase transition: either one or the other.

Continuous phase transition:

- $\phi^0_{p_c} = \phi^1_{p_c}$;
- *in* ϕ_{p_c} *connections decrease polynomially;*
- *no infinite cluster for* ϕ_{p_c};
- *strong RSW type estimates.*

. . . or discontinuous:

- $\phi^0_{p_c} \neq \phi^1_{p_c}$;
- *in* $\phi^0_{p_c}$ *connections decrease exponentially,*
- *infinite cluster in* $\phi^1_{p_c}$.



Theorem (Duminil-Copin, Sidoravicius, Tassion 2015)

Phase transition: either one or the other.

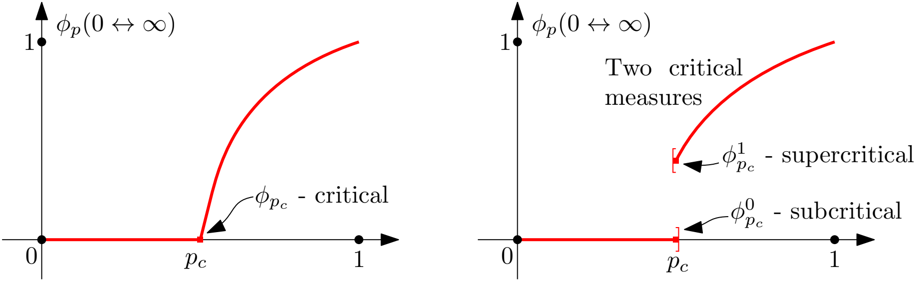
Continuous phase transition:

- $\phi^0_{p_c} = \phi^1_{p_c}$;
- *in* ϕ_{p_c} *connections decrease polynomially;*
- *no infinite cluster for* ϕ_{p_c};
- *strong RSW type estimates.*

...or discontinuous:

- $\phi^0_{p_c} \neq \phi^1_{p_c}$;
- *in* $\phi^0_{p_c}$ *connections decrease exponentially,*
- *infinite cluster in* $\phi^1_{p_c}$.

When $q \in [1, 4]$



Theorem (Duminil-Copin, Sidoravicius, Tassion 2015)

Phase transition: either one or the other.

Continuous phase transition:

- $\phi^0_{p_c} = \phi^1_{p_c}$;
- *in* ϕ_{p_c} *connections decrease polynomially;*
- *no infinite cluster for* ϕ_{p_c};
- *strong RSW type estimates.*

When $q \in [1, 4]$

. . . or discontinuous:

- $\phi^0_{p_c} \neq \phi^1_{p_c}$;
- *in* $\phi^0_{p_c}$ *connections decrease exponentially,*
- *infinite cluster in* $\phi^1_{p_c}$.

(HDC,MH,MG,IM,VT 2017):

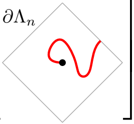
. . . when $q > 4$

Theorem (H. Duminil-Copin, M. Gagnebin, M. Harel, I.M., V. Tassion)

The phase transition of RCM on the square lattice with $q > 4$ **is discontinuous**.

Moreover, if $\lambda > 0$ satisfies $\cosh(\lambda) = \sqrt{q}/2$, then

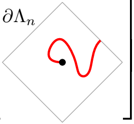
$$\xi(q)^{-1} = \lim_{n\to\infty} -\frac{1}{n} \log \phi^0_{p_c,q} \left[\quad \right]$$



$$= \lambda + 2\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \tanh(k\lambda) > 0.$$

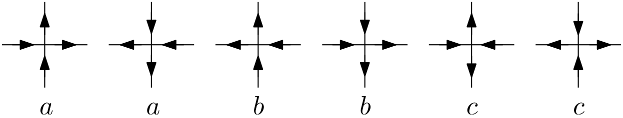
As $q \searrow 4$, $\quad \xi(q)^{-1} \sim 8 \exp\left(-\frac{\pi^2}{\sqrt{q-4}}\right).$

$$\phi^0_{p_c,q} \left[\quad \right]$$



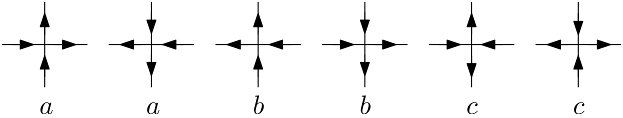
$$= \exp\Big(-\frac{n}{\xi(q)} + o(n)\Big)$$

Relation the six vertex model.



A brief introduction to the six vertex model.

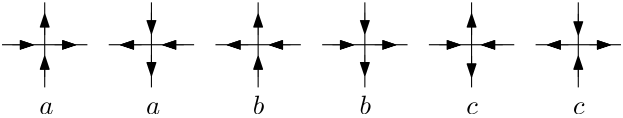
Configurations: On a part of $\mathbb{Z}^2$: orient each edge s.t. each vertex has **exactly two incoming edges**.



Weight: $a^{n_1+n_2} \cdot b^{n_3+n_4} \cdot c^{n_5+n_6}$

A brief introduction to the six vertex model.

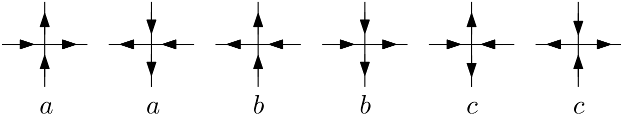
Configurations: On a part of $\mathbb{Z}^2$: orient each edge s.t. each vertex has **exactly two incoming edges**.



Probability: $$\frac{1}{Z_{6V}} a^{n_1+n_2} \cdot b^{n_3+n_4} \cdot c^{n_5+n_6}$$

A brief introduction to the six vertex model.

Configurations: On a part of $\mathbb{Z}^2$: orient each edge s.t. each vertex has **exactly two incoming edges**.

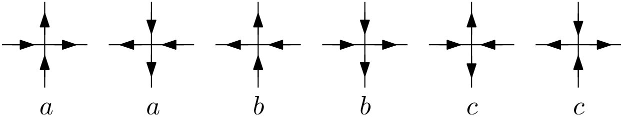


Probability: $$\frac{1}{Z_{6V}} a^{n_1+n_2} \cdot b^{n_3+n_4} \cdot c^{n_5+n_6}$$

We limit ourselves to $a = b = 1$ and $c \geq 2$
($\Delta = \frac{a^2+b^2-c^2}{2ab} < -1$: anti-ferroelectric phase).

A brief introduction to the six vertex model.

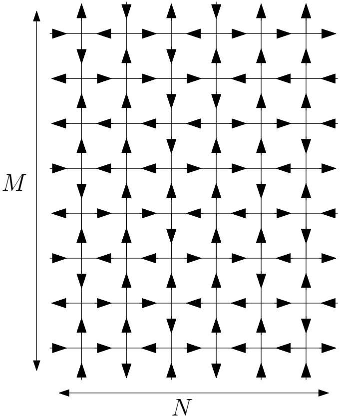
Configurations: On a part of $\mathbb{Z}^2$: orient each edge s.t. each vertex has **exactly two incoming edges**.



Probability: $$\frac{1}{Z_{6V}} c^{n_5+n_6}$$

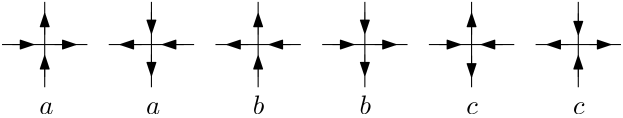
We limit ourselves to $a = b = 1$ and $c \geq 2$ ($\Delta = \frac{a^2+b^2-c^2}{2ab} < -1$: anti-ferroelectric phase).

Torus $\mathbb{T}_{N,M}$, with $M \to \infty$ then $N \to \infty$.



A brief introduction to the six vertex model.

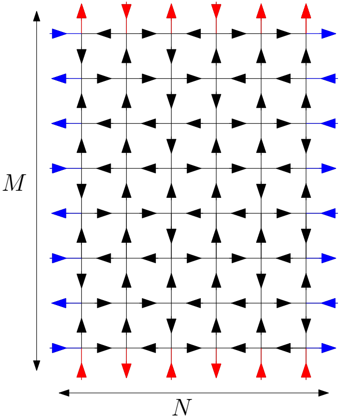
Configurations: On a part of $\mathbb{Z}^2$: orient each edge s.t. each vertex has **exactly two incoming edges**.



Probability: $$\frac{1}{Z_{6V}} c^{n_5+n_6}$$

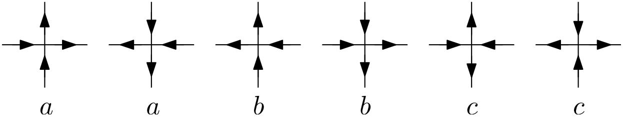
We limit ourselves to $a = b = 1$ and $c \geq 2$ ($\Delta = \frac{a^2+b^2-c^2}{2ab} < -1$: anti-ferroelectric phase).

Torus $\mathbb{T}_{N,M}$, with $M \to \infty$ then $N \to \infty$.



A brief introduction to the six vertex model.

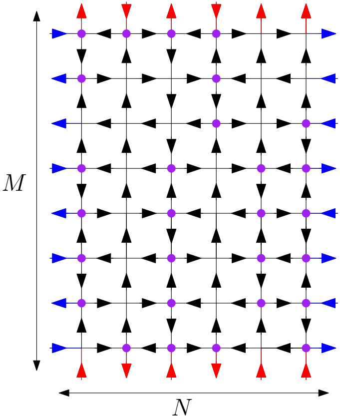
Configurations: On a part of $\mathbb{Z}^2$: orient each edge s.t. each vertex has **exactly two incoming edges**.



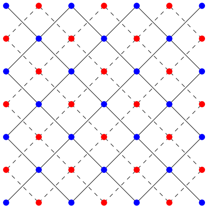
Probability: $$\frac{1}{Z_{6V}} c^{n_5+n_6}$$

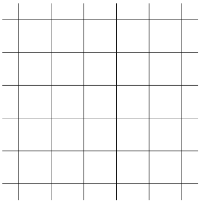
We limit ourselves to $a = b = 1$ and $c \geq 2$
($\Delta = \frac{a^2+b^2-c^2}{2ab} < -1$: anti-ferroelectric phase).

Torus $\mathbb{T}_{N,M}$, with $M \to \infty$ then $N \to \infty$.

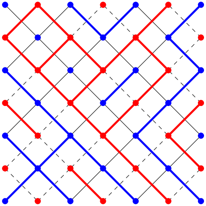


From random cluster to six vertex.

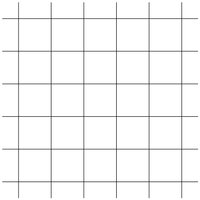




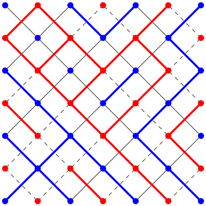
From random cluster to six vertex.



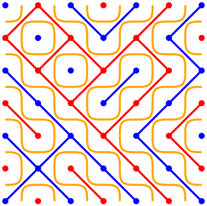
ω



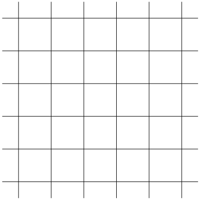
From random cluster to six vertex.



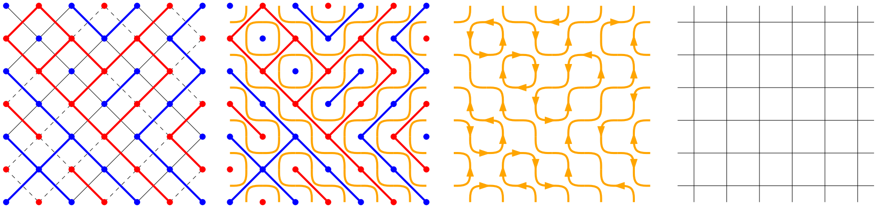
ω



$\omega^{(\ell)}$

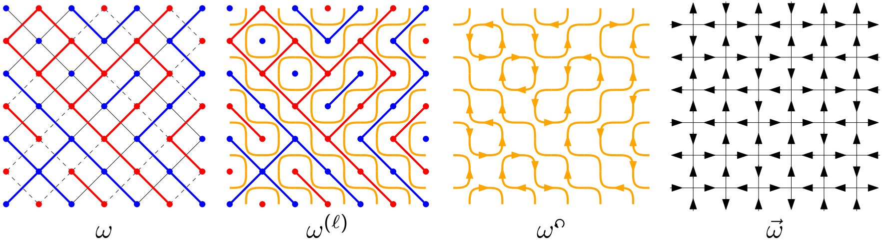


From random cluster to six vertex.

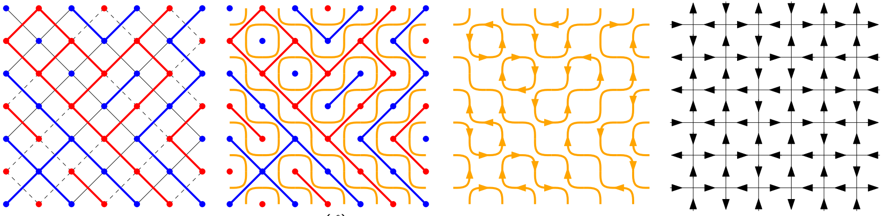


ω $\omega^{(\ell)}$ $\omega^{\curvearrowleft}$

From random cluster to six vertex.



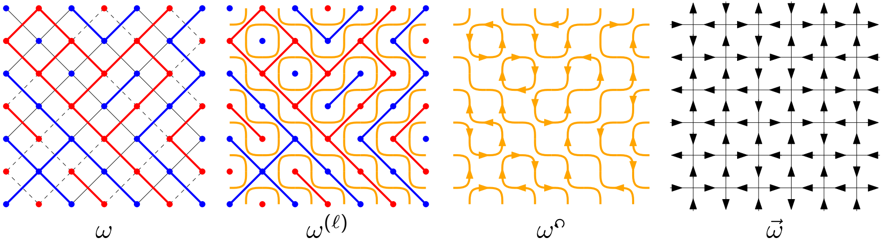
From random cluster to six vertex.



ω $\quad$ $\omega^{(\ell)}$ $\quad$ $\omega^{\circlearrowleft}$ $\quad$ $\vec{\omega}$

$$w_{\mathrm{RC}}(\omega) = p_{sd}^{o(\omega)}(1-p_{sd})^{c(\omega)}q^{k(\omega)}$$

From random cluster to six vertex.

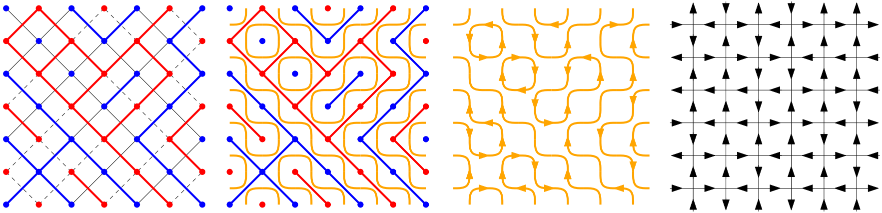


$$w_{\mathrm{RC}}(\omega) = p_{sd}^{o(\omega)}(1-p_{sd})^{c(\omega)}q^{k(\omega)}$$

$$= (1-p_{sd})^{|E|}\left(\frac{p_{sd}}{1-p_{sd}}\right)^{o(\omega)} q^{k(\omega)}$$

$$p_{sd} = p_c = \frac{\sqrt{q}}{1+\sqrt{q}}$$

From random cluster to six vertex.

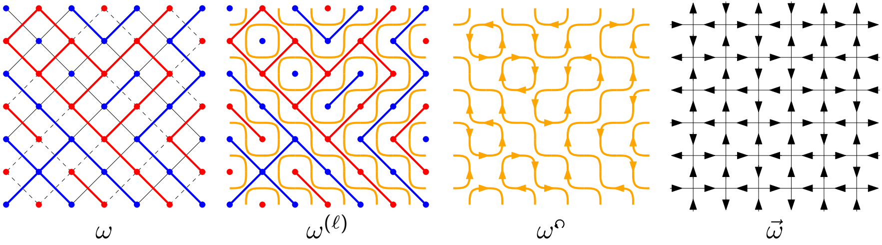


ω $\qquad$ $\omega^{(\ell)}$ $\qquad$ $\omega^{\circlearrowleft}$ $\qquad$ $\vec{\omega}$

$$
\begin{aligned}
w_{\mathrm{RC}}(\omega) &= p_{sd}^{o(\omega)}(1-p_{sd})^{c(\omega)}q^{k(\omega)} \\
&= (1-p_{sd})^{|E|}\left(\tfrac{p_{sd}}{1-p_{sd}}\right)^{o(\omega)} q^{k(\omega)} \\
&= \left(\tfrac{1}{1+\sqrt{q}}\right)^{|E|}\sqrt{q}^{\,o(\omega)}\; q^{k(\omega)}
\end{aligned}
$$

$$p_{sd} = p_c = \tfrac{\sqrt{q}}{1+\sqrt{q}}$$

From random cluster to six vertex.

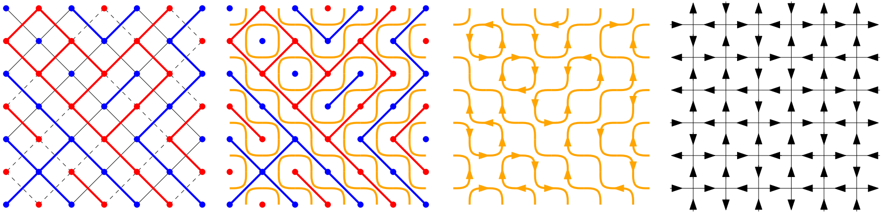


$$w_{\mathrm{RC}}(\omega) = p_{sd}^{o(\omega)}(1-p_{sd})^{c(\omega)}q^{k(\omega)}$$

$$= (1-p_{sd})^{|E|}\left(\frac{p_{sd}}{1-p_{sd}}\right)^{o(\omega)} q^{k(\omega)}$$

$$= \left(\frac{1}{1+\sqrt{q}}\right)^{|E|} \sqrt{q}^{2k(\omega)+o(\omega)}$$

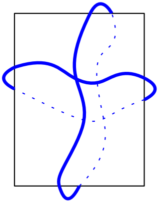
From random cluster to six vertex.



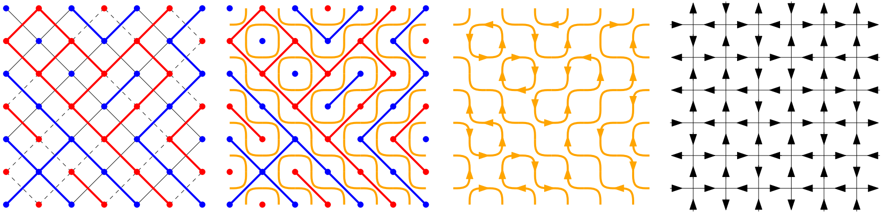
ω $\qquad$ $\omega^{(\ell)}$ $\qquad$ $\omega^{\circlearrowleft}$ $\qquad$ $\vec{\omega}$

$$
\begin{aligned}
w_{\mathrm{RC}}(\omega) &= p_{sd}^{o(\omega)}(1-p_{sd})^{c(\omega)}q^{k(\omega)} \\
&= (1-p_{sd})^{|E|}\left(\tfrac{p_{sd}}{1-p_{sd}}\right)^{o(\omega)} q^{k(\omega)} \\
&= \left(\tfrac{1}{1+\sqrt{q}}\right)^{|E|}\ \sqrt{q}^{2k(\omega)+o(\omega)}
\end{aligned}
$$

$$2k(\omega)+o(\omega)=\ell(\omega)+2s(\omega)+|V|$$



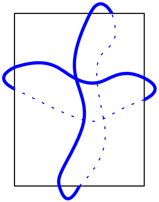
From random cluster to six vertex.



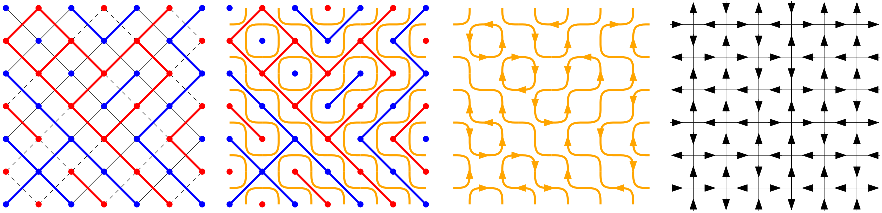
ω $\omega^{(\ell)}$ $\omega^{\circlearrowleft}$ $\vec{\omega}$

$$
\begin{aligned}
w_{\mathrm{RC}}(\omega) &= p_{sd}^{o(\omega)}(1-p_{sd})^{c(\omega)}q^{k(\omega)} \\
&= (1-p_{sd})^{|E|}\left(\tfrac{p_{sd}}{1-p_{sd}}\right)^{o(\omega)} q^{k(\omega)} \\
&= \left(\tfrac{1}{1+\sqrt{q}}\right)^{|E|} \sqrt{q}^{2k(\omega)+o(\omega)} \\
&= \left(\tfrac{1}{1+\sqrt{q}}\right)^{|E|} \sqrt{q}^{|V|}\sqrt{q}^{\ell(\omega)+2s(\omega)}
\end{aligned}
$$

$$2k(\omega)+o(\omega) = \ell(\omega)+2s(\omega)+|V|$$



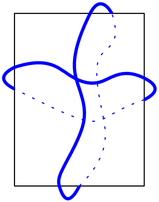
From random cluster to six vertex.



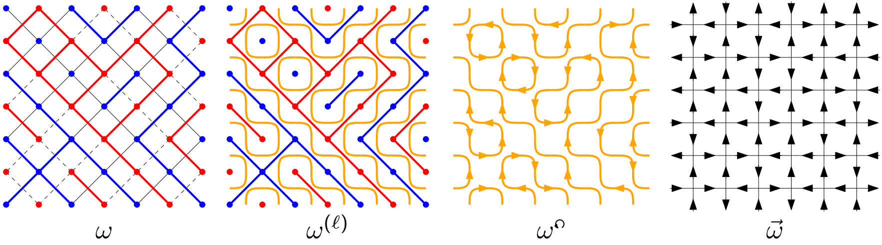
ω $\omega^{(\ell)}$ $\omega^{\circlearrowleft}$ $\vec{\omega}$

$$
\begin{aligned}
w_{\mathrm{RC}}(\omega) &= p_{sd}^{o(\omega)}(1-p_{sd})^{c(\omega)} q^{k(\omega)} \\
&= (1-p_{sd})^{|E|} \left(\tfrac{p_{sd}}{1-p_{sd}}\right)^{o(\omega)} q^{k(\omega)} \\
&= \left(\tfrac{1}{1+\sqrt{q}}\right)^{|E|} \sqrt{q}^{2k(\omega)+o(\omega)} \\
&= \left(\tfrac{1}{1+\sqrt{q}}\right)^{|E|} \sqrt{q}^{|V|} \sqrt{q}^{\ell(\omega)+2s(\omega)} \\
&= C \sqrt{q}^{\ell(\omega)+2s(\omega)}
\end{aligned}
$$

$$2k(\omega)+o(\omega) = \ell(\omega)+2s(\omega)+|V|$$

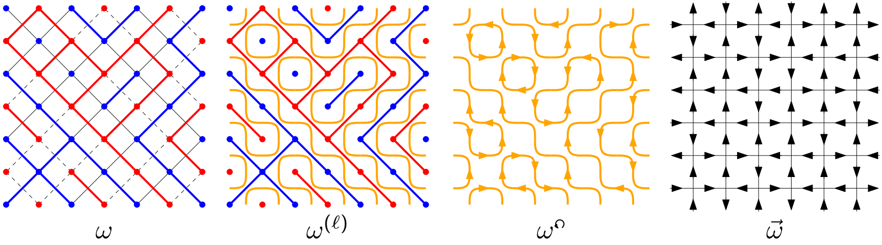


From random cluster to six vertex.



$$w_{\mathrm{RC}}(\omega) = C\ \sqrt{q}^{\ell(\omega)+2s(\omega)}$$

From random cluster to six vertex.

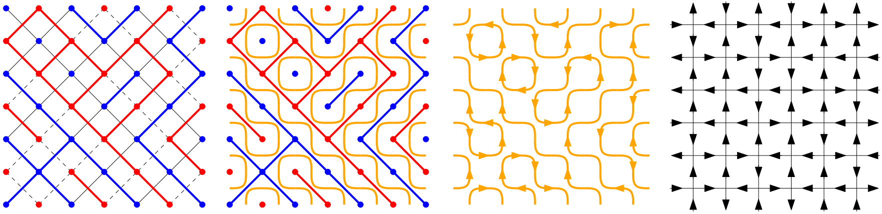


$w_{\mathrm{RC}}(\omega) = C\ \sqrt{q}^{\ell(\omega)+2s(\omega)}$

$w_{\circlearrowleft}(\omega^{\circlearrowleft}) = \exp\left[\frac{\lambda}{2\pi} \times \text{total winding}\right]$,

where $\cosh\lambda = \frac{\sqrt{q}}{2}$

From random cluster to six vertex.



ω $\qquad$ $\omega^{(\ell)}$ $\qquad$ $\omega^{\curvearrowright}$ $\qquad$ $\vec{\omega}$

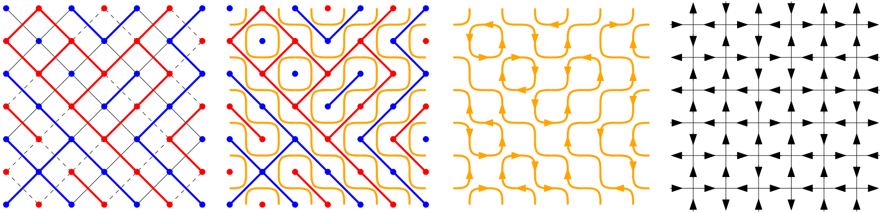
$$w_{\mathrm{RC}}(\omega) = C\ \sqrt{q}^{\ell(\omega)+2s(\omega)}$$
$$= C\ q^{s(\omega)}\left(\tfrac{\sqrt{q}}{2}\right)^{\ell_0(\omega)} \sum_{\omega^{\curvearrowright}} w_{\curvearrowright}(\omega^{\curvearrowright})$$

$$w_{\curvearrowright}(\omega^{\curvearrowright}) = \exp\left[\tfrac{\lambda}{2\pi} \times \text{total winding}\right],$$

where $\cosh\lambda = \frac{\sqrt{q}}{2}$

$$\sqrt{q} = e^{\lambda} + e^{-\lambda}$$

From random cluster to six vertex.



ω $\quad$ $\omega^{(\ell)}$ $\quad$ $\omega^{\curvearrowleft}$ $\quad$ $\vec{\omega}$

$$w_{\mathrm{RC}}(\omega) = C\ \sqrt{q}^{\ell(\omega)+2s(\omega)}$$
$$= C\ q^{s(\omega)}\Big(\tfrac{\sqrt{q}}{2}\Big)^{\ell_0(\omega)} \sum_{\omega^{\curvearrowleft}} w_{\curvearrowleft}(\omega^{\curvearrowleft})$$

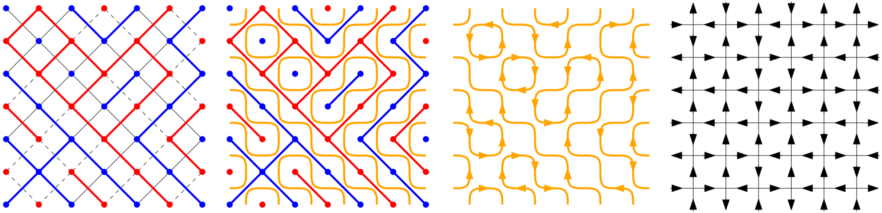
$$w_{6\mathrm{V}}(\vec{\omega}) = c^{n_5(\vec{\omega})+n_6(\vec{\omega})}$$

$$w_{\curvearrowleft}(\omega^{\curvearrowleft}) = \exp\Big[\tfrac{\lambda}{2\pi} \times \text{total winding}\Big],$$

where $\cosh\lambda = \frac{\sqrt{q}}{2}$

$$c = e^{\frac{\lambda}{2}} + e^{-\frac{\lambda}{2}} = \sqrt{2+\sqrt{q}}$$

From random cluster to six vertex.



ω $\qquad$ $\omega^{(\ell)}$ $\qquad$ $\omega^{\frown}$ $\qquad$ $\vec{\omega}$

$$w_{\mathrm{RC}}(\omega) = C\ \sqrt{q}^{\,\ell(\omega)+2s(\omega)}$$
$$= C\ q^{s(\omega)}\left(\tfrac{\sqrt{q}}{2}\right)^{\ell_0(\omega)} \sum_{\omega^{\frown}} w_{\frown}(\omega^{\frown})$$

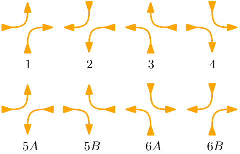
$$w_{6\mathrm{V}}(\vec{\omega}) = c^{n_5(\vec{\omega})+n_6(\vec{\omega})} = \sum_{\omega^{\frown}} w_{\frown}(\omega^{\frown}).$$

$$w_{\frown}(\omega^{\frown}) = e^{\frac{\lambda}{2}\#A - \frac{\lambda}{2}\#B}$$

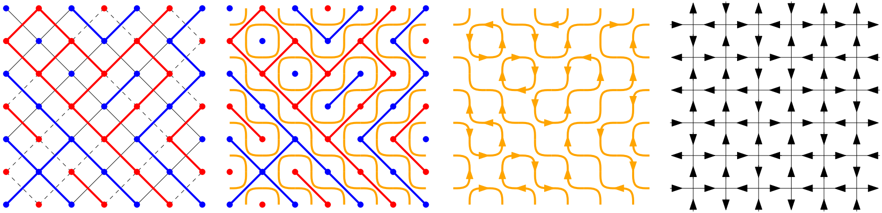
$$w_{\frown}(\omega^{\frown}) = \exp\left[\tfrac{\lambda}{2\pi} \times \text{total winding}\right],$$

where $\cosh\lambda = \frac{\sqrt{q}}{2}$

$$c = e^{\frac{\lambda}{2}} + e^{-\frac{\lambda}{2}} = \sqrt{2+\sqrt{q}}$$



From random cluster to six vertex.



ω $\qquad$ $\omega^{(\ell)}$ $\qquad$ $\omega^{\curvearrowleft}$ $\qquad$ $\vec{\omega}$

$$
\begin{aligned}
w_{\mathrm{RC}}(\omega) &= C \ \sqrt{q}^{\ell(\omega)+2s(\omega)} \\
&= C \ q^{s(\omega)} \left(\tfrac{\sqrt{q}}{2}\right)^{\ell_0(\omega)} \sum_{\omega^{\curvearrowleft}} w_{\curvearrowleft}(\omega^{\curvearrowleft})
\end{aligned}
$$

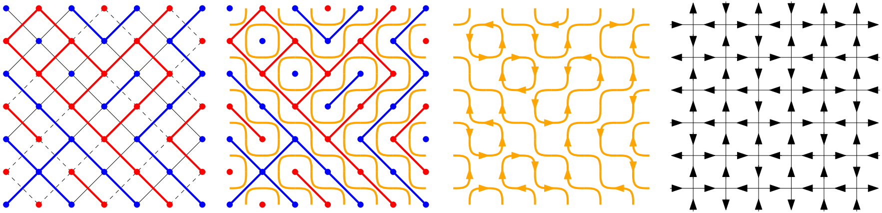
$$w_{6\mathrm{V}}(\vec{\omega}) = c^{n_5(\vec{\omega})+n_6(\vec{\omega})} \ = \sum_{\omega^{\curvearrowleft}} w_{\curvearrowleft}(\omega^{\curvearrowleft}).$$

$$w_{\curvearrowleft}(\omega^{\curvearrowleft}) = \exp\left[\tfrac{\lambda}{2\pi} \times \text{total winding}\right],$$

where $\cosh \lambda = \frac{\sqrt{q}}{2}$

$$c = e^{\frac{\lambda}{2}} + e^{-\frac{\lambda}{2}} = \sqrt{2+\sqrt{q}}$$

From random cluster to six vertex.



ω $\quad$ $\omega^{(\ell)}$ $\quad$ $\omega^{\circlearrowleft}$ $\quad$ $\vec{\omega}$

$$w_{\mathrm{RC}}(\omega) = C\ \sqrt{q}^{\ell(\omega)+2s(\omega)}$$
$$= C\ q^{s(\omega)}\left(\tfrac{\sqrt{q}}{2}\right)^{\ell_0(\omega)} \sum_{\omega^{\circlearrowleft}} w_{\circlearrowleft}(\omega^{\circlearrowleft})$$

$$w_{\circlearrowleft}(\omega^{\circlearrowleft}) = \exp\left[\tfrac{\lambda}{2\pi} \times \text{total winding}\right],$$

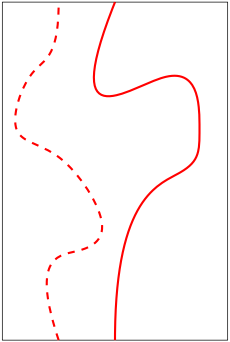
where $\cosh\lambda = \frac{\sqrt{q}}{2}$

$$c = e^{\frac{\lambda}{2}} + e^{-\frac{\lambda}{2}} = \sqrt{2+\sqrt{q}}$$

$$w_{6\mathrm{V}}(\vec{\omega}) = c^{n_5(\vec{\omega})+n_6(\vec{\omega})} = \sum_{\omega^{\circlearrowleft}} w_{\circlearrowleft}(\omega^{\circlearrowleft}).$$

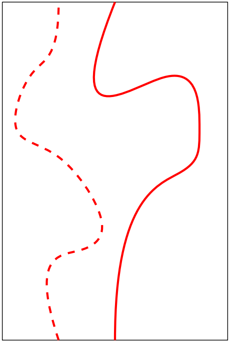
Conclusion: $$\sum_{\omega\in\Omega_{\mathrm{RC}}} w_{\mathrm{RC}}(\omega)\left(\tfrac{2}{\sqrt{q}}\right)^{\ell_0(\omega)} q^{-s(\omega)} = C \sum_{\vec{\omega}\in\Omega_{6\mathrm{V}}} w_{6\mathrm{V}}(\vec{\omega}).$$

Correlation length for s.d. RCM

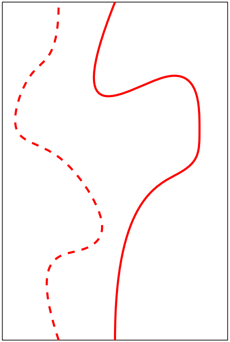


$$\sum_{\omega \in \Omega_{\mathrm{RC}}} w_{\mathrm{RC}}(\omega)\Big(\frac{2}{\sqrt{q}}\Big)^{\ell_0(\omega)} q^{-s(\omega)} = C \sum_{\vec{\omega} \in \Omega_{6\mathrm{V}}} w_{6\mathrm{V}}(\vec{\omega}).$$

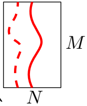
Correlation length for s.d. RCM



$$\sum_{\omega \in \Omega_{\mathrm{RC}}} w_{\mathrm{RC}}(\omega)\Big(\tfrac{2}{\sqrt{q}}\Big)^{\ell_0(\omega)} q^{-s(\omega)} = C\ Z_{6V}(N, M).$$

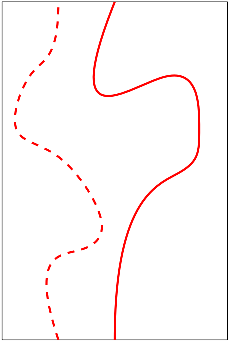


$$\sum_{\omega \in \Omega_{\mathrm{RC}}} w_{\mathrm{RC}}(\omega) \Big(\tfrac{2}{\sqrt{q}}\Big)^{\ell_0(\omega)} q^{-s(\omega)} = C \; Z_{6V}(N, M).$$

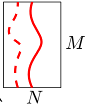


$$\mathbb{P}\Big(\text{[figure]} \Big) \sim \exp\Big(-\frac{M}{\xi(N)}\Big), \quad \text{as } M \to \infty.$$

Correlation length for s.d. RCM

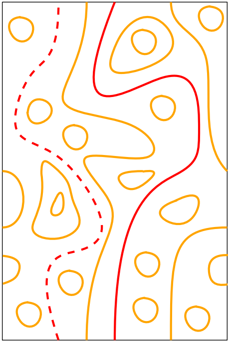


$$\sum_{\omega \in \Omega_{\mathrm{RC}}} w_{\mathrm{RC}}(\omega) \left(\frac{2}{\sqrt{q}}\right)^{\ell_0(\omega)} q^{-s(\omega)} = C \; Z_{6V}(N, M).$$

$\mathbb{P}\Bigg($  $\Bigg) \sim \exp\left(-\frac{M}{\xi(N)}\right), \quad \text{as } M \to \infty.$

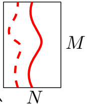
$$\xi(N) \to \xi(q), \quad \text{as } N \to \infty.$$

Correlation length for s.d. RCM



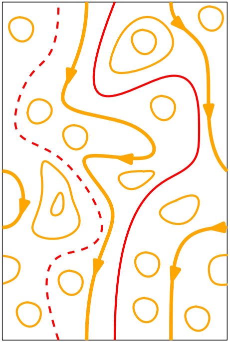
$$\sum_{\omega \in \Omega_{\mathrm{RC}}} w_{\mathrm{RC}}(\omega)\Big(\tfrac{2}{\sqrt{q}}\Big)^{\ell_0(\omega)} q^{-s(\omega)} = C\ Z_{6V}(N, M).$$

$$\mathbb{P}\Bigg(\ \Bigg) \sim \exp\Big(-\frac{M}{\xi(N)}\Big), \quad \text{as } M \to \infty.$$

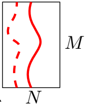


$$\xi(N) \to \xi(q), \quad \text{as } N \to \infty.$$

Correlation length for s.d. RCM



$$\sum_{\omega\in\Omega_{\mathrm{RC}}} w_{\mathrm{RC}}(\omega)\Big(\tfrac{2}{\sqrt{q}}\Big)^{\ell_0(\omega)} q^{-s(\omega)} = C\ Z_{6V}(N,M).$$

$\mathbb{P}\Bigg($  $\Bigg)$

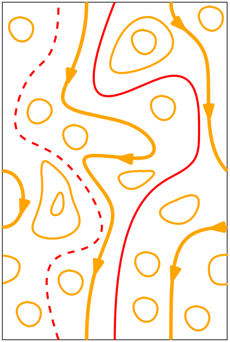
$$\sim \exp\Big(-\frac{M}{\xi(N)}\Big), \quad \text{as } M \to \infty.$$

$$\xi(N) \to \xi(q), \quad \text{as } N \to \infty.$$

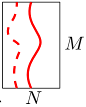
$$C\ Z_{6V}^{[N/2-1]}(N,M) \le \sum_{\substack{\omega\in\Omega_{\mathrm{RC}} \\ \text{vertically winding loop}}} w_{\mathrm{RC}}(\omega)\Big(\tfrac{2}{\sqrt{q}}\Big)^{\ell_0(\omega)} q^{-s(\omega)} \le 4C\ Z_{6V}^{[N/2-1]}(N,M)$$

$$\text{where } Z_{6V}^{[k]}(N,M) = \sum_{\substack{\vec{\omega}: \\ \text{with } k \text{ up arrows}}} w_{6\mathrm{V}}(\vec{\omega})$$

Correlation length for s.d. RCM

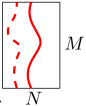


$$\sum_{\omega \in \Omega_{\mathrm{RC}}} w_{\mathrm{RC}}(\omega) \Big(\tfrac{2}{\sqrt{q}}\Big)^{\ell_0(\omega)} q^{-s(\omega)} = C \; Z_{6V}(N, M).$$

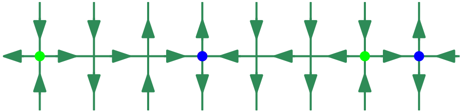
$\mathbb{P}\Bigg($  $\Bigg) \sim \exp\Big(-\dfrac{M}{\xi(N)}\Big), \quad \text{as } M \to \infty.$

$$\xi(N) \to \xi(q), \quad \text{as } N \to \infty.$$

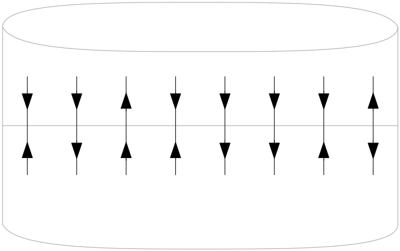
$$C \; Z_{6V}^{[N/2-1]}(N, M) \leq \sum_{\substack{\omega \in \Omega_{\mathrm{RC}} \\ \text{vertically winding loop}}} w_{\mathrm{RC}}(\omega) \Big(\tfrac{2}{\sqrt{q}}\Big)^{\ell_0(\omega)} q^{-s(\omega)} \leq 4C \; Z_{6V}^{[N/2-1]}(N, M)$$

Conclusion: $\mathbb{P}\Bigg($  $\Bigg) \sim \dfrac{Z_{6V}^{[N/2-1]}(N, M)}{Z_{6V}(N, M)} \sim \dfrac{Z_{6V}^{[N/2-1]}(N, M)}{Z_{6V}^{[N/2]}(N, M)}.$

The transfer matrix of the six vertex model.



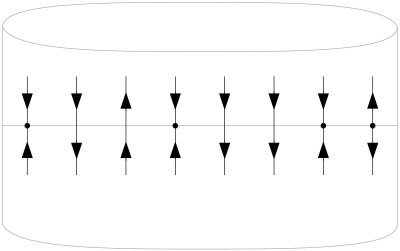
Definition of the transfer matrix



Two rows of vertical arrows: Ψ_1, Ψ_2, how to complete the line between?

If $\Psi_1 \neq \Psi_2$, start by the differences. . .

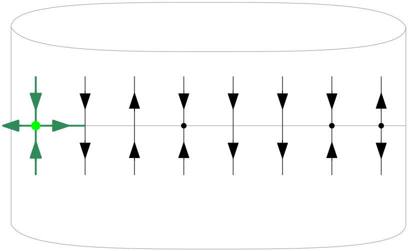
Definition of the transfer matrix



Two rows of vertical arrows: Ψ_1, Ψ_2, how to complete the line between?

If $\Psi_1 \neq \Psi_2$, start by the differences...

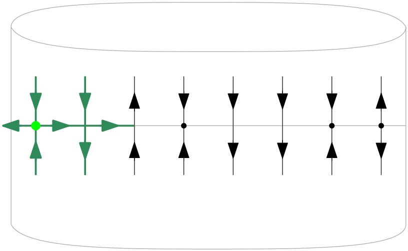
Definition of the transfer matrix



Two rows of vertical arrows: Ψ_1, Ψ_2, how to complete the line between?

If $\Psi_1 \neq \Psi_2$, start by the differences...

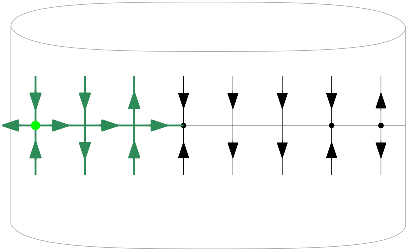
Definition of the transfer matrix



Two rows of vertical arrows: Ψ_1, Ψ_2, how to complete the line between?

If $\Psi_1 \neq \Psi_2$, start by the differences...

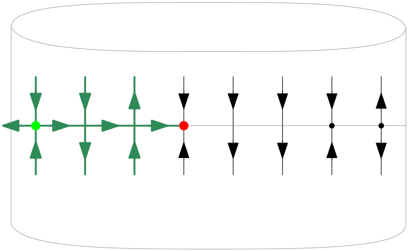
Definition of the transfer matrix



Two rows of vertical arrows: Ψ_1, Ψ_2, how to complete the line between?

If $\Psi_1 \neq \Psi_2$, start by the differences...

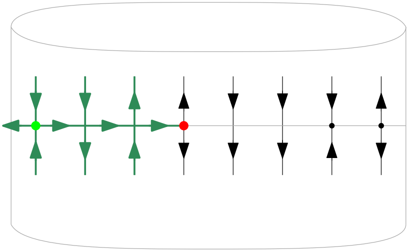
Definition of the transfer matrix



Two rows of vertical arrows: Ψ_1, Ψ_2, how to complete the line between?

If $\Psi_1 \neq \Psi_2$, start by the differences...

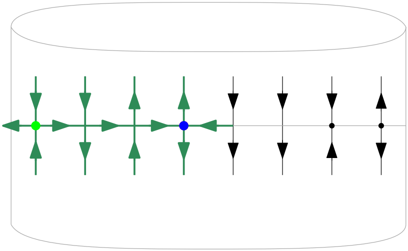
Definition of the transfer matrix



Two rows of vertical arrows: Ψ_1, Ψ_2, how to complete the line between?

If $\Psi_1 \neq \Psi_2$, start by the differences...

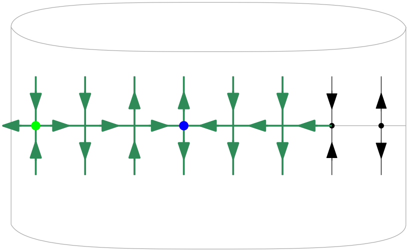
Definition of the transfer matrix



Two rows of vertical arrows: Ψ_1, Ψ_2, how to complete the line between?

If $\Psi_1 \neq \Psi_2$, start by the differences...

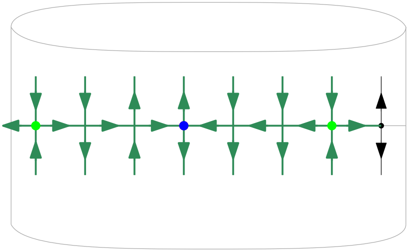
Definition of the transfer matrix



Two rows of vertical arrows: Ψ_1, Ψ_2, how to complete the line between?

If $\Psi_1 \neq \Psi_2$, start by the differences...

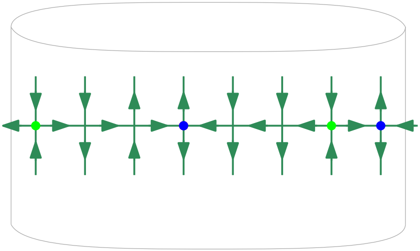
Definition of the transfer matrix



Two rows of vertical arrows: Ψ_1, Ψ_2, how to complete the line between?

If $\Psi_1 \neq \Psi_2$, start by the differences...

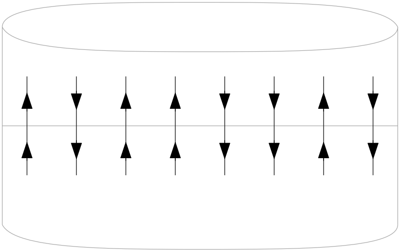
Definition of the transfer matrix



Two rows of vertical arrows: Ψ_1, Ψ_2, how to complete the line between?

If $\Psi_1 \neq \Psi_2$, start by the differences...
At most one possible completion (when Ψ_1 and Ψ_2 are **interlaced**).

Definition of the transfer matrix

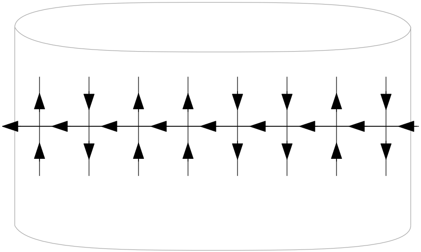


Two rows of vertical arrows: Ψ_1, Ψ_2, how to complete the line between?

If $\Psi_1 \neq \Psi_2$, start by the differences. . .
At most one possible completion (when Ψ_1 and Ψ_2 are **interlaced**).

If $\Psi_1 = \Psi_2$,

Definition of the transfer matrix

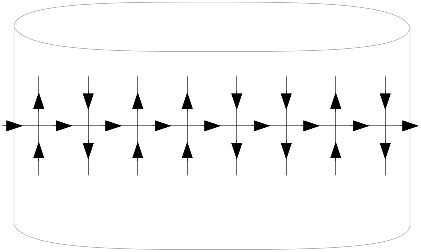


Two rows of vertical arrows: Ψ_1, Ψ_2, how to complete the line between?

If $\Psi_1 \neq \Psi_2$, start by the differences...
At most one possible completion (when Ψ_1 and Ψ_2 are **interlaced**).

If $\Psi_1 = \Psi_2$,

Definition of the transfer matrix

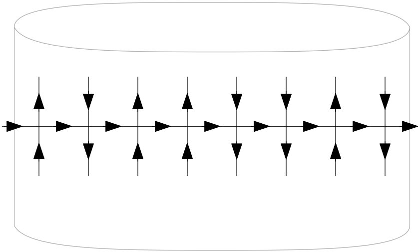


Two rows of vertical arrows: Ψ_1, Ψ_2, how to complete the line between?

If $\Psi_1 \neq \Psi_2$, start by the differences...
At most one possible completion
(when Ψ_1 and Ψ_2 are **interlaced**).

If $\Psi_1 = \Psi_2$, two possible completions.

Definition of the transfer matrix



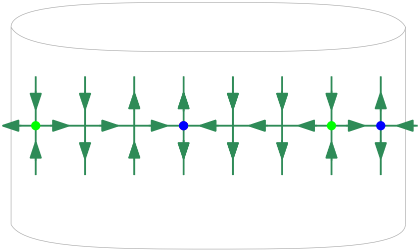
Two rows of vertical arrows: Ψ_1, Ψ_2, how to complete the line between?

If $\Psi_1 \neq \Psi_2$, start by the differences...
At most one possible completion (when Ψ_1 and Ψ_2 are **interlaced**).

If $\Psi_1 = \Psi_2$, two possible completions.

$$V(\Psi_1, \Psi_2) = \begin{cases} 2 & \text{if } \Psi_1 = \Psi_2, \\ c^{\#\text{ differences}} & \text{if } \Psi_1 \neq \Psi_2 \text{ and } \Psi_1 \text{ and } \Psi_2 \text{ are interlaced}, \\ 0 & \text{otherwise}, \end{cases}$$

Definition of the transfer matrix



Two rows of vertical arrows: Ψ_1, Ψ_2, how to complete the line between?

If $\Psi_1 \neq \Psi_2$, start by the differences...
At most one possible completion (when Ψ_1 and Ψ_2 are **interlaced**).

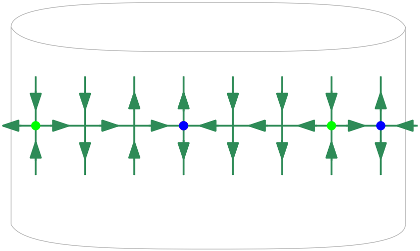
If $\Psi_1 = \Psi_2$, two possible completions.

$$V(\Psi_1, \Psi_2) = \begin{cases} 2 & \text{if } \Psi_1 = \Psi_2, \\ c^{\#\text{ differences}} & \text{if } \Psi_1 \neq \Psi_2 \text{ and } \Psi_1 \text{ and } \Psi_2 \text{ are interlaced}, \\ 0 & \text{otherwise}, \end{cases}$$

Total weight of configuration with vertical arrows $\Psi_1, \ldots, \Psi_M$:

$$V(\Psi_1, \Psi_2) \cdot \ldots \cdot V(\Psi_{M-1}, \Psi_M)$$

Definition of the transfer matrix



Two rows of vertical arrows: Ψ_1, Ψ_2, how to complete the line between?

If $\Psi_1 \neq \Psi_2$, start by the differences...
At most one possible completion (when Ψ_1 and Ψ_2 are **interlaced**).

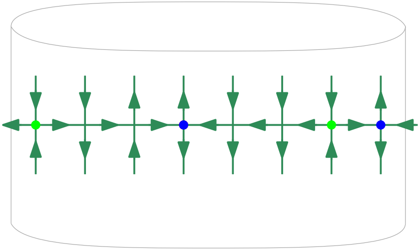
If $\Psi_1 = \Psi_2$, two possible completions.

$$V(\Psi_1, \Psi_2) = \begin{cases} 2 & \text{if } \Psi_1 = \Psi_2, \\ c^{\#\text{ differences}} & \text{if } \Psi_1 \neq \Psi_2 \text{ and } \Psi_1 \text{ and } \Psi_2 \text{ are interlaced}, \\ 0 & \text{otherwise}, \end{cases}$$

Total weight of configuration with vertical arrows $\Psi_1, \ldots, \Psi_M$ on torus :

$$V(\Psi_1, \Psi_2) \cdot \ldots \cdot V(\Psi_{M-1}, \Psi_M) V(\Psi_M, \Psi_1)$$

Definition of the transfer matrix



Two rows of vertical arrows: Ψ_1, Ψ_2, how to complete the line between?

If $\Psi_1 \neq \Psi_2$, start by the differences... At most one possible completion (when Ψ_1 and Ψ_2 are **interlaced**).

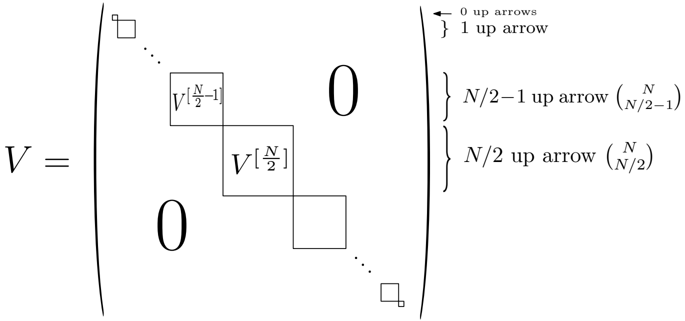
If $\Psi_1 = \Psi_2$, two possible completions.

$$V(\Psi_1, \Psi_2) = \begin{cases} 2 & \text{if } \Psi_1 = \Psi_2, \\ c^{\#\text{ differences}} & \text{if } \Psi_1 \neq \Psi_2 \text{ and } \Psi_1 \text{ and } \Psi_2 \text{ are interlaced,} \\ 0 & \text{otherwise,} \end{cases}$$

Total weight of configuration on torus: $V \in \mathcal{M}_{2^N,2^N}$

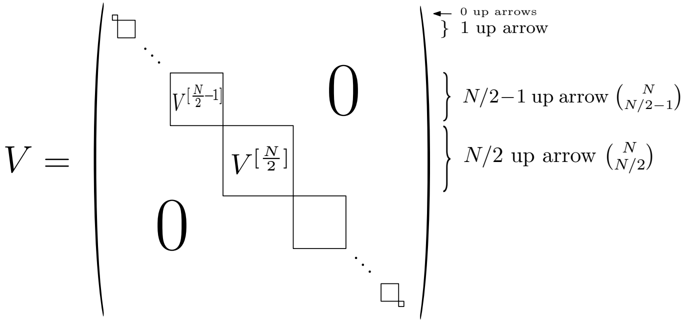
$$Z_{6V}(N, M) = \sum_{\Psi_1,\dots,\Psi_M} V(\Psi_1, \Psi_2) \cdot \dots \cdot V(\Psi_{M-1}, \Psi_M) V(\Psi_M, \Psi_1) = \mathrm{Tr}(V^M).$$

Block structure of V:



Blocks $V^{[k]}$ corresponding to lines with k up arrows

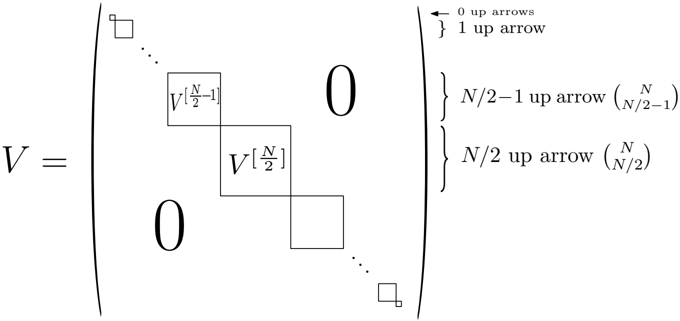
Block structure of V:



Blocks $V^{[k]}$ corresponding to lines with k up arrows

$V^{[k]}$ non-negative entries & irreducible $\Rightarrow$ the Perron Frobenius theorem applies

Block structure of V:



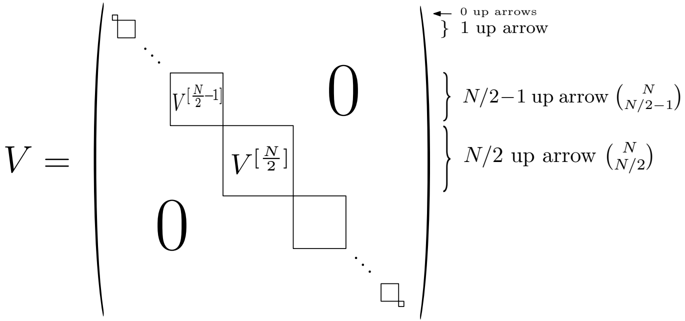
Blocks $V^{[k]}$ corresponding to lines with k up arrows

$V^{[k]}$ non-negative entries & irreducible $\Rightarrow$ the Perron Frobenius theorem applies

Let $\Lambda_0^{[k]} > \Lambda_1^{[k]} \geq \Lambda_2^{[k]} \dots$ be the eigenvalues of $V^{[k]}$

$$\mathrm{Tr}(V^{[k]})^M = \left(\Lambda_0^{[k]}\right)^M + \left(\Lambda_1^{[k]}\right)^M + \left(\Lambda_2^{[k]}\right)^M + \dots$$

Block structure of V:



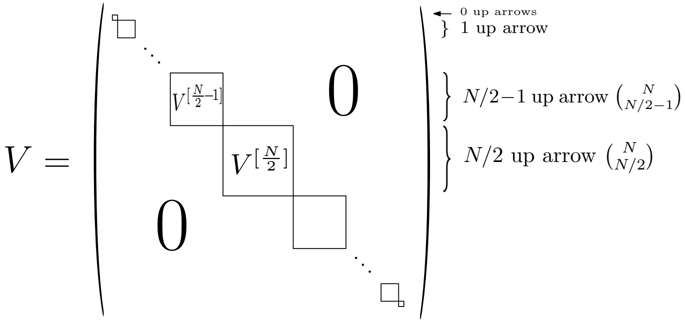
Blocks $V^{[k]}$ corresponding to lines with k up arrows

$V^{[k]}$ non-negative entries & irreducible $\Rightarrow$ the Perron Frobenius theorem applies

Let $\Lambda_0^{[k]} > \Lambda_1^{[k]} \geq \Lambda_2^{[k]} \dots$ be the eigenvalues of $V^{[k]}$

$$\mathrm{Tr}(V^{[k]})^M = \left(\Lambda_0^{[k]}\right)^M (1 + O(1-\epsilon)^M)$$

Block structure of V:



Blocks $V^{[k]}$ corresponding to lines with k up arrows

$V^{[k]}$ non-negative entries & irreducible $\Rightarrow$ the Perron Frobenius theorem applies

Let $\Lambda_0^{[k]} > \Lambda_1^{[k]} \geq \Lambda_2^{[k]} \dots$ be the eigenvalues of $V^{[k]}$

$$Z_{6V}^{[k]}(N,M) = \sum_{\substack{\vec{\omega}: \\ \text{with } k \text{ up arrows}}} w_{6\mathrm{V}}(\vec{\omega}) = \mathrm{Tr}(V^{[k]})^M = \left(\Lambda_0^{[k]}\right)^M (1 + O(1-\epsilon)^M)$$

Conclusion

Free energy (6V model):

$$f(1,1,c) = \lim_{\substack{N\to\infty \\ M\to\infty}} \frac{1}{MN} \log Z_{6V}(N,M)$$

Conclusion

Free energy (6V model):

$$f(1,1,c) = \lim_{\substack{N\to\infty \\ M\to\infty}} \frac{1}{MN} \log Z_{6V}(N,M) = \lim_{N\to\infty} \lim_{M\to\infty} \frac{1}{MN} \log Z_{6V}^{[N/2]}(N,M)$$

Conclusion

Free energy (6V model):

$$f(1,1,c) = \lim_{\substack{N\to\infty \\ M\to\infty}} \frac{1}{MN} \log Z_{6V}(N,M) = \lim_{N\to\infty} \lim_{M\to\infty} \frac{1}{MN} \log(\Lambda_0^{[N/2]})^M$$

Conclusion

Free energy (6V model):

$$f(1,1,c) = \lim_{\substack{N\to\infty \\ M\to\infty}} \frac{1}{MN} \log Z_{6V}(N,M) = \lim_{N\to\infty} \frac{1}{N} \log \Lambda_0^{[N/2]}$$

Conclusion

Free energy (6V model):

$$f(1,1,c) = \lim_{\substack{N\to\infty \\ M\to\infty}} \frac{1}{MN} \log Z_{6V}(N,M) = \lim_{N\to\infty} \frac{1}{N} \log \Lambda_0^{[N/2]}(N)$$

Conclusion

Free energy (6V model):

$$f(1,1,c) = \lim_{\substack{N\to\infty \\ M\to\infty}} \frac{1}{MN} \log Z_{6V}(N,M) = \lim_{N\to\infty} \frac{1}{N} \log \Lambda_0^{[N/2]}(N)$$

$$= \frac{\lambda}{2} + \sum_{k=1}^{\infty} \frac{e^{-k\lambda}\tanh(k\lambda)}{k}$$

Conclusion

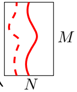
Free energy (6V model):

$$f(1,1,c) = \lim_{\substack{N\to\infty \\ M\to\infty}} \frac{1}{MN} \log Z_{6V}(N,M) = \lim_{N\to\infty} \frac{1}{N} \log \Lambda_0^{[N/2]}(N)$$

$$= \frac{\lambda}{2} + \sum_{k=1}^{\infty} \frac{e^{-k\lambda} \tanh(k\lambda)}{k}$$

Correlation length (Random Cluster model)

$$\xi^{-1}(q) = \lim_{N\to\infty} \lim_{M\to\infty} -\frac{1}{M} \log \mathbb{P}\left(\quad \right)$$



Conclusion

Free energy (6V model):

$$f(1,1,c) = \lim_{\substack{N\to\infty \\ M\to\infty}} \frac{1}{MN} \log Z_{6V}(N,M) = \lim_{N\to\infty} \frac{1}{N} \log \Lambda_0^{[N/2]}(N)$$

$$= \frac{\lambda}{2} + \sum_{k=1}^{\infty} \frac{e^{-k\lambda}\tanh(k\lambda)}{k}$$

Correlation length (Random Cluster model)

$$\xi^{-1}(q) = \lim_{N\to\infty} \lim_{M\to\infty} -\frac{1}{M} \log \frac{Z_{6V}^{[N/2-1]}(N,M)}{Z_{6V}^{[N/2]}(N,M)}$$

Conclusion

Free energy (6V model):

$$f(1,1,c) = \lim_{\substack{N\to\infty \\ M\to\infty}} \frac{1}{MN} \log Z_{6V}(N,M) = \lim_{N\to\infty} \frac{1}{N} \log \Lambda_0^{[N/2]}(N)$$

$$= \frac{\lambda}{2} + \sum_{k=1}^{\infty} \frac{e^{-k\lambda} \tanh(k\lambda)}{k}$$

Correlation length (Random Cluster model)

$$\xi^{-1}(q) = \lim_{N\to\infty} \lim_{M\to\infty} -\frac{1}{M} \log \frac{Z_{6V}^{[N/2-1]}(N,M)}{Z_{6V}^{[N/2]}(N,M)} = \lim_{N\to\infty} -\frac{1}{M} \log \frac{\Lambda_0^{[N/2-1]}(N)^M}{\Lambda_0^{[N/2]}(N)^M}$$

Conclusion

Free energy (6V model):

$$f(1,1,c) = \lim_{\substack{N\to\infty \\ M\to\infty}} \frac{1}{MN} \log Z_{6V}(N,M) = \lim_{N\to\infty} \frac{1}{N} \log \Lambda_0^{[N/2]}(N)$$

$$= \frac{\lambda}{2} + \sum_{k=1}^{\infty} \frac{e^{-k\lambda}\tanh(k\lambda)}{k}$$

Correlation length (Random Cluster model)

$$\xi^{-1}(q) = \lim_{N\to\infty} \lim_{M\to\infty} -\frac{1}{M} \log \frac{Z_{6V}^{[N/2-1]}(N,M)}{Z_{6V}^{[N/2]}(N,M)} = \lim_{N\to\infty} -\log \frac{\Lambda_0^{[N/2-1]}(N)}{\Lambda_0^{[N/2]}(N)}$$

Conclusion

Free energy (6V model):

$$f(1,1,c) = \lim_{\substack{N\to\infty \\ M\to\infty}} \frac{1}{MN} \log Z_{6V}(N,M) = \lim_{N\to\infty} \frac{1}{N} \log \Lambda_0^{[N/2]}(N)$$

$$= \frac{\lambda}{2} + \sum_{k=1}^{\infty} \frac{e^{-k\lambda}\tanh(k\lambda)}{k}$$

Correlation length (Random Cluster model)

$$\xi^{-1}(q) = \lim_{N\to\infty} \lim_{M\to\infty} -\frac{1}{M} \log \frac{Z_{6V}^{[N/2-1]}(N,M)}{Z_{6V}^{[N/2]}(N,M)} = \lim_{N\to\infty} -\log \frac{\Lambda_0^{[N/2-1]}(N)}{\Lambda_0^{[N/2]}(N)}$$

$$= \lambda + 2\sum_{k=1}^{\infty} \tfrac{(-1)^k}{k} \tanh(k\lambda) = \sum_{k=0}^{\infty} \frac{4}{(2k+1)\sinh\left(\frac{\pi^2(2k+1)}{2\lambda}\right)} > 0$$

Thank you!