Linear rational finite differences and applications

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Outline

1. Barycentric rational interpolation
2. Differentiation of barycentric rational interpolants
3. Linear barycentric rational finite differences
4. Application: Extended Floater–Hormann interpolation
Introduction and notation

Barycentric rational interpolation

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One-dimensional interpolation

Given:

\[ a \leq x_0 < x_1 < \ldots < x_n \leq b, \quad n + 1 \text{ distinct nodes and} \]
\[ f(x_0), f(x_1), \ldots, f(x_n), \quad \text{corresponding values}. \]

Find a function \( g \) from a finite-dimensional linear subspace of \((C[a, b], \| \cdot \|_\infty)\) such that \( g \) interpolates \( f \) at the nodes,

\[ g(x_j) = f(x_j) = f_j, \quad j = 0, \ldots, n. \]
Choose an integer \( d \in \{0, 1, \ldots, n\} \),

for \( j = 0, \ldots, n - d \), define \( p_j(x) \), the polynomial of degree \( \leq d \) interpolating \( f_j, f_{j+1}, \ldots, f_{j+d} \).

The \( d \)-th interpolant of the family is given by

\[
    r_n[f](x) = \frac{\sum_{j=0}^{n-d} \lambda_j(x)p_j(x)}{\sum_{j=0}^{n-d} \lambda_j(x)}, \quad \text{where} \quad \lambda_j(x) = \frac{(-1)^j}{(x - x_j) \ldots (x - x_{j+d})}. 
\]
Construction presented by Floater and Hormann

- Choose an integer \( d \in \{0, 1, \ldots, n\} \),
- for \( j = 0, \ldots, n - d \), define \( p_j(x) \), the polynomial of degree \( \leq d \) interpolating \( f_j, f_{j+1}, \ldots, f_{j+d} \).

The \( d \)-th interpolant of the family is given by

\[
r_n[f](x) = \frac{1}{\sum_{j=0}^{n-d} \lambda_j(x)} \sum_{j=0}^{n-d} \lambda_j(x) p_j(x),
\]

where \( \lambda_j(x) = \frac{(-1)^j}{(x - x_j) \ldots (x - x_{j+d})} \).
Construction presented by Floater and Hormann

- Choose an integer $d \in \{0, 1, \ldots, n\}$,
- for $j = 0, \ldots, n - d$, define $p_j(x)$, the polynomial of degree $\leq d$ interpolating $f_j, f_{j+1}, \ldots, f_{j+d}$.

The $d$-th interpolant of the family is given by

$$r_n[f](x) = \frac{\sum_{j=0}^{n-d} \lambda_j(x)p_j(x)}{\sum_{j=0}^{n-d} \lambda_j(x)},$$

where

$$\lambda_j(x) = \frac{(-1)^j}{(x - x_j) \ldots (x - x_{j+d})}.$$
Lemma

Let \( \{x_j\}_{j=0}^n \) be a set of \( n + 1 \) distinct nodes, \( \{f_j\}_{j=0}^n \) corresponding real numbers and let \( \{v_j\}_{j=0}^n \) be any nonzero real numbers. Then

(a) the **barycentric** rational function

\[
R(x) = \frac{\sum_{j=0}^{n} \frac{v_j}{x - x_j} f_j}{\sum_{j=0}^{n} \frac{v_j}{x - x_j}},
\]

interpolates \( f_k \) at \( x_k \):
\[
\lim_{x \to x_k} R(x) = f_k;
\]

(b) conversely, every rational interpolant of the \( f_j \) may be written in barycentric form for some weights \( v_j \).
Write $r_n[f]$ in barycentric form

$$r_n[f](x) = \frac{\sum_{j=0}^{n} \frac{w_j}{x-x_j} f_j}{\sum_{j=0}^{n} \frac{w_j}{x-x_j}}.$$ 

For equispaced nodes, the weights $w_j$ oscillate in sign with absolute values

- $1, 1, \ldots, 1, 1$, \quad d = 0, \quad (Berrut)$
- $\frac{1}{2}, 1, 1, \ldots, 1, 1, \frac{1}{2}$, \quad d = 1, \quad (Berrut)
- $\frac{1}{4}, \frac{3}{4}, 1, 1, \ldots, 1, 1, \frac{3}{4}, \frac{1}{4}$, \quad d = 2, \quad (Floater–Hormann)
- $\frac{1}{8}, \frac{4}{8}, \frac{7}{8}, 1, 1, \ldots, 1, 1, \frac{7}{8}, \frac{4}{8}, \frac{1}{8}$, \quad d = 3, \quad (Floater–Hormann)
Theorem (Floater–Hormann (2007))

Let $0 \leq d \leq n$ and $f \in C^{d+2}[a, b]$, $h = \max_{0 \leq i \leq n-1} (x_{i+1} - x_i)$, then

- the rational function $r_n[f]$ has no poles in $\mathbb{R}$,
- if $d \geq 1$,

$$\|r_n[f] - f\| = \max_{a \leq x \leq b} |r_n[f](x) - f(x)| \leq Ch^{d+1},$$

- if $d = 0$,

$$\|r_n[f] - f\| \leq C\beta h,$$

where $\beta$ is a mesh ratio and $C$ is a generic constant, independent of $n$. 
Differentiation of barycentric rational interpolants
Proposition (Schneider-Werner (1986))

Let \( R \) be a rational function given in its barycentric form with non-vanishing weights. Assume that \( x \) is not a pole of \( R \). Then for \( k \geq 1 \)

\[
\frac{1}{k!} R^{(k)}(x) = \sum_{j=0}^{n} \frac{v_j}{x - x_j} R[(x)^k, x_j], \quad x \text{ not a node},
\]

\[
\frac{1}{k!} R^{(k)}(x_i) = -\left( \sum_{j=0}^{n} v_j R[(x_i)^k, x_j] \right) / v_i, \quad i = 0, \ldots, n.
\]
Define the matrix $D^{(1)}$ (Baltensperger–Berrut–Noël (1999)):

$$D_{ij}^{(1)} := \begin{cases} 
\frac{v_j}{v_i} \frac{1}{x_i - x_j}, & i \neq j, \\
- \sum_{k=0}^{n} D_{ik}^{(1)}, & i = j,
\end{cases}$$

and $D^{(k)}$, $k > 1$, with the “hybrid formula” (Tee (2006)),

$$D_{ij}^{(k)} := \begin{cases} 
\frac{k}{x_i - x_j} \left( \frac{v_j}{v_i} D_{ii}^{(k-1)} - D_{ij}^{(k-1)} \right), & i \neq j, \\
- \sum_{\ell=0}^{n} D_{i\ell}^{(k)}, & i = j.
\end{cases}$$
Evaluation of derivatives of $\mathcal{R}$ at the nodes

With

$$f := (f_0, \ldots, f_n)^T,$$

the product

$$D^{(k)} \cdot f$$

returns

$$(\mathcal{R}^{(k)}(x_0), \ldots, \mathcal{R}^{(k)}(x_n))^T,$$

the vector of the $k$-th derivative of $\mathcal{R}$ at the nodes.
Let us now investigate the convergence rate of the $k$-th derivative, $k = 1, \ldots, d$, of $r_n[f]$ at equispaced or quasi-equispaced nodes. By quasi-equispaced nodes (Elling 2007) we shall mean here sequences of points whose minimal spacing $h_{\text{min}}$ satisfies

$$h_{\text{min}} \geq ch,$$

where $c$ is a constant.
Convergence rates for the derivatives at the nodes

**Theorem**

Suppose $n$, $d \leq n$, and $k \leq d$ are positive integers and $f \in C^{d+1+k}[a, b]$. If the nodes $x_j$, $j = 0, \ldots, n$, are equispaced or quasi-equispaced, then

$$|r_n^{(k)}[f](x_j) - f^{(k)}(x_j)| \leq Ch^{d+1-k}, \quad 0 \leq j \leq n,$$

where $C$ only depends on $d$, $k$ and derivatives of $f$. 


Linear barycentric rational finite differences
We introduce **rational finite difference** (RFD) formulas for the approximation, at a node $x_i$, of the $k$-th derivative of a $C^{d+1+k}$ function,

$$
\left. \frac{d^k f}{dx^k} \right|_{x=x_i} \approx \left. \frac{d^k}{dx^k} r_n[f] \right|_{x=x_i} = \sum_{j=0}^{n} D_{ij}^{(k)} f_j.
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$$
Errors in centered FD, resp. RFD for $d = 4$

Figure: Errors in the approximation at $x = 0$ of the second and fourth order derivatives of $1/(1 + 25x^2)$ sampled in $[-5, 5]$ with $4 \leq n \leq 1000$
Errors in one-sided FD, resp. RFD for $d = 4$

Figure: Errors in the approximation at $x = -5$ of the second and fourth order derivatives of $1/(1 + x^2)$ sampled in $[-5, 5]$ with $4 \leq n \leq 1000$
Extended Floater–Hormann interpolation
The **Lebesgue constant**

\[ \Lambda_n = \max_{a \leq x \leq b} \Lambda_n(x) = \max_{a \leq x \leq b} \frac{n}{\sum_{j=0}^{n} \left| \frac{w_j}{x - x_j} \right|}, \]

is the condition number of the interpolation method. It is the maximum of the **Lebesgue function** \( \Lambda_n(x) \) in \([a, b]\).
Figure: Lebesgue function for Floater–Hormann interpolation in equispaced nodes in $[-1, 1]$ with $d = 2, 3, 4, 5$ and $n = 40$
Theorem (Bos–De Marchi–Hormann–K. (2011))

Let $0 \leq d \leq n$ and the nodes $x_j$, $j = 0, \ldots, n$, be equispaced. Then

$$\frac{2^{d-2}}{d+1} \ln \left( \frac{n}{d} - 1 \right) \leq \Lambda_n \leq 2^{d-1}(2 + \ln(n)).$$

Logarithmic growth with $n$ and exponential growth with $d$. 

Lebesgue constant for Floater–Hormann interpolation
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Theorem (Bos–De Marchi–Hormann–K. (2011))

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Logarithmic growth with $n$ and exponential growth with $d$. 
Extended Floater–Hormann interpolation

It turns out that for given $d$, $\Lambda_n(x)$ has at most $d$ high oscillations at the ends and is much smaller in the remaining part of the interval if the nodes are equispaced.

Idea: we add $2d$ new data values $\tilde{f}_{-d}, \ldots, \tilde{f}_{-1}$ and $\tilde{f}_{n+1}, \ldots, \tilde{f}_{n+d}$, corresponding to additional nodes $x_{-d}, \ldots, x_{-1}$ and $x_{n+1}, \ldots, x_{n+d}$. The global data set is then interpolated and evaluated only in the interval $[a, b]$.

To be precise, we choose positive integers $\tilde{n} \ll n$ and $\tilde{d} \leq \tilde{n}$, and compute $r_{\tilde{n}}^{(k)}[f](x_0)$ and $r_{\tilde{n}}^{(k)}[f](x_n)$, $k = 1, \ldots, \tilde{d}$, where $r_{\tilde{n}}[f](x_0)$ is the rational interpolant of the values $f_0, \ldots, f_{\tilde{n}}$, evaluated in $x_0$, and $r_{\tilde{n}}[f](x_n)$ is the rational interpolant of the values $f_{n-\tilde{n}}, \ldots, f_n$, evaluated in $x_n$, both with parameter $\tilde{d}$.
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To be precise, we choose positive integers \( \tilde{n} \ll n \) and \( \tilde{d} \leq \tilde{n} \), and compute \( r_{\tilde{n}}^{(k)}[f](x_0) \) and \( r_{\tilde{n}}^{(k)}[f](x_n), \ k = 1, \ldots, \tilde{d} \), where \( r_{\tilde{n}}[f](x_0) \) is the rational interpolant of the values \( f_0, \ldots, f_{\tilde{n}} \), evaluated in \( x_0 \), and \( r_{\tilde{n}}[f](x_n) \) is the rational interpolant of the values \( f_{n-\tilde{n}}, \ldots, f_{n} \), evaluated in \( x_n \), both with parameter \( \tilde{d} \).
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To be precise, we choose positive integers $\tilde{n} \ll n$ and $\tilde{d} \leq \tilde{n}$, and compute $r_{\tilde{n}}^{(k)}[f](x_0)$ and $r_{\tilde{n}}^{(k)}[f](x_n)$, $k = 1, \ldots, \tilde{d}$, where $r_{\tilde{n}}[f](x_0)$ is the rational interpolant of the values $f_0, \ldots, f_{\tilde{n}}$, evaluated in $x_0$, and $r_{\tilde{n}}[f](x_n)$ is the rational interpolant of the values $f_{n-\tilde{n}}, \ldots, f_n$, evaluated in $x_n$, both with parameter $\tilde{d}$. 

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We set

\[
\widetilde{f}_j := \begin{cases} 
  f_0 + \sum_{k=1}^{\tilde{d}} r^{(k)}_n [f](x_0) \frac{(x_j - x_0)^k}{k!}, & -d \leq j \leq -1, \\
  f_j, & 0 \leq j \leq n, \\
  f_n + \sum_{k=1}^{\tilde{d}} r^{(k)}_n [f](x_n) \frac{(x_j - x_n)^k}{k!}, & n + 1 \leq j \leq n + d.
\end{cases}
\]

Our extension of the Floater–Hormann family then is

\[
\tilde{r}_n[f](x) := \frac{\sum_{j=-d}^{n+d} w_j \frac{x - x_j}{\sum_{j=-d}^{n+d} w_j}}{\sum_{j=-d}^{n+d} w_j} \frac{x - x_j}{\sum_{j=-d}^{n+d} w_j}.
\]
Extended Floater–Hormann interpolation

We set

\[
\tilde{f}_j := \begin{cases} 
  f_0 + \sum_{k=1}^{\tilde{d}} r_n^{(k)}[f](x_0) \frac{(x_j - x_0)^k}{k!}, & -d \leq j \leq -1, \\
  f_j, & 0 \leq j \leq n, \\
  f_n + \sum_{k=1}^{\tilde{d}} r_n^{(k)}[f](x_n) \frac{(x_j - x_n)^k}{k!}, & n + 1 \leq j \leq n + d. 
\end{cases}
\]

Our extension of the Floater–Hormann family then is

\[
\tilde{r}_n[f](x) := \frac{\sum_{j=-d}^{n+d} \frac{w_j}{x - x_j} \tilde{f}_j}{\sum_{j=-d}^{n+d} \frac{w_j}{x - x_j}}.
\]
Convergence of extended Floater–Hormann interpolation

Notation:

\[ D := \min\{d, \tilde{d}\} \].

**Theorem**

Suppose \( n, d, \tilde{n} < n \) and \( \tilde{d} \leq \tilde{n} \) are positive integers and assume that \( f \in C^{d+2}[a - dh, b + dh] \cap C^{2\tilde{d}+1}([a, a + \tilde{n}h] \cup [b - \tilde{n}h, b]) \) is sampled at \( n + 1 \) equispaced nodes in \([a, b]\). Then

\[ \|\tilde{r}_n[f] - f\| := \max_{x \in [a,b]} |\tilde{r}_n[f](x) - f(x)| \leq Ch^{D+1} \].

\( \tilde{r}_n \) has no real poles.
For positive integers $n$ and $d$, the Lebesgue constant $\tilde{\Lambda}_n$ associated with extended Floater–Hormann interpolation is bounded as

$$\tilde{\Lambda}_n \leq 2 + \ln(n + 2d).$$

Logarithmic growth with $n$ and $d$. 
Lebesgue constant for extended FH interpolation

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$$\tilde{\Lambda}_n \leq 2 + \ln(n + 2d).$$

Logarithmic growth with $n$ and $d$. 
Lebesgue functions

Figure: Lebesgue function with $n = 50$ for extended FH interpolation in equispaced nodes with $d = 3$ (left), for polynomial interpolation in Chebyshev points of the second kind (center) and first kind (right)
Lebesgue constants and bound

Figure: Lebesgue constants associated with FH and EFH interpolation with $d = 8$ and $8 \leq n \leq 1000$, together with the upper bound on the EFH Lebesgue constant.
Lebesgue constants

Figure: Lebesgue constants associated with FH and EFH interpolation in equispaced nodes with $n = 200$ and $1 \leq d \leq 25$
Interpolation of Runge’s example $1/(1 + x^2)$ in $[-5, 5]$

**Figure:** Error behaviour of spline, FH and EFH interpolation of $1/(1 + x^2)$ with $d = 4$ and $20 \leq n \leq 1000$
Perturbed Runge’s example $1/(1 + x^2)$ in $[-5, 5]$
Thank you for your attention!