Geometric And Functional Analysis 1016-443X/95/0x001-xx$1.50+0.20/0
Vol. 5, No. x (1995)

VOLUMES IN HYPERBOLIC 5-SPACE

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0. Introduction

In this work, we study the problem of determining volumes of five-dimensional hyperbolic polytopes. Generally, the non-Euclidean volume problem has not progressed very much since its origin last century with the works of N.I. Lobachevsky [Lo] in hyperbolic 3-space H^3 and of L. Schläfli [Sc] on the n-sphere S^n. While concrete results are available only for small dimensions n, which, for $n=3$, were reinterpreted and unified by H.S.M. Coxeter [Co1], their methods however are of timeless value.

A first observation is that each simplex in an n-dimensional space X^n of constant curvature is equidissectable into orthoschemes (see 1.2). An orthoscheme $R \subset X^n$ is a simplex bounded by hyperplanes $H_0, \dots, H_n$ such that $H_i \perp H_j$ for $|i-j|>1$. It is, up to congruence, uniquely determined by its (at most n) non-right dihedral angles, and, as the nomenclature indicates, it is described very conveniently by means of schemes or weighted graphs (see 1.1). For hyperbolic orthoschemes R with vertices p_i opposite to H_i, $i=0,\dots,n$, at most the endpoints p_0 and p_n of the orthogonal edge path $p_0p_1,\dots,p_{n-1}p_n$ may be points at infinity. In such cases, R is called *simply* or *doubly asymptotic*.

Secondly, by Schläfli's differential formula (cf. [Sc, No. 22]; see 2.1, Theorem 1), there is the following very simple, but fundamental expression for the volume differential of an orthoscheme $R \subset H^n$ in terms of infinitesimal angle variations:

$$d\,\mathrm{vol}_n(R) = \frac{1}{1-n}\sum_{j=1}^{n} \mathrm{vol}_{n-2}(F_j)\,d\alpha_j \ , \qquad \mathrm{vol}_0(F_j) := 1 \ , \tag{0}$$

wherein $F_j = R \cap H_{j-1} \cap H_j$ denotes the codimension two ridge of the dihedral angle α_j of R. This principle separates the odd dimensional volume problem from the even dimensional one (by Schläfli's reduction formula, the volume of an even dimensional orthoscheme is expressible in terms of the volumes of certain lower and odd dimensional ones (cf. [Sc, No. 26] and [K2, §14.2.2])); for $n=3$, the differential formula (0) was also known to Lobachevsky who used it to derive a volume formula for hyperbolic

3-orthoschemes by introducing a new function, the so-called Lobachevsky function, which is related to Euler's dilogarithm $\mathrm{Li}_2(z) = \sum_{r=1}^{\infty} \frac{z^r}{r^2}$ (cf. [Lo]; see 2.2).

The inductive nature of (0) allows in combination with Lobachevsky's result to attack the volume problem of higher dimensional hyperbolic orthoschemes.

First results about volumes in hyperbolic 5-space were obtained by P. Müller [Mü] in 1954 and by J. Böhm [B] in 1960. Using different approaches, their main result shows that, apart from lower order logarithms, the trilogarithm $\mathrm{Li}_3(z) = \sum_{r=1}^{\infty} \frac{z^r}{r^3}$, as a function of a single variable, suffices to express volumes of hyperbolic polytopes of dimension five. Both contributions, however, show some disadvantages as to explicitness and generalizability to higher dimensions. Böhm considered compact 5-orthoschemes R and adopted Coxeter's integration method, that is, he extended suitably the Schläfli differential (0) by the differentials of certain additional parameters associated to R (see also [BHe, §§5.7, 5.8]). Although this procedure allows to represent the volume of R as a single integral, the integrand is so involved that the formal relation to trilogarithms is worked out only in implicit form. Müller simplified the setting by representing each orthoscheme as an algebraic sum of simply asymptotic orthoschemes R. His integration method is based on a coordinate description for R in the upper half space model for hyperbolic space. The resulting volume formula for R depends on complicated arguments and is, expressed only in fragments, difficult to survey. As he pointed out, his approach is not extendable to dimensions six and higher (see also [BHe, Anmerkung [1], p. 257]).

In this paper, we present an explicit volume formula for doubly asymptotic 5-orthoschemes which is comparatively easy to survey; the formula is expressed in terms of polylogarithms of orders at most three in the dihedral angles, and it solves in principle the volume problem in hyperbolic 5-space. Indeed, by a result of C.H. Sah [S] and H. Debrunner [D], every hyperbolic polytope of *odd* dimension can be equidissected into doubly asymptotic orthoschemes (see 1.2, Proposition 1.2).

Doubly asymptotic orthoschemes are characterized by some nice and very useful properties (see 1.2). For example, a doubly asymptotic n-orthoscheme gives rise to a cycle of $n+1$ of such orthoschemes, and each cycle is parametrizable by $n+3$ points on $\mathrm{P}_1(\mathbf{R})$ (see Proposition 1.3 and the remarks thereafter). By forming circular graphs out of them, doubly asymptotic orthoschemes enable the construction of new polytopes in higher dimensional spaces (see Proposition 1.4).

For the solution of the volume problem, we make use of Schläfli's differential (0) and Lobachevsky's result in the three-dimensional case. In order to perform the remaining single integration of (0) for doubly asymptotic 5-orthoschemes, we distinguish two cases. By expressing the cycle property of a doubly asymptotic 5-orthoscheme R in terms of its dihedral angles $\alpha_i = \angle(H_{i-1}, H_i)$, $1 \le i \le 5$, we obtain the relation

$$\lambda := \cot\alpha_0 \tan\alpha_3 = \tan\alpha_1 \cot\alpha_4 = \cot\alpha_2 \tan\alpha_5 \ ,$$

wherein the additional angle α_0 can be seen as some angle in R.

The integration of the volume differential in the case $\lambda = 1$ was already performed in [K3]; there, we made use only of the so-called Trilobachevsky function $\text{Л}_3(\omega) = \frac{1}{4}\,\mathrm{Re}(\mathrm{Li}_3(e^{2i\omega}))$, $\omega \in \mathbf{R}$ (see 2.3, Theorem 2). The case $\lambda \neq 1$ is much more difficult; nevertheless, Theorem 3 of 2.3 provides a volume formula for an arbitrary doubly asymptotic 5-orthoscheme in terms of trilogarithm functions in its dihedral angles. It would be interesting to know whether the formula can be simplified in terms of Trilobachevsky functions.

The volume of a doubly asymptotic orthoscheme in hyperbolic 6-space is known to be expressible in terms of the volumes of certain spherical orthoschemes of dimensions at most three (see [K2, Theorem 14.4]). Therefore, by using our method and the result of Theorem 3, the next step would be to study volumes in hyperbolic 7-space (see 2.3, Remark (ii)).

Volumes of hyperbolic simplices are of interest in several other contexts. One reason is that the known volume functions, the polylogarithms of lower orders, satisfy certain cocyle equations which explain their importance for instance for group cohomology and Hilbert's third problem on scissors congruence, for Zagier's conjectures about values of zeta functions, and for regulators in algebraic K-theory (cf. [C] and the references therein). They also give rise to volumes of hyperbolic manifolds which characterize their topological type. In 2.4, we study implications of Theorem 3 for the volume spectrum $\mathbf{Vol}_5$ of five-dimensional hyperbolic manifolds. By computing covolumes of hyperbolic Coxeter groups and deriving estimates, we gain some informations about $\mathbf{Vol}_5$ (cf. also [K3]).

This work is organized as follows: In section 1, we discuss the algebrogeometrical properties of doubly asymptotic orthoschemes making use of the language of weighted graphs. Section 2 contains the analytical part with all volume computations. We conclude the paper with two appendices; in Appendix A, we collect some useful determinant identities. The part B1 of Appendix B contains a summary on polylogarithms, in particular, for orders two and three. In B2, we show how the characteristic volume integral can be represented in terms of trilogarithmic functions.

Acknowledgement. I would like to thank Herbert Gangl for helpful discussions about trilogarithm identities.

1. The Geometry of Polytopes in Hyperbolic Space

1.1 Description of polytopes in spaces of constant curvature. Let X^n be either the sphere S^n, Euclidean space E^n, or hyperbolic space $\overline{H^n} = H^n \cup \partial H^n$ extended by the set ∂H^n of points at infinity. Let S^n be embedded in E^{n+1} and $\overline{H^n}$ be realized in Minkowski space $E^{n,1}$ of signature $(n, 1)$.

An n-dimensional convex polytope $P \subset X^n$ is the non-empty intersection of finitely many closed half-spaces bounded by hyperplanes H_i with outer unit normal vectors e_i, $i \in I$, say. We always assume that P is indecomposable (i.e. $\{e_i\}_{i\in I}$ does not split into two mutually orthogonal subsets) and of finite volume.

To a polytope $P \subset X^n$ we can associate its Gram matrix $G(P) = (\langle e_i, e_j\rangle)_{i,j\in I}$ of the vectors $\{e_i\}_{i\in I}$. We assume from now on that P is acute-angled, which means, by abuse of language, that all non-right dihedral angles $\alpha_{ij} = \angle(H_i, H_j)$ are strictly less than $\frac{\pi}{2}$. Then P is determined by $G(P)$ in the following way (cf. [V, §2]):

Proposition 1.1. *Let $G = (g_{ij})$ be an indecomposable symmetric $m \times m$-matrix of rank $n+1$ with $g_{ii} = 1$ and $g_{ij} \le 0$ for $i \ne j$. Then G is the Gram matrix $G(P)$ of an acute-angled polytope $P \subset X^n$ of finite volume defined uniquely up to congruence. In particular,*

(1) *if G is positive definite (elliptic), then $m = n+1$, and P is a simplex on the sphere S^n;*
(2) *if G is positive semidefinite (parabolic), then $m = n+2$, and P is a simplex in E^{n+1};*
(3) *if G is of signature $(n, 1)$ (hyperbolic), then P is a convex polytope in $\overline{H^n}$ with m facets.*

If P has many right dihedral angles, then P can be better visualized through its scheme $\Sigma(P)$: In general, a scheme Σ is a weighted graph whose nodes i, j are either joined by an edge with positive weight ω_{ij}, or i, j are disjoint with weight $\omega_{ij} = 0$. The number $|\Sigma|$ of nodes is called the order of Σ. To every (connected) scheme Σ of order m there corresponds an (indecomposable) symmetric matrix $A(\Sigma)$ of order m with entries $a_{ii} = 1$ and $a_{ij} = -\omega_{ij} \le 0$ for $i \ne j$. Rank, determinant and character of definiteness of Σ are defined by the corresponding data of $A(\Sigma)$. In particular, Σ is elliptic, or parabolic, or hyperbolic if either all its components are elliptic, or – beside of elliptic ones – there is at least one parabolic component, or precisely one component is hyperbolic.

The scheme $\Sigma(P)$ of an acute-angled polytope $P \subset X^n$ is the scheme whose matrix coincides with the Gram matrix $G(P)$: The nodes i correspond to the bounding hyperplanes $H_i = e_i^\perp$ of P and the weights equal $-\langle e_i, e_j\rangle$, $i, j \in I$. $\Sigma(P)$ describes P uniquely up to congruence.

As for Coxeter polytopes in X^n (all the dihedral angles look like $\frac{\pi}{p}$, $p \in \mathbf{N}$, $p \geq 2$), we join two nodes related by the weight $\cos\alpha$ ($\alpha \in \left(0, \frac{\pi}{2}\right)$ or $\alpha = \frac{p\pi}{q}$, $p, q \in \mathbf{N}$ coprime with $1 \leq p < q$) by a single line marked α or $\frac{q}{p}$ (for $\alpha = \frac{\pi}{3}$, the weight is dropped). If two bounding hyperplanes of $P \subset X^n$, $X^n \neq S^n$, are parallel, their nodes are connected by an edge marked ∞; if they are divergent in hyperbolic space, we join them by a dotted line discarding the weight ≥ 1.

In the following we consider acute-angled polytopes in $\overline{H^n}$. Their geometry is particularly rich since their Gram matrices can be of arbitrarily high order as long as the index of inertia is one; moreover, depending on whether a vertex is an ordinary point, or a point at infinity, or ultrainfinite (that is, lying outside the cone in $E^{n,1}$ defining hyperbolic space and forcing its truncation to reach finite volume), the scheme of its vertex polytope (see below) is elliptic, or parabolic, or hyperbolic, encoding therefore all three geometries of constant curvature (cf. [V, §3]). For the purpose of volume computations, we restrict ourselves to appropriate families of hyperbolic polytopes, to the ones represented by simplest schemes, which, as we shall see, are simultaneously the most important ones.

1.2 Doubly asymptotic orthoschemes. The most basic and important family of polytopes are n-orthoschemes $R \subset X^n$, that is, simplices bounded by hyperplanes $H_0, \ldots, H_n$ such that $H_i \perp H_j$ for $|i-j| > 1$; their schemes $\Sigma(R)$ are linear of order $n+1$ with weights $\alpha_i = \angle(H_{i-1}, H_i)$, $1 \leq i \leq n$:

$$\Sigma(R) \qquad : \qquad \circ\overset{\alpha_1}{\text{———}}\circ\text{—}\cdots\text{—}\circ\overset{\alpha_n}{\text{———}}\circ \quad .$$

Denote by p_i the vertex of R opposite the facet $H_i \cap R$. Then, the vertices $p_0, \ldots, p_n$ form a totally orthogonal edge path $p_0p_1, \ldots, p_{n-1}p_n$, whose starting-point p_0 and final point p_n are called *principal vertices* of R. The vertex polytope of each vertex p_i in R is the simplex described by the subscheme of order n of $\Sigma(R)$ arising by discarding the node i together with the edges emanating from it. In hyperbolic space, among all vertices of R, at most p_0 and p_n may be points at infinity (the only parabolic subschemes of rank $n-1$ of $\Sigma(R)$ may be

$$\circ\overset{\alpha_1}{\text{———}}\circ\text{—}\cdots\text{—}\circ\overset{\alpha_{n-1}}{\text{————}}\circ \quad \text{and} \quad \circ\overset{\alpha_2}{\text{———}}\circ\text{—}\cdots\text{—}\circ\overset{\alpha_n}{\text{———}}\circ \quad ;$$

see also [BHe, Satz 15, p. 188]). If p_0 or/and p_n are at infinity, R is said to be *simply* or *doubly asymptotic*. Notice also that the dihedral angles α_i, $1 \leq i \leq n$, are always acute (see [BHe, Hilfssatz 2, p. 155]) and form a complete set of invariants for $R \subset \overline{H^n}$.

The notion of orthoschemes was introduced and systematically studied by Schläfli (cf. [Sc]), however, in spherical space only. These simplices are generalizations of right-angled triangles and arise in a very natural manner out of general polytopes by successive dropping of perpendiculars to lower dimensional faces.

This amounts to say that the scissors congruence groups $\mathcal{P}(X^n)$ (that is, the abelian groups generated by $[P]$ for each polytope P in X^n equipped with the relations (i) $[P \sqcup Q] = [P] + [Q]$ ($\sqcup$ denotes disjoint interior union) and (ii) $[P] = [Q]$ for P isometric to Q) are generated by the classes of orthoschemes.

In the hyperbolic case, there is an isomorphism between the groups $\mathcal{P}(H^n)$ and $\mathcal{P}(\overline{H^n})$ for $n > 0$ (cf. [DuS, Theorem 2.1, p. 162]). For later purposes, the following property is very important (cf. [D, Prop. 6.4, p. 142], [S, Prop. 3.7, p. 195]):

PROPOSITION 1.2. *For $n > 1$ odd, $\mathcal{P}(\overline{H^n})$ is generated by the classes of doubly asymptotic orthoschemes.*

The proof of Proposition 1.2 is based on two different cutting and pasting procedures for orthoschemes. First, each orthoscheme in H^n, $n \geq 2$ arbitrary, can be written as an alternating sum of $n+1$ simply asymptotic ones (cf. [Mü, p. 9], [BHe, p. 191 ff]). Secondly, and by a different method, each simply asymptotic n-orthoscheme can be represented as an alternating sum of precisely one simply asymptotic and n doubly asymptotic n-orthoschemes (cf. [D, Theorem, (i), p. 127]). Finally, in the combination of the two cutting and pasting processes for a simply asymptotic n-orthoscheme R, one can show that for n *odd* the two simply asymptotic orthoscheme summands in the scissors congruence relation for R cancel.

Doubly asymptotic orthoschemes $R \subset \overline{H^n}$ are represented by schemes

$$\Sigma(R) \qquad : \qquad \circ \overset{\alpha_1}{\text{——}} \circ - \cdots - \circ \overset{\alpha_n}{\text{——}} \circ$$

with the parabolic subschemes

$$\circ \overset{\alpha_1}{\text{——}} \circ - \cdots - \circ \overset{\alpha_{n-1}}{\text{——}} \circ \quad \text{and} \quad \circ \overset{\alpha_2}{\text{——}} \circ - \cdots - \circ \overset{\alpha_n}{\text{——}} \circ \quad .$$

DEFINITION. *Let the angle $0 < \alpha_0 < \frac{\pi}{2}$ be such that the graph $\circ \overset{\alpha_0}{\text{——}} \circ - \cdots - \circ \overset{\alpha_{n-2}}{\text{——}} \circ$ is parabolic of rank $n-1$.*

By (A1), Appendix A, we can write

$$\cos^2\alpha_0 = \det(\circ\overset{\alpha_1}{\text{———}}\circ-\cdots-\circ\overset{\alpha_{n-2}}{\text{———}}\circ)/\det(\circ\overset{\alpha_2}{\text{———}}\circ-\cdots-\circ\overset{\alpha_{n-2}}{\text{———}}\circ)$$

$$=1-\cfrac{\cos^2\alpha_1}{1-\cfrac{\cos^2\alpha_2}{\ddots-\cfrac{\cos^2\alpha_{n-3}}{1-\cos^2\alpha_{n-2}}}} =:\ 1-\frac{\cos^2\alpha_1|}{|1}-\cdots-\frac{\cos^2\alpha_{n-2}|}{|1}\ ,$$

using Pringsheim's notation for continued fractions.

Proposition 1.3. *The schemes* $\Sigma_i\ :\ \circ\overset{\alpha_i}{\text{———}}\circ-\cdots-\circ\overset{\alpha_{n+i-1}}{\text{———}}\circ$, $i\in\mathbf{Z}$ *modulo* $n+1$, *form a cycle of* $n+1$ *doubly asymptotic orthoschemes in* $\overline{H^n}$ *wherein two neighbours* Σ_i,Σ_{i+1} *have a principal vertex in common. Moreover,* $\det\Sigma_i=\det\Sigma_j$ *for* $i,j\in\mathbf{Z}$ *modulo* $n+1$.

Proof: We prove the periodicity of the weights, that means, $\alpha_{n+i+1}=\alpha_i$, and the determinant property for $i=j'-1=j''-2=0$, only.

Let $0<\alpha_{n+1}<\frac{\pi}{2}$ be such that $\circ\overset{\alpha_3}{\text{———}}\circ-\cdots-\circ\overset{\alpha_{n+1}}{\text{———}}\circ$ is parabolic. By (A1), this means that

$$\cos^2\alpha_{n+1}=\det(\circ\overset{\alpha_3}{\text{———}}\circ-\cdots-\circ\overset{\alpha_n}{\text{———}}\circ)\ /\ \det(\circ\overset{\alpha_3}{\text{———}}\circ-\cdots-\circ\overset{\alpha_{n-1}}{\text{———}}\circ)\ .$$

We have to show that $\alpha_{n+1}=\alpha_0$. For this, look at the extended schemes

$$\Sigma'\ :\ \circ\overset{\alpha_0}{\text{———}}\circ\overset{\alpha_1}{\text{———}}\circ-\cdots-\circ\overset{\alpha_{n-1}}{\text{———}}\circ\overset{\alpha_n}{\text{———}}\circ\quad\text{and}$$

$$\Sigma''\ :\ \circ\overset{\alpha_1}{\text{———}}\circ\overset{\alpha_2}{\text{———}}\circ-\cdots-\circ\overset{\alpha_n}{\text{———}}\circ\overset{\alpha_{n+1}}{\text{———}}\circ\ ,$$

whose determinants equal, by parabolicity,

$$\det\ \Sigma'=\det\ \Sigma_1=\det\ \Sigma_0=\det\,(\circ\overset{\alpha_0}{\text{———}}\circ-\cdots-\circ\overset{\alpha_{n-1}}{\text{———}}\circ)\ ,$$

$$\det\ \Sigma''=\det\ \Sigma_1=\det\ \Sigma_2=\det\,(\circ\overset{\alpha_2}{\text{———}}\circ-\cdots-\circ\overset{\alpha_{n+1}}{\text{———}}\circ)\ .$$

Therefore, $\det\Sigma'=\det\Sigma''=\det\Sigma_0=\det\Sigma_1=\det\Sigma_2$ which implies that

$$\begin{aligned}\det\Sigma_1&=-\cos^2\alpha_0\ \det(\circ\overset{\alpha_2}{\text{———}}\circ-\cdots-\circ\overset{\alpha_{n-1}}{\text{———}}\circ)\\&=-\cos^2\alpha_{n+1}\ \det(\circ\overset{\alpha_2}{\text{———}}\circ-\cdots-\circ\overset{\alpha_{n-1}}{\text{———}}\circ)\ .\end{aligned}$$

Hence, $\alpha_0=\alpha_{n+1}$. Since all Σ_i, $i\in\mathbf{Z}$ modulo $n+1$, have (equal) negative determinant, two parabolic subschemes of rank $n-1$ and elliptic subschemes of order less or equal to n, Σ_i are of signature $(n,1)$ and, being linear, describe therefore doubly asymptotic n-orthoschemes. Q.E.D.

Remarks: (i) The hyperbolic orthoscheme cycle of Proposition 1.3 is a by-product of Schläfli's generalization of Napier's rule (embodied in the Pentagramma Mirificum) for *spherical* triangles (cf. [Sc, p. 259-260], [Co2], [Co3], [IH]):

Start with a *spherical* $(n-2)$-orthoscheme $\circ\overset{\alpha_2}{\text{——}}\circ-\cdots-\circ\overset{\alpha_{n-1}}{\text{——}}\circ$, bounded by hyperplanes $H_1,\dots,H_{n-1}$, say. Denote by $p_1,\dots,p_{n-1}$ its vertices and by H_0, H_n the polar hyperplanes of the principal vertices p_1, p_{n-1}. By polarity, the additional positive weights are given by (use the linear dependence of $H_i,\dots,H_{n+i-1}$, i modulo $n+1$)

$$\begin{aligned}
\cos^2\alpha_n &= \cos^2\big(\angle(H_{n-1},H_n)\big) = \frac{\det(\circ\overset{\alpha_2}{\text{——}}\circ-\cdots-\circ\overset{\alpha_{n-1}}{\text{——}}\circ)}{\det(\circ\overset{\alpha_2}{\text{——}}\circ-\cdots-\circ\overset{\alpha_{n-2}}{\text{——}}\circ)}\ ,\\
\cos^2\alpha_0 &= \cos^2\big(\angle(H_n,H_0)\big) = \frac{\det(\circ\overset{\alpha_3}{\text{——}}\circ-\cdots-\circ\overset{\alpha_{n}}{\text{——}}\circ)}{\det(\circ\overset{\alpha_3}{\text{——}}\circ-\cdots-\circ\overset{\alpha_{n-1}}{\text{——}}\circ)}\ ,\\
\cos^2\alpha_1 &= \cos^2\big(\angle(H_0,H_1)\big) = \frac{\det(\circ\overset{\alpha_2}{\text{——}}\circ-\cdots-\circ\overset{\alpha_{n-1}}{\text{——}}\circ)}{\det(\circ\overset{\alpha_3}{\text{——}}\circ-\cdots-\circ\overset{\alpha_{n-2}}{\text{——}}\circ)}\ .
\end{aligned}\tag{1}$$

Analogously to the two-dimensional case, and by induction, the angles $\alpha_n,\alpha_0,\alpha_1$ can easily be seen as edge lengths in the spherical orthoscheme

$$\circ\overset{\alpha_2}{\text{——}}\circ-\cdots-\circ\overset{\alpha_{n-1}}{\text{——}}\circ\ .$$

More precisely,

$$\begin{aligned}
\alpha_n &= \angle(H_{n-1},H_n) = \frac{\pi}{2} - l(p_{n-2},p_{n-1})\ ,\\
\alpha_0 &= \angle(H_n,H_0) = l(p_{n-1},p_1)\ ,\\
\alpha_1 &= \angle(H_0,H_1) = \frac{\pi}{2} - l(p_1,p_2)\ ,
\end{aligned}$$

where $l(p_i,p_j)$ denotes the length of the edge p_ip_j.

Finally, $H_0,\dots,H_n$ form a cycle wherein non-consecutive hyperplanes, by polarity, are mutually perpendicular, and any consecutive set of $n-1$ hyperplanes bound a spherical $(n-2)$-orthoscheme (for $n=4$, this is Napier's Pentagramma Mirificum).

(ii) Let $\Sigma(R)\ :\ \circ\overset{\alpha_1}{\text{——}}\circ-\cdots-\circ\overset{\alpha_{n}}{\text{——}}\circ$ be a doubly asymptotic n-orthoscheme. Then, $\Sigma(R)$ is (up to congruence) uniquely determined by $n-2$ of its n dihedral angles and can be parametrized through $\mathrm{P}_1(\mathbf{R})$ in the following way (cf. [Sc, Nr. 27, p. 256ff], [Co3, §3]): Let $c_i := \cos^2\alpha_i$, $1\le i\le n-2$. Set, for $k=0,1,\dots$,

$$\begin{aligned}
c_{n-1+k} := \cos^2\alpha_{n-1+k} &= \frac{\det(\circ\overset{\alpha_{k+1}}{\text{——}}\circ-\cdots-\circ\overset{\alpha_{n+k-2}}{\text{——}}\circ)}{\det(\circ\overset{\alpha_{k+1}}{\text{——}}\circ-\cdots-\circ\overset{\alpha_{n+k-3}}{\text{——}}\circ)}\\
&= 1 - \frac{c_{n-2+k}|}{|1} - \cdots - \frac{c_{k+1}|}{|1}\ .
\end{aligned}\tag{2}$$

By Proposition 1.3, the sequence $\{c_l\}_{l\ge1}$ has period $n+1$, that is, $c_{n+2}=c_1$.

So, let x_0, x_1, x_2 be three distinct points on $\mathrm{P}_1(\mathbf{R})$ and choose $n-2$ further points $x_3, \dots, x_{n+1}$ such that their cross-ratios give

$$\{x_{l-1}, x_{l+2}; x_l, x_{l+1}\} = \frac{x_{l-1}-x_l}{x_{l+2}-x_l} : \frac{x_{l-1}-x_{l+1}}{x_{l+2}-x_{l+1}} = c_l \ , \quad l = 1, \dots, n-2 \ . \qquad (3)$$

Combining some properties of cross-ratios one can check that

$$\{x_{n-2}, x_{n+1}; x_{n-1}, x_n\} = 1 - \frac{c_{n-2}|}{1|} - \cdots - \frac{c_1|}{1|} \quad .$$

Hence, $\{x_{n-2}, x_{n+1}; x_{n-1}, x_n\} = c_{n-1}$ (see (2)). Therefore, by identifying $x_{l+n+2} = x_l$, (3) holds for all $l = 1, \dots, n$ and we have a cycle of $n+3$ points on $\mathrm{P}_1(\mathbf{R})$ parametrizing the Napier cycle of Proposition 1.3.

Starting with a Napier cycle of doubly asymptotic orthoschemes in $\overline{H^m}$, we can construct new families of polytopes in $\overline{H^{lm+k}}$. Look at full periods of doubly asymptotic orthoschemes in $\overline{H^{2n+1}}$, $n \geq 1$, which split into identical halves (cf. [Sc, No. 28, p. 261 ff]); that is, their (extended) schemes (of order $2n+3$) are of the form

$$\begin{aligned} \Sigma_0^{2n+3} \ &: \ \circ\overset{\alpha_0}{\text{———}}\circ - \cdots - \circ\overset{\alpha_n}{\text{———}}\circ\overset{\alpha_0}{\text{———}}\circ\overset{\alpha_1}{\text{———}}\circ - \cdots - \circ\overset{\alpha_n}{\text{———}}\circ \ \text{ such that} \\ \Sigma_i^{2n+1} \ &: \ \circ\overset{\alpha_i}{\text{———}}\circ - \cdots - \circ\overset{\alpha_n}{\text{———}}\circ\overset{\alpha_0}{\text{———}}\circ\overset{\alpha_1}{\text{———}}\circ - \cdots - \circ\overset{\alpha_{n-2+i}}{\text{————}}\circ \ , \end{aligned} \qquad (4)$$

for $i = 0, 1, 2$, are parabolic subschemes. By (A2), this means that

$$\begin{aligned} &\det \Sigma_0^n \cdot \{\det \Sigma_0^{n+1} - \cos^2\alpha_n \det \Sigma_1^{n-1}\} = \\ &\det(\circ\overset{\alpha_0}{\text{———}}\circ - \cdots - \circ\overset{\alpha_{n-2}}{\text{————}}\circ) \cdot \Big\{ \det(\circ\overset{\alpha_0}{\text{———}}\circ - \cdots - \circ\overset{\alpha_{n-1}}{\text{————}}\circ) - \\ &\quad - \cos^2\alpha_n \ \det(\circ\overset{\alpha_1}{\text{———}}\circ - \cdots - \circ\overset{\alpha_{n-2}}{\text{————}}\circ) \Big\} \ = \ 0 \quad . \end{aligned} \qquad (5)$$

Since Σ_0^n is elliptic, we obtain:

$$\begin{aligned} \cos^2\alpha_n &= \frac{\det \Sigma_0^{n+1}}{\det \Sigma_1^{n-1}} = \frac{\det \Sigma_0^n \cdot \det \Sigma_0^{n+1}}{\det \Sigma_1^{n-1} \cdot \det \Sigma_0^n} \\ &= \left\{ 1 - \frac{\cos^2\alpha_0|}{|1} - \cdots - \frac{\cos^2\alpha_{n-1}|}{|1} \right\} \cdot \left\{ 1 - \frac{\cos^2\alpha_{n-1}|}{|1} - \cdots - \frac{\cos^2\alpha_0|}{|1} \right\} \end{aligned}$$

and cyclic permutations of it. Apart from the $n+1$ (usually) different doubly asymptotic $(2n+1)$-orthoschemes Σ_i^{2n+2}, $0 \leq i \leq n$, in the Napier cycle, we can construct the following hyperbolic polytopes:

PROPOSITION 1.4. *Let $m, n \in \mathbf{N}$ be such that $(m,n) \neq (1,1)$. Suppose Σ_0^{2n+3} to be as in* (4)*, and denote by Ω_n^m the cyclic scheme of m repetitions of Σ_0^{n+2} : $\circ\overset{\alpha_0}{\text{———}}\circ - \cdots - \circ\overset{\alpha_n}{\text{———}}\circ$. Then, Ω_n^m is hyperbolic and of finite volume for $m = 1, 2, 4$ and arbitrary $n \geq 1$. In particular,*

(a) Ω_n^1 *describes a compact simplex in* H^n*;*
(b) Ω_n^2 *describes a totally asymptotic simplex in* $\overline{H^{2n+1}}$*;*
(c) Ω_n^4 *describes a doubly truncated orthoscheme in* $\overline{H^{4n+1}}$*.*

Proof: First, we compute the determinants $\det \Omega_n^m$ for $m, n \geq 1$, $(m, n) \neq (1, 1)$. By Lemma A, Appendix A, we obtain

$$\det \Omega_n^m = \begin{cases} 0 & \text{for } m \equiv 0(4) \ ; \\ -2 \prod\limits_{i=0}^{n} \cos^m \alpha_i & \text{for } m \equiv 1, 3(4) \ ; \\ -4 \prod\limits_{i=0}^{n} \cos^m \alpha_i & \text{for } m \equiv 2(4) \ . \end{cases}$$

Ad (a): Suppose that $n > 1$. Since Ω_n^1 is of order $n + 1$ and contains elliptic subschemes of order n, $\det \Omega_n^1 < 0$ implies that $\mathrm{sign}\Omega_n^1 = (n, 1)$. Hence, by 1.1, Ω_n^1 is a compact hyperbolic n-simplex.

Ad (b): Ω_n^2 is of order $2(n + 1)$ with elliptic subschemes of order $2n$. Again, $\det \Omega_n^2 < 0$ guarantees that $\mathrm{sign}\Omega_n^2 = (2n + 1, 1)$. Therefore, Ω_n^2 yields a non-compact $(2n+1)$-simplex of finite volume *all* of whose vertices are at infinity.

Ad (c): Here, $\det \Omega_n^4 = 0$ for $n \geq 1$. Ω_n^4 is of order $4n + 4$ and contains, by discarding any two antipodal nodes, two (disjoint) parabolic subschemes of rank $2n$, each; by discarding two non-adjacent non-antipodal nodes in Ω_n^4, we are left with one hyperbolic and one elliptic subscheme whose signatures add up to $(4n + 1, 1)$. Together with $\det \Omega_n^4 = 0$ this implies that $\mathrm{sign}\Omega_n^4 = (4n + 1, 1)$. It is easy to see that Ω_n^4 is an orthoscheme in $\overline{H^{4n+1}}$ whose ultrainfinite principal vertices are cut off by means of their polar hyperplanes (cf. [IH, §3, p. 530]; see also Remark (i)).

It remains to show that, for $m = 3$ and $m \geq 5$, $n \geq 1$, Ω_n^m cannot describe a hyperbolic polytope of finite volume.

Let $m = 3$. Since Ω_n^3 is of order $3(n + 1)$, has negative determinant and contains elliptic subschemes of order $3n + 1$, we obtain $\mathrm{sign}\Omega_n^3 = (3n + 2, 1)$. Therefore, Ω_n^3 is a $(3n + 2)$-simplex containing an open subset of $\overline{H^{3n+2}}$; but it is of infinite volume since all its vertices are ultrainfinite (described by hyperbolic subschemes of order $3n + 2$).

For $m \geq 5$, Ω_n^m is superhyperbolic, that means, of index of inertia bigger than one. To prove this, we look first at the case $m = 5$. It is easy to see that Ω_n^5 contains the two disjoint subschemes

$$\circ \overset{\alpha_0}{\text{———}} \circ \!-\! \cdots \!-\! \circ \overset{\alpha_n}{\text{———}} \circ \overset{\alpha_0}{\text{———}} \circ \!-\! \cdots \!-\! \circ \overset{\alpha_{n-1}}{\text{———}} \circ \quad \text{and}$$
$$\circ \overset{\alpha_0}{\text{———}} \circ \!-\! \cdots \!-\! \circ \overset{\alpha_n}{\text{———}} \circ \overset{\alpha_0}{\text{———}} \circ \!-\! \cdots \!-\! \circ \overset{\alpha_n}{\text{———}} \circ \ ,$$

which are both hyperbolic of signature $(2n + 1, 1)$. Therefore, Ω_n^5 contains a subscheme of signature $(4n+2,2)$. The same reasoning works also for $m>5$. Q.E.D.

1.3 The totally asymptotic regular simplex. A regular simplex $S_{reg}(2\alpha) \subset \overline{H^n}$, $n \geq 2$, with dihedral angles 2α satisfying $\frac{1}{n} < \cos(2\alpha) \leq \frac{1}{n-1}$ can be dissected into orthoschemes; by drawing perpendiculars starting from its center or from a vertex, $S_{reg}(2\alpha)$ admits the subdivisions

$$\left[S_{reg}(2\alpha)\right] = (n+1)![\sigma_{n+1}] = n![\nu_{n+1}] \ ,$$

where the simplices σ_{n+1} and ν_{n+1} are defined by the schemes

$$\begin{array}{lcl} \sigma_{n+1} & : & \circ \overset{\alpha}{\text{——}} \circ \text{——} \circ \text{—} \cdots \text{—} \circ \text{——} \circ \text{——} \circ \ ; \\ \nu_{n+1} & : & \circ \overset{2\alpha}{\text{———}} \circ \overset{\alpha}{\text{———}} \circ \text{——} \circ \text{—} \cdots \text{—} \circ \text{——} \circ \text{——} \circ \ . \end{array}$$

If $S^{\infty}_{reg}(2\alpha)$ is a totally asymptotic regular n-simplex, that is, $\cos(2\alpha) = \frac{1}{n-1}$, then σ_{n+1} is simply asymptotic, and ν_{n+1} is doubly asymptotic. Put $\nu^0_{n+1} := \nu_{n+1}$ and define, for $0 < i \leq [\frac{n-1}{2}]$,

$$\nu^i_{n+1} \quad : \quad \underset{0}{\circ} \text{——} \circ \text{—} \cdots \text{—} \circ \text{——} \circ \overset{\alpha}{\text{———}} \underset{i}{\circ} \overset{2\alpha}{\text{———}} \circ \overset{\alpha}{\text{———}} \circ \text{——} \circ \text{—} \cdots \text{—} \circ \text{——} \circ \text{——} \underset{n}{\circ} \ .$$

Then, $\nu^i_{n+1} = \nu^{n-1-i}_{n+1}$, and there are the following identities:

$$[\,\nu^i_{n+1}\,] = \binom{n+1}{i+1} [\,\sigma_{n+1}\,] \qquad \text{for } i = 0, 1, \ldots, \left[\frac{n-1}{2}\right] \ .$$

These relations are consequences of different dissections of $S_{reg}(2\alpha)$ as Schläfli observed for their spherical analogues (cf. [Sc, section I, p. 271] and [D, (7.4), p. 147]). On the other hand, in the hyperbolic totally asymptotic case, the dissecting doubly asymptotic orthoschemes ν^i_{n+1}, $i = 0, \ldots, n-1$, belong to a Napier cycle (see Proposition 1.3). For the remaining Napier neighbor

$$\nu^n_{n+1} \quad : \quad \circ \overset{\alpha}{\text{——}} \circ \text{——} \circ \text{—} \cdots \text{—} \circ \text{——} \circ \overset{\alpha}{\text{——}} \circ \ ,$$

however, it is an open problem whether there is a scissors congruence relation connecting $[\nu^n_{n+1}]$ to $[\nu^i_{n+1}]$, $i = 0, \ldots, n-1$.

The regular simplex with its volume is a distinguished object and appears in different contexts, not only in geometry. Its importance stems also from the extremality property that a hyperbolic simplex is of maximal volume if and only if it is regular and totally asymptotic (see 2.4).

2. Volumes of Doubly Asymptotic 5-orthoschemes

2.1 Schläfli's volume differential formula. Our aim is to derive a volume formula for a hyperbolic 5-simplex. For this, by means of Proposition 1.2, it is sufficient to consider doubly asymptotic orthoschemes. We shall make use of Schläfli's formula expressing the volume differential of a simplex in terms of infinitesimal angle perturbations (cf. [Sc, Satz, p. 235]). In the version for hyperbolic orthoschemes, it says (cf. [AVSo, 2.2, p. 118]):

THEOREM 1. *For a family of hyperbolic n-orthoschemes R, $n \geq 2$, with dihedral angles α_j attached at the codimension two faces F_j $(1 \leq j \leq n)$ of R, the volume differential $d\,\mathrm{vol}_n$ can be represented by*

$$d\,\mathrm{vol}_n(R) = \frac{1}{1-n} \sum_{j=1}^{n} \mathrm{vol}_{n-2}(F_j) d\alpha_j \ , \quad \mathrm{vol}_0(F_j) := 1 \ . \tag{6}$$

Formula (6) holds also for a family of simply or doubly asymptotic (and even truncated) orthoschemes in $\overline{H^n}$, $n > 3$ (cf. [K2, §14.2.1, p. 310]), whereas for $n = 3$, (6) remains valid provided that each vertex at infinity is cut off by means of a small horosphere before measuring edge lengths (cf. [M, p. 310]).

2.2 The three dimensional case. In order to integrate Schläfli's differential for hyperbolic 5-orthoschemes, we have to express the volume coefficients in (6) in terms of dihedral angles. This can be achieved by means of Lobachevsky's formula for hyperbolic 3-orthoschemes (cf. [Lo]): Let R denote a hyperbolic 3-orthoscheme,

$$\Sigma(R) \qquad : \qquad \circ \overset{\alpha_1}{\text{———}} \circ \overset{\alpha_2}{\text{———}} \circ \overset{\alpha_3}{\text{———}} \circ \ .$$

Since $\det \Sigma(R) < 0$, we obtain the realization condition $\cos\alpha_2 > \sin\alpha_1 \sin\alpha_3$ for R. It is very convenient to introduce an additional angle $0 \leq \theta \leq \frac{\pi}{2}$ defined by

$$\tan^2\theta = \frac{|\det \Sigma(R)|}{\cos^2\alpha_1 \cos^2\alpha_3} = \frac{\cos^2\alpha_2 - \sin^2\alpha_1 \sin^2\alpha_3}{\cos^2\alpha_1 \cos^2\alpha_3} \ ,$$

the so-called *principal parameter* of R. In terms of the imaginary part of Euler's Dilogarithm $\mathrm{Li}_2(z) = \sum_{r=1}^{\infty} \frac{z^r}{r^2}$, $|z| \leq 1$, (see Appendix B1)

$$Л_2(\omega) = \tfrac{1}{2} \mathrm{Im}\left(\mathrm{Li}_2(e^{2i\omega})\right) = \tfrac{1}{2} \sum_{r=1}^{\infty} \frac{\sin(2r\omega)}{r^2} \ ,$$

Lobachevsky derived the formula

$$\begin{aligned}\mathrm{vol}_3(R) = \tfrac{1}{4}\Big\{ & \text{Л}_2(\alpha_1+\theta) - \text{Л}_2(\alpha_1-\theta) + \text{Л}_2\left(\frac{\pi}{2}+\alpha_2-\theta\right) + \\ & + \text{Л}_2\left(\frac{\pi}{2}-\alpha_2-\theta\right) + \text{Л}_2(\alpha_3+\theta) - \text{Л}_2(\alpha_3-\theta) + 2\text{Л}_2\left(\frac{\pi}{2}-\theta\right)\Big\} \,.\end{aligned} \tag{7}$$

This volume formula for a hyperbolic 3-orthoscheme $R = p_0p_1p_2p_3$ is invariant with respect to polar truncation of an ultrainfinite principal vertex p_0 or p_3 of R as long as the line through its (longest) hypotenuse p_0p_3 is hyperbolic; otherwise, (7) has to be slightly modified (cf. [K1]).

By means of (7) and obvious dissections, we obtain further results. For example:

For a doubly asymptotic orthoscheme

$$\Sigma(R) \;:\; \circ\overset{\alpha}{\text{———}}\circ\overset{\alpha'}{\text{———}}\circ\overset{\alpha}{\text{———}}\circ \;,\; \alpha' := \frac{\pi}{2}-\alpha \;:\; \mathrm{vol}_3(R) = \frac{1}{2}\text{Л}_2(\alpha) \;. \tag{8}$$

Therefore, since $\mathcal{P}(\overline{H^3})$ is generated by doubly asymptotic orthoschemes, the Lobachevsky function $\text{Л}_2(\omega)$, $\omega \in \mathbf{R}$, is characterizable as hyperbolic 3-volume.

For a totally asymptotic simplex Q with scheme $\Sigma(Q) = \Omega_1^2$ (see Proposition 1.4):

$$\Sigma(Q) \quad : \quad \begin{array}{c} \alpha \\ \alpha' \; \square \; \alpha' \\ \alpha \end{array} \quad : \quad \mathrm{vol}_3(Q) = \text{Л}_2(\alpha) + \text{Л}_2(\alpha') \;. \tag{9}$$

For a general totally asymptotic simplex $S^\infty(\alpha,\beta,\gamma)$, $\alpha+\beta+\gamma=\pi$:

$$\mathrm{vol}_3\big(S^\infty(\alpha,\beta,\gamma)\big) = \text{Л}_2(\alpha) + \text{Л}_2(\beta) + \text{Л}_2(\gamma) \;. \tag{10}$$

In particular, for the regular simplex $S^\infty_{reg}(\frac{\pi}{3})$, we get $\mathrm{vol}_3(S^\infty_{reg}(\frac{\pi}{3})) = 3\,\text{Л}_2(\frac{\pi}{3}) \simeq 1.0149$.

2.3 The volume formula. Let $R \subset \overline{H^5}$ denote a doubly asymptotic orthoscheme represented by

$$\Sigma(R) \qquad : \qquad \circ\overset{\alpha_1}{\text{———}}\circ\overset{\alpha_2}{\text{———}}\circ\overset{\alpha_3}{\text{———}}\circ\overset{\alpha_4}{\text{———}}\circ\overset{\alpha_5}{\text{———}}\circ \;.$$

By Proposition 1.3, $R =: R_1$ is part of a 6-cycle of doubly asymptotic orthoschemes R_i, $i = 0,\dots,5$,

$$\Sigma(R_i) \;:\; \circ\overset{\alpha_i}{\text{———}}\circ\overset{\alpha_{i+1}}{\text{———}}\circ\overset{\alpha_{i+2}}{\text{———}}\circ\overset{\alpha_{i+3}}{\text{———}}\circ\overset{\alpha_{i+4}}{\text{———}}\circ \;, \quad i \in \mathbf{Z} \text{ modulo } 6 \;.$$

This cycle property can be put into the following analytical form (cf. also [K3, Lemma, p. 652]):

PROPOSITION 2.1. *The Napier cycle of doubly asymptotic 5-orthoschemes R_i with graphs*

$$\Sigma(R_i) \;:\; \circ\overset{\alpha_i}{\text{———}}\circ\overset{\alpha_{i+1}}{\text{———}}\circ\overset{\alpha_{i+2}}{\text{———}}\circ\overset{\alpha_{i+3}}{\text{———}}\circ\overset{\alpha_{i+4}}{\text{———}}\circ \;, \quad i \in \mathbf{Z} \textit{ modulo } 6 \;,$$

associated to $R = R_1$ satisfies the relation

$$\cot\alpha_0 \tan\alpha_3 = \tan\alpha_1 \cot\alpha_4 = \cot\alpha_2 \tan\alpha_5 = \tan\Theta\ , \tag{11}$$

where $0 \le \Theta \le \frac{\pi}{2}$ is the angle given by

$$\tan^2\Theta = \frac{|\det\,\Sigma(R)|}{\cos^2\alpha_1\cos^2\alpha_3\cos^2\alpha_5}\ .$$

Proof: First, we remark that the parabolicity condition for R_i yields

$$\det\,\Sigma(R_i) = -\cos^2\alpha_{i-2}\det(\circ\overset{\alpha_i}{\text{——}}\circ\overset{\alpha_{i+1}}{\text{——}}\circ\overset{\alpha_{i+2}}{\text{——}}\circ)\ ,\ i \in \mathbf{Z} \text{ modulo } 6\ .$$

Moreover, by Proposition 1.3, $\det\Sigma(R_i) = \det\Sigma(R_j)$ for $i, j \in \mathbf{Z}$ modulo 6, $i \ne j$. Therefore,

$$\Theta = \theta_{2i+1} = \frac{\pi}{2} - \theta_{2i}\ , \tag{12}$$

where $0 \le \theta_i \le \frac{\pi}{2}$ is defined by

$$\tan^2\theta_i = \frac{\det(\circ\overset{\alpha_i}{\text{——}}\circ\overset{\alpha_{i+1}}{\text{——}}\circ\overset{\alpha_{i+2}}{\text{——}}\circ)}{\cos^2\alpha_i\cos^2\alpha_{i+2}}\ , \quad i, j \in \mathbf{Z} \text{ modulo } 6\ .$$

The relations (11) follow now from properties of the Napier cycle for spherical 3-orthoschemes (see 1.2, Remark (i)) and (12): The length l_i of the edge where the dihedral angle α_i sits equals $\frac{\pi}{2} - \alpha_{i+3}$, and, by 1.2, (1), there is the correspondence

$$\tan l_i = \cot\alpha_{i+3} = \tan\theta_i \cot\alpha_i\ , \quad i, j \in \mathbf{Z} \text{ modulo } 6\ . \tag{13}$$

Q.E.D.

Notice that for $\Theta = 0$, that is, $\det\Sigma(R) = 0$, R is degenerated in dimension implying $\mathrm{vol}_5(R) = 0$.

Let R be the convex hull of p_i (see Figure 1), which are opposite to the bounding hyperplanes H_i, $0 \le i \le 5$, as usual; denote by F_i the apex face of R associated to α_i, $1 \le i \le 5$. Notice that the angle α_0 (see (13) and 1.2, Remark (i)) can be seen as a dihedral angle of R; more precisely, $\alpha_0' = \frac{\pi}{2} - \alpha_0 = p_0p_4p_1 = p_4p_1p_5$ (see Figure 1).

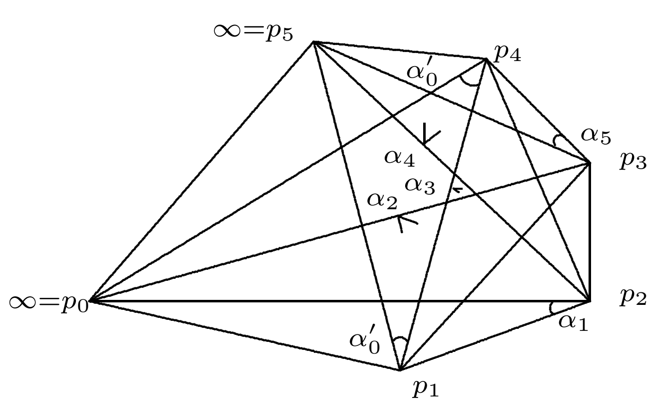


Figure 1

By Theorem 1, the volume differential $d\,\mathrm{vol}_5(R)$ of R takes the form

$$(-4)\,d\,\mathrm{vol}_5(R)=\sum_{j=1}^{5}\mathrm{vol}_3(F_j)\,d\alpha_j\ .$$

Notice that $\alpha_1,\ldots,\alpha_5$ are not independent parameters of R, wherefore the coefficients $\mathrm{vol}_3(F_j)$ are not all partial derivatives $\frac{\partial\,\mathrm{vol}_5(R)}{\partial\alpha_j}$. Since p_0, p_5 are vertices of R at infinity, the faces

$$F_j=R\cap H_{j-1}\cap H_j=p_0\cdots p_{j-2}\widehat{p_{j-1}}\widehat{p_j}p_{j+1}\cdots p_5\ ,\quad 1\le j\le 5\ ,$$

are asymptotic 3-orthoschemes with the schemes

$$\begin{aligned}
\Sigma(F_1)\quad &:\quad \circ\overset{\alpha_4'}{\text{———}}\circ\overset{\alpha_4}{\text{———}}\circ\overset{\alpha_5}{\text{———}}\circ\ ,\\
\Sigma(F_2)\quad &:\quad \circ\overset{\alpha_5}{\text{———}}\circ\overset{\alpha_5'}{\text{———}}\circ\overset{\alpha_5}{\text{———}}\circ\ ,\\
\Sigma(F_3)\quad &:\quad \circ\overset{\alpha_0'}{\text{———}}\circ\overset{\alpha_0}{\text{———}}\circ\overset{\alpha_0'}{\text{———}}\circ\ ,\\
\Sigma(F_4)\quad &:\quad \circ\overset{\alpha_1}{\text{———}}\circ\overset{\alpha_1'}{\text{———}}\circ\overset{\alpha_1}{\text{———}}\circ\ ,\\
\Sigma(F_5)\quad &:\quad \circ\overset{\alpha_1}{\text{———}}\circ\overset{\alpha_2}{\text{———}}\circ\overset{\alpha_2'}{\text{———}}\circ\ .
\end{aligned}$$

Their volumes can be computed by Lobachevsky's formula (7) in the following way:

$$\begin{aligned}
&\mathrm{vol}_3(F_1)=\tfrac{1}{4}\left\{\text{Л}_2\left(\frac{\pi}{2}-\alpha_4+\alpha_5\right)-\text{Л}_2\left(\frac{\pi}{2}+\alpha_4+\alpha_5\right)+2\text{Л}_2(\alpha_4)\right\}\ ;\\
&\mathrm{vol}_3(F_2)=\tfrac{1}{2}\text{Л}_2(\alpha_5)\ ;\ \mathrm{vol}_3(F_3)=\tfrac{1}{2}\text{Л}_2(\alpha_0')\ ;\ \mathrm{vol}_3(F_4)=\tfrac{1}{2}\text{Л}_2(\alpha_1)\ ;\qquad(14)\\
&\mathrm{vol}_3(F_5)=\tfrac{1}{4}\left\{\text{Л}_2\left(\frac{\pi}{2}-\alpha_2+\alpha_1\right)-\text{Л}_2\left(\frac{\pi}{2}+\alpha_2+\alpha_1\right)+2\text{Л}_2(\alpha_2)\right\}\ ,
\end{aligned}$$

with the dependences (see Proposition 2.1)

$$\lambda:=\tan\Theta=\cot\alpha_0\tan\alpha_3=\tan\alpha_1\cot\alpha_4=\cot\alpha_2\tan\alpha_5\ . \qquad (15)$$

Hence, we need to integrate the differential

$$\begin{aligned}
(-8)\,d\,\mathrm{vol}_5(R)=&-\tfrac{1}{2}\left\{\text{Л}_2\left(\frac{\pi}{2}-\alpha_5+\alpha_4\right)+\text{Л}_2\left(\frac{\pi}{2}+\alpha_5+\alpha_4\right)\right\}d\alpha_1+\\
&+\text{Л}_2(\alpha_4)d\alpha_1+\text{Л}_2(\alpha_5)d\alpha_2+\text{Л}_2(\alpha_0')d\alpha_3+\text{Л}_2(\alpha_1)d\alpha_4+\qquad(16)\\
&+\text{Л}_2(\alpha_2)d\alpha_5-\tfrac{1}{2}\left\{\text{Л}_2\left(\frac{\pi}{2}-\alpha_1+\alpha_2\right)+\text{Л}_2\left(\frac{\pi}{2}+\alpha_1+\alpha_2\right)\right\}d\alpha_5
\end{aligned}$$

subject to the relation (15), that is,

$$\begin{aligned}
\tan\alpha_2&=\tan\theta_2\tan\alpha_5\ ,\\
\tan\alpha_0&=\tan\theta_2\tan\alpha_3\ ,\qquad(15')\\
\tan\alpha_4&=\tan\theta_2\tan\alpha_1\ ,
\end{aligned}$$

with

$$\tan^2\theta_2 = \frac{\sin^2\alpha_2\sin^2\alpha_4 - \cos^2\alpha_3}{\cos^2\alpha_2\cos^2\alpha_4} = \cot^2\Theta\ .$$

Observe that we can choose a path of integration along which the parameter Θ is constant: R is characterized by three independent parameters. If we fix α_1, α_4 and let α_2 vary, for example, then, by (15′), Θ is constant. Hence, in the sequel, we may assume $\lambda = \tan\Theta$ to be constant.

Suppose $\alpha_1, \alpha_3, \alpha_5$ to be the free parameters among the five dihedral angles $\alpha_1, \dots, \alpha_5$ with the mutual dependences (15), and integrate the (complete) volume differential beginning from the (collapsed) orthoscheme R_{deg} of volume zero with dihedral angles $\alpha_1 = \alpha_2 = \alpha_4 = \alpha_5 = \frac{\pi}{2}$, $\alpha_0 = \alpha_3 = 0$. This yields, together with a symmetrization argument,

$$\begin{aligned}(-8)\,\mathrm{vol}_5(R) = {} & I(\lambda^{-1}, 0; \alpha_1) + \tfrac{1}{2} I(\lambda, 0; \alpha_2) - I(\lambda^{-1}, 0; \alpha_0') + \\ & + \tfrac{1}{2} I(\lambda, 0; \alpha_4) + I(\lambda^{-1}, 0; \alpha_5) - \tfrac{1}{4}\Big\{ I\left(\lambda, -\left(\frac{\pi}{2} + \alpha_1\right); \frac{\pi}{2} + \alpha_1 + \alpha_2\right) + \\ & + I\left(\lambda, -\left(\frac{\pi}{2} - \alpha_1\right); \frac{\pi}{2} - \alpha_1 + \alpha_2\right) - I\left(\lambda, -\left(\frac{\pi}{2} + \alpha_1\right); \pi + \alpha_1\right) - \\ & - I\left(\lambda, -\left(\frac{\pi}{2} - \alpha_1\right); \pi - \alpha_1\right) - I\left(\lambda, -\left(\frac{\pi}{2} + \alpha_5\right); \pi + \alpha_5\right) - \\ & - I\left(\lambda, -\left(\frac{\pi}{2} - \alpha_5\right); \pi - \alpha_5\right) + I\left(\lambda, -\left(\frac{\pi}{2} + \alpha_5\right); \frac{\pi}{2} + \alpha_5 + \alpha_4\right) + \\ & + I\left(\lambda, -\left(\frac{\pi}{2} - \alpha_5\right); \frac{\pi}{2} - \alpha_5 + \alpha_4\right)\Big\}\ .\end{aligned} \tag{17}$$

Here, $\lambda = \tan\Theta$ is as in (15), and $I(a, b; x)$ is the function in the variable x defined by

$$I(a, b; x) = \int_{\frac{\pi}{2}}^{x} \text{Л}_2(y)\, d\arctan\big(a\tan(b + y)\big)\ , \quad a, b \in \mathbf{R} \text{ fixed}\ . \tag{18}$$

Introducing the abbreviations

$$I_\delta(a, b; x) := I\left(a, -\left(\frac{\pi}{2} + b\right); \frac{\pi}{2} + b + x\right) - I\left(a, -\left(\frac{\pi}{2} + b\right); \frac{\pi}{2} + b + \frac{\pi}{2}\right)\ ; \tag{19}$$

$$I_{alt}(a, b; x) := I_\delta(a, b; x) + I_\delta(a, -b; x)\ , \tag{20}$$

formula (17) can be written more economically in the form

$$\begin{aligned}(-8)\,\mathrm{vol}_5(R) = {} & I(\lambda^{-1}, 0; \alpha_1) + \tfrac{1}{2} I(\lambda, 0; \alpha_2) - I(\lambda^{-1}, 0; \alpha_0') + \\ & + \tfrac{1}{2} I(\lambda, 0; \alpha_4) + I(\lambda^{-1}, 0; \alpha_5) - \tfrac{1}{4}\big\{I_{alt}(\lambda, \alpha_1; \alpha_2) + I_{alt}(\lambda, \alpha_5; \alpha_2)\big\}\ .\end{aligned} \tag{21}$$

The aim is now to express the integral (18) in terms of polylogarithms of orders less than or equal to three, related or even simpler functions. There are special cases which can be treated easier. In fact, there is a basic difference between $a = 1$ and $a \neq 1$; for the volume problem, the case

$\lambda = 1$, that means, $\alpha_1 = \alpha_4$, $\alpha_2 = \alpha_5$, $\alpha_3 = \alpha_0$, was already solved in [K3, Theorem, p. 659]. The result in terms of the Trilobachevsky function $\text{Л}_3(\omega)$, $\omega \in \mathbf{R}$, (see Appendix B1) is as follows:

THEOREM 2. *Let R denote a doubly asymptotic 5-orthoscheme with $\lambda = \tan\Theta = 1$, that is,*

$$\Sigma(R)\ :\ \circ\overset{\alpha_1}{\text{———}}\circ\overset{\alpha_2}{\text{———}}\circ\overset{\alpha_3}{\text{———}}\circ\overset{\alpha_1}{\text{———}}\circ\overset{\alpha_2}{\text{———}}\circ\ ,\quad \cos^2\alpha_1+\cos^2\alpha_2+\cos^2\alpha_3 = 1\ .$$

Then,

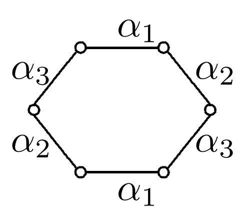
$$\begin{aligned}\mathrm{vol}_5(R) = \frac{1}{4}\Big\{ \text{Л}_3(\alpha_1) + \text{Л}_3(\alpha_2) - \frac{1}{2}\text{Л}_3\left(\frac{\pi}{2}-\alpha_3\right)\Big\} - \\ - \frac{1}{16}\Big\{\text{Л}_3(\frac{\pi}{2}+\alpha_1+\alpha_2) + \text{Л}_3\left(\frac{\pi}{2}-\alpha_1+\alpha_2\right)\Big\} + \frac{3}{64}\,\zeta(3)\,.\end{aligned} \tag{22}$$

Remarks: The orthoscheme

$$\Sigma(R)\ :\ \circ\overset{\alpha_1}{\text{———}}\circ\overset{\alpha_2}{\text{———}}\circ\overset{\alpha_3}{\text{———}}\circ\overset{\alpha_1}{\text{———}}\circ\overset{\alpha_2}{\text{———}}\circ\ ,\quad \cos^2\alpha_1+\cos^2\alpha_2+\cos^2\alpha_3 = 1\ ,$$

of Theorem 2 belongs to a Napier period splitting into identical halves and generating further hyperbolic polytopes (see Proposition 1.4). For example, we obtain the totally asymptotic 5-simplex Q with diagram

$\Sigma(Q) = \Omega_2^2\quad :$



$,\quad \cos^2\alpha_1 + \cos^2\alpha_2 + \cos^2\alpha_3 = 1\ .$

Figure 2

Its volume can be computed in the following way (cf. [K4, 3.4, Theorem 3]):

$$\begin{aligned}\mathrm{vol}_5(Q) = \tfrac{1}{2}\Big\{\text{Л}_3(\alpha_1) + \text{Л}_3(\alpha_2) + \text{Л}_3(\alpha_3) - \text{Л}_3\left(\frac{\pi}{2}-\alpha_1\right) - \\ - \text{Л}_3\left(\frac{\pi}{2}-\alpha_2\right) - \text{Л}_3\left(\frac{\pi}{2}-\alpha_1\right)\Big\} + \frac{7}{32}\,\zeta(3)\ .\end{aligned} \tag{23}$$

By a similar construction (see Proposition 1.4), we obtain another asymptotic polytope $R_2 \subset \overline{H^5}$ described by

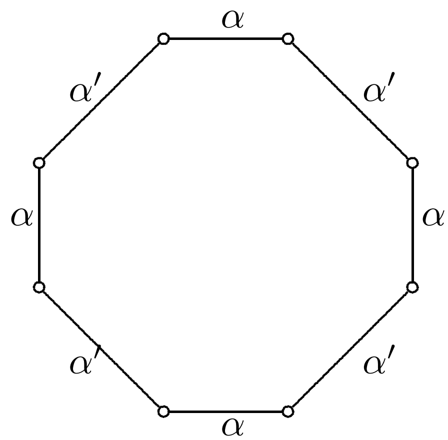
$\Sigma(R_2) = \Omega_1^4 \quad :$ 
$, \qquad \alpha' = \frac{\pi}{2} - \alpha \; .$

Figure 3

R_2 is a doubly truncated 5-orthoscheme whose dihedral angles α resp. α' are attached at the (doubly asymptotic) orthoscheme faces

$$\circ \overset{\alpha'}{\text{———}} \circ \overset{\alpha}{\text{———}} \circ \overset{\alpha'}{\text{———}} \circ \quad \text{resp.} \quad \circ \overset{\alpha}{\text{———}} \circ \overset{\alpha'}{\text{———}} \circ \overset{\alpha}{\text{———}} \circ \; .$$

For its volume, we obtain

$$\mathrm{vol}_5(R_2) = -\tfrac{1}{2}\big\{\text{Л}_3(\alpha) + \text{Л}_3(\alpha')\big\} + \tfrac{1}{32}\zeta(3) \; . \tag{24}$$

In comparison with the result for $\lambda = 1$ as presented in Theorem 2, the case $\lambda \neq 1$ is much more difficult; one reason for this is hidden behind the function theoretical behavior of the inverse tangent function

$$\arctan(x) = \frac{1}{2i} \log \frac{1+ix}{1-ix} \; ,$$

which does not allow the transformation of the integral $I(a,b;x)$ into $\text{Л}_3(\omega)$, $\omega \in \mathbf{R}$. In order to derive a formula for $\mathrm{vol}_5(R)$ in the case $\lambda \neq 1$ in terms of trilogarithmic functions with arguments connected to the dihedral angles, we have to transform suitably the integrals

$$I(a,b;x) = \int_{\frac{\pi}{2}}^{x} \text{Л}_2(y)\, d \arctan(a \tan(b+y)) \; , \quad a,b \in \mathbf{R} \text{ fixed} \; .$$

According to Appendix B2, (B16) and (B17), $I(a,b;x)$ can be expressed in terms of polylogarithms of orders less or equal to three as follows:

$$\begin{aligned} I(a,b;x) = {} & \Big(x - \frac{\pi}{2}\Big) \arctan(a \cot b) \log 2 + \text{Л}_2(x) \big\{ \arctan(a \tan(b+x)) + \\ & - \arctan(a \cot b)\big\} + \frac{1}{2} \log 2 \Big\{ \mathrm{Li}_2\Big((1+a)\sin(b+x), \frac{\pi}{2} - (b+x)\Big) - \\ & - \mathrm{Li}_2\Big((1-a)\sin(b+x), \frac{\pi}{2} - (b+x)\Big) \Big\} - \\ & - \frac{1}{2} \log 2 \Big\{ \mathrm{Li}_2\Big((1+a)\sin\Big(b + \frac{\pi}{2}\Big), b\Big) - \mathrm{Li}_2\Big((1-a)\sin\Big(b+\frac{\pi}{2}\Big), b\Big) \Big\} - \\ & - H(a,b;x) \; , \end{aligned} \tag{25}$$

where $\mathrm{Li}_2(r,\phi)$ denotes $\mathrm{Re}(\mathrm{Li}_2(re^{i\phi}))$. Moreover, for $\tan\omega = ia$, $c = c(\omega,b) = \cot(b+\omega)$, and $u = u(x) := \cot x$, $H(a,b;x)$ is given by

$$\begin{aligned} H(a,b;x) = \frac{1}{4}\mathrm{Re}\Bigg[& 2i\left(\frac{\pi}{2}-x\right)\log|\sin x|\cdot\log(1-cu) + F\left(\frac{ic}{1+ic};1-iu\right) - \\ & - F\left(\frac{-ic}{1-ic};1+iu\right) + F\left(\frac{1}{1-ic};1-cu\right) - F\left(\frac{1}{1+ic};1-cu\right) + \\ & + F\left(-\frac{1}{ic};\frac{1-cu}{1-iu}\right) - F\left(\frac{1}{ic};\frac{1-cu}{1+iu}\right) + F\left(\frac{2}{1+ic};\frac{1-cu}{1+iu}\right) - \\ & - F\left(\frac{2}{1-ic};\frac{1-cu}{1-iu}\right)\Bigg] , \end{aligned} \tag{26}$$

where, for $s,z\in\mathbf{C}$, $z\notin\mathbf{R}_{\le 0}$,

$$F(s;z) = \mathrm{Li}_3(sz) - \mathrm{Li}_3(s) - \log z\cdot\mathrm{Li}_2(sz) + \tfrac{1}{2}\log^2(z)\,\mathrm{Li}_1(sz)\ . \tag{27}$$

Summarizing, we obtain

THEOREM 3. *Denote by $R\subset\overline{H^5}$ a doubly asymptotic 5-orthoscheme represented by*

$$\Sigma(R)\quad : \quad \circ\overset{\alpha_1}{\text{———}}\circ\overset{\alpha_2}{\text{———}}\circ\overset{\alpha_3}{\text{———}}\circ\overset{\alpha_4}{\text{———}}\circ\overset{\alpha_5}{\text{———}}\circ \qquad \textit{with}$$

$$\lambda = \tan\Theta = \frac{|\det\Sigma(R)|^{1/2}}{\cos\alpha_1\cos\alpha_3\cos\alpha_5}\ , \qquad 0\le\Theta\le\frac{\pi}{2}\ .$$

Let $0\le\alpha_0\le\frac{\pi}{2}$ such that $\tan\alpha_0 = \cot\Theta\tan\alpha_3$. Then,

$$\begin{aligned} \mathrm{vol}_5(R) = -\frac{1}{8}\Big\{ & I(\lambda^{-1},0;\alpha_1) + \frac{1}{2}I(\lambda,0;\alpha_2) - I(\lambda^{-1},0;\alpha_0') + \frac{1}{2}I(\lambda,0;\alpha_4) + \\ & + I(\lambda^{-1},0;\alpha_5)\Big\} + \frac{1}{32}\big\{I_{alt}(\lambda,\alpha_1;\alpha_2) + I_{alt}(\lambda,\alpha_5;\alpha_2)\big\}\ , \end{aligned} \tag{28}$$

where $I(a,b;x)$ is the trilogarithmic function according to (25)–(27) *with the property $I(1,b;x) = -\text{Л}_3(x) - \frac{3}{16}\zeta(3)$, and $I_{alt}(a,b;x)$ is related to $I(a,b;x)$ by* (19) *and* (20)*.*

Remarks: (i) Theorem 3 provides the complete solution of the volume problem for five-dimensional non-Euclidean polytopes, at least in principle. Namely, by Proposition 1.2, the volume of a *compact* hyperbolic 5-orthoscheme is expressible as sum and difference of volumes of doubly asymptotic 5-orthoschemes. According to the trigonometric principle (that is, hyperbolic k-volume is i^k times spherical k-volume (cf. [BHe, p. 20-21, p. 210])), the formula in the compact case can be dualized by means of analytical continuation to yield a volume formula for spherical 5-orthoschemes. Finally, any non-Euclidean polytope is equidissectable to orthoschemes (see 1.2).

(ii) By the inductive character of Schläfli's volume differential representation (6), formula (28) in its explicit form is indispensable for tackling seven and higher dimensional volume problems by similar methods as presented here. The case of hyperbolic 7-volume will be of special interest because there are indications that the ordinary Tetralogarithm $\mathrm{Li}_4(z) = \sum_{r=1}^{\infty} \frac{z^r}{r^4}$ may not suffice anymore to express the volume of a hyperbolic 7-simplex (cf. [We, 8.5, p. 181]).

(iii) By volume comparison of equidissectable 5-polytopes (see, for example, [K3, (18), p. 656]), Theorem 3 allows to produce geometrical functional equations for the Trilogarithm. This function is not only of purely function theoretical interest; it appears, among other things, also in the context of Zagier's conjectures about values of Dedekind zeta functions $\zeta_F(s)$ at the place $s = 3$ and Borel's regulator for algebraic K-groups (see [C] for a survey on Goncharov's proof for $s = 3$ and further references).

(iv) By means of certain dissections performed on five-dimensional Coxeter orthoschemes which satisfy the angle condition of Theorem 2, we could compute the contents of the quasicrystallographic simplices (cf. [K3, (43), p. 663])

$$\nu_1 \quad : \quad \circ\overset{5}{\text{——}}\circ\overset{\frac{5}{2}}{\text{——}}\circ\overset{5}{\text{——}}\circ\overset{\frac{5}{2}}{\text{——}}\circ\overset{5}{\text{——}}\circ \quad ,$$

$$\nu_2 \quad : \quad \circ\overset{\frac{5}{2}}{\text{——}}\circ\overset{5}{\text{——}}\circ\overset{\frac{5}{2}}{\text{——}}\circ\overset{5}{\text{——}}\circ\overset{\frac{5}{2}}{\text{——}}\circ \quad .$$

These orthoschemes do *not* satisfy anymore the conditions of Theorem 2. Therefore, their volumes

$$\begin{aligned} \mathrm{vol}_5(\nu_1) &= \frac{1}{96} \text{Л}_3\left(\frac{\pi}{5}\right) + \frac{\zeta(3)}{800} \simeq 0.0020 \ , \\ \mathrm{vol}_5(\nu_2) &= \frac{1}{96} \text{Л}_3\left(\frac{\pi}{5}\right) \simeq 0.0005 \ , \end{aligned}$$

serve as test objects for (28).

2.4 Applications. Based on Theorem 3 of 2.3, there are various applications and further directions to study. We restrict our attention to the volume spectrum of hyperbolic manifolds of dimension five. A hyperbolic n-manifold is a complete Riemannian manifold of constant sectional curvature -1 and can be written in the form $M^n = H^n/\Gamma$, where Γ is a discrete and torsionfree group of isometries of H^n. The volumes $\mathrm{vol}_n(M^n)$ form the n-th volume spectrum denoted by

$$\mathbf{Vol}_n := \left\{ \mathrm{vol}_n(M^n) \;\middle|\; M^n = H^n/\Gamma \quad \text{hyperbolic manifold} \right\} .$$

By a result of Wang [W], $\mathbf{Vol}_n$ is a discrete subset of $\mathbf{R}_+$ for $n \neq 3$; for n even, this is a consequence of the theorem of Gauss-Bonnet. While the theorem of Gauss-Bonnet reflects essentially the structure of $\mathbf{Vol}_n$ and, in particular, yields the bound $\mathrm{vol}_n(M^n) \geq \frac{\mathrm{vol}_n(S^n)}{2}$ for n even, very little is known about $\mathbf{Vol}_n$ for $n \geq 5$ odd.

Hyperbolic Coxeter groups provide examples of hyperbolic orbifolds and, by passing over to subgroups of finite index acting without fixed points, of hyperbolic manifolds. Coxeter groups with (truncated) orthoschematic fundamental domains were classified by Im Hof [IH]. In dimension five, their covolumes can be computed by means of Theorem 2 and Theorem 3 (see [K3] and [K4] for the covolumes of all arithmetic hyperbolic Coxeter groups with linear and cyclic diagrams of order six). For example (cf. [K3, (39), p. 662]),

$$\mathrm{vol}_5(\circ - \circ = \circ - \circ - \circ - \circ) = \frac{7\,\zeta(3)}{46,080} \; . \qquad (29)$$

Ratcliffe and Tschantz [RT] constructed a non-orientable arithmetic hyperbolic 5-manifold M with positive first Betti number by glueing together the facets of a non-compact polytope P which is dissectable into 184,320 copies of $\circ - \circ = \circ - \circ - \circ - \circ$. Hence, by (29), $\mathrm{vol}_5(M) = 28\zeta(3)$, and

$$28\zeta(3) \cdot \mathbf{N} \subset \mathbf{Vol}_5 \quad .$$

This is all the information we have about $\mathbf{Vol}_5$ so far. In order to obtain estimates for elements in $\mathbf{Vol}_5$, the totally asymptotic regular n-simplex $S^{\infty}_{reg}(2\alpha)$, $\cos(2\alpha) = \frac{1}{n-1}$, is of importance:

THEOREM 4. *In $\overline{H^n}$, $n \geq 2$, a simplex is of maximal volume if and only if it is totally asymptotic and regular.*

This result was proved by Haagerup and Munkholm [HMu] by purely functional analytical methods.

By 1.3, we know that $\mathrm{vol}_n(S^{\infty}_{reg}) = n!\,\mathrm{vol}_n(\nu_{n+1})$, where ν_{n+1} is the doubly asymptotic n-orthoscheme

$$\nu_{n+1} \quad : \quad \circ \overset{2\alpha}{\text{———}} \circ \overset{\alpha}{\text{———}} \circ - \circ - \cdots - \circ - \circ - \circ \quad .$$

For $n = 5$, we get $\mathrm{vol}_5(S^{\infty}_{reg}) = 5!\,\mathrm{vol}_5(\nu_6)$, where

$$\nu_6 \quad : \quad \circ \overset{2\alpha}{\text{———}} \circ \overset{\alpha}{\text{———}} \circ - \circ - \circ - \circ \quad \text{with} \quad \lambda = \frac{\sqrt{\det(\circ - \circ - \circ - \circ)}}{\cos^2 \frac{\pi}{3}} = \sqrt{5} \quad .$$

By Theorem 3 of 2.3, we therefore obtain the following bound for the volume of any hyperbolic 5-simplex S:

$$\mathrm{vol}_5(S) \leq \mathrm{vol}_5(S^{\infty}_{reg}) \simeq 0.0578 \; .$$

Appendix A. Some Useful Determinant Identities

Let $\Sigma_n \ : \ \circ\frac{\alpha_1}{\qquad}\circ - \cdots - \circ\frac{\alpha_n}{\qquad}\circ$ be a scheme of order $n+1$ with weights $\cos\alpha_i$, $1 \le i \le n$, $\det\Sigma_0 := 1$ and $\det\Sigma_1 = \sin^2\alpha_1$. Put

$$\sigma_i^j \quad : \quad \circ\frac{\alpha_i}{\qquad}\circ - \cdots - \circ\frac{\alpha_j}{\qquad}\circ \qquad \text{for } 1 \le i \le j \le n \ .$$

Apart from $\sigma_1^n = \Sigma_n$, σ_i^j is a proper subscheme of Σ_n. There are the following recursion formulae (cf. [Sc, (1), p. 258, and (1), p. 261]):

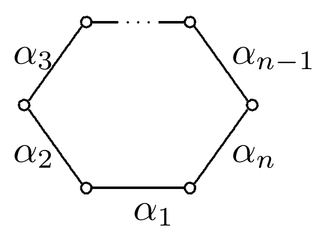
$$\det\Sigma_n = \det\Sigma_{n-1} - \cos^2\alpha_n \det\Sigma_{n-2} \ , \qquad n \ge 2 \quad ; \tag{A1}$$

$$\det\Sigma_n = \det\sigma_1^{k-1}\det\sigma_{k+1}^n - \cos^2\alpha_k \det\sigma_1^{k-2}\det\sigma_{k+2}^n \qquad \text{for } 2 < k \le n-2 \ . \tag{A2}$$

Moreover, by [Sc, (2), p. 259],

$$\det\sigma_1^{n-1}\det\sigma_2^n - \det\sigma_1^n\det\sigma_2^{n-2} = \prod_{i=1}^n \cos^2\alpha_i \ . \tag{A3}$$

For a cyclic scheme Δ_n of order n with weights $\cos\alpha_i$, $1 \le i \le n$,

$\Delta_n \quad :$ 

.

Figure 4

one can show (for one choice of indices) that (cf. [Sc, p. 262])

$$\det\Delta_n = -2\prod_{i=1}^n \cos\alpha_i + \det\sigma_1^{n-1} - \cos^2\alpha_n \det\sigma_2^{n-2} \ . \tag{A4}$$

Finally, by induction, we obtain the following

LEMMA A. *Let $m, n \in \mathbf{N}$, $(m,n) \ne (1,1)$. Suppose Σ_0^{2n+3} to be as in (4), and denote by Ω_n^m the cyclic scheme of m repetitions of Σ_0^{n+2} : $\circ\frac{\alpha_0}{\qquad}\circ - \cdots - \circ\frac{\alpha_n}{\qquad}\circ$. Then,*

$$\det\Omega_n^m = \begin{cases} 0 & \textit{for } m \equiv 0(4) \ ; \\ -2\prod_{i=0}^n \cos^m\alpha_i & \textit{for } m \equiv 1,3(4) \ ; \\ -4\prod_{i=0}^n \cos^m\alpha_i & \textit{for } m \equiv 2(4) \ . \end{cases} \tag{A5}$$

Appendix B. Trilogarithms and the Integral $I(a,b;x)$

B1. The Trilogarithm $\mathrm{Li}_3(z)$, as every polylogarithm function $\mathrm{Li}_k(z)$, arises as an iterated logarithm in the following way: Let

$$\mathrm{Li}_1(z) := -\log(1-z)\ , \quad \text{with} \quad \mathrm{Li}_1(z) = \sum_{r=1}^{\infty} \frac{z^r}{r} \quad \text{for} \quad |z| < 1\ ,$$

and define the k-logarithm or polylogarithm of order $k \geq 2$ by

$$\mathrm{Li}_k(z) := \int_0^z \frac{\mathrm{Li}_{k-1}(t)}{t} dt\ , \quad \text{with} \quad \mathrm{Li}_k(z) = \sum_{r=1}^{\infty} \frac{z^r}{r^k} \quad \text{for} \quad |z| \leq 1\ . \tag{B1}$$

Then, $\mathrm{Li}_k(1) = \zeta(k)$, $k \geq 2$, and there is the identity

$$\frac{1}{m^{k-1}} \mathrm{Li}_k(z^m) = \mathrm{Li}_k(z) + \mathrm{Li}_k(\omega z) + \cdots + \mathrm{Li}_k(\omega^{m-1} z)\ , \tag{B2}$$

where $\omega = e^{2\pi i/m}$, $m \geq 1$. Moreover, $\overline{\mathrm{Li}_k(z)} = \mathrm{Li}_k(\overline{z})$ for $k \geq 1$.

For arguments $z = e^{2i\alpha}$, $\alpha \in \mathbf{R}$, on the unit circle, real and imaginary part of $\mathrm{Li}_k(z)$ play a particular role. Define the higher Lobachevsky functions by

$$\begin{aligned} \text{Л}_{2k}(\alpha) &= \frac{1}{2^{2k-1}} \operatorname{Im}\left(\mathrm{Li}_{2k}(e^{2i\alpha})\right) = \frac{1}{2^{2k-1}} \sum_{r=1}^{\infty} \frac{\sin(2r\alpha)}{r^{2k}}\ , \\ \text{Л}_{2k+1}(\alpha) &= \frac{1}{2^{2k}} \operatorname{Re}\left(\mathrm{Li}_{2k+1}(e^{2i\alpha})\right) = \frac{1}{2^{2k}} \sum_{r=1}^{\infty} \frac{\cos(2r\alpha)}{r^{2k+1}}\ , \end{aligned} \tag{B3}$$

generalizing Lobachevsky's function (see 2.2)

$$\text{Л}_2(\alpha) = \tfrac{1}{2} \operatorname{Im}\left(\mathrm{Li}_2(e^{2i\alpha})\right) = -\int_0^{\alpha} \log|2\sin t|\, dt\ .$$

There are the relations

$$\text{Л}_{2k}(\alpha) = \int_0^{\alpha} \text{Л}_{2k-1}(t) dt\ , \quad \text{Л}_{2k+1}(\alpha) = \frac{1}{2^{2k}} \zeta(2k+1) - \int_0^{\alpha} \text{Л}_{2k}(t) dt\ . \tag{B4}$$

Moreover, $\text{Л}_k(\alpha)$ is π-periodic, even (odd) for k odd (even) and, by (B2), distributes according to

$$\frac{1}{m^{k-1}} \text{Л}_k(m\alpha) = \sum_{r=0}^{m-1} \text{Л}_k\left(\alpha + \frac{r\pi}{m}\right)\ . \tag{B5}$$

Now, $\mathrm{Li}_k(e^{2i\alpha})$ splits into one part consisting of the Lobachevsky function $\text{Л}_k(\alpha)$, while the other part is always elementary (cf. [L, (16), (17), (22), p. 300]).

For $z = re^{i\phi}$, $r > 0$, $0 \leq \phi < 2\pi$, put $\mathrm{Li}_k(r,\phi) := \operatorname{Re}(\mathrm{Li}_k(re^{i\phi}))$ in the standard way. Then, $\mathrm{Li}_k(r,\phi) = \mathrm{Li}_k(r,-\phi)$, and (cf. [L, (5.5), p. 121])

$$\mathrm{Li}_2(re^{i\phi}) = \mathrm{Li}_2(r,\phi) + i\big[\omega \log r + \text{Л}_2(\omega) + \text{Л}_2(\phi) - \text{Л}_2(\omega+\phi)\big] \ , \qquad \text{(B6)}$$

where $\tan\omega = \frac{r\sin\phi}{1-r\cos\phi}$. For $\mathrm{Li}_3(re^{i\phi})$, however, there is no equivalent to (B6).

B2. In order to express the integral (see 2.3, (23))

$$I(a,b;x) = \int_{\frac{\pi}{2}}^{x} \text{Л}_2(y) d\arctan\big(a\tan(b+y)\big) \ , \quad a,b\in\mathbf{R} \quad \text{fixed} \ ,$$

in terms of polylogarithms of orders less than or equal to three, we make use of the following integral expressions

$$\mathrm{Li}_2(z) = \int_1^\infty \log t \left\{\frac{1}{t-z} - \frac{1}{t}\right\} dt \ , \quad \mathrm{Li}_3(z) = \frac{1}{2}\int_1^\infty \log^2 t \left\{\frac{1}{t-z} - \frac{1}{t}\right\} dt \ .$$

Then, we can deduce the identities

$$\begin{aligned}\frac{1}{2}\int_1^\infty \log^2(t-a)\left\{\frac{1}{t-b} - \frac{1}{t-c}\right\} dt &= \mathrm{Li}_3\left(\frac{b-a}{1-a}\right) - \mathrm{Li}_3\left(\frac{c-a}{1-a}\right) + \\ &+ \log(1-a)\left\{\mathrm{Li}_2\left(\frac{b-a}{1-a}\right) - \mathrm{Li}_2\left(\frac{c-a}{1-a}\right)\right\} - \frac{1}{2}\log^2(1-a)\log\left(\frac{1-b}{1-c}\right) \ ;\end{aligned}$$

$$\begin{aligned}\int_1^\infty \log t\log(t-a)\left\{\frac{1}{t-b} - \frac{1}{t}\right\} dt &= \mathrm{Li}_3(b) + \mathrm{Li}_3\left(\frac{b-a}{b}\right) + \\ &+ \mathrm{Li}_3\left(\frac{b-a}{1-a}\right) - \mathrm{Li}_3\left(\frac{b-a}{b(1-a)}\right) - \mathrm{Li}_3\left(\frac{a}{a-1}\right) + \log(1-a)\Big\{\mathrm{Li}_2\left(\frac{b-a}{1-a}\right) - \\ &- \mathrm{Li}_2\left(\frac{b-a}{b(1-a)}\right) - \mathrm{Li}_2\left(\frac{a}{(a-1)}\right)\Big\} + \frac{1}{2}\log^2(1-a)\log\frac{a}{b} - \frac{1}{3}\log^3(1-a) \ .\end{aligned}$$

In the combination, they give rise to the following very useful integral expression, for $c \neq a,b$ (notice that [Mü, (A33)] is incomplete and has a misprint; see also [L, 8.4.3, p. 271]):

$$\begin{aligned}J(a,b,c;z) &= \int_0^z \log(1+at)\log(1+bt)d\log(1+ct) = \\ &= \mathrm{Li}_3\left(\frac{b}{a}\right) - \mathrm{Li}_3\left(\frac{c-b}{c-a}\right) + \mathrm{Li}_3\left(\frac{c}{c-a}\right) + \mathrm{Li}_3\left(\frac{c}{c-b}\right) + \\ &+ \mathrm{Li}_3\left(\frac{(c-b)(1+az)}{(c-a)(1+bz)}\right) - \mathrm{Li}_3\left(\frac{b(1+az)}{a(1+bz)}\right) - \mathrm{Li}_3\left(\frac{c(1+az)}{c-a}\right) - \\ &- \mathrm{Li}_3\left(\frac{c(1+bz)}{c-b}\right) + \log\left(\frac{1+az}{1+bz}\right)\Big\{\mathrm{Li}_2\left(\frac{b(1+az)}{a(1+bz)}\right) -\end{aligned}$$

$$- \mathrm{Li}_2\left(\frac{(c-b)(1+az)}{(c-a)(1+bz)}\right)\Bigg\} + \log(1+az)\,\mathrm{Li}_2\left(\frac{c(1+az)}{c-a}\right) +$$
$$+ \log(1+bz)\,\mathrm{Li}_2\left(\frac{c(1+bz)}{c-b}\right) + \log(1+az)\log(1+bz)\log\frac{a(1+cz)}{a-c} -$$
$$- \frac{1}{2}\log^2(1+bz)\log\frac{a(c-b)}{b(c-a)}\ , \tag{B7}$$

or, in order to abbreviate,

$$\begin{aligned} J(a,b,c;z) = F\left(\frac{c-b}{c-a};\frac{1+az}{1+bz}\right) - F\left(\frac{b}{a};\frac{1+az}{1+bz}\right) - \\ - F\left(\frac{c}{c-a};1+az\right) - F\left(\frac{c}{c-b};1+bz\right)\ , \end{aligned} \tag{B8}$$

where

$$F(s;z) = G(s;z) - G(s;1) \quad \text{with}$$
$$G(s;z) = \mathrm{Li}_3(sz) - \log z\cdot\mathrm{Li}_2(sz) + \tfrac{1}{2}\log^2 z\cdot\mathrm{Li}_1(sz)\ .$$

It remains to be checked under which conditions the identity (B8) is well-defined. Instead of studying the monodromy behavior of the functions involved in general, we consider only the case relevant for volume computations. Assume therefore that $a,b,c\in\mathbf{C}\setminus\mathbf{R}$ with $c\neq a,b$, and that $z=:x\in\mathbf{R}$. Then, the integrand in $J(a,b,c;x)$ (see (B7)) is a well-defined holomorphic function along the straight integration path $t\mapsto tx$, $t\in[0,1]$. On the other hand side, each summand of the form $F(s;z)=G(s;z)-G(s,1)$ can also be put in integral form since

$$G(s;z) = \frac{s}{2}\int_0^z \log^2 t\,\frac{dt}{1-st}$$

by twice integrating by parts the defining equation $\mathrm{Li}_3(sz)=\int_0^{sz}\frac{\mathrm{Li}_2(t)}{t}\,dt$. Hence, by the same remark as above, $F(s;z)$ and therefore equation (B8) make sense for the parameter range under consideration.

Now, we relate $I(a,b;x)$ to integrals of the form (B7). Since $I(1,b;x)=-\mathrm{Л}_3(x)-\frac{3}{16}\zeta(3)$, it is sufficient to consider the case $a\neq 1$. First, integration by parts together with [L, (28), p. 307] yields

$$I(a,b;x) = \mathrm{Л}_2(x)\arctan\big(a\tan(b+x)\big) + \frac{\log 2}{2}\Big\{\,\mathrm{Li}_2\Big((1+a)\sin(b+x),\frac{\pi}{2}-$$
$$-(b+x)\Big) - \mathrm{Li}_2\Big((1-a)\sin(b+x),\frac{\pi}{2}-(b+x)\Big)\Big\}-$$
$$-\frac{\log 2}{2}\left\{\mathrm{Li}_2\left((1+a)\sin\left(b+\frac{\pi}{2}\right),b\right) - \mathrm{Li}_2\left((1-a)\sin\left(b+\frac{\pi}{2}\right),b\right)\right\}+$$
$$+\int_{\frac{\pi}{2}}^{x}\arctan\big(a\tan(b+y)\big)\log\sin y\,dy\ , \tag{B9}$$

where $\mathrm{Li}_2(r,\phi)=\mathrm{Re}\big(\mathrm{Li}_2(re^{i\phi})\big)$. Next, we write

$$\begin{aligned}\int_{\frac{\pi}{2}}^{x}\arctan\big(a\tan(b+y)\big)\log\sin y\,dy &= \frac{1}{2i}\int_{\frac{\pi}{2}}^{x}\log\frac{1+ia\tan(b+y)}{1-ia\tan(b+y)}\log\sin y\,dy\\ &=\arctan(a\cot b)\Big\{\Big(x-\frac{\pi}{2}\Big)\log 2+\text{Л}_2(x)\Big\}+\\ &\quad+\frac{1}{2i}\int_{\frac{\pi}{2}}^{x}\log\frac{1+\cot(\omega-b)\cot y}{1-\cot(\omega+b)\cot y}\log\sin y\,dy\ ,\end{aligned}\tag{B10}$$

where $\omega\in\mathbf{C}$ is such that $\tan\omega=ia$ (that is, if $a=:\tanh A<1$, we have $\omega=iA$, while for $a=:\coth\tilde{A}>1$, we obtain $\omega=i(\tilde{A}+i\frac{\pi}{2})$). In the integral

$$i(\omega,b;x):=\frac{1}{2i}\int_{\frac{\pi}{2}}^{x}\log\frac{1+\cot(\omega-b)\cot y}{1-\cot(\omega+b)\cot y}\log\sin y\,dy\ ,$$

we make the variable change $t:=\cot y$ and decompose it such that

$$\begin{aligned}i(a,b;x)=&h(a,b;x)-h(-a,b;x)\quad\text{with}\\ h(a,b;x):=&-\frac{1}{4i}\int_0^{\cot x}\log\big(1-\cot(\omega+b)t\big)\log(1+t^2)\frac{dt}{1+t^2}\ .\end{aligned}\tag{B11}$$

Let $c=c(\omega,b):=\cot(\omega+b)$. Then, $c(\omega,b)=\overline{c(-\omega,b)}$ which implies that $h(a,b;x)=\overline{-h(-a,b;x)}$. Therefore,

$$i(a,b;x)=2\,\mathrm{Re}\big(h(a,b;x)\big)\ .\tag{B12}$$

By developing into partial fractions, we obtain

$$-8ih(\omega,b;x)=H_1^++H_1^-+H_2^++H_2^-\ ,\tag{B13}$$

wherein $H_i^\pm=H_i^\pm(\omega,b;x)$, $i=1,2$, stand for

$$\begin{aligned}H_1^\pm:=&\int_0^{\cot x}\log(1-ct)\log(1\pm it)\frac{dt}{1\pm it}\\ &=\pm\frac{1}{2i}\log^2(1\pm iu)\log(1-cu)\pm\frac{i}{2}J(\pm i,\pm i,-c;\cot x)\ ;\end{aligned}\tag{B14}$$

$$H_2^\pm:=\int_0^{\cot x}\log(1-ct)\log(1\pm it)\frac{dt}{1\mp it}=\pm iJ(-c,\pm i,\mp i;\cot x)\ .\tag{B15}$$

Combining equations (B7) up to (B15), we finally obtain

$$
\begin{aligned}
I(a,b;x) &= \left(x-\frac{\pi}{2}\right)\log 2\arctan(a\cot b) + \text{Л}_2(x)\Big\{\arctan(a\tan(b+x))+ \\
&+\arctan(a\cot b)\Big\} + \tfrac{1}{2}\log 2\Big\{\mathrm{Li}_2\left((1+a)\sin(b+x),\frac{\pi}{2}-(b+x)\right)- \\
&-\mathrm{Li}_2\left((1-a)\sin(b+x),\frac{\pi}{2}-(b+x)\right)\Big\}-\frac{1}{2}\log 2\left\{\mathrm{Li}_2\left((1+a)\sin\left(b+\frac{\pi}{2}\right),b\right)-\right. \\
&\left.-\mathrm{Li}_2\left((1-a)\sin(b+\frac{\pi}{2}),b\right)\right\} - H(a,b;x)\ , \qquad \text{(B16)}
\end{aligned}
$$

wherein, for $\tan\omega = ia$, $c = c(\omega,b) = \cot(b+\omega)$, and $u = u(x) := \cot x$,

$$
\begin{aligned}
H(a,b;x) &= -2\,\mathrm{Re}\,\big(h(a,b;x)\big) = \frac{1}{4}\,\mathrm{Re}\left[2i\left(\frac{\pi}{2}-x\right)\log|\sin x|\cdot\log(1-cu)+\right. \\
&+F\left(\frac{ic}{1+ic};1-iu\right)-F\left(\frac{-ic}{1-ic};1+iu\right)+F\left(\frac{1}{1-ic};1-cu\right)- \\
&-F\left(\frac{1}{1+ic};1-cu\right)+F\left(-\frac{1}{ic};\frac{1-cu}{1-iu}\right)-F\left(\frac{1}{ic};\frac{1-cu}{1+iu}\right)+ \\
&\left.+F\left(\frac{2}{1+ic};\frac{1-cu}{1+iu}\right)-F\left(\frac{2}{1-ic};\frac{1-cu}{1-iu}\right)\right]. \qquad \text{(B17)}
\end{aligned}
$$

For $s,z\in\mathbf{C}$, $z\notin\mathbf{R}_{\le 0}$, $F(s;z)$ denotes the function

$$
F(s;z) = G(s;z)-G(s;1) = \mathrm{Li}_3(sz)-\mathrm{Li}_3(s)-\log z\cdot\mathrm{Li}_2(sz)+\tfrac{1}{2}\log^2(z)\,\mathrm{Li}_1(sz)\ ,
$$

which satisfies $F(0;z) = F(s;1) = 0$ and $F(\overline{s};\overline{z}) = \overline{F(s;z)}$.

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Submitted: March 1994
Revised Version: November 1994