

EXPECTED DEPTH OF RANDOM WALKS ON GROUPS

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ABSTRACT. For G a finitely generated group and $g \in G$, we say g is detected by a normal subgroup $N \triangleleft G$ if $g \notin N$. The depth $D_G(g)$ of g is the lowest index of a normal, finite index subgroup N that detects g . In this paper we study the expected depth, $\mathbb{E}[D_G(X_n)]$, where X_n is a random walk on G . We give several criteria that imply that

$$\mathbb{E}[D_G(X_n)] \xrightarrow{n \rightarrow \infty} 2 + \sum_{k \geq 2} \frac{1}{[G : \Lambda_k]},$$

where Λ_k is the intersection of all normal subgroups of index at most k . We explain how the right-hand side above appears as a natural limit and also give an example where the convergence does not hold.

1. INTRODUCTION

Let G be a finitely generated group. The *depth* of an element in G encodes how well approximated that element is by finite quotients of the group. The goal of this article is to find the average depth of an element of G . As such, this question is ill-posed, and a more precise one is: *what is the asymptotic expected depth of a random walk on the Cayley graph of G ?* This question arises naturally when quantifying *residual finiteness*, or in other words, when studying statistics surrounding the *depth function*.

For $g \in G$ and N a normal subgroup of G , we say g is *detected by N* if $g \notin N$ (in other words, if g is mapped onto a non-trivial element of G/N). The *depth* of g is the lowest index of a normal, finite index subgroup N that detects g . Formally, for $g \in G$, $g \neq e$, set

$$D_G(g) := \min\{|G/N| : N \triangleleft_{\text{finite index}} G \text{ and } g \notin N\}.$$

For $g = e$ the above definition would produce a depth equal to $\min \emptyset = \infty$. In the context of random walks, this singularity would produce trivial results. To circumvent this triviality, we instead define $D_G(e) := 0$. With this definition, G is *residually finite* if and only if $D_G(g) < \infty$ for all elements $g \in G$.

Let G be residually finite and S be a finite generating set, which will be always considered symmetric. Then the *residual finiteness growth function* is

$$F_G^S(n) = \max_{g \in B_G^S(n)} D_G(g),$$

where $B_G^S(n)$ is the ball of radius n in the Cayley graph $\text{Cay}(G, S)$. This notion was introduced in [BR10] and has been studied for various classes of groups; the relevant results to this paper are listed in §2.1.

While the residual finiteness growth function reflects the largest depth of an element in the ball of radius n , the question of interest in this paper is what we can say about the “average” depth of a “uniform” element of the group. If G is discrete but infinite, there is no natural

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definition of a uniform probability measure on G , hence no good notion of a uniform element. However, we may try to approach the desired “average” by averages of well defined measures. Two approaches come to mind:

- For $n \geq 0$, let Z_n be a uniform element in $B_G^S(n)$, and let

$$a_n = \mathbb{E}[D_G(Z_n)] = \frac{1}{|B_G^S(n)|} \sum_{g \in B_G^S(n)} D_G(g).$$

We could then say that the average depth of an element of G is $\lim_n a_n$, provided that this limit exists.

- Alternatively, one may define a random walk $(X_n)_{n \in \mathbb{N}}$ on a Cayley graph $\text{Cay}(G, S)$ of G , starting from the neutral element e , and set

$$b_n = \mathbb{E}[D_G(X_n)].$$

Then define the average depth as $\lim_n b_n$, again under the condition that the limit exist.

One expects that for compliant groups, both limits exist and are equal. We will focus on the second situation; but will make reference to the first to stress similarities. The exact definition of $(X_n)_{n \geq 0}$ as well as a discussion on random walks on groups is deferred to §2.2. We mention here only that $(X_n)_{n \geq 0}$ is a *lazy* random walk, that is a process that at every step remains unchanged with probability 1/2 and takes a step otherwise.

It is a general fact that for an integer valued non-negative random variable Y ,

$$\mathbb{E}(Y) = \sum_{k \geq 0} \mathbb{P}(Y > k).$$

It may therefore be interesting to study $\mathbb{P}[D_G(X_n) > k]$ for $(X_n)_{n \geq 0}$ as above.

For $k \geq 2$, let Λ_k be the intersection of all normal subgroups of G of index at most k . (For $k = 0, 1$, set $\Lambda_k = G$). Then, for $g \in G \setminus \{e\}$, $D_G(g) > k$ if and only if $g \in \Lambda_k$. Thus

$$(1) \quad \mathbb{E}[D_G(X_n)] = \sum_{k \geq 0} \mathbb{P}(X_n \in \Lambda_k \setminus \{e\}).$$

As we will see in Corollary 2.5,

$$\mathbb{P}(X_n \in \Lambda_k \setminus \{e\}) \xrightarrow{n \rightarrow \infty} \frac{1}{[G : \Lambda_k]}.$$

One may therefore expect that

$$(2) \quad \mathbb{E}[D_G(X_n)] \xrightarrow{n \rightarrow \infty} 2 + \sum_{k \geq 2} \frac{1}{[G : \Lambda_k]}.$$

For this reason, we call the right hand side of the above the *presumed limit*. However, the convergence above is far from obvious. The main goal of this paper is to provide criteria for G under which (2) holds. We will also provide an example where this is not valid.

The finiteness of $\sum_{k \geq 2} \frac{1}{[G : \Lambda_k]}$ in (2) depends on the group G and is related to the *intersection growth* $i_G(k) = [G : \Lambda_k]$ of G . It follows from Lemma 3.1 and Proposition 3.2 that $\sum_{k \geq 2} \frac{1}{[G : \Lambda_k]} < \infty$ for finitely generated linear groups. Moreover, finitely generated nilpotent groups enjoy this property as it is a classical result that they are linear (see Segal [Seg83, Chapter 5, §B, Theorem 2] or Hall [Hal69, p. 56, Theorem 7.5]).

Our two results ensuring (2) are the following.

Theorem 1.1. *Let G be a linear group with Kazhdan Property (T). Then*

$$\lim_{n \rightarrow \infty} \mathbb{E}[D_G(X_n)] = 2 + \sum_{k \geq 2} \frac{1}{[G : \Lambda_k]} < \infty$$

for any finite generating set S of G . In particular, $\lim_{n \rightarrow \infty} \mathbb{E}[D_G(X_n)]$ is finite for the special linear groups $SL_k(\mathbb{Z})$ with $k \geq 3$.

Theorem 1.2. *Let G be a finitely generated nilpotent group. Then*

$$\lim_{n \rightarrow \infty} \mathbb{E}[D_G(X_n)] = 2 + \sum_{k \geq 2} \frac{1}{[G : \Lambda_k]} < \infty$$

for any finite generating set S of G .

In §4.1, it will also be shown that the convergence holds whenever the presumed limit is infinite. Considering these examples, one may think that the convergence in (2) is always valid. However, in Proposition 4.9, we exhibit a 3-generated group for which the presumed limit is finite but $\lim_{n \rightarrow \infty} \mathbb{E}[D_G(X_n)] = \infty$.

Henceforth, when no ambiguity is possible, we drop the index G from the notation $D_G(\cdot)$.

2. PRELIMINARIES

2.1. Depth function and residual finiteness growth. This short subsection includes some results on the residual finiteness growth function that we will use in the sequel.

Theorem 2.1 ([BR10]). *Let G be a finitely generated nilpotent group with a generating set S . Then*

$$F_G(n) \leq C \log(n)^{h(G)}, \quad \forall n \geq 2,$$

where $h(G)$ is the Hirsch length of G and $C = C(G, S)$ is a constant independent of n .

The Prime Number Theorem and Hall's Embedding Theorem play key roles in the proof of Theorem 2.1. In [BRM15], the following is proved using Gauss's Counting Lemma to help quantify Mal'cev's classical proof of residual finiteness of finitely generated linear groups.

Theorem 2.2 ([BRM15]). *Let K be a field. Let G be a finitely generated subgroup of $GL(m, K)$ with a generating set S . Then there exists a positive integer b such that*

$$F_G(n) \leq Cn^b, \quad \forall n \geq 1,$$

where $C=C(G,S)$ is a constant independent of n .

The above results bound from above the residual finiteness growth. Conversely, the following states that there exist groups with arbitrary large residual finiteness growth.

Theorem 2.3 ([BRS16]). *For any function $f: \mathbb{N} \rightarrow \mathbb{N}$, there exists a residually finite group G and a two element generating set S for G , such that $F_G^S(n) \geq f(n)$ for all $n \geq 8$.*

The proof of Theorem 2.3 in [BRS16] involves an explicit construction of a finitely generated group embedded in an infinite product of finite simple groups.

2.2. Random walks on groups. Let G be a finitely generated group with a finite symmetric generating set $S = \{s_1, \dots, s_k\}$, *i.e.* such that $S^{-1} = S$. A random walk $(X_n)_{n \geq 0}$ on G is a Markov chain with state space G and such that $X_0 = e_G$ and $X_{n+1} = X_n \cdot Y_n$ for $n \geq 0$ where Y_0, Y_1, \dots are independent and uniform in $\{s_1, \dots, s_k\}$.

If G is finite with $|G| = m$, one may consider the transition matrix P of the random walk $(X_n)_{n \geq 0}$ on G defined by

$$P(x, y) = \frac{1}{|S|} \sum_{s \in S} \mathbf{1}_{\{y=xs\}},$$

where $\mathbf{1}_{\{y=xs\}} = 1$ if $y = xs$ and 0 otherwise. It is simply the adjacency matrix of the Cayley graph $\text{Cay}(G, S)$, normalized by $\frac{1}{|S|}$. The generating set is considered symmetric so as to have an un-oriented Cayley graph, or equivalently to have P symmetric.

Let $1 = \lambda_1 \geq \dots \geq \lambda_m \geq -1$ be the eigenvalues of P and x_1, \dots, x_m be a basis of orthonormal eigenvectors of P (such a basis necessarily exists since P is real and symmetric). Let σ be an initial distribution on G seen as a probability vector of dimension m , and let $p_u = (\frac{1}{m}, \dots, \frac{1}{m})$ be the uniform distribution on G . It is well-known that the distribution of such random walk converges to the uniform distribution whenever the graph is assumed to not be bipartite. For a general convergence statement one considers a *lazy* random walk instead; that is a walk with transition matrix $L = \frac{1}{2}I + \frac{1}{2}P$. The lazy random walk at time n takes a step of the original random walk with probability $\frac{1}{2}$ and stays at the current vertex with probability $\frac{1}{2}$. Notice that the eigenvectors of L are x_1, \dots, x_m and the corresponding eigenvalues are all non-negative $\mu_1 = \frac{1}{2} + \frac{1}{2}\lambda_1 = 1 > \mu_2 = \frac{1}{2} + \frac{1}{2}\lambda_2 \geq \dots \geq \mu_m = \frac{1}{2} + \frac{1}{2}\lambda_m \geq 0$.

Lemma 2.4. *Let G be a finite group with a finite symmetric generating set S . With the above notation $\|\sigma L^n - p_u\|_2 \leq \mu_2^n$. In particular, $|\sigma L^n(g) - \frac{1}{m}| \leq \mu_2^n$ for every $g \in G$.*

Proof. For $n \geq 1$, the matrix $\sigma \cdot L^n$ is a probability distribution and it represents the distribution of the n -th step of the lazy random walk on G that starts at a random vertex selected according to σ .

We write $\sigma = \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_m x_m$, with $\alpha_1, \dots, \alpha_m \in \mathbb{R}$. Since x_1, \dots, x_m are eigenvectors, we have $\sigma L^n = \alpha_1 \mu_1^n x_1 + \alpha_2 \mu_2^n x_2 + \dots + \alpha_m \mu_m^n x_m$. Notice that $\mu_1 = 1$, $x_1 = \frac{1}{\sqrt{m}}(1, \dots, 1)$ and $\alpha_1 = \sigma \cdot x_1^T = \frac{1}{\sqrt{m}}$ which implies that $\alpha_1 x_1 = (\frac{1}{m}, \dots, \frac{1}{m}) = p_u$. We deduce that

$$\|\sigma L^n - p_u\|_2 = \|\alpha_2 \mu_2^n x_2 + \dots + \alpha_m \mu_m^n x_m\|_2 \leq \max_{i=2, \dots, m} |\mu_i|^n \cdot \sqrt{\alpha_2^2 + \dots + \alpha_m^2} \leq \mu_2^n.$$

□

Corollary 2.5. *Let G be a finitely generated infinite group with a finite symmetric generating set S and let N be a normal subgroup of G of finite index. Consider the lazy random walk $(X_n)_{n \geq 0}$ on $\text{Cay}(G, S)$. Then*

$$(3) \quad \left| \mathbb{P}(X_n \in N) - \frac{1}{|G : N|} \right| \leq \mu^n,$$

where μ is the second largest eigenvalue of the transition matrix of \tilde{X}_n , the lazy random walk on $\text{Cay}(G/N, S)$ induced by $(X_n)_{n \geq 0}$. Moreover,

$$(4) \quad \mathbb{P}(X_n \in N \setminus \{e\}) \xrightarrow{n \rightarrow \infty} \frac{1}{|G : N|}.$$

Proof. To prove (3) it suffices to observe that $\mathbb{P}(X_n \in N) = \mathbb{P}(\tilde{X}_n = e_N)$ and conclude by Lemma 2.4. Let us now show (4).

It is a standard fact (see for instance [VSCC93, Thm. VI.3.3, Thm. VI.5.1]) that, since G is infinite,

$$\mathbb{P}(X_n = e) \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Moreover, as discussed above, the eigenvalue μ appearing in (3) is strictly smaller than 1. These two facts, together with (3), imply (4). \square

In the rest of the paper, we will always consider lazy random walks as described above. Straightforward generalisations are possible, such as to random walks with non-uniform symmetric transition probabilities – that is, walks taking steps according to a finitely supported, symmetric probability on G , with e having a positive probability (which is to say that the walk has, at any given step, a positive probability of staying at the same place). Certain transition probabilities with infinite support (but finite first moment) may also be treated, but small complications arise in specific parts of the proof. For the sake of legibility, we limit ourselves to the simple framework of uniform probabilities on symmetric generating sets.

2.3. Asymptotic density. For a given group G generated by a finite set S , consider the so-called *asymptotic density* (as defined in [BV02]) of a subset X in G defined as follows

$$(5) \quad \rho_S(X) = \limsup_{n \rightarrow \infty} \frac{|X \cap B_G^S(n)|}{|B_G^S(n)|}.$$

If G satisfies

$$(6) \quad \lim_{n \rightarrow \infty} \frac{|B_G^S(n+1)|}{|B_G^S(n)|} = 1$$

then by [BV02] ρ_S is left- and right-invariant; and in particular, $\rho(H) = \frac{1}{|G:H|}$ for a finite index subgroup H . Moreover, under this condition, the limsup in (5) is actually a limit.

Condition (6) holds for groups of polynomial growth (as a consequence of [Pan08]). Thus, for all $k \geq 2$, recalling the random variables X_n and Z_n defined in the introduction,

$$\lim_{n \rightarrow \infty} \mathbb{P}(X_n \in \Lambda_k \setminus \{e\}) = \lim_{n \rightarrow \infty} \mathbb{P}(Z_n \in \Lambda_k \setminus \{e\}) = \frac{1}{|G : \Lambda_k|}.$$

For groups with exponential growth however condition (6) fails, and the second limit in the above display does not necessarily exist. We give next an example for $G = F_{\{a,b\}}$, the free group generated by two elements $\{a, b\}$ and for a normal subgroup $N \triangleleft F_{\{a,b\}}$. Take $S = \{a, b, a^{-1}, b^{-1}\}$. For $g \in F_{\{a,b\}}$, let $\|g\|$ be the word-length of g , that is, the graph distance from g to e in $\text{Cay}(F_{\{a,b\}}, S)$. Set

$$N = \{g \in F_{\{a,b\}} : \|g\| \in 2\mathbb{N}\}.$$

It is straightforward to check that N is a normal subgroup of $F_{\{a,b\}}$ of index 2. However,

$$\frac{|N \cap B_{F_{\{a,b\}}, S}(n)|}{|B_{F_{\{a,b\}}, S}(n)|} = \begin{cases} \frac{3^{n+1} - 1}{4 \cdot 3^n - 2}, & \text{for } n \text{ even,} \\ \frac{3^n - 1}{4 \cdot 3^n - 2}, & \text{for } n \text{ odd.} \end{cases}$$

It is immediate from the above that $\mathbb{P}(Z_n \in N)$ does not converge when $n \rightarrow \infty$.

The above example, together with Corollary 2.5, explains the choice of the random walk $(X_n)_{n \geq 0}$ rather than of the uniform variables $(Z_n)_{n \geq 0}$ on $B_G^S(n)$.

Another reason for this choice relates to sampling. Suppose we have sampled an instance of the variable Z_n for some $n \geq 0$. In order to then obtain a sample of Z_{n+1} , one needs to restart the relatively costly process of sampling a uniform point in $B_G^S(n+1)$. For the random walk however, if X_n is simulated for some $n \geq 0$, X_{n+1} is easily obtained by multiplying X_n with a random element in S . This makes the sampling of a sequence (X_1, X_2, \dots) much easier than that of a sequence (Z_1, Z_2, \dots) .

3. RESIDUAL AVERAGE

We mentioned in the introduction that our goal is to compute the ‘‘average’’ depth of an element in G . In addition to the two methods proposed above, that is taking the limit of $\mathbb{E}[D_G(X_n)]$ or $\mathbb{E}[D_G(Z_n)]$, one may compactify G so that it has a Haar probability measure and take the average depth with respect to it. The natural way to render G compact is by considering its *profinite completion*, which we will denote by \widehat{G} . It is a compact group, with a unique uniform Haar measure which we denote by μ . The depth function D_G may be extended by continuity to \widehat{G} , and we call $D_{\widehat{G}}$ this extension. Then the *residual average* of G , denoted by $\text{Ave}(G)$, is

$$\text{Ave}(G) := \int_{\widehat{G}} D_{\widehat{G}} d\mu.$$

For details of the profinite completion construction see [Wil98]. For further details of the residual average construction see [BRM10].

Lemma 3.1. *For any linear group G ,*

$$(7) \quad \text{Ave}(G) = 2 + \sum_{k=2}^{\infty} \frac{1}{|G : \Lambda_k|}.$$

Note that $\text{Ave}(G)$ in (7) is equal to the limit in (2).

Proof. Recall the fact that we have conveniently defined $\Lambda_0 = \Lambda_1 = G$ and therefore that $\mu(\Lambda_0) = \mu(\Lambda_1) = 1$. The residual average is then

$$\begin{aligned} \text{Ave}(G) &= \sum_{k=2}^{\infty} k \cdot [\mu(\Lambda_{k-1}) - \mu(\Lambda_k)] = \sum_{k \geq 2} \sum_{\ell=0}^{k-1} [\mu(\Lambda_{k-1}) - \mu(\Lambda_k)] \\ &= \sum_{\ell \geq 0} \sum_{k > \ell} [\mu(\Lambda_{k-1}) - \mu(\Lambda_k)] \\ &= \sum_{\ell \geq 0} \mu(\Lambda_{\ell}) \end{aligned}$$

We are authorized to change the order of summation in the third equality, since all the terms in the sum are non-negative. In the last equality, we have used the telescoping sum and the fact that $\mu(\Lambda_k) \rightarrow 0$ as $k \rightarrow \infty$. The latter convergence is due to G being residually finite.

The first two terms in the last sum above are equal to 1; for $\ell \geq 2$, $\mu(\Lambda_{\ell}) = \frac{1}{|G : \Lambda_{\ell}|}$. The lemma follows immediately. \square

The following theorem, taken from [BRM10], will be necessary when proving Theorem 1.1.

Theorem 3.2 (Theorem 1.4 [BRM10]). *Let Γ be any finitely generated linear group. Then the residual average of Γ is finite.*

For completeness, we give a proof of the above. The present proof is based on the one in the original paper, with some adjustments meant to render certain points clearer. The main difference with the original proof is that here we focus on the connection to intersection growth. One has to be especially careful in proving this result, as the residual finiteness growth may vary when passing to subgroups of finite index (see [BRK12, Example 2.5]).

Proof. We follow the proof [BRM10, Theorem 1.4], with some changes and expansions. According to [BRM10, Proposition 5.2], there exists an infinite representation

$$\rho : \Gamma \rightarrow GL(n, K)$$

for some n and K/\mathbb{Q} finite. By [BRM10, Lemma 2.5], it suffices to show that the normal residual average of $\rho(\Gamma)$ is finite. Set $\Lambda = \rho(\Gamma)$ and set S to be the coefficient ring of Λ .

For each $\delta > 0$, from the proof of [BRM10, Proposition 5.1], there exists a normal residual system \mathfrak{F}_δ on Λ given by $\Delta_j = \Lambda \cap \ker r_j$, where

$$r_j : GL(n, S) \rightarrow GL(n, S/\mathfrak{p}_{S,j}^{k_j}),$$

and $[\Gamma : \Delta_j] \leq [\Gamma : \Delta_{j+1}] \leq [\Gamma : \Delta_j]^{1+\delta}$. In addition, we have

$$|r_j(\Lambda)| = O_j p_j^{\ell_j}$$

where

$$(8) \quad 1 \leq O_j < p_j^{n^2}.$$

We also have for constants $N > (n^2)!$ and $C > 4$ that

$$(9) \quad \ell_j > N + C j n^2$$

and

$$\ell_j + C n^2 < \ell_{j+1} \leq \ell_j + (C + 1)n^2.$$

For each $i < j$, we claim that the largest power of p_j that divides $[\Lambda : \Delta_i]$ is $p_j^{n^2}$. To see this claim, note that if p_j^m divides $O_i p_i^{\ell_i}$, since p_i, p_j are distinct primes, p_j^m must divide O_i , however $O_i < p_i^{n^2}$ and $p_i < p_j$. The claim follows. Since

$$[\Delta_i : \Delta_i \cap \Delta_j] = [\Lambda : \Delta_j][\Delta_j : \Delta_i \cap \Delta_j]/[\Lambda : \Delta_i] = O_j p_j^{\ell_j} [\Delta_j : \Delta_i \cap \Delta_j]/[\Lambda : \Delta_i],$$

the aforementioned claim implies that

$$[\Delta_i : \Delta_i \cap \Delta_j] \geq p_j^{\ell_j - n^2}.$$

Set $\Lambda_i := \bigcap_{n=1}^i \Delta_n$, then

$$[\Lambda_{j-1} : \Lambda_j] = O_j p_j^{\ell_j} [\Delta_j : \Lambda_j]/[\Lambda : \Lambda_{j-1}],$$

and $[\Lambda : \Lambda_{j-1}]$ divides $[\Lambda : \Delta_1] \cdots [\Lambda : \Delta_{j-1}]$ for which $p_j^{(j-1)n^2}$ is the largest power of p_j that appears as a factor. Hence, we obtain

$$[\Lambda_{j-1} : \Lambda_j] \geq p_j^{\ell_j - (j-1)n^2}.$$

From this, we obtain the important inequality

$$(10) \quad [\Lambda : \Lambda_k] \geq \prod_{j=1}^k p_j^{\ell_j - (j-1)n^2}.$$

To employ 10, we need a comparison function. This is where we deviate from the proof in [BRM10]. Define, for $g \in \Lambda \setminus \{1\}$,

$$M(g) = \min\{[\Lambda : \Delta_i] : g \notin \Delta_i\}.$$

Let \hat{M} be the unique continuous extension of M to $\hat{\Gamma}$. Then clearly, $D_\Gamma(g) \leq M(g)$, and so

$$\int \hat{D}_\Gamma(g) d\mu \leq \int \hat{M}(g) d\mu.$$

Moreover, applying definitions and doing some calculus yields:

$$\begin{aligned} \int \hat{M}(g) d\mu &= \sum_{k=1}^{\infty} [\Lambda : \Delta_k] \mu(\Lambda_{k-1} \setminus \Lambda_k) \\ &= \sum_{k=1}^{\infty} [\Lambda : \Delta_k] \left(\frac{1}{[\Lambda : \Lambda_{k-1}]} - \frac{1}{[\Lambda : \Lambda_k]} \right) \\ &= \sum_{k=1}^{\infty} \frac{[\Lambda : \Delta_{k+1}] - [\Lambda : \Delta_k]}{[\Lambda : \Lambda_k]} \\ &< \sum_{k=1}^{\infty} \frac{[\Lambda : \Delta_k]^{1+\delta}}{[\Lambda : \Lambda_k]}. \end{aligned}$$

The last inequality follows from the conclusion of [BRM10, Proposition 5.1] that $[\Lambda : \Delta_{k+1}] \leq [\Lambda : \Delta_k]^{1+\delta}$. Applying (8) and (10) while plugging in the value for $[\Lambda : \Lambda_k]$ yields

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{[\Lambda : \Delta_k]^{1+\delta}}{[\Lambda : \Lambda_k]} &\leq \sum_{k=1}^{\infty} \frac{O_k p_k^{(1+\delta)\ell_k}}{\prod_{j=1}^k p_j^{\ell_j - (j-1)n^2}} \\ &\leq \sum_{k=1}^{\infty} \frac{p_k^{n^2 + (1+\delta)\ell_k}}{\prod_{j=1}^k p_j^{\ell_j - (j-1)n^2}} \\ &= \sum_{k=1}^{\infty} \frac{p_k^{(2-k)n^2 + (\delta)\ell_k}}{\prod_{j=1}^{k-1} p_j^{\ell_j - (j-1)n^2}}. \end{aligned}$$

We apply the ratio test to this series:

$$\frac{k\text{-th term}}{(k+1)\text{-th term}} = \frac{\frac{p_k^{(2-k)n^2 + (\delta)\ell_k}}{\prod_{j=1}^{k-1} p_j^{\ell_j - (j-1)n^2}}}{\frac{p_k^{(1-k)n^2 + (\delta)\ell_{k+1}}}{\prod_{j=1}^k p_j^{\ell_j - (j-1)n^2}}} = \frac{p_k^{n^2 + \delta(\ell_k - \ell_{k+1})}}{p_k^{\ell_k - (k-1)n^2}} = p_k^{kn^2 + (\delta-1)\ell_k - \delta\ell_{k+1}}$$

By choice of ℓ_k , if $\delta > 0$ to be sufficiently small we have for large enough k and Inequality (9),

$$kn^2 + (\delta-1)\ell_k - \delta\ell_{k+1} < kn^2 + (\delta-1)\ell_k < 0.$$

Hence, as p_k is an increasing sequence of integers, the ratio test implies that the series above converges, and $\int \hat{M}(g) d\mu$ is finite. We conclude that the normal residual average of Λ is finite, as desired. \square

4. EXPECTED DEPTH OF RANDOM WALKS ON GROUPS

Fix for the whole section a finitely generated residually finite group G and a finite symmetric generating set S . Consider the simple lazy random walk $(X_n)_{n \geq 0}$ on the Cayley graph $\text{Cay}(G, S)$, as defined in §2.2. Recall that we are interested in

$$\mathbb{E}[D_G(X_n)] = \sum_{k \geq 2} k \mathbb{P}[D(X_n) = k] = \sum_{k \geq 0} \mathbb{P}[D(X_n) > k] = \sum_{k \geq 0} \mathbb{P}(X_n \in \Lambda_k \setminus \{e\}).$$

The second equality is obtained through the same double-sum argument as in the proof of Lemma 3.1.

4.1. First estimates.

Proposition 4.1. *We have*

$$\liminf_{n \rightarrow \infty} \mathbb{E}[D(X_n)] \geq 2 + \sum_{k \geq 2} \frac{1}{|G : \Lambda_k|}.$$

Proof. Recall the expression (1) for $E[D(X_n)]$:

$$\mathbb{E}[D(X_n)] = \sum_{k \geq 0} \mathbb{P}(X_n \in \Lambda_k \setminus \{e\}).$$

Also recall from Corollary 2.5 that

$$\mathbb{P}(X_n \in \Lambda_k \setminus \{e\}) \xrightarrow{n \rightarrow \infty} \begin{cases} \frac{1}{|G : \Lambda_k|} & \text{if } k \geq 2, \\ 1 & \text{if } k = 0, 1. \end{cases}$$

The result follows from Fatou's lemma. □

Corollary 4.2. *Suppose G is such that $\sum_{k \geq 2} \frac{1}{|G : \Lambda_k|}$ diverges. Then $\lim_{n \rightarrow \infty} \mathbb{E}(D(X_n)) = \infty$.*

Proof. This is a direct consequence of Proposition 4.1. □

Proposition 4.3. *Suppose there exists a sequence of positive numbers $\{p_k\}_{k \geq 2}$ with*

- $\sum_{k \geq 2} p_k < \infty$;
- $\mathbb{P}(X_n \in \Lambda_k \setminus \{e\}) \leq p_k$ for all $n \geq 1$ and $k \geq 2$.

Then

$$\lim_{n \rightarrow \infty} \mathbb{E}[D(X_n)] = 2 + \sum_{k \geq 2} \frac{1}{|G : \Lambda_k|} < \infty.$$

Proof. Fix a sequence $(p_k)_{k \geq 2}$ as above, and set $p_0 = p_1 = 1$. Then the convergence of $\mathbb{P}(X_n \in \Lambda_k \setminus \{e\})$ to $\frac{1}{|G : \Lambda_k|}$ is dominated by p_k . Since p_k is summable, the dominated convergence theorem implies the desired result. □

Below, when applying Proposition 4.3, we will do so using the sequence

$$p_k = \sup_{n \geq 0} \mathbb{P}(X_n \in \Lambda_k \setminus \{e\}), \quad \text{for } k \geq 2.$$

This sequence obviously satisfies the domination criterion; one needs to show it is summable in order to apply the proposition.

4.2. Sufficient condition using spectral properties. When writing

$$\mathbb{E}[D(X_n)] = \sum_{k \geq 0} \mathbb{P}(X_n \in \Lambda_k \setminus \{e\}),$$

all terms with $k \geq F_G^S(n)$ are null, because $X_n \in B_G^S(n)$ and $\Lambda_{F_G^S(n)} \cap B_G^S(n) = \{e\}$, by definition of $F_G^S(n)$. Thus, we may only sum up to $F_G^S(n) - 1$ in the formula above.

Theorem 4.4.

$$\left| \mathbb{E}[D(X_n)] - 2 - \sum_{k=2}^{F_G^S(n)-1} \frac{1}{|G : \Lambda_k|} \right| \leq 2F_G^S(n)\mu_{F_G^S(n)}^n + c,$$

where $c = c(G)$ is a constant and $\mu_{F_G^S(n)}$ is the second largest eigenvalue of the transition matrix of the induced lazy random walk \tilde{X}_n on $\text{Cay}(G/\Lambda_{F_G^S(n)}, S)$.

Proof of Theorem 4.4. We have

$$\mathbb{E}[D(X_n)] = \sum_{k=0}^{F_G^S(n)-1} \mathbb{P}(X_n \in \Lambda_k \setminus \{e\}) = 2 + \sum_{k \geq 2}^{F_G^S(n)-1} \mathbb{P}(X_n \in \Lambda_k) - F_G^S(n)\mathbb{P}(X_n = e).$$

Notice that $\mathbb{P}(X_n = e)$ may be rewritten $\mathbb{P}(X_n \in \Lambda_{F_G^S(n)})$. Also recall from Corollary 2.5 that, for any $k \geq 2$, $|\mathbb{P}(X_n \in \Lambda_k) - \frac{1}{|G:\Lambda_k|}| \leq \mu_k^n$, where μ_k is the second largest eigenvalue of the transition matrix of the induced lazy random walk on $\text{Cay}(G/\Lambda_k, S)$. Thus we obtain

$$\left| \mathbb{E}(D(X_n)) - 2 - \sum_{k=2}^{F_G^S(n)-1} \frac{1}{|G : \Lambda_k|} \right| \leq 1 + F_G^S(n)\mu_{F_G^S(n)}^n + \sum_{k=2}^{F_G^S(n)-1} \mu_k^n$$

Moreover the graph morphism

$$\text{Cay}(G/\Lambda_k) \rightarrow \text{Cay}(G/\Lambda_{k-1})$$

is a covering. It follows that the respective second largest eigenvalues satisfy $1 > \mu_k \geq \mu_{k-1}$ and thus

$$F_G^S(n)\mu_{F_G^S(n)}^n + \sum_{k=2}^{F_G^S(n)-1} \mu_k^n \leq 2F_G^S(n)\mu_{F_G^S(n)}^n.$$

□

Proof of Theorem 1.1. The group G has Property (T), thus there exists a constant $0 < \theta < 1$ such that for any normal finite index subgroup $N \triangleleft G$ the second largest eigenvalue of the Cayley graph of G/N is bounded above by $\theta < 1$. (The exact value of θ does depend on the generating set S of G .)

We use Theorem 4.4 to conclude: by [BR10] we have, for some $b \in \mathbb{N}$,

$$\lim_{n \rightarrow \infty} F_G^S(n)\mu_{F_G^S(n)}^n \leq \lim_{n \rightarrow \infty} Cn^b \cdot \theta^n = 0.$$

Moreover, $\sum_{k \geq 2} \frac{1}{|G:\Lambda_k|}$ is finite by Lemma 3.1 and Proposition 3.2. □

4.3. Sufficient condition: abelian groups.

Lemma 4.5. *Let $(X_n)_{n \geq 0}$ be a lazy random walk on \mathbb{Z} (that is on a Cayley graph of \mathbb{Z} , as in §2.2). Then there exists a constant $C > 0$ such that, for all $m \geq 1$*

$$(11) \quad \sup_{n \geq 0} \mathbb{P}(X_n \in m\mathbb{Z} \setminus \{0\}) \leq \frac{C}{\sqrt{m}}.$$

In particular, there exists $c > 0$ such that, for $k \geq 2$,

$$(12) \quad \sup_{n \geq 0} \mathbb{P}(X_n \in \Lambda_k(\mathbb{Z}) \setminus \{0\}) \leq e^{-ck}.$$

Proof. We start with the proof of (11). Let $c_0 > 0$ be such that $|X_1| < \frac{1}{2c_0}$ almost surely. Then, for $n \leq c_0m$, $\mathbb{P}(X_n \in m\mathbb{Z} \setminus \{0\}) = 0$. For $n \geq c_0m$, write

$$(13) \quad \mathbb{P}(X_n \in m\mathbb{Z} \setminus \{0\}) = \sum_{\ell \in \mathbb{Z}} \mathbb{P}(X_n \in m\mathbb{Z} \setminus \{0\} | X_{n-c_0m} = \ell) \mathbb{P}(X_{n-c_0m} = \ell).$$

Now notice that, due to the choice of c_0 , $|X_n - X_{n-c_0m}| < m/2$ almost surely. However, for any fixed $\ell \in \mathbb{Z}$, there exists at most one element $m(\ell) \in m\mathbb{Z}$ with $|\ell - m(\ell)| < m/2$. If no such element exists, choose $m(\ell) \in m\mathbb{Z}$ arbitrarily. Thus

$$\begin{aligned} \mathbb{P}(X_n \in m\mathbb{Z} \setminus \{0\} | X_{n-c_0m} = \ell) &\leq \mathbb{P}(X_n = m(\ell) | X_{n-c_0m} = \ell) \\ &= \mathbb{P}(X_{c_0m} = m(\ell) - \ell) \leq \frac{C}{\sqrt{c_0m}}, \end{aligned}$$

where the last inequality is due to [VSCC93, Thm. VI.5.1] and $C > 0$ is some fixed constant depending only on the transition probability of the random walk. When injecting the above in (13), we find

$$\mathbb{P}(X_n \in m\mathbb{Z} \setminus \{0\}) \leq \sum_{\ell \in \mathbb{Z}} \frac{C}{\sqrt{c_0m}} \mathbb{P}(X_{n-c_0m} = \ell) = \frac{C}{\sqrt{c_0m}}.$$

Since the right-hand side does not depend on n , this implies (11) with an adjusted value of C .

We move on to proving (12). The (normal) subgroups of \mathbb{Z} are of the form $k\mathbb{Z}$, with k being their index. Thus, for $k \geq 2$,

$$\Lambda_k(\mathbb{Z}) = m_k\mathbb{Z},$$

where m_k is the least common multiple of $1, \dots, k$ (see [IBM]). It follows from the Prime Number Theorem that there exists a constant $c > 0$ such that

$$m_k \geq \exp(ck), \quad \forall k \geq 2.$$

The above bound, together with (11), implies (12) with an adjusted value of c . \square

Corollary 4.6 (Expected depth for \mathbb{Z}). *Let $(X_n)_{n \geq 0}$ be a lazy random walk on \mathbb{Z} (that is, on a Cayley graph of \mathbb{Z} , as in §2.2). Then*

$$(14) \quad \mathbb{E}[D_{\mathbb{Z}}(X_n)] \xrightarrow{n \rightarrow \infty} 2 + \sum_{k \geq 2} \frac{1}{|\mathbb{Z} : \Lambda_k|} < \infty.$$

Proof. For $k \geq 2$, set $p_k = \sup_{n \geq 0} \mathbb{P}(X_n \in \Lambda_k(\mathbb{Z}) \setminus \{0\})$. By Lemma 4.5, $\sum_{k \geq 2} p_k < \infty$.

Moreover, the sequence $(p_k)_{k \geq 2}$ dominates the convergence of $\mathbb{P}(X_n \in \Lambda_k(\mathbb{Z}) \setminus \{0\})$, as required in Proposition 4.3. The conclusion follows. \square

Proposition 4.7. *Let G and H be two finitely generated residually finite groups. Let $(X_n)_{n \geq 0}$ be a random walk on a Cayley graph of $G \times H$, as in §2.2. Then*

$$\mathbb{P}[D_{G \times H}(X_n) > k] \leq \mathbb{P}[D_G(Y_n) > k] + \mathbb{P}[D_H(Z_n) > k] \quad \text{for all } k \geq 0,$$

where $(Y_n)_{n \geq 0}$ and $(Z_n)_{n \geq 0}$ are the random walks on G and H , respectively, induced by $(X_n)_{n \geq 0}$.

Proof. Recall that $\mathbb{E}[D_{G \times H}(X_n)] = \sum_{k \geq 0} \mathbb{P}[D_{G \times H}(X_n) > k]$. Notice that for $g = (g_1, g_2) \in G \times H$ we have estimates

$$D_{G \times H}(g) \leq D_G(g_1) \text{ if } g_1 \neq 0$$

$$D_{G \times H}(g) \leq D_G(g_2) \text{ if } g_2 \neq 0$$

Therefore $\mathbb{P}[D_{G \times H}(X_n) > k] \leq \mathbb{P}[D_G(Y_n) > k] + \mathbb{P}[D_H(Z_n) > k]$. \square

Corollary 4.8. *Let G be a finitely generated abelian group and let $(X_n)_{n \geq 0}$ be a lazy random walk on its Cayley graph, as in §2.2. Then*

$$\mathbb{E}[D_G(X_n)] \xrightarrow{n \rightarrow \infty} 2 + \sum_{k \geq 2} \frac{1}{|G : \Lambda_k|} < \infty,$$

for any finite generating set S of G .

Proof. Let G be a finitely generated abelian group. Then it may be written as $G = \mathbb{Z} \times \cdots \times \mathbb{Z} \times H$, where the product contains j copies of \mathbb{Z} and H is a finite abelian group. The depth of elements of H is bounded by $|H|$. By Proposition 4.7 and Lemma 4.5, there exists $c > 0$ such that

$$\mathbb{P}(X_n \in \Lambda_k(G) \setminus \{0\}) \leq j e^{-ck}, \quad \forall k > |H|.$$

We conclude using the same domination argument as in the proof of Corollary 4.6. \square

4.4. Nilpotent groups: proof of Theorem 1.2.

Torsion free case. Suppose first that G is a finitely generated and torsion free nilpotent group, different from \mathbb{Z} . Observe that G is a poly- C_∞ group, and in particular, $G = \mathbb{Z} \rtimes H$ where H is a non-trivial finitely generated nilpotent group. Let S be a system of generators of G and consider the lazy random walk on G with steps taken uniformly in S . We will write it in the product form $(X_n, Y_n)_{n \geq 0}$, where $X_n \in \mathbb{Z}$ and $Y_n \in H$, for all $n \geq 0$. Notice then that $(X_n)_{n \geq 0}$ is a lazy random walk on \mathbb{Z} , as treated in Lemma 4.5. This is not true on the second coordinate: $(Y_n)_{n \geq 0}$ is not a random walk on H , it is not even a Markov process.

For $k \geq 2$, let

$$(15) \quad p_k(G) = \sup_{n \geq 0} \mathbb{P}[D_G(X_n, Y_n) > k],$$

and recall from Proposition 4.3 (and the commentary below it) that our goal is to prove that $\sum_k p_k < \infty$.

One may easily check that, for any $m \in \mathbb{N}$, $m\mathbb{Z} \rtimes H$ is a normal subgroup of G . Thus, for all $x \in \mathbb{Z} \setminus \{0\}$ and $y \in H$,

$$D_G(x, y) \leq D_{\mathbb{Z}}(x).$$

We may therefore bound p_k by

$$\begin{aligned} p_k(G) &\leq \sup_{n \geq 0} \mathbb{P}[D_G(X_n) > k] + \mathbb{P}[X_n = 0 \text{ and } D_G(0, Y_n) > k] \\ &\leq p_k(\mathbb{Z}) + \sup_{n \geq 0} \mathbb{P}[X_n = 0 \text{ and } D_G(0, Y_n) > k]. \end{aligned}$$

In the above, $p_k(\mathbb{Z}) = \sup_{n \geq 0} \mathbb{P}[D_{\mathbb{Z}}(X_n) > k]$. We have shown in Lemma 4.5 that $\sum_k p_k(\mathbb{Z}) < \infty$, and we may focus on whether the second supremum is summable.

Fix $k \geq 2$. Since G is nilpotent, there exists $c > 0$ such that

$$F_G^S(n) \leq c(\log n)^{h(G)},$$

where $h(G)$ is the Hirsch length of G (see Theorem 2.1). This should be understood as follows. In order for an element $g \in G$ to have $D_G(g) \geq k$, it is necessary that $\|g\|_S \geq \exp(Ck^{1/h(G)})$, where $\|g\|_S$ denotes the length of g with respect to the generating set S and $C > 0$ is a constant independent of g .

In particular, we conclude that

$$\begin{aligned} \mathbb{P}[X_n = 0 \text{ and } D_G(0, Y_n) > k] &= 0 & \text{if } n < \exp(Ck^{1/h(G)}); \\ \mathbb{P}[X_n = 0 \text{ and } D_G(0, Y_n) > k] &\leq \mathbb{P}(X_n = 0) & \text{if } n \geq \exp(Ck^{1/h(G)}). \end{aligned}$$

In treating the second case, observe that, since $(X_n)_{n \geq 0}$ is a random walk on \mathbb{Z} ,

$$\mathbb{P}(X_n = 0) \leq c_0 n^{-1/2},$$

for some constant $c_0 > 0$ (see [VSCC93, Thm. VI.5.1]). Thus

$$\mathbb{P}[X_n = 0 \text{ and } D_G(0, Y_n) > k] \leq c_0 \exp\left(-\frac{C}{2}k^{1/h(G)}\right), \quad \forall n \in \mathbb{N}.$$

We conclude that

$$\sum_{k \geq 2} \sup_{n \geq 0} \mathbb{P}[X_n = 0 \text{ and } D_G(0, Y_n) > k] < \infty,$$

and therefore that $\sum_{k \geq 0} p_k(G) < \infty$.

General nilpotent case. Let now G be a finitely generated nilpotent group. Consider the set $T(G)$ of all torsion elements of G . Since G is nilpotent, the set $T(G)$ is a finite normal subgroup in G . Consider an epimorphism $\pi: G \rightarrow G/T(G)$. Denote by H the quotient $G/T(G)$ and notice that H is a finitely generated torsion-free nilpotent group.

Observe that for any non-trivial element in H detected by a normal subgroup in H of index k there exists a normal subgroup in G of index at most k that detects its preimage. In other words, for all $g \in G \setminus T(G)$,

$$D_G(g) \leq D_H(\pi(g)).$$

The random walk $(X_n)_{n \geq 0}$ on G induces a random walk $(\pi(X_n))_{n \geq 0}$ on H . Let $d = \max\{D_G(g), g \in T(G)\}$. Due to the observation above, for all $k > d$,

$$\mathbb{P}[D_G(X_n) \geq k] \leq \mathbb{P}[D_H(\pi(X_n)) \geq k].$$

By the previous case we deduce that $\sum_{k > d} \sup_{n \geq 0} \mathbb{P}[D_G(X_n) \geq k] < \infty$ and by Proposition 4.3 we conclude. \square

4.5. A counter example.

Proposition 4.9 (Groups with infinite expected depth). *There exists a finitely generated residually finite group G such that*

$$\lim_{n \rightarrow \infty} \mathbb{E}(D(X_n)) = \infty,$$

but for which the ‘‘presumed’’ limit $\sum_{k \geq 2} \frac{1}{|G:\Lambda_k|}$ is finite.

Proof. The existence of finitely generated residually finite groups with arbitrary large residual finiteness growth was shown in [BRS16].

Let H be a two-generated group (with generators a, b) such that, for any $n \geq 8$, there exists an element h_n in the ball of radius n of H with $D_H(h_n) \geq 24^n$. Let $(X_n)_{n \geq 0}$ be a lazy simple random walk on the Cayley graph of $G = H \times \mathbb{Z}$ with the natural choice of 3 generators (that is $(a, 0)$, $(b, 0)$ and $(e_H, 1)$, where a and b are the two generators of H mentioned above) and their inverses.

Then, for any $n \geq 1$, $\mathbb{P}[X_n = (h_n, 0)] \geq 12^{-n}$. Therefore

$$\mathbb{E}[D_G(X_n)] \geq \mathbb{P}[X_n = (h_n, 0)] \cdot D_G[(h_n, 0)] = \mathbb{P}[X_n = (h_n, 0)] \cdot D_H(h_n) \geq 2^n,$$

Hence the expectation of the depth of X_n tends to infinity.

Furthermore, observe that $\Lambda_k(G)$ is a subgroup of $H \times \Lambda_k(\mathbb{Z})$ and hence

$$|G : \Lambda_k(G)| \geq |\mathbb{Z} : \Lambda_k(\mathbb{Z})|.$$

It follows that $\sum_{k \geq 2} \frac{1}{|G : \Lambda_k(G)|} \leq \sum_{k \geq 2} \frac{1}{|\mathbb{Z} : \Lambda_k(\mathbb{Z})|} < \infty$. □

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