Exploring the phase transition of planar FK-percolation

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September 14, 2023

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1 Bernoulli percolation: the basics

1.1 Definitions

Fix $G = (V, E)$ a graph. In this document, G will be the hypercubic lattice $\mathbb{Z}^d$ or subgraphs of it, but for now we can consider the model on general graphs.

Definition 1.1. For $p \in [0, 1]$, let $(\omega(e))_{e\in E}$ be i.i.d. Bernoulli random variables of parameter p. Write $\mathbb{P}_p$ for the law of ω, it is a measure on $\Omega = \{0, 1\}^E$.

We identify the *configuration* ω with the sub-graph of G with vertices V and edges $\{e \in E : \omega(e) = 1\}$.

We call an edge e with $\omega(e) = 1$ *open* (in the configuration ω), or *closed* if $\omega(e) = 0$. We will also identify ω with the subset of E formed of the open edges.

Connections in ω will be denoted by $\leftrightarrow$, or $\overset{\omega}{\leftrightarrow}$ when ω needs to be specified.

Question of interest When studying percolation, the questions of interest revolve around the geometry of the connected components (or *clusters*) of ω, specifically the large ones. As such, when $G = \mathbb{Z}^d$, the most basic question is whether ω contains an infinite cluster.

Fix henceforth $G = \mathbb{Z}^d$ for some $d \geq 1$. It is immediate that $\mathbb{P}_p$ is translation invariant and ergodic for all p. We conclude that

$$\mathbb{P}_p[\text{there exists an infinite cluster}] \in \{0, 1\} \qquad \text{for all } p \in [0, 1].$$

Furthermore, the existence of an infinite cluster under $\mathbb{P}_p$ is equivalent to the positivity of

$$\theta(p) := \mathbb{P}_p[0 \text{ is in an infinite cluster}] = \mathbb{P}_p[0 \leftrightarrow \infty].$$

Monotonicity in p, definition of p_c Notice that the measures $\mathbb{P}_p$ may be coupled in an increasing fashion. More precisely, if P is the probability measure on $[0, 1]^E$ produced by sampling i.i.d. uniforms $(U_e)_{e\in E}$ on $[0, 1]$, and if we set

$$\omega_p(e) = \begin{cases} 0 & \text{if } U_e \leq 1 - p \\ 1 & \text{if } U_e > 1 - p, \end{cases} \qquad \text{for all } e \in E \text{ and } p \in [0, 1]$$

then ω_p has law $\mathbb{P}_p$ for all p and

$$\omega_p(e) \leq \omega_{p'}(e) \qquad \text{for all } e \in E \text{ and } p \leq p',\ P\text{-a.s.}.$$

Due to these properties, we call P an *increasing coupling* of the measures $\mathbb{P}_p$.

The events $\{$there exists an infinite cluster$\}$ and $\{0 \leftrightarrow \infty\}$ are increasing, in that they are stable by the addition of open edges. Due to the increasing coupling above, the probabilities of increasing events are increasing functions of p. In particular $p \mapsto \theta(p)$ is an increasing function.

Definition 1.2. The *critical point* (or point of phase transition) of Bernoulli percolation is $p_c = p_c(\mathbb{Z}^d) \in [0, 1]$ be defined by

$$p_c = \sup\{p \in [0, 1] : \theta(p) = 0\} = \inf\{p \in [0, 1] : \theta(p) > 0\}.$$

The equality of the two expression defining p_c is due to the monotonicity of $\theta(p)$. Moreover, as discussed above, we immediately conclude that

$$\mathbb{P}_p[\text{there exists an infinite cluster}] = \begin{cases} 0 \text{ if } p < p_c \\ 1 \text{ if } p > p_c. \end{cases}$$

The questions that come to mind next are

- Is the phase transition non-trivial, i.e. do we have $0 < p_c < 1$?
- How do clusters behave away from p_c?

  In the *sub-critical phase* $p < p_c$, all clusters are finite; we expect them to have an exponential decay of radii:

  $$\mathbb{P}_p[0 \leftrightarrow \partial\Lambda_n] \leq e^{-c(p)n} \qquad \text{for all } n \text{ and } p < p_c, \tag{1}$$

  where $c(p) > 0$ is a constant depending on p and $\Lambda_n := \{-n, \dots, n\}^d$.

  In the *super-critical phase* $p > p_c$, there exists at least one infinite cluster; we expect it to be unique and all other clusters to have an exponential decay of radii:

  $$\mathbb{P}_p[0 \leftrightarrow \partial\Lambda_n \text{ but } 0 \not\leftrightarrow \infty] \leq e^{-c(p)n} \qquad \text{for all } n \text{ and } p > p_c, \tag{2}$$

  where $c(p) > 0$ is a constant depending on p and $\Lambda_n := \{-n, \dots, n\}^d$.
- Is there percolation at p_c, that is do we have $\theta(p_c) > 0$, or not? The same question may be rephrased as whether the phase transition is continuous ($\theta(p_c) = 0$) or discontinuous ($\theta(p_c) > 0$). For $d = 3$, this remains one of the main open question in the field.
- If the phase transition is continuous, and all clusters are finite at p_c, what is the decay of $\mathbb{P}_{p_c}[0 \leftrightarrow \partial\Lambda_n]$ as $n \to \infty$? Furthermore, what is the geometry of the large, but finite clusters?

For now, we will answer only the first question.

1.2 Non-triviality of p_c

For the whole of this section, we will work on $\mathbb{Z}^d$ with $d \geq 2$ (in the case of $d = 1$ we trivially have $p_c = 1$ – see exercises). The goal of this section is to prove the following.

Theorem 1.3. *For all $d \geq 2$ we have $0 < p_c < 1$.*

Both bounds use the celebrated Peierls argument, named after the German-British physicist Rudolf Peierls. This argument, most clearly illustrated in the proof of Proposition 1.4, is a generic way of identifying trivial behaviour for models in perturbative regimes (that is when the parameters are close to their extremes). It studies the competition between energy and entropy using corse estimates.

Proof. The proof follows directly from Propositions 1.4, which shows that $p_c \geq \frac{1}{2d-1}$ and Proposition 1.6. ☐

1.2.1 Lower bound on p_c

Proposition 1.4. *For all $d \geq 2$ and $p < \frac{1}{2d-1}$, there exists $c(p) > 0$ such that*

$$\mathbb{P}_p[0 \leftrightarrow \partial\Lambda_n] \leq e^{-c(p)n} \qquad \textit{for all } n \geq 1.$$

Proof. Let A_n be the set of simple paths on $\mathbb{Z}^d$ of length n (i.e. containing n edges) starting from 0. Observe that, for all n,

$$\begin{aligned}
\mathbb{P}_p[0 \leftrightarrow \partial\Lambda_n] &\leq \mathbb{P}_p[\exists \gamma \in A_n \text{ formed only of open edges}] \\
&\leq \mathbb{E}_p[\#\{\gamma \in A_n \text{ formed only of open edges}\}] \\
&= \sum_{\gamma \in A_n} \mathbb{P}_p[\gamma \text{ is formed only of open edges}\}] \\
&= |A_n| \cdot p^n.
\end{aligned}$$

Finally, it is immediate to check that $|A_n| \leq 2d \cdot (2d-1)^{n-1}$. Inserting this estimate in the above, we obtain the desired conclusion. $\square$

1.2.2 Duality of percolation

For the upper bound on p_c we will work with the model in two dimensions. The advantage of the two dimensional setting is the dual model, which we define here.

The dual of $\mathbb{Z}^2$ is the lattice $(\mathbb{Z}^2)^* = \mathbb{Z}^2 + (\frac{1}{2}, \frac{1}{2})$. Each face of $\mathbb{Z}^2$ contains a vertex of $(\mathbb{Z}^2)^*$ at is center, and each edge e of $\mathbb{Z}^2$ has a dual edge e^* intersecting it and jointing the two faces separated by e. This relation allows to define the dual for any planar graph; we only focus here on $\mathbb{Z}^2$ for convenience.

If $\omega \in \{0,1\}^E$ denotes a percolation configuration on $\mathbb{Z}^2$, we define its dual configuration ω^* by

$$\omega^*(e^*) = 1 - \omega(e) \qquad \text{for all } e \in E.$$

The following observations are immediate but essential.

Fact 1.5. *If ω is sampled according to $\mathbb{P}_p$, then ω^* has law $\mathbb{P}_{1-p}$.*

Moreover, the clusters of ω are surrounded by paths of ω^*, and vice-versa. We will generally call everything that has to do with the percolation ω^* or the lattice $(\mathbb{Z}^2)^*$ *dual*, while those related to ω and $\mathbb{Z}^2$ are called *primal*.

1.2.3 Upper bound for p_c

Proposition 1.6. *For all $d \geq 2$ and $p > 2/3$,*

$$\mathbb{P}_p[0 \leftrightarrow \infty] > 0.$$

Proof. Since $p_c(\mathbb{Z}^d) \leq p_c(\mathbb{Z}^2)$ for all $d \geq 2$, is suffices to treat the case $d = 2$. We focus on $\mathbb{Z}^2$ for the rest of the proof.

Fix some $r \geq 0$. For Λ_r not to be connected to infinity, there needs to exist a dual circuit in ω^* surrounding 0. This circuit intersect the axis $\mathbb{N} \times \{0\}$ at some point $(k + \frac{1}{2}, \frac{1}{2})$ with $k \geq r$ and needs to have length at least k. We conclude that

$$\begin{aligned}
\mathbb{P}_p[\Lambda_r \not\leftrightarrow \infty] &\leq \sum_{k \geq r} \mathbb{P}_p[\text{cluster of } (k + \tfrac{1}{2}, \tfrac{1}{2}) \text{ in } \omega^* \text{ has radius at least } k\}] \\
&\leq \sum_{k \geq r} \sum_{\gamma \in A_k} \mathbb{P}_{1-p}[\gamma \text{ is formed only of open edges}\}] \\
&= \sum_{k \geq r} 4 \cdot 3^{k-1} \cdot (1-p)^k.
\end{aligned}$$

The series above is convergent when $1 - p < \frac{1}{3}$, which is to say $p > \frac{2}{3}$. It follows that one may choose $r = r(p)$ such that

$$\mathbb{P}_p[\Lambda_r \leftrightarrow \infty] \geq \tfrac{1}{2}.$$

Finally, the event above is independent of the configuration inside Λ_r. As the probability that all edges of Λ_r are open is positive, we conclude that

$$\mathbb{P}_p[0 \leftrightarrow \infty] \geq \mathbb{P}_p[\Lambda_r \leftrightarrow \infty \text{ and all edges of } \Lambda_r \text{ open}] \geq \tfrac{1}{2}\mathbb{P}_p[\text{all edges of } \Lambda_r \text{ open}] > 0.$$

$\square$

As a byproduct of the proof, we also find that, for $d = 2$ and $p > \frac{2}{3}$, there exists $c(p) > 0$ such that

$$\mathbb{P}_p[0 \leftrightarrow \partial\Lambda_n \text{ but } 0 \not\leftrightarrow \infty] \leq e^{-c(p)n} \qquad \text{for all } n \geq 1.$$

1.3 Sharpness of phase transition

Theorem 1.7. *Fix $d \geq 2$. For all $p < p_c$ there exists $c(p) > 0$ such that*

$$\mathbb{P}_p[0 \leftrightarrow \partial\Lambda_n] \leq e^{-c(p)n} \qquad \textit{for all } n \geq 1. \tag{3}$$

The proof given below is beautiful and surprisingly simple; it is taken from [?].

1.3.1 Derivatives of increasing events

Let A be an increasing event. We say that an edge e is *pivotal* for A (in a configuration ω) if $\omega \cup \{e\} \in A$ but $\omega \setminus \{e\} \notin A$.

Proposition 1.8. *Suppose that A depends only on finitely many edges. Then $p \mapsto \mathbb{P}_p[A]$ is a $\mathcal{C}^\infty$ function and*

$$\frac{d\mathbb{P}_p[A]}{dp} = \sum_{e \in E} \mathbb{P}_p[e \textit{ is pivotal for } A] = \mathbb{E}_p[\# \textit{ pivotal edges for } A]. \tag{4}$$

Proof. Suppose that A depends only on what happens inside Λ_n. Consider then percolation limited to Λ_n and write $|\omega|$ for the number of open edges and $|\omega^c|$ for the number of closed edges of a configuration ω. Then

$$\mathbb{P}_p[A] = \sum_\omega p^{|\omega|}(1-p)^{|\omega^c|}\mathbf{1}_{\{\omega \in A\}}.$$

Differentiating this we find,

$$\begin{aligned}\frac{d\mathbb{P}_p[A]}{dp} &= \sum_\omega \Big(|\omega| p^{|\omega|-1}(1-p)^{|\omega^c|} - |\omega^c| p^{|\omega|}(1-p)^{|\omega^c|-1}\Big)\mathbf{1}_{\{\omega \in A\}} \\ &= \sum_e \sum_\omega \Big(\tfrac{1}{p}\mathbf{1}_{\{\omega(e)=1\}} - \tfrac{1}{1-p}\mathbf{1}_{\{\omega(e)=0\}}\Big)\mathbb{P}_p[\omega]\mathbf{1}_{\{\omega \in A\}}\end{aligned}$$

For ω such that $\omega \cup \{e\}, \omega \setminus \{e\} \in A$, the contribution of these two configuration to the above sum is 0. The same is true when $\omega \cup \{e\}, \omega \setminus \{e\} \notin A$. When $\omega \cup \{e\} \in A$, but $\omega \setminus \{e\} \notin A$, the contribution of the two configurations to the above is

$$\tfrac{1}{p}\mathbb{P}_p[\omega \cup \{e\}] = \mathbb{P}_p[\omega \cup \{e\}] + \mathbb{P}_p[\omega \setminus \{e\}].$$

This concludes the proof. □

1.3.2 The crucial quantity $\varphi_p(S)$

For S a finite connected set of edges containing 0, write

$$\partial S = \{u \in V : u \text{ is adjacent to edges in and outside } S\}.$$

It is also authorised to take $S = \emptyset$, in which case $\partial S = \{0\}$. Let

$$\varphi_p(S) = \mathbb{E}_p\big[\#\{u \in \partial S : 0 \overset{S}{\longleftrightarrow} u\}\big].$$

Above, $\overset{S}{\longleftrightarrow}$ refers to connections using only edges of S. See Fig. 1 for an illustration.

The following two lemmas will imply Theorem 1.7 directly.

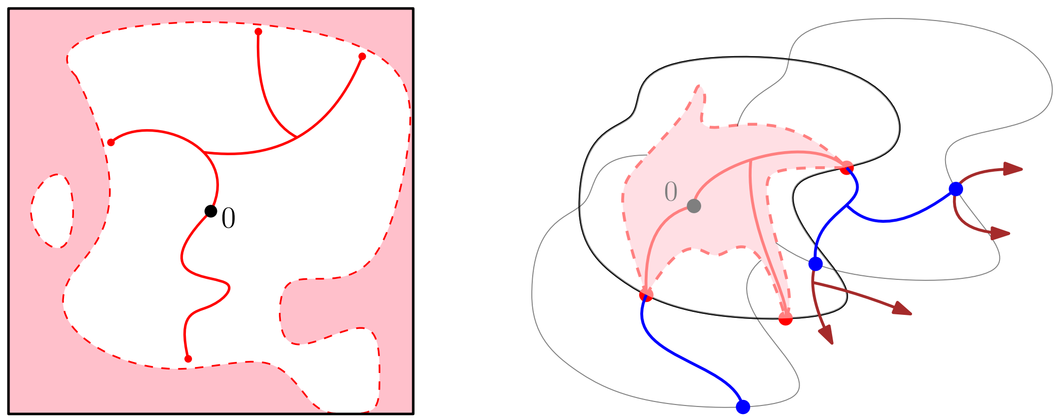


Figure 1: *Left:* The white region containing 0 is the set of edges S. Here 0 is connected to four vertices on ∂S. When proving Lemma 1.10, S denotes the connected component of 0 in the complement of the cluster of $\partial\Lambda_n$. *Right:* An illustration of the argument in the proof of Lemma 1.9.

Lemma 1.9. *For any $p \in [0,1]$ and S a finite connected set of edges containing* 0,

$$\mathbb{P}_p[0 \leftrightarrow \partial\Lambda_n] \le \varphi_p(S)^{\lfloor n/\mathrm{diam}(S)\rfloor} \tag{5}$$

Lemma 1.10. *For any $p \in [0,1]$ and $n \ge 1$.*

$$\frac{d\mathbb{P}_p[0 \leftrightarrow \partial\Lambda_n]}{dp} \ge \inf_S \varphi_p(S) \cdot (1-\theta(p)) \tag{6}$$

where the infimum is over all finite connected sets of edges S containing 0.

The above should be understood as

$$\frac{d\theta(p)}{dp} \ge \inf_S \varphi_p(S) \cdot (1-\theta(p)),$$

even though the differential is not well defined.

Proof of Theorem 1.7. Set

$$\tilde{p}_c = \inf\{p : \inf_S \varphi_p(S) \ge 1/2\}.$$

Then, for $p < \tilde{p}_c$, there exists S such that $\varphi_p(S) < 1/2$. Applying Lemma 1.9, we conclude that

$$\mathbb{P}_p[0 \leftrightarrow \partial\Lambda_n] \le 2^{-\lfloor n/\mathrm{diam}(S)\rfloor} \le e^{-c(p)n},$$

with S one of realisations such that $\varphi_p(S) < 1/2$.

Conversely, for $p > \tilde{p}_c$, we claim the existence of an infinite cluster. First, notice that $\inf_S \varphi_p(S)$ is increasing in p, and therefore $\inf_S \varphi_p(S) \ge 1/2$ for all $p > \tilde{p}_c$. Moreover, either $\theta(p) > 1/2$ or, for any $u \in (\tilde{p}_c, p]$ (6) implies that

$$\frac{d\mathbb{P}_u[0 \leftrightarrow \partial\Lambda_n]}{du} \ge \inf_S \phi_u(S) \cdot (1-\theta(u)) \ge \frac{1}{4}.$$

Integrating the above and taking n to infinity, we conclude that

$$\theta(p) \ge \tfrac{1}{4}(p - \tilde{p}_c) > 0.$$

The two cases above allow us to conclude that $p_c = \tilde{p}_c$ and therefore that (3) holds for all $p < p_c$. □

1.3.3 The sub-critical regime via $\varphi_p(S)$: proof of Lemma 1.9

Fix S. The lemma is only meaningful when $\varphi(S) < 1$, so when S contains all edges adjacent to 0. We suppose this henceforth.

We will explore the cluster of 0 and compare its growth to that of a Galton Watson tree with offspring law

$$\nu(i) = \mathbb{P}_p\big[\#\{u \in \partial S : 0 \overset{S}{\leftrightarrow} u\} = i\big] \quad \text{for } i \geq 0.$$

We describe the exploration step by step; each step corresponding to revealing the number of offspring of one individual in the Galton-Watson tree. Fig. 1 (right side) illustrates this process. Fix a configuration ω.

Step 0: Let C_0 be the connected component of 0 in the configuration $\omega \cap S$ (we view this both as a set of open edges and also as the set of endpoints of these edges). Call the points of $\partial S \cap \mathsf{C}_0$ the offspring of 0; they are the points of generation 1.

At this stage, $\mathsf{Active}_0 = \partial S \cap \mathsf{C}_0$ are the *active* vertices, and $\mathsf{Exp}_0 = \mathsf{C}_0$ are the set of *explored edges*.

Step j: Pick some vertex $u \in \mathsf{Active}_{j-1}$ of minimal generation (according to some pre-defined order), and let C_u be the connected component of u in the configuration $\omega \cap (S+u) \setminus \mathsf{Exp}_{j-1}$. Let

$$\mathsf{Active}_j = \mathsf{Active}_{j-1} \cup (\partial(S+u) \cap \mathsf{C}_u) \setminus \{u\}.$$

The points in $\mathsf{Active}_j \setminus \mathsf{Active}_{j-1}$ are called offspring of u and their generation is that of u plus 1. Finally, set $\mathsf{Exp}_j = \mathsf{Exp}_{j-1} \cup \mathsf{C}_u$.

The process stops when no more active vertices exist.

Notice that the number of new active vertices $\mathsf{Active}_j \setminus \mathsf{Active}_{j-1}$, conditionally on $(\mathsf{Active}_i, \mathsf{Exp}_i)_{i<j}$ is bounded by a variable of law ν. Indeed, when sapling C_u, we only use edges which were not previously explored or which were previously revealed to be closed. Thus, C_u has the law of the cluster of 0 in $S \setminus B$, where B is a set of edges that depends on $(\mathsf{Active}_i, \mathsf{Exp}_i)_{i<j}$. This cluster is clearly smaller than the cluster of 0 in S, which produces offspring of law ν.

As such, the variables X_k defined as the number of vertices of generation k are indeed bounded from above by a Galton Watson tree of offspring law ν, whose expectation is φ. We conclude that

$$\mathbb{P}_p[X_k \geq 1] \leq \mathbb{E}_p[X_k] = \varphi(S)^k.$$

Finally, we claim that this procedure explores the whole cluster of 0. Indeed, at every stage j of the process, any vertex adjacent to at least one edge in Exp_j is either active, or is surrounded by explored or closed edges – this may easily be proved by induction on j. Thus, if at any stage $\mathsf{Active}_j = \emptyset$, it follows that all edges adjacent to Exp_j are closed, and the cluster of 0 is Exp_j.

As a consequence of this observation, and of the fact that the vertices at generation k are within graph-distance $k \cdot \mathrm{diam}(S)$ of 0, we conclude that

$$\mathbb{P}_p[0 \leftrightarrow \partial\Lambda_n] \leq \mathbb{P}_p[X_{\lfloor n/\mathrm{diam}(S)\rfloor} \geq 1] \leq \varphi_p(S)^{\lfloor n/\mathrm{diam}(S)\rfloor},$$

as claimed □

Remark 1.11. A much easier proof of the above may written using the BK-inequality [**?**, Thm 2.12].

Remark 1.12. Applying the above to the minimal admissible S, that is the $2d$ edges adjacent to 0, we retrieve (almost) Peierls' argument of Proposition 1.4.

1.3.4 The super-critical regime via $\varphi_p(S)$: proof of Lemma 1.10

Fix $n \geq 1$. Recall from (4) that $\frac{d\mathbb{P}_p[0 \leftrightarrow \partial\Lambda_n]}{dp}$ is equal to the expected number of pivotals for the event $\{0 \leftrightarrow \partial\Lambda_n\}$. Pivotals may be open or closed; we will lower bound here the number of closed pivotals. We will work exclusively on Λ_n, and therefore restrict ourselves to the edges in this graph.

Write C for the set of edges connected to $\partial\Lambda_n$ and $\partial_{\text{ext}}\mathsf{C}$ for all edges of Λ_n adjacent, but not contained in C. If C contains 0 (that is, contains an edge adjacent to 0) then $0 \leftrightarrow \partial\Lambda_n$, and there are no closed pivotals. When C does not contain 0, let S denote the connected component of 0 in $\Lambda_n \setminus (\mathsf{C} \cup \partial_{\text{ext}}\mathsf{C})$ – see Fig. 1 (left side). Notice that any vertex u of ∂S is separated from C by a closed edge, which is part of $\partial_{\text{ext}}\mathsf{C}$). Moreover, when $u \leftrightarrow 0$ (a connection which necessarily occurs in S), the edge separating u from C is a closed pivotal for $\{0 \leftrightarrow \partial\Lambda_n\}$. Thus

$$\begin{aligned}\frac{d\mathbb{P}_p[0 \leftrightarrow \partial\Lambda_n]}{dp} &\geq \sum_C \mathbb{E}_p[\#\{u \in \partial S : 0 \overset{S}{\leftrightarrow} u\} | \mathsf{C} = C] \cdot \mathbb{P}_p[\mathsf{C} = C] \\ &= \sum_C \varphi_p(S) \cdot \mathbb{P}_p[\mathsf{C} = C] \\ &\geq \inf_S \varphi_p(S) \cdot \mathbb{P}_p[0 \not\leftrightarrow \partial\Lambda_n],\end{aligned}$$

where the sum is over all possible realisations C of C with C not containing 0 and where S is determined by C. In the equality we used the fact that the conditioning on $\mathsf{C} = C$ only gives information on the edges of C and $\partial_{\text{ext}}C$, but not on those in S. The spatial independence of Bernoulli percolation is essential here.

Finally, conclude with the simple observation that $\mathbb{P}_p[0 \not\leftrightarrow \partial\Lambda_n] \geq 1 - \theta(p)$. □

Exercises: Bernoulli percolation

Observe that p_c may be defined on any vertex-transitive graph in the same way as on $\mathbb{Z}^d$.

Exercise 1.1. Show that $p_c(\mathbb{Z}) = 1$.

Exercise 1.2. Show that for the "ladder" graph $\mathbb{Z} \times \{0, 1\}$, $p_c = 1$.

Exercise 1.3. Let T_d denote the $d+1$-regular tree (with the root having degree d rather than $d+1$). Prove that $p_c(T_d) = \frac{1}{d}$ and observe that the Peierls argument works all the way up to p_c.

Prove that $\mathbb{P}_{p_c}(T_d) = 0$. Show that $\mathbb{P}_{p_c}[0 \text{ is connected to distance } n] = n^{-\alpha_1 + o(1)}$ for an exponent $\alpha_1 > 0$ to be determined.

Exercise 1.4. Show that $p_c(\mathbb{Z}^d)$ is decreasing in d. Show that it is strictly decreasing. Show that $p_c(\mathbb{Z}^d) \to 0$ as $d \to \infty$.

Exercise 1.5. For $d = 2$, use the self-duality of percolation to prove that, for all $n \geq 1$,

$$\mathbb{P}_{\frac{1}{2}}\Big[\{0\} \times [0, n] \text{ connected to } \{n+1\} \times [0, n] \text{ inside } [0, n+1] \times [0, n]\Big] = \tfrac{1}{2}.$$

Using the sharpness of the phase transition to conclude that $p_c(\mathbb{Z}^2) \geq \frac{1}{2}$.

Exercise 1.6. Show that $p \mapsto \theta(p)$ is right-continuous.

Indication: $\theta(p)$ is the decreasing limit of the continuous and increasing functions $p \mapsto \mathbb{P}_p[0 \leftrightarrow \partial\Lambda_n]$.

2 FK-percolation

2.1 FK-percolation on finite graphs & monotonicity

We start by defining the FK-percolation measure on finite graphs. Fix $G = (V, E)$ some finite subgraph of $\mathbb{Z}^2$. Define its boundary by

$$\partial G = \{v \in V(G) : \text{ with neighbour outside of } G\}.$$

A boundary condition ξ on G is a partition of ∂G. Vertices in the same set of the partition are said to be wired together.

Two specific boundary conditions play a special role, these are the free boundary conditions, written $\xi = 0$, where no vertices are wired together, and the wired boundary conditions, written $\xi = 1$, where all vertices are wired together.

Definition 2.1. For G as above, $q \geq 1$, $p \in [0, 1]$ and ξ a boundary condition on G, define the FK-percolation measure $\phi^{\xi}_{G,p,q}$ as the probability measure on $\{0, 1\}^E$ with

$$\phi^{\xi}_{G,p,q}[\omega] = \frac{1}{Z^{\xi}_{G,p,q}} p^{|\omega|}(1-p)^{|E\setminus\omega|} q^{k(\omega^{\xi})},$$

where $k(\omega^{\xi})$ is the number of connected components of ω where all vertices of each component of ξ are considered connected. The constant $Z^{\xi}_{G,p,q} = \sum_{\omega} p^{|\omega|}(1-p)^{|E\setminus\omega|} q^{k(\omega^{\xi})}$ is chosen to that $\phi^{\xi}_{G,p,q}$ is a probability measure; it is called the *partition function* of the model.

Observe that Bernoulli percolation is a particular case of the above, obtained when $q = 1$. For $q \neq 1$, edges are not independent under $\phi^{\xi}_{G,p,q}$; we call this a *dependent percolation* model. The questions of interest remain the same as in the Bernoulli case.

The first and most basic property of FK-percolation is the Spatial Markov property.

Proposition 2.2 (Spatial Markov property)**.** *For H a subgraph of G,*

$$\phi^{\xi}_{G,p,q}[\omega \text{ on } H \,|\, \omega \text{ on } G \setminus H] = \phi^{\zeta}_{H,p,q}[\omega \text{ on } H]$$

where ζ are the boundary conditions induced by ω^{ξ} on $G \setminus H$, that is the wiring produced by $\omega^{\xi} \setminus H$ between the vertices of ∂H.

The proof is a direct computation which we omit.

We now turn to the question of monotonicity. Let us make a brief aside on the general topic of monotonicity of measures.

Ordering of measures: generalities There are two ways to view *stochastic ordering.* Consider two probability measures μ, ν on $\{0, 1\}^E$, where E denotes some finite set. We say that $\mu \leq_{\text{st}} \nu$ (ν stochastically dominates μ) if the two following equivalent conditions are satisfied

(a) there exists probability measure P producing two configurations ω, ω' such that ω has law μ, ω' has law ν, and $\omega(e) \leq \omega'(e)$ for all $e \in E$ P-a.s. We call P an increasing coupling of μ and ν;

(b) $\mu[A] \leq \nu[A]$ for all increasing events A.

That (a) and (b) are equivalent is the content of Strassen's theorem [].

A second, related notion is that of positive association. We say that μ is *positively associated* if $\mu[A \cap B] \geq \mu[A]\mu[B]$ for all A, B increasing events. This may be understood as

$$\mu[. \mid A] \geq_{\text{st}} \mu,$$

for all increasing events A.

The following criteria are particularly convenient for proving stochastic monotonicity and positive association.

Theorem 2.3 (Holley & FKG). *For positive measures μ, ν on $\{0,1\}^E$,*

(i) if $\mu(\omega \cap \omega')\nu(\omega \cup \omega') \geq \mu(\omega)\nu(\omega')$ for all $\omega, \omega' \in \{0,1\}^E$, then $\mu \leq_{\text{st}} \nu$;

(ii) if $\mu(\omega \cap \omega')\mu(\omega \cup \omega') \geq \mu(\omega)\mu(\omega')$ for all $\omega, \omega' \in \{0,1\}^E$, then μ is positively associated.

Moreover, it suffices to check conditions for ω and ω' differing only for two edges.

The proof of the above is very beautiful and we direct the reader to [?] for details. It is worth mentioning that the conditions in (ii) (which is called the FKG lattice condition) is stronger that positive association; measures satisfying it are sometimes called monotonic and have additional convenient properties.

Monotonicity properties of FK-percolation

Proposition 2.4. *For G a finite subgraph of $\mathbb{Z}^d$, $q \geq 1$, $p \in [0,1]$ and ξ a boundary condition on G*

(i) $\phi^{\xi}_{G,p,q}$ is positively associated;

(ii) for $p' \geq p$ and $\zeta \geq \xi$ (in the sense that any vertices wired in ξ are also wired in ζ),

$$\phi^{\xi}_{G,p,q} \leq_{\text{st}} \phi^{\zeta}_{G,p',q}. \tag{7}$$

In other words, $\phi^{\xi}_{G,p,q}$ increasing in p and ξ.

For the above it is crucial that $q \geq 1$. This is the main reason why the regime $0 < q < 1$ is much less studied.

This proposition, together with the Spatial Markov property will be used very often, sometimes in implicit ways; the novice reader will need some time to discover the full strength of these tools combined.

A immediate consequence is that the free and wired boundary conditions produce the minimal and maximal measures, respectively:

$$\phi^{0}_{G,p,q} \leq_{\text{st}} \phi^{\xi}_{G,p,q} \leq_{\text{st}} \phi^{1}_{G,p,q}, \qquad \text{for any b.c. } \xi.$$

To illustrate Proposition 2.4 let us compute the probabilities for an edge to be open in the simplest setting: when G is formed of a single edge e:

$$\phi^{\xi}_{\{e\},p,q}[e \text{ open}] = \begin{cases} p & \text{if } \xi = 1 \\ \frac{p}{p+(1-p)q} & \text{if } \xi = 0. \end{cases}$$

Notice that the probabilities are indeed increasing in p and the boundary conditions. From the above, we deduce the following domination of and by Bernoulli percolation.

Corollary 2.5. *For G, q, p and ξ as above,*

$$\mathbb{P}_{\frac{p}{p+(1-p)q}} \leq_{\rm st} \phi^{\xi}_{G,p,q} \leq_{\rm st} \mathbb{P}_p, \tag{8}$$

where $\mathbb{P}_{\cdot}$ denotes the Bernoulli percolation on G.

Proof. We do not give a full proof, but limit ourselves to mentioning that $\phi^{\xi}_{G,p,q}$ may be obtained by sequentially sampling edges, using coin tosses with probabilities that depend on the edge and the previously sampled edges. Throughout the process, all coin tosses have parameters between $\frac{p}{p+(1-p)q}$ and p.

Alternatively, one may use the Holley criterion. □

2.2 Infinite-volume measures

Another consequence of Proposition 2.4 is the monotonicity the measures in G. Specifically, if H denotes a subgraph of G, then

$$\phi^0_{H,p,q} \leq_{\rm st} \phi^0_{G,p,q}\Big|_H \qquad \text{and} \qquad \phi^1_{H,p,q} \geq_{\rm st} \phi^1_{G,p,q}\Big|_H,$$

where $|_H$ indicates the restriction to H. Indeed, if we focus on the first inequality, $\phi^0_{G,p,q}\Big|_H$ is a mixture of measures $\phi^{\xi}_{H,p,q}$, with ξ potential boundary conditions induced on H by the configuration on $G \setminus H$, and all these measures dominate $\phi^0_{H,p,q}$.

This monotonicity properties allow us to construct infinite volume measures via thermodynamical limits.

Fact 2.6. *For all p and $q \geq 1$, the following limits exist for the weak convergence*[1]

$$\phi^0_{p,q} = \lim_{n\to\infty} \phi^0_{\Lambda_n,p,q} \qquad \textit{and} \qquad \phi^1_{p,q} = \lim_{n\to\infty} \phi^1_{\Lambda_n,p,q}. \tag{9}$$

The same limits are obtained for general graphs G_n that increase to $\mathbb{Z}^d$. Furthermore, both limits above are translation invariant and ergodic probability measures on $\{0,1\}^{E(\mathbb{Z}^2)}$.

These infinite volume measures satisfy the so-called DLR condition, which essentially states that the Spatial Markov property also holds in infinite volume. The DLR formalism allows one to define the general notion of infinite volume measures, but we will not go further in that direction in these notes.

Note that it is not generally clear whether $\phi^0_{p,q}$ and $\phi^1_{p,q}$ are equal or not. In other words, while sending the boundary conditions to infinity, do they still manage to influence what happens locally?

It is a direct consequence of Proposition 2.4 that

$$\phi^0_{p,q} \leq_{\rm st} \phi^1_{p,q} \qquad \text{and} \qquad \phi^i_{p,q} \leq_{\rm st} \phi^i_{p',q} \quad \text{for all } p < p' \text{ and } i \in \{0,1\}. \tag{10}$$

Also, any limits (or infinite volume measures) of $\phi^{\xi_n}_{\Lambda_n,p,q}$ for sequences of boundary conditions ξ_n are always sandwiched between $\phi^0_{p,q}$ and $\phi^1_{p,q}$.

Observe that we have not yet managed to compare $\phi^1_{p,q}$ and $\phi^0_{p',q}$. The following result allows us to do this.

Proposition 2.7. *Fix $q \geq 1$. There exist at most countably many values of $p \in [0,1]$ such that $\phi^0_{p,q} \neq \phi^1_{p,q}$. As a consequence, for all $p < p'$*

$$\phi^0_{p,q} \leq_{\rm st} \phi^1_{p,q} \leq_{\rm st} \phi^0_{p',q}. \tag{11}$$

[1] We say that $\mu_n \to \nu$ if for any event A depending on a finite set of edges $\mu_n(A) \to \nu(A)$.

Essentially the proposition states that the influence of decreasing the boundary conditions can not compensate that of increasing the parameter.

Proof. We only sketch the proof here as it is a very general approach. Define the *free energy* of FK-percolation with parameters p and q as

$$f(p,q) = \lim_n f_n^{\xi_n}(p,q) = \lim_n \tfrac{1}{n^d} \log Z_{\Lambda_n,p,q}^{\xi_n}, \tag{12}$$

where the limit may be taken for any sequence of boundary conditions (ξ_n) and will not depend on this sequence.

Explicit computation show that

$$\frac{d}{dp} f_n^{\xi_n}(p,q) = \sum_{e\in E(\Lambda_n)} \phi_{\Lambda_n,p,q}^{\xi_n}[e \text{ open}], \tag{13}$$

which is increasing in p. We conclude that the functions $p \mapsto f_n^{\xi_n}(p,q)$ are all convex, and therefore so is $f(.,q)$.

As a convex function, $f(.,q)$ has left- and right-derivatives at all points, and is differentiable at all except at most countably many points. Furthermore, taking the limit as $n\to\infty$ in (13), we conclude that for all p for which $f(.,q)$ is differentiable,

$$\frac{d}{dp} f(p,q) = \phi_{p,q}^0[e \text{ open}] = \phi_{p,q}^1[e \text{ open}] \qquad \text{for any edge } e.$$

The above, together with the stochastic ordering between $\phi_{p,q}^0$ and $\phi_{p,q}^1$ implies that $\phi_{p,q}^0 = \phi_{p,q}^1$. Then (11) follows directly from the above and (10). ☐

2.3 Phase transition

We are now ready to define the phase transition of FK-percolation, as we did for Bernoulli percolation.

Definition 2.8. Fix $d\geq 2$ and $q \geq 1$. Set

$$p_c = p_c(q) = \sup\{p : \ \phi_{p,q}^0[0 \leftrightarrow \infty] = 0\}.$$

As for Bernoulli percolation, for $p < p_c$ $\phi_{p,q}^1[0\leftrightarrow\infty] = 0$ and $\phi_{p,q}^1$-a.s. there exists no infinite cluster; for $p > p_c$ $\phi_{p,q}^0[0\leftrightarrow\infty] > 0$ and $\phi_{p,q}^0$-a.s. there exists at least one infinite cluster.

In addition, the domination (8) by Bernoulli percolation allow us to deduce that

$$0 < p_c(q) < 1 \qquad \text{for all } q\geq 1 \text{ and } d \geq 2.$$

We close this part with a application of the properties described above.

Proposition 2.9. *If p is such that $\phi_{p,q}^1[0\leftrightarrow\infty] = 0$, then $\phi_{p,q}^0 = \phi_{p,q}^1$.*

Proof. This is a very instructive exercise, see Exercise 2.6 ☐

2.4 Some known facts not discussed here

We wish to mention some important known results which we do not discuss here. These are valid also for the particular case of Bernoulli percolation ($q = 1$); some only apply to this case.

Uniqueness of the infinite cluster A general approach based on the ergodicity of the measure and on the amenability of $\mathbb{Z}^d$ allows one to prove that if an infinite cluster exists, it is unique.

Theorem 2.10 (Burton-Keane []). *Fix $d \geq 1$, $q \geq 1$. Then, for all $p \in [0,1]$ and $i \in \{0,1\}$, either $\phi^i_{p,q}$[there exists an infinite cluster] $= 0$ or*

$$\phi^i_{p,q}[\textit{there exists exactly one infinite cluster}] = 1.$$

That the number of infinite clusters is a.s. constant under $\phi^i_{p,q}$ is a simple matter of ergodicity. It follows then that this number may only be 0, 1 or infinity. The main difficulty in the proof of Theorem 2.10 is to exclude the latter.

Sharpness of the phase transition for FK-percolation in all dimensions The sharpness result of Theorem 1.7 was extended to general FK-percolation in [] via a revolutionary use of the OSSS inequality.

Theorem 2.11. *Fix $d \geq 2$ and $q \geq 1$. For all $p < p_c(q)$ there exists $c(p) > 0$ such that*

$$\phi^1_{p,q}[0 \leftrightarrow \partial\Lambda_n] \leq e^{-c(p)n} \qquad \textit{for all } n \geq 1.$$

We will not prove this result in these notes.

Supercritical sharpness of the phase transition for Bernoulli percolation Up to now, we only discussed the subcritical sharpness, that is the exponential decay of connection probabilities in the subcritical regime. Recall that in the supercritical regime, we also expect trivial large-scale behaviour, in that connection probabilities converge exponentially to their limits (see also Exercise 2.3).

Theorem 2.12. *Fix $d \geq 2$ consider Bernoulli percolation on $\mathbb{Z}^d$. Then, for all $p > p_c$, there exists $c(p) > 0$ such that*

$$\mathbb{P}_p[0 \leftrightarrow \partial\Lambda_n \textit{ but } 0 \not\leftrightarrow \infty] \leq e^{-c(p)n} \qquad \textit{for all } n \geq 1.$$

In two dimensions, this theorem is easily deduced from the subcritical sharpness of the dual model. For dimensions $d \geq 3$, the key ingredient in this proof is the celebrated Grimmett-Marstrand theorem [] which states that the critical point of Bernoulli percolation on a slab $\mathbb{S}_k = \mathbb{Z}^2 \times \{0, \dots, k\}^{d-2}$ tends to that of $\mathbb{Z}^d$ when $k \to \infty$. At the time of writing, the equivalent result is not available for FK-percolation with $q > 1$.

Exercises: Bernoulli and FK-percolation

Exercise 2.1. Consider Bernoulli percolation on $\mathbb{Z}^2$. Assuming that $\theta(p) > 0$, use the FKG inequality to prove that

$$\mathbb{P}_p[\{n\} \times [-n, n] \xleftrightarrow{\Lambda_n^c} \infty] \to 1 \quad \text{as } n \to \infty.$$

Using the uniqueness of the infinite cluster and the self-duality of $\mathbb{P}_{1/2}$ prove that

$$\theta(1/2) = 0.$$

Combine this with the lower bound on $p_c \geq 1/2$ (Exercise 1.5) to conclude that $p_c = 1/2$ for Bernoulli percolation on $\mathbb{Z}^2$, and that the phase transition is continuous.

Exercise 2.2. Consider Bernoulli percolation on $\mathbb{Z}^d$. The goal of this exercice is to prove continuity of $p \mapsto \theta(p)$ for all $p \neq p_c$. Recall from Exercise 1.6 that this function is right-continuous. Thus, we only need to prove left-continuity for $p \neq p_c$.

Fix p is such that $\theta(p) > \lim_{u \nearrow p} \theta(u)$. In particular $\theta(p) > 0$ and $p \geq p_c$.

(a) Consider the increasing coupling P of Bernoulli percolation using uniforms $(U_e)_{e \in E}$. Argue that

$$P[0 \xleftrightarrow{\omega_p} \infty \text{ but } 0 \overset{\omega_u}{\not\longleftrightarrow} \infty \text{ for all } u < p] > 0.$$

(b) Argue that, conditionally on ω_p, $(U_e)_{e \in \omega_p}$ are i.i.d. uniforms on $[0, 1-p]$. Conclude that a.s. for any n there exists $u < p$ such that $\omega_u = \omega_p$ on Λ_n.

(c) Call a vertex v *fragile* (for some configuration $(U_e)_e$) if $v \xleftrightarrow{\omega_p} \infty$ but $v \overset{\omega_u}{\not\longleftrightarrow} \infty$ for all $u < p$. Prove that if 0 is fragile, then a.s. all vertices of the infinite cluster are fragile.

(d) Deduce that, for any $u < p$, $\mathbb{P}_p$[there exists no infinite cluster] > 0. Show that this implies $p \leq p_c$ and conclude.

Exercise 2.3. Consider Bernoulli percolation on $\mathbb{Z}^d$ and $p > p_c(\mathbb{Z}^d)$. Using the supercritical sharpness and the uniqueness of the infinite cluster, prove that

$$\theta(p)^2 \leq \mathbb{P}_p[x \leftrightarrow y] \leq \theta(p)^2 + e^{-c(p)\|x-y\|} \qquad \text{for all } x, y \in V,$$

for some $c(p) > 0$.

Exercise 2.4. Fix $d \geq 2$ and $q \geq 1$. Prove that, for all p,

$$\lim_{u \nearrow p} \phi^0_{u,q} = \lim_{u \nearrow p} \phi^1_{u,q} = \phi^0_{p,q} \qquad \text{and} \qquad \lim_{u \searrow p} \phi^0_{u,q} = \lim_{u \searrow p} \phi^1_{u,q} = \phi^1_{p,q},$$

in the sense that, for all A depending on finitely many edges $\lim_{u \nearrow p} \phi^0_{u,q}[A] = \phi^0_{p,q}[A]$.

Exercise 2.5. Fix $d \geq 2$, $q \geq 1$ and $p \in (0,1)$.

(a) Show that for any sequence of boundary conditions $(\xi_n)_n$, the following limit exists and does not depend on the sequence

$$f(p,q) = \lim_{N \to \infty} f^\xi_N(p,q) = \lim_N \tfrac{1}{N^2} \log Z^{\xi_N}_{\Lambda_N,p,q}.$$

(b) Show that $p \mapsto f^\xi_N(p,q)$ is convex, and conclude that $p \mapsto f(p,q)$ is also a convex function.

Indication: compute the differential of $f^\xi_N(p,q)$.

(c) Prove that $\partial^-_p f(p,q) = \phi^0_{p,q}[e \text{ open}]$ and $\partial^+_p f(p,q) = \phi^1_{p,q}[e \text{ open}]$.

(d) Conclude that, when $p \mapsto f(p,q)$ is differentiable, $\phi^0_{p,q}[e \text{ open}] = \phi^1_{p,q}[e \text{ open}]$. Using $\phi^0_{p,q} \leq_{\text{st}} \phi^1_{p,q}$, deduce that $\phi^0_{p,q} = \phi^1_{p,q}$.

Exercise 2.6. Consider FK-percolation on $\mathbb{Z}^d$ with $q > 1$ and some p. Prove that for any $n < N$ and A an increasing event depending only on the edges in Λ_n,

$$0 \leq \phi^1_{\Lambda_N}[A] - \phi^0_{\Lambda_N}[A] \leq \phi^1_{\Lambda_N}[\Lambda_n \leftrightarrow \partial\Lambda_N],$$

Indication: explore the cluster of $\partial\Lambda_N$ under $\phi^1_{\Lambda_N}$. If it does not reach Λ_n, prove that the probability of A is then smaller $\phi^0_{\Lambda_N}[A]$.

Deduce that if p is such that $\phi^1[0 \leftrightarrow \infty] = 0$, then $\phi^0 = \phi^1$.

Exercise 2.7. Consider FK-percolation on $\mathbb{Z}^2$ with $q > 1$. Use duality to show that

$$\phi^\xi_{\Lambda_n}[\Lambda_n \text{ crossed from left to right by open path}] = c,$$

for all n, where ξ is the b.c. where the left and right sides are wired, while the top and bottom are free. Why is $c \neq 1/2$?

Assuming the sharpness of the phase transition and the uniqueness of the infinite cluster, proceed as in Exercise 2.1 to prove that $p_c(d) = \frac{\sqrt{q}}{1+\sqrt{q}}$.

Why can't we conclude that the phase transition is continuous?

3 Dichotomy / quadrichotomy

3.1 Statement of quadrichotomy theorem

Write H_n for the event that there exists a open circuit in $\Lambda_{2n} \setminus \Lambda_n$ that surrounds Λ_n; also denote by H_n^* the same event for the dual model.

Theorem 3.1 (Duminil-Copin, Tassion 2019). *Fix $q > 1$ and $p \in [0,1]$. Then exactly one of the four following cases occurs (for constants $c > 0$ depending on p, q)*

(a) for all $n \geq 1$, $\phi^1_{\Lambda_n}[0 \leftrightarrow \partial\Lambda_n] \leq e^{-cn}$;

(b) for all $n \geq 1$, $\phi^0_{\Lambda_{2n}\setminus\Lambda_n}[H_n] \geq c$ and $\phi^1_{\Lambda_{2n}\setminus\Lambda_n}[H_n^*] \geq c$;

(c) for all $n \geq 1$, $\phi^0[0 \leftrightarrow \partial\Lambda_n] \leq e^{-cn}$ and $\phi^1[0 \overset{*}{\leftrightarrow} \partial\Lambda_n] \leq e^{-cn}$;

(d) for all $n \geq 1$, $\phi^0_{\Lambda_n}[0 \overset{*}{\leftrightarrow} \partial\Lambda_n] \leq e^{-cn}$.

The above generally implies that FK-percolation has a sharp phase transition, for which either (b) or (c) occurs. Indeed, these regimes correspond to

(a) the subcritical phase;
(b) a point of continuous phase transition;
(c) a point of discontinuous phase transition;
(d) the supercritical phase.

The theorem above was proved on general two dimensional lattices with certain symmetries. The proof is slightly more complicated than on the square lattice $\mathbb{Z}^2$. In the case of $\mathbb{Z}^2$, the self-duality allows us to determine the point of phase transition for $\mathbb{Z}^2$.

Corollary 3.2. *For each $q \geq 1$, we have*

$$p_c = p_{\rm sd}(\mathbb{Z}^2) = \frac{\sqrt{q}}{1+\sqrt{q}}.$$

Moreover, for $p < p_c(q)$ case (a) occurs, while for $p > p_c(q)$, (d) occurs. For $p = p_c(q)$ either (b) or (c) occur, depending on q.

Section 3.2 discusses how Theorem 3.1 implies Corollary 3.2, and more generally, the behaviour of the model in each of the four regimes.

3.2 Consequences for phase transition

Here are a few observations and consequences of Theorem 3.1. In particular, these elements are enough to prove Corollary 3.2.

- The infinite-volume measure is unique ($\phi^0 = \phi^1$) in all cases except (c).
- Point (a) is equivalent to the apparently weaker condition

$$\phi^1[0 \leftrightarrow \partial\Lambda_n] \leq e^{-cn} \qquad \text{for all } n \text{ and some } c > 0. \tag{14}$$

Indeed, if we assume the above, we may fix n and N so that

$$\phi^1_{\Lambda_N}[H_n^*] \geq e^{-c/20},$$

with c being the constant given by (14). Then for $R \geq 4N$, we claim that

$$\phi^1_{\Lambda_{2R}\setminus\Lambda_R}[H_R^* \,|\, 0 \leftrightarrow \Lambda_R] \geq \phi^1_{\Lambda_N}[H_n^*]^{10R/n} \geq e^{-c\,R/2}. \tag{15}$$

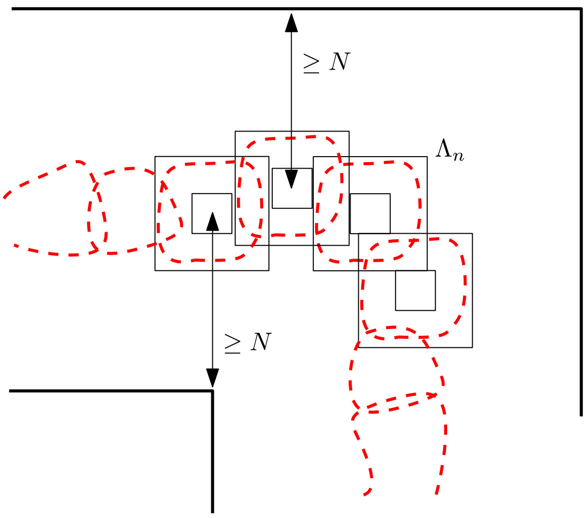


Figure 2: Combining events H_n^* to create H_R^* for the proof of (15).

Indeed, the event H_R^* may be constructed by the intersection of $10N/n$ translations of H_n^*, centred at points inside the annulus $\Lambda_{2R}\setminus\Lambda_R$, at a distance at least N from its boundary, as illustrated in Fig 2. Then, for any $R\ge 2N$, by the comparison of boundary conditions,

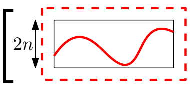
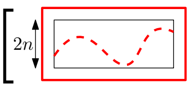
$$\begin{aligned}\phi^1_{\Lambda_{2R}}[0\leftrightarrow\Lambda_{2R}] &\le \phi^1_{\Lambda_{2R}}[0\leftrightarrow\Lambda_R\,|\,H_R^*]\big/\phi^1_{\Lambda_{2R}}[H_R^*\,|\,0\leftrightarrow\Lambda_R]\\ &\le \phi^0_{\Lambda_{2R}}[0\leftrightarrow\Lambda_R]\big/\phi^1_{\Lambda_{2R}\setminus\Lambda_R}[H_R^*]\\ &\le e^{-cR/2}.\end{aligned}$$

The second inequality uses the Spatial Markov property, while the last is a consequence of (14) and (15).

- Point (b) has several equivalent formulations such as

  $$\inf_n \phi^0_{[-2n(1+\varepsilon),2n(1+\varepsilon)]\times[-n(1+\varepsilon),n(1+\varepsilon)]}\big[[-2n,2n]\times[-n,n]\text{ has horizontal crossing}\big]>0,$$

  for any $\varepsilon>0$, and the same for the dual. This is illustrated by

  $\phi^0\Big[$  $\Big] > c$ $\qquad$ $\phi^1\Big[$  $\Big] > c$

  for some $c>0$ and all $n\ge 1$.

- If (b) holds for some p,q, then there exist constants $c,C>0$ such that

  $$n^{-C}\le\phi^1_{\Lambda_n,p,q}[0\leftrightarrow\partial\Lambda_n]\le n^{-c}\qquad\text{for all } n\ge 1. \tag{16}$$

  In particular $\phi^1_{p,q}=\phi^0_{p,q}$ and under this measure, there exists a.s. no infinite cluster This is a standard, but informative use of the RSW theory, which we leave as an exercice to the reader.

- If (c) holds at some p, then $u\mapsto\theta(u)=\phi^0[0\leftrightarrow\infty]$ is discontinuous at this value of p.

- If (d) holds, then, under the infinite volume measure ϕ, there exists a.s. an infinite cluster. Moreover there exists $c > 0$ such that

$$0 < \phi[0 \leftrightarrow \infty] \le \phi[0 \leftrightarrow \partial\Lambda_n] \le \phi[0 \leftrightarrow \infty] + e^{-cn} \qquad \text{for all } n \ge 1. \tag{17}$$

- For each q, exactly one of (b) or (c) occurs, and exactly for one value of p which we call p_c. As a consequence, we say that the phase transition is sharp. We sketch this next.

  We already mentioned that (a) occurs for p close enough to 0 and (d) occurs for p close to 1; the two are mutually exclusive. Moreover, by monotonicity (a) and (d) each occur for intervals of p. Since the values of p for which (a) and (d) respectively occur are open sub-sets of $[0,1]$, (a) and (d) both fail on the interval $[\sup\{p : (a)\}, \inf\{p : (d)\}]$, which contains at least one point.

  We will now argue that the interval is actually reduced to a single point. Fix $p_c = \inf\{p : (d)\}$.

  If (c) occurs at p_c, monotonicity implies that $\phi^1_{p,q}[H_n] \to 0$ for all $p < p_c$ (recall that $\phi^1 = \phi^0$ for all but countably may edge-intensities). We already explained that this is equivalent to (a) occurring for all $p < p_c$.

  If (b) occurs at p_c, a more involved argument is necessary: it requires the use of some form of sharp threshold technique. One easy way do to this is to observe that (16) applies to all $p \le p_c$, and bounds the influence of any individual edge on the event H_n. More precisely, we have

$$\phi^0_{\Lambda_{2n},p,q}[H_n \,|\, \omega_e = 1] - \phi^0_{\Lambda_{2n},p,q}[H_n] \le \phi^0_{\Lambda_{4n},p,q}[0 \leftrightarrow \partial\Lambda_n] \le n^{-c}$$

  with $c > 0$ given by (16).

  The BKKKL inequality then states that $p \mapsto \phi^0_{\Lambda_{2n},p,q}[H_n]$ satisfies a sharp-threshold principle below p_c, and thus that

$$\phi^0_{\Lambda_{2n},p,q}[H_n] \xrightarrow[n\to\infty]{} 0 \qquad \text{for all } p < p_c. \tag{18}$$

  Finally observe that

$$\phi^1_{\Lambda_{4n},p,q}[H_n] \le \frac{\phi^1_{\Lambda_{4n},p,q}[H_n \,|\, H^*_{2n}]}{\phi^1_{\Lambda_{4n},p,q}[H^*_{2n} \,|\, H_n]} \le \tfrac{1}{c'}\phi^0_{\Lambda_{2n},p,q}[H_n \cap H^*_{2n}], \tag{19}$$

  with $c' = \inf_{p \le p_c; n \ge 1} \phi^1_{\Lambda_{4n},p,q}[H^*_{2n} \,|\, H_n] \ge \inf_{n\ge 1} \phi^1_{\Lambda_{4n}\setminus\Lambda_{2n},p,q}[H^*_{2n}] > 0$ due to our assumption that (b) occurs at p_c.

  The two above displays combine to prove that $\phi^1_{\Lambda_{4n},p,q}[H_n] \to 0$ as $n \to \infty$ for all $p < p_c$. The finite size criterion of Proposition 3.3 below allows us to conclude that (a) occurs below p_c.

- On $\mathbb{Z}^2$ the sharpness of the phase transition combined with the duality imply that $p_c = p_{sd}$.

- The phase transition is called *continuous* if (b) occurs at p_c (indeed, observables such as $\phi^0_{p,q}[e \text{ open}]$ are continuous in p at p_c) and *discontinuous* if (c) occurs at p_c. The latter is also called a first order phase transition as $p \mapsto f(p,q)$ is not differentiable at p_c; the former is a second- or higher order phase transition corresponding to the order of derivative of $p \mapsto f(p,q)$ that diverges at p_c.

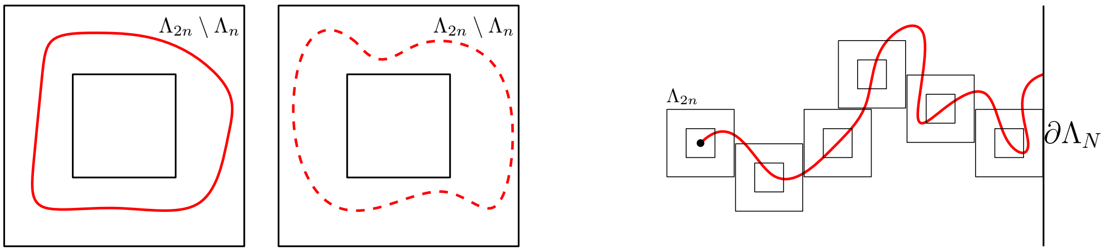


Figure 3: *Left:* The events H_n and H_n^*, respectively. *Right:* If 0 is connected to $\partial\Lambda_N$, then there exists a path of at least $\lfloor N/4n \rfloor$ disjoint but neighbouring translates of $\Lambda_{2n} \setminus \Lambda_n$ by points of $(n\mathbb{Z})^2$ for which H_n^* fails.

3.3 Finite size criterion for (a) and (d)

We will focus here on point (a). Point (d) is obtained in exactly the same way for the dual model.

Proposition 3.3. *There exists $\delta > 0$ such that,*

$$\sup\{\phi^1_{\Lambda_{2n}}[H_n^*] : n \geq 1\} > 1 - \delta \Leftrightarrow \big(\exists c > 0 \text{ s.t. } \phi^1_{\Lambda_n}[0 \leftrightarrow \partial\Lambda_n] \leq e^{-cn} \ \ \forall n \geq 1\big). \tag{20}$$

The same hold for the dual model.

Notice that the proposition above immediately implies the openness of the sets of p for which regimes (a) and (d), respectively, occur. Indeed, if we assume that (a) occurs at some p, then there exists n such that $\phi^1_{\Lambda_{2n},p,q}[H_n^*] > 1 - \delta$. As n is fixed, the probability above is continuous in p, hence the inequality is also valid in a neighbourhood of p. This implies (a) in said neighbourhood.

Proof of Proposition 3.3. The implication from right to left is obvious; we will focus on the opposite one. The idea of the proof is described in Fig. 3.

Fix some n. For $N = 4kn$, the occurence of $0 \leftrightarrow \partial\Lambda_N$ implies the existence of a family of k points $x_1, \dots, x_k \in (n\mathbb{Z})^2$ so that the translate $\Lambda_n(x_i) \leftrightarrow \Lambda_{2n}(x_i)^c$ of $(H_n^*)^c$ occurs for each $i = 1, \dots, k$ and so that x_i is at a L^∞-distance $4n$ from x_{i-1}. The number of such families of points may be bounded above by C^k for some fixed constant C, while

$$\phi^1_{\Lambda_N}\Big[\bigcap_{i=1}^{k}\{\Lambda_n(x_i) \leftrightarrow \Lambda_{2n}(x_i)^c\}\Big] \leq (1 - \phi^1_{\Lambda_{2n}}[H_n^*])^k.$$

Combining the two estimates, we obtain exponential decay for $\phi^1_{\Lambda_N}[0 \leftrightarrow \partial\Lambda_N]$ as soon as $1 - \phi^1_{\Lambda_{2n}}[H_n^*] < \frac{1}{2C} =: \delta$. □

3.4 RSW in strips

The following is a crucial technical result called the Russo-Seymour-Welsh (RSW) theorem. Generally RSW theorems lower-bound probabilities to cross rectangles in the long direction by functions of probabilities to cross rectangles in the short (easy) direction. Crucially, these bounds are independent of the scale of the rectangles.

RSW results combine well with a-priori estimates of crossing probabilities for rectangles in the easy direction. The most basic situation is for Bernoulli percolation on $\mathbb{Z}^2$ with $p = 1/2$, for which it is known that the probability of crossing a square of any size is 1/2. The original RSW theory developed in [?, ?] allows one to deduce the existence of $c > 0$

$$\mathbb{P}_{1/2}\Big[[0,2n]\times[0,n] \text{ contains horizontal open crossing}\Big] \geq c \qquad \text{for all } n \geq 1.$$

The original RSW theory of [?, ?] takes advantage of the spatial independence of percolation and the many symmetries of the lattice and of the model (translation, reflection, rotation by $\pi/2$). This theory proved to be extremely useful in studying percolation models in two dimensions, and was therefore generalised in many directions.

The version below was proved in [?]. Set $\mathrm{Strip}_n = \mathbb{Z} \times [-n, n]$.

Theorem 3.4 (RSW in strips)**.** *There exist constants $c, C > 0$ such that, for any $\rho \geq 1$ and $m \geq n$,*

$$\phi^{\xi}_{\mathrm{Strip}_m}[[-\rho n, \rho n]\times[-n,n] \text{ crossed horizontally}] \\ \geq c(\phi^{\xi}_{\mathrm{Strip}_m}[[-\rho n, \rho n]\times[-n,n] \text{ crossed vertically}])^{C\rho}, \tag{21}$$

where ξ are any boundary conditions invariant under horizontal shift (in particular wired on the top and free on the bottom).

Notice that, by taking $m \to \infty$, (21) also applies in infinite volume. We will not prove the above, as the proof is quite technical. The interested reader may consult the proofs of [?], or its more more streamlined and general proof [?].

We will be particularly interested in the following corollary of Theorem 3.4.

Corollary 3.5. *Suppose $p \leq p_{\mathrm{sd}}(q) = \frac{\sqrt{q}}{1+\sqrt{q}}$. There exists $c > 0$ such that, for any $\rho \geq 1$*

$$\phi^{1/0}_{\mathrm{Strip}_{2n}}[[-\rho n, \rho n]\times[-n,n] \text{ crossed horizontally in } \omega^*] \geq c^{\rho}, \tag{22}$$

where $1/0$ are the boundary conditions wired on the top and free on the bottom of Strip_{2n}*.*

Proof. By duality, either $\phi^{1/0}_{\mathrm{Strip}_{2n}}[[-\rho n, \rho n]\times[-n,n]$ crossed horizontally in $\omega^*] \geq 1/2$ or $\phi^{1/0}_{\mathrm{Strip}_{2n}}[[-\rho n, \rho n]\times[-n,n]$ crossed vertically in $\omega] \geq 1/2$. If the former occurs, the proof is complete. If the second occurs, since $p < p_{\mathrm{sd}}$ and the dual of the boundary conditions 1/0 are simply the vertical reflection of themselves,

$$\phi^{1/0}_{\mathrm{Strip}_{2n}}[[-\rho n, \rho n]\times[-n,n] \text{ crossed vertically in } \omega^*] \geq 1/2.$$

Then (21) allows us to conclude. □

3.5 Distinguishing between (b) and (c)

In the whole section we work at a fix set of parameters p, q such that

$$\textbf{(a) and (d) both fail for } (p,q). \tag{23}$$

This assumption will ensures that, under favorable boundary conditions, we may create circuits in the primal/dual with positive probability. Indeed, by Proposition 3.3 and Theorem 3.4, we conclude that

$$\phi^1[H_n] \geq c \qquad \text{and} \qquad \phi^0[H_n^*] \geq c \tag{24}$$

for all n and $c > 0$ some constant.

However, when the boundary conditions are adverse, the events H_n and H_n^* may have much smaller probabilities. Write, for $n \geq 1$,

$$u_n = \phi^0_{\Lambda_{20n}}[H_n] \qquad \textit{and} \qquad u_n^* = \phi^0_{\Lambda_{20n}}[H_n^*].$$

Proposition 3.6. *There exists $\delta > 0$ such that, if $\min\{u_{n_0}, u^*_{n_0}\} < \delta$, then*

$$u_{10kn_0} \le 2^{-k} \quad \text{and} \quad u_{10kn_0} \le 2^{-k} \qquad \text{for all } k \ge 1. \tag{25}$$

This proposition, together with the finite size criterion of Proposition 3.3 suffices to imply the quadrichotomy theorem, as we will see in the next subsection. Indeed, the above proves that the sequences (u_n) and (u^*_n) are either both bounded away from 0 uniformly, or both decrease exponentially fast.

The proof of Proposition 3.6 is based on the following two lemmas. The first is not particularly difficult. It ultimately states that the behaviours of the sequences (u_n) and (u^*_n) are the same.

Lemma 3.7. *There exists a constant $C \ge 1$ such that, for all $n \ge 1$,*

$$u_n \ge \tfrac{1}{C}(u^*_{n/4})^C \qquad \text{and} \qquad u^*_n \ge \tfrac{1}{C}(u_{n/4})^C \qquad \text{for all } k \ge 1.$$

Proof. We focus on the first inequality; the second is the same applied to the dual model. By standard arguments of exploration and comparison of boundary conditions,

$$\phi^0_{\Lambda_{10n}}[H_n] \ge \phi^1_{\Lambda_{10n}}[H_n | \Lambda_n \not\leftrightarrow \partial\Lambda_{10n}] \ge \phi^1_{\Lambda_{10n}}[H_n]\phi^1_{\Lambda_{10n}}[\Lambda_{2n} \not\leftrightarrow \partial\Lambda_{10n} | H_n] \tag{26}$$

From (24) and the comparison of boundary conditions we conclude that $\phi^1_{\Lambda_{20n}}[H_n] \ge c$. Using a strategy similar to that of Fig. 2 and the FKG inequality for the dual model, we may prove that

$$\phi^1_{\Lambda_{10n}}[\Lambda_{2n} \not\leftrightarrow \partial\Lambda_{10n} | H_n] \ge \phi^1_{\Lambda_{10n}\setminus\Lambda_{2n}}[\Lambda_{2n} \not\leftrightarrow \partial\Lambda_{10n}] \ge (u^*_{n/4})^C$$

for some $C \ge 1$ ($C = 1000$ should suffice). Note that it is important in the above that the instances of $H^*_{n/4}$ used to create a dual circuit around Λ_{2n} should be at distance $10n/4$ from the wired boundary conditions, namely from $\partial\Lambda_{10n}$ and $\partial\Lambda_{2n}$.

These last two observations, inserted in (26) allow us to conclude. □

The second lemma is the key.

Lemma 3.8. *Assume $p \le p_{\rm sd}$. There exists $c > 0$ such that, for every $n \ge 1$,*

$$u_{10kn} \le (c\, u_n)^k \qquad \text{for all } k \ge 1.$$

Proof. The constants $c_.$ below are positive and independent of n and k. Figure 4 may be useful in understanding the proof.

Fix $k, n \ge 1$ and define the event $\mathcal{C}$ as the intersection of the translates of H_n by the $(20jn, 0)$ with $j = -k, \dots, k$. Due to our assumption (23) – or more precisely its consequence (24)

$$\phi^0_{\Lambda_{200kn}}[\mathcal{C}] \ge c^k_{\mathcal{C}}\, \phi^0_{\Lambda_{200kn}}[H_{kn}] = c^k_{\mathcal{C}}\, u_{20kn},$$

for some $c_{\mathcal{C}} > 0$.

Write $\mathcal{E}$ for the event that there exists a dual horizontal crossing of $[-200kn, 200kn] \times [2n, 4n]$ and $\tilde{\mathcal{E}}$ for the vertical reflection of this event. Then, repeated applications (22) at scales $N = 2^j n$ with $\ell = 200\, k \cdot 2^{-j}$ yield

$$\phi^0_{\Lambda_{200kn}}[\mathcal{E} \cap \tilde{\mathcal{E}} \,|\, \mathcal{C}] \ge \prod_{j=1}^{\log k} c_{\mathcal{E}}^{k/2^j} \le c_{\mathcal{E}}^{2k}.$$

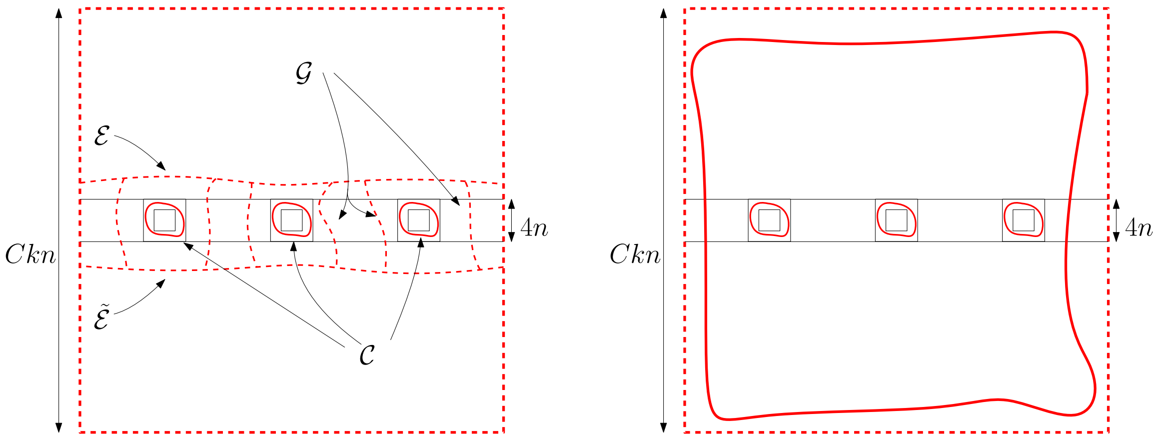


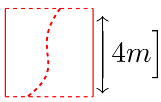
Figure 4: *Left:* The event $\mathcal{C} \cap \mathcal{E} \cap \tilde{\mathcal{E}} \cap \mathcal{G}$. Its probability may be bounded from above by the probability of the primal circuits of $\mathcal{C}$ occurring, conditionally on the dual paths. This produces an upper bound of u_n^k.
Conversely, its probability is bounded from below in several steps. Conditionally on $\mathcal{C}$, the dual paths forming $\mathcal{E} \cap \tilde{\mathcal{E}}$ and $\mathcal{G}$ appear with probability at least $(c_{\mathcal{G}}\, c_{\mathcal{E}})^k$. *Right:* Constructing $\mathcal{C}$ by first requiring that H_{kn} occurs; conditionally on this event $\mathcal{C}$ occurs with probability at least $c_{\mathcal{C}}^k$. As such, $\mathcal{C}$ has probability at least $c_{\mathcal{C}}^k\, u_{20kn}$.

Finally, when $\mathcal{E} \cap \mathcal{E}^c$ occurs, write $\mathcal{G}$ for the event that there exist dual crossing between the top-most dual path realising $\mathcal{E}$ and the bottom-most path realising $\mathcal{E}^c$, in each of the squares $[20jn - 10n, 20jn - 2n] \times [-4n, 4n]$ and $[20jn + 2n, 20jn + 10n] \times [-4n, 4n]$ with $j = -k, \dots, k$. By pushing of boundary conditions and FKG,

$$\phi^0_{\Lambda_{200kn}}[\mathcal{G} \,|\, \mathcal{E} \cap \tilde{\mathcal{E}} \cap \mathcal{C}] \geq c_{\mathcal{G}}^k,$$

where

$$c_{\mathcal{G}} = \inf_m \phi^{\mathrm{alt}}_{\Lambda_{4m}} \Big[\quad \Big] > 0, \qquad (27)$$



where the boundary conditions of the measure are alternating as described by the picture: wired on the left- and right sides, free on the top and bottom. The positivity of $c_{\mathcal{G}}$ is due to duality and the assumption that $p \leq p_{\mathrm{sd}}$.

Combing the three displays above to deduce that

$$\phi^0_{\Lambda_{200kn}}[\mathcal{C} \,|\, \mathcal{E} \cap \tilde{\mathcal{E}} \cap \mathcal{G}] \phi^0_{\Lambda_{200kn}}[\mathcal{C} \cap \mathcal{E} \cap \tilde{\mathcal{E}} \cap \mathcal{G}] \geq c_{\mathcal{G}}^k c_{\mathcal{E}}^k c_{\mathcal{C}}^k\, u_{20kn}.$$

Now, the dual paths forming $\mathcal{E} \cap \tilde{\mathcal{E}} \cap \mathcal{G}$ shield each of the events H_n from each other. Thus, we conclude

$$\phi^0_{\Lambda_{200kn}}[\mathcal{C} \,|\, \mathcal{E} \cap \tilde{\mathcal{E}} \cap \mathcal{G}] \leq u_n^k.$$

The last two displays combine to prove the desired inequality. □

3.6 Concluding: proof of Theorem 3.1

It is immediate to see that cases (a)-(d) are mutually exclusive.

By Proposition 3.3, the sets of parameters where (a) and (d) are indeed open, as they are characterised by a finite size criterion, which is stable under perturbation of parameters.

Assume now that p, q are such that (a) and (d) fail. Then Proposition 3.6 implies that the sequences (u_n) and (u_n^*) are either both uniformly bounded away from 0, or both converge exponentially fast to 0.

Basic applications of FKG such as in Figure 2 show that the former case implies (b). The comparison of boundary conditions (7) may be used to deduce that the latter case implies (c). □

Exercises: FK-percolation on $\mathbb{Z}^2$, fine properties

Exercise 3.1. Consider FK-percolation on $\mathbb{Z}^2$ with some $q \geq 1$ and $p \in [0, 1]$. Assume that we are in the case (b) of Theorem 3.1, that is the RSW regime. Show that there exists $c > 0$ such that

$$c\,n^{-1} \leq \phi_{\Lambda_n}^{\xi}[0 \leftrightarrow \partial\Lambda_n] \leq n^{-c} \qquad \text{for all } n \geq 1.$$

Deduce that $\phi[|\mathcal{C}_0|] = \infty$, where $\mathcal{C}_0$ is the cluster of 0, and ϕ is the unique infinite volume measure with this set of parameters (here it is used as an expectation).

Exercise 3.2. Consider FK-percolation on $\mathbb{Z}^2$ with some $q \geq 1$ and $p \in [0, 1]$. Assume that we are in the case (c) of Theorem 3.1, that is the discontinuous phase transition regime, where $\phi^0 \neq \phi^1$.

(a) Using $\phi^0 = (\phi^1)^*$, show that, for any fixed edge e, $\phi^0[e \text{ open}] < 1/2$.

(b) Write $\phi_{\Lambda_n}^{\text{per}}$ for the FK-percolation measure on the square torus of side-length $2n$. Prove that, for any fixed edge e, $\phi_{\Lambda_n}^{\text{per}}[e \text{ open}] = 1/2$.

(c) Assume that the only Gibbs measures[2] of FK-percolation on $\mathbb{Z}^2$ which are invariant under translations and rotations by $\pi/2$ are the linear combinations of ϕ^0 and ϕ^1 (this may be proved using relatively soft tools). Prove that

$$\phi_{\Lambda_n}^{\text{per}} \longrightarrow \tfrac{1}{2}\phi^0 + \tfrac{1}{2}\phi^1 \qquad \text{as } n \to \infty,$$

in the sense that the probabilities of any event depending on finitely many edges converges.

(d) Is $\lim_n \phi_{\Lambda_n}^{\text{per}}$ ergodic?

Exercise 3.3. Fix $q \geq 1$ and $p \in [0, 1]$. Let ϕ be an **ergodic** Gibbs measure for FK-percolation on $\mathbb{Z}^2$ which is invariant under translations and rotations by $\pi/2$. Use the same construction as in Exercise 2.1 to prove that a.s. there exists no infinite primal cluster or a.s. there exists no infinite dual cluster.

Deduce that the only Gibbs measures of FK-percolation on $\mathbb{Z}^2$ which are invariant under translations and rotations by $\pi/2$ are the linear combinations of ϕ^0 and ϕ^1.

[2] For the purpose of this exercise, Gibbs measures should be understood as the potential limits of finite volume mesures

Exercise 3.4. Fix $q \geq 1$ and $p \in [0,1]$. Let ϕ be a Gibbs measure for FK-percolation on $\mathbb{Z}^2$ which is invariant under translations and rotations by $\pi/2$. Assume that there exists $c > 0$ such that

$$\begin{aligned} &\phi\big[\omega \text{ contains a horizontal crossing of } [0,2n]\times[0,n]\big] \geq c \qquad \text{and} \\ &\phi\big[\omega^* \text{ contains a horizontal crossing of } [0,2n]\times[0,n]\big] \geq c \qquad \text{for all } n \geq 1. \end{aligned}$$

Prove (without using the quadrichotomy theorem) that

$$\begin{aligned} &\phi\big[\omega \text{ contains a horizontal crossing of } [0,kn]\times[0,n]\big] \geq c^{2k} \qquad \text{and} \\ &\delta < \phi[H_n] < 1-\delta \qquad \text{for all } n \geq 1, \end{aligned}$$

and the same for the dual, for some $\delta > 0$.

Does this imply that $\phi\big[0 \leftrightarrow \partial\Lambda_n\big] \to 0$?

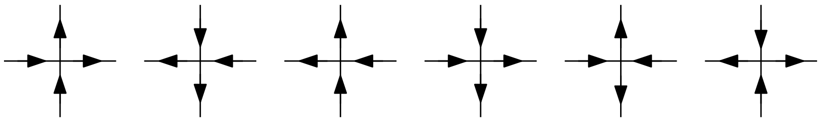


Figure 5: The six possible vertex configurations obeying the ice rule. The weights are chosen to be invariant under total arrow reversal.

4 Continuous/discontinuous phase transition

The goal of this section is to show the following theorem.

Theorem 4.1. *The phase transition of FK-percolation on* $\mathbb{Z}^2$ *is continuous if* $1 \leq q \leq 4$*, and discontinuous if* $q > 4$*.*

The theorem above was proved in a series of papers. Originally, the continuity was proved in [?] and the discontinuity in [?]. Alternative proofs of these two regimes were obtained in [?] and [?].

We will present here a proof of both the continuity and discontinuity regimes inspired by the method of [?]. It consist in the explicit computation of the rate of decay of a certain event, that allows us to distinguish cases (b) and (c). The computation is done by relating critical FK-percolation to the six-vertex model, which we define below. The free energy of the six-vertex model is then estimated by applying the Bethe ansatz to its transfer matrix and computing its leading eigenvalues.

This section is meant to highlight the links between FK-percolation and the six-vertex model, and the very different techniques that may be used to analyse these. A more specific take-home message is that the continuity/discontinuity of the phase transition of FK-percolation corresponds to the delocalisation/localisation of the height function of the corresponding six-vertex model, which in turn corresponds to the differentiability/non-differentiability of the free energy of the six-vertex model in terms of the "slope", at slope 0.

4.1 Six vertex model on torus

For $L, M \in 2\mathbb{N}$, write $\mathbb{T}_{L,M} = (\mathbb{Z}/L\mathbb{Z}) \times (\mathbb{Z}/M\mathbb{Z})$ for the torus of width L and height M. A six vertex configuration on $\mathbb{T}_{L,M}$ is an assignment of directions (or arrows) to each edge of $\mathbb{T}_{L,M}$ with the restriction that any vertex has exactly two incoming and two outgoing edges; we call this restriction the *ice rule*. As a result, there are only six possible configurations at each vertex, whence the name of the model.

The six configuration each carry a weight (see Figure 5); we will consider always $a, b, c > 0$. Then, the weight of a configuration $\vec{\omega}$ is

$$w_{6V}(\vec{\omega}) = a^{\#\text{vertices type } a} \cdot b^{\#\text{vertices type } b} \cdot c^{\#\text{vertices type } c}.$$

It is standard to parametrise the model via

$$\Delta = \frac{a^2 + b^2 - c^2}{2ab}, \tag{28}$$

as models with constant Δ behave similarly. In this section we will focus exclusively on the case $a = b = 1$, when $\Delta = 1 - c^2/2 \in (-\infty, 1)$.

Preservation of arrows. We will partition the torus into horizontal rows of vertical edges. It is a direct but crucial consequence of the ice rule that in any six-vertex configuration, the number of up-arrows is the same for each row. We call this the *preservation of up-arrows.*

Define the partition functions

$$\vec{Z}^{(k)}_{L,M} = \sum_{\substack{\vec{\omega} \text{ with } L/2+k \\ \text{up-arrows per row}}} w_{6V}(\omega) \qquad \text{and} \qquad \vec{Z}_{L,M} = \sum_{k=-L/2}^{L/2} \vec{Z}^{(k)}_{L,M}.$$

Finally, for $\alpha \in [-1/2, 1/2]$, define the free energy of the (sloped) model as

$$f_{6V} = \lim_{L\to\infty} \lim_{M\to\infty} \tfrac{1}{LM} \log Z_{6\mathrm{V}}(\mathbb{T}^{\times}_{L,M}) \quad \text{and} \quad f_{6V}(\alpha) = \lim_{L\to\infty} \lim_{M\to\infty} \tfrac{1}{LM} \log Z^{(\alpha L)}_{6\mathrm{V}}(\mathbb{T}^{\times}_{L,M}).$$

Simple combinatorial tricks (see for instance [?, proof of Cor 1.4]) show that the limits exists no matter the order in which L and M are send to infinity. Moreover, they show that

$$f_{6V} = f_{6V}(0) = \lim_{L\to\infty} \lim_{M\to\infty} \tfrac{1}{LM} \log Z^{(0)}_{6\mathrm{V}}(\mathbb{T}^{\times}_{L,M}).$$

4.2 Relation to FK-percolation: BKW correspondence

We next present a correspondence between critical FK-percolation on (a $\pi/4$-rotated version) of the torus and the six vertex model described above. It is sometimes called the Baxter-Kelland-Wu (or BKL) correspondence — 1979. We start by describing several the FK-percolation related to the six vertex model on $\mathbb{T}_{L,M}$.

FK-percolation setting. Notice that $\mathbb{T}_{L,M}$ is a bipartite graph; consider a bi-partite colouring in black and white of its vertices. Let $\mathbb{T}^{\times}_{L,M}$ be the graph containing only the black vertices of $\mathbb{T}_{L,M}$, with edges between vertices at distance $\sqrt{2}$. Write Ω_{FK} for the set of FK-percolation configurations on $\mathbb{T}^{\times}_{L,M}$.

We will work here with $q \geq 1$ fixed and $p = p_c(q) = \frac{\sqrt{q}}{1+\sqrt{q}}$. Thus

$$w_{\mathrm{FK}}(\omega) = p^{|\omega|}(1-p)^{|E(\mathbb{T}_{L,M})|-|\omega|} q^{\#\text{ clusters}} = (1-p)^{|E(\mathbb{T}_{L,M})|} \cdot \sqrt{q}^{|\omega|} q^{\#\text{ clusters}}.$$

Write $\mathcal{E}_k$ for the event that ω contains at least k vertically crossing clusters, where clusters are counted on the cylinder obtained by cutting $\mathbb{T}_{L,M}$ horizontally at height 0.

Proposition 4.2 (BKW correspondence)**.** *We have*

$$f_{\mathrm{FK}} = f_{6V} + \tfrac{1}{4}\log q + \log(1+\sqrt{q}) \qquad \textit{and} \tag{29}$$

$$\phi_{\mathbb{T}^{\times}_{L,M}}[\mathcal{E}_{\alpha L}]^{1/LM} = \exp\big(f_{6V}(\alpha) - f_{6V}(0) + o(1)\big) \qquad \textit{for any } \alpha > 0 \tag{30}$$

The rest of the section is dedicated to proving this result.

Parameters of the correspondence First let us define a useful series of parameters. For $q \geq 1$, let λ be such that

$$e^{\lambda} + e^{-\lambda} = \sqrt{q}. \tag{31}$$

Notice that λ is real for $q \geq 4$ and purely imaginary for $1 \leq q < 4$. In both cases there are two possible choices for λ (up to multiplication by ± 1); we do not impose a canonical choice.

When attempting to relate to six-vertex model to FK-percolation, c and q need to be related. We choose c so that

$$c = e^{\frac{\lambda}{2}} + e^{-\frac{\lambda}{2}} = \sqrt{2+\sqrt{q}}. \tag{32}$$

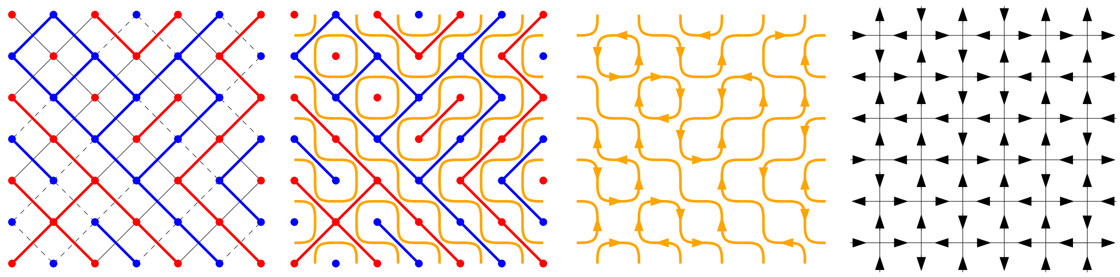


Figure 6: The different steps in the correspondence between the random-cluster and six-vertex models on a torus. From left to right: A random-cluster configuration and its dual, the corresponding loop configuration, an orientation of the loop configuration, the resulting six-vertex configuration. Note that in the first picture, there exist both a primal and dual cluster winding vertically around the torus; this leads to two loops that wind vertically (see second picture); if these loops are oriented in the same direction (as in the third picture), then the number of up arrows on every row of the six-vertex configuration is equal to $\frac{N}{2} \pm 1$.

Loop configurations We define two more types of configurations needed in describing the BKW correspondence. An *oriented loop* on $\mathbb{T}_{L,M}$ is a cycle on $\mathbb{T}_{L,M}$ which is edge-disjoint and non-self-intersecting. We may view oriented loops as ordered collections of edges of $E(\mathbb{T}_{L,M})$, quotiented by cyclic permutations of the indices. Un-oriented loops (or simply loops) are oriented loops considered up to reversal of the indices. A (oriented) loop configuration on $\mathbb{T}_{L,M}$ is a partition of $E(\mathbb{T}_{L,M})$ into (oriented) loops. Write Ω_{Loop} and $\Omega^{\circ}_{\text{Loop}}$ for the set of configurations of un-oriented and oriented loops, respectively.

Associate the following weight to un-oriented and oriented loop configurations. For an un-oriented loop configuration ω°, write $\ell(\omega^\circ)$ for the number of different loops of ω° and $\ell_0(\omega)$ the number of such loops that are not retractable (on the torus) to a point. For an oriented loop configuration $\omega^{\curvearrowright}$, set $\ell_-(\omega^{\curvearrowright})$ and $\ell_+(\omega^{\curvearrowright})$ for the number of retractable loops of $\omega^{\curvearrowright}$ which are oriented clockwise and counterclockwise, respectively.

Set

$$w_\circ(\omega^\circ) = \sqrt{q}^{\ell(\omega^\circ)} \cdot \left(\tfrac{2}{\sqrt{q}}\right)^{\ell_0(\omega^\circ)} \qquad \text{for all } \omega^\circ \in \Omega^\circ_{\text{Loop}};$$
$$w_{\curvearrowright}(\omega^{\curvearrowright}) = e^{\lambda(\ell_+(\omega^{\curvearrowright}) - \ell_-(\omega^{\curvearrowright}))} \qquad \text{for all } \omega^{\curvearrowright} \in \Omega^\circ_{\text{Loop}}.$$

Correspondence between configurations. The correspondence between configurations is best descried in Figure 6.

Un-oriented loop configurations are in bijection to FK-percolation configurations: associate to any FK-percolation configuration ω the unique loop configuration whose loops intersect no not intersect any primal or dual edges of ω.

An oriented loop configuration $\omega^{\curvearrowright}$ is said to be coherent with a the un-oriented loop configuration containing the same loops. It is also said to be coherent with the six-vertex configuration whose edge-orientations are given by the orientations of the loops.

Notice that for any un-oriented loop configuration ω°, there are several oriented loop configurations coherent with it; namely $2^{\ell(\omega^\circ)}$. Also, for any six vertex configuration $\vec{\omega}$, there are $2^{\#\text{vertices type } c}$ oriented loop configurations coherent with $\vec{\omega}$.

Correspondence for weights. The following lemmas related the weights of the different configurations. Both results are obtained via fairly direct computations, full proofs are available in [?].

Lemma 4.3. *For any* $\omega \in \Omega_{\mathrm{FK}}$*,*

$$w_{\mathrm{FK}}(\omega) = C\,\sqrt{q}^{\ell(\omega^\circ)} q^{s(\omega)} = C\left(\tfrac{\sqrt{q}}{2}\right)^{\ell_0(\omega)} q^{s(\omega)} \sum_{\omega^{\curvearrowright}} w_{\curvearrowright}(\omega^{\curvearrowright}), \tag{33}$$

where ω° *is the loop configuration corresponding to* ω *and the sum is over the* $2^{\ell(\omega)}$ *oriented loop configurations coherent with* ω°*. The term* $s(\omega^\circ) \in \{0,1\}$ *is the indicator that* ω *contains a cluster that winds around* $\mathbb{T}_{L,M}$ *in both the vertical and horizontal directions. Finally* $C = \frac{\sqrt{q}^{LM/2}}{(1+\sqrt{q})^{LM}}$*.*

The first equality is proved by induction on the number of open edges of ω. The second is obtained by observing that in the sum on the right-hand side, every retractible loop of ω° appears with its two possible orientations, thus with a total weight of $e^\lambda + e^{-\lambda} = \sqrt{q}$; non-retractible loops appear with a weight of 2, compensated by the term $\left(\frac{\sqrt{q}}{2}\right)^{\ell_0(\omega)}$.

Lemma 4.4. *For any six vertex configuration* $\vec{\omega}$

$$w_{6V}(\vec{\omega}) = \sum_{\omega^{\curvearrowright} \text{ coherent w. } \vec{\omega}} w_{\curvearrowright}(\omega^{\curvearrowright}). \tag{34}$$

The key here is to see $w_{\curvearrowright}(\omega^{\curvearrowright})$ as $e^{\lambda \mathrm{wind}(\omega^{\curvearrowright})/2\pi}$, where $\mathrm{wind}(\omega^{\curvearrowright})$ is the total winding of all loops of $\omega^{\curvearrowright}$. This winding may be computed locally: vertices of type a and b produce a total winding of 0, while vertices of type c produce a total winding of $\pm\pi$, depending on how the loops are split at the vertex. See also Figure 7.

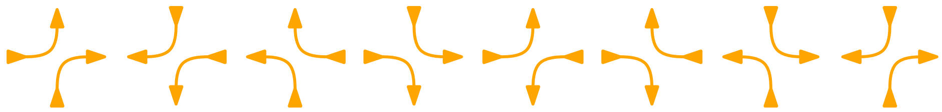


Figure 7: The 8 different types of vertices encountered in an oriented loop configuration.

Consequences for the partition functions: proof of Proposition 4.2 Summing (33) and (34) over all configurations, we easily find that

$$Z_{\mathrm{FK}}\,\phi_{\mathbb{T}^\times_{L,N}}\left[\left(\tfrac{2}{\sqrt{q}}\right)^{\ell_0} q^{-s}\right] = \sum_{\omega \in \Omega_{\mathrm{FK}}} \left(\tfrac{2}{\sqrt{q}}\right)^{\ell_0(\omega)} q^{-s(\omega)} \cdot w_{\mathrm{FK}}(\omega) = C\, Z_{6V}. \tag{35}$$

This is not exactly an equality between the two partition function, due to the term $\phi_{\mathbb{T}^\times_{L,N}}\left[\left(\tfrac{2}{\sqrt{q}}\right)^{\ell_0} q^{-s}\right]$. Note however that this term is of no significance for the free energy. Indeed, ℓ_0 is bounded by $L+M$, so

$$f_{\mathrm{FK}} := \lim_{L\to\infty} \lim_{M\to\infty} \frac{1}{LM} \log Z_{\mathrm{FK}}(\mathbb{T}^\times_{L,M}) = f_{6V} + \tfrac{1}{4}\log q - \log(1+\sqrt{q}). \tag{36}$$

This proves (29).

We can even do better. Fix $k \geq 0$. For a six-vertex configuration $\vec{\omega}$, write $K(\vec{\omega})$ for the excess of up-arrows on each row, that is the number of up-arrows on any one row minus $L/2$. Use the same notation for oriented loop configuration $\omega^{\curvearrowright}$.

When $\vec{\omega}$ and ω° are coherent $K(\vec{\omega}) = K(\omega^\circ)$. In particular, the oriented loop configurations coherent with $\{\vec{\omega} : K(\vec{\omega}) = k\}$ are exactly those with $K(\omega^\circ) = k$.

For an un-oriented loop configuration ω°, write $J_k(\omega)$ for the number of ways in which its vertically-winding loops may be oriented to produce $\omega^\circlearrowleft$ with $K(\omega^\circlearrowleft) = k$. For most ω° we have $J_k(\omega^\circ) = 0$. Indeed, to have $J_k(\omega^\circ) > 0$, ω° needs to have at least $2k$ loops winding vertically around the torus (where a loop winding vertically more than once is counted with multiplicity). Finally, these loops are all counted in $\ell_0(\omega^\circlearrowleft)$. Indeed, any retractible or horizontally winding loop contributes the same number of up- and down-arrows to each row; unbalances come only from loops winding vertically around $\mathbb{T}_{L,M}$.

For an un-oriented loop configuration ω°, write $\ell_h(\omega^\circ)$ for the number of non-retractible loops of ω that do not wind vertically around the torus (in particular they do wind horizontally). Observe that re-orienting the retractible loops of ω°, or those contributing to ℓ_h, does not change $K(\omega^\circlearrowleft)$. Thus, applying (34), then (33), we find

$$Z_{6V}^{(k)} = \sum_{\omega^\circlearrowleft:\, K(\omega^\circlearrowleft)=k} w_\circlearrowleft(\omega^\circlearrowleft) = \sum_{\omega^\circ} J_k(\omega) \frac{2^{\ell_h(\omega^\circ)}}{\sqrt{q}^{\ell_0(\omega^\circ)}} \sqrt{q}^{\ell(\omega)}$$
$$= \tfrac{1}{C} \sum_{\omega} J_k(\omega) \frac{2^{\ell_h(\omega^\circ)}}{\sqrt{q}^{\ell_0(\omega^\circ)}} q^{-s(\omega)} w_{\mathrm{FK}}(\omega) = \frac{Z_{\mathrm{FK}}}{C}\, \phi_{\mathbb{T}_{L,N}^\times}\Big[J_k(\omega^\circ)\, \tfrac{2^{\ell_h(\omega^\circ)}}{\sqrt{q}^{\ell_0(\omega^\circ)}} q^{-s(\omega)}\Big].$$

All terms in the probability on the right-hand side, except $J_k(\omega^\circ)$, are ultimately unimportant. Moreover, most of the probability will come from configurations with $J_k(\omega^\circ) = 1$. The important observation is that $J_k(\omega^\circ) \neq 0$ only if there are at least k primal clusters crossing $\mathbb{T}_{L,M}^\times$ vertically (where clusters are actually counted on the cylinder obtained by cutting $\mathbb{T}_{L,M}^\times$ horizontal at height 0).

Recall that we are interested in $f_{6V}(\alpha)$ which corresponds to the partition function over configurations with an excess $K(\vec{\omega}) = \alpha L$. Crude upper and lower bounds on the terms appearing in the probability above suffice to prove that

$$f_{6V}(\alpha) = f_{\mathrm{FK}} + \lim_{L\to\infty} \lim_{M\to\infty} \tfrac{1}{LM} \log \phi_{\mathbb{T}_{L,M}^\times}[\mathcal{E}_{\alpha L}] - \tfrac{1}{4}\log q - \log(1+\sqrt{q})$$

Together with (29), this proves (30) (also recall that $f_{6V} = f_{6V}(0)$).

4.3 Solving six vetex via transfer matrix

Theorem 4.5. *Recall that $f_{6V}(\alpha)$ denotes the free energy of the six vertex model "with a density α of excess up-arrows". Then*

$$f_{6V}(\alpha) = \begin{cases} C\alpha + o(\alpha) & \text{if } c > 2, \text{ with } C = C(c), \\ C\alpha^2 + o(\alpha^2) & \text{if } 0 < c \le 2, \text{ with } C = C(c), \end{cases} \qquad \text{as } \alpha \to 0. \tag{37}$$

The theorem above is obtained via the estimation of the leading eigenvalues of blocks of the transfer matrix of the six vertex model via the Bethe ansatz. We describe these objects next.

Transfer matrix formalism We identify the possible configurations of vertical arrows on one row by $\{\pm\}^L$. Consider the matrix $V = V_L$ defined by

$$V(\psi, \psi') = \sum_{\text{possible completions}} c^{\#\text{type } c \text{ vertices}} \qquad \forall \psi, \psi' \in \{\pm\}^L. \tag{38}$$

where the completions refer to the possible assignments of horizontal arrows between ψ and ψ' that obey the ice rule. See Figure 8 for an example.

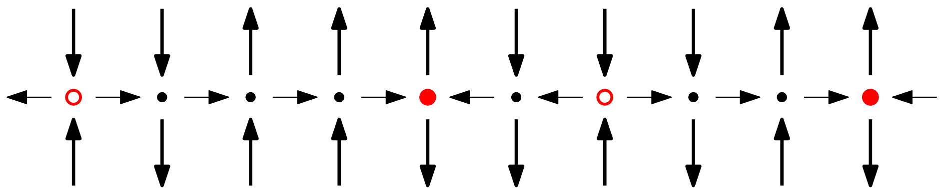


Figure 8: The unique possible completion between ψ (bottom config) and ϕ' (top config) that obeys the ice rule. The weight here is c^4.

The transfer matrix is split into blocks corresponding to the number of up-arrows on each row (recall that this number is conserved). Write $V^{(k)}$ for the block with $N/2+k$ up-arrow. Then, it is immediate to check that

$$Z_{6V}^{(k)}(\mathbb{T}(L,M)) = \mathrm{Tr}(V^{(k)})^M. \tag{39}$$

Furthermore, each such block has non-negative entries and may be shown to be irreducible. It follows by the Perron-Frobenius theorem that each $V^{(k)}$ has a single eigenvector of maximal eigenvalue (with all other eigenvalues being of strictly smaller modulus). We call these the Perron-Frobenius eigenvector and eigenvalue, and write $\Lambda^{(k)} = \Lambda^{(k)}(L)$ for the latter.

Then (39) implies

$$Z_{6V}^{(k)}(\mathbb{T}(L,M)) = (\Lambda^{(k)}(L))^M(1+e^{-\varepsilon M}), \tag{40}$$

for some $\varepsilon = \varepsilon(L) > 0$. Since $Z_{6V}^{(k)}(\mathbb{T}(L,M))$ is used to determine the free energy with slope α, we find

$$f_{6V}(\alpha) = \lim_{L} \tfrac{1}{L} \log \Lambda^{(k)}(L). \tag{41}$$

Statement of the Bethe Ansatz for the six-vertex model. The Bethe Ansatz is a method for producing eigenvectors and eigenvalues of the transfer matrix. We state it below.

Recall that $\Delta = (2-c^2)/2$ and set

$$\mu := \begin{cases} \arccos(-\Delta) & \text{if } c \le 2 \\ = 0 & \text{if } c > 2 \end{cases} \qquad \text{and} \qquad \mathcal{D} := (-\pi+\mu, \pi-\mu).$$

Let $\Theta : \mathcal{D}^2 \to \mathbb{R}$ to be the unique continuous function which satisfies $\Theta(0,0) = 0$ and

$$\exp(-i\Theta(x,y)) = e^{i(x-y)} \cdot \frac{e^{-ix}+e^{iy}-2\Delta}{e^{-iy}+e^{ix}-2\Delta}.$$

For $z \neq 1$, we set

$$L(z) := 1 + \frac{c^2 z}{1-z}, \qquad M(z) := 1 - \frac{c^2}{1-z}.$$

Theorem 4.6 (Bethe Ansatz for V)**.** *Fix $n = N/2 - k$. Let $(p_1, p_2, \ldots, p_n) \in \mathcal{D}^n$ be distinct and satisfy the equations*

$$Np_j = 2\pi I_j - \sum_{k=1}^{n} \Theta(p_j, p_k), \qquad \forall j \in \{1, \ldots, n\}. \tag{BE}$$

Then, $\psi = \sum_{|\vec{x}|=n} \psi(\vec{x})\, \Psi_{\vec{x}}$ *, where* $\psi(\vec{x})$ *is given by*

$$\psi(\vec{x}) := \sum_{\sigma \in \mathfrak{S}_n} A_\sigma \prod_{k=1}^{n} \exp\left(ip_{\sigma(k)} x_k\right), \quad \textit{where} \quad A_\sigma := \varepsilon(\sigma) \prod_{1 \le k < \ell \le n} e^{ip_{\sigma(k)}}\,(e^{-\mathrm{i}p_{\sigma(k)}} + e^{\mathrm{i}p_{\sigma(\ell)}} - 2\Delta),$$

(for σ *an element of the symmetry group* $\mathfrak{S}_n$*) satisfies the equation* $V\psi = \Lambda\psi$*, where*

$$\Lambda = \begin{cases} \prod_{j=1}^{n} L(e^{ip_j}) + \prod_{j=1}^{n} M(e^{ip_j}) & \textit{if } p_1, \dots, p_n \textit{ are non zero,} \\ \Big[2 + c^2(N-1) + c^2 \sum_{j \neq \ell} \partial_1 \Theta(0, p_j)\Big] \cdot \prod_{j \neq \ell} M(e^{ip_j}) & \textit{if } p_\ell = 0 \textit{ for some } \ell. \end{cases}$$

It is expected that there exists exactly one solution to (BE), which produces the Perron-Frobenius eigenvector and eigenvalue of $V^{(k)}$.

Let us assume that one may find solutions $p_1, \dots, p_n$ to (BE), and that their distribution in $\mathcal{D}$ ($\rho_N = \frac{1}{N} \sum_{i=1}^{n} \delta_{p_i}$) converges to a measure $\rho(x)dx$ admitting a density – this is sometimes called *condensation* of the Bethe roots. Then (BE) may be re-written as

$$2\pi\rho(x) = 1 + \int_{\mathcal{D}} \partial_1 \Theta(x,y)\rho(y)dy \qquad \forall x \in \mathcal{D} \tag{42}$$

and

$$\lim_{N \to \infty} \frac{1}{N} \log \Lambda = \int_{\mathcal{D}} \log \left| M\left(e^{ix}\right) \right| \rho(x) dx. \tag{43}$$

The condition (42) allows one to determine ρ, which may then be injected in (43) to obtain asymptotics for Λ.

We will not prove Theorem 4.6, nor how it is applies to our six vertex model. We limit ourselves to mentioning that

- The difficulties in applying this method are: prove that (BE) has solutions and that these solutions do condensate as $L \to \infty$. Show that the resulting vector is not null, and therefore that the resulting Λ is an eigenvalue. Prove that Λ is the Perron-Frobenius eigenvalue of the corresponding block of V. Finally, when this is done, one needs to compute sufficiently precise estimates on the resulting eigenvalue, using the continuous version of the Bethe equations (42) and (43).

- The Bethe ansatz changes form at $\Delta = -1$, corresponding to $c = 2$. Outside of this value, it appears to depend smoothly on Δ.

- These difficulties were overcome in [?] and [?]. The general technique to prove that Λ is indeed the Perron-Frobenius eigenvalue was done by considering the model (for finite L) at the “trivial” cases $\Delta = 0$ and $\Delta = -\infty$, where the computations are explicit, then proving that $\Delta \mapsto \psi, \Lambda$ is an analytic function, hence preserving the positivity of ψ. The condensation is also proved using this type of evolution in Δ: it is proved that $\psi(\Delta)$ can not “escape” continuously a region of condensation.

- The estimates obtained $\Lambda^{(k)}/\Lambda^{(0)}$ are not sufficiently fine to apply them to $k = 1$ (or small values of k). For our purpose $k = \alpha L$, where the estimates are more robust.

- The ultimate computation of $\Lambda^{(k)}/\Lambda^{(0)}$ proves (37).

4.4 Deducing the type of phase transition

A direct consequence of Proposition 4.2 and Theorem 4.6 is the following estimate on $\phi_{\mathbb{T}^{\times}_{L,M}}[\mathcal{E}_{\alpha L}]$.

Corollary 4.7. *We have*

$$\lim_{L\to\infty}\lim_{M\to\infty}\tfrac{1}{LM}\log\phi_{\mathbb{T}^{\times}_{L,M}}[\mathcal{E}_{\alpha L}] = \begin{cases} C\alpha + o(\alpha) & \text{if } c > 2, \text{ with } C = C(c), \\ C\alpha^2 + o(\alpha^2) & \text{if } 0 < c \leq 2, \text{ with } C = C(c). \end{cases} \tag{44}$$

This corollary implies directly that (b) of Theorem 3.1 occurs for $0 < c \leq 2$ and point (c) occurs for $c > 2$. Indeed, one way to see this is to see that the correlation length

$$\xi^{-1} = \xi^{-1}(p_c, q) = \lim_n \frac{1}{n}\log\phi^0[0 \leftrightarrow \partial\Lambda_n], \tag{45}$$

may be related to $\phi_{\mathbb{T}^{\times}_{L,M}}[\mathcal{E}_k]$ by

$$e^{-kM/\xi} \leq \phi_{\mathbb{T}^{\times}_{L,M}}[\mathcal{E}_k] \leq e^{-(k-1)M/\xi}. \tag{46}$$

When combining this with (44) we conclude that $\xi(p_c, q) = \infty$ if an only if $1 \leq q \leq 4$. Theorem 3.1 allows us to conclude.