Convergence of linear barycentric rational interpolation for analytic functions

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Outline

1. Linear barycentric rational interpolation
2. Polynomial interpolation of analytic functions
3. Barycentric rational interpolation of analytic functions
Introduction and notation

Linear barycentric rational interpolation

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Convergence of LBRI for analytic functions
One-dimensional interpolation

Given:

\[ a \leq x_0 < x_1 < \ldots < x_n \leq b, \quad n + 1 \text{ distinct nodes and} \]
\[ f(x_0), f(x_1), \ldots, f(x_n), \quad \text{corresponding values}. \]

We study functions \( g \) from a finite-dimensional linear subspace of \( (C[a, b], \| \cdot \|_\infty) \) which interpolate \( f \) between the nodes,

\[ g(x_i) = f(x_i) = f_i, \quad i = 0, \ldots, n. \]
Construction presented by Floater and Hormann

- Given $n$, choose an integer $d \in \{0, 1, \ldots, n\}$, the “blending parameter”,
- for $i = 0, \ldots, n - d$, define $p_i(x)$, the polynomial of degree $\leq d$ interpolating $f_i, f_{i+1}, \ldots, f_{i+d}$.

The $d$-th interpolant of the family is a “blend” of the $p_i$,

$$r_n(x) = \frac{\sum_{i=0}^{n-d} \lambda_i(x) p_i(x)}{\sum_{i=0}^{n-d} \lambda_i(x)}, \quad \text{with} \quad \lambda_i(x) = \frac{(-1)^i}{(x - x_i) \ldots (x - x_{i+d})}.$$

Note that for $d = n$, $r_n$ simplifies to $p_n$. 
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\]

Note that for \( d = n \), \( r_n \) simplifies to \( p_n \).
For its evaluations, we write \( r_n \) in **barycentric form**

\[
r_n(x) = \frac{\sum_{i=0}^{n-d} \lambda_i(x) p_i(x)}{\sum_{i=0}^{n-d} \lambda_i(x)} = \frac{\sum_{i=0}^{n} \frac{w_i}{x - x_i} f_i}{\sum_{i=0}^{n} \frac{w_i}{x - x_i}}.
\]

For equispaced nodes, the weights \( w_i \) oscillate in sign with absolute values

\[
1, 1, \ldots, 1, 1, \quad d = 0, \quad \text{(Berrut)}
\]

\[
\frac{1}{2}, 1, 1, \ldots, 1, 1, \frac{1}{2}, \quad d = 1, \quad \text{(Berrut)}
\]

\[
\frac{1}{4}, \frac{3}{4}, 1, 1, \ldots, 1, 1, \frac{3}{4}, \frac{1}{4}, \quad d = 2, \quad \text{(Floater–Hormann)}
\]

\[
\frac{1}{8}, \frac{4}{8}, \frac{7}{8}, 1, 1, \ldots, 1, 1, \frac{7}{8}, \frac{4}{8}, \frac{1}{8}, \quad d = 3. \quad \text{(Floater–Hormann)}
\]
Properties of Floater–Hormann interpolation

Theorem (Floater–Hormann (2007))

Let $0 \leq d \leq n$ and $f \in C^{d+2}[a, b]$, $h = \max_{0 \leq i \leq n-1} (x_{i+1} - x_i)$, then

- the rational function $r_n$ has no poles in $\mathbb{R}$,
- if $d \geq 1$,

$$
\|f - r_n\|_\infty = \max_{a \leq x \leq b} |f(x) - r_n(x)| \leq Kh^{d+1},
$$

- if $d = 0$,

$$
\|f - r_n\|_\infty \leq K \beta h,
$$

where $\beta$ is a mesh ratio and $K$ is a constant, independent of $n$. 
The **Lebesgue constant** associated with linear barycentric interpolation,

\[
\Lambda_n = \max_{a \leq x \leq b} \left| \sum_{i=0}^{n} \frac{w_i}{x - x_i} \right|,
\]

is the condition number of the interpolation scheme.

**Theorem (Bos–De Marchi–Hormann–K. (2012))**

Let \(0 \leq d \leq n\) and the nodes \(x_i, i = 0, \ldots, n\), be equispaced. Then

\[
\frac{2^{d-2}}{d+1} \log \left( \frac{n}{d} - 1 \right) \leq \Lambda_n \leq 2^{d-1}(2 + \log n).
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Convergence/divergence of polynomial interpolation for analytic functions
Node density, node measure and logarithmic potential

Let the nodes \( x_i \) be distributed according to a node measure \( \mu \) with support \([a, b]\) and positive piecewise continuous node density

\[
\phi(x) = \frac{d\mu}{dx}(x) > 0, \quad \text{for } x \in [a, b].
\]

Associated with \( \mu \) is a logarithmic potential

\[
U^\mu(z) := - \int_a^b \log |z - x| \, d\mu(x) = - \int_a^b \phi(x) \log |z - x| \, dx.
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Let the nodes $x_i$ be distributed according to a **node measure** $\mu$ with support $[a, b]$ and positive piecewise continuous **node density**

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Associated with $\mu$ is a **logarithmic potential**

$$U^\mu(z) := -\int_a^b \log |z - x| \, d\mu(x) = -\int_a^b \phi(x) \log |z - x| \, dx.$$
Theorem

For a given node measure $\mu$ and the associated potential $U^\mu$, let $f$ be analytic inside $C_s$, the level line of $U^\mu$ which passes through a singularity $s$ of $f$. The polynomial interpolant $p_n$ of $f$ then converges to $f$ inside $C_s$, diverges outside and

$$\lim_{n \to \infty} |f(z) - p_n(z)|^{1/n} = \exp\left(U^\mu(s) - U^\mu(z)\right).$$
Figure: Level lines of $\exp(U^\mu(s) - \min_{-1 \leq x \leq 1} U^\mu(x))$ for polynomial interpolation with equispaced nodes (left) and Chebyshev points (right).
Convergence/divergence of linear barycentric rational interpolation for analytic functions
Variable blending parameter

Aim: We generalize the potential theory from polynomial interpolation to linear rational interpolation.

From now on, the blending parameter $d$ is a variable nonnegative integer $d(n)$ such that

$$d(n)/n \rightarrow C, \quad n \rightarrow \infty,$$

for $C \in (0, 1]$ fixed. In practice, e.g., $d(n) = \text{round}(Cn)$. 
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We suppose that \( j(n) \) is a sequence of indices such that \( j(n) \leq n - d(n) \) and \( x_{j(n)} \to \alpha \) for some \( \alpha \in [a, b] \).

One can show that the nodes \( x_{j(n)}, \ldots, x_{j(n)+d(n)} \) of \( p_{j(n)}(x) \), are then asymptotically contained in an interval \( [\alpha, \beta(C)] \), and distributed according to the node density \( \phi(x) \), restricted and normalized to that interval, and a node measure \( \nu_\alpha \).
Two important bounds

Recall that

\[ r_n(x) = \frac{\sum_{i=0}^{n-d} \lambda_i(x) p_i(x)}{\sum_{i=0}^{n-d} \lambda_i(x)}, \quad \lambda_i(x) = \frac{(-1)^i}{(x - x_i) \ldots (x - x_{i+d})}. \]

Lemma

For any \( C \in (0, 1], \ z \in \mathbb{C} \setminus [a, b] \) and \( x \in [a, b] \), we have

\[ \limsup_{n \to \infty} \left| \sum_{i=0}^{n-d(n)} \lambda_i(z) \right|^{1/(n+1)} \leq \max_{\alpha} \exp(CU^{\nu}\alpha(z)) \]

and

\[ \liminf_{n \to \infty} \left| \sum_{i=0}^{n-d(n)} \lambda_i(x) \right|^{1/(n+1)} \geq \max_{\alpha} \exp(CU^{\nu}\alpha(x)). \]
If $f$ is analytic inside a simple, closed and rectifiable curve $C$, which is contained in a closed simply connected region around the nodes, then the interpolation error may be written as

$$f(x) - r_n(x) = \frac{1}{2\pi i} \int_C \frac{f(s)}{x-s} \cdot \frac{\sum_{i=0}^{n-d} \lambda_i(s)}{\sum_{i=0}^{n-d} \lambda_i(x)} \, ds,$$

which is a Hermite-type error formula.

We define the new “potential function”

$$V^{C,\mu}(z) := \max_{\alpha} C U^{\nu\alpha}(z),$$

and the contours

$$C_R := \left\{ z \in \mathbb{C} : \frac{\exp(V^{C,\mu}(z))}{\min_{x \in [a,b]} \exp(V^{C,\mu}(x))} = R \right\}.$$
Hermite-type error formula

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Main convergence/divergence theorem

Theorem (Güttel–K. (2012))

Let $f$ be a function analytic in an open neighbourhood of $[a, b]$ and let $R > 0$ be the smallest number such that $f$ is analytic in the interior of $C_R$, then

$$\limsup_{n \to \infty} \|f - r_n\|_\infty^{1/n} \leq R.$$  

In the case of equispaced nodes, further simplifications occur in $\sum_{i=0}^{n-d} \lambda_i(z)$ due to symmetries.
Main convergence/divergence theorem

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Let $f$ be a function analytic in an open neighbourhood of $[a, b]$ and let $R > 0$ be the smallest number such that $f$ is analytic in the interior of $C_R$, then

$$\limsup_{n \to \infty} \frac{\|f - r_n\|}{n^{1/n}} \leq R.$$ 

In the case of equispaced nodes, further simplifications occur in $\sum_{i=0}^{n-d} \lambda_i(z)$ due to symmetries.
Effects of the symmetry in $\sum \lambda_i$

Figure: Levels of $|\sum_{i=0}^{n-d} \lambda_i(z)|^{1/(n+1)}$ with $d = 20$ for $n = 100$ equispaced nodes (left) and perturbed equispaced nodes (right) on a log_{10} scale.
**Convergence/divergence behaviour**

Figure: Level lines of convergence for barycentric rational interpolation for $C = 0.2$ with equispaced nodes (left) and (right) relative error curves for the interpolation of $1/(x - 0.3i)$ with both node sequences, asymptotic relative error bound and upper bound on $\epsilon \cdot \Lambda_n$. 
The choice of $C$ and $d$

The numerically observed error, which depends on $n$ and $C$, is a superposition of exponential convergence or divergence of the interpolant in exact arithmetic (e.a.) and the amplification of rounding errors, i.e., in equispaced nodes,

$$\text{observed error}(C, n) \approx \text{interpolation error in e.a.} + \text{imprecision} \times \text{condition number}$$

$$\approx DR^n + \varepsilon \|f\|_{\infty} \Lambda_n$$

$$=: \text{predicted error}(C, n).$$

Aim: given $n$ and the closest singularity $s$ of $f$, determine $C \in (0, 1]$ such that the predicted error is minimal.
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\[ \log(1.2 - x)(x + 2)^2 \text{ in } [-1, 1] \]

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**Figure:** Relative errors for \( f(x) = \log(1.2 - x)(x + 2)^2 \) with \( 2 \leq n \leq 250 \) equispaced nodes in \([-1, 1]\) with \( d = \text{round}(Cn) \), asymptotic convergence rates and nearly optimal values of \( C \) and \( d \).
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**Figure**: Relative errors for \( f(x) = \log(1.2 - x)(x + 2)^2 \) with \( 2 \leq n \leq 250 \) equispaced nodes in \([-1, 1]\) with \( d = \text{round}(Cn) \), asymptotic convergence rates and nearly optimal values of \( C \) and \( d \).
arctan(\(\pi x\)) in \([-1, 1]\)

Figure: Relative errors for \(f(x) = \arctan(\pi x)\) with \(2 \leq n \leq 250\) equispaced nodes in \([-1, 1]\) with \(d = \text{round}(Cn)\) and nearly optimal values of \(C\) and \(d\) (\(s = \pm i/\pi\)), and asymptotic convergence rates.
Figure: Relative errors for \( f(x) = \Gamma(x + 1.1) \) with \( 2 \leq n \leq 250 \) equispaced nodes in \([-1, 1]\) with \( d = \text{round}(Cn) \) and nearly optimal values of \( C \) and \( d \) (\( s = -1.1 \)), and asymptotic convergence rates.
Figure: Relative errors for \( f(x) = \sin(x) \) with \( 2 \leq n \leq 1000 \) equispaced nodes in \([-5, 5]\) with \( d = \text{round}(Cn) \) and nearly optimal values of \( C \) and \( d \) (taking \( s = 10 \)), and asymptotic convergence rates.
Thank you!