Volumes of cusped hyperbolic manifolds

RUTH KELLERHALS

Dedicated to Prof. Friedrich Hirzebruch
on the occasion of his 70th birthday

Abstract. For $n-$dimensional hyperbolic manifolds of finite volume with $m \geq 1$ cusps a new lower volume bound is presented which is sharp for $n = 2, 3$. The estimate depends upon m and the ideal regular simplex volume. The proof makes essential use of a density argument for ball packings in Euclidean and hyperbolic spaces and explicit formulae for the simplicial density function. Examples, consequences for the Gromov invariant, and – for n even – the maximal number of cusps are discussed.

0. Introduction

Let M be a hyperbolic manifold of dimension $n \geq 2$, that is, a complete Riemannian $n-$manifold of constant sectional curvature -1. Assume that M is non-compact but of finite volume. Then, M has finitely many disjoint unbounded ends of finite volume, the cusps of M. Each cusp is diffeomorphic to $N \times (0, \infty)$, where N is a compact Euclidean $(n-1)-$manifold. This comparatively simple geometric structure at infinity allows to investigate the size of M, as expressed by the volume for example, with much more success than in the compact manifold case (cf. [K3]).

For $n = 3$, a first lower volume bound for oriented cusped 3$-$manifolds was obtained by R. Meyerhoff [M1]. His methods consist in measuring the size of each individual cusp $C \subset M$. To this end, by making use of Jørgensen's trace inequality for discrete non-elementary subgroups of $PSL(2, \mathbb{C})$, a particular horoball in the universal cover H^3 can be associated to C. In a second step, the volume left out by the cusp C is estimated by means of a density argument with respect to the induced horoball packing in H^3.

Subsequently, C. C. Adams [A2] refined and extended these ideas by taking into account the tangency between the cusps of M. In this way, he obtained a clearly improved volume bound for cusped hyperbolic $3-$manifolds M of the form

$$\mathrm{vol}_3(M) \geq m \cdot \nu_3 \quad , \tag{0.1}$$

where ν_3 denotes the ideal regular simplex volume. Moreover, the estimate (0.1) is sharp for $m = 1, 2$. For example, the non-orientable 1–cusped Gieseking manifold is the unique hyperbolic $3-$manifold of minimal volume (cf. [A1]).

Inspired by Adams' approach [A2], we are able to generalize (0.1) for $m-$cusped hyperbolic manifolds of arbitrary dimension $n \geq 2$ such that the result is sharp for $n = 2, 3$. In its most accessible, yet weaker form our volume bound is expressed by (cf. §3, Theorem 3.5, Corollary 3.6)

$$\mathrm{vol}_n(M) \geq m \cdot \frac{2^n}{n(n+1)} \cdot \nu_n \quad , \tag{0.2}$$

where, again, ν_n equals the ideal regular simplex volume in H^n. Our results considerably improve previous work of S. Hersonsky [He] whose methods imitate Meyerhoff's procedure in n dimensions. For completeness, we review the results [He] by introducing the notion of canonical cusp in **3.1**.

An important but in (0.2) hidden role is played by the geometry of ball and horoball packings in Euclidean and hyperbolic spaces (cf. §2). More precisely, the notion of simplicial density function and an explicit formula for it are essential (cf. **2.2**). Some preliminaries about hyperbolic geometry are summarized in §1.

As an immediate consequence of (0.2), in **4.1**, we obtain a simple lower bound for the Gromov invariant of M. Another application deals with cusped hyperbolic manifolds of even dimensions. In **4.2**, we present an upper bound for the number of cusps in terms of the Euler–Poincaré characteristic by making use of the generalized Gauss–Bonnet–Chern theorem. This estimate is sharp for $n = 2$ while, for $n \geq 4$, this problem is unresolved since up to now we do not have sufficiently many different constructions of cusped $n-$manifolds at hand. Finally, in **4.3**, we discuss examples and further results about the volume spectrum of non-compact hyperbolic $n-$manifolds.

Acknowledgement. This paper covers a part of the author's habilitation thesis [K4]. In the present form, this work arose and was written up during a research stay at I.H.E.S. in Bures-sur-Yvette (France) in March 1997. The author expresses her thanks to the Director, Prof. J.-P. Bourguignon, and the staff for the invitation and the hospitality.

1. Preliminaries

1.1. The hyperbolic space H^n

Let H^n denote the hyperbolic $n-$space, that is, the simply connected, complete Riemannian $n-$manifold of constant sectional curvature -1. As realization for H^n we choose the conformal model of Poincaré,

$$H^n = \left(E^n_+ \, , \, ds^2 = \frac{dx_1^2 + \cdots + dx_n^2}{x_n^2} \right) \quad , \tag{1.1}$$

in the upper half space $E^n_+ = \{ x = (x_1, \ldots , x_n) \in E^n \,|\, x_n > 0 \}$ of Euclidean $n-$space E^n. The compactification $\overline{H^n} = H^n \cup \partial H^n$ consists of H^n together with the set $\partial H^n = \widehat{E}^{n-1} := E^{n-1} \cup \{\infty\}$ of its points at infinity.

Hyperbolic $r-$spheres $S_r(p)$ centered at ordinary points $p = (p_1, \ldots , p_n) \in H^n$ are Euclidean $(p_n \cdot \sinh r)-$spheres centered at $(p_1, \ldots , p_{n-1}, p_n \cdot \cosh r)$ and contained in E^n_+. They bound $r-$balls $B_r(p)$ with center p of volume

$$\mathrm{vol}_n(B_r(p)) = \Omega_{n-1} \int\limits_0^r \sinh^{n-1} t \, dt \quad ,$$

where $\Omega_{n-1} = 2\pi^{\frac{n}{2}} / \Gamma(\frac{n}{2})$ denotes the volume of the standard unit $(n-1)-$sphere S^{n-1}.
Horospheres $S_\infty(q)$ based at infinite points $q \in \partial H^n$ are either Euclidean spheres in E^n_+ internally tangent to E^{n-1} at $q \neq \infty$, or Euclidean hyperplanes in E^n_+ parallel to E^{n-1} for $q = \infty$. Horospheres are all congruent and carry a Euclidean metric in a natural way. For example, by (1.1), the horosphere $S_\infty(\infty)$ at distance $\rho > 0$ from the ground space E^{n-1} becomes a Euclidean $(n-1)-$space with respect to the metric

$$ds^2\big|_{S_\infty(\infty)} = \rho^{-2} \, (\, dx_1^2 + \cdots + dx_{n-1}^2 \,) \quad .$$

Horospheres $S_\infty(q)$ bound horoballs $B_\infty(q)$ of infinite volume.

1.2. Isometries of H^n

Let $I(H^n)$ be the group of isometries of H^n. The subgroup of orientation preserving isometries of H^n is denoted by $I^+(H^n)$. Each element of $I(H^n)$ can be written as a finite product of reflections in spheres or hyperplanes of $\widehat{E}^n$ leaving the upper half space E^n_+ invariant. More precisely, the group $I(H^n)$ is isomorphic to the subgroup $M(E^n_+) \subset M(\widehat{E}^n)$ of Möbius transformations of $\widehat{E}^n$ that leave E^n_+ invariant. By means of Poincaré extension, we obtain the isomorphisms

$$I(H^n) \simeq M(E^n_+) \simeq M(\widehat{E}^{n-1}) \quad .$$

According to the Brouwer fixed point theorem every Möbius transformation of $\widehat{E}^n$ has at least one fixed point. This leads to the well-known characterization of conjugacy classes of elements $\varphi \in M(E^n_+)\,,\, \varphi \neq id\big|_{E^n_+}$:

(a) φ has a fixed point in E^n_+, and φ is elliptic ;

(b) φ has precisely one fixed point in $\widehat{E}^{n-1}$, say q, and φ is parabolic ;

(c) φ has precisely two fixed points in $\widehat{E}^{n-1}$, and φ is loxodromic .

Parabolic Möbius transformations are of particular interest. Every parabolic element $\varphi \in M(E^n_+)$ is conjugate to the Poincaré extension of a fixed point free isometry of E^{n-1} (cf. [Ra, Theorem 4.7.2]). Among them, there are parabolic translations of the form $\varphi(x) = x + b$ for some vector $b \in E^n$. Geometrically, every parabolic Möbius transformation $\varphi \in M(E^n_+)$ with fixed point $q \in \widehat{E}^{n-1}$ gives rise to a pencil $\mathcal{P}_q$ of all (asymptotically) parallel geodesics in H^n with limiting point $q \in \partial H^n$. The mapping φ leaves the complementary set $\mathcal{C}_q$ consisting of all horospheres based at q invariant and acts isometrically on each horosphere $S_\infty(q) \in \mathcal{C}_q$ with respect to its intrinsic Euclidean geometry.

Finally, a subgroup $G \subset M(E^n_+)$ is elementary if G has a finite orbit Gp for some point $p \in \overline{H^n}$. In particular, an elementary subgroup $G \subset M(E^n_+)$ is of parabolic type if G fixes one point $q \in \partial H^n$ and has no further finite orbits in $\overline{H^n}$. It is known [Ra, Theorem 5.5.5] that G is discrete and of parabolic type if and only if G is conjugate to an infinite discrete subgroup of the isometry group $I(E^{n-1})$ of E^{n-1}. Moreover, if $\varphi, \psi \in M(E^n_+)$ are such that ψ is loxodromic with one fixed point in common with φ, then the subgroup $< \varphi, \psi >$ generated by φ, ψ is not discrete (cf. [Ra, Theorem 5.5.4]). Hence, a discrete torsion-free elementary group G containing a parabolic (loxodromic) element, consists of parabolic (loxodromic) elements, only, and they all have the same fixed point(s).

2. The density of a ball packing

2.1. Ball packings in the standard geometries

Let $n \geq 2$, and denote by X^n either the Euclidean space E^n, or hyperbolic space $\overline{H^n}$. A ball packing $\mathcal{B} = \mathcal{B}_{X^n}(r)$ of X^n is an arrangement of non-overlapping balls $B = B(r)$ of radius r. In the sequel, we summarize the most important definitions and results about ball packings in X^n. For more details and proofs, we refer to [Bö], [K4], [K5] and [Ro].

There are different notions of packing density. For later purposes, the local density measure is best suited. Consider the Dirichlet–Voronoĭ cell of B

$$D = D(B, \mathcal{B}) := \{\, p \in X^n \mid \operatorname{dist}(p, B) \leq \operatorname{dist}(p, B')\ ,\ \forall B' \in \mathcal{B}\,\} \quad ,$$

where $\mathrm{dist}(p,B)$ is assumed to be negative for $p \in B$. Since D is the intersection of a locally finite collection of half spaces in X^n, it is a convex polyhedron, eventually of infinite volume. The family $\{\, D(B,\mathcal{B}) \mid B \in \mathcal{B}\}$ covers X^n without overlappings or gaps. The local density $ld_n(B,\mathcal{B})$ of B with respect to $\mathcal{B}$ is given by the density of B with respect to its Dirichlet–Voronoĭ cell D, that is,

$$ld_n(B,\mathcal{B}) := \frac{\mathrm{vol}_n(B)}{\mathrm{vol}_n(D)} \quad .$$

It follows that $ld_n(B,\mathcal{B}) < 1$. More precisely, the local density can be estimated from above by the simplicial density function $d_n(r)$. For its definition, consider $n+1$ balls $B = B(r)$ of radius r mutually touching one another. Their centers give rise to a regular $n-$simplex $S_{reg} = S_{reg}(2r) \subset X^n$ of edge length $2r$. The simplicial density function $d_n(r)$ on X^n is now given by

$$d_n(r) = (n+1)\,\frac{\mathrm{vol}_n(B \cap S_{reg})}{\mathrm{vol}_n(S_{reg})} \quad , \tag{2.1}$$

which satisfies $d_1(r) = 1$.

For $X^n = E^n$, the simplicial density function $d_n(r)$ does not depend on r, and we write $d_n = d_n(r)$. Indeed, one can interpret d_n as limiting density $d_n = \lim_{r\to 0} d_n(r)$ on H^n by looking at the curvature dependence of the volume element for H^n. As an example, one easily computes

$$d_2 = \frac{\pi}{2\sqrt{3}} \simeq 0.90690 \quad . \tag{2.2}$$

By a result of A. Thue, this value is the maximal density for disc packings of E^2, and it is attained by the density δ_2 of the lattice packing associated to the root lattice A_2 (cf. [FT, p. 94–95]).

For ball packings of E^n, $n > 2$, the simplicial density function d_n remains an upper density bound. This was shown by C. A. Rogers [Ro, Theorem 7.1]. Even more generally, for a packing $\mathcal{B}$ of X^n with balls B of radius r, K. Böröczky [Bö, Theorem 1] proved that

$$ld_n(B,\mathcal{B}) \le d_n(r) \quad , \quad \forall\ B \in \mathcal{B} \quad . \tag{2.3}$$

The estimate (2.3) is sharp if the Dirichlet–Voronoĭ cell D of a ball $B \in \mathcal{B}$ forms a regular polytope in X^n. If this holds for each cell D of the packing $\mathcal{B}$, then the balls B are the in-balls (inscribed balls) of a regular honeycomb, and $\mathcal{B}$ is a regular ball packing of X^n. The regular honeycombs of X^n are all classified. A list of them in terms of their Schläfli symbols $\{r,3,\ldots,3\}$ with $r \in \mathbb{N}$ $(r \ge 3)$ can be found in [Co].

A horoball packing $\mathcal{B}_\infty$ of $\overline{H^n}$ is an arrangement of non-overlapping horoballs B_∞ in $\overline{H^n}$. The notion of local density can be extended for horoball packings $\mathcal{B}_\infty$ of $\overline{H^n}$. Let $B_\infty \in \mathcal{B}_\infty$, and $p \in H^n$ arbitrary. Then, $\mathrm{dist}(p,B_\infty)$ is defined to be the length of the unique perpendicular from p to the horosphere S_∞ bounding B_∞, where again $\mathrm{dist}(p,B_\infty)$

is taken negative for $p \in B_\infty$. The Dirichlet–Voronoĭ cell $D(B_\infty)$ of B_∞ is defined to be the convex body

$$D_\infty = D(B_\infty) = \{\, p \in H^n \mid \mathrm{dist}(p, B_\infty) \le \mathrm{dist}(p, B'_\infty) \ , \ \forall B'_\infty \in \mathcal{B}_\infty \,\} \quad .$$

Since both, B_∞ and D_∞, are of infinite volume, the concept of local density has to be modified. Let $q \in \partial H^n$ denote the base point of B_∞, and interpret S_∞ as Euclidean $(n-1)-$space (cf. **1.1**). Let $B_{n-1}(R) \subset S_\infty$ be a ball with center $c \in S_\infty$. Then, $q \in \partial H^n$ and $B_{n-1}(R)$ determine a convex cone $C_n(R) :=\mathrm{cone}_q(B_{n-1}(R)) \subset \overline{H^n}$ with apex q consisting of all hyperbolic geodesics through $B_{n-1}(R)$ with limiting point q. With these preparations, the local density $ld_n(B_\infty, \mathcal{B}_\infty)$ of B_∞ with respect to $\mathcal{B}_\infty$ is defined by

$$ld_n(B_\infty, \mathcal{B}_\infty) := \overline{\lim_{R\to\infty}} \ \frac{\mathrm{vol}_n(B_\infty \cap C_n(R))}{\mathrm{vol}_n(D_\infty \cap C_n(R))} \quad ,$$

and it is independent of the choice of the center c of $B_{n-1}(R)$. By analytical continuation, the simplicial density function $d_n(r)$ on $\overline{H^n}$ can be extended easily for the case $r = \infty$, too. Consider $n+1$ horoballs B_∞ which are mutually tangent. The convex hull of their base points at infinity is a totally asymptotic or ideal regular simplex $S^\infty_{reg} \subset \overline{H^n}$ of finite volume. Hence, it is legitimate to write

$$d_n(\infty) = (n+1)\, \frac{\mathrm{vol}_n(B_\infty \cap S^\infty_{reg})}{\mathrm{vol}_n(S^\infty_{reg})} \quad . \tag{2.4}$$

For a horoball packing $\mathcal{B}_\infty$ of $\overline{H^n}$, there is an analogue of (2.3), namely (cf. [Bö, Theorem 4])

$$ld_n(B_\infty, \mathcal{B}_\infty) \le d_n(\infty) \quad , \quad \forall\ B_\infty \in \mathcal{B}_\infty \quad . \tag{2.5}$$

The upper bound $d_n(\infty)$ in (2.5) is attained for a regular horoball packing, that is, a packing by horoballs which are inscribable in the cells of a regular honeycomb of $\overline{H^n}$. For $n = 2$, there is only one such packing. It belongs to the regular tesselation $\{\,\infty, 3\,\}$. Its dual $\{\,3, \infty\,\}$ is the regular tesselation by ideal triangles all of whose vertices are surrounded by infinitely many triangles. This packing has in-circle density $d_2(\infty) = \frac{\pi}{3}$. For $n > 2$, there is precisely one horoball packing left whose Dirichlet–Voronoĭ cells give rise to a regular honeycomb. This honeycomb is described by the Schläfli symbol $\{\,6, 3, 3\,\}$. Its dual consists of ideal regular simplices $S^\infty_{reg} \subset \overline{H^3}$ with dihedral angle $\frac{\pi}{3}$ building up a $6-$cycle around each edge of the tesselation.

2.2. A formula for the simplicial horoball density

First, we present a formula for the simplicial density function d_n on E^n in terms of the regular simplex volume (cf. [K4], [K5]). Let $S_0 \subset S^{n-1}$ denote a spherical regular simplex of dihedral angle $2\alpha_0 = \arccos\left(\frac{1}{n}\right)$ (note that a regular simplex $S_{reg}(2\alpha)$ exists in S^{n-1} if its dihedral angle 2α satisfies $-1 < \cos(2\alpha) < \frac{1}{n-1}$). Then,

$$d_n = \frac{1}{n} \cdot \prod_{k=2}^{n} \left(\frac{k+1}{k-1}\right)^{\frac{n-k+1}{2}} \cdot \mathrm{vol}_{n-1}(S_0) \quad . \tag{2.6}$$

For $n \leq 7$, the expression (2.6) for d_n can be evaluated by using the existing volume formulae for $\mathrm{vol}_{n-1}(S_0)$ in terms of its dihedral angle (cf. [K2]). For $n > 7$, $\mathrm{vol}_{n-1}(S_0)$ can be at least estimated in an elementary way (cf. [K3, Lemma 4]).

n	$d_n \simeq$
2	0.90690
3	0.77964
4	0.64782
5	0.52571
6	0.41924
7	0.32999

Table 1. The Euclidean simplicial density d_n

Asymptotically, d_n behaves according to (cf. [Ro, (11), p. 90])

$$d_n \sim \frac{(n+1)!\, e^{\frac{n}{2}-1}}{\sqrt{2} \cdot \Gamma(\frac{n}{2}+1) \cdot (4n)^{\frac{n}{2}}} \sim \frac{n}{e} \cdot \frac{1}{2^{\frac{n}{2}}} \quad .$$

Let us turn to horoball packings of $\overline{H^n}$ (cf. K4], [K5]). We already know that $d_2(\infty) = \frac{3}{\pi}$.

THEOREM 2.1.

Let $n \geq 3$, and denote by $\nu_n = vol_n(S^\infty_{reg})$ the ideal regular simplex volume in $\overline{H^n}$. Then, the simplicial horoball density $d_n(\infty)$ is given by

$$d_n(\infty) = \frac{n+1}{n-1} \cdot \frac{n}{2^{n-1}} \cdot \prod_{k=2}^{n-1} \left(\frac{k-1}{k+1}\right)^{\frac{n-k}{2}} \cdot \frac{1}{\nu_n} \quad . \tag{2.7}$$

For hyperbolic simplicial $n-$volumes, there are explicit formulas in terms of dihedral angles only for $n \le 6$ (cf. [K1], [K2]). For the volume ν_n of an ideal regular simplex S^∞_{reg}, however, there is a representation of ν_n as power series for all $n \ge 2$, which is due to J. Milnor [Mi, *How to compute volume in hyperbolic space*, §4].

COROLLARY 2.2.

The simplicial horoball density $d_n(\infty)$ is given by

$$d_n(\infty) = \frac{n+1}{n-1} \cdot \frac{\sqrt{n}}{2^{n-1}} \cdot \frac{\prod_{k=2}^{n-1} \left(\frac{k-1}{k+1}\right)^{\frac{n-k}{2}}}{\sum_{k=0}^{\infty} \frac{\beta(\beta+1)\cdots(\beta+k-1)}{(n+2k)!} A_{n,k}} \quad , \tag{2.8}$$

where $\beta = \frac{1}{2}(n+1)$ *and* $A_{n,k} = \sum_{\substack{i_0+\cdots+i_n=k \\ i_\mu \ge 0}} \frac{(2i_0)!\cdots(2i_n)!}{i_0!\cdots i_n!}$.

n	$d_n(\infty) \simeq$
2	0.95493
3	0.85328
4	0.73046
5	0.60695
6	0.49339
7	0.39441
8	0.31114

Table 2. The simplicial horoball density $d_n(\infty)$

3. A lower volume bound for cusped hyperbolic manifolds

3.1. Structure of hyperbolic manifolds

Let $n \ge 2$, and denote by M a hyperbolic $n-$manifold, that is, a complete Riemannian $n-$manifold of constant sectional curvature -1. Equivalently, M is a Clifford-Klein space form

$$M = H^n/\Gamma \quad ,$$

where $\Gamma \subset I(H^n)$ is a discrete, torsion-free subgroup. In the sequel, we assume M to be of finite volume. The Margulis lemma yields informations about the global structure of M (cf. [BGS, §10], [Ra, §12]). In particular, there is a compact $n-$manifold M_0 with (possibly empty) boundary such that $M - M_0$ consists of at most finitely many disjoint unbounded ends of finite volume, the cusps of M. Each cusp is diffeomorphic to $N \times (0, \infty)$, where N is a compact Euclidean $(n-1)-$manifold. By a more detailed analysis of Γ [BGS, 10.3], [Ra, proof of Theorem 12.6.6], a cusp C can be identified with

$$C = C_q = V_q/\Gamma_q$$

for some point $q \in \partial H^n$, where $\Gamma_q < \Gamma$ is of parabolic type with fixed point q, and where $V_q \subset H^n$ is some precisely invariant unbounded region in H^n with $\overline{V_q} \ni q$. Actually, V_q is a horoball based at q: Since C is of finite volume, Γ_q – as discrete subgroup of $I(E^{n-1})$ – acts cocompactly on E^{n-1} and is therefore crystallographic. By a theorem of Bieberbach (cf. [Bu]), the free abelian group $\Lambda = \Lambda(\Gamma_q)$ of parabolic translations in Γ_q is of finite index and of rank $n-1$. Therefore, by [Ra, p. 594 and Theorem 5.4.6], V_q is a horoball based at q. The point q is called a cusped point of M.

By expanding a cusp C until it intersects itself or another cusp of M in a finite number of points on its boundary but such that C is still covered by horoballs, we continue to call C a cusp.

Finally, there is a universal constant $v_n > 0$ such that for each hyperbolic $n-$manifold M

$$\mathrm{vol}_n(M) \geq v_n \quad . \tag{3.1}$$

3.2. Canonical cusps

Let $M = H^n/\Gamma$ be an oriented hyperbolic manifold, that is, $\Gamma < I^+(H^n)$. Assume that M is non-compact but of finite volume. Therefore, M has at least one cusp $C = C_q = V_q/\Gamma_q$.

Following [He, 36], by interpreting $I^+(H^n)$ as group of Clifford matrices, one can associate to Γ_q a particular horoball $B_q \subset H^n$ based at q such that B_q/Γ_q embeds in M. Let $S \in I^+(H^n)$ with $S(\infty) = q$. Consider in $(S^{-1}\Gamma S)_\infty$ the subgroup $\Lambda = \Lambda((S^{-1}\Gamma S)_\infty)$ of all parabolic translations. As above, Λ is of finite index and free abelian of rank $n-1$. Interpret Λ as lattice of vectors in E^{n-1} and denote by $\mu \in \Lambda - \{0\}$ a shortest vector. Then, the canonical horoball B_q based at the cusped point q is defined to be the horoball $S(B_\infty(\mu))$ based at q, where $B_\infty(\mu) = \{\, x \in E^n_+ \mid x_{n+1} > |\mu| \,\}$. By results of [He], $S(B_\infty(\mu))$ is well defined and precisely invariant with respect to Γ. Therefore, B_q/Γ_q embeds in M.

The region $U = B_q/\Gamma_q \subset M$ is called the canonical cusp associated to q. Let $\mathcal{U} = \{\, U \mid U$ canonical cusp of $M\,\}$. By [He, Proposition 3.3], the elements of $\mathcal{U}$ are pairwise disjoint. Write $\mathrm{vol}_n(\mathcal{U}) = \sum_{U\in\mathcal{U}} \mathrm{vol}_n(U)$. Then,

$$\mathrm{vol}_n(M) \geq \mathrm{vol}_n(\mathcal{U}) \quad . \tag{3.2}$$

Our next aim is to estimate $\mathrm{vol}_n(\mathcal{U})$ universally from below. We do this by first considering a single element of $\mathcal{U}$. Let $U = B_q/\Gamma_q \in \mathcal{U}$ be a canonical cusp for some cusped point q of M. We know that Γ_q is a crystallographic subgroup of $I^+(E^{n-1})$. Denote by

$$i_{n-1} := \max\{\, [\,\Gamma : \Lambda(\Gamma)\,] \mid \Gamma < I^+(E^{n-1}) \text{ crystallographic } \} \quad ,$$

which is a finite number by the theorems of Bieberbach (cf. [Bu]). In particular, one has (cf. [BBNWZ, Table 8C, p. 408], [Sz])

$$i_2 = 1 \quad ; \quad i_3 \leq 6 \quad ; \quad i_4 \leq 12 \quad , \quad i_5 \leq 24 \quad , \tag{3.3}$$

and, for arbitrary $k \geq 6$ (cf. [Bu]),

$$i_k \leq 3^{k^2} \quad . \tag{3.4}$$

LEMMA 3.1.

Let $U \in \mathcal{U}$ denote a canonical cusp. Then,

$$\mathrm{vol}_n(U) \geq \frac{c(n)}{i_{n-1} \cdot d_{n-1}} \quad , \tag{3.5}$$

where d_{n-1} is the Euclidean simplicial density, and the constant $c(n)$ is given by

$$c(n) = \frac{\Omega_{n-2}}{2^{n-1} \cdot (n-1)^2} \quad .$$

Proof. Our proof is very similar to [He, proof of Proposition 3.4]. Let $U = B_q/\Gamma_q$ for some cusped point $q \in \partial H^n$. Assume without loss of generality that $q = \infty$. Associate to the stabilizer Γ_∞ its translational lattice Λ with shortest vector $\mu \neq 0$. As usually, let $B_\infty = B_\infty(\mu)$ be the canonical horoball in E^n_+ based at ∞. A fundamental domain of the translation group Λ acting on E^{n-1} is a Dirichlet domain $P \subset E^{n-1}$ which contains a ball $B_0 := B(\frac{|\mu|}{2})$ of radius $\frac{|\mu|}{2}$. Therefore, we obtain a lattice packing

$$\mathcal{B} = \{\, \gamma B_0 \mid \gamma \in \Lambda \,\}$$

of E^{n-1} with balls of radius $\frac{|\mu|}{2}$ and Dirichlet–Voronoĭ cells $\{\,\gamma P \mid \gamma \in \Lambda\,\}$. It follows (cf. **2.1**) that

$$\mathrm{vol}_{n-1}(P) = \frac{\mathrm{vol}_{n-1}(B_0)}{d_\Lambda} = \frac{\Omega_{n-2}\cdot|\mu|^{n-1}}{2^{n-1}\cdot(n-1)\cdot d_\Lambda} \quad , \tag{3.6}$$

where d_Λ is the Euclidean $(n-1)$-dimensional packing density for Λ. By (2.3), $d_\Lambda \le d_{n-1}$, where d_{n-1} is the Euclidean simplicial density (cf. (2.2)).

For the action of the Poincaré extension of Λ on $B_\infty(\mu)$, a fundamental domain is obviously of the form

$$G = \{\, x = (x_1,\dots,x_n) \in H^n \mid (x_1,\dots,x_{n-1}) \in P\ ;\ x_n > |\mu|\,\} \quad ,$$

whose volume is given by

$$\mathrm{vol}_n(G) = \int\limits_G \frac{dx_1\cdots dx_n}{x_n^n} = \mathrm{vol}_{n-1}(P)\cdot\int\limits_{|\mu|}^{\infty}\frac{dx_n}{x_n^n} = \frac{\mathrm{vol}_{n-1}(P)}{(n-1)\cdot|\mu|^{n-1}} \quad . \tag{3.7}$$

By (3.6) and (3.7), we obtain

$$\mathrm{vol}_n(G) = \frac{\Omega_{n-2}}{2^{n-1}\cdot(n-1)^2\cdot d_\Lambda} \ge \frac{\Omega_{n-2}}{2^{n-1}\cdot(n-1)^2\cdot d_{n-1}} \quad .$$

For the canonical cusp neighborhood $U = B_\infty(\mu)/\Gamma_\infty$, we deduce

$$\begin{aligned}\mathrm{vol}_n(U) = \frac{\mathrm{vol}_n(G)}{[\,\Gamma_\infty : \Lambda\,]} &= \frac{\Omega_{n-2}}{2^{n-1}\cdot(n-1)^2\cdot d_{n-1}\cdot[\,\Gamma_\infty:\Lambda\,]}\\ &\ge \frac{\Omega_{n-2}}{2^{n-1}\cdot(n-1)^2\cdot d_{n-1}\cdot i_{n-1}} \quad .\end{aligned}$$

□

Remark.

(a) According to the proof of Lemma 3.1, we derived an even better lower volume bound, namely,

$$\mathrm{vol}_n(U) \ge \frac{c(n)}{i_{n-1}\cdot\delta_{n-1}} \quad ,$$

where δ_{n-1} denotes the density of an optimal lattice packing in E^{n-1}. The values of δ_{n-1} are known for $1 \le n \le 8$; for $10 \le n \le 13$, there are still explicit lower bound for δ_{n-1} (cf. [K4], [K5]).

3.3. A universal lower volume bound

Let M be a hyperbolic $n-$manifold of finite volume with $m \ge 1$ cusps. Denote by $\mathcal{C} = \{\,C_1,\dots,C_m\,\}$ a set of cusps of M. Write $\mathrm{vol}_n(\mathcal{C}) = \sum_{i=1}^m \mathrm{vol}_n(C_i)$. Then,

$$\mathrm{vol}_n(M) \geq \mathrm{vol}_n(\mathcal{C}) \quad . \tag{3.8}$$

We can improve (3.8) as follows (for $n = 3$, see also [A2, Lemma 2.1]).

LEMMA 3.2.

Let $\mathcal{C}$ denote a set of cusps of M. Then,

$$\mathrm{vol}_n(M) \geq \frac{\mathrm{vol}_n(\mathcal{C})}{d_n(\infty)} \quad , \tag{3.9}$$

where $d_n(\infty)$ is the simplicial horoball density.

Proof. Let $M = H^n/\Gamma$. Since the elements of $\mathcal{C}$ are pairwise disjoint, it suffices to prove (3.9) for $\mathcal{C} = \{\, C \,\}$. By definition, C is of the form V_q/Γ_q where q is some cusped point of M. Assume without loss of generality that $q = \infty$. Then, V_∞ is a horoball $B \subset E^n_+$ with basis ∞ and provides a horoball packing (cf. **2.1**)

$$\mathcal{B}_\infty = \{\, \gamma B \mid \gamma \in \Gamma - \Gamma_\infty \,\} \quad ,$$

whose Dirichlet–Voronoĭ cells D are all congruent. If Γ_∞ would be trivial, then each D would be a Dirichlet fundamental domain for the action of Γ on H^n (cf. [Ra, §6.5]). Since $\Gamma_\infty \neq \{\, id \,\}$, consider a fundamental domain G for the action of Γ_∞ on D. Then, $D = \cup_{\gamma \in \Gamma_\infty} \gamma G$. Since G is also a fundamental domain for Γ, one has $\mathrm{vol}_n(M) = \mathrm{vol}_n(G)$. For the local density $ld_n(B, \mathcal{B}_\infty)$, we deduce

$$ld_n(B, \mathcal{B}_\infty) = \frac{\mathrm{vol}_n(B \cap G)}{\mathrm{vol}_n(G)} \quad ,$$

and by (2.5),

$$\frac{\mathrm{vol}_n(B \cap G)}{\mathrm{vol}_n(G)} \leq d_n(\infty) \quad .$$

Since $\mathrm{vol}_n(B \cap G) = \mathrm{vol}_n(C)$ and $\mathrm{vol}_n(G) = \mathrm{vol}_n(M)$, the lemma follows. □

Remark.

(b) It follows from the proof and **2.1** that the inequality (3.9) is sharp if the lift of each element of $\mathcal{C}$ to H^n induces a regular horoball packing. Since the latter exist only for $n \leq 3$, we deduce

$$\mathrm{vol}_n(M) > \frac{\mathrm{vol}_n(\mathcal{C})}{d_n(\infty)} \quad \text{for} \quad n > 3 \quad .$$

By combining (3.2), (3.5) and (3.9), we get a first, rough volume bound for cusped hyperbolic manifolds (cf. [K4, Satz 3.2.5]).

PROPOSITION 3.3.

Let M denote an oriented hyperbolic $n-$manifold of finite volume with $m \geq 1$ cusps. Then,

$$\mathrm{vol}_n(M) \geq m \cdot \frac{c(n)}{i_{n-1} \cdot d_{n-1} \cdot d_n(\infty)} \quad . \tag{3.10}$$

Remark.

(c) The inequality (3.10) remains valid for non-orientable manifolds M if the right hand side of (3.10) is multiplied by a factor $\frac{1}{2}$. This comes from the passage to the orientable double cover $\widetilde{M}$ of M with $\mathrm{vol}_n(\widetilde{M}) = 2 \cdot \mathrm{vol}_n(M)$.

Example.

Let $M = H^3/\Gamma$ be an oriented hyperbolic $3-$manifold of finite volume with one cusp. Then, by (2.7), (3.3), (3.10), Remark (a) and (2.2), we obtain the volume estimate

$$\mathrm{vol}_3(M) \geq \frac{\sqrt{3}}{4 \cdot d_3(\infty)} = \frac{\nu_3}{2} \simeq 0.50747 \quad ,$$

which was already discovered by R. Meyerhoff [M1], [M2]. His proof relies on the estimate $\mathrm{vol}_3(C) \geq \frac{\sqrt{3}}{4}$ for a cusp $C \subset M$ based on Jørgensen's trace inequality for discrete non-elementary subgroups of $PSL(2, \mathbb{C})$ and an observation similar to Lemma 3.2. On the other hand, consider the Gieseking manifold N_1 which arises from the ideal regular simplex S^∞_{reg} with dihedral angle $\frac{\pi}{3}$ and volume ν_3 by identifying suitably its faces. N_1 is non-orientable and has exactly one cusp. C. C. Adams [A1] showed that N_1 is the unique hyperbolic $3-$manifold with one cusp of minimal volume. Therefore, in the orientable case, Adams obtained the better estimate $\mathrm{vol}_3(M) > \nu_3$.

Indeed, Proposition 3.3 can be improved considerably by taking into account the tangency in boundary points of cusps with themselves or other cusps of M (cf. [A2, §2] for the case $n = 3$). A set $\mathcal{C}$ of m cusps of M is called a maximal disjoint set of cusps if the interiors of the cusps are pairwise disjoint and if none of the cusps in $\mathcal{C}$ can be enlarged without having its interior intersect with the interior of itself or some other cusp of $\mathcal{C}$. Each of these intersection points is termed tangency point. The total number $k = k(\mathcal{C})$ of tangency points between cusps of $\mathcal{C}$ is called the tangency number of $\mathcal{C}$. Finally, write $\mathrm{vol}_n(\mathcal{C}) := \sum_{C \in \mathcal{C}} \mathrm{vol}_n(C)$.

LEMMA 3.4.

Let M be a hyperbolic $n-$manifold of finite volume. Denote by $\mathcal{C}$ a set of maximal disjoint cusps of M with tangency number $k = k(\mathcal{C})$. Then,

$$\mathrm{vol}_n(\mathcal{C}) \geq 2 \cdot k \cdot \frac{c(n)}{d_{n-1}} \quad . \tag{3.11}$$

Proof. Let $M = H^n/\Gamma$, and fix an element $C \in \mathcal{C}$ with cusped point q. For simplicity, assume that $q = \infty$. Write $C = B/\Gamma_\infty$, where $B = B_\infty(\rho)$ is a horoball with basis ∞ and at distance $\rho > 0$, say, from the ground space $\{ x_n = 0 \}$. Consider a fundamental polytope $P_\infty \subset \{ x_n = 0 \}$ for the action of Γ_∞ on horospheres based at ∞.

The tangency points of C give rise to a set of $\Gamma_\infty-$inequivalent Euclidean $(n-1)-$balls of radius $\frac{\rho}{2}$ in $\{ x_n = 0 \}$ as follows. Let r denote the number of tangency points of C with any other cusp $C' \in \mathcal{C}$, and let s be the number of tangency points of C with itself. A tangency point of C with a cusp C' gives rise to a horoball B' in H^n covering C' which touches B and which is based in a point of P_∞ modulo the action of Γ_∞ on ∂P_∞ . When C touches itself, two points on its boundary are identified. In H^n, they correspond to two points on ∂B which project to P_∞. Moreover, they are the touching points of B with two distinct horoballs based in points of P_∞ . All together, there are $r + 2s$ horoballs based in P_∞ and touching B all distinct under the action of Γ_∞. Observe that they form $n-$dimensional Euclidean balls of radius $\frac{\rho}{2}$. Projected to $\{ x_n = 0 \}$, we obtain a collection of disjoint balls $B_1, \ldots, B_{r+2s} \subset E^{n-1}$ of radius $\frac{\rho}{2}$ all of whose centers lie in P_∞ . Consider the ball packing

$$\mathcal{B} := \{ \gamma(B_1), \ldots, \gamma(B_{r+2s}) \mid \gamma \in \Gamma_\infty \} \quad .$$

It is easy to see that its local density equals

$$\frac{\mathrm{vol}_{n-1}(\cup_{i=1}^{r+2s} B_i)}{\mathrm{vol}_{n-1}(P_\infty)} = (r+2s) \cdot \frac{\mathrm{vol}_{n-1}(B_1)}{\mathrm{vol}_{n-1}(P_\infty)} \quad .$$

By (2.3), we obtain

$$\mathrm{vol}_{n-1}(P_\infty) \geq (r+2s) \cdot \frac{\mathrm{vol}_{n-1}(B_1)}{d_{n-1}} \quad .$$

Hence (cf. also (3.7)),

$$\begin{aligned}
\mathrm{vol}_n(C) &= \mathrm{vol}_{n-1}(P_\infty) \cdot \int\limits_\rho^\infty \frac{dx_n}{x_n^n} \\
&= \frac{\mathrm{vol}_{n-1}(P_\infty)}{(n-1) \cdot \rho^{n-1}} \\
&\geq (r+2s) \cdot \frac{\mathrm{vol}_{n-1}(B_1)}{(n-1) \cdot \rho^{n-1} \cdot d_{n-1}} \\
&= (r+2s) \cdot \frac{\Omega_{n-2}}{(n-1)^2 \cdot 2^{n-1} \cdot d_{n-1}} \\
&= (r+2s) \cdot \frac{c(n)}{d_{n-1}} \quad .
\end{aligned}$$

Since a tangency point of C with a cusp C' contributes the same additional amount of volume $\frac{c(n)}{d_{n-1}}$ to $\mathrm{vol}_n(C')$, we finally obtain

$$\mathrm{vol}_n(\mathcal{C}) \geq 2 \cdot k \cdot \frac{c(n)}{d_{n-1}} \quad . \qquad \square$$

With these preparations, we can quantify v_n in (3.1) for the case of cusped hyperbolic $n-$manifolds.

THEOREM 3.5.

Let M denote a hyperbolic $n-$manifold of finite volume with $m \geq 1$ cusps. Then,

$$\mathrm{vol}_n(M) \geq 2 \cdot m \cdot \frac{c(n)}{d_{n-1} \cdot d_n(\infty)} = m \cdot \frac{\Omega_{n-2}}{2^{n-2} \cdot (n-1)^2 \cdot d_{n-1} \cdot d_n(\infty)} \quad . \tag{3.12}$$

For $n > 3$, the inequality (3.12) is strict.

Proof. The proof is similar to [A2, Lemma 2.4]. Let $\mathcal{C} = \{ C_1, \ldots, C_m \}$ denote a set of cusps of M. We associate to $\mathcal{C}$ a maximal disjoint set of cusps. For this, expand C_1 until it just touches itself, by shrinking the other cusps if necessary. Then, we obtain at least one point of tangency. Expand successively each of the remaining cusps until it touches itself or one of the previously enlarged cusps. In this way, we obtain a maximal disjoint set of cusps with $k \geq m$ tangency points. The assertion follows now from Lemma 3.2, Lemma 3.4 and Remark (b). □

Remarks.

(d) The volume bound (3.12) can be made explicit for each n: Formula (2.8) in Corollary 2.2 expresses the simplicial horoball density $d_n(\infty)$ as a function of $n \geq 2$ (cf. also Table 2). The Euclidean simplicial density d_{n-1} is known explicitly for $n \leq 8$ while for arbitrary n there are elementary estimates improving $d_{n-1} < 1$ (cf. **2.2**).

(e) For hyperbolic $n-$manifolds M with geodesic boundary and of finite volume, a result analogous to Theorem 3.5 was obtained by Y. Miyamoto (cf. [Miy] and also [K4]). By introducing the notions of $r-$hyperball packing and the simplicial hyperdensity $\rho_n(r)$, he showed [Miy, Theorem 4.2] that

$$\frac{\mathrm{vol}_n(M)}{\mathrm{vol}_{n-1}(\partial M)} \geq \rho_n(0) \quad ,$$

which is sharp for $n=3,4$. The limiting density $\rho_n(0)$ can be expressed in the form

$$\rho_n(0) = \frac{\mathrm{vol}_n(T_{reg})}{(n+1)\cdot \mathrm{vol}_{n-1}(S^{\infty}_{reg})} \quad ,$$

where $T_{reg} \subset \overline{H^n}$ is a (polarly) truncated regular simplex all of its vertices are at infinity.

COROLLARY 3.6.

Let M denote a hyperbolic $n-$manifold of finite volume with $m \geq 1$ cusps. Let $S_0 \subset S^{n-2}$ be a regular simplex with dihedral angle $2\alpha_0 = \arccos(\frac{1}{n-1})$. Denote by ν_n the volume of an ideal regular simplex in $\overline{H^n}$. Then,

$$\mathrm{vol}_n(M) \geq m \cdot \frac{2}{n(n+1)} \cdot \frac{\Omega_{n-2}}{\mathrm{vol}_{n-2}(S_0)} \cdot \nu_n \geq m \cdot \frac{2^n}{n(n+1)} \cdot \nu_n \quad . \tag{3.13}$$

For $n>3$, the inequalities in (3.13) are strict.

Proof. The first inequality in (3.13) follows from Theorem 3.5 by expressing the Euclidean simplicial density d_{n-1} by means of (2.6) and the simplicial horoball density $d_n(\infty)$ by means of (2.7). The second, strict inequality is obtained by observing that the dihedral angle of S_0 satisfies $2\alpha_0 < \frac{\pi}{2}$. Moreover, for an arbitrary regular simplex $S_{reg}(2\alpha) \subset S^k$, the volume $\mathrm{vol}_k(S_{reg}(2\alpha))$ is a strictly monotonely increasing function in α (cf. [K3, §4, (A2)]). Since S^k is dissected into 2^{k+1} copies of $S_{reg}(\frac{\pi}{2})$, we obtain, for $n>2$,

$$\mathrm{vol}_{n-2}(S_0) < \mathrm{vol}_{n-2}(S_{reg}(\frac{\pi}{2})) = \frac{\Omega_{n-2}}{2^{n-1}} \quad .$$

For $n=2$, we have $\mathrm{vol}_0(S_{reg}) = 1 = \Omega_0/2$.

□

Remark.

(f) The regular simplex volume $\mathrm{vol}_{n-2}(S_0)$ is commensurable with Ω_{n-2} if the dihedral angle $2\alpha_0 = \arccos(\frac{1}{n-1})$ of S_0 is commensurable with π. This is the case precisely for $n=2$ and $n=3$ (cf. **4.2**, **4.3**).

4. Applications

4.1. A lower bound for the Gromov invariant

Let X denote an oriented closed connected $n-$manifold. The Gromov invariant (or the simplicial volume) $||\,X||$ of X is defined to be the simplicial ℓ^1-norm of the fundamental class $[X]$ of X in $H_n(X;\mathbb{R})$, that is,

$$||\,X|| = \inf\{\,||\,c\,||\mid c \text{ is a singular } n-\text{cycle representing } [X]\,\} \quad .$$

This definiton can be extended for non–orientable manifolds X by passing to the double cover $\widetilde{X}$ of X and setting

$$||\,X|| = \frac{1}{2}||\,\widetilde{X}|| \quad .$$

For a closed $n-$manifold X which supports an affine flat bundle of dimension n, a result of J. Milnor–D. Sullivan–J. Smillie (cf. [G2, §0.3]) says that

$$||\,X|| \geq 2^n\cdot|\chi| \quad ,$$

where χ is the Euler number of the bundle. This result is meaningful only for n even since otherwise χ vanishes.

For an oriented closed spherical or Euclidean Clifford-Klein space form, the Gromov invariant vanishes. This follows from the fact that $||\,f_*(\alpha)|| \leq ||\,\alpha||$ for a continuous map $f\,:\,X\longrightarrow Y$ and an element $\alpha\in H_k(X;\mathbb{R})$.

Let $n\geq 2$, and consider an oriented closed hyperbolic $n-$manifold M. An oriented closed Riemannian surface M_g of genus $g>1$ has Gromov invariant

$$||\,M_g|| = 2\,|\,\chi(M_g)\,| = 4(g-1) \quad .$$

For n arbitrary, W. Thurston [Th, Corollary 6.1.7] proved that $||\,M||$ is always strictly positive and satisfies $||\,M||\geq \mathrm{vol}_n(M)/\nu_n$. For the wider class of hyperbolic manifolds M of finite volume, M. Gromov [G2, §0.4] sharpened Thurston's result by showing

$$||\,M|| = \frac{\mathrm{vol}_n(M)}{\nu_n} \quad . \tag{4.1}$$

His proof is based on a different but equivalent definition of $||\,M||$ in the sense of bounded cohomology and the observation that M is concave relative to infinity (cf. [G2, §1, Appendix 3]). By means of Corollary 3.6, we can estimate the Gromov invariant of cusped hyperbolic manifolds universally from below using (4.1) (for $n=3$, cf. also [A2, Corollary 5.1]).

COROLLARY 4.1.

Let M denote a hyperbolic $n-$manifold of finite volume with $m \geq 1$ cusps. Then, for $n = 3$, $||\,M\,|| \geq m$. For $n > 3$,

$$||\,M\,|| > m \cdot \frac{2^n}{n(n+1)} \quad . \tag{4.2}$$

4.2. Cusped hyperbolic manifolds of even dimension

Let $n = 2l \geq 2$, and consider a cusped hyperbolic $n-$manifold M of finite volume. By the theorem of Gauss–Bonnet–Chern, which was generalized by G. Harder and M. Gromov (cf. [G2, Theorem (C')]) to the non-compact case, the volume of M is proportional to the Euler–Poincaré characteristic $\chi(M)$ according to

$$\mathrm{vol}_{2l}(M) = (-1)^l\, \frac{\Omega_{2l}}{2} \cdot \chi(M) \quad . \tag{4.3}$$

By (4.3) and Corollary 3.6, we can estimate the maximal number of cusps of M.

COROLLARY 4.2.

Let $n \geq 2$ be even, and denote by M an $n-$dimensional hyperbolic manifold of finite volume with $m \geq 1$ cusps. Then,

$$m \leq \frac{\pi}{2} \cdot \frac{n(n+1)}{n-1} \cdot \frac{\mathrm{vol}_{n-2}(S_0)}{\nu_n} \cdot |\,\chi(M)\,| \leq \frac{n(n+1)}{2^{n+1}} \cdot \frac{\Omega_n}{\nu_n} \cdot |\,\chi(M)\,| \quad , \tag{4.4}$$

where $S_0 \subset S^{n-2}$ is a regular simplex with dihedral angle $2\alpha_0 = \arccos(\frac{1}{n-1})$, and ν_n denotes hyperbolic ideal regular $n-$simplex volume. For $n > 2$, the inequalities in (4.4) are strict.

Example 1.

Let $n = 2$. Denote by M a hyperbolic Riemannian surface, that is, $\chi(M) < 0$. Then, (4.3) yields

$$\mathrm{vol}_2(M) = \frac{\Omega_2}{2}\,|\,\chi(M)\,| \in 2\pi \cdot \mathbb{N} \quad .$$

Assume that M is non-compact with m cusps. By the weaker estimate in (4.4), m is bounded from above by

$$m \leq 3\,|\,\chi(M)\,| \quad .$$

Hence, a non-compact hyperbolic surface M of minimal volume has at most 3 cusps. It is known that there are exactly 4 non-homeomorphic Riemannian surfaces of minimal volume 2π. Among them, there is one hyperbolic surface with 3 cusps, the 3–punctured sphere. It is obtained by glueing 2 ideal triangles of area π each.

Example 2.

Let $n=4$. J. Ratcliffe and S. Tschantz [RT] constructed several hundreds of non-compact hyperbolic 4−manifolds as quotients by congruence 2 subgroups of $O(4,1;\mathbb{Z})$. These manifolds are of minimal volume $4\pi^2/3$ with up to 6 cusps and arise all by glueing suitably together the facets of the ideal 24−cell (an ideal regular hyperbolic 4−polytope all of whose 24 facets are octahedra).

A computation of ν_4 (cf. [K1]) gives

$$\nu_4=\frac{4\pi}{3}\left(\pi-5\alpha_0\right) \quad , \tag{4.5}$$

where $\cos(2\alpha_0)=1/3$, that is, $\pi-5\alpha_0=\arccos\left(\sqrt{\frac{242}{243}}\right)$. Moreover, $\mathrm{vol}_2(S_0)=6\alpha_0-\pi$. Hence, by Corollary 3.6 and (4.5), an arbitrary cusped hyperbolic 4−manifold M satisfies the strict inequality

$$\mathrm{vol}_4(M)>m\cdot\frac{8\pi^2}{15}\cdot\frac{\pi-5\alpha_0}{6\alpha_0-\pi}\simeq m\cdot 0.61293 \quad . \tag{4.6}$$

This result improves the bound of S. Hersonsky [He, Theorem 2] which, in the oriented manifold case, gives

$$\mathrm{vol}_4(M)\geq m\cdot\frac{\sqrt{3}}{36}\simeq m\cdot 0.04811 \quad .$$

By (4.3), $\mathrm{vol}_4(M)\in 4\pi^2/3\cdot\mathbb{N}$. Therefore, a manifold M of minimal volume such that (4.6) is close to being sharp would need to have 21 cusps. On the other hand, by the first inequality in Corollary 4.2, the number m of cusps of a manifold M is bounded from above by

$$m<\frac{10\pi}{3}\cdot\frac{\mathrm{vol}_2(S_0)}{\nu_4}\cdot\chi(M) \quad , \tag{4.7}$$

that is, for $\chi(M)=1$,

$$m\leq\left[\ \frac{5}{2}\cdot\frac{6\alpha_0-\pi}{\pi-5\alpha_0}\ \right]=21 \quad .$$

Here, $[u]$ denotes the biggest integer smaller than or equal to u. Therefore, if one could find a 4−manifold with Euler–Poincaré characteristic equal to 1 and having 21 cusps, for example, then our estimates (4.6), (4.7) would be rather accurate ! To our knowledge, the existence of such a manifold is as yet not known.

4.3. Further results

Let $n \geq 3$ be odd, and consider the $n-$th hyperbolic volume spectrum

$$\mathbf{Vol}_n := \{ \, \mathrm{vol}_n(M) \mid M \text{ hyperbolic } n-\text{manifold} \, \} \subset \mathbb{R}_+ \quad ,$$

and its subset $\mathbf{Vol}_n^\infty \subset \mathbf{Vol}_n$ of volumes formed by cusped manifolds.

Let $n = 3$. By work of Thurston and T. Jørgensen, the structure of the spectrum $\mathbf{Vol}_3$ is very particular (cf. [G1]). For example, it is well-ordered, finite-to-one, and its smallest element v_3 must be realized by compact manifolds. Despite many research efforts, it is still an open question which manifolds are of minimal volume. A candidate is the example due to J. Weeks and S. Matveev–A. Fomenko. It is obtained by Dehn surgery on the figure eight knot complement on S^3. Its volume is approximatively equal to 0.94272.

Consider the spectrum $\mathbf{Vol}_3^\infty$ with smallest element $v_3^\infty > 0$. By Theorem 3.5, we know that a hyperbolic $3-$manifold M with $m \geq 1$ cusps satisfies

$$\mathrm{vol}_3(M) \geq m \cdot \nu_3 \simeq m \cdot 1.01494 \quad . \tag{4.8}$$

More concretely, Adams [A1, Theorem 2.5] showed that $v_3^\infty = \nu_3$, and that this volume is attained exclusively by the Gieseking manifold N_1 (cf. Example, **3.3**). Furthermore, he proved [A2, Theorem 3.2] that the manifold N_2 arising by glueing together two copies of $S_{reg}^\infty(\frac{\pi}{3})$ is the unique (non-orientable) hyperbolic $3-$manifold with 2 cusps, while – for $m > 2$ – the inequality (4.8) is strict.

Finally, let $n = 5$. By Theorem 3.5 and Tables 1 and 2, the volume of any $m-$cusped hyperbolic $5-$manifold M is bounded from below by

$$\mathrm{vol}_5(M) > m \cdot \frac{\Omega_3}{128 \cdot d_4 \cdot d_5(\infty)} \simeq m \cdot 0.39220 \quad . \tag{4.9}$$

However, to our knowledge, there is only one geometric construction of a cusped hyperbolic $5-$manifold known. It is due to Ratcliffe and Tschantz [RT]. Their manifold is of positive first Betti number and has 10 cusps. It is obtained by glueing the facets of a polytope $P \subset \overline{H^5}$ which in turn consists of 184,320 copies of the Coxeter simplex R whose symmetry group is given by the reflection group with Coxeter-Dynkin diagram

$$\Sigma(R) \quad : \qquad \circ\!\!-\!\!\!-\!\!\circ\!\!=\!\!\!=\!\!\circ\!\!-\!\!\!-\!\!\circ\!\!-\!\!\!-\!\!\circ\!\!-\!\!\!-\!\!\circ \qquad .$$

The volume of R was computed in [K2, (29)] and equals

$$\mathrm{vol}_5(R) = \frac{7}{46,080}\,\zeta(3) \quad .$$

Therefore, one obtains

$$\mathrm{vol}_5(M) = 28 \cdot \zeta(3) \simeq 33.65759 \quad , \tag{4.10}$$

which should be compared with (4.9). Finally, one deduces that

$$28 \cdot \zeta(3) \cdot \mathbb{N} \subset \mathbf{Vol}_5^{\infty} \quad .$$

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Mathematisches Institut
Georg-August-Universität Göttingen
Bunsenstraße 3–5
37073 Göttingen
Germany

ruth@cfgauss.uni-math.gwdg.de