Bond percolation on isoradial graphs: criticality and universality

Geoffrey R. Grimmett · **Ioan Manolescu**

Abstract In an investigation of percolation on isoradial graphs, we prove the criticality of canonical bond percolation on isoradial embeddings of planar graphs, thus extending celebrated earlier results for homogeneous and inhomogeneous square, triangular, and other lattices. This is achieved via the star–triangle transformation, by transporting the box-crossing property across the family of isoradial graphs. As a consequence, we obtain the universality of these models *at* the critical point, in the sense that the one-arm and $2j$-alternating-arm critical exponents (and therefore also the connectivity and volume exponents) are constant across the family of such percolation processes. The isoradial graphs in question are those that satisfy certain weak conditions on their embedding and on their track system. This class of graphs includes, for example, isoradial embeddings of periodic graphs, and graphs derived from rhombic Penrose tilings.

Keywords Bond percolation · isoradial graph · rhombic tiling · Penrose tiling · inhomogeneous percolation · universality · critical exponent · arm exponent · scaling relations · box-crossing · star–triangle transformation · Yang–Baxter equation

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1 Introduction

Two-dimensional disordered systems, when critical, are presumed to have properties of conformality and universality. Rigorous evidence for this classic paradigm

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from conformal field theory includes the recent analysis of the critical Ising model
by Chelkak and Smirnov [12,42] on a family of graphs known as 'isoradial'. On
the one hand, such isoradial graphs are especially harmonious in a theory of discrete
holomorphic functions (introduced by Duffin, see [11,14,32]), and on the other they
are well adapted to transformations of star–triangle type (explained by Kenyon [24]).
These two properties resonate with the intertwined concepts of conformality and uni-
versality.

It is shown here that, for a broad class $\mathscr{G}$ of isoradial graphs, the associated bond
percolation model is critical, and furthermore certain of its critical exponents are con-
stant across $\mathscr{G}$. The class $\mathscr{G}$ includes many specific models studied earlier, but is much
more extensive than this restricted class. In earlier related work of Kesten [26,27],
Wierman [46] and others, it has been assumed that the graph under study has certain
invariances under, for example, translation, rotation and/or reflection. Such assump-
tions of regularity play no role in the current work, where the key assumptions are that
the graph under study is isoradial, and satisfies two weak conditions on its embedding
and on its so-called track system, namely the so-called bounded-angles property and
the square-grid property. The first of these is a condition of non-degeneracy, and the
second requires the graph to contain a square grid within its track system. (Formal
definitions are deferred to Sections 2.1 and 4.2.)

In advance of the formalities of Section 2, we explain the term 'isoradial'. Let G
be a planar graph embedded in the plane $\mathbb{R}^2$. It is called *isoradial* if there exists $r > 0$
such that, for every face F of G, the vertices of F lie on a circle of radius r with centre
in the interior of F. Note that isoradiality is a property of the planar embedding of G
rather than of G itself. By rescaling the embedding of G, we may assume $r = 1$.

It was noted by Duffin [14] that isoradial graphs are in two–one correspondence
with rhombic tilings of the plane (the name 'isoradial' was coined later by Kenyon).
While details of this correspondence are deferred to Section 2, we highlight one fact
here. Let $G = (V, E)$ be isoradial. An edge $e \in E$ lies in two faces, and therefore two
circumcircles. As illustrated in Figure 1.1, e subtends the same angle $\theta_e \in (0, \pi)$ at
the centres of these circumcircles, and we define $p_e \in (0, 1)$ by

$$\frac{p_e}{1 - p_e} = \frac{\sin(\frac{1}{3}[\pi - \theta_e])}{\sin(\frac{1}{3}\theta_e)}. \tag{1.1}$$

We consider bond percolation on G with edge-probabilities $\mathbf{p} = (p_e : e \in E)$. Formula
(1.1) appeared for the first time in this explicit form in the work of Kenyon [24].

Our main results are that (subject to two assumptions on G) these percolation
models are critical, and their critical exponents (*at* the critical point) are universal
across the class of such models. More precisely, the exponents ρ, η, δ and the $2j$-
alternating arm exponents ρ_{2j} are constant across the class (assuming they exist). See
[15] for an account of the general theory of percolation.

Our methods work only *at* the critical point, and we are unable to extend such
universality to exponents *near* the critical point, such as γ and β. Neither have we
any progress to report on the *existence* of critical exponents for these models. Essen-
tially the only model for which existence has been proved is site percolation on the
triangular lattice (see [43]), and our numerical knowledge of the exponents is based
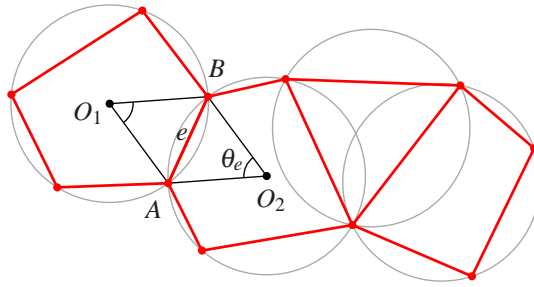
Fig. 1.1 Part of an isoradial graph. Each face is inscribed in a circle of radius 1. With the edge e, we associate the angle θ_e. This and later figures are best viewed in colour.

in this case on the proof by Smirnov [41] of the convergence in the scaling limit to SLE$_6$ (see also the more recent works [10,45]).

The principal tool in the current work is the star–triangle transformation, otherwise known amongst physicists as the Yang–Baxter equation. This was discovered by Kennelly [23] in 1899 in the context of electrical networks, and was adapted in 1944 by Onsager [34] to the Ising model in conjunction with Kramers–Wannier duality. It is a key element in the work of Baxter [3] on exactly solvable models in statistical mechanics (see [31,37] for scientific and historical accounts). Sykes and Essam [44] used the star–triangle transformation to predict the critical surfaces of inhomogeneous bond percolation on triangular and hexagonal lattices, and it is also a tool in the study of the random-cluster model [16], and of the dimer model [24].

The star–triangle transformation is especially well matched to isoradiality. The exchange of a star with a triangle is equivalent to a local move in the associated rhombic tiling. In the case of percolation, this move replaces one triple of edge-parameters by another triple given according to (1.1), and these are precisely the triples that satisfy the star–triangle transformation. This general observation is valid for several models of statistical physics, including the Ising and Potts models and the random-cluster model (see [24]). Note, however, that the star–triangle transformation contributes also to the analysis of models on non-isoradial graphs (see [3]).

The overall approach of this paper is to use the star–triangle transformation to move between models, in a manner closely related to that developed earlier by Baxter, Kenyon, and others. Using a coupling of probability measures on different isoradial graphs, one may show as in [19,20] that the star–triangle transformation preserves open connections in the percolation model, and may therefore be used to transport the so-called 'box-crossing property' from one model to another. The *box-crossing property* is the statement that any rectangle in $\mathbb{R}^2$ of given aspect-ratio is crossed by an open path, with probability bounded away from 0 uniformly in the size, position, and rotation of the box. This property has played an important role in the theory of criticality and scaling in two-dimensional percolation, as described in Kesten's book [27], and more recently in [45].

This is a continuation and extension of the work begun in [19,20]. The main idea is to use the star–triangle transformation to transport the box-crossing property

between graphs. By the Russo–Seymour–Welsh (RSW) lemma of [38,40], homogeneous percolation on the square lattice (with $p = \frac{1}{2}$) has the box-crossing property, and this property may be transported, by repeated applications of star–triangle transformations, to the isoradial graphs considered here. Once we know that an isoradial graph and its (isoradial) dual graph have the box-crossing property, criticality follows as in [19, Props 4.1–4.2]. The above universality follows by related arguments.

We return briefly to the star–triangle transformation and the Ising model. The Ising model differs from percolation and the random-cluster model in the following significant regard. Whereas the Ising model may be transformed thus for any triple of edge-interactions, the percolation and random-cluster models may be transformed only when the interaction-triple satisfies a certain equation connected to the critical surface in the translation-invariant lattice model (see [19, Thm 1.1]). For this reason, the star–triangle transformation enables a deeper understanding of the Ising model than of its cousins. Its application to the Ising model has been developed in a series of papers by Baxter and others under the title 'Z-invariant Ising model' beginning with [4] and continued in later works. We do not attempt a full bibliography here, but mention the paper [2] by Au-Yang and Perk, the recent papers of Chelkak and Smirnov [12,42], as well as the dimer analysis [6] of Boutillier and de Tilière.

This paper is organized as follows. Isoradial graphs and the box-crossing property are summarised in Section 2, and our main results concerning criticality and universality are stated in Section 3. Section 4 contains a fuller description of isoradiality and its connections to rhombic tilings. The star–triangle transformation is described in Section 5. The box-crossing property, Theorem 3.1, is then proved in two steps: first in Section 6 for the special case of isoradial square lattices, and then in Section 7 where it is explained how this case may be extended to general isoradial graphs satisfying the given conditions. Theorem 3.4, concerning universality, is proved in Section 8.

The set of integers is denoted by $\mathbb{Z}$, of natural numbers by $\mathbb{N}$, and of non-negative integers by $\mathbb{N}_0$.

2 Isoradiality and the box-crossing property

2.1 Isoradial graphs

Let $G = (V, E)$ be a planar graph embedded in the plane $\mathbb{R}^2$ (it is assumed that the edges are embedded as straight-line segments, with intersections only at vertices). It is called *isoradial* if, for every bounded face F of G, the vertices of F lie on a circle of (circum)radius 1 with centre in the interior of F. In the absence of a contrary assumption, we shall assume that isoradial graphs are infinite with bounded faces. While isoradiality is a property of the planar embedding of G rather than of G itself, we shall sometimes speak of an isoradial graph.

Let $G = (V, E)$ be isoradial. Each edge $e = \langle A, B \rangle$ of G lies in two faces, with circumcentres O_1 and O_2. Since the two circles have equal radii, the quadrilateral AO_1BO_2 is a rhombus. Therefore, the angles AO_1B and BO_2A are equal, and we write $\theta_e \in (0, \pi)$ for their common value. See Figure 1.1.

Definition 2.1. *Let $\varepsilon > 0$. The isoradial graph G is said to have the* bounded-angles *property* $\mathrm{BAP}(\varepsilon)$ *if*

$$\theta_e \in [\varepsilon, \pi - \varepsilon], \qquad e \in E. \tag{2.1}$$

It is said to have, simply, the bounded-angles *property if it satisfies* $\mathrm{BAP}(\varepsilon)$ *for some* $\varepsilon > 0$.

All isoradial graphs of this paper will be assumed to have the bounded-angles property. The area of the rhombus AO_1BO_2 equals $\sin\theta_e$ and, under $\mathrm{BAP}(\varepsilon)$,

$$\sin\varepsilon \leq \mathrm{Area}(AO_1BO_2) \leq 1. \tag{2.2}$$

We shall study isoradial graphs satisfying a further property called the 'square-grid property'. We defer a discussion of the square-grid property until Section 4.2, since it necessitates a fuller account of isoradial graphs and their relationship to rhombic tilings of the plane and the associated 'track systems' introduced by de Bruijn [7, 8]. Although one may construct isoradial graphs that do not have the square-grid property, the property is satisfied by many graphs arising from the rhombic tilings that we have encountered in the literature on plane tilings.

From the many examples of isoradial graphs with the square-grid property, we select for illustration the set of isoradial embeddings of the square lattice, and more generally isoradial embeddings of any connected periodic graph. It is not the automorphism group of the graph that is relevant here, but rather the geometry of its track system. For example, isoradial graphs arising from rhombic Penrose tilings have the square-grid property. For definitions and further discussion, see Section 4.

2.2 Percolation on isoradial graphs

Bond percolation on a graph $G = (V, E)$ has sample space $\Omega := \{0,1\}^E$, to which is assigned a product measure $\mathbb{P}_{\mathbf{p}}$ with $\mathbf{p} = (p_e : e \in E) \in [0,1]^E$. When G is isoradial, there is a canonical product measure, denoted $\mathbb{P}_G$, associated with its embedding, namely that with $p_e = p_{\theta_e}$ where θ_e is given in the last section and illustrated in Figure 1.1, and

$$\frac{p_\theta}{1 - p_\theta} = \frac{\sin(\frac{1}{3}[\pi - \theta])}{\sin(\frac{1}{3}\theta)}. \tag{2.3}$$

Note that $p_\theta + p_{\pi-\theta} = 1$, and that G has the bounded-angles property $\mathrm{BAP}(\varepsilon)$ if and only if

$$p_{\pi-\varepsilon} \leq p_e \leq p_\varepsilon, \qquad e \in E. \tag{2.4}$$

The graph G has a dual graph $G^* = (V^*, E^*)$. Since G is isoradial, so is G^*. This fact is discussed in greater depth in Section 4, but it is clear from Figure 1.1 that G^* may be embedded in $\mathbb{R}^2$ with vertices at the centres of circumcircles, and edges between circumcentres of abutting faces. Let $e^* \in E^*$ be the dual edge (in this embedding) crossing the primal edge $e \in E$. Then $\theta_{e^*} = \pi - \theta_e$, so that $p_e + p_{e^*} = 1$ by (2.3). In conclusion, the canonical measure $\mathbb{P}_{G^*}$ is dual to the primal measure $\mathbb{P}_G$. By (2.4),

$$G^* \text{ satisfies } \mathrm{BAP}(\varepsilon) \text{ if and only if } G \text{ satisfies } \mathrm{BAP}(\varepsilon). \tag{2.5}$$

See, for example, [15, Sect. 11.2] for an account of graphical duality.

Here is some basic notation. Let $G = (V,E)$ be a graph, not necessarily isoradial or even planar, and let $\omega \in \Omega := \{0,1\}^E$. An edge e is called *open* (or ω-*open*) if $\omega(e) = 1$, and *closed* otherwise. A path of G is called *open* if all its edges are open. For $u,v \in V$, we say *u is connected to v* (in ω), written $u \leftrightarrow v$ (or $u \xleftrightarrow{G,\omega} v$), if G contains an open path from u and v; if they are not connected, we write $u \xleftrightarrow{G,\omega} v$. An *open cluster* of ω is a maximal set of pairwise-connected vertices. Let $C_v = \{u \in V : u \leftrightarrow v\}$ denote the open cluster containing the vertex v, and write $v \leftrightarrow \infty$ if $|C_v| = \infty$.

For $\omega \in \Omega$ and $e \in E$, let $\omega^*(e^*) = 1 - \omega(e)$, so that e^* is open in the dual graph G^* (written *open**) if e is closed in the primal graph G.

2.3 The box-crossing property

Let $G = (V,E)$ be a countably infinite, connected graph embedded in the plane, and let $\mathbb{P}$ be a probability measure on $\Omega := \{0,1\}^E$. The 'box-crossing property' is concerned with the probabilities of open crossings of domains in $\mathbb{R}^2$. This has proved to be a very useful property indeed for the study of infinite open clusters in G; see, for example, [17,20,27].

A *(planar) domain* $\mathscr{D}$ is an open, simply connected subset of $\mathbb{R}^2$ which, for simplicity, we assume to be bounded by a Jordan curve $\partial \mathscr{D}$. Most domains of this paper are the interiors of polygons. Let $\mathscr{D}$ be a domain, and let A, B, C, D be distinct points on its boundary in anticlockwise order. Let $\omega \in \Omega$. We say that $\mathscr{D}$ has an open crossing from DA to BC if G contains an open path using only edges intersecting $\mathscr{D}$ which, when viewed as an arc in $\mathbb{R}^2$, intersects $\partial \mathscr{D}$ exactly twice, once between D and A (including the endpoints) and once between B and C.

A *rectangular domain* is a set $\mathscr{B} = f((0,x) \times (0,y)) \subseteq \mathbb{R}^2$, where $x,y > 0$ and $f : \mathbb{R}^2 \to \mathbb{R}^2$ comprises a rotation and a translation. The *aspect-ratio* of this rectangle is $\max\{x/y, y/x\}$. We say $\mathscr{B}$ has *open crossings* in a configuration $\omega \in \Omega$ if it has open crossings both from $f(\{0\} \times [0,y])$ to $f(\{x\} \times [0,y])$ and from $f([0,x] \times \{0\})$ to $f([0,x] \times \{y\})$.

Definition 2.2. *A probability measure $\mathbb{P}$ on Ω is said to have the* box-crossing property *if, for $\rho > 0$, there exist $l_0 = l_0(\rho) > 0$ and $\delta = \delta(\rho) > 0$ such that, for all $l > l_0$ and all rectangular domains $\mathscr{B}$ with side-lengths l and ρl,*

$$\mathbb{P}(\mathscr{B} \text{ has open crossings}) \geq \delta. \qquad (2.6)$$

An isoradial graph G is said to possess the box-crossing property if $\mathbb{P}_G$ possesses it.

In a standard application of the Harris–FKG inequality (see [15, Sect. 2.2]), it suffices for the box-crossing property to consider boxes with aspect-ratio 2, and moreover only such boxes with horizontal/vertical orientation (see, for example, [18, Prop. 3.2] and [19, Prop. 3.1]). If (2.6) holds for this restricted class of boxes with $\rho = 2$ and $\delta = \delta(2)$, we say that G satisfies $\mathrm{BXP}(l_0, \delta)$. All graphs considered here are isoradial with circumradius 1, and for such graphs one may take $l_0 = 3$. We thus abbreviate $\mathrm{BXP}(3,\delta)$ to $\mathrm{BXP}(\delta)$.

It was proved by Russo [38] and Seymour–Welsh [40] that the isotropic embedding of the square lattice (with $p = \frac{1}{2}$) has the box-crossing property, and more generally in [19] that certain inhomogeneous embeddings of the square, triangular, and hexagonal lattices have the property.

3 Main results

Let $\mathscr{G}$ be the class of isoradial graphs with the bounded-angles property and the square-grid property (we recall that the square-grid property is formulated in Section 4.2). The main technical result of this paper is the following. Criticality and universality will follow.

Theorem 3.1. *For $G \in \mathscr{G}$, $\mathbb{P}_G$ possesses the box-crossing property.*

A more precise statement holds. In discussing the square-grid property in Section 4.2, we will introduce a more specific property denoted SGP(I) for $I \in \mathbb{N}$. We shall show that, for $\varepsilon > 0$ and $I \in \mathbb{N}$, there exists $\delta = \delta(\varepsilon, I) > 0$ such that:

$$\text{if } G \text{ satisfies BAP}(\varepsilon) \text{ and SGP}(I), \mathbb{P}_G \text{ satisfies BXP}(\delta). \tag{3.1}$$

Let $G = (V, E)$ be a graph, and let $\mathbb{P}$ be a product measure on $\{0,1\}^E$ with intensities $(p_e : e \in E)$. For $\delta \in \mathbb{R}$, we write $\mathbb{P}^\delta$ for the percolation measure with intensities $p_e^\delta := (0 \vee (p_e + \delta)) \wedge 1$. [As usual, $x \vee y = \max\{x, y\}$ and $x \wedge y = \min\{x, y\}$.] If G is embedded in $\mathbb{R}^2$, the *radius* rad(C_v) of the open cluster at $v \in V$ is the supremum of $k \geq 0$ such that C_v contains a vertex outside the box $v + (-k, k)^2 \subseteq \mathbb{R}^2$.

The proofs of the following Theorems 3.2 and 3.4 rely heavily on the box-crossing property of Theorem 3.1.

Theorem 3.2 (Criticality). *Let $G = (V, E) \in \mathscr{G}$, and let $v > 0$.*

(a) *There exist $a, b, c, d > 0$ such that, for $v \in V$,*

$$ak^{-b} \leq \mathbb{P}_G\big(\text{rad}(C_v) \geq k\big) \leq ck^{-d}, \qquad k \geq 1.$$

(b) *There exists, $\mathbb{P}_G$-a.s., no infinite open cluster.*

(c) *There exist $f, g > 0$ such that, for $v \in V$,*

$$\mathbb{P}_G^{-v}(|C_v| \geq k) \leq fe^{-gk}, \qquad k \geq 0.$$

(d) *There exists $h > 0$ such that, for $v \in V$,*

$$\mathbb{P}_G^v(v \leftrightarrow \infty) > h.$$

(e) *There exists, $\mathbb{P}_G^v$-a.s., exactly one infinite open cluster.*

More precisely, if $G \in \mathscr{G}$ satisfies BXP(ε) and SGP(I), the claims of the theorem hold with constants that depend only on ε, I, and not further on G. The proof of Theorem 3.2 is summarised at the end of this section.

Turning to critical exponents and universality, we write $f(t) \asymp g(t)$ as $t \to t_0 \in [0, \infty]$ if there exist strictly positive constants A, B such that

$$Ag(t) \leq f(t) \leq Bg(t) \tag{3.2}$$

in some neighbourhood of t_0 (or for all large t in the case $t_0 = \infty$). For functions $f^u(t)$, $g^u(t)$ indexed by $u \in U$, we say that $f^u \asymp g^u$ *uniformly in u* if (3.2) holds with constants A, B not depending on u. We write $f(t) \approx g(t)$ if $\log f(t)/\log g(t) \to 1$, and $f^u \approx g^u$ *uniformly in u* if the convergence is uniform in u.

The critical exponents of interest here are those denoted conventionally as ρ, η, δ, and the alternating arm-exponents ρ_{2j}. We begin by defining the so-called *arm-events*. Let B_n denote the box $[-n,n]^2$ of $\mathbb{R}^2$, with boundary ∂B_n. For $N < n$, let $\mathscr{A}(N,n)$ be the *annulus* $[-n,n]^2 \setminus (-N,N)^2$ with inner radius N and outer radius n. The *inner* (respectively, *outer*) *boundary* of the annulus is ∂B_N (respectively, ∂B_n). For $u \in \mathbb{R}^2$, write $\mathscr{A}^u(N,n)$ for the translate $\mathscr{A}(N,n) + u$. A *primal* (respectively, *dual*) *crossing* of $\mathscr{A}(N,n)$ is an open (respectively, open*) path whose intersection with $\mathscr{A}(N,n)$ is an arc with an endpoint in each boundary of the annulus. Primal crossings are said to have colour 1, and dual crossings colour 0.

Let $k \in \mathbb{N}$. A sequence $\sigma \in \{0,1\}^k$ is called a *colour sequence* of length k. For such σ, the arm-event $A_\sigma(N,n)$ is the event that there exist k vertex-disjoint crossings $\gamma_1, \ldots, \gamma_i, \ldots, \gamma_k$ of $\mathscr{A}(N,n)$ with colours σ_i taken in anticlockwise order. The corresponding event on the translated annulus $\mathscr{A}^u(N,n)$ is denoted $A_\sigma^u(N,n)$ and is said to be 'centred at u'. As in [20], the value of N is largely immaterial to what follows, but $N = N(\sigma)$ is taken sufficiently large that the events $A_\sigma(N,n)$ are non-empty for $n \geq N$.

A colour sequence σ is called *monochromatic* if either $\sigma = (1,1,\ldots,1)$ or $\sigma = (0,0,\ldots,0)$, and *bichromatic* otherwise. It is called *alternating* if it has even length and either $\sigma = (1,0,1,0,\ldots)$ or $\sigma = (0,1,0,1,\ldots)$. When $\sigma = (1)$, $A_\sigma(N,n)$ is called the *one-arm event* and denoted $A_1(N,n)$. When σ is alternating with length $k = 2j$, the corresponding event is denoted $A_{2j}(N,n)$.

Let $G \in \mathscr{G}$ be an isoradial graph with vertex-set V, and let C_v be the open cluster of the vertex $v \in V$, under the canonical measure $\mathbb{P}_G$. We concentrate here on the following exponents given in terms of $\mathbb{P}_G$, with limits that are uniform in the choice of v:

(a) volume exponent: $\mathbb{P}_G(|C_v| = n) \approx n^{-1-1/\delta}$ as $n \to \infty$,
(b) connectivity exponent: $\mathbb{P}_G(v \leftrightarrow w) \approx |w-v|^{-\eta}$ as $|w-v| \to \infty$,
(c) one-arm exponent: $\mathbb{P}_G[A_1^v(N,n)] \approx n^{-\rho_1}$ as $n \to \infty$,
(d) $2j$-alternating-arms exponents: $\mathbb{P}_G[A_\sigma^v(N,n)] \approx n^{-\rho_{2j}}$ as $n \to \infty$, for each alternating colour sequence σ of length $2j$, with $j \geq 1$.

It is believed that the above uniformly asymptotic relations hold for suitable exponent-values, and indeed with $\approx$ replaced by the stronger relation $\asymp$.

The conventional one-arm exponent ρ is given by $\rho = 1/\rho_1$, as in [15, Sect. 9.1]. Parts (c) and (d) above are parts of the following more extensive conjecture.

Conjecture 3.3. *Let G be an isoradial graph with the bounded-angles property, and let $k \in \mathbb{N}$ and $\sigma \in \{0,1\}^k$.*

(a) *There exists $\rho(\sigma, G) > 0$ such that*

$$\mathbb{P}_G[A_\sigma^u(N,n)] \approx n^{-\rho(\sigma,G)} \qquad as\ n \to \infty,$$

uniformly in $u \in \mathbb{R}^2$.

(b) *The exponent $\rho(\sigma, G)$ does not depend on the choice of $G \in \mathscr{G}$.*

Essentially the only two-dimensional percolation process for which the above arm-exponents ρ_σ are proved to exist (and, furthermore, many of their values known explicitly) is site percolation on the triangular lattice (see [5,41,43]). In this special case (not belonging to the class of models considered in this paper), the ρ_σ are constant for all bichromatic colour sequences of given length (see [1]), and the monochromatic arm-exponents have been studied in [5].

A critical exponent π is said to *exist* for a graph $G \in \mathscr{G}$ if the appropriate asymptotic relation holds. It is called $\mathscr{G}$-*invariant* if it exists for all $G \in \mathscr{G}$ and its value is independent of the choice of G.

Our universality theorem is presented next. Part (a) amounts to a verification of Conjecture 3.3(b) for isoradial graphs $G \in \mathscr{G}$, and for colour sequences which are either of length one or alternating.

Theorem 3.4 (Universality).

(a) *Let $\pi \in \{\rho\} \cup \{\rho_{2j} : j \geq 1\}$. If π exists for some $G \in \mathscr{G}$, then it is $\mathscr{G}$-invariant.*
(b) *If either ρ or η exists for some $G \in \mathscr{G}$, then ρ, η, δ are $\mathscr{G}$-invariant and satisfy the scaling relations $\eta\rho = 2$ and $2\rho = \delta + 1$.*

Kesten showed in [30] (see also [33]) that certain properties of a *critical* percolation process imply properties of the *near-critical* process, when the underlying graph has a sufficiently rich automorphism group. In particular, knowledge of certain critical exponents *at criticality* implies knowledge of exponents *away from criticality*. Only certain special isoradial graphs have sufficient homogeneity for such arguments to hold without new ideas of substance. Therefore, further discussion is omitted, and the reader is referred instead to [20, Sect. 1.4].

Finally, we make some comments on the proofs. There are two principal steps in the proof of Theorem 3.1. Firstly, using a technique involving star–triangle transformations, the box-crossing property is transported from the homogeneous square lattice to an arbitrary isoradial embedding of the square lattice (with the bounded-angles property). Secondly, the square-grid property is used to transport the box-crossing property to general isoradial graphs. This method may be used also to show the invariance of certain arm exponents across the class of such isoradial graphs, as in Theorem 3.4. The basic approach is that of [19,20], but the geometrical constructions used here differ in substantial regards from those papers.

Proof of Theorem 3.2. By (2.5), (4.1), and Theorem 3.1, both $\mathbb{P}_G$ and $\mathbb{P}_{G^*}$ have the box-crossing property. The claims then follow as in [19, Props 4.1, 4.2] (see also [19, Remark 4.3]), and the details are omitted. It suffices to check that the conditions of the Remark hold under the bounded-angles property. The constants in the theorem may be tracked through the proofs, and are found to depend only on the values of ε and I. $\square$

4 Isoradial graphs and rhombic tilings

4.1 Rhombic tilings

A *rhombic tiling* is a planar graph embedded in $\mathbb{R}^2$ such that every face is a rhombus of side-length 1. Rhombic tilings have featured prominently in the theory of planar tilings, both periodic and aperiodic. A famous example is the aperiodic rhombic tiling of Penrose [36], and the generalizations of de Bruijn [7,8] and others. The reader is referred to [21,39] for general accounts of the theory of tiling.

There is a two–one correspondence between isoradial graphs and rhombic tilings of the plane, which we review next. Let $G = (V, E)$ be an isoradial graph. The *diamond graph* $G^{\diamond}$ is defined as follows. The vertex-set of $G^{\diamond}$ is $V^{\diamond} := V \cup C$ where C is the set of circumcentres of faces of G; elements of V shall be called *primal* vertices, and elements of C *dual* vertices. Edges are placed between pairs $v \in V$, $c \in C$ if and only if c is the centre of a circumcircle of a face containing v. Thus $G^{\diamond}$ is bipartite. Since G is isoradial, the diamond graph $G^{\diamond}$ is a rhombic tiling, and is illustrated in Figure 4.1.
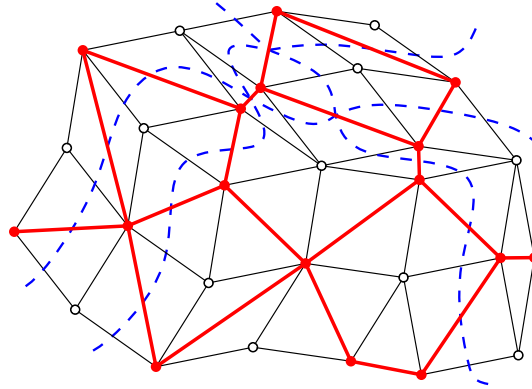
Fig. 4.1 The isoradial graph G is drawn in red, and the associated diamond graph $G^{\diamond}$ in black. The primal vertices of $G^{\diamond}$ are those of G; the dual vertices are centres of faces of G. A track is a doubly infinite sequence of adjacent rhombi sharing a common vector, and may be represented by a path, drawn in blue. Two tracks meet in an edge of G lying in some face of $G^{\diamond}$.

From the diamond graph $G^{\diamond}$ may be found both G and its planar dual G^*. Write V_1 and V_2 for the two sets of vertices in the bipartite $G^{\diamond}$. For $i = 1, 2$, let G_i be the graph with vertex-set V_i, two points of which are joined by an edge if and only if they lie in the same face of $G^{\diamond}$. One of the graphs G_1, G_2 is G and the other is its dual G^*. It follows in particular that G^* is isoradial. Let $e \in E$ and let e^* denote its dual edge. The pair e, e^* are diagonals of the same rhombus of $G^{\diamond}$ and are thus perpendicular.

The above construction may be applied to any rhombic tiling T to obtain a primal/dual pair of isoradial graphs.

4.2 Track systems

Rhombic tilings have attracted much interest, especially since the discovery by Penrose [35,36] of his celebrated aperiodic tiling. Penrose's rhombic tiling was elaborated by de Bruijn [7,8], who developed the following representation in terms of 'ribbons' or '(train) tracks'. Let $G = (V,E)$ be isoradial. An edge e_0 of $G^\diamond$ belongs to two rhombi r_0, r_1 of $G^\diamond$. Write e_{-1} (respectively, e_1) for the edge of r_0 (respectively, r_1) opposite e_0, so that e_{-1}, e_0, e_1 are parallel unit-line-segments. The edge e_{-1} (respectively, e_1) belongs to a further rhombus r_{-1} (respectively, r_2) that is distinct from r_0 (respectively, r_1). By iteration of this procedure, we obtain a doubly-infinite sequence of rhombi $(r_i : i \in \mathbb{Z})$ such that the intersections $(r_i \cap r_{i+1} : i \in \mathbb{Z})$ are distinct, parallel unit-line-segments. We call such a sequence a *(train) track*. We write $\mathscr{T}(G)$ for the set of tracks of G, and note that $\mathscr{T}(G) = \mathscr{T}(G^*)$. The track construction is illustrated in Figure 4.1.

A track $(r_i : i \in \mathbb{Z})$ is sometimes illustrated as an arc joining the midpoints of the line-segments $r_i \cap r_{i+1}$ in sequence. The set $\mathscr{T}$ may therefore be represented as a family of doubly-infinite arcs which, taken together with the intersections of arcs, defines a graph. We shall denote this graph by $\mathscr{T}$ also. A vertex v of $G^\diamond$ is said to be *adjacent* to a track $(r_i : i \in \mathbb{Z})$ if it is a vertex of one of the rhombi r_i.

It was pointed out by de Bruijn, and is easily checked, that the rhombi in a track are distinct. Furthermore, two distinct tracks may have no more than one rhombus in common. Since each rhombus belongs to exactly two tracks, it is the unique intersection of these two tracks.

Kenyon and Schlenker [25] have showed a converse theorem. Let Q be an infinite planar graph embedded in the plane with the property that every face has four sides. One may define the tracks of Q by an adaptation of the above definition: a track exits a face across the edge opposite to its entry. Then Q may be deformed continuously into a rhombic tiling if and only if (i) no track intersects itself, and (ii) no two tracks intersect more than once.

A track t is said to be *oriented* if it is endowed with a direction. As an oriented track t is followed in its given direction, it crosses sides of rhombi which are parallel. Viewed as vectors from right to left, these sides constitute a unit vector $\tau(t)$ of $\mathbb{R}^2$ called the *transverse vector* of t. The transverse vector makes an angle with the x-axis called the *transverse angle* of t, with value in the interval $[0, 2\pi)$.

Let $I \in \mathbb{N}$. We say that an isoradial graph G has the *square-grid property* SGP(I) if its track-set $\mathscr{T}$ may be partitioned into three sets $\mathscr{T} = S \cup T_1 \cup T_2$ satisfying the following.

(a) For $k = 1, 2$, T_k is a set $(t_k^i : i \in \mathbb{Z})$ of distinct non-intersecting tracks indexed by $\mathbb{Z}$.

(b) For $k = 1, 2$ and $s \in \mathscr{T} \setminus T_k$, every track of T_k intersects s, and these intersections occur in their lexicographic order.

(c) For $k = 1, 2$, $i \in \mathbb{Z}$, and $s \in T_{3-k}$, the number of track-intersections on s between its intersections with t_k^i and t_k^{i+1} is strictly less than I.

Two tracks belonging to the same T_k are said to be *parallel*. We refer to $T_1 \cup T_2$ as a *square grid* of G, assumed implicitly to satisfy (c) above. A square grid is a subset of tracks with the topology of the square lattice (and satisfying (c)).

Since the square-grid property pertains to the diamond graph $G^\diamond$ rather than to G itself,

$$G \text{ satisfies SGP}(I) \text{ if and only if } G^* \text{ satisfies SGP}(I). \qquad (4.1)$$

An isoradial graph G is said to have the square-grid property (SGP) if it satisfies SGP(I) for some $I \in \mathbb{N}$. As before, $\mathscr{G}$ denotes the set of all isoradial graphs with the bounded-angles property and the square-grid property. More specifically, we write $\mathscr{G}(\varepsilon, I)$ for the set of G satisfying BAP(ε) and SGP(I).

Let $G \in \mathscr{G}$ have square grid $T_1 \cup T_2$. It may be seen by the bounded-angles property that, for $k = 1, 2$, every $x \in \mathbb{R}^2$ lies either in some track of T_k or in the region of $\mathbb{R}^2$ 'between' two consecutive elements of T_k.

4.3 Examples

Here are three families of isoradial graphs with the square-grid property, and one without.

4.3.1 Isoradial square lattices

An isoradial embedding of the square lattice is called an *isoradial square lattice* The track-system of such a graph is simply a square grid, and *vice versa*.

4.3.2 Periodic graphs

A planar graph H, embedded in $\mathbb{R}^2$, is said to be *periodic* if there exist distinct non-zero vectors $\tau_1, \tau_2 \in \mathbb{R}^2$ such that H is invariant under shifts by either τ_i. Let G be an isoradial embedding of a periodic connected graph H (the embedding itself need not be periodic). The track system $\mathscr{T}$ of G (viewed as a set of arcs) is determined by the structure of H. Since H is periodic, so is $\mathscr{T}$ (viewed as a graph). Therefore, $\mathscr{T}$ may be embedded homeomorphically into $\mathbb{R}^2$ in a periodic manner. After re-scaling, we may assume that $\mathscr{T}$ is invariant under any unit shift of $\mathbb{R}^2$ in the direction of a coordinate vector. In fact, $\mathscr{T}$ may be thought of as the lifting to the universal cover of a track-system on a torus.

As observed in [25, Sect. 5.2], any oriented track t has an asymptotic angle $\alpha(t) \in S^1$, and in addition the reversed track has direction $\pi + \alpha(t)$. Let $t \in \mathscr{T}$, viewed as a subset of $\mathbb{R}^2$. There exists $(a, b) \in \mathbb{Z}^2$, $(a, b) \neq (0, 0)$, such that t is invariant under the shift $\tau_{a,b} : z \mapsto z + (a, b)$. We have that $\tan \alpha(t) = b/a$. By periodicity, the set of all angles (modulo π) of $\mathscr{T}$ is finite, and we write it as $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ with $m \geq 1$.

Let T_k be the set of tracks with asymptotic angle (modulo π) α_k. By periodicity, each T_k is a set of tracks indexed by $\mathbb{Z}$, and may be ordered according to their crossings of the line with polar coordinates $\theta = \theta_0$ with $\theta_0 \neq \alpha_k$ for all k. Since tracks $t_k \in T_k$, $t_l \in T_l$ (with $k \neq l$) have different asymptotic angles, they must intersect.

It remains to show that any $t, t' \in T_k$ do not intersect (whence, in particular, $m \geq 2$). Suppose the converse, that there exist $k \in \{1, 2, \ldots, m\}$ and $t, t' \in T_k$ such that t and t' intersect at some point $J \in \mathbb{R}^2$. Since t and t' have the same angle α_k, there exists $(a, b) \in \mathbb{Z}^2$ such that t and t' are invariant under $\tau_{a,b}$. Therefore, they intersect at $J + n(a, b)$ for all $n \in \mathbb{Z}$, in contradiction of the fact that they may have at most one intersection.

For any distinct pair T_k, T_l, part (c) of the square-grid property holds by periodicity.

We have proved not only that G has the square-grid property, but the stronger fact that its track-set may be partitioned into m classes of parallel tracks.

4.3.3 Rhombic tilings via multigrids

The following 'multigrid' construction was introduced and studied by de Bruijn [7, 8, 9]. A *grid* is a set of parallel lines in $\mathbb{R}^2$ with some common perpendicular unit-vector v. A *multigrid* is a family of grids with pairwise non-parallel perpendiculars. Suppose there are $m \geq 2$ grids, with perpendiculars $v_1, v_2, \ldots, v_m$. The kth grid is given in terms of a set $C_k = \{c_k^i : i \in \mathbb{Z}\}$ of reals, specifically as the set of all $z \in \mathbb{R}^2$ with $z \cdot v_k = c_k^i$ as i ranges over $\mathbb{Z}$. It is assumed that the c_k^i are strictly increasing in i, with $c_k^i / i \to 1$ as $i \to \pm\infty$.

With the lines of the kth grid, duly oriented, we associate a unit vector w_k. It is explained in [9] how, under certain conditions on the C_k, v_k, w_k, one may 'dualize' the multigrid to obtain a rhombic tiling of $\mathbb{R}^2$. The track-set of the ensuing tiling is a homeomorphism of the multigrid with transverse vectors w_k. Under the additional assumption that the differences $|c_k^{i+1} - c_k^i|$ are uniformly bounded away from 0 and ∞, all such tilings have both the bounded-angles property and the square-grid property. The results of this paper apply to the associated isoradial graphs.

Penrose's rhombic tiling may be obtained thus with $m = 5$, the v_k being vectors forming a regular pentagon, with $w_k = v_k$, and $C_k = \{i + \gamma_k : i \in \mathbb{Z}\}$ with an appropriate vector (γ_k). Other choices of the parameters yield a broader class of aperiodic rhombic tilings of the plane. See [7, 8]. Percolation on Penrose tilings has been considered in [22].
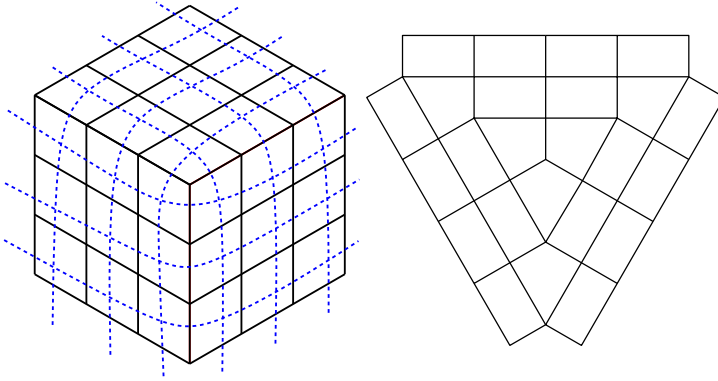
4.3.4 A track-system with no square grid

Figure 4.2 is an illustration of a track-system without the square-grid property.

4.4 Equivalence of metrics

Let G be an isoradial graph. It will be convenient to use both the Euclidean metric $|\cdot|$ and the graph-metric $d^\diamond$ on $G^\diamond$. For $n \in N$ and $u \in G^\diamond$ we write $\Lambda_u^\diamond(n)$ for the ball of $d^\diamond$-radius n centred at u:

$$\Lambda_u^\diamond(n) = \{v \in G^\diamond : d^\diamond(u, v) \leq n\}.$$

Fig. 4.2 Part of a rhombic tiling without the square-grid property, and one of the two corresponding isoradial graphs.

Proposition 4.1. *Let $\varepsilon > 0$. There exists $c_d = c_d(\varepsilon) > 0$ such that, for any isoradial graph $G = (V, E)$ satisfying* BAP(ε),

$$c_d^{-1}|v - v'| \le d^\diamond(v, v') \le c_d|v - v'|, \qquad v, v' \in G^\diamond. \tag{4.2}$$

Proof. Let u, v be distinct vertices of $G^\diamond$. Since each edge of $G^\diamond$ has length 1, $d^\diamond(u, v) \ge |u - v|$. Conversely, let S_{uv} be the set of all faces of $G^\diamond$ (viewed as closed sets of $\mathbb{R}^2$) that intersect the straight-line segment uv of $\mathbb{R}^2$ joining u to v. Since the diameter of any such face is less than 2, every point of the union of S_{uv} is within Euclidean distance 2 of uv. By BAP(ε) and (2.2), every face has area at least $\sin\varepsilon$, and therefore $|S_{uv}| \le 4(|u - v| + 4)/\sin\varepsilon$. Similarly, there exists $\delta = \delta(\varepsilon) > 0$ such that $|u - v| \ge \delta$. The edge-set of elements of S_{uv} contains a path of edges of $G^\diamond$ from u to v, whence

$$d^\diamond(u, v) \le \frac{8}{\sin\varepsilon}(|u - v| + 4) \le \frac{8(\delta + 4)}{\delta \sin\varepsilon}|u - v|,$$

as required. $\square$

4.5 The box-crossing property for graphs in $\mathscr{G}$

This section begins with a definition of the rectangular domains of an isoradial graph $G \in \mathscr{G}$, using the topology of its square grid.

Let (t, t') be an ordered pair of non-intersecting tracks of G. A point $x \in \mathbb{R}^2$ is said to be 'strictly between' t and t' if, with these tracks viewed as arcs of $\mathbb{R}^2$, there exists an unbounded path of $\mathbb{R}^2$ from x that intersects t but not t', and *vice versa*. A face F of $G^\diamond$ is said to be *between* t and t' if: either F is a rhombus of t, or every point of F is strictly between t and t'. Note that this usage of 'between' is not symmetric: there are faces between t and t' that are not between t' and t. A vertex or edge of $G^\diamond$ is said to be 'between' t and t' if it belongs to some face between t and t'. The *domain* between t and t' is the union of the (closed) faces between t and t'. It is useful to think

of a domain as either a subgraph of $G^\diamond$, or (unlike the domains of Section 2.3) as a closed region of $\mathbb{R}^2$.

Suppose $G \in \mathscr{G}$ has a square grid $S \cup T$, with $S = (s_j : j \in \mathbb{Z})$ and $T = (t_i : i \in \mathbb{Z})$. We call tracks in S (respectively, T) *horizontal* (respectively, *vertical*). For $i_1, i_2, j_1, j_2 \in \mathbb{Z}$ we define $\mathscr{D} = \mathscr{D}(t_{i_1}, t_{i_2}; s_{j_1}, s_{j_2})$ to be the intersection of the domains between t_{i_1} and t_{i_2} and between s_{j_1} and s_{j_2}.

We say that $\mathscr{D}$ is crossed horizontally if G contains an open path π such that: (i) every edge of π lies in $\mathscr{D}$, and (ii) the first edge crosses t_{i_1} and the last vertex is adjacent to t_{i_2}. Write $C_{\mathrm{h}}(\mathscr{D}) = C_{\mathrm{h}}(t_{i_1}, t_{i_2}; s_{j_1}, s_{j_2})$ for the event that $\mathscr{D}$ is crossed horizontally, with a similar definition of the vertical-crossing event $C_{\mathrm{v}}(\mathscr{D})$. See Figure 4.3 for an illustration of the above notions.
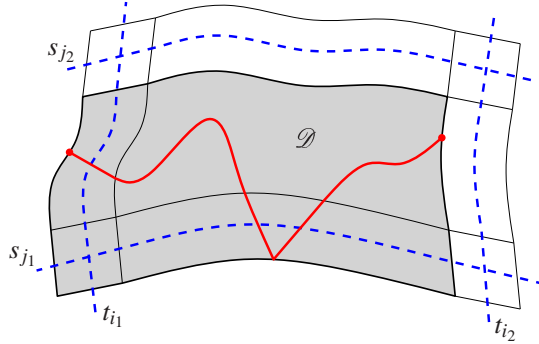

Fig. 4.3 The shaded domain $\mathscr{D} = \mathscr{D}(t_{i_1}, t_{i_2}; s_{j_1}, s_{j_2})$ is crossed horizontally.

The purpose of the following proposition is to restate the box-crossing property in terms of the geometry of the square grid.

Proposition 4.2. *Let $\varepsilon > 0$, $I \in \mathbb{N}$, and let $G \in \mathscr{G}(\varepsilon, I)$. The graph G has the box-crossing property if and only if there exists $\delta > 0$ such that, for $N \in \mathbb{N}$ and $i, j \in \mathbb{Z}$,*

$$\mathbb{P}_G\big[C_{\mathrm{h}}(t_i, t_{i+2N}; s_j, s_{j+N})\big], \mathbb{P}_G\big[C_{\mathrm{v}}(t_i, t_{i+N}; s_j, s_{j+2N})\big] \geq \delta. \tag{4.3}$$

Moreover, if (4.3) *holds, then G satisfies* $\mathrm{BXP}(\delta')$ *with δ' depending on δ, ε, I and not further on G.*

Proof. We prove only the final sentence of the proposition. The converse (that the box-crossing property implies (4.3) for some $\delta > 0$) holds by similar arguments, and will not be used this paper. Let $G \in \mathscr{G}(\varepsilon, I)$, and assume (4.3) with $\delta > 0$.

Let $N \in \mathbb{N}$. For $i, j \in \mathbb{Z}$, the *cell* $C_{i,j}$ is defined to be the domain $\mathscr{D}(t_{iN}, t_{(i+1)N}; s_{jN}, s_{(j+1)N})$. The cells have disjoint interiors and cover the plane. Two distinct cells $C = C_{i,j}$, $C' = C_{k,l}$ are said to be *adjacent* if (i, j) and (k, l) are adjacent vertices of the square lattice, in which case we write $C \sim C'$. More specifically, we write $C \sim_{\mathrm{h}} C'$ (respectively, $C \sim_{\mathrm{v}} C'$) if $|i - k| = 1$ (respectively, $|j - l| = 1$). With the adjacency relation $\sim$, the graph having the set of cells as vertex-set is isomorphic to the square lattice.

Each cell has perimeter at most $4IN$, and therefore diameter not exceeding $2IN$. A cell contains at least N^2 faces of $G^\diamond$, and thus (by (2.2)) has total area at least $N^2 \sin \varepsilon$.

For $\mu \in \mathbb{N}$ with $\mu \geq 2I$, let $u = (-\mu N, 0)$ and $v = (\mu N, 0)$ viewed as points in the plane. Let S_{uv}^N be the set of cells that intersect the straight-line segment uv with endpoints u, v, and let U_{uv}^N be the union of such cells. Let R be the tube $uv + [-2IN, 2IN]^2$. Thus R has area $8IN(\mu N + 2IN)$, and $U_{uv}^N \subseteq R$. Since each cell has area at least $N^2 \sin \varepsilon$, the cardinality of S_{uv}^N satisfies

$$|S_{uv}^N| \leq \frac{8IN(\mu N + 2IN)}{N^2 \sin \varepsilon} = \frac{8I(\mu + 2I)}{\sin \varepsilon}. \tag{4.4}$$
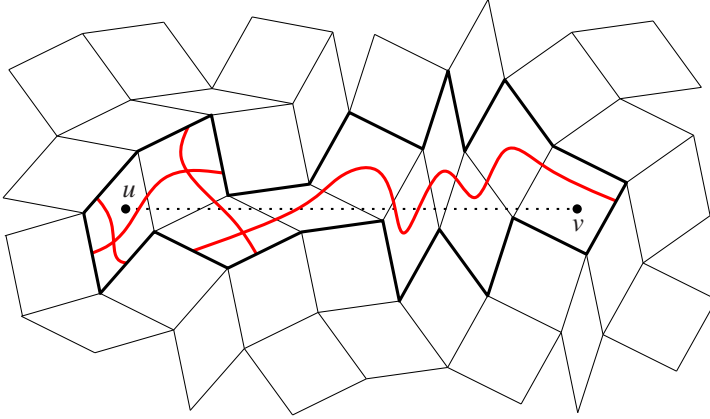
Fig. 4.4 The region U_{uv}^N is outlined in bold, and contains a chain of cells joining u to v. The events H_k are drawn explicitly for the first two contiguous pairs of cells.

There exists a chain of cells $C_1, \ldots, C_K \in S_{uv}^N$ such that $u \in C_1$, $v \in C_K$ and $C_k \sim C_{k+1}$ for $k = 1, 2, \ldots, K-1$; see Figure 4.4. Let $k \in \{1, 2, \ldots, K-1\}$, and assume $C_k \sim_{\mathrm{h}} C_{k+1}$. Let H_k be the event that C_k and C_{k+1} are crossed vertically, and $C_k \cup C_{k+1}$ is crossed horizontally. A similar definition holds when $C_k \sim_{\mathrm{v}} C_{k+1}$, with vertical and horizontal interchanged. By (4.3) and the Harris–FKG inequality, $\mathbb{P}_G(H_k) \geq \delta^3$.

By the Harris–FKG inequality, the fact that $K \leq |S_{u,v}^N|$, and (4.4),

$$\mathbb{P}_G\left(\bigcap_{k=1}^{K-1} H_k \right) \geq \delta^{3K} \geq \delta^{24I(\mu+2I)/\sin \varepsilon}. \tag{4.5}$$

If the event on the left side occurs, the rectangle

$$S_{\mu,N} := \left[-(\mu - 2I)N, (\mu - 2I)N \right] \times [-2IN, 2IN]$$

of $\mathbb{R}^2$ is crossed horizontally.

Let $R_k = [-k,k] \times [-\frac{1}{2}k, \frac{1}{2}k]$ where $k \geq 8I$. Pick N such that $4IN \leq k \leq 8IN$, so that R_k is 'higher' and 'shorter' than $S_{10I,N}$. By (4.5) with $\mu = 10I$,

$$\mathbb{P}_G \left(R_k \text{ is crossed horizontally} \right) \geq \delta'', \qquad (4.6)$$

where $\delta'' = \delta^{288I^2/\sin \varepsilon}$. Smaller values of k are handled by adjusting δ'' accordingly.

The same argument is valid for translates and rotations of the line-segment uv, and the proof is complete. $\qquad\square$

4.6 Isoradial square lattices

An *isoradial square lattice* is an isoradial embedding of the square lattice $\mathbb{Z}^2$. Isoradial square lattices, and only these graphs, have a square grid as track-system.

Let G be an isoradial square lattice. The diamond graph $G^\diamond$ possesses two families of parallel tracks, namely the horizontal tracks $(s_j : j \in \mathbb{Z})$ and the vertical tracks $(t_i : i \in \mathbb{Z})$. The graph $G^\diamond$, and hence the pair (G, G^*) also, may be characterized in terms of two vectors of angles linked to the transverse vectors. First, we orient s_0 in an arbitrary way (interpreted as 'rightwards'). As we proceed in the given direction along s_0, the crossing tracks t_i are numbered in increasing sequence, and are oriented from right to left (interpreted as 'upwards'). Similarly, as we proceed along t_0, the crossing tracks s_j are numbered in increasing sequence and oriented from left to right. Using the notation of Section 4.2, the transverse vector $\tau(s_j)$ has some transverse angle β_j, and similarly $\tau(t_i)$ has some transverse angle γ_i. Rather than working with the γ_i, we work instead with $\alpha_i := \gamma_i - \pi$ as illustrated in Figure 4.5. Write $\alpha = (\alpha_i : i \in \mathbb{Z})$ and $\beta = (\beta_j : j \in \mathbb{Z})$, and note that $\alpha_i \in [-\pi, \pi)$, $\beta_j \in [0, 2\pi)$. We will generally assume that G is rotated in such a way that $\alpha_0 = 0$, so that $\beta_j \in [0, \pi]$ and $\beta_j - \pi \leq \alpha_i \leq \beta_j$ for $i, j \in \mathbb{Z}$.

The vertex of $G^\diamond$ adjacent to the four tracks $t_{i-1}, t_i, s_{j-1}, s_j$ is denoted $v_{i,j}$. If not otherwise stated, we shall assume that the tracks are labelled in such a way that the vertex $v_{0,0}$ is a *primal* vertex of $G^\diamond$.

Tracks t_i, s_j intersect in a rhombus of $G^\diamond$ with sides $\tau(t_i)$, $\tau(s_j)$, $-\tau(t_i)$, $-\tau(s_j)$ in clockwise order, and thus its internal angles are $\beta_j - \alpha_i$ and $\pi - (\beta_j - \alpha_i)$. Thus, G satisfies the bounded-angles property $\mathrm{BAP}(\varepsilon)$ if and only if

$$\beta_j - \alpha_i \in [\varepsilon, \pi - \varepsilon], \qquad i, j \in \mathbb{Z}. \qquad (4.7)$$

Conversely, for two vectors α, β satisfying (4.7), we may construct the diamond graph denoted $G^\diamond_{\alpha,\beta}$ as in Figure 4.5. This gives rise to an isoradial square lattice denoted $G_{\alpha,\beta}$ (and its dual) satisfying $\mathrm{BAP}(\varepsilon)$. We write $\mathbb{P}_{\alpha,\beta}$ for the canonical measure of $G_{\alpha,\beta}$.

We introduce now some notation to be used later. For a set W of vertices of $G^\diamond$, we define the *height* of W by

$$h(W) = \sup\{ j : \exists i \text{ with } v_{i,j} \in W \}.$$

This definition extends in an obvious way to sets of edges.
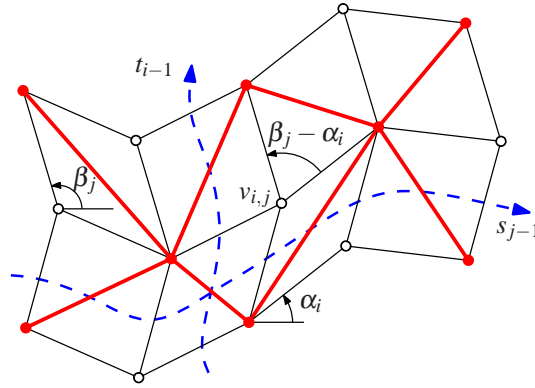
Fig. 4.5 An isoradial square lattice (in red) with the associated diamond graph. The diamond graph is isomorphic to $\mathbb{Z}^2$, and its embedding is characterized by two sequences α, β of angles.

In Section 5.2 is described an operation of so-called 'track-exchange' on isoradial square lattices. This introduces a potential for confusion between the *label* and the *level* of a track. In the $G_{\alpha,\beta}$ above, we say that s_j is (initially) at *level j*. The level of s_j may change under track-exchange, but $v_{i,j}$ shall always refer to the vertex between levels $j-1$ and j in the new graph.

Due to this potential confusion, we may use a different notation for domains in square lattices than for general graphs. For $M_1, M_2, N_1, N_2 \in \mathbb{Z}$ with $M_1 \leq M_2$, $N_1 \leq N_2$, let $B(M_1, M_2; N_1, N_2)$ be the subgraph of G induced by the subset of vertices lying in $\{v_{i,j} : M_1 \leq i \leq M_2, \ N_1 \leq j \leq N_2\}$. For $M, N \in \mathbb{N}$, we use the abbreviated notation $B(M,N) = B(-M,M;0,N)$. A *horizontal crossing* of $B = B(M_1, M_2; N_1, N_2)$ is an open path of B linking some vertex v_{M_1,n_1} to some vertex v_{M_2,n_2}; a *vertical crossing* links some v_{m_1,N_1} to some v_{m_2,N_2}. We write $C_{\mathrm{h}}[B]$ (respectively, $C_{\mathrm{v}}[B]$) for the event that a box B contains a horizontal (respectively, vertical) crossing. For a vertex $v = v_{i,j}$ of G, we write B^v for the translate $\{v_{r,s} : v_{r-i,s-j} \in B\}$.

When applied to G, we have that

$$B(M_1, M_2; N_1, N_2) = \mathscr{D}(t_{M_1}, t_{M_2}; s_{N_1}, s_{N_2}),$$

since s_{N_1} and s_{N_2} are the tracks at levels N_1 and N_2 respectively. As mentioned before, the latter will not always be the case. Use of the notation B emphasizes that domains are defined in terms of tracks at specific levels, rather than of tracks with specific labels.

The following lemma will be used in Section 6.

Lemma 4.3. *Let $G = (V, E)$ be an isoradial square lattice satisfying the bounded-angles property* $\mathrm{BAP}(\varepsilon)$ *and the following.*

(a) *For $\rho \geq 1$, there exists $\eta(\rho) > 0$ such that*

$$\mathbb{P}_G\big(C_{\mathrm{h}}[B^v(\lfloor \rho N \rfloor, N)]\big) \geq \eta(\rho), \qquad N \in \mathbb{N}, \ v \in V.$$

(b) *There exist $\rho_0, \eta_0 > 0$ such that*

$$\mathbb{P}_G\big(C_v[B^v(N, \lfloor \rho_0 N \rfloor)]\big) \geq \eta_0, \qquad N \geq \rho_0^{-1}, \; v \in V.$$

Then there exists $\delta = \delta(\rho_0, \eta_0, \eta(1), \eta(2\rho_0^{-1}), \varepsilon) > 0$ such that G has the box-crossing property $\mathrm{BXP}(\delta)$.

Outline proof. Assume (a) and (b) hold. Just as in the proof of [19, Prop. 3.1] (see also [19, Remark 3.2]), the crossing probabilities of boxes of G with aspect-ratio 2 and horizontal/vertical orientations are bounded away from 0 by a constant that depends only on the aspect-ratios of the boxes illustrated in [19, Fig. 3.1]. (Here, the boxes in question are those of G viewed as an isoradial square lattice, that is, boxes of the form $B(\cdot; \cdot)$ defined before the lemma.) Therefore, the hypothesis of Proposition 4.2 holds with suitable constants, and the claim follows from its conclusion. $\square$

5 The star–triangle transformation

We review the basic action of the star–triangle transformation, and show its harmony with isoradial embeddings. It is shown in Section 5.3 how a sequence of star–triangle transformations may be used to exchange two tracks of an isoradial square lattice.

5.1 Star–triangle transformation

The following material is standard but is included for completeness. For proofs and details see, for example, [19].

Consider the triangle $\Delta = (V, E)$ and the star $\Delta' = (V', E')$ of Figure 5.1. Let $\mathbf{p} = (p_0, p_1, p_2) \in [0, 1)^3$ be a triplet of parameters. Write $\Omega = \{0, 1\}^E$ with associated product probability measure $\mathbb{P}_{\mathbf{p}}^{\triangle}$ with intensities p_i (as in the left diagram of Figure 5.1), and $\Omega' = \{0, 1\}^{E'}$ with associated measure $\mathbb{P}_{1-\mathbf{p}}^{\bigcirc}$, with intensities $1 - p_i$ (as in the right diagram of Figure 5.1). Let $\omega \in \Omega$ and $\omega' \in \Omega'$. For each graph we may consider open connections between its vertices, and we abuse notation by writing, for example, $x \xleftrightarrow{\Delta, \omega} y$ for the *indicator function* of the event that x and y are connected in Δ by an open path of ω. Thus connections in Δ are described by the family $(x \xleftrightarrow{\Delta, \omega} y : x, y \in V)$ of random variables, and similarly for Δ'.

Proposition 5.1 (Star–triangle transformation). *Let $\mathbf{p} \in [0, 1)^3$ be such that*

$$p_0 + p_1 + p_2 - p_0 p_1 p_2 = 1. \tag{5.1}$$

The families

$$\left(x \xleftrightarrow{\Delta, \omega} y : x, y = A, B, C\right), \quad \left(x \xleftrightarrow{\Delta', \omega'} y : x, y = A, B, C\right),$$
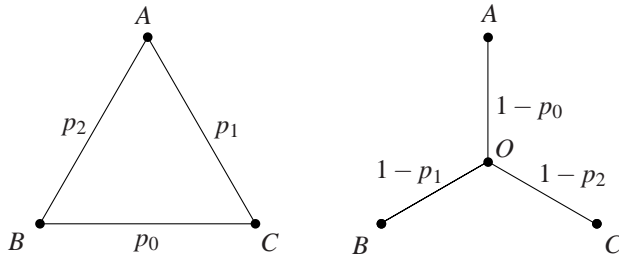
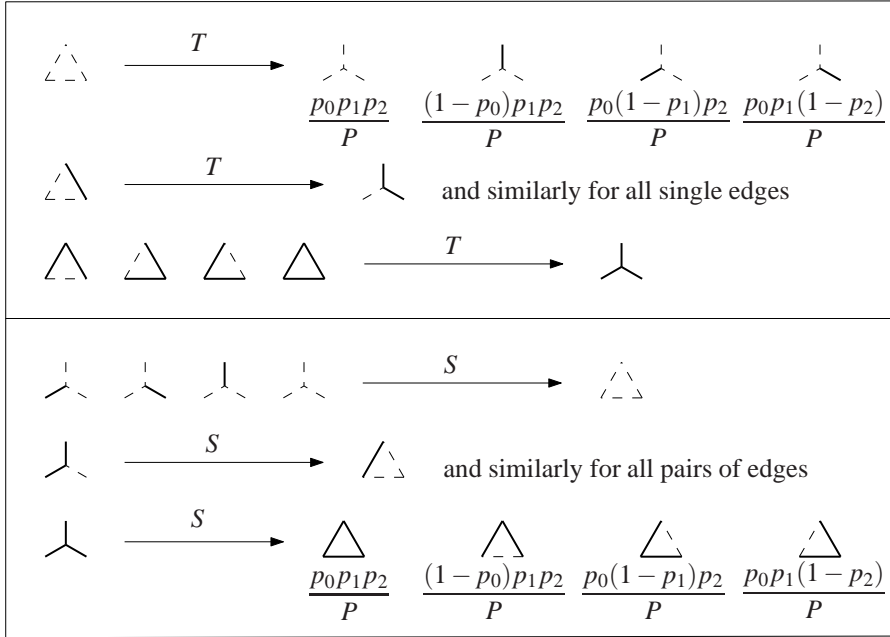*have the same law.*

Fig. 5.1 The star–triangle transformation

**Fig. 5.2** The random maps T and S. Note that $P := (1-p_0)(1-p_1)(1-p_2)$.

Next we explore couplings of the two measures. Let $\mathbf{p} \in [0,1)^3$ satisfy (5.1), and let Ω (respectively, Ω') have associated measure $\mathbb{P}^{\triangle}_{\mathbf{p}}$ (respectively, $\mathbb{P}^{\bigcirc}_{1-\mathbf{p}}$) as above. There exist random mappings $T : \Omega \to \Omega'$ and $S : \Omega' \to \Omega$ such that $T(\omega)$ has law $\mathbb{P}^{\bigcirc}_{1-\mathbf{p}}$, and $S(\omega')$ has law $\mathbb{P}^{\triangle}_{\mathbf{p}}$. Such mappings are given in Figure 5.2, and we shall not specify them more formally here. Note from the figure that $T(\omega)$ is deterministic for seven of the eight elements of Ω; only in the eighth case does $T(\omega)$ involve further randomness. Similarly, $S(\omega')$ is deterministic except for one special ω'. Each probability in the figure is well defined since $P := (1-p_0)(1-p_1)(1-p_2) > 0$.

Proposition 5.2 (Star–triangle coupling). *Let* $\mathbf{p} \in [0,1)^3$ *satisfy* (5.1) *and let S and T be as in Figure 5.2. With ω and ω' sampled as above,*

(a) $T(\omega)$ *has the same law as* ω',
(b) $S(\omega')$ *has the same law as* ω,
(c) *for* $x, y \in \{A, B, C\}$, $x \xleftrightarrow{\Delta, \omega} y$ *if and only if* $x \xleftrightarrow{\Delta', T(\omega)} y$,
(d) *for* $x, y \in \{A, B, C\}$, $x \xleftrightarrow{\Delta', \omega'} y$ *if and only if* $x \xleftrightarrow{\Delta, S(\omega')} y$.

5.2 The star–triangle transformation for isoradial graphs

Let $G = (V, E)$ be an isoradial graph, and let Δ be a triangle of G with vertices A, B, C. Seen as a transformation between graphs, the star–triangle transformation changes Δ into a star Δ' with a new central vertex $O \in \mathbb{R}^2$. It turns out that O may be chosen in such a way that the new graph, denoted G', is isoradial also. The right way of seeing this is via the diamond graph $G^\diamond$, as illustrated in Figure 5.3. This construction has its roots in the Z-invariant Ising model of Baxter [3,4], studied in the context of isoradial graphs by Mercat [32], Kenyon [24], and Costa-Santos [13] (see also [6]).
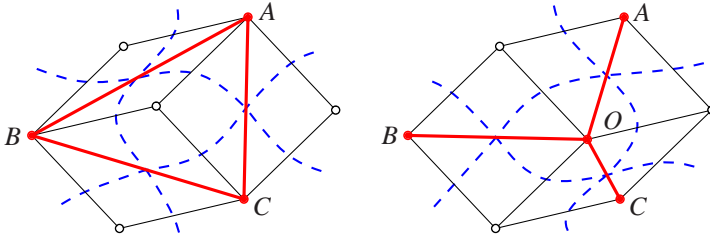
Fig. 5.3 The triangle on the left is replaced by the star on the right. The new vertex O is the circumcentre of the three dual vertices of the surrounding hexagon of $G^\diamond$.

The triangle Δ comprises the diagonals of three rhombi of $G^\diamond$. These rhombi form the interior of a *hexagon* with primary vertices A, B, C and three further dual vertices. Let O be the circumcentre of these dual vertices. Three new rhombi are formed from the hexagon augmented by O (as shown). The star Δ' has edges AO, BO, CO, and the ensuing graph is isoradial (since it stems from a rhombic tiling).

By an examination of the angles in the figure, the canonical measure on Δ' is that obtained from Δ by the star–triangle transformation of Section 5.1. That is, the star–triangle transformation maps $\mathbb{P}_G$ to $\mathbb{P}_{G'}$. Furthermore, for $\varepsilon > 0$,

$$G \text{ satisfies BAP}(\varepsilon) \text{ if and only if } G' \text{ satisfies BAP}(\varepsilon). \qquad (5.2)$$

We shall sometimes view the star–triangle transformation as acting on the rhombic tiling $G^\diamond$ rather than on G, and thereby it acts simultaneously on G and its dual G^*.

The star–triangle transformation of Figure 5.3 is said to act on the *track-triangle* formed by the tracks on the left side, and to *slide* one of the tracks illustrated there *over the intersection* of the other two, thus forming the track-triangle on the right side.
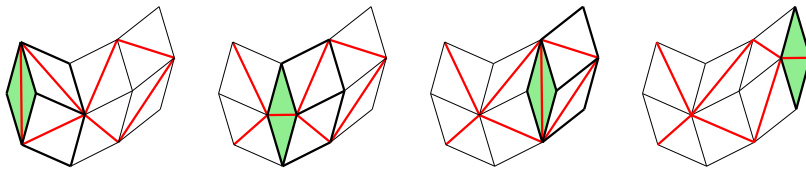
A star–triangle transformation maps an open path of G to an open path of G'. We shall not spell this out in detail, but recall the ideas from [19]. Let π be an open path of G that intersects some hexagon H of $G^\diamond$, and consider the star–triangle transformation σ acting in H. Since σ preserves open connections within H, it maps π to some $\sigma(\pi)$ containing an open path. A minor complication arises if H contains a star of G and π ends at the centre of this star. In this case, the endpoint of $\sigma(\pi)$ is a vertex of the resulting triangle.

5.3 Track-exchange in an isoradial square lattice

Let G be an isoradial square lattice. The tracks of G are to be viewed as doubly-infinite sequences of rhombi with a common vector. In this section, we describe a procedure for interchanging two consecutive parallel tracks.

Consider a vertical strip $G = G_{\alpha,\beta}$ of the square lattice, where $\alpha = (\alpha_i : -M \le i \le N)$ and $\beta = (\beta_j : j \in \mathbb{Z})$ are vectors of angles satisfying BAP(ε), (4.7). Thus every *finite* face of G has circumradius 1. (There are also *infinite* faces to the left and right of the strip.) There are two types of tracks in G, the finite horizontal tracks (s_j), and the infinite vertical tracks (t_i). We explain next how to exchange two adjacent horizontal tracks by a sequence of star–triangle transformations, employing a process that is implicit in [24]. Track s_j has transverse angle β_j, as illustrated in Figure 4.5, and the 'exchange' of two tracks may be interpreted as the interchange of their transverse angles.

We write Σ_j for the operation that exchanges the tracks at levels $j-1$ and j. When applied to G, Σ_j exchanges s_{j-1} and s_j, and we describe Σ_j by reference to $G^\diamond$. If $\beta_j = \beta_{j-1}$, there is nothing to do, and Σ_j interchanges the labels of the tracks without changing the transverse angles. Assume $\beta_j > \beta_{j-1}$. We insert a new rhombus on the left side of the strip formed of s_{j-1} and s_j, marked in green in Figure 5.4. This creates a hexagon in $G^\diamond$, containing either a triangle or a star of G. The star–triangle transformation is applied within this hexagon, thereby moving the new rhombus to the right. By repeated star–triangle transformations, we 'slide' the new rhombus along the two tracks from left to right. When it reaches the right side, it is removed. In the new graph, the original tracks s_{j-1} and s_j have been exchanged (or, more precisely, the transverse angles of the tracks at levels $j-1$ and j have been interchanged). Let Σ_j be the transformation thus described, and say that Σ_j 'goes from left to right' when $\beta_j > \beta_{j-1}$. If $\beta_j < \beta_{j-1}$, we construct Σ_j 'from right to left'.

Fig. 5.4 A new rhombus is introduced on the left (marked in green). This is then 'slid' along the pair of tracks by a sequence of star–triangle transformations, until it reaches the right side where it is removed.

Viewed as an operation on graphs, Σ_j replaces an isoradial graph G by another isoradial graph $\Sigma_j(G)$. It operates on configurations also, as follows. Let ω be an edge-configuration of G, and assign a random state to the new 'green' edge with the distribution appropriate to the isoradial embedding. The star–triangle transformations used in Σ_j are independent applications of the kernels T and S of Figure 5.2. The ensuing configuration on $\Sigma_j(G)$ is written $\Sigma_j(\omega)$. Thus Σ_j is a random operator on ω, with randomness stemming from the extra edge and the star–triangle transformations. Note that Σ_j is not a local transformation, in that the state of an edge in $\Sigma_j(G)$ depends on the states of certain distant edges.
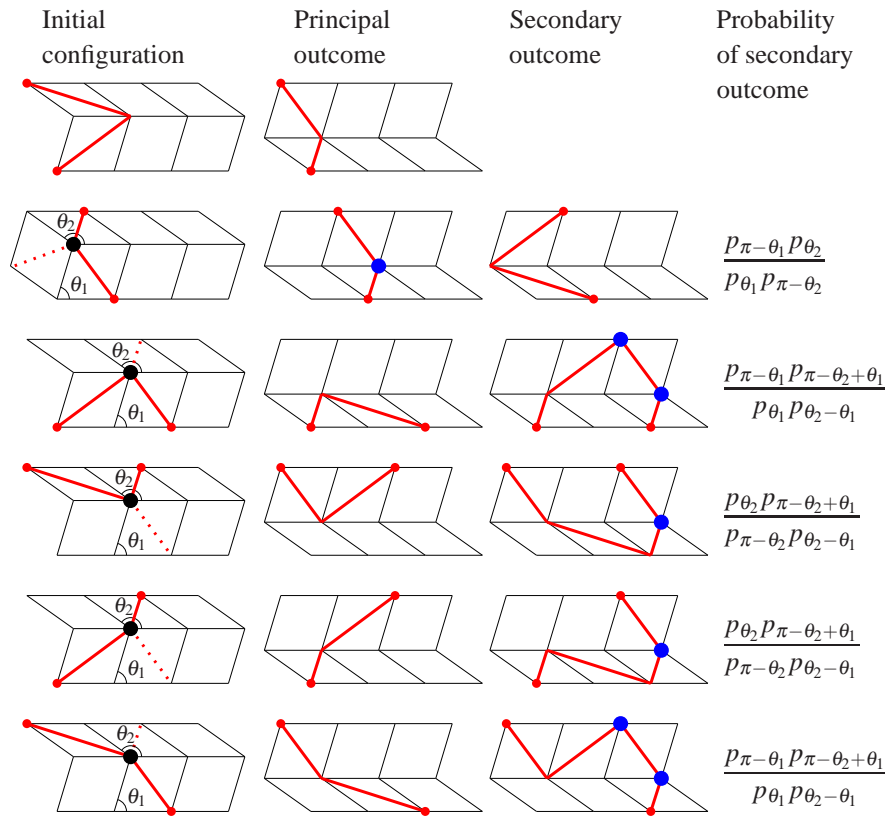

Fig. 5.5 The six possible ways in which γ may intersect the strip in two edges between height $j-1$ and $j+1$, and the corresponding actions of Σ_j. In five cases, the resulting configuration can be non-deterministic. If the dotted edge is closed, the resulting configuration is in the second column. If it is open, the resulting configuration is that of the third column with the given probability (recall from (2.3) that $p_{\pi-\alpha} = 1 - p_\alpha$). The movement of black vertices can cause the height increases marked in blue. The tracks s_k are drawn as horizontal for simplicity, and $\theta_1 = \beta' - \alpha_m$, $\theta_2 = \beta - \alpha_m$, where $v_{m,j}$ denotes the black vertex, and β' (respectively, β) is the transverse angle of the lower (respectively, upper) track.

Let σ_j denote the transposition of the $(j-1)$th and jth terms of a sequence. We may write

$$\Sigma_j(G_{\alpha,\beta}, \mathbb{P}_{\alpha,\beta}) = (G_{\alpha,\sigma_j\beta}, \mathbb{P}_{\alpha,\sigma_j\beta}).$$

When applying the Σ_j in sequence, we distinguish between the *label* s_j of a track and its *level*. Thus, Σ_j interchanges the tracks currently at levels $j-1$ and j.
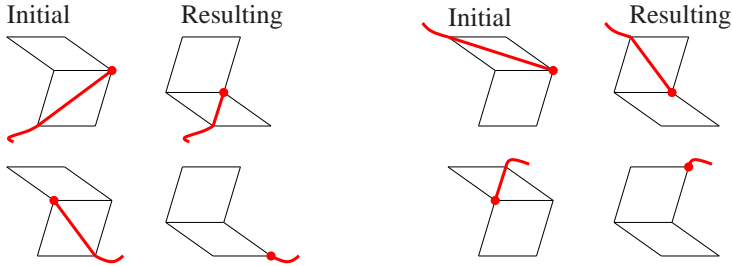
Fig. 5.6 If an endpoint of γ lies between the two tracks, the corresponding edge is sometimes contracted to a single point.

We consider next the transportation of open paths. Let ω be a configuration on $G_{\alpha,\beta}$, and let γ be an ω-open path. The action of a star–triangle transformation on γ is discussed in detail in [19, Sect. 2.3]. The transformation Σ_j comprises three steps: the addition of an edge to $G_{\alpha,\beta}$, a series of star–triangle transformations, and the removal of an edge. The first step does not change γ, and the effect of the second step is discussed in Section 5.2 and the following paragraphs. If the removed edge is in the image of the path γ at the moment of removal, we say that Σ_j *breaks* γ. Thus, $\Sigma_j(\gamma)$ is an open path of $\Sigma_j(G)$ whenever Σ_j does not break γ. In applying the Σ_j, we shall choose the strip-width $M+N$ sufficiently large that open paths of the requisite type do not reach the boundary, and are therefore not broken.

Finally, we summarise in Figures 5.5–5.6 the action of Σ_j on the path γ, with M and N chosen sufficiently large. Consider two tracks s', s at respective levels $j-1$ and j, with transverse angles β' and β. Edges of γ lying outside levels $j-1$ and j are unchanged by Σ_j. The intersection of γ with these two tracks forms a set of open sub-paths of length either 1 or 2; there are four possible types of length 1, and six of length 2. We do not describe this in detail, but refer the reader to the figures, which are drawn for the case $\beta > \beta'$. The path γ may cross the tracks in more than one of the diagrams on the left of Figure 5.5, and the image path contains an appropriate subset of the edges in the listed outcomes. Note that, if the intersections of γ with s and s' are at distance at least 2 from the lateral boundaries, then Σ_j does not break γ,

In the special case when $\beta = \beta'$, Σ_j interchanges the labels of s' and s but alters neither embedding nor configuration. In this degenerate case, we set $\Sigma_j(\gamma) = \gamma$, and note that Figure 5.5 remains accurate.

6 Proof of Theorem 3.1: Isoradial square lattices

6.1 Outline of proof

The proof for isoradial square lattices is based on Proposition 6.1, following. For $\xi \in [0, 2\pi)$, we write $G_{\alpha,\xi}$ for the isoradial square lattice generated by the angle-sequence α and the constant sequence (ξ).

Proposition 6.1. *Let $\delta, \varepsilon > 0$. There exists $\delta' = \delta'(\delta, \varepsilon) > 0$ such that the following holds. Let $G_{\alpha,\beta}$ be an isoradial square lattice satisfying* BAP(ε), *and let $\xi \in [0, 2\pi)$ be such that α and the constant sequence (ξ) satisfy* BAP(ε), (4.7). *If $G_{\alpha,\xi}$ satisfies* BXP(δ), *then $G_{\alpha,\beta}$ satisfies* BXP(δ').

Corollary 6.2. *Let $\varepsilon > 0$. There exists $\delta = \delta(\varepsilon) > 0$ such that every isoradial square lattice satisfying* BAP(ε) *has the box-crossing property* BXP(δ).

Since $\mathscr{G}(\varepsilon, 1)$ is the set of isoradial square lattices satisfying BAP(ε), the corollary is equivalent to (3.1) with $I = 1$.

Proof of Corollary 6.2. Let $\varepsilon > 0$ and let $G_{\alpha,\beta}$ satisfy BAP(ε).

First, assume that one of the two sequences α, β is constant. Without loss of generality we may take α to be constant, and by rotation of the graph, we shall assume $\alpha \equiv 0$. There exists $\delta > 0$ such that the homogeneous square lattice $G_{0,\pi/2}$ satisfies BXP(δ) (see, for example, [15, Sect. 1.7]). By Proposition 6.1 with $\xi = \frac{1}{2}\pi$, $G_{\alpha,\beta}$ satisfies BXP(δ') for some $\delta' = \delta'(\delta, \varepsilon) > 0$.

Consider now the case of general α, β. By the above, G_{α,β_0} satisfies BXP(δ'). By Proposition 6.1 with $\xi = \beta_0$, $G_{\alpha,\beta}$ satisfies BXP(δ'') for some $\delta'' = \delta''(\delta', \varepsilon) > 0$. $\qquad\square$

By Lemma 4.3, Proposition 6.1 follows from the forthcoming Propositions 6.4 and 6.8, dealing respectively with horizontal and vertical crossings.

The proofs are outlined in the remainder of this section. The basic idea is that pieces of $G_{\alpha,\beta}$ and $G_{\alpha,\xi}$ may be glued together along a horizontal track. Track-exchanges may then be performed repeatedly in order to swap parts of $G_{\alpha,\xi}$ and $G_{\alpha,\beta}$ while maintaining the existence of certain open paths.

Let α, β, and ξ be as in Proposition 6.1. Fix N, to be chosen separately in the two proofs, and define

$$\widetilde{\beta}_j = \begin{cases} \xi & \text{if } j < N, \\ \beta_{j-N} & \text{if } j \geq N. \end{cases}$$

We refer to the part of $G = G_{\alpha,\widetilde{\beta}}$ above height N as the *irregular block*, and that with height between 0 and N as the *regular* block. The regular block may be viewed as part of $G_{\alpha,\xi}$, and the irregular block as part of $G_{\alpha,\beta}$. We will only be interested in the graph above height 0.

In Section 6.2, we explain how to transport *horizontal box-crossings* from the regular block to the irregular block. Consider an open horizontal crossing of a wide rectangle in the regular block of G. A sequence of track-exchanges is made from the

top to the bottom of the regular block in such a way that the regular block moves upwards. See Figure 6.1.

The evolution of the open crossing is observed throughout the transformations. Its endpoints are pinned to the lowest track, and therefore do not change. The open path may itself drift upwards, and the core of the proof of Proposition 6.4 lies in a probabilistic control of its rate of drift.

Vertical box-crossings are studied in Section 6.3. This time, the tracks of the regular block are moved upwards by a process of track-exchange, as illustrated in Figure 6.6. Consider an open vertical crossing of some rectangle of the regular block. During the exchanges of tracks, the highest point of the path may drift downwards, and as before one needs some control on its rate of drift. It is key that, after all tracks of the regular block have been moved upwards, the irregular block contains an open vertical crossing of some (wider and lower) rectangle. This is achieved by showing that the height of the upper endpoint of the crossing decreases by at most one at each step, and does not decrease with some uniformly positive probability.

The arguments and illustrations of Section 5.3 will be central to the proofs of Propositions 6.4 and 6.8.

Separate considerations of horizontal and vertical crossings are required since α and β do not play equal roles. Consider the proof of Corollary 6.2. When passing from $G_{0,\pi/2}$ to $G_{\alpha,\beta}$ with one of the sequences α, β constant, both and horizontal and vertical box-crossings need be transported. On the other hand, Proposition 6.4 suffices in the second part of the proof, as explained in the following remark.

Remark 6.3. *The material in Section 6.3, and specifically Proposition 6.8, may be circumvented by use of [19, Thm 1.5], where the box-crossing property is proved for so-called 'highly inhomogeneous' square lattices. We do not take this route here since it would reduce the integrity of the current proof, and would require the reader to be familiar with a different method of applying the star–triangle transformation to square (and triangular) lattices.*

Here is an outline of the alternative approach. In the language of [19], an isoradial square lattice $G_{\alpha,\xi}$ satisfying BAP(ε) *is a highly inhomogeneous square lattice satisfying [19, eqn (1.5)] (with an adjusted value of ε). By [19, Thm 1.5], such a lattice has the box-crossing property. By Proposition 6.4, horizontal box-crossings may be transported from $G_{\alpha,\xi}$ to the more general isoradial square lattice $G_{\alpha,\beta}$. Similarly, by interchanging the roles of the horizontal and vertical tracks of $G_{\alpha,\beta}$, we obtain the existence of vertical box-crossings in that lattice. Such crossing probabilities are now combined, using Lemma 4.2, to obtain Theorem 3.1.*

The following is fixed for the rest of this section. Let $\varepsilon > 0$, and let α, β be sequences of angles satisfying BAP(ε), (4.7). Let ξ be an angle such that α and (ξ) satisfy BAP(ε), (4.7). All constants in this section may depend on ε, but not further on α, β, ξ unless otherwise stated.

6.2 Horizontal crossings

Proposition 6.4. *There exist* $\lambda, N_0 \in \mathbb{N}$, *depending on* ε *only, such that, for* $\rho \in \mathbb{N}$ *and* $N \geq N_0$,

$$
\begin{aligned}
\mathbb{P}_{\alpha,\beta}\big(C_{\mathrm{h}}[B((\rho-1)N,\lambda N)]\big) \\
\geq (1 - \rho e^{-N})\mathbb{P}_{\alpha,\xi}\big(C_{\mathrm{h}}[B(\rho N,N)]\big) \\
\times \mathbb{P}_{\alpha,\xi}\big(C_{\mathrm{v}}[B(-\rho N,-(\rho-1)N;0,N)]\big) \\
\times \mathbb{P}_{\alpha,\xi}\big(C_{\mathrm{v}}[B((\rho-1)N,\rho N;0,N)]\big).
\end{aligned}
\tag{6.1}
$$

Proof. Fix $\rho \in \mathbb{N}$ with $\rho > 1$, and $\lambda, N_0 \in \mathbb{N}$ to be chosen later, and let $N \geq N_0$. Recall the graph $G = G_{\alpha,\widetilde{\beta}}$ given in Section 6.1. We work on a vertical strip $\{v_{i,j} : -M \leq i \leq M\}$ of G with width $2M$, where

$$
M = (\rho + 2\lambda + 1)N,
\tag{6.2}
$$

and we truncate α to a finite sequence $(\alpha_i : -M \leq i \leq M - 1)$.

We will work with graphs obtained from G by a sequential application of the transformations Σ_j of Section 5.3, and to this end we let

$$
U_k = \Sigma_k \circ \Sigma_{k+1} \circ \cdots \circ \Sigma_{N+k-1}, \qquad k \geq 1.
\tag{6.3}
$$

Note that U_k moves the track at level $N+k-1$ to level $k-1$, while raising the tracks at levels $k-1, \ldots, N+k-2$ by one level each (see Figure 6.1). We propose to apply $U_1, U_2, \ldots, U_{\lambda N}$ to G in turn, thereby moving part of the irregular block beneath the regular block.
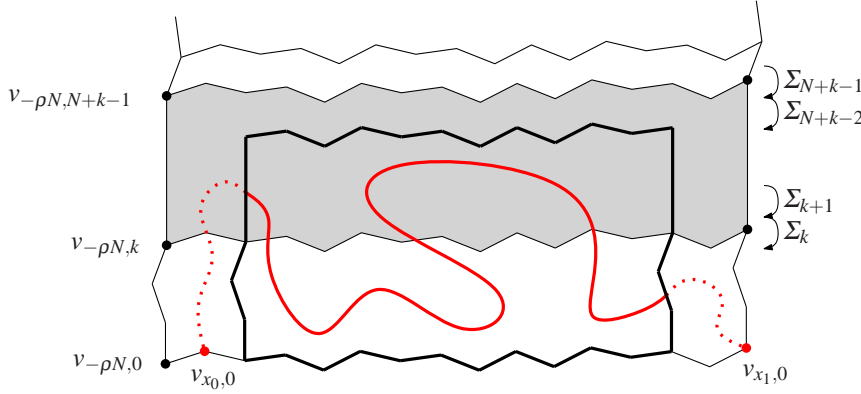
Fig. 6.1 The transformation U_k raises the (shaded) regular block by one unit, and moves the track above by N units downwards.

Let E_N be the event that there exists an open path of G within $B(\rho N, N)$, with endpoints $v_{x_0,0}$ and $v_{x_1,0}$ for some $x_0 \in [-\rho N, -(\rho-1)N]$ and $x_1 \in [(\rho-1)N, \rho N]$.

By the definition of $\widetilde{\beta}$, $B(\rho N, N)$ is entirely contained in the regular block of G. By the Harris–FKG inequality,

$$
\begin{aligned}
\mathbb{P}_{\alpha,\widetilde{\beta}}(E_N) \geq{} & \mathbb{P}_{\alpha,\xi}\big(C_{\mathrm{h}}[B(\rho N, N)]\big) \\
& \times \mathbb{P}_{\alpha,\xi}\big(C_{\mathrm{v}}[B(-\rho N, -(\rho-1)N; 0, N)]\big) \\
& \times \mathbb{P}_{\alpha,\xi}\big(C_{\mathrm{v}}[B((\rho-1)N, \rho N; 0, N)]\big).
\end{aligned}
\tag{6.4}
$$

Let ω^0 be a configuration on G, chosen according to $\mathbb{P}_G$. For $k \in \mathbb{N}$, let $G^0 = G$ and

$$
G^k = U_k \circ \cdots \circ U_1(G), \quad \omega^k = U_k \circ \cdots \circ U_1(\omega^0).
$$

The family $(\omega^k : k \geq 0)$ is a sequence of configurations on the G^k with associated law denoted $\mathbb{P}$. Note that $\mathbb{P}$ is given in terms of the law of ω^0, and of the randomizations contributing to the U_i. The marginal law of ω^k under $\mathbb{P}$ is $\mathbb{P}_{G^k}$.

Let $\omega^0 \in E_N$, and let γ^0 be a path in $B(\rho N, N)$ with endpoints $v_{x_0,0}$ and $v_{x_1,0}$ for some $x_0 \in [-\rho N, -(\rho-1)N]$ and $x_1 \in [(\rho-1)N, \rho N]$. Let $\gamma^k = U_k \circ \cdots \circ U_1(\gamma^0)$. The path evolves as we apply the U_k sequentially, and most of this proof is directed at studying the sequence $\gamma^0, \gamma^1, \ldots, \gamma^{\lambda N}$.

First we show that the path is not broken by the track-exchanges. For $0 \leq k \leq \lambda N$, set

$$
D^k = \big\{ v_{x,y} \in (G^k)^\diamond : |x| \leq (\rho+1)N + 2k - y,\ 0 \leq y \leq N + k \big\}.
$$

The proof of the following elementary lemma is summarised at the end of this section.

Lemma 6.5. *For* $0 \leq k \leq \lambda N$, γ^k *is an open path contained in* D^k.

The set $\{v_{x,0} : x \in \mathbb{Z}\}$ of vertices of $G^\diamond$ is invariant under the U_k, whence the endpoints of the γ^k are constant for all k. It follows that the horizontal span of $\gamma^{\lambda N}$ is at least $2(\rho-1)N$.

If $\gamma^{\lambda N}$ has maximal height not exceeding λN, then it contains a $\omega^{\lambda N}$-open horizontal crossing of $B((\rho-1)N, \lambda N)$. The graph $G^{\lambda N}$ agrees with $G_{\alpha,\beta}$ within $B((\rho-1)N, \lambda N)$, so that

$$
\mathbb{P}_{\alpha,\beta}\big(C_{\mathrm{h}}[B((\rho-1)N, \lambda N)]\big) \geq \mathbb{P}\big(h(\gamma^{\lambda N}) \leq \lambda N \,\big|\, E_N\big)\mathbb{P}(E_N).
$$

By (6.4), it suffices to show the existence of $\lambda, N_0 \in \mathbb{N}$ such that,

$$
\mathbb{P}\big(h(\gamma^{\lambda N}) \leq \lambda N \,\big|\, \omega^0\big) \geq 1 - \rho e^{-N}, \qquad N \geq N_0,\ \omega^0 \in E_N,
\tag{6.5}
$$

and the rest of the proof is devoted to this. The basic idea is similar to the corresponding step of [19], but the calculation is more elaborate.

Let $\omega^0 \in E_N$ and let γ^0 be as above. We observe the evolution of the heights of the images of γ^0 within each column. For $n \in \mathbb{Z}$ and $0 \leq k \leq \lambda N$, set

$$
\mathscr{C}_n = \{v_{n,y} : y \in \mathbb{Z}\}, \quad h_n^k = \begin{cases} h(\gamma^k \cap \mathscr{C}_n) & \text{if } \gamma^k \cap \mathscr{C}_n \neq \varnothing, \\ -\infty & \text{otherwise.} \end{cases}
$$

Thus, $h(\gamma^{\lambda N}) = \sup\{h_n^{\lambda N} : n \in \mathbb{Z}\}$.

The process $(h_n^k : n \in \mathbb{Z})$, $k = 0, 1, \ldots, \lambda N$, has some lateral drift depending on the directions of the track-exchanges Σ_j. We will modify it in order to relate it to the growth process of [19]. The track above the regular block is transported by U_k through the regular block, and thus all Σ_j contributing to U_k are in the same direction (either all move from right to left, or from left to right). Let $(d_k : k \geq 0)$ be given by $d_0 = 0$ and

$$d_{k+1} = \begin{cases} d_k + 1 & \text{if } \beta_k > \xi, \\ d_k & \text{if } \beta_k = \xi, \\ d_k - 1 & \text{if } \beta_k < \xi, \end{cases}$$

and set $H_n^k = h_{n+d_k}^k$. The term d_k is included to compensate for the asymmetric lateral drift induced by the directions of the track-exchanges (see Figure 5.5). Thus H_n^k has a roughly symmetric evolution as n increases. The rest of the proof is devoted to the process $\mathbf{H}^k = (H_n^k : n \in \mathbb{Z})$, $k = 0, 1, \ldots, \lambda N$.

We introduce some notation to be used in the proof. A sequence $\mathbf{R} = (R_n : n \in \mathbb{Z}) \in (\mathbb{Z} \cup \{-\infty\})^{\mathbb{Z}}$ is termed a *range*. The *height in column n* of $\mathbf{R}$ is the value R_n, and the *height* of the range is $\sup\{R_n : n \in \mathbb{Z}\}$. For two ranges $\mathbf{R}^1$, $\mathbf{R}^2$, we write $\mathbf{R}^1 \leq \mathbf{R}^2$ if $R_n^1 \leq R_n^2$ for $n \in \mathbb{Z}$. The *maximum* of a family of ranges is the pointwise supremum sequence. The range $\mathbf{R}$ is called *regular* if

$$|R_{n+1} - R_n| \leq 1, \qquad n \in \mathbb{Z}. \tag{6.6}$$

The *mountain* at a point $(n, r) \in \mathbb{Z}^2$ is defined to be the range $\mathbf{M}(n, r) = (M(n, r)_l : l \in \mathbb{Z})$ given by

$$M(n, r)_l = \begin{cases} r - |n - l| + 1 & \text{for } l \neq n, \\ r & \text{for } l = n. \end{cases}$$

Note that mountains have flat tops of width 3 centred at (n, r), and sides with gradient ± 1. The *covering* of a range $\mathbf{R}$ is the range $C(\mathbf{R})$ formed as the union of the mountains of each of its elements:

$$C(\mathbf{R}) = \max\{\mathbf{M}(n, R_n) : n \in \mathbb{Z}\}.$$

We note that $\mathbf{R} \leq C(\mathbf{R})$ with sometimes strict inclusion, and also that $\mathbf{R}$ and $C(\mathbf{R})$ have the same height. If $\mathbf{R}$ is regular, the heights of $\mathbf{R}$ and $C(\mathbf{R})$ in any given column differ by at most 1. See Figure 6.2 for an illustration of these notions and their role in the evolution of $\mathbf{H}$. We return to the study of $(\mathbf{H}^k)$.

Lemma 6.6. *There exists $\eta = \eta(\varepsilon) \in (0, 1)$, and a family of independent Bernoulli random variables $(Y_n^k : n \in \mathbb{Z}, 0 \leq k < \lambda N)$ with common parameter η such that*

$$H_n^{k+1} \leq \max\{C(\mathbf{H}^k)_n, H_n^k + Y_n^k\}, \quad n \in \mathbb{Z}, 0 \leq k < \lambda N. \tag{6.7}$$
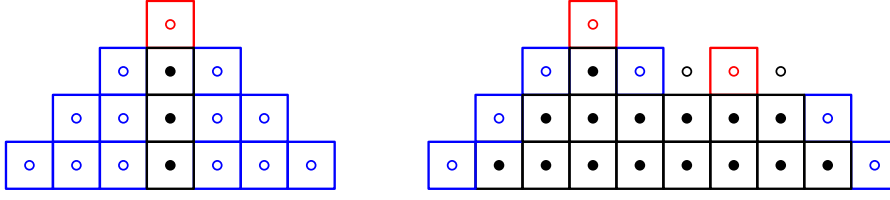
Fig. 6.2 *Left*: One step in the evolution of **H**. The initial range $\mathbf{H}^0$ has only one occupied column (black). The blue/black squares form the mountain of the black column. The red square is added at random. *Right*: One step in the evolution of the $\mathbf{X}^k$ (or $\mathbf{H}^k$ when regular). The black squares are the configuration at step k, the blue squares are the additions at time $k+1$ due to the covering, and the red squares are the random additions.

The (Y_n^k) are random variables used in the star–triangle transformations, and the probability space may be enlarged to accommodate these variables.

The proof of the lemma is deferred until later in the section. Meanwhile, we continue the proof of (6.5) by following that of [19, Prop. 3.7]. Let (Y_n^k) be as in Lemma 6.6, and let $\mathbf{X}^k := (X_n^k : n \in \mathbb{Z})$, $k = 0, 1, \dots, \lambda N$, be the Markov chain given as follows.

(a) The initial value $\mathbf{X}^0$ is the regular range given by

$$X_n^0 = \begin{cases} N & \text{for } n \in [-\rho N, \rho N], \\ N + \rho N - |n| & \text{for } n \notin [-\rho N, \rho N]. \end{cases}$$

(b) For $k \geq 0$, conditional on $\mathbf{X}^k$, the range $\mathbf{X}^{k+1}$ is given by

$$X_n^{k+1} = \max\{X_{n-1}^k, X_n^k + Y_n^k, X_{n+1}^k\}, \qquad n \in \mathbb{Z}. \tag{6.8}$$

We show first, by induction, that $\mathbf{X}^k \geq \mathbf{H}^k$ for all k. It is immediate that $\mathbf{X}^0$ is regular, and that $\mathbf{X}^0 \geq \mathbf{H}^0$. Suppose that $\mathbf{X}^k \geq \mathbf{H}^k$. By (6.8), each range $\mathbf{X}^k$ is regular and

$$\mathbf{X}^{k+1} \geq C(\mathbf{X}^k) \geq C(\mathbf{H}^k). \tag{6.9}$$

By (6.7), $H_n^{k+1} > C(\mathbf{H}^k)_n$ only if $Y_n^k = 1$. Since $X_n^k \geq H_n^k$, we have in this case that

$$X_n^{k+1} \geq X_n^k + 1 \geq H_n^k + 1 = H_n^{k+1}. \tag{6.10}$$

By (6.9)–(6.10), $\mathbf{X}^{k+1} \geq \mathbf{H}^{k+1}$, and the induction step is complete.

The $\mathbf{X}^k$ are controlled via the following lemma.

Lemma 6.7 ([19]). *There exist $\lambda, N_0 \in \mathbb{N}$, depending on η only, such that*

$$\mathbb{P}\left(\max_n X_n^{\lambda N} \leq \lambda N\right) \geq 1 - \rho e^{-N}, \qquad \rho \in \mathbb{N}, \ N \geq N_0.$$

Sketch proof. This follows that of [19, Lemma 3.11]. A small difference arises through the minor change of the initial value $\mathbf{X}^0$, but this is covered by the inclusion of smaller-order terms in [19, eqn (3.31)]. $\qquad\square$

Let λ and N_0 be given thus. For $N \geq N_0$ and $\omega^0 \in E_N$,

$$\mathbb{P}\big(h(\gamma^{\lambda N}) \leq \lambda N \,|\, \omega^0\big) \geq \mathbb{P}\Big(\max_n X_n^{\lambda N} \leq \lambda N\Big)$$
$$\geq 1 - \rho e^{-N}.$$

This concludes the proof of Proposition 6.4. $\qquad\square$

Proof of Lemma 6.6. Let $k \geq 0$ and let ω be a configuration on G^k. Let γ be an open path on G^k that visits no vertex within distance 2 of the sides of G^k and with $h(\gamma) \leq N + k$. We abuse notation slightly by defining $\mathbf{H}^k$ and $\mathbf{H}^{k+1}$ as in the proof of Proposition 6.4 with γ and $U_{k+1}(\gamma)$ instead of γ^k and γ^{k+1}, respectively. That is,

$$H_n^k = h(\gamma \cap \mathscr{C}_{n+d_k}), \quad H_n^{k+1} = h(U_{k+1}(\gamma) \cap \mathscr{C}_{n+d_{k+1}}), \qquad n \in \mathbb{Z}.$$

We will prove that there exists a family of independent Bernoulli random variables $(Y_n : n \in \mathbb{Z})$, independent of ω, with some common parameter $\eta = \eta(\varepsilon) > 0$ to be specified later, such that

$$H_n^{k+1} \leq \max\{C(\mathbf{H}^k)_n, H_n^k + Y_n\}, \qquad n \in \mathbb{Z}. \tag{6.11}$$

Once this is proved, the i.i.d. family $(Y_n^k : n \in \mathbb{Z}, 0 \leq k < \lambda N)$ may be constructed step by step, by applying the above to the pair ω^k, γ^k for $0 \leq k < \lambda N$. By Lemma 6.5, the assumptions on γ are indeed satisfied by each γ^k. By the independence of $(Y_n : n \in \mathbb{Z})$ and ω above, the family $(Y_n^k : n, k)$ satisfies the conditions of the lemma.

It remains to prove (6.11) for fixed k. If $\beta_k = \xi$, no track-exchange takes place, hence $\mathbf{H}^{k+1} = \mathbf{H}^k$ and (6.11) holds. Suppose $\beta_k \neq \xi$. Without loss of generality we may suppose $\beta_k > \xi$, so that $d_{k+1} = d_k + 1$ and the track-exchanges in the application of $U := U_{k+1}$ are all from left to right. To simplify notation we shall assume $d_k = 0$.

Equation (6.11) is proved in two steps. First, we will show that

$$H_n^{k+1} \leq \max\{H_{n-i}^k - |i| + 1 : i \in \mathbb{Z}\}. \tag{6.12}$$

This equation is a weaker version of (6.11) in which each Y_n is replaced by 1.

We prove (6.12) by analysing the individual track-exchanges of which U is composed. For $k \leq j \leq N + k$, let $\Psi_j = \Sigma_{j+1} \circ \cdots \circ \Sigma_{N+k}$. Thus, Ψ_{N+k} is the identity, $\Psi_k = U$, and $\Psi_{j-1} = \Sigma_j \circ \Psi_j$. Recall that the diamond graph is bipartite, with the primal and dual vertices as vertex-sets. A vertex $v_{n,r}$ is said to be contained in a range $\mathbf{R}$ if $r \leq R_n$. A set of vertices is contained in $\mathbf{R}$ if every member is thus contained.

Let the sequence $(\mathbf{L}^j : j = N+k, N+k-1, \ldots, k)$ of ranges be defined recursively as follows. First, $\mathbf{L}^{N+k} = \mathbf{H}^k$. We obtain $\mathbf{L}^{j-1}$ from $\mathbf{L}^j$ by increasing its height in certain columns: for each primal vertex $v_{n,j}$ contained in $\mathbf{L}^j$, the heights in columns $n+1$ and $n+2$ increase to $j+1$ and j, if not already at that height or greater.

We claim that $\Psi_j(\gamma)$ is contained in $\mathbf{L}^j$ for $N+k \geq j \geq k$, which is to say that

$$h(\Psi_j(\gamma) \cap \mathscr{C}_n) \leq L_n^j, \qquad n \in \mathbb{Z}. \tag{6.13}$$

The above holds for $j = N+k$ by the definition of $\mathbf{L}^{N+k}$, and we proceed by (decreasing) induction on j as follows. The path $\Psi_{j-1}(\gamma)$ is obtained by applying Σ_j to $\Psi_j(\gamma)$,

as illustrated in Figure 5.5. Possible increases in column heights are marked in blue. Since the black vertices in Figure 5.5 are contained in $\mathbf{L}^j$, the blue ones are contained in $\mathbf{L}^{j-1}$. This concludes the induction.

Therefore, $U(\gamma) = \Psi_k(\gamma)$ is contained in $\mathbf{L}^k$, and hence inequality (6.12) follows once we have proved that

$$L^k_{n+1} \le \max\{H^k_{n-i} - |i| + 1 : i \ge -1\}. \tag{6.14}$$

This we shall do by observing that the sequence $(\mathbf{L}^j)$ is, in a certain sense, additive with respect to its initial state. We think of $\mathbf{L}^{N+k}$ as a union of columns, each of whose evolutions may be followed individually.
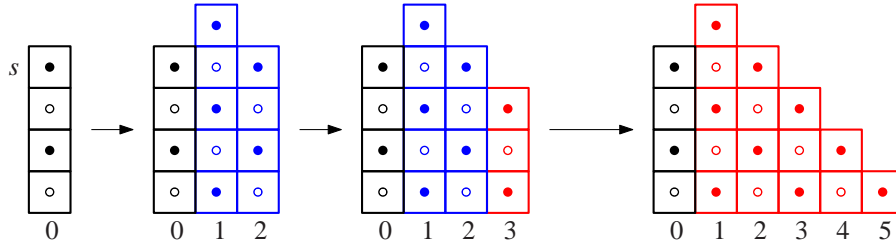
Fig. 6.3 An illustration of the sequence $\widetilde{\mathbf{L}}(0,s)^j$ beginning with the initial column $\widetilde{\mathbf{L}}(0,s)^{N+k} = \Delta(0,s)$. This column is unchanged up to and including $j = s$, and then it evolves as illustrated.

Let $r,s \in \mathbb{Z}$ be such that $s \le N+k$ and $r+s$ is even, so that $v_{r,s}$ is a primal vertex. Let $\Delta(r,s)$ be the range comprising a single column of height s at position r. Consider the sequence $(\widetilde{\mathbf{L}}(r,s)^j)$ with the same dynamics as $(\mathbf{L}^j)$ but with initial state $\widetilde{\mathbf{L}}(r,s)^{N+k} = \Delta(r,s)$. The evolution of $\widetilde{\mathbf{L}}(r,s)^j$ is illustrated in Figure 6.3. We have that $\widetilde{\mathbf{L}}(r,s)^j = \Delta(r,s)$ for $N+k \ge j \ge s$ and, for $s > j \ge k$,

$$\widetilde{L}(r,s)^j_m = \begin{cases} -\infty & \text{if } m < r \text{ or } m > r+s-j+1, \\ s & \text{if } m = r, \\ s-(m-r)+2 & \text{if } r < m \le r+s-j+1. \end{cases}$$

The range $\mathbf{L}^j$ is obtained by combining the contributions of the columns of $\mathbf{H}^k$, in that

$$\mathbf{L}^j = \max\{\widetilde{\mathbf{L}}(r,H^k_r)^j : r \in \mathbb{Z}\}, \qquad N+k \ge j \ge k. \tag{6.15}$$

A rearrangement of the above with $j = k$ implies (6.14); (6.12) follows by extending the maximum in (6.14) over $i \in \mathbb{Z}$.

Let $n \in \mathbb{Z}$ be such that

$$H^k_n + 1 \le \max\{H^k_{n-i} - |i| + 1 : i \in \mathbb{Z} \setminus \{0\}\}. \tag{6.16}$$

Then (6.12) implies $H^{k+1}_n \le C(\mathbf{H}^k)_n$, whence (6.11) holds for this particular value of n.

It remains to prove (6.11) when (6.16) fails. Assume n does not satisfy (6.16), so that (6.12) implies $H_n^{k+1} \leq H_n^k + 1$. We shall prove that

$$H_n^{k+1} \leq H_n^k + Y_n, \tag{6.17}$$

where the Y_n are independent Bernoulli random variables with respective parameters

$$\eta_k(n) := \frac{p_{\pi-\xi+\alpha_n} p_{\pi-\beta_k+\xi}}{p_{\xi-\alpha_n} p_{\beta_k-\xi}}, \tag{6.18}$$

(with p_θ given in (2.3)), and which are independent of ω.

Let $l = H_n^k$. We first analyse the action of $\Psi_l = \Sigma_{l+1} \circ \cdots \circ \Sigma_{N+k}$, and then that of Σ_l.
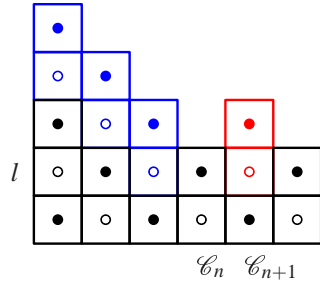
The vertex $v_{n,l}$ is necessarily primal. Since (6.16) fails,

$$H_{n-i}^k \leq l + |i| - 1, \qquad i \in \mathbb{Z} \setminus \{0\}.$$

Since each $v_{n-i,l+|i|-1}$ is a dual vertex, we have the strengthened inequality

$$H_{n-i}^k \leq l + |i| - 2, \qquad i \in \mathbb{Z} \setminus \{0\}. \tag{6.19}$$

See Figure 6.4 for an illustration of the environment around $v_{n,l}$.

Fig. 6.4 The environment around $v_{n,l}$. By (6.19), the black blocks contain $\mathbf{H}^k$. The range $\mathbf{L}^l$, and hence the path $\Psi_l(\gamma)$, is contained in the aggregate range shown. The height in $\mathscr{C}_{n+1}$ increases only if the red block appears when applying Σ_l.

By (6.13), and (6.19) substituted into (6.15),

$$h(\Psi_l(\gamma) \cap \mathscr{C}_{n-i}) \leq L_{n-i}^l \leq l + i, \qquad i \geq -1. \tag{6.20}$$

Note that Σ_l is the final track-exchange with the potential to add vertices to the path at height $l + 1$. Hence, $H_n^{k+1} = l + 1$ only if $v_{n+1,l+1}$ is contained in $\Psi_{l-1}(\gamma)$, or, equivalently, only if the height in $\mathscr{C}_{n+1}$ increases to $l + 1$ when applying Σ_l to $\Psi_l(\gamma)$.

By (6.20) with $i = 0, 1$, the only cases in which this may happen are those of the third and sixth lines of Figure 5.5 (with $v_{n,l}$ the black vertex). (See Figure 6.5 for a more detailed illustration of the third case.) Moreover, the height in $\mathscr{C}_{n+1}$ increases only if the secondary outcome occurs. In both cases, the secondary outcome occurs with probability $\eta_k(n)$ if the edge $e = \langle v_{n,l}, v_{n+1,l+1} \rangle$ is open, and does not occur if e
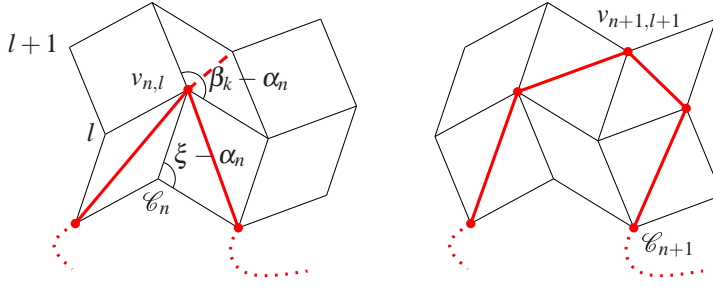
Fig. 6.5 The third case of Figure 5.5. If the dashed edge in the initial configuration (left) is open then, with probability $\eta_k(n)$, the resulting configuration is that on the right side.

is closed. We therefore provide ourselves with a Bernoulli random variable Y_n, with parameter $\eta_k(n)$, for use in the former situation. We have that $H_n^{k+1} = H_n^k + 1$ only if $Y_n = 1$, and (6.17) follows.

Let $A = \xi - \alpha_n$ and $B = \beta_k - \xi$. By (6.18),

$$\eta_k(n) = \frac{p_{\pi-A} p_{\pi-B}}{p_A p_B} = \frac{\sin(\frac{1}{3}A) \sin(\frac{1}{3}B)}{\sin(\frac{1}{3}[\pi - A]) \sin(\frac{1}{3}[\pi - B])}$$

$$= \frac{\cos(\frac{1}{3}[A-B]) - \cos(\frac{1}{3}[A+B])}{\cos(\frac{1}{3}[A-B]) - \cos(\frac{1}{3}[2\pi - A - B])} =: g(A, B).$$

By assumption, $B > 0$, and so by (4.7),

$$\varepsilon \le A \le A + B \le \pi - \varepsilon. \tag{6.21}$$

There exists $c(\varepsilon) > 0$ such that, subject to (6.21),

$$\cos(\tfrac{1}{3}[A-B]) \ge \cos(\tfrac{1}{3}[A+B]) \ge \cos(\tfrac{1}{3}[2\pi - A - B]) + c(\varepsilon).$$

Therefore,

$$\eta := \sup\{g(A,B) : \varepsilon \le A \le A + B \le \pi - \varepsilon\}$$

satisfies $\eta < 1$, and this concludes the proof of the lemma.   □

Proof of Lemma 6.5. We sketch this. Since $B(\rho N, N) \subseteq D^0$, we have that $\gamma^0 \subseteq D^0$. It suffices to show that, for $0 \le k < \lambda N$ and γ an open path in D^k, U_k does not break γ and $U_k(\gamma) \subseteq D^{k+1}$.

By considering the individual track-exchanges of which U_k is composed, it may be seen that $\Psi_j(\gamma)$ is an open path contained in D^{k+1} for all j (with $\Psi_j = \Sigma_{j+1} \circ \Psi_{j+1}$ as in the last proof). In considering how $\Psi_j(\gamma)$ is obtained from $\Psi_{j+1}(\gamma)$, it is useful to inspect the different cases of Figure 5.5, and in particular those involving blue points. The path may be displaced laterally and, during the sequential application of track-exchanges, the drift may be extended laterally as it is propagated downwards. The shapes of the D^i have been chosen in such a way that $\Psi_j(\gamma)$ is contained in D^{k+1} for all j. The argument is valid regardless of the direction of Σ_j.   □

6.3 Vertical crossings

Proposition 6.8. *Let* $\delta = \frac{1}{2}p_{\pi-\varepsilon}^4 \in (0, \frac{1}{2})$. *There exists* $c_N = c_N(\delta) > 0$ *satisfying* $c_N \to 1$ *as* $N \to \infty$ *such that*

$$\mathbb{P}_{\alpha,\beta}\left(C_v[B(4N, \delta N)]\right) \geq c_N \mathbb{P}_{\alpha,\xi}\left(C_v[B(N, N)]\right), \qquad N \in \mathbb{N}.$$

Proof. The notation of Section 6.1 will be used. Let $N \in \mathbb{N}$, and, as in Section 6.2, we will work with a vertical strip of width $2M$ of the graph $G = G_{\alpha,\widetilde{\beta}}$. For this proof we take $M = 5N$. For $k \in \{0, 1, \ldots, N-1\}$, set

$$V_k = \Sigma_{2N-k} \circ \cdots \circ \Sigma_{N-k+1}. \tag{6.22}$$

The map V_k exchanges the track at level $N - k$ with the N tracks immediately above it. The sequential action of $V_0, V_1, \ldots, V_{N-1}$ moves the regular block upwards track by track, see Figure 6.6.
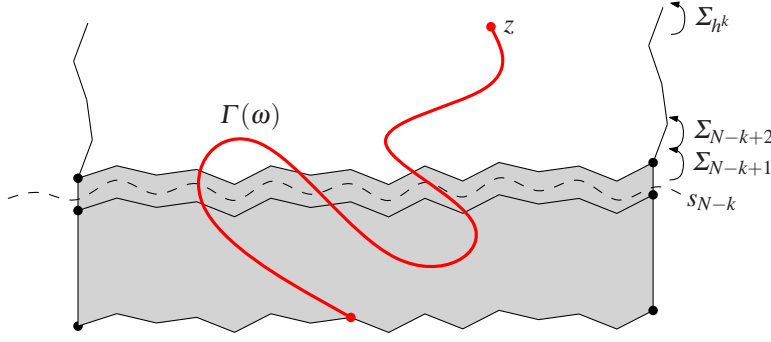
Fig. 6.6 The transformation V_k moves s_{N-k} upwards by N units.

Let ω^0 be a configuration on $G^0 := G_{\alpha,\widetilde{\beta}}$ chosen according to its canonical measure $\mathbb{P}_{\alpha,\widetilde{\beta}}$, and let

$$G^k = V_{k-1} \circ \cdots \circ V_0(G_{\alpha,\widetilde{\beta}}),$$
$$\omega^k = V_{k-1} \circ \cdots \circ V_0(\omega^0),$$
$$D^k = \left\{v_{x,y} \in (G^k)^\diamond : |x| \leq N + 2k + y, \ 0 \leq y \leq 2N\right\},$$
$$h^k = \sup\left\{h \leq N : \exists x_1, x_2 \in \mathbb{Z} \text{ with } v_{x_1,0} \xleftrightarrow{D^k, \omega^k} v_{x_2,h}\right\}.$$

That is, h^k is the greatest height of an open path of G^k starting in $\{v_{x,0} : x \in \mathbb{Z}\}$ and lying in the trapezium D^k. The law $\mathbb{P}$ of the sequence $(\omega^k : k \in \mathbb{N}_0)$ is a combination of the law of ω^0 with those of the star–triangle transformations comprising the V_k.

The box $B(N, N)$ is contained in D^0, and lies entirely in the regular block of G^0. The box $B(4N, \delta N)$ contains the part of D^N between heights 0 and δN, and lies in

the irregular section of G^N ($\delta < \frac{1}{2}$ is given in the proposition). Therefore, it suffices to prove the existence of $c_N = c_N(\delta) > 0$ such that $c_N \to 1$ and

$$\mathbb{P}(h^N \geq \delta N) \geq c_N \mathbb{P}(h^0 \geq N). \tag{6.23}$$

The remainder of this section is devoted to the proof of (6.23).

Let $(\Delta_i : i \in \mathbb{N})$ be independent random variables with common distribution

$$P(\Delta = 0) = 2\delta, \quad P(\Delta = -1) = 1 - 2\delta. \tag{6.24}$$

The Δ_i are independent of all random variables used in the construction of the percolation processes of this paper. We set

$$H^k = H^0 + \sum_{i=1}^{k} \Delta_i, \tag{6.25}$$

where H^0 is an independent copy of h^0, independent of the Δ_i. The inequalities $\leq_{\text{st}}$, $\geq_{\text{st}}$ refer to stochastic ordering.

Lemma 6.9. *Let* $0 \leq k < N$. *If* $h^k \geq_{\text{st}} H^k$ *then* $h^{k+1} \geq_{\text{st}} H^{k+1}$.

Inequality (6.23) is deduced as follows. Evidently, $h^0 \geq_{\text{st}} H^0$ and, by Lemma 6.9, $h^N \geq_{\text{st}} H^N$. In particular,

$$\mathbb{P}(h^N \geq \delta N) \geq P(H^N \geq \delta N).$$

Since h^0 and H^0 have the same distribution,

$$\frac{\mathbb{P}(h^N \geq \delta N)}{\mathbb{P}(h^0 \geq N)} \geq \frac{P(H^N \geq \delta N)}{P(H^0 \geq N)}$$
$$\geq P(H^N \geq \delta N \mid H^0 \geq N) =: c_N(\delta).$$

Now, (H^k) is a random walk with mean step-size $2\delta - 1$. By the law of large numbers, $c_N \to 1$ as $N \to \infty$. In addition, $c_N > 0$, and (6.23) follows. $\qquad\square$

Proof of Lemma 6.9. Let $0 \leq k < N$. We apply V_k to G^k, and study the effects of the track-exchanges in V_k. For $N - k \leq j \leq 2N - k$, let $\Psi_j = \Sigma_j \circ \cdots \circ \Sigma_{N-k+1}$, and let D_j^k be the subgraph of $\Psi_j(G^k)^\diamond$ induced by vertices $v_{x,y}$ with $0 \leq y \leq 2N$ and

$$|x| \leq \begin{cases} N + 2k + y + 2 & \text{if } y \leq j, \\ N + 2k + y + 1 & \text{if } y = j + 1, \\ N + 2k + y & \text{if } y > j + 1. \end{cases} \tag{6.26}$$

The D_j^k increase with j, and $D^k \subseteq D_{N-k}^k$, $D_{2N-k}^k \subseteq D^{k+1}$.

Let $\omega_j^k = \Psi_j(\omega^k)$ and

$$h_j^k = \sup\{h \leq N : \exists x_1, x_2 \in \mathbb{Z} \text{ with } v_{x_1,0} \xleftrightarrow{D_j^k, \omega_j^k} v_{x_2,h}\},$$

noting that

$$h^k \le h^k_{N-k}, \quad h^{k+1} \ge h^k_{2N-k}. \tag{6.27}$$

First, we prove that, for $N - k \le j < 2N - k$,

$$h^k_{j+1} \ge h^k_j - 1, \tag{6.28}$$

$$h^k_{j+1} \ge h^k_j \quad \text{if } h^k_j \ne j+1, \tag{6.29}$$

$$\mathbb{P}(h^k_{j+1} \ge h \mid h^k_j = h) \ge 2\delta \quad \text{if } h = j+1. \tag{6.30}$$

Fix j such that $N - k \le j < 2N - k$. Let γ be an ω^k_j-open path of $\Psi_j(G^k)$, lying in D^k_j, with one endpoint at height 0 and the other at height h^k_j.

By consideration of Figure 5.5, $\Sigma_{j+1}(\gamma)$ is a ω^k_{j+1}-open path contained in D^k_{j+1}. The lower endpoint of γ is not affected by Σ_{j+1}. The upper endpoint is affected only if it is at height $j+1$, in which case its height decreases by at most 1 (see Figure 5.6). This proves (6.28) and (6.29), and we turn to (6.30).

Let $\mathscr{P}_j$ be the set of paths γ of $\Psi_j(G^k)$, contained in D^k_j, with one endpoint at height 0, the other endpoint at height $h(\gamma)$ and all other vertices with heights between 1 and $h(\gamma) - 1$.

We perform a preliminary computation. Let $\gamma, \gamma' \in \mathscr{P}_j$. We write $\gamma' < \gamma$ if $\gamma' \ne \gamma$, $h(\gamma') = h(\gamma)$, and γ' contains no edge strictly to the right of γ within $\{v_{x,y} : |x| \le M, \, 0 \le y \le h(\gamma)\}$. Note that

$$h^k_j = \sup\{h(\gamma) : \gamma \in \mathscr{P}_j, \, \gamma \text{ is } \omega^k_j\text{-open}\},$$

and denote by $\Gamma = \Gamma(\omega^k_j)$ the ω^k_j-open path of $\mathscr{P}_j$ that is the minimal element of $\{\gamma \in \mathscr{P}_j : h(\gamma) = h^k_j, \, \gamma \text{ is } \omega^k_j\text{-open}\}$ with respect to the order $<$. Thus, Γ is the leftmost path of $\mathscr{P}_j$ reaching height h^k_j.

We have that

$$\{\Gamma(\omega^k_j) = \gamma\} = \{\gamma \text{ is } \omega^k_j\text{-open}\} \cap N_\gamma, \qquad \gamma \in \mathscr{P}_j, \tag{6.31}$$

where N_γ is the decreasing event that:

(a) there is no $\gamma' \in \mathscr{P}_j$ with $h(\gamma') > h(\gamma)$, all of whose edges not belonging to γ are ω^k_j-open,

(b) there is no $\gamma' < \gamma$ with $h(\gamma') = h(\gamma)$, all of whose edges not belonging to γ are ω^k_j-open.

Note that N_γ is independent of the event $\{\gamma \text{ is } \omega^k_j\text{-open}\}$.

Let F be a set of edges of $\Psi_j(G^k)$, disjoint from γ, and let C_F be the event that every edge in F is ω^k_j-*closed*. Let $\mathbb{P}^k_j$ denote the marginal law of ω^k_j, and p_e the edge-

probability of the edge e of $\Psi_j(G^k)$. By (6.31) and the Harris–FKG inequality,

$$
\begin{aligned}
\mathbb{P}(C_F \mid \Gamma = \gamma) &= \frac{\mathbb{P}_j^k(\Gamma = \gamma \mid C_F)}{\mathbb{P}_j^k(\Gamma = \gamma)} \mathbb{P}_j^k(C_F) \\
&= \frac{\mathbb{P}_j^k(N_\gamma \mid C_F)}{\mathbb{P}_j^k(N_\gamma)} \mathbb{P}_j^k(C_F) \\
&\geq \mathbb{P}_j^k(C_F) = \prod_{f \in F}(1 - p_f),
\end{aligned}
\tag{6.32}
$$

where we have extended the domain of $\mathbb{P}$ to include the intermediate subsequence of $\omega^k = \omega_{N-k}^k, \omega_{N-k+1}^k, \ldots, \omega_{2N-k}^k = \omega^{k+1}$.

Let $\gamma \in \mathscr{P}_j$ with $h(\gamma) = j+1$ and suppose $\Gamma(\omega_j^k) = \gamma$. Without loss of generality, we may suppose that Σ_{j+1}, applied to $\Psi_j(G^k)$, goes from left to right; a similar argument holds otherwise.

Let $z = v_{x,j+1}$ denote the upper endpoint of γ and let z' denote the other endpoint of the unique edge of γ leading to z. Either $z' = v_{x+1,j}$ or $z' = v_{x-1,j}$. In the second case, it is automatic as in Figure 5.6 that $h(\Sigma_{j+1}(\gamma)) \geq j+1$.

Assume that $z' = v_{x+1,j}$, as illustrated in Figure 6.7. Let $F = \{e_1, e_2, e_3, e_4\}$ where

$$
\begin{aligned}
e_1 &= \langle v_{x,j+1}, v_{x-1,j+2} \rangle, \quad & e_2 &= \langle v_{x-1,j+2}, v_{x-2,j+1} \rangle, \\
e_3 &= \langle v_{x-2,j+1}, v_{x-1,j} \rangle, \quad & e_4 &= \langle v_{x-1,j}, v_{x,j+1} \rangle,
\end{aligned}
$$

are the edges of the face of $\Psi_j(G^k)$ to the left of z. By definition of $\mathscr{P}_j$, F is disjoint from γ. By studying the three relevant star–triangle transformations contributing to Σ_{j+1} as illustrated in Figure 6.7, we find as in Figure 5.2 that

$$
\begin{aligned}
\mathbb{P}\big(h(\Sigma_{j+1}(\gamma)) \geq j+1 \,\big|\, \Gamma = \gamma\big) &\geq \frac{p_{e_1} p_{e_4}}{(1 - p_{e_1})(1 - p_{e_4})} \mathbb{P}(C_F \mid \Gamma = \gamma) \\
&\geq \frac{p_{e_1} p_{e_4}}{(1 - p_{e_1})(1 - p_{e_4})} \mathbb{P}(C_F),
\end{aligned}
$$

by (6.32).
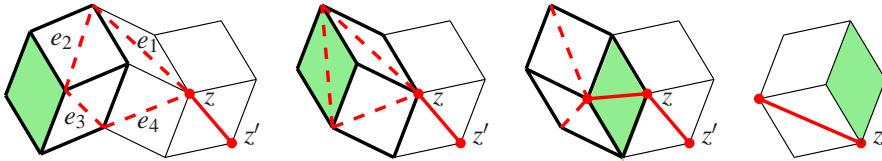
Fig. 6.7 Three star–triangle transformations contributing to Σ_{j+1}, from left to right. The dashed edges are closed, the bold edges are open. The first and last passages occur with probability 1, and the second with probability $p_{e_1} p_{e_4} / (1 - p_{e_1})(1 - p_{e_4})$.

In summary, we have that

$$\mathbb{P}(h^k_{j+1} \geq h^k_j \mid \Gamma = \gamma) \geq \frac{p_{e_1} p_{e_4}}{(1 - p_{e_1})(1 - p_{e_4})} \prod_{f \in F}(1 - p_f)$$

$$= p_{e_1} p_{e_4}(1 - p_{e_2})(1 - p_{e_3})$$

$$\geq p^4_{\pi-\varepsilon} = 2\delta, \qquad (6.33)$$

by (2.4). The proof of (6.30) is complete.

It remains to show that (6.28)–(6.30) imply the lemma. This may seem obvious, but, since the sequence $(h^k_j)_j$ is not Markovian, some technical details are needed.

Suppose $h^k \geq_{\mathrm{st}} H^k$. We shall bound (stochastically) the sequence $(h^k_j)_j$ by a Markov chain, as follows. Let $(X_j : j = N - k, \ldots, 2N - k)$ be an inhomogeneous Markov chain taking values in $\mathbb{N}_0$, with transition probabilities given by

$$X_{j+1} = X_j \quad \text{if } X_j \neq j + 1,$$

$$P(X_{j+1} = x \mid X_j = j + 1) = \begin{cases} 2\delta & \text{if } x = j + 1, \\ 1 - 2\delta & \text{if } x = j. \end{cases}$$

By (6.28)–(6.30), for all j,

$$\mathbb{P}(h^k_{j+1} \geq x \mid h^k_j = y) \geq P(X_{j+1} \geq x \mid X_j = z), \quad x, y, z \in \mathbb{N}_0, \ z \leq y.$$

Let $X_{N-k} = H^k$. By the induction hypothesis, $H^k \leq_{\mathrm{st}} h^k \leq h^k_{N-k}$, whence $X_{2N-k} \leq_{\mathrm{st}} h^k_{2N-k}$ as in [19, Sect. 3.3] (by [19, Lemma 3.7], iterated). Moreover $X_{2N-k} - X_{N-k} =_{\mathrm{st}} \Delta_{k+1}$. Therefore,

$$h^{k+1} \geq h^k_{2N-k} \geq_{\mathrm{st}} X_{2N-k} =_{\mathrm{st}} H^k + \Delta_{k+1} = H^{k+1},$$

as claimed.

□

7 Proof of Theorem 3.1: The general case

Let $G \in \mathscr{G}(\varepsilon, I)$. By SGP($I$), there exist two families $(s_j : j \in \mathbb{Z})$ and $(t_i : i \in \mathbb{Z})$ of tracks forming a square grid of G. A star–triangle transformation is said to act 'between s_k and s_0' if the three faces of $G^\diamond$ on which it acts are between s_k and s_0. (Recall from Section 4.5 that such faces may belong to s_k but not to s_0). A path is said to be between s_0 and s_k if it comprises only edges between s_0 and s_k (that is, edges belonging to faces between s_0 and s_k). A vertex of $G^\diamond$ is said to be 'just below' s_0 if it is adjacent to s_0 and between s_{-1} and s_0.

Let $E_N = E_N(G)$ be the event that there exists an open path γ on G such that:

(a) γ is between s_0 and s_N,
(b) the endpoints of γ are just below s_0,
(c) one endpoint is between t_{-2N} and t_{-N} and the other between t_N and t_{2N}.

There is no further condition on the horizontal extent of γ. We claim that there exists $\delta = \delta(\varepsilon, I) > 0$, independent of G and N, such that

$$\mathbb{P}_G(E_N) \geq \delta, \qquad N \geq 1. \tag{7.1}$$

Since γ contains a horizontal crossing of the domain $\mathscr{D} = \mathscr{D}(t_{-N}, t_N; s_0, s_N)$, (7.1) implies

$$\mathbb{P}_G\big[C_{\mathrm{h}}(t_{-N}, t_N; s_0, s_N)\big] \geq \delta.$$

Since δ depends only on ε and I, the corresponding inequality holds for crossings of translations of $\mathscr{D}$, and also with the roles of the (s_j) and (t_i) reversed. By Proposition 4.2, the claim of the theorem (together with the stronger statement (3.1)) follows from (7.1), and we turn to its proof.

The method is as follows. Consider the graph G between s_0 and s_N. By making a finite sequence of star–triangle transformations between s_N and s_0, we shall move the s_j downwards in such a way that the section of the resulting graph, lying both between t_{-2N} and t_{2N} and between the images of s_0 and s_N, forms a box of an isoradial square lattice. By Corollary 6.2, this box is crossed horizontally with probability bounded away from 0. The above star–triangle transformations are then reversed to obtain a horizontal crossing of $\mathscr{D}$ in the original graph G.

Since a finite sequence of star–triangle transformations changes G at only finitely many places, we may retain the track-notation s_j, t_i throughout their application. We say $s_j, s_{j+1}, \ldots, s_{j+k}$ are *adjacent between* t_{N_1} *and* t_{N_2} if there exists no track-intersection in the domain $\mathscr{D}(t_{N_1}, t_{N_2}; s_j, s_{j+k})$ except those on $s_j, s_{j+1}, \ldots, s_{j+k}$. The proof of the next lemma is deferred until later in this section.

Lemma 7.1. *There exists a finite sequence $(T_k : 1 \leq k \leq K)$ of star–triangle transformations, each acting between s_N and s_0, such that, in $T_K \circ \cdots \circ T_1(G)$, the tracks $s_0, \ldots, s_N$ are adjacent between t_{-2N} and t_{2N}.*

Let $(T_k : 1 \leq k \leq K)$ be given thus, and write $G^0 = G$ and $G^k = T_k \circ \cdots \circ T_1(G^0)$. Let S_k be the inverse transformation of T_k, as in Section 5.1, so that $S_k(G^k) = G^{k-1}$. Since the track notation is retained for each G^k, the event E_N is defined on each such graph. By a careful analysis of its action, we may see that S_k preserves E_N for $k = K, K-1, \ldots, 1$. The details of the argument (which requires no estimate of the value of K) are provided in the next paragraph.

Let $1 \leq k \leq K$ and let γ be an open path of G^k satisfying (a)–(c) above. Since T_k acts between s_N and s_0, it does not move s_0, and $S_k(\gamma)$ has the same endpoints as γ. Moreover the three faces of $(G^k)^\diamond$ on which S_k acts are either all strictly below s_N, or two of them are part of s_N and the third is above. In the first case $S_k(\gamma)$ may differ from γ but is still contained between s_0 and s_N; in the second case S_k does not influence γ. In conclusion, $S_k(\gamma)$ is an open path on G^{k-1} that satisfies (a)–(c).

Since the canonical measure is conserved under a star–triangle transformation, the remark above implies

$$\mathbb{P}_G(E_N) \geq \mathbb{P}_{G^K}(E_N). \tag{7.2}$$

It remains to prove a lower bound for $\mathbb{P}_{G^K}(E_N)$.

Write $(r_i : i \in \mathbb{Z})$ for the sequence of all tracks other than the s_j, indexed and oriented according to their intersections with s_0, with $r_0 = t_0$, and including the t_i in

increasing order. Let β_j be the transverse angle of s_j, and $\pi + \alpha_i$ that of r_i. Since each r_i intersects each s_j, the vectors $\alpha = (\alpha_i : i \in \mathbb{Z})$, $\beta = (\beta_j : j \in \mathbb{Z})$ satisfy (4.7), and hence $G_{\alpha,\beta}$ is an isoradial square lattice satisfying BAP(ε). By Corollary 6.2, there exists $\delta' = \delta'(\varepsilon) > 0$ such that $G_{\alpha,\beta}$ satisfies the box-crossing property BXP(δ').

The track-system of G^K inside $\mathscr{D}(t_{-2N}, t_{2N}; s_0, s_N)$ is isomorphic to a rectangle of $\mathbb{Z}^2$, and comprises the horizontal tracks $s_0, s_1, \ldots, s_N$, crossed in order by those r_i between (and including) t_{-2N} and t_{2N}. Thus, G^K agrees with $G_{\alpha,\beta}$ inside this domain.

Consider the following boxes of $(G^K)^\diamond$:

$$V_1 = \mathscr{D}(t_{-2N}, t_{-N}; s_0, s_N),$$
$$V_2 = \mathscr{D}(t_N, t_{2N}; s_0, s_N),$$
$$H = \mathscr{D}(t_{-2N}, t_{2N}; s_0, s_N).$$

By the Harris–FKG inequality,

$$\begin{aligned}
\mathbb{P}_{G^K}(E_N) &\geq \mathbb{P}_{G^K}\big[C_{\mathrm{v}}(V_1) \cap C_{\mathrm{v}}(V_2) \cap C_{\mathrm{h}}(H)\big] \\
&\geq \mathbb{P}_{G^K}[C_{\mathrm{v}}(V_1)]\,\mathbb{P}_{G^K}[C_{\mathrm{v}}(V_2)]\,\mathbb{P}_{G^K}[C_{\mathrm{h}}(H)].
\end{aligned} \tag{7.3}$$

The boxes V_1, V_2 in $(G^K)^\diamond$ may be regarded as boxes in $G_{\alpha,\beta}^\diamond$, and have height N and width at least N. Similarly, the box H has height N and width at most $4IN$. By BXP(δ') and (7.3), there exists $\delta = \delta(\varepsilon, I) > 0$ such that $\mathbb{P}_{G^K}(E_N) \geq \delta$, and (7.2) is proved.

Proof of Lemma 7.1. We shall prove the existence of a finite sequence $(T_k : 1 \leq k \leq K)$ of star–triangle transformations, each acting between s_1 and s_0, such that, in $T_K \circ \cdots \circ T_1(G)$, the tracks s_0, s_1 are adjacent between t_{-2N} and t_{2N}. The general claim follows by iteration.

In this proof we work with the graph G only through its track-set $\mathscr{T}$. Tracks will be viewed as arcs in $\mathbb{R}^2$. A *point* of $\mathscr{T}$ is the intersection of two tracks, and we write $\mathscr{P}$ for the set of points.

Let $\mathscr{N}$ be the set of tracks that are not parallel to s_0. Any $r \in \mathscr{N}$ intersects both s_0 and s_1 exactly once, and we orient such r in the direction from its intersection with s_0 to that with s_1.

An oriented path γ on the track-set $\mathscr{T}$ is called *increasing* if it uses only tracks in $\mathscr{N}$ and it conforms to their orientations. For points $y_1, y_2 \in \mathscr{P}$, we write $y_1 \geq y_2$ if there exists an increasing path γ from y_2 to y_1. By the properties of $\mathscr{T}$ given in Section 4.2, the relation $\geq$ is reflexive, antisymmetric, and transitive, and is thus a partial order on $\mathscr{P}$.

Let $\mathscr{R}_k$ be the closed region of $\mathbb{R}^2$ delimited by t_{-k}, t_k, s_0, s_1, illustrated in Figure 7.1. A point $y \in \mathscr{P}$ is coloured *black* if it is strictly between s_0 and s_1, and in addition $y \geq y'$ for some y' in $\mathscr{R} := \mathscr{R}_{2N}$ or on its boundary. In particular, any point in the interior of $\mathscr{R}$ or of its left/right boundaries is black. We shall see that the black points are precisely those to be 'moved' above s_1 by the star–triangle transformations T_k.

We prove first that the number B of black points is finite. By BAP(ε), the number of tracks intersecting $\mathscr{R}$ is finite. Let y^+ (respectively, y^-) be the rightmost (respectively, leftmost) point on s_1 that is the intersection of s_1 with a track r that intersects
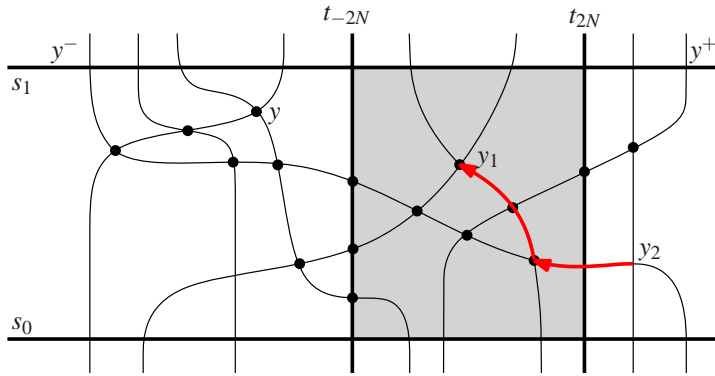
Fig. 7.1 The black points are indicated. The path γ from y_2 to y_1 is drawn in red. The points y and y_1 are maximal, and are not comparable. The region $\mathscr{R}$ is shaded.

$\mathscr{R}$. We claim that

$$\text{if } r \in \mathscr{T} \text{ has a black point, it intersects } s_1 \text{ between } y^- \text{ and } y^+. \qquad (7.4)$$

Assume (7.4) for the moment. Since a black point is the unique intersection of two tracks, and since (7.4) implies that there are only finitely many tracks with black points, we have that $B < \infty$.

We prove (7.4) next. Let y be a black point. If $y \in \mathscr{R}$, (7.4) follows immediately. Thus we may suppose, without loss of generality, that y is strictly to the left of $\mathscr{R}$. There exists an increasing path γ, starting at a point on the left boundary of $\mathscr{R}$ and ending at y. Take γ to be the 'highest' such path. Let $(r_l : 1 \leq l \leq L)$ be the tracks used by γ in order, where $L < \infty$. We will prove by induction that, for $l \geq 1$,

$$r_l \text{ intersects } s_1 \text{ between } y^- \text{ and } y^+. \qquad (7.5)$$

Clearly (7.5) holds with $l = 1$ since r_1 intersects $\mathscr{R}$.
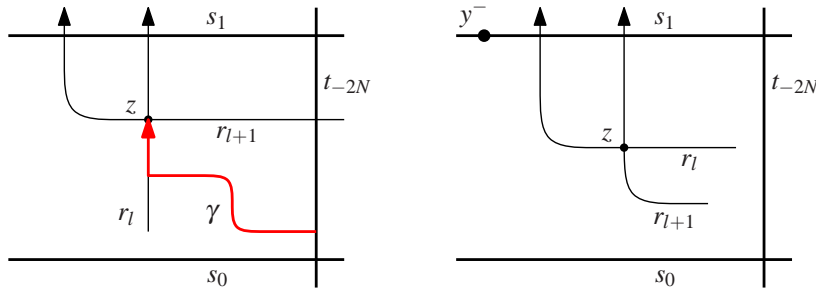
Fig. 7.2 *Left*: The oriented track r_{l+1} crosses r_l from right to left, in contradiction of the choice of γ as highest. *Right*: The track r_{l+1} crosses r_l from left to right.

Suppose $1 \le l < L$ and (7.5) holds for r_l, and let $z = r_l \cap r_{l+1}$. If r_{l+1} intersects $\mathscr{R}$ (before or after z), (7.5) follows trivially. Suppose r_{l+1} does not intersect $\mathscr{R}$. There are two possibilities: either r_{l+1} crosses r_l from right to left, or from left to right. The first case is easily seen to be impossible, since it contradicts the choice of γ as highest. Hence, r_{l+1} crosses r_l from left to right (see Figure 7.2). The part of the oriented track r_{l+1} after z is therefore above the corresponding part of r_l. Since r_{l+1} intersects s_1 after z, and does not intersect $\mathscr{R}$, the intersection of r_{l+1} and s_1 lies between y^- and y^+, and the induction step is complete.

In conclusion r_L intersects s_1 between y^+ and y^-. Let r denote the other track containing y. By the same reasoning, r intersects r_L from left to right, whence it also intersects s_1 between y^+ and y^-. This concludes the proof of (7.4), and we deduce that $B < \infty$.

If $B = 0$, there is no point in the interior of either $\mathscr{R}$ or its left/right sides, whence s_0, s_1 are adjacent between t_{-2N} and t_{2N}.

Suppose $B \ge 1$. We will show that B may be reduced by one by a star–triangle transformation acting between s_0 and s_1, and the claim of the lemma will follow by iteration.

Since $B < \infty$, there exists a black point that is maximal in the partial order $\ge$, and we pick such a point $y = r_1 \cap r_2$. By the maximality of y, the tracks r_1, r_2, s_1 form a track-triangle. By applying the star–triangle transformation to this track-triangle as in Figure 5.3, the point y is moved above s_1, and the number of black points is decreased. This concludes the proof of Lemma 7.1. $\qquad\square$

8 Arm exponents

8.1 Outline of proof

We recall the isoradial embedding $G_{0,\pi/2}$ of the homogeneous square lattice, with associated measure denoted $\mathbb{P}_{0,\pi/2}$.

Proposition 8.1. *Let $k \in \{1,2,4,\dots\}$, $\varepsilon > 0$, and $I \in \mathbb{N}$. There exist constants $c_i = c_i(k,\varepsilon,I) > 0$ and $N_0 = N_0(k,\varepsilon,I) \in \mathbb{N}$ such that, for $N \ge N_0$, $n \ge c_0 N_0$, $G \in \mathscr{G}(\varepsilon,I)$, and any vertex u of $G^{\diamond}$,*

$$c_1 \mathbb{P}_{0,\pi/2}[A_k(N,n)] \le \mathbb{P}_G[A_k^u(N,n)] \le c_2 \mathbb{P}_{0,\pi/2}[A_k(N,n)].$$

Part (a) of Theorem 3.4 is an immediate consequence; part (b) is discussed in Section 8.6.

Sections 8.2–8.5 are devoted to the proof of Proposition 8.1. In Section 8.2 is presented a modified definition of the arm-events, adapted to the context of an isoradial graph. This is followed by Proposition 8.2, which asserts in particular the equivalence of the two types of arm-events. The proof of Proposition 8.1 follows, using the techniques of the proof of Theorem 3.1; the proof for isoradial square lattices is in Section 8.3, and for general graphs in Section 8.4. Section 8.5 contains the so-called separation theorem, together with the proof of Proposition 8.2.

For the remainder of this section, $\varepsilon > 0$ and $I \in \mathbb{N}$ shall remain fixed. Unless otherwise stated, constants $c_i > 0$, $N_0 \in \mathbb{N}$ depend only on ε, I, and on the number k

of arms in the event under study. We use the expression 'for $n > N$ sufficiently large' to mean: for $n \geq c_0 N$ and $N > N_0$.

8.2 Arm-events

Let $G \in \mathscr{G}(\varepsilon, I)$, $k \in \{1, 2, 4, \dots\}$, and let s be a track and u be a vertex of $G^{\diamond}$, adjacent to s. For $n \geq N$, we define the 'modified arm-event' $\widetilde{A}_k^{u,s}(N, n)$ as follows. For simplicity of notation, we omit explicit reference to u and s when no ambiguity results, but in such a case we say that $\widetilde{A}_k(N, n)$ is 'centred at u'.

A certain technical assumption will be useful in Section 8.3, when applying star–triangle transformations to isoradial square lattices. Recall the notation $\Lambda_u^{\diamond}(n)$ from Section 4.4, and the constant c_d of (4.2). We shall frequently require a primal vertex u and dual vertex u^* of $G^{\diamond}$ to satisfy the appropriate inclusion of the following:

$$C_u \subseteq \Lambda_u^{\diamond}(3c_d^2 n), \qquad C_{u^*}^* \subseteq \Lambda_{u^*}^{\diamond}(3c_d^2 n). \tag{8.1}$$

The modified arm-events $\widetilde{A}_k(N, n) = \widetilde{A}_k^{u,s}(N, n)$ are defined thus:

(i) For $k = 1$, $\widetilde{A}_1(N, n)$ is the event that there exist vertices $x_1 \in \Lambda_u^{\diamond}(N)$ and $y_1 \notin \Lambda_u^{\diamond}(n)$, both adjacent to s, on the same side of s as u and satisfying (8.1), such that $x_1 \longleftrightarrow y_1$.

(ii) For $k = 2$, $\widetilde{A}_2(N, n)$ is the event that there exist vertices $x_1, x_1^* \in \Lambda_u^{\diamond}(N)$ and $y_1, y_1^* \notin \Lambda_u^{\diamond}(n)$, all adjacent to s and on the same side of s as u, such that:
    (a) x_1, x_1^* and y_1, y_1^* satisfy (8.1),
    (b) $x_1 \longleftrightarrow y_1$ and $x_1^* \longleftrightarrow^* y_1^*$.

(iii) For $k = 2j \geq 4$, $\widetilde{A}_k(N, n)$ is the event that there exist vertices $x_1, \dots, x_j \in \Lambda_u^{\diamond}(N)$ and $y_1, \dots, y_j \notin \Lambda_u^{\diamond}(n)$, all adjacent to s and on the same side of s as u, such that:
    (a) each x_i and y_i satisfies (8.1),
    (b) $x_i \longleftrightarrow y_i$ and $x_i \nleftrightarrow x_{i'}$ for $i \neq i'$.

The following proposition contains three statements, the third of which relates the modified arm-events to those of Section 3. All arm-events A_k and $\widetilde{A}_k$ considered here are centred at the same vertex $u \in G^{\diamond}$. The event $\widetilde{A}_k(N, n)$ is to be interpreted in terms of any of the tracks to which u is adjacent.

Proposition 8.2. *There exist constants $c_i > 0$ such that, for $n > N$ sufficiently large,*

$$\mathbb{P}_G[A_k(N, 2n)] \leq \mathbb{P}_G[A_k(N, n)] \leq c_1 \mathbb{P}_G[A_k(N, 2n)], \tag{8.2}$$

$$\mathbb{P}_G[A_k(N, n)] \leq \mathbb{P}_G[A_k(2N, n)] \leq c_2 \mathbb{P}_G[A_k(N, n)], \tag{8.3}$$

$$c_3 \mathbb{P}_G[A_k(N, n)] \leq \mathbb{P}_G[\widetilde{A}_k(N, n)] \leq c_4 \mathbb{P}_G[A_k(N, n)]. \tag{8.4}$$

By (8.4), for $n > N$ sufficiently large, there exist constants $c_5, c_6 > 0$ such that, if u is adjacent to the tracks s and t,

$$c_5 \mathbb{P}_G[\widetilde{A}_k^{u,s}(N, n)] \leq \mathbb{P}_G[\widetilde{A}_k^{u,t}(N, n)] \leq c_6 \mathbb{P}_G[\widetilde{A}_k^{u,s}(N, n)].$$

The proof of Proposition 8.2 is deferred to Section 8.5. It relies on the so-called separation theorem, an account of which may be found in that section.

8.3 Proof of Proposition 8.1: Isoradial square lattices

Let G be an isoradial square lattice satisfying the bounded-angles property $\mathrm{BAP}(\varepsilon)$, and let u be a vertex of $G^\diamond$. As usual, the horizontal tracks are labelled $(s_j : j \in \mathbb{Z})$ and the vertical tracks $(t_i : i \in \mathbb{Z})$.

As explained in Section 4.6, $G = G_{\alpha,\beta}$ for angle-sequences $\alpha = (\alpha_i : i \in \mathbb{Z})$, $\beta = (\beta_j : j \in \mathbb{Z})$ satisfying (4.7). We label α and β in such a way that $u = v_{0,0}$, whence u is adjacent to t_0 and s_0 (here we do not require $v_{0,0}$ to be primal). The horizontal track at level 0, initially s_0, may change its label through track-exchanges. Let ξ be such that α and the constant sequence (ξ) satisfy $\mathrm{BAP}(\varepsilon)$, (4.7). All arm-events in the following are centred at $u = v_{0,0}$.

Lemma 8.3. *There exist constants $c_i > 0$ such that, for $n > N$ sufficiently large,*

$$c_1 \mathbb{P}_{\alpha,\xi}[A_k(N,n)] \le \mathbb{P}_G[A_k(N,n)] \le c_2 \mathbb{P}_{\alpha,\xi}[A_k(N,n)].$$

Proof. Let $N, n \in \mathbb{N}$ be picked (later) such that N and n/N are large, and write $M = \lceil 3c_d^2 n \rceil$. For $0 \le m \le M$, let G^m be the isoradial square lattice with angle-sequences $\widetilde{\alpha} = (\alpha_i : -4M \le i \le 4M)$ and $\widetilde{\beta}^m$, with

$$\widetilde{\beta}_j^m = \begin{cases} \xi & \text{if } -m \le j < m, \\ \beta_{j+m} & \text{if } -(m+M) \le j < -m, \\ \beta_{j-m} & \text{if } m \le j < m+M, \\ \xi & \text{if } j < -(m+M) \text{ or } j \ge m+M. \end{cases} \tag{8.5}$$

Thus G^m is obtained from G by taking the horizontal tracks s_j, $-M \le j < M$, splitting them with a band of height $2m$, and filling the rest of space with horizontal tracks having transverse angle ξ. By the choice of ξ, each G^m satisfies $\mathrm{BAP}(\varepsilon)$. Moreover, inside $\Lambda_u^\diamond(M)$, G^0 is identical to G and G^M is identical to $G_{\alpha,\xi}$.

For $0 \le m < M$, let

$$U_m = (\Sigma_{m+1} \circ \cdots \circ \Sigma_{m+M}) \circ (\Sigma_{-(m+1)} \circ \cdots \circ \Sigma_{-(m+M)}),$$

where the Σ_j are given in Section 5.3. Under U_m, the track at level $m+M$ is moved to the position directly above that at level $m-1$, and the level $-(m+M+1)$ track below the level $-m$ track. We have that

$$U_m(G^m) = G^{m+1}.$$

Let ω^0 be a configuration on G^0 such that $\widetilde{A}_k^{u,s_0}(N,n)$ occurs. Set $j = 1$ when $k = 1$, and $j = k/2$ when $k \ge 2$. There exist vertices $x_1, \ldots, x_j, y_1, \ldots, y_j$ and, when $k = 2$, x_1^*, y_1^*, all lying in the set $\{v_{m,0} : m \in \mathbb{Z}\}$ of vertices of $G^\diamond$, such that:

(a) $x_i \xleftrightarrow{\ G^0,\omega^0\ } y_i$ and $x_i \xleftarrow{\ G^0,\omega^0\ }\!\!\!\!/\!\!\!\!\longrightarrow x_{i'}$ for $i \ne i'$,

(b) $x_1^* \xleftrightarrow{\ G^0,\omega^0\ }{}^* y_1^*$, when $k = 2$,

(c) $d^\diamond(v_{0,0}, x_i) \le N$, $d^\diamond(v_{0,0}, y_i) > n$,

(d) $d^\diamond(v_{0,0}, x_1^*) \le N$, $d^\diamond(v_{0,0}, y_1^*) > n$, when $k = 2$,

(e) $C_{x_i} \subseteq \Lambda_u^\diamond(M)$ and, when $k = 2$, $C_{x_1^*}^* \subseteq \Lambda_u^\diamond(M)$.

As we apply $U_{M-1} \circ \cdots \circ U_0$ to (G^0, ω^0), the images of paths from each of x_i, y_i, and x_1^*, y_1^* retain their starting points.

Each $\Lambda_u^\diamond(r)$ has a diamond shape. By an argument similar to that of Lemma 6.5, for $0 \leq m \leq M$,

$$C_{x_i}(\omega^m) \subseteq \Lambda_u^\diamond(M + 2m), \quad C_{x_1^*}^*(\omega^m) \subseteq \Lambda_u^\diamond(M + 2m).$$

Moreover, since $C_{x_i}(\omega^m)$ and $C_{x_1^*}^*(\omega^m)$ do not extend to the left/right boundaries of G^m, these clusters neither break nor merge with one another. Therefore,

(a) $x_i \xleftrightarrow{\ G^M, \omega^M\ } y_i$ and $x_i \xleftrightarrow{\ G^M, \omega^M\ \not{}\ } x_{i'}$ for $i \neq i'$,

(b) $x_1^* \xleftrightarrow{\ G^M, \omega^M\ }^* y_1^*$, when $k = 2$,

so that $\omega^M \in A_k(c_d N, c_d^{-1} n)$. (A related discussion may be found in [20, Sect. 3.2].) In conclusion, there exists $c_3 > 0$ such that

$$\mathbb{P}_G[A_k(N,n)] \leq c_3 \mathbb{P}_{G^0}[\widetilde{A}_k(N,n)] \qquad \text{by (8.4)}$$
$$\leq c_3 \mathbb{P}_{G^M}[A_k(c_d N, c_d^{-1} n)].$$

Since the intersection of any G^m with $\mathscr{A}(c_d N, c_d^{-1} n)$ is contained in $\Lambda_u^\diamond(M)$, we have by the discussion after (8.5) that there exists $c_4 > 0$ with

$$\mathbb{P}_G[A_k(N,n)] \leq c_3 \mathbb{P}_{\alpha,\xi}[A_k(c_d N, c_d^{-1} n)]$$
$$\leq c_3 c_4 \mathbb{P}_{\alpha,\xi}[A_k(N,n)],$$

by (8.2) and (8.3), iterated. The second inequality of Lemma 8.3 is proved.

Turning to the first inequality, let ω^M be a configuration on G^M such that $\widetilde{A}_k(N,n)$ occurs (the arm event is defined in terms of $v_{0,0}$ and the horizontal track at level 0). It may be seen as above that $\omega^0 = U_{M-1} \circ \cdots \circ U_0(\omega^M)$ is a configuration on G^0 contained in $A_k(c_d N, c_d^{-1} n)$. Furthermore,

$$\mathbb{P}_{\alpha,\xi}[A_k(N,n)] \leq c_3 \mathbb{P}_{G^M}[\widetilde{A}_k(N,n)] \leq c_3 c_4 \mathbb{P}_G[A_k(N,n)].$$

The proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Corollary 8.4. *There exist constants $c_i > 0$ such that, for $n > N$ sufficiently large and any isoradial square lattice $G_{\alpha,\beta} \in \mathscr{G}(\varepsilon, I)$,*

$$c_1 \mathbb{P}_{0,\pi/2}[A_k(N,n)] \leq \mathbb{P}_{\alpha,\beta}[A_k(N,n)] \leq c_2 \mathbb{P}_{0,\pi/2}[A_k(N,n)]. \tag{8.6}$$

Proof. If α is a constant vector (α_0), (8.6) follows by Lemma 8.3 with $\xi = \alpha_0 + \pi/2$.

For α non-constant, we apply Lemma 8.3 with $\xi = \beta_0$, thus bounding the arm-event probabilities for $G_{\alpha,\beta}$ by those for G_{α,β_0}. Now, G_{α,β_0} is of the type analysed above, and the conclusion follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

8.4 Proof of Proposition 8.1: The general case

Let $G \in \mathscr{G}(\varepsilon, I)$, and let $(s_j : j \in \mathbb{Z})$ and $(t_i : i \in \mathbb{Z})$ be two families of tracks forming a square grid of G, duly oriented. Write $(r_i : i \in \mathbb{Z})$ for the sequence of all tracks other than the s_j, indexed and oriented according to their intersections with s_0, with $r_0 = t_0$, and including the t_i in increasing order. Let β_j be the transverse angle of s_j, and $\pi + \alpha_i$ that of r_i. Since each r_i intersects each s_j, the vectors $\alpha = (\alpha_i : i \in \mathbb{Z})$, $\beta = (\beta_j : j \in \mathbb{Z})$ satisfy (4.7), and hence $G_{\alpha,\beta}$ is an isoradial square lattice satisfying BAP(ε). As in Lemma 7.1, we may retain the labelling of tracks throughout the proof. Let u be the vertex adjacent to s_0 and t_0, below and respectively left of these tracks. All arm-events in the following are centred at the vertex u and expressed in terms of the track s_0.

Lemma 8.5. *There exist constants $c_1, c_2 > 0$ such that, for $n > N$ sufficiently large,*

$$c_1 \mathbb{P}_{\alpha,\beta}[A_k(N,n)] \leq \mathbb{P}_G[A_k(N,n)] \leq c_2 \mathbb{P}_{\alpha,\beta}[A_k(N,n)].$$

This lemma, together with Corollary 8.4, implies Proposition 8.1 for arm events centred at u. By the square-grid property, any vertex is within bounded distance of one of the tracks $(s_j : j \in \mathbb{Z})$. This allows us to extend the conclusion to arm events centred at any vertex.

Proof. Let $n \in \mathbb{N}$ and $M = \lceil c_d n \rceil$. By Lemma 7.1, applied in two stages above and below s_0, there exists a finite sequence R^+ of star–triangle transformations such that, in $G^M := R^+(G)$, the tracks $s_{-M}, \ldots, s_M$ are adjacent between t_{-M} and t_M. Moreover, no star–triangle transformation in R^+ involves a rhombus lying in s_0. The sequence R^+ has an inverse sequence denoted R^-. Note that G^M agrees with $G_{\alpha,\beta}$ inside $B_n + u \subseteq \Lambda_u^\diamond(M)$.

Let ω be a configuration on G belonging to $\widetilde{A}_k(N,n)$, and let vertices x_i, y_i be given accordingly. Consider the image configuration $\omega^M = R^+(\omega^0)$ on G^M. By considering the action of the transformation R^+, we may see that

(a) $x_i \xleftrightarrow{\ G^M, \omega^M\ } y_i$ and $x_i \xleftarrow{\ G^M, \omega^M\ }\!\!\!\!\!/\!\!\longrightarrow x_{i'}$ for $i \neq i'$,

(b) $x_1^* \xleftrightarrow{\ G^M, \omega^M\ }{}^* y_1^*$, when $k = 2$.

Taken together with (4.2), this implies that $\omega^M \in A_k(c_d N, c_d^{-1} n)$. Therefore, there exist $c_i > 0$ such that

$$\begin{aligned}
\mathbb{P}_G[A_k(N,n)] &\leq c_3 \mathbb{P}_G[\widetilde{A}_k(N,n)] && \text{by (8.4)} \\
&\leq c_3 \mathbb{P}_{G^M}[A_k(c_d N, c_d^{-1} n)] && \\
&= c_3 \mathbb{P}_{\alpha,\beta}[A_k(c_d N, c_d^{-1} n)] && \\
&\leq c_3 c_4 \mathbb{P}_{\alpha,\beta}[A_k(N,n)] && \text{by (8.2) and (8.3),}
\end{aligned}$$

and the second inequality of the lemma is proved.

Conversely, let ω^M be a configuration on G^M belonging to $\widetilde{A}_k(N,n)$. By applying the inverse transformation, we obtain the configuration $\omega = R^-(\omega^M)$ on G. As above,

$$
\begin{aligned}
\mathbb{P}_{\alpha,\beta}[A_k(N,n)] &= \mathbb{P}_{G^M}[A_k(N,n)] \\
&\leq c_3 \mathbb{P}_{G^M}[\widetilde{A}_k(N,n)] &&\text{by (8.4)} \\
&\leq c_3 \mathbb{P}_G[A_k(c_dN, c_d^{-1}n)] \\
&\leq c_3 c_4 \mathbb{P}_G[A_k(N,n)] &&\text{by (8.2) and (8.3).}
\end{aligned}
$$

This concludes the proof of the first inequality of the lemma.   □

8.5 Proof of Proposition 8.2

This section is devoted to the proof of Proposition 8.2, and is not otherwise relevant to the rest of the paper. The two main ingredients of the proof are the separation theorem (Theorem 8.6) and the equivalence of metrics, (4.2).

8.5.1 Separation theorem

The so-called separation theorem was proved by Kesten in [30] (and elaborated in [33]) in the context of homogeneous site percolation. The theorem is set more generally in [20, Thm 3.5], and this gives rise to the current version, Theorem 8.6, to be used in the proof of Proposition 8.2. The formal statement of Theorem 8.6 requires some notation from [20,30,33] which, for completeness, is presented below.

Let $G \in \mathscr{G}(\varepsilon, I)$. By Theorem 3.1 (see also (3.1)), $\mathbb{P}_G$ satisfies the box-crossing property $\mathrm{BXP}(\delta)$ with $\delta = \delta(\varepsilon, I)$. Let σ be a colour sequence of length k. The event $A_\sigma(N,n)$ requires the existence of a number of 'arms' crossing an annulus. Roughly speaking, the separation theorem implies that the extremities of these arms may be taken to be distant from one another.

We shall consider open and (dual) open* crossings between the interior and exterior boundaries of $\mathscr{A}(N,n)$. For clarity, we concentrate first on the behaviour of crossings at their exterior endpoints. Let $\eta \in (0,1)$. A primal (respectively, dual) η-*exterior-fence* is a set Γ of connected open (respectively, open*) paths comprising the union of:

   (i) a crossing of $\mathscr{A}(N,n)$ from its interior to its exterior boundary, with exterior endpoint denoted $\mathrm{ext}(\Gamma)$,

together with certain further paths which we describe thus under the assumption that $\mathrm{ext}(\Gamma) = (n,y)$ is on the right side of ∂B_n:

   (ii) a vertical crossing of the box $[n,(1+\sqrt{\eta})n] \times [y-\eta n, y+\eta n]$,
   (iii) a connection between the above two crossings, contained in $\mathrm{ext}(\Gamma) + B_{\sqrt{\eta}n}$.

If $\mathrm{ext}(\Gamma)$ is on a different side of ∂B_n, the event of condition (ii) is replaced by an appropriately rotated and translated event. This definition is illustrated in Figure 8.1.

One may similarly define an η-*interior*-fence by considering the behaviour of the crossing near its interior endpoint. We introduce also the concept of a primal
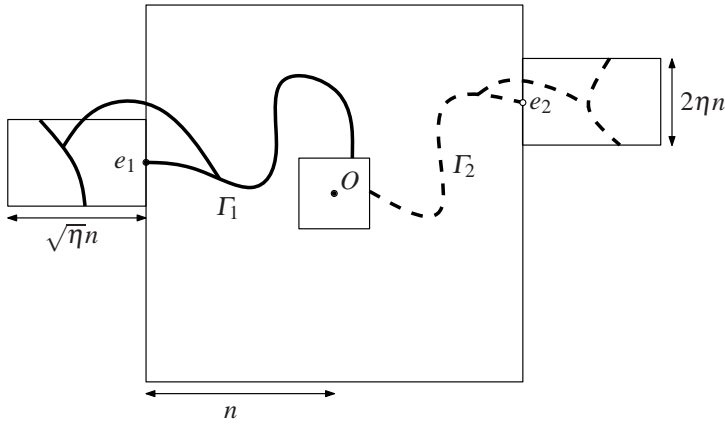
Fig. 8.1 A primal η-exterior-fence Γ_1 with exterior endpoint e_1, and a dual η-exterior-fence Γ_2.
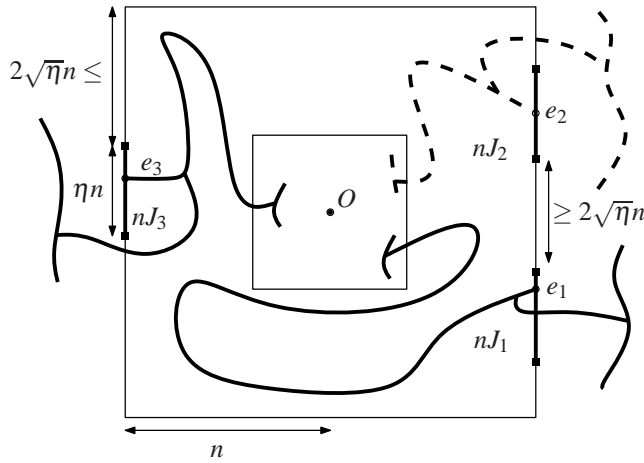
Fig. 8.2 The event $A_\sigma^{J,J}(N,n)$ with $\sigma = (1,0,1)$ and η-landing-sequence J. Each crossing Γ_i is an η-fence with exterior endpoint $e_i \in nJ_i$.

(respectively, dual) η-*fence*; this is a union of an open (respectively, open*) crossing of $\mathscr{A}(N,n)$ together with further paths in the vicinities of both interior and exterior endpoints along the lines of the above definitions.

An η-*landing-sequence* is a sequence of closed sub-intervals $J = (J_i : i = 1,2,\ldots,k)$ of ∂B_1, taken in anticlockwise order, such that each J_i has length η, and the minimal distance between any two intervals, and between any interval and a corner of B_1, is greater than $2\sqrt{\eta}$. We shall assume that

$$0 < k(\eta + 2\sqrt{\eta}) < 8, \qquad (8.7)$$

so that η-landing-sequences exist.

Let η, η' satisfy (8.7), and let J (respectively, J') be an η-landing-sequence (respectively, η'-landing-sequence). Write $A_\sigma^{J,J'}(N,n)$ for the event that there exists a sequence of η-fences $(\Gamma_i : i = 1, 2, \ldots, k)$ in the annulus $\mathscr{A}(N,n)$, with colours prescribed by σ, such that, for all i, the interior (respectively, exterior) endpoint of Γ_i lies in NJ_i (respectively, nJ_i'). These definitions are illustrated in Figure 8.2.

We now state the separation theorem. The proof is omitted, and may be constructed via careful readings of the appropriate sections of [30, 33].

Theorem 8.6 (Separation theorem). *For $\eta_0 > 0$ and a colour sequence σ, there exist constants $c > 0$ and $N_1 \in \mathbb{N}$ depending only on η_0, σ, ε, and I such that: for all $\eta, \eta' \geq \eta_0$ satisfying (8.7), all η-landing-sequences J and η'-landing-sequences J', and all $N \geq N_1$ and $n \geq 2N$, we have*

$$\mathbb{P}_G[A_\sigma^{J,J'}(N,n)] \geq c\mathbb{P}_G[A_\sigma(N,n)].$$

8.5.2 Proof of Proposition 8.2

Inequalities (8.2) and (8.3) follow from Theorem 8.6 and the box-crossing property. The proof, omitted here, is essentially that of [33, Prop. 12], and uses the extension of paths by judiciously positioned box-crossings. We turn therefore to the proof of (8.4).

Consider the first inequality of (8.4) (the second is easier to prove). The idea is as follows. Suppose that $A_k(N,n)$ occurs (together with some additional assumptions). One may construct a bounded number of open or open* box-crossings in order to obtain $\widetilde{A}_k(N,n)$. These two arm-events are given in terms of annuli defined via different metrics — the Euclidean metric and $d^\diamond$, respectively — but the radii of these annuli are comparable by (4.2).

Assume $n/N \geq 2$, and let

$$M = c_d^{-1}N, \quad m = c_d n, \tag{8.8}$$

with c_d as in (4.2). Let $k \in \{1, 2, 4, \ldots\}$, $\sigma = (1, 0, 1, 0, \ldots)$, and consider the corresponding arm-event $A_k(M,m)$. All constants in the following proof may depend on k, ε, and I but, unless otherwise specified, on nothing else. All arm-events that follow are assumed centred at the vertex u adjacent to a track s. By translation, we may assume that u is the origin of $\mathbb{R}^2$. In order to gain some control over the geometry of s, we may assume, without loss of generality, that its transverse angle β satisfies $\beta \in [\frac{1}{4}\pi, \frac{3}{4}\pi]$.

Let $\eta = \eta(k) > 0$ satisfy (8.7), and let J be an η-landing sequence of length k, entirely contained in $\{1\} \times [0,1]$, with J_1 being the lowest interval. Henceforth assume $M \geq N_1$, where N_1 is given in Theorem 8.6 with $\eta_0 = \eta$. By that theorem, there exists $c_0 > 0$ such that

$$\mathbb{P}_G[A_k^{J,J}(M,m)] \geq c_0\mathbb{P}_G[A_k(M,m)]. \tag{8.9}$$

Let (M, v_i) be the lower endpoint of MJ_i, and (M, w_i) the upper. Let H_M be the event that, for $i \in \{1, 2, \ldots, k\}$, the following crossings of colour σ_i exist:

(a) a horizontal crossing of $[-w_i, M] \times [v_i, w_i]$,
(b) a vertical crossing of $[-w_i, -v_i] \times [-w_i, w_i]$,
(c) for i odd, a horizontal crossing of $[-w_i, w_i] \times [-w_i, -v_i]$,
(d) for i even, a horizontal crossing of $[-w_i, w_k] \times [-w_i, -v_i]$.

If $k \geq 4$, we require also an open* vertical crossing of $[v_k, w_k] \times [-w_k, 0]$. The event H_M depends only on the configuration inside B_M, and is illustrated in Figure 8.3.
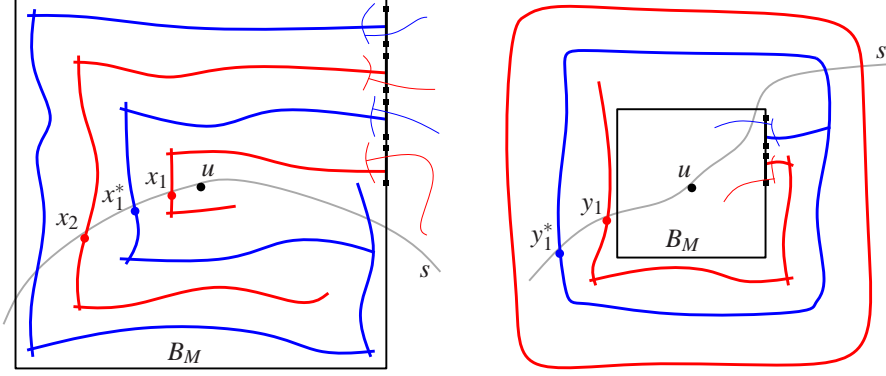
Fig. 8.3 *Left*: The event H_M for $k = 4$. The red paths are open, the blue paths are open*. The thin coloured paths are parts of the interior fences of $A_k^{J,J}(M, m)$. *Right*: The event K_m for $k = 2$, together with parts of the exterior fences of the arm-event. The track s intersects the open/open* crossings just above the points labelled x_i and y_i.

Let (m, v_i) be the lower endpoint of mJ_i, and (m, w_i) the upper. Let K_m be the event that, for $i \in \{1, 2, \ldots, k\}$, the following crossings of colour σ_i exist:

(a) a horizontal crossing of $[m, (m + w_i)] \times [v_i, w_i]$,
(b) a vertical crossing of $[(m + v_i), m + w_i] \times [-(m + w_i), w_i]$,
(c) a horizontal crossing of $[-(m + w_i), m + w_i] \times [-(m + w_i), -(m + v_i)]$,
(d) if i is odd, a vertical crossing of $[-(m + w_i), -(m + v_i)] \times [-(m + w_i), m + w_i]$,
(e) if i is even, a vertical crossing of $[-(m + w_i), -(m + v_i)] \times [-(m + w_i), m + w_k]$.

We require in addition the following:

(f) when $k = 1$, an open* circuit in $\mathscr{A}(2m, 3m)$,
(g) when $k = 2$, an open circuit in $\mathscr{A}(2m, 3m)$,
(h) when $k \geq 2$, an open* circuit in $\mathscr{A}(m + v_k, m + w_k)$.

The event K_m depends only on the configuration inside $\mathscr{A}(m, 3m)$, and is illustrated in Figure 8.3.

Set $j = 1$ when $k = 1$, and $j = k/2$ when $k \geq 2$. We claim that, on $H_M \cap K_m \cap A_k^{J,J}(M, m)$, there exist vertices $x_1, \ldots, x_j, y_1, \ldots, y_j$ and, when $k = 2$, x_1^*, y_1^*, all adjacent to s and on the same side of s as u, such that:

(a) $x_i \in B_M$, $y_i \notin B_m$ and, when $k = 2$, $x_1^* \in B_M$, $y_1^* \notin B_M$,
(b) $x_i \longleftrightarrow y_i$ and $x_i \nleftrightarrow x_{i'}$ for $i \neq i'$,

(c) $x_1^* \longleftrightarrow^* y_1^*$ when $k = 2$,

(d) $C_{x_i} \subseteq B_{3m}$ and, when $k = 2$, $C_{x_1^*}^* \subseteq B_{3m}$.

This claim holds as follows. The crossings in the definition of H_M (respectively, K_m) may be regarded as extensions of the arms of $A_k^{J,J}(M,m)$ inside B_M (respectively, outside B_m). Let λ be the straight line with inclination $\beta \in [\frac{1}{4}\pi, \frac{3}{4}\pi]$, passing through u. Since s corresponds to a chain of rhombi with common sides parallel to λ, it intersects λ only in the edge of $G^\diamond$ crossing s and containing u. Therefore, the part of s to the left of λ necessarily intersects all the above extensions. These intersections provide the x_i, y_i and, when $k = 2$, x_1^*, y_1^*. The remaining statements above are implied by the definitions of the relevant events.

By (4.2), (8.1), and (8.8),

$$H_M \cap K_m \cap A_k^{J,J}(M,m) \subseteq \widetilde{A}_k(c_d M, c_d^{-1} m).$$

By the extended Harris–FKG inequality of [30, Lemma 3] (see also [33, Lemma 12]),

$$\mathbb{P}_G\big(H_M \cap K_m \cap A_k^{J,J}(M,m)\big) \geq \mathbb{P}_G(H_M)\mathbb{P}_G(K_m)\mathbb{P}_G[A_k^{J,J}(M,m)].$$

The events H_M and K_m are given in terms of crossings of boxes with aspect-ratios independent of M and m. Therefore, there exists $c_1 > 0$ such that, for m and M sufficiently large, $\mathbb{P}_G(H_M) \geq c_1$ and $\mathbb{P}_G(K_m) \geq c_1$. In conclusion, by (8.8), there exists $c_5 > 0$ such that, for $n/N \geq 2$,

$$\begin{aligned}
\mathbb{P}_G[\widetilde{A}_k(N,n)] &\geq \mathbb{P}_G(H_M)\mathbb{P}_G(K_m)\mathbb{P}_G[A_k^{J,J}(M,m)] && \\
&\geq c_1^2 c_0 \mathbb{P}_G[A_k(M,m)] && \text{by (8.9)} \\
&\geq c_1^2 c_0 c_5 \mathbb{P}_G[A_k(c_d M, c_d^{-1} m)] && \\
&= c_1^2 c_0 c_5 \mathbb{P}_G[A_k(N,n)] && \text{by (8.8)}
\end{aligned}$$

where the third inequality holds by iteration of (8.2)–(8.3). The first inequality of (8.4) follows.

The second inequality is simpler. Set $M = c_d N$ and $m = c_d^{-1} n$. By (4.2), $\widetilde{A}_k(c_d^{-1} M, c_d m) \subseteq A_k(M,m)$. By iteration of (8.2)–(8.3), there exists $c_6 > 0$ such that, for $m > M$ sufficiently large,

$$\begin{aligned}
\mathbb{P}_G[\widetilde{A}_k(c_d^{-1} M, c_d m)] &\leq \mathbb{P}_G[A_k(M,m)] \\
&\leq c_6 \mathbb{P}_G[A_k(c_d^{-1} M, c_d m)].
\end{aligned}$$

This concludes the proof of Proposition 8.2.

8.6 Proof of Theorem 3.4(b)

Theorem 3.4(b) follows from Theorem 3.4(a) and the following proposition.

Proposition 8.7. *Let $G \in \mathcal{G}$. If either ρ or η exists for G, then ρ, η, and δ exist for G, and they satisfy $\eta\rho = 2$ and $2\rho = \delta + 1$.*

Kesten [29] proved this statement for the special case of homogeneous bond percolation on the square lattice, using arguments that rely heavily on estimates from the earlier [28]. As noted in these papers, the conclusions may be extended to periodic models satisfying the box-crossing property. They may in fact be extended still further, to planar percolation processes $(G, \mathbb{P})$ embedded in $\mathbb{R}^2$ that satisfy the following:

(a) a uniform upper bound on the lengths of both primal and dual edges,
(b) a (strictly positive) lower and upper bound on the density of vertices of G,
(c) $\mathbb{P}$ and the measure $\mathbb{P}^*$ of the dual process have the box-crossing property,
(d) the existence of a constant $c > 0$ such that, for n sufficiently large and any vertices u, v,

$$c^{-1}\mathbb{P}\big(\mathrm{rad}(C_u) > n\big) \le \mathbb{P}\big(\mathrm{rad}(C_v) > n\big) \le c\mathbb{P}\big(\mathrm{rad}(C_u) > n\big).$$

Conditions (a) and (b) are satisfied by any isoradial graph with the bounded-angles property. By Theorem 3.1 and Proposition 8.1, conditions (c) and (d) are satisfied by isoradial graphs with the bounded-angles property and the square-grid property. Proposition 8.7 follows for general $G \in \mathscr{G}$.

No major changes are required in adapting the proofs of [28, 29], and we include no further details here.

Acknowledgements

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