# Arithmetic property of growth rates of hyperbolic Coxeter groups （双曲コクセター群の増大度の数論的性質） <br> BY <br> TOMOSHIGE YUKITA 

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## Chapter 1

## Introduction

The upper half-space $\mathbb{H}^{d}=\left\{x=\left(x_{1}, \cdots, x_{d}\right) \in \mathbb{R}^{d} \mid x_{d}>0\right\}$ with the metric $|d x| / x_{d}$ is a model of hyperbolic $d$-space, so called the upper half-space model. In the upper half-space model $\mathbb{H}^{d}$, a hyperplane $H$ is defined to be a Euclidean hemisphere or half-plane orthogonal to $\mathbb{E}^{d-1}=\left\{x \in \mathbb{R}^{d} \mid x_{d}=0\right\}$. Every hyperplane $H$ subdivide $\mathbb{H}^{d}$ into two closed components $H^{+}$and $H^{-}$bounded by $H$, then $H^{ \pm}$is called a closed half-space bounded by $H$. According to the position of two hyperplanes $H_{1}$ and $H_{2}$, we define the dihedral angle between the closed half-spaces $H_{1}^{-}$and $H_{2}^{-}$as follows ; (i) if $H_{1}$ and $H_{2}$ are intersecting in $\mathbb{H}^{d}$, let us choose a point $x \in H_{1} \cap H_{2}$ and consider the outer normal vector $u_{1}$ and $u_{2}$ of $H_{1}^{-}$ and $H_{2}^{-}$at $x$. Then the dihedral angle between $H_{1}^{-}$and $H_{2}^{-}$is defined as the real number $\theta \in[0, \pi)$ satisfying $\cos \theta=-\left(u_{1}, u_{2}\right)$ where $(\cdot, \cdot)$ denotes the Euclidean inner product on $\mathbb{R}^{d}$. (ii) if $H_{1}$ and $H_{2}$ meet at a point on the boundary $\partial \mathbb{H}^{d}$, then we define the dihedral angle between $H_{1}^{-}$and $H_{2}^{-}$to be equal to zero.

We call $P \subset \mathbb{H}^{d}$ a hyperbolic d-polytope if $P$ can be written as the intersection of finitely many closed half-spaces and has a non-empty interior. In particular, a hyperbolic 2-(resp. 3-)polytopes are called a hyperbolic polygon (resp. polyhedron). A hyperbolic $d$-polytope $P \subset \mathbb{H}^{d}$ of finite volume is called a Coxeter polytope if all of its dihedral angles are of the form $\frac{\pi}{k}$ for an integer $k \geq 2$ or $k=\infty$ i.e., the intersection of the respective facets is a point on the boundary $\partial \mathbb{H}^{d}$. The set $S$ of reflections with respect to the facets of $P$ generates a discrete group $\Gamma<\operatorname{Isom}\left(\mathbb{H}^{d}\right)$, called a hyperbolic Coxeter group, and the pair $(\Gamma, S)$ is called the Coxeter system associated with $P$. Then $P$ becomes a fundamental domain for $\Gamma$. If $P$ is compact (resp. non-compact), the hyperbolic Coxeter group $\Gamma$ is called cocompact (resp. cofinite). The growth series $f_{S}(t)$ of a Coxeter system $(\Gamma, S)$ is the formal power series $\sum_{l=0}^{\infty} a_{l} t^{l}$ where $a_{l}$ is the number of elements of $\Gamma$ whose word length with respect to $S$ is equal to $l$. Then $\tau_{\Gamma}:=\limsup _{l \rightarrow \infty} \sqrt[l]{a_{l}}$ is called the growth rate of the Coxeter system $(\Gamma, S)$. By means of the CauchyHadamard theorem, $\tau_{\Gamma}$ is equal to the reciprocal of the radius of convergence $R$ of
$f_{S}(t)$. The growth series and the growth rate of a hyperbolic Coxeter polyhedron $P$ is defined to be the growth series and the growth rate of the Coxeter system $(\Gamma, S)$ associated with $P$, respectively.

It is known that the growth rate of a hyperbolic Coxeter polyhedron is a real algebraic integer bigger than 1 [8]. Certain classes of real algebraic integers show up in the study of the growth rates of hyperbolic Coxeter polytopes: Salem numbers, Pisot numbers, and Perron numbers. We recall the definitions of them; a real algebraic integer $\tau>1$ is called (i) a Salem number if $\tau^{-1}$ is an algebraic conjugate of $\tau$ and all other algebraic conjugates lie on the unit circle. (ii) a Pisot number if all its algebraic conjugates are less than 1 in absolute value. (iii) a Perron number if all other algebraic conjugates are less than $\tau$ in absolute value. By definition, Salem numbers and Pisot numbers are Perron numbers.

For compact hyperbolic Coxeter polygons and polyhedra, Cannon-Wagreich and Parry showed that their growth rates are Salem numbers [2], [16]. Floyd also showed that the growth rates of non-compact hyperbolic Coxeter polygons are Pisot numbers [6]. In the case of non-compact hyperbolic Coxeter simplices and pyramids, Komori and Umemoto showed that their growth rates are Perron numbers [12], [13]. Kellerhals-Nonaka and Komori-Yukita proved that the growth rates of ideal hyperbolic Coxeter polyhedra are Perron numbers, independently [14], [15]. By results of [10] and [22], the growth rates of certain families of compact hyperbolic Coxeter 4-polytopes are Perron numbers. In general, Kellerhals and Perren conjectured that the growth rates of hyperbolic Coxeter polytopes are Perron numbers [10]. Moreover, they conjectured about the distribution of poles of the growth functions of compact hyperbolic Coxeter $d$-polytopes as follows; (i) for $d$ is odd number, the growth function has a pole at 1 and precisely $\frac{d-1}{2}$ poles in an open unit interval $(0,1)$. (ii) for $d$ is even number, the growth function has precisely $\frac{d}{2}$ poles in the unit interval. (iii) in both cases, there exists a real pole $t \in(0,1)$ such that any non-real pole $z$ is contained in the annulus $\left\{t<|z|<t^{-1}\right\}$.

In this dissertation, we study the arithmetic property of growth rates of noncompact hyperbolic Coxeter polyhedra and 4-polytopes, and prove the following Theorems:

Theorem A. [24], [25] The growth rates of non-compact hyperbolic Coxeter polyhedra are Perron numbers.

By combining with the results of Parry, Theorem A can be summarized in Theorem B.
Theorem B. The growth rates of hyperbolic Coxeter polyhedra are Perron numbers.

In the study of the growth rates of non-compact hyperbolic Coxeter 4-polytopes, we provide the first example of an infinite sequence of non-compact hyperbolic Coxeter 4-polytopes whose growth rates are Perron numbers.

Theorem C. [26] Let $P_{1}$ be an ideal hyperbolic Coxeter 4-pyramid over a 3dimensional cube. Then, (i) We can glue $n$ copies of $P_{1}$ along their isometric facets and construct ideal hyperbolic Coxeter 4-polytope $P_{n}$ with $n+6$ facets. (ii) For any $n \geq 1$, the growth rate of $P_{n}$ is Perron number.

This dissertation is organized as follows. In Chapter 2, we provide the necessary background and review useful formulas which allow us to calculate the growth function of a hyperbolic Coxeter polytope. In Chapter 3, we establish the growth function of a non-compact hyperbolic Coxeter polyhedron and prove Theorem A. The proof of Theorem A is subdivided into two parts. In Section 3.1, we consider non-compact hyperbolic Coxeter polyhedra whose dihedral angles are of the form $\frac{\pi}{k}$ for $k=2,3,4,5,6$. In Section 3.2, we calculate the growth functions of non-compact hyperbolic Coxeter polyhedra having at least one dihedral angle of the form $\frac{\pi}{k}$ for some integer $k \geq 7$. In Chapter 4 , we construct an infinite sequence of ideal hyperbolic Coxeter 4-polytopes and prove Theorem C.

## Chapter 2

## Preliminaries

In this chapter, we introduce the relevant notations and review some useful identities in order to calculate the growth functions of hyperbolic Coxeter polytopes.

### 2.1 Coxeter groups, growth series and growth rate

Definition 2.1.1. [9, Coxeter system, Coxeter graph, growth rate]
(i) A Coxeter system $(\Gamma, S)$ consists of a group $\Gamma$ and a finite set of generators $S \subset \Gamma, S=\left\{s_{i}\right\}_{i=1}^{N}$, with relations $\left(s_{i} s_{j}\right)^{m_{i j}}$ for each $i, j$, where $m_{i i}=1$ and $m_{i j} \geq 2$ or $m_{i j}=\infty$ for $i \neq j$. We call $\Gamma$ a Coxeter group. For any subset $T \subset S$, we define $\Gamma_{T}$ to be the subgroup of $\Gamma$ generated by $\left\{s_{i}\right\}_{t \in T}$. Then $\Gamma_{T}$ is called the Coxeter subgroup of $\Gamma$ generated by $T$.
(ii) The Coxeter graph of $(\Gamma, S)$ is constructed as follows:

Its vertex set is $S$. If $m_{i j} \geq 3\left(s_{i} \neq s_{j} \in S\right)$, we join the pair of vertices by an edge and label it with $m_{i j}$.
(iii) The growth series $f_{S}(t)$ of a Coxeter system $(\Gamma, S)$ is the formal power series $\sum_{l=0}^{\infty} a_{l} t^{l}$ where $a_{l}$ is the number of elements of $\Gamma$ whose word length with respect to $S$ is equal to $l$. Then $\tau=\limsup _{l \rightarrow \infty} \sqrt[l]{a_{l}}$ is called the growth rate of $(\Gamma, S)$.

A Coxeter system $(\Gamma, S)$ is irreducible if the Coxeter graph of $(\Gamma, S)$ is connected. Irreducible finite Coxeter systems are completely classified by Coxeter [4].
Theorem 2.1.1. [4] Let $(\Gamma, S)$ be a finite Coxeter system. Then, the Coxeter graph of $(\Gamma, S)$ is isomorphic to one of Table 2.1.

We recall Solomon's formula and Steinberg's formula which are very useful for calculating growth series.

| $A_{n}$ |  |
| :---: | :---: |
| $B_{n}$ |  |
| $D_{n}$ |  |
| $E_{6}$ |  |
| $E_{7}$ |  |
| $E_{8}$ |  |
| $F_{4}$ | $\stackrel{S}{1}^{-}$ |
| $\mathrm{H}_{3}$ | $\underbrace{\circ}_{s_{1}} \underset{s_{2}}{-5} \underset{s_{3}}{0}$ |
| $H_{4}$ | ${ }_{s_{1}}^{\circ}-\underset{s_{2}}{-}-S_{3}^{-}-\stackrel{5}{S_{4}}$ |
| $I_{2}(n)$ | $\stackrel{n_{0}}{{ }_{S_{1}}}{ }_{\circ} s_{2}$ |

Table 2.1: The Coxeter graphs of irreducible finite Coxeter systems

Theorem 2.1.2. [19, Solomon's formula] The growth series $f_{S}(t)$ of an irreducible finite Coxeter system $(\Gamma, S)$ can be written as $f_{S}(t)=\left[m_{1}+1 ; m_{2}+1 ; \cdots ; m_{p}+1\right]$ where $[n]=1+t+\cdots+t^{n-1},[m ; n]=[m][n]$, etc., and where $\left\{m_{1}, m_{2}, \cdots, m_{p}\right\}$ is the set of exponents of $(\Gamma, S)$.

The exponents of irreducible finite Coxeter groups are shown in Table 2.2 (see [9] for details).

Theorem 2.1.3. [20, Steinberg's formula] Let $(\Gamma, S)$ be a Coxeter system. Denote by $\Gamma_{T}$ the Coxeter subgroup of $\Gamma$ generated by the subset $T \subseteq S$, and denote by $f_{T}(t)$ the growth series of the Coxeter system $\left(\Gamma_{T}, T\right)$. Set $\mathcal{F}=\left\{T \subseteq S: \Gamma_{T}\right.$ is finite \}. Then

$$
\frac{1}{f_{S}\left(t^{-1}\right)}=\sum_{T \in \mathcal{F}} \frac{(-1)^{|T|}}{f_{T}(t)}
$$

By Theorem 2.1.2 and Theorem 2.1.3, the growth series of $(\Gamma, S)$ is represented by a rational function $\frac{p(t)}{q(t)}(p, q \in \mathbb{Z}[t])$. The rational function $\frac{p(t)}{q(t)}$ is called the

Table 2.2: Exponents

| Coxeter group | Exponents | growth series |
| :---: | :---: | :---: |
| $A_{n}$ | $1,2, \cdots, n$ | $[2 ; 3 ; \cdots ; n+1]$ |
| $B_{n}$ | $1,3, \cdots, 2 n-1$ | $[2 ; 4 ; \cdots ; 2 n]$ |
| $D_{n}$ | $1,3, \cdots, 2 n-3, n-1$ | $[2 ; 4 ; \cdots ; 2 n-2 ; n]$ |
| $E_{6}$ | $1,4,5,7,8,11$ | $[2 ; 5 ; 6 ; 8 ; 9 ; 12]$ |
| $E_{7}$ | $1,5,7,9,11,13,17$ | $[2 ; 6 ; 8 ; 10 ; 12 ; 14 ; 18]$ |
| $E_{8}$ | $1,7,11,13,17,19,23,29$ | $[2 ; 8 ; 12 ; 14 ; 18 ; 20 ; 24 ; 30]$ |
| $F_{4}$ | $1,5,7,11$ | $[2 ; 6 ; 8 ; 12]$ |
| $H_{3}$ | $1,5,9$ | $[2 ; 6 ; 10]$ |
| $H_{4}$ | $1,11,19,29$ | $[2 ; 12 ; 20 ; 30]$ |
| $I_{2}(n)$ | $1, n-1$ | $[2 ; n]$ |

growth function of $(\Gamma, S)$. Since the coefficients of the growth series are positive, $f_{S}(t)$ diverges at $t=R$, where $R$ is the radius convergence of the series. Therefore, the positive real root of $q(t)$ which has the smallest absolute value among all the roots of $q(t)$ is equal to R .

### 2.2 Hyperbolic Coxeter polytopes

In this dissertation, we are interested in Coxeter groups which act discontinuously on the hyperbolic space $\mathbb{H}^{d}$.

Definition 2.2.1. [18, Upper half-space model of hyperbolic d-space] The upper half-space $\mathbb{H}^{d}=\left\{\left(x_{1}, \cdots, x_{d}\right) \in \mathbb{R}^{d} \mid x_{d}>0\right\}$ equipped with the metric $\frac{|d x|}{x_{d}}$ is a model of the d-dimensional hyperbolic geometry, so called the upper half-space model. The boundary $\partial \mathbb{H}^{d}$ of $\mathbb{H}^{d}$ in the one-point compactification $\mathbb{R}^{d} \cup\{\infty\}$ of the Euclidean d-space $\mathbb{R}^{d}$ is called the boundary at infinity. We denote the closure of a subset $A \subset \mathbb{R}^{d} \cup\{\infty\}$ by $\bar{A}$.

By identifying $\mathbb{R}^{d-1}$ with $\mathbb{R}^{d-1} \times\{0\}$ in $\mathbb{R}^{d}$, the boundary at infinity $\partial \mathbb{H}^{d}$ is equal to $\mathbb{R}^{d-1} \cup\{\infty\}$. A subset $H \subset \mathbb{H}^{d}$ is called a hyperplane of $\mathbb{H}^{d}$ if and only if it is a Euclidean hemisphere or a half-plane orthogonal to $\mathbb{R}^{d-1}$. Every hyperplane $H$ subdivide $\mathbb{H}^{d}$ into two closed components $H^{+}$and $H^{-}$bounded by $H$, then $H^{ \pm}$ is called a closed half-space bounded by $H$.
Definition 2.2.2. [23, Hyperbolic d-polytope] $A$ subset $P \subset \overline{\mathbb{H}}^{d}$ is called a hyperbolic d-polytope if $P$ can be written as the intersection of finitely many closed half-spaces: $P=\cap H_{i}^{-}$, where $H_{i}^{-}$is the closed domain of $\mathbb{H}^{d}$ bounded by a hyperplane $H_{i}$. We also assume that none of the closed half-spaces $H_{i}^{-}$contains
the intersection of all the others. A hyperbolic 2- (resp. 3-)polytope is called a hyperbolic polygon (resp. hyperbolic polyhedron).

Suppose that $H_{i} \cap H_{j} \neq \emptyset$ in $\mathbb{H}^{d}$. Then we define the dihedral angle between $H_{i}^{-}$and $H_{j}^{-}$as follows: let us choose a point $x \in H_{i} \cap H_{j}$ and consider the outer normal vectors $u_{i}$ and $u_{j}$ of $H_{i}^{-}$and $H_{j}^{-}$at $x$. Then the dihedral angle between $H_{i}^{-}$and $H_{j}^{-}$is defined as the real number $\theta \in[0, \pi)$ satisfying $\cos \theta=-\left(u_{i}, u_{j}\right)$ where $(\cdot, \cdot)$ denotes the Euclidean inner product on $\mathbb{R}^{d}$ at $x$.

If $\overline{H_{i}} \cap \overline{H_{j}} \in \overline{\mathbb{H}}^{d}$ is a point at the boundary $\partial \mathbb{H}^{d}$ of $\mathbb{H}^{d}$, we define the dihedral angle between $H_{i}^{-}$and $H_{j}^{-}$to be equal to zero.

Theorem 2.2.1. [18, Theorem 6.3.1 and Theorem 6.3.4] Let $P=\cap_{i=1}^{N} H_{i}^{-}$be a hyperbolic d-polytope. Set $F_{i}=P \cap H_{i}$ for $i=1, \cdots, N$. Then, $F_{i}$ is a hyperbolic (d-1)-polytope.

We call $F_{1}, \cdots, F_{N}$ the facets of $P$.
Definition 2.2.3. [18, Faces of hyperbolic polytope] The $(d-k)$-faces of $P$ for $k=1, \cdots, d$ are defined inductively as follows: The $(d-1)$-faces are defined to be the facets of $P$. By Theorem 2.2.1, $(d-1)$-faces are hyperbolic $(d-1)$-polytopes. Suppose that all the $(d-k)$-faces of $P$ have been defined and that are hyperbolic $(d-k)$-polytopes. Then, the $(d-k-1)$-faces of $P$ are defined to be the facets of the $(d-k)$-faces of $P$. We call a 0- (resp. 1-)face vertex (resp. edge).

A horosphere $\Sigma$ based at a point at infinity $u$ is defined to be a $(d-1)$ dimensional Euclidean sphere in $\mathbb{H}^{d}$ tangent to $\mathbb{R}^{d-1}$ at $u$ (resp. a Euclidean hyperplane parallel to $\mathbb{R}^{d-1}$ ) if $u$ is situated on $\mathbb{R}^{d-1}$ (resp. $u=\infty$ ). If we restrict the hyperbolic metric to the horosphere $\Sigma$, it makes a model of $(d-1)$-dimensional Euclidean geometry.

Definition 2.2.4. [23, Vertex at infinity] A point at infinity $p \in \partial \mathbb{H}^{d}$ is called a vertex at infinity of $P$ if $p \in \bar{P}$ and there exists a horosphere $\Sigma$ based at $p$ such that the intersection of $\Sigma$ and $P$ is bounded subset of $\Sigma$.

A hyperbolic polytope is called ideal if all of its vertices are vertices at infinity.
Definition 2.2.5. [23, Hyperbolic Coxeter polytope] A hyperbolic d-polytope $P \subset$ $\mathbb{H}^{d}$ of finite volume is called a hyperbolic Coxeter d-polytope if all of its dihedral angles have the form $\frac{\pi}{k}$ for an integer $k \geq 2$ or $k=\infty$ if the intersection of the respective bounding hyperplanes is a point on the boundary $\partial \mathbb{H}^{d}$.

Notice that a hyperbolic polytope in $\mathbb{H}^{d}$ is of finite volume if and only if it is the convex hull of finitely many points in $\overline{\mathbb{H}}^{d}$.

Theorem 2.2.2 (Theorem 7.1.2 and Theorem 7.1.4 [18]). Let $P$ be a hyperbolic Coxeter d-polytope and $\left\{F_{i}\right\}_{i=1}^{N}$ be the set of facets of $P$. Denote the reflection in the facet $F_{i}$ by $s_{i}$. Then, (i) the group $\Gamma$ generated by the reflections in the facets of $P$ is a discrete subgroup of $\operatorname{Isom}\left(\mathbb{H}^{d}\right)$, and (ii)the pair $(\Gamma, S), S=\left\{s_{i}\right\}_{i=1}^{N}$, is a Coxeter system, with relations $\left(s_{i} s_{j}\right)^{k_{i j}}$ for each pair of intersecting two facets $F_{i}$ and $F_{j}$, where $\frac{\pi}{k_{i j}}$ is the dihedral angle between $F_{i}$ and $F_{j}$.

We call $\Gamma$ (resp. $(\Gamma, S)$ ) the d-dimensional hyperbolic Coxeter group (resp. system) associated with $P$. Moreover, if $P$ is compact (resp. non-compact), $\Gamma$ is called cocompact (resp. cofinite). The growth series, growth function and growth rate of the hyperbolic Coxeter system $(\Gamma, S)$ associated with $P$ is called the growth series, growth function and growth rate of $P$, respectively. We denote the growth function and the growth rate of $P$ by $f_{P}(t)$ and $\tau_{P}$.

Definition 2.2.6. [23, Gram matrix, Coxeter scheme] Let $P=\cap_{i=1}^{N} H_{i}^{-}$be a hyperbolic Coxeter polytope. For every pair of facets $F_{i}$ and $F_{j}$, define

$$
c_{i j}= \begin{cases}1 & \text { if } i=j \\ -\cos \frac{\pi}{k_{i j}} & \text { if they intersect at the dihedral angle } \frac{\pi}{k_{i j}} \\ -1 & \text { if its intersection is a point on } \partial \mathbb{H}^{d} \\ -\cosh d\left(F_{i}, F_{j}\right) & \text { if they do not intersect }\end{cases}
$$

where, $d\left(F_{i}, F_{j}\right)$ is the hyperbolic distance between them. The $N \times N$ symmetric matrix $M(P)=\left(c_{i j}\right)$ is called the Gram matrix of $P$. The Coxeter scheme $X(P)$ of $P$ is defined as follows; Its vertex set is $\left\{F_{1}, \cdots, F_{N}\right\}$. If the dihedral angle $\frac{\pi}{k_{i j}}$ between two facets $F_{i}$ and $F_{j}$ is less than $\frac{\pi}{2}$, we join the pair of vertices by an edge. For each edge, we label it with $k_{i j}$ if $k_{i j} \geq 4$. Two vertices are joined by a dotted edge labeled with the hyperbolic distance between corresponding hyperplanes if they do not intersect.

Note that the Coxeter graph of the hyperbolic Coxeter system associated with $P$ can be obtained by changing dotted edges of the Coxeter scheme into edges labeled with $\infty$.

A subscheme of a Coxeter scheme $X(P)$ is called elliptic (resp. parabolic) if the corresponding submatrix of the Gram matrix $M(P)$ is positive definite (resp. positive semi-definite and its rank equals $d-1$ ). It is known that elliptic (resp. parabolic) subschemes correspond to finite (resp. affine) Coxeter systems.

Theorem 2.2.3. [23, Theorem 2.2, p. 109 and Theorem 2.5, p.110] Let $P$ be a hyperbolic Coxeter d-polytope and $f$ be a $(d-k)$-face of $P$. (i) Suppose that $f$ is not a vertex at infinity and the $k$ facets $F_{1}, \cdots, F_{k}$ are adjacent to $f$. Then, the subscheme of $X(P)$ spanned by the vertices corresponding to $F_{1}, \cdots, F_{k}$ is elliptic.


Figure 2.1: hyperbolic Coxeter pyramid and its Coxeter scheme
(ii) If $f$ is a vertex at infinity of $P$, then the corresponding subscheme is parabolic. (iii) For any elliptic (resp. parabolic) subscheme of $X(P)$, the intersection of the facets corresponding to the vertices of the subscheme is a face (resp. vertex at infinity) of $P$.

Combining Theorem 2.1.2, Theorem 2.1.3, and Theorem 2.2.3, we can compute the growth function of a hyperbolic Coxeter polytope in terms of its combinatorial structure. For example, we calculate the growth function of the hyperbolic Coxeter pyramid $P$ depicted in Fig 2.1. Since $P$ is ideal, vertices do not contribute to the growth function of $P$. By Theorem 2.2.3, the 5 facets of $P$ are correspond to the finite Coxeter system $A_{1}$, and the 4 edges whose dihedral angles are $\frac{\pi}{2}$ (resp. $\frac{\pi}{4}$ ) correspond to the finite Coxeter system $A_{1} \times A_{1}$ (resp. $\left.I_{2}(4)\right)$. Therefore, by Theorem 2.1.2 and Theorem 2.1.3, the growth function $f_{P}(t)$ of $P$ can be calculated as

$$
\frac{1}{f_{P}\left(t^{-1}\right)}=1-\frac{5}{[2]}+\frac{4}{[2,2]}+\frac{4}{[2,4]} .
$$

Parry calculated the growth functions of compact hyperbolic Coxeter polygons and polyhedra, and by expressing the growth functions in suitable forms, clarified the arithmetic nature of the growth rates.
Theorem 2.2.4. [16] The growth rates of compact hyperbolic Coxeter polygons and polyhedra are Salem numbers, where a real algebraic integer $\tau>1$ is called a Salem number if $\tau^{-1}$ is an algebraic conjugate of $\tau$ and all other algebraic conjugates lie on the unit circle (see Fig 2.2).

Floyd considered the growth rates of non-compact hyperbolic Coxeter polygons.
Theorem 2.2.5. [6] The growth rates of non-compact hyperbolic Coxeter polygons are Pisot numbers, where a real algebraic integer $\tau>1$ is called a Pisot number if all its algebraic conjugate are less than 1 in absolute value (see Fig 2.3).


Figure 2.2: Salem num- Figure 2.3: Pisot number


Figure 2.4: Perron number ber

For the case of non-compact hyperbolic Coxeter polyhedra, Komori and Umemoto studied the growth rates of non-compact hyperbolic Coxeter tetrahedra.

Theorem 2.2.6. [12] The growth rates of non-compact hyperbolic Coxeter tetrahedra are Perron numbers, where a real algebraic integer $\tau>1$ is called a Perron number if all its algebraic conjugates are less than $\tau$ in absolute value (see Fig 2.4).

We would like to comment here that Kellerhals and Perren conjectured that the growth rates of hyperbolic Coxeter polytopes are Perron numbers [10].

### 2.3 Andreev's Theorem

In this section, we restrict our attention to the 3-dimensional case and review the complete classification of hyperbolic polyhedra by Andreev.

Definition 2.3.1. [1, abstract polyhedron] An abstract polyhedron $C$ is a simple graph on the 2-dimensional sphere $S^{2}$ all of its vertices are 3-valent or 4-valent. If each edge $e$ of an abstract polyhedron $C$ is labeled with $0<\alpha_{e} \leq \frac{\pi}{2}$, $C$ is called an abstract acute-angled polyhedron.

Theorem 2.3.1. [1, Andreev's theorem] Let $C$ be an abstract acute-angled polyhedron not a tetrahedron. There is a hyperbolic polyhedron $P$ of finite volume in $\mathbb{H}^{3}$ whose 1-skeleton provides $C$ if and only if the following conditions are satisfied:
(a) if three distinct edges of $C$ meet at a vertex, then the sum of the labels is greater than or equal to $\pi$;
(b) if four distinct edges of $C$ meet at a vertex, then all the labels equal $\frac{\pi}{2}$
(c) if three faces of $C$ are pairwise adjacent but do not meet at a vertex, then the sum of the labels on the edges formed by adjacent faces is less than $\pi$;
(d) if four faces of $C$ are cyclically adjacent but do not meet at a vertex, then the sum of the labels on the edges formed by adjacent faces is less than $2 \pi$;
(e) if a face $F_{i}$ is adjacent to faces $F_{j}$ and $F_{k}$, while $F_{j}$ and $F_{k}$ are not adjacent but have a common vertex which $F_{i}$ does not share, then at least one of the labels on the edges formed by $F_{i}$ with $F_{j}$ or with $F_{k}$ is different from $\frac{\pi}{2}$.
(f) if $C$ is a triangular prism, then the sum of the labels on edges contained in one of the triangular bases is strictly less than $3 \pi$.


Figure 2.5: An exapmle of Andreev's theorem; the labels 2 and 4 in the left handside figure mean that the labels on edges are $\frac{\pi}{2}$ and $\frac{\pi}{4}$. The red coloured facet in the left hand-side figure is realized by the red coloured hemisphere in the right hand-side figure.

Suppose that $P$ is a Coxeter polyhedron in $\mathbb{H}^{3}$, and let $v$ be a vertex of $P$. Let $F_{1}, \cdots, F_{n}$ be adjacent facets of $P$ incident to $v$ and $\frac{\pi}{k_{i}}$ be the dihedral angle between $F_{i}$ and $F_{i+1}$. By Theorem 2.3.1, the number of facets of $P$ incident to $v$ is at most 4 and $k_{1}, \cdots, k_{n}$ satisfy the following conditions.

$$
\begin{array}{ll}
k_{1}=k_{2}=k_{3}=k_{4}=2 & \text { if } n=4 \\
\frac{1}{k_{1}}+\frac{1}{k_{2}}+\frac{1}{k_{3}} \geq 1 & \text { if } n=3 \tag{2.2}
\end{array}
$$

Note that a vertex $v$ of $P$ belongs to $\partial \mathbb{H}^{3}$ if and only if $k_{1}=k_{2}=k_{3}=k_{4}=2$ or $\frac{1}{k_{1}}+\frac{1}{k_{2}}+\frac{1}{k_{3}}=1$, and we call such a vertex a cusp, for short.

We shall use the following notation and terminology in this section and Chapter 3:

- If a vertex $v$ of $P$ satisfies the identity (2.1), we call $v$ a cusp of type $(2,2,2,2)$.
- If a vertex $v$ of $P$ satisfies the inequality (2.2), we call $v$ a vertex of type $\left(k_{1}, k_{2}, k_{3}\right)$.
- $v_{2,2,2,2}$ denotes the number of cusps of type ( $2,2,2,2$ ).
- $v_{k_{1}, k_{2}, k_{3}}$ denotes the number of vertices of type $\left(k_{1}, k_{2}, k_{3}\right)$.
- $V, E, F$ denotes the number of vertices, edges and facets of $P$.
- If an edge $e$ of $P$ has dihedral angle $\frac{\pi}{k}$, we call it $\frac{\pi}{k}$-edge.
- $e_{k}$ denotes the number of $\frac{\pi}{k}$-edges.

Lemma 2.3.1. Let $P \subset \overline{\mathbb{H}}^{3}$ be a non-compact hyperbolic Coxeter polyhedron. Then the following identities and inequality hold.

$$
\begin{align*}
& V-E+F=2  \tag{2.3}\\
& V=v_{2,2,2,2}+\sum_{n \geq 2} v_{2,2, n}+v_{2,3,3}+v_{2,3,4}+v_{2,3,5}+v_{2,3,6}+v_{2,4,4}+v_{3,3,3}  \tag{2.4}\\
& E=\sum_{n \geq 2} e_{n}  \tag{2.5}\\
& 2 e_{2}=4 v_{2,2,2,2}+3 v_{2,2,2}+2 \sum_{n \geq 3} v_{2,2, n}+v_{2,3,3}+v_{2,3,4}+v_{2,3,5}+v_{2,3,6}+v_{2,4,4}  \tag{}\\
& 2 e_{3}=3 v_{3,3,3}+2 v_{2,3,3}+v_{2,2,3}+v_{2,3,4}+v_{2,3,5}+v_{2,3,6}  \tag{2.7}\\
& 2 e_{4}=2 v_{2,4,4}+v_{2,2,4}+v_{2,3,4}  \tag{2.8}\\
& 2 e_{5}=v_{2,2,5}+v_{2,3,5}  \tag{2.9}\\
& 2 e_{6}=v_{2,2,6}+v_{2,3,6}  \tag{2.10}\\
& 2 e_{n}=v_{2,2, n} \quad \text { for } n \geq 7  \tag{2.11}\\
& v_{2,2,2,2}+v_{2,3,6}+v_{2,4,4}+v_{3,3,3} \geq 1 \tag{2.12}
\end{align*}
$$

Proof. The identity (2.3) is Euler's polyhedral formula. By the definitions of $V, E$ and $e_{n}$, the identities (2.4) and (2.5) hold for $P$. Non-compactness of $P$ implies the inequality (2.12). The other identities are obtained by counting the number of edges which is adjacent to each vertex. For example, the identity (2.8) is obtained as follows. Any $\frac{\pi}{4}$-edge has strictly two vertices of type $(2,2,4)$ or $(2,3,4)$ or $(2,4,4)$ (see Fig 2.6). On the other hand, any vertex of type $(2,2,4)$ or $(2,3,4)$ has one $\frac{\pi}{4}$-edge and any vertex of type $(2,4,4)$ has two $\frac{\pi}{4}$-edges. Hence, we obtain the identity (2.8).

We use these identities and the last inequality to express growth functions of the hyperbolic Coxeter polyhedra under consideration.

Theorem 2.3.2. Let $\sigma$ be the sum of the $\frac{\pi}{k}$-edges for $k \geq 7$ of a non-compact hyperbolic Coxeter polyhedron $P$, that is,

$$
\sigma=\sum_{k \geq 7} e_{k}
$$

Then we obtain the following inequality.

$$
\sigma \leq F-3
$$



Figure 2.6:

Moreover, if the equality $\sigma=F-3$ holds, then $P$ has a unique cusp of type $(2,2,2,2)$, and all other vertices of $P$ are of type $(2,2, k)$ for $k \geq 7$.

In order to prove Theorem 2.3.2, we use the following deformation argument for Coxeter polyhedra studied by Kolpakov in [11]. We present it in a modified form which is more suitable for further account.

Theorem 2.3.3. [11, Proposition 1 and 2] (i) Suppose that a non-compact hyperbolic Coxeter polyhedron $P \subset \mathbb{H}^{3}$ has some $\frac{\pi}{k}$-edges for $k \geq 7$. Then all of the $\frac{\pi}{k}$-edges can be contracted to cusps of type (2,2,2,2). The hyperbolic Coxeter polyhedron $\hat{P}$ which is obtained from $P$ by contracting all $\frac{\pi}{k}$-edges for $k \geq 7$ of $P$ is called the pinched Coxeter polyhedron of $P$.
(ii) If a hyperbolic Coxeter polyhedron $P$ has some cusps of type (2, 2, 2, 2), then there exists a unique Coxeter polyhedron which is obtained from $P$ by opening one cusp of type (2, 2, 2, 2). (see Fig 2.7)


Figure 2.7: Deformation argument
Proof of Theorem 2.3.2. Suppose that $P$ is a non-compact hyperbolic Coxeter polyhedron and the sum of the numbers of the $\frac{\pi}{k}$-edges for $k \geq 7$ of $P$ is $\sigma$. In this proof, $\hat{P}$ denotes the pinched Coxeter polyhedron obtained from $P$ and $\hat{V}, \hat{E}, \hat{F}, \hat{v}_{2,2,2,2}, \hat{v}_{k_{1}, k_{2}, k_{3}}$ and $\hat{e}_{k}$ denote respectively the number of vertices, edges, facets, cusps of type $(2,2,2,2)$, vertices of type $\left(k_{1}, k_{2}, k_{3}\right)$ and $\frac{\pi}{k}$-edges of $\hat{P}$.

By substituting the identities (2.4)-(2.11) for the identity (2.3), we can see the following identity for $\hat{P}$.

$$
\begin{equation*}
\hat{F}-2=\hat{v}_{2,2,2,2}+\frac{1}{2}(\text { the number of vertices of } \hat{P} \text { with valency } 3) . \tag{2.13}
\end{equation*}
$$

Even if we contract the all $\frac{\pi}{k}$-edges for $k \geq 7$ of $P$, the number of facets of $\hat{P}$ is equal to the number of faces of $P$, so that we obtain the following relations for $\hat{P}$.

$$
\begin{align*}
F & =\hat{F} .  \tag{2.14}\\
\hat{v}_{2,2,2,2} & =v_{2,2,2,2}+\sigma . \tag{2.15}
\end{align*}
$$

Then, by substituting the identities (2.14) and (2.15) for (2.13), we see that

$$
\begin{equation*}
F-2=v_{2,2,2,2}+\sigma+\frac{1}{2} \text { (the number of vertices of } \hat{P} \text { with valency } 3 \text { ). } \tag{2.16}
\end{equation*}
$$

The identity (2.16) implies that

$$
\sigma \leq F-2
$$

Moreover, if $P$ satisfies the identity $\sigma=F-2$, then all of the vertices of $\hat{P}$ are cusps of type $(2,2,2,2)$ obtained from $P$ by contracting all $\frac{\pi}{k}$-edges for $k \geq 7$ of $P$. This observation means that all of the vertices of $P$ are of type $(2,2, k)$ for $k \geq 7$. Therefore, $P$ has no cusps. This fact contradicts the assumption that $P$ is non-compact. Thus, we obtain the following inequality.

$$
\sigma \leq F-3
$$

Suppose that $\sigma=F-3$. Then, the identity (2.16) is rewritten as

$$
\begin{align*}
F-2 & =v_{2,2,2,2}+F-3 \\
& +\frac{1}{2}(\text { the number of vertices of } \hat{P} \text { with valency } 3) \tag{2.17}
\end{align*}
$$

Since any $\frac{\pi}{k}$-edge for $k \geq 3$ is adjacent to two vertices with valency 3 , if $P$ has at least one cusp of type $(2,3,6)$ or $(2,4,4)$ or $(3,3,3)$, then $P$ has at least three vertices with valency 3 .

Therefore, by the identity (2.17), we obtain the following inequality.

$$
F-2 \geq v_{2,2,2,2}+F-3+\frac{3}{2}=v_{2,2,2,2}+F-\frac{3}{2} .
$$

Hence if $P$ has at least one cusp of type $(2,3,6)$ or $(2,4,4)$ or $(3,3,3)$, we arrive at a contradiction. This implies that if $\sigma=F-3, P$ has a unique cusp of type $(2,2,2,2)$, and all other vertices of $P$ are of type $(2,2, k)$ for $k \geq 7$.

### 2.4 Computing the root distribution of a real polynomial

In this Section, we review Sturm's theorem and Kronecker's theorem. Sturm's theorem allows one to determine the distribution of the real roots of a real polynomial and Kronecker's theorem tells us how to count roots of a real polynomial contained in a closed disk of radius $r$ centered at the origin 0 in the complex plane $\mathbb{C}$. For references, see [3], [11] and [17].

### 2.4.1 Sturm's theorem

Let $f$ and $g$ be real polynomials. We may assume that $\operatorname{deg} f \geq \operatorname{deg} g$. By the Euclidean algorithm, we can define polynomials $f_{2}, \cdots, f_{r}$ as follows:

$$
\begin{gathered}
f=q_{1} g-f_{2}, \quad \operatorname{deg} g>\operatorname{deg} f_{2} . \\
g=q_{2} f_{2}-f_{3}, \quad \operatorname{deg} f_{2}>\operatorname{deg} f_{3} . \\
f_{2}=q_{3} f_{3}-f_{4}, \quad \operatorname{deg} f_{3}>\operatorname{deg} f_{4} . \\
\vdots \\
f_{r-2}=q_{r-1} f_{r-1}-f_{r}, \quad \operatorname{deg} f_{r-1}>\operatorname{deg} f_{r} . \\
f_{r-1}=q_{r} f_{r} .
\end{gathered}
$$

Then, the finite sequence $f_{0}:=f, f_{1}:=g, f_{2}, \cdots, f_{r}$ of real polynomials is called the Sturm sequence $S(f, g)$ of $f$ and $g$. Note that $f_{r}$ is the greatest common divisor of polynomials $f$ and $g$. For any $t_{0} \in \mathbb{R}$, the number of sign changes in $S(f, g)$ at $t_{0}$ is denoted by $w\left(t_{0}\right)$, that is, $w\left(t_{0}\right)$ is the number of sign changes in the sequence $f\left(t_{0}\right), g\left(t_{0}\right), f_{2}\left(t_{0}\right), \cdots, f_{r}\left(t_{0}\right)$ ignoring zeros.
Example 2.4.1. Let $f(z):=z^{5}-3 z-1$ and $g(z):=f^{\prime}(z)=5 z^{4}-3$. Then, $S(f, g)$ can be calculated as follows:

$$
\begin{aligned}
f(z) & =z^{5}-3 z-1 \\
g(z) & =5 z^{4}-3 \\
f_{2}(z) & =12 z+5 \\
f_{3}(z) & =1
\end{aligned}
$$

We consider the number of sign changes in the Sturm sequence at -2 . We have $f(-2)=-27, g(-2)=77, f_{2}(-2)=-19, f_{3}(-2)=1$, so that $w(-2)$ is equal to 3.

Theorem 2.4.1. [3, Theorem 8.8.15, Sturm's theorem] Let $f$ be a real polynomial and $S\left(f, f^{\prime}\right)=\left\{f_{0}, f_{1}, \cdots, f_{r}\right\}$. Suppose that $a, b \in \mathbb{R}$, $a<b$, are not roots of $f$.

Then the number of distinct real roots of $f$ in the closed interval $[a, b]$ is equal to $w(a)-w(b)$.

From now on, we assume that the real polynomials $f$ and $g$ have no common roots. For each real root $t_{0}$ of $f$, the number of sign changes in $f, g$ satisfies one of the following three conditions;
(i) the number of sign changes in $f(t), g(t)$ decreases by 1 when $t$ passes through $t_{0}$.
(ii) the number of sign changes in $f(t), g(t)$ increases by 1 when $t$ passes through $t_{0}$.
(iii) the number of sign changes in $f(t), g(t)$ does not vary when $t$ passes through $t_{0}$.

We assign the number $\varepsilon_{t_{0}}=1,-1$ and 0 to each root $t_{0}$ of $f$ when the number of sign changes of $f$ and $g$ satisfies the condition (i), (ii) and (iii), respectively. The following well-known theorem is proved analogously to Sturm's theorem.

Theorem 2.4.2. Suppose that the real numbers $a$ and $b$ are not roots of $f$. Then, the following identity holds for $S(f, g)$.

$$
\sum_{t_{0} \in[a, b]: f\left(t_{0}\right)=0} \varepsilon_{t_{0}}=w(a)-w(b) .
$$

### 2.4.2 Separation of complex roots

We use the following notation:

- $\mathbb{C}_{z}$ and $\mathbb{C}_{w}$ denote respectively the complex planes with coordinates $z=$ $x+i y$ and $w=u+i v$.
- $S_{r} \subset \mathbb{C}_{z}$ is a circle of radius $r>0$ centered at the origin $0 \in \mathbb{C}_{z}$.
- $B_{r} \subset \mathbb{C}_{z}$ is an open disk of radius $r>0$ centered at 0 .
- A parametrization for $S_{r}$ is given as follows:

$$
z(t)=r \frac{t^{2}-1}{t^{2}+1}-i r \frac{2 t}{t^{2}+1} .
$$

- $f(z)$ is a real polynomial of a complex variable $z$.
- Expanding $f(z(t))$, it can be represented as

$$
f(z(t))=\frac{\varphi_{r}(t)+i \psi_{r}(t)}{\left(t^{2}+1\right)^{\operatorname{deg} f}} \text { on } S_{r}
$$

where $\varphi_{r}(t)$ and $\psi_{r}(t)$ are real polynomials of a real variable $t$.
Lemma 2.4.1. Suppose that $f(z)$ has no roots on $S_{r}$. Given $M>0$ such that the closed interval $[-M, M]$ contains all real roots of $\varphi_{r}$, the following identity holds for $S\left(\varphi_{r}, \psi_{r}\right)$.

$$
\sum_{t_{0} \in[-M, M]: \varphi_{r}\left(t_{0}\right)=0} \varepsilon_{t_{0}}=w(-M)-w(M)
$$

Proof. The assumption that $f(z)$ has no roots on $S_{r}$ implies that the real polynomials $\varphi_{r}(t)$ and $\psi_{r}(t)$ do not have common real roots. Therefore, we can apply Theorem 2.4.2 to $\varphi_{r}(t)$ and $\psi_{r}(t)$.

By considering $f(z)$ as a holomorphic function from $\mathbb{C}_{z}$ to $\mathbb{C}_{w}$, we parameterize the closed curve $f\left(S_{r}\right)$ as $w(t)=\frac{\varphi_{r}(t)}{\left(t^{2}+1\right)^{\operatorname{deg} f}}+i \frac{\psi_{r}(t)}{\left(t^{2}+1\right)^{\operatorname{deg} f}}$. In order to calculate the winding number of $f\left(S_{r}\right)$, we divide $f\left(S_{r}\right)$ into closed curves $C_{1}, \cdots, C_{m}$ as follows; trace $f\left(S_{r}\right)$ from the initial point $f(r)=\lim _{t \rightarrow-\infty} w(t)$, and if the curve crosses the $v$-axis twice, then we mark each crossing point with $\alpha_{1}$ and $\alpha_{2}$ and go back to the initial point $f(r)$ along the straight line from the point $\alpha_{2}$ to the initial point $f(r)$. This locus makes the closed curve $C_{1}$. After that, we go back to $f\left(S_{r}\right)$ along the straight line from $f(r)$ to $\alpha_{2}$. By repeating this procedure, the closed curve $f\left(S_{r}\right)$ is divided into closed curves $C_{1}, \cdots, C_{m}$ (see Fig 2.8). Under the division


Figure 2.8: Division of the closed curve $f\left(S_{r}\right)$
of $f\left(S_{r}\right)$, the winding number of $f\left(S_{r}\right)$ equals the sum of the winding numbers of
closed curves $C_{1}, \cdots, C_{m}$. To calculate the winding number of each closed curve $C_{i}$, we assign the number $\chi_{\alpha_{k}}=1$ (resp. $\chi_{\alpha_{k}}=-1$ ) to a crossing point $\alpha_{k}$ of the $v$-axis and $C_{i}$ if the argument of $C_{i}$ is increasing (resp. decreasing) around the crossing point $\alpha_{k}$ (see Fig 2.9). Then, the winding number of $C_{i}$ is equal to


Figure 2.9: Assigning the number to crossing points
the sum of $\frac{1}{2} \chi_{\alpha_{k}}$ at each crossing point $\alpha_{k}$. Note that if $C_{i}$ has no crossing points of the $v$-axis and $C_{i}$, then the winding number of $C_{i}$ equals 0 . For example, the winding number of $C_{1}, C_{2}$ and $C_{3}$ in Fig 2.9 is equal to $0,-1$ and 0 , respectively. This observation shows that the winding number of $f\left(S_{r}\right)$ equals the sum of the number $\frac{1}{2} \chi_{\alpha}$ on each crossing point $\alpha$ of the $v$-axis and $f\left(S_{r}\right)$.

Let us now consider the Sturm sequence of polynomials $\varphi_{r}(t)$ and $\psi_{r}(t)$. Every crossing point of the curve $f\left(S_{r}\right)$ corresponds to a root of $\varphi_{r}(t)$. For any root $t_{0} \in \mathbb{R}$ of $\varphi_{r}(t)$, the argument of $f\left(S_{r}\right)$ is increasing (resp. decreasing) if $\varepsilon_{t_{0}}=-1$ (resp. $\varepsilon_{t_{0}}=1$ ). This observation, together with Theorem 2.4.2 and the argument principle, implies the following equalities.

$$
\begin{aligned}
\#\left\{z \in B_{r} \mid z \text { is a root of } f(z)\right\} & =\text { the winding number of } f\left(S_{r}\right) \\
& =\frac{1}{2} \sum_{\alpha_{k}: \operatorname{a~mark} \text { on } f\left(S_{r}\right)} \chi_{\alpha_{k}} \\
& =\frac{1}{2} \sum_{t_{0}: \varphi_{r}\left(t_{0}\right)=0}-\varepsilon_{t_{0}} .
\end{aligned}
$$

By Lemma 2.4.1, we obtain Kronecker's theorem.
Theorem 2.4.3. [17, Theorem 1.4.6, Kronecker's theorem] Suppose that $f(z)$ has no roots on $S_{r}$. Then the number of roots of $f$ contained in $B_{r}$ equals to $\frac{w(M)-w(-M)}{2}$, where $M>0$ is a real number such that $[-M, M]$ contains all roots of $\varphi_{r}(t)$.

If we substitute $z(t)=r \frac{t-i}{t+i}$ for $f(z)$, then $f(z(t))$ can be rewritten according to

$$
f(z(t))=\frac{\Phi(t)+i \Psi(t)}{(t+i)^{\operatorname{deg} f}}
$$

Since $\frac{1}{2 \pi} \int_{f\left(S_{r}\right)} d \log w=\frac{1}{2 \pi} \int_{f\left(S_{r}\right)} d \arg w$ (see [7]), the winding number of $f\left(S_{r}\right)$ equals $\frac{1}{2 \pi} \int_{-\infty}^{\infty} \arg \{\Phi(t)+i \Psi(t)\} d t-\frac{1}{2 \pi} \int_{-\infty}^{\infty} \arg (t+i)^{\operatorname{deg} f} d t$. For abbreviation, we denote the quantities $\frac{1}{2 \pi} \int_{-\infty}^{\infty} \arg \{\Phi(t)+i \Psi(t)\} d t$ and $\frac{1}{2 \pi} \int_{-\infty}^{\infty} \arg (t+i)^{\operatorname{deg} f} d t$ by $\Theta(\Phi(t)+i \Psi(t))$ and $\Theta\left((t+i)^{\operatorname{deg} f}\right)$. $\Theta(\Phi(t)+i \Psi(t))$ and $\Theta\left((t+i)^{\operatorname{deg} f}\right)$ measure the increases of arguments of the curves $\Phi(t)+i \Psi(t)$ and $(t+i)^{\operatorname{deg} f}$, respectively (see [7] for details). Applying the previous argument to the curve $\Phi(t)+i \Psi(t)$, we obtain

$$
\Theta(\Phi(t)+i \Psi(t))=\frac{w(M)-w(-M)}{2}
$$

By substituting $t=\tan \theta$ to change the variable $t$ into $\theta$, we have

$$
\Theta\left((t+i)^{\operatorname{deg} f}\right)=-\frac{\operatorname{deg} f}{2}
$$

Therefore, we get the following corollary with the help of Kronecker's theorem.
Corollary 2.4.1. Suppose that $f(z)$ has no roots on $S_{r}$. Let $w(t)$ denote the number of sign changes in the Sturm sequence of $\Phi(t)$ and $\Psi(t)$. Then, the number of roots of $f$ contained in $B_{r}$ equals to $\frac{w(M)-w(-M)+\operatorname{deg} f}{2}$.

For any real polynomial $f$, the sign of $f(t)$ for sufficiently large (resp. small) $t \in \mathbb{R}$ is determined by the leading coefficient (resp. multiplied by $(-1)^{\operatorname{deg} f}$ ). Therefore, in order to determine $w(M)$, we only have to consider the leading coefficients of the Sturm sequence of $\Phi(t)$ and $\Psi(t)$. For the rest of the paper, $w(\infty)$ (resp. $w(-\infty)$ ) denotes the number of sign changes of the leading coefficient (resp. multiplied by $(-1)^{\operatorname{deg} f_{i}}$ ) of the Sturm sequence.

### 2.4.3 Method for deciding about the root distribution of a real polynomial.

Suppose $f(z)$ is a real polynomial of one complex variable $z$. Then, we can determine its roots as follows.

In order to count the number of real roots of $f$ contained in the closed interval $[a, b]$ :

1. Check that $a$ and $b$ are not roots of $f$.
2. Calculate the Sturm sequence of $f(t)$ and $f^{\prime}(t)$.
3. By using Sturm's theorem, $w(a)-w(b)$ is equal to the number of real roots of $f$ contained in $[a, b]$.

In order to count the number of roots of $f$ contained in $B_{r}$ :

1. Calculate the two real polynomials $\Phi(t)$ and $\Psi(t)$ by substituting $z(t)=$ $r \frac{t-i}{t+i}$ into $f(z)$.
2. Check that $f(z)$ has no roots on $S_{r}$. For example, if the resultant of $\Phi(t)$ and $\Psi(t)$ is not 0 , then $f(z)$ has no roots on $S_{r}$.
3. Calculate the Sturm sequence of $\Phi(t)$ and $\Psi(t)$.
4. By Corollary 2.4.1 and the definition of $w(\infty)$ and $w(-\infty)$, the number of roots of $f$ contained in $B_{r}$ is equal to $\frac{w(\infty)-w(-\infty)+\operatorname{deg} f}{2}$.

## Chapter 3

## Growth rates of 3-dimensional hyperbolic Coxeter groups

Komori and Umemoto proved that the growth rates of non-compact hyperbolic Coxeter simplices are Perron numbers [12] (see Theorem 2.2.6). Therefore, we may assume that the number of faces of the hyperbolic Coxeter polyhedra under consideration is at least 5. The following proposition due to Komori and Umemoto [12] will be of fundamental importance when showing that the growth rates of non-compact hyperbolic Coxeter polyhedra are Perron numbers. In this chapter, a facet of $P$ is called a face.

Proposition 3.0.1 (Lemma 1 [12]). Let $g(t)$ be a polynomial of degree $n \geq 2$ having the form

$$
g(t)=\sum_{k=1}^{n} n_{k} t^{k}-1
$$

where $n_{k}$ are non-negative integers. We assume that the greatest common divisor of $\left\{k \in \mathbb{N} \mid n_{k} \neq 0\right\}$ is 1 . Then there exists a real number $r_{0}, 0<r_{0}<1$ which is the unique zero of $g(t)$ having the smallest absolute value among all zeros of $g(t)$.

Our aim is to express the growth functions of non-compact hyperbolic Coxeter polyhedra as rational functions whose denominator polynomials satisfy the conditions of Proposition 3.0.1. This will be done by using Steinberg's formula (see Theorem 2.1.3) and the relations (2.3)-(2.12) (see Lemma 2.3.1). By applying this strategy to the growth functions of ideal hyperbolic Coxeter polyhedra, Komori and Yukita showed the following theorem.
Theorem 3.0.1. [14] Let $P$ be an ideal hyperbolic Coxeter polyhedron with $F$ faces. Then, (i) $\tau_{P}$ is a Perron number. (ii) $F-3 \leq \tau_{P} \leq F-1$ and $\tau_{P}=F-3$ if and only if all of the dihedral angles of $P$ are $\frac{\pi}{2}$. (iiii) The set of the growth rates of ideal hyperbolic Coxeter polyhedra is unbounded above.

### 3.1 The growth rates of non-compact hyperbolic Coxeter polyhedra whose dihedral angles are of the form $\frac{\pi}{m}$ for $m=2,3,4,5,6$

In this section, we prove the following theorem.
Theorem 3.1.1. [24] Let $P$ be a non-compact hyperbolic Coxeter polyhedron whose dihedral angles are $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{5}$ and $\frac{\pi}{6}$. Then the growth rate of $P$ is a Perron number.

The proof of Theorem 3.1.1 will be divided into two steps.

### 3.1.1 The growth rates of right-angled non-compact hyperbolic polyhedra

Let $P$ be a right-angled non-compact hyperbolic polyhedron, that is, all of its dihedral angles are $\frac{\pi}{2}$. By means of Steinberg's formula, we can calculate the growth function $f_{P}(t)$ of $P$ as

$$
\frac{1}{f_{P}\left(t^{-1}\right)}=1-\frac{F}{[2]}+\frac{e_{2}}{[2,2]}-\frac{v_{2,2,2}}{[2,2,2]} .
$$

By using the identities (2.3), (2.4), (2.5), and (2.6), it can be rewritten as

$$
\begin{aligned}
\frac{1}{f_{P}(t)} & =1-\frac{F t}{[2]}+\frac{e_{2} t^{2}}{[2,2]}-\frac{v_{2,2,2} t^{3}}{[2,2,2]} \\
& =\frac{(t-1)}{[2,2,2]}\left\{\left(v_{2,2,2,2}-1\right) t^{2}+\left(v_{2,2,2,2}+\frac{1}{2} v_{2,2,2}-2\right) t-1\right\} \\
& =\frac{(t-1)}{[2,2,2]}\left\{\left(v_{2,2,2,2}-1\right) t^{2}+(F-4) t-1\right\} \\
& =\frac{(t-1)}{[2,2,2]} H_{2}(t)
\end{aligned}
$$

where we put

$$
H_{2}(t)=\left(v_{2,2,2,2}-1\right) t^{2}+(F-4) t-1 .
$$

Proposition 3.1.1. All the coefficients of $\mathrm{H}_{2}(t)$ is nonnegative except its constant term. Moreover, the growth rate of $P$ is a Perron number.

Proof. Put $n_{2}=v_{2,2,2,2}-1$ and $n_{1}=F-4 . \quad F \geq 5$ and the inequality (2.12) imply that $n_{1}, n_{2} \geq 0$. If $v_{2,2,2,2} \geq 2$, by using Proposition 3.0.1, we conclude that the growth rate of $P$ is a Perron number. If $v_{2,2,2,2}=1$, by Andreev's theorem (see Theorem 2.3.1) and the result of Federico [5], we can see that such a hyperbolic Coxeter polyhedron has at least 6 faces. Therefore, $H_{2}(t)$ has only one real positive root which is less than 1 . Hence, we conclude that the growth rate of $P$ is a Perron number.

### 3.1.2 The growth rates of non-compact Coxeter polyhedra whose dihedral angles are $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{5}$ and $\frac{\pi}{6}$

Let $P$ be a non-compact hyperbolic Coxeter polyhedron whose dihedral angels are $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{5}$ and $\frac{\pi}{6}$. In order to prove Theorem 3.1.1, by Proposition 3.1.1, we may assume that $P$ is not a right-angled polyhedron. By means of Steinberg's formula, we calculate the growth function $f_{P}(t)$ of $P$ as

$$
\begin{aligned}
\frac{1}{f_{P}(t)} & =1-\frac{F t}{[2]}+\frac{e_{2} t^{2}}{[2 ; 2]}+\frac{e_{3} t^{3}}{[2 ; 3]}+\frac{e_{4} t^{4}}{[2 ; 4]}+\frac{e_{5} t^{5}}{[2 ; 5]}+\frac{e_{6} t^{6}}{[2 ; 6]}-\frac{v_{2,2,2} t^{3}}{[2 ; 2 ; 2]}-\frac{v_{2,2,3} t^{4}}{[2 ; 2 ; 3]} \\
& -\frac{v_{2,2,4} t^{5}}{[2 ; 2 ; 4]}-\frac{v_{2,2,5} t^{6}}{[2 ; 2 ; 5]}-\frac{v_{2,2,6} t^{7}}{[2 ; 2 ; 6]}-\frac{v_{2,3,3} t^{6}}{[2 ; 3 ; 4]}-\frac{v_{2,3,4} t^{9}}{[2 ; 4 ; 6]}-\frac{v_{2,3,5} t^{15}}{[2 ; 6 ; 10]}
\end{aligned}
$$

By using Mathematica, we can see that the polynomial $[2 ; 4 ; 6 ; 10]$ is a common multiple of the denominator polynomials of the above identity. Therefore, the growth function $f_{P}(t)$ can be written as

$$
\frac{1}{f_{P}(t)}=\frac{h_{2,3,4,5,6}(t)}{[2 ; 4 ; 6 ; 10]},
$$

where $h_{2,3,4,5,6}(t)$ is a integer polynomial of at most degree 18. Substituting the identities $(2.3) \sim(2.10)$ into the coefficients of $h_{2,3,4,5,6}(t)$, we obtain that $h(1)=0$. By using Mathematica, the growth function $f_{P}(t)$ is expressed as

$$
\frac{1}{f_{P}(t)}=\frac{(t-1)}{[2 ; 4 ; 6 ; 10]} H_{2,3,4,5,6}(t)
$$

where

$$
\begin{aligned}
H_{2,3,4,5,6}(t) & =\left(v_{2,3,6}+v_{2,4,4}+v_{3,3,3}+v_{2,2,2,2}-1\right) t^{17} \\
& +\left(2 v_{2,3,6}+2 v_{2,4,4}+2 v_{3,3,3}+v_{2,2,2,2}+F-5\right) t^{16} \\
& +\left(\frac{1}{2} v_{2,2,3}+\frac{1}{2} v_{2,2,4}+\frac{1}{2} v_{2,2,5}+\frac{1}{2} v_{2,2,6}+v_{2,3,3}+v_{2,3,4}+v_{2,3,5}+4 v_{2,3,6}+4 v_{2,4,4}+\frac{7}{2} v_{3,3,3}+3 v_{2,2,2,2}+F-8\right) t^{15} \\
& +\left(\frac{1}{2} v_{2,2,4}+\frac{1}{2} v_{2,2,5}+\frac{1}{2} v_{2,2,6}+v_{2,3,3}+\frac{3}{2} v_{2,3,4}+\frac{3}{2} v_{2,3,5}+\frac{11}{2} v_{2,3,6}+5 v_{2,4,4}+5 v_{3,3,3}+3 v_{2,2,2,2}+3 F-16\right) t^{14} \\
& +\left(v_{2,2,3}+v_{2,2,4}+\frac{3}{2} v_{2,2,5}+\frac{3}{2} v_{2,2,6}+2 v_{2,3,3}+3 v_{2,3,4}+\frac{7}{2} v_{2,3,5}+\frac{15}{2} v_{2,3,6}+7 v_{2,4,4}+7 v_{3,3,3}+5 v_{2,2,2,2}+3 F-20\right) t^{13}
\end{aligned}
$$

$$
\begin{aligned}
& +\left(\frac{1}{2} v_{2,2,3}+v_{2,2,4}+v_{2,2,5}+\frac{3}{2} v_{2,2,6}+2 v_{2,3,3}+\frac{7}{2} v_{2,3,4}+\frac{9}{2} v_{2,3,5}+8 v_{2,3,6}+8 v_{2,4,4}+\frac{15}{2} v_{3,3,3}+5 v_{2,2,2,2}+5 F-28\right) t^{12} \\
& +\left(v_{2,2,3}+\frac{3}{2} v_{2,2,4}+2 v_{2,2,5}+2 v_{2,2,6}+3 v_{2,3,3}+\frac{9}{2} v_{2,3,4}+6 v_{2,3,5}+9 v_{2,3,6}+9 v_{2,4,4}+9 v_{3,3,3}+6 v_{2,2,2,2}+5 F-31\right) t^{11} \\
& +\left(v_{2,2,3}+\frac{3}{2} v_{2,2,4}+\frac{3}{2} v_{2,2,5}+2 v_{2,2,6}+3 v_{2,3,3}+\frac{9}{2} v_{2,3,4}+\frac{13}{2} v_{2,3,5}+9 v_{2,3,6}+9 v_{2,4,4}+9 v_{3,3,3}+6 v_{2,2,2,2}+6 F-35\right) t^{10} \\
& +\left(v_{2,2,3}+\frac{3}{2} v_{2,2,4}+2 v_{2,2,5}+2 v_{2,2,6}+3 v_{2,3,3}+\frac{9}{2} v_{2,3,4}+7 v_{2,3,5}+9 v_{2,3,6}+9 v_{2,4,4}+9 v_{3,3,3}+6 v_{2,2,2,2}+6 F-36\right) t^{9} \\
& +\left(v_{2,2,3}+\frac{3}{2} v_{2,2,4}+2 v_{2,2,5}+2 v_{2,2,6}+3 v_{2,3,3}+\frac{9}{2} v_{2,3,4}+7 v_{2,3,5}+9 v_{2,3,6}+9 v_{2,4,4}+9 v_{3,3,3}+6 v_{2,2,2,2}+6 F-36\right) t^{8} \\
& +\left(v_{2,2,3}+\frac{3}{2} v_{2,2,4}+\frac{3}{2} v_{2,2,5}+2 v_{2,2,6}+3 v_{2,3,3}+\frac{9}{2} v_{2,3,4}+\frac{13}{2} v_{2,3,5}+8 v_{2,3,6}+8 v_{2,4,4}+8 v_{3,3,3}+5 v_{2,2,2,2}+6 F-35\right) t^{7} \\
& +\left(v_{2,2,3}+\frac{3}{2} v_{2,2,4}+2 v_{2,2,5}+2 v_{2,2,6}+3 v_{2,3,3}+\frac{9}{2} v_{2,3,4}+6 v_{2,3,5}+7 v_{2,3,6}+7 v_{2,4,4}+7 v_{3,3,3}+5 v_{2,2,2,2}+5 F-31\right) t^{6} \\
& +\left(\frac{1}{2} v_{2,2,3}+v_{2,2,4}+v_{2,2,5}+\frac{3}{2} v_{2,2,6}+2 v_{2,3,3}+\frac{7}{2} v_{2,3,4}+\frac{9}{2} v_{2,3,5}+5 v_{2,3,6}+5 v_{2,4,4}+\frac{11}{2} v_{3,3,3}+3 v_{2,2,2,2}+5 F-28\right) t^{5} \\
& +\left(v_{2,2,3}+v_{2,2,4}+\frac{3}{2} v_{2,2,5}+\frac{3}{2} v_{2,2,6}+2 v_{2,3,3}+3 v_{2,3,4}+\frac{7}{2} v_{2,3,5}+\frac{7}{2} v_{2,3,6}+4 v_{2,4,4}+4 v_{3,3,3}+3 v_{2,2,2,2}+3 F-20\right) t^{4} \\
& +\left(\frac{1}{2} v_{2,2,4}+\frac{1}{2} v_{2,2,5}+\frac{1}{2} v_{2,2,6}+v_{2,3,3}+\frac{3}{2} v_{2,3,4}+\frac{3}{2} v_{2,3,5}+\frac{3}{2} v_{2,3,6}+2 v_{2,4,4}+2 v_{3,3,3}+v_{2,2,2,2}+3 F-16\right) t^{3} \\
& +\left(\frac{1}{2} v_{2,2,3}+\frac{1}{2} v_{2,2,4}+\frac{1}{2} v_{2,2,5}+\frac{1}{2} v_{2,2,6}+v_{2,3,3}+v_{2,3,4}+v_{2,3,5}+v_{2,3,6}+v_{2,4,4}+\frac{3}{2} v_{3,3,3}+v_{2,2,2,2}+F-8\right) t^{2} \\
& +(F-5) t-1 .
\end{aligned}
$$

Let us write $H_{2,3,4,5,6}(t)=\sum_{k=1}^{17} n_{k} t^{k}-1$.
Lemma 3.1.1. All the coefficients of $H_{2,3,4,5,6}(t)$ are non-negative except the coefficients $n_{2}, n_{4}, n_{6}$, and its constant term. Moreover, the coefficients $n_{16}$ and $n_{15}$ are positive.

Proof. Consider the sum of the terms $v_{2,3,6}, v_{2,4,4}, v_{3,3,3}, v_{2,2,2,2}$, and $F$ of the coefficients of $H_{2,3,4,5,6}(t)$. By the inequality (2.12) and $F \geq 5$, we obtain $n_{k} \geq 0$ for $k \neq 2,4,6$. Moreover, we have $n_{16} \geq\left(v_{2,3,6}+v_{2,4,4}+v_{3,3,3}+v_{2,2,2,2}\right)+F-5>0$. Substituting the identities (2.7), (2.8), (2.9), and (2.10) into the coefficient $n_{15}$, we get

$$
\begin{aligned}
n_{15} & \geq \frac{1}{2} v_{2,2,3}+\frac{1}{2} v_{2,2,4}+\frac{1}{2} v_{2,2,5}+\frac{1}{2} v_{2,2,6}+v_{2,3,3}+v_{2,3,4}+v_{2,3,5}+\frac{1}{2} v_{3,3,3}+v_{2,4,4}+v_{2,3,6} \\
& =\frac{1}{3} e_{3}+e_{4}+e_{5}+e_{6}+\frac{1}{3} v_{2,2,3}+\frac{2}{3} v_{2,3,3}+\frac{1}{3} v_{2,3,4}+\frac{1}{3} v_{2,3,5}+\frac{1}{3} v_{2,3,6} .
\end{aligned}
$$

The assumption that $P$ is not a right-angled polyhedron implies that $n_{15}$ is positive.

By Lemma 3.1.1, it is sufficient to prove that the coefficients $n_{2}, n_{4}$, and $n_{6}$ are non-negative.
Lemma 3.1.2. Suppose that $P$ has at least two cusps. Then, the coefficients $n_{2}, n_{4}$ and $n_{6}$ of $H_{2,3,4,5,6}(t)$ are non-negative.
Proof. In a manner similar to the proof of Lemma 3.1.1, we obtain that $n_{4}$ and $n_{6}$ are non-negative. Substituting the identities (2.7), (2.8), (2.9), and (2.10) into
the coefficient $n_{2}$, we can write $n_{2}$ as follows:
$n_{2}=\left\{\begin{array}{l}\frac{1}{2} v_{2,2,4}+\frac{1}{2} v_{2,2,5}+\frac{1}{2} v_{2,2,6}+\frac{1}{2} v_{2,3,4}+\frac{1}{2} v_{2,3,5}+\frac{1}{2} v_{2,3,6}+v_{2,4,4}+v_{2,2,2,2}+e_{3}+F-8 \\ \frac{1}{2} v_{2,2,3}+\frac{1}{2} v_{2,2,5}+\frac{1}{2} v_{2,2,6}+v_{2,3,3}+\frac{1}{2} v_{2,3,4}+v_{2,3,5}+v_{2,3,6}+\frac{3}{2} v_{3,3,3}+v_{2,2,2,2}+e_{4}+F-8 \\ \frac{1}{2} v_{2,2,3}+\frac{1}{2} v_{2,2,4}+\frac{1}{2} v_{2,2,6}+v_{2,3,3}+v_{2,3,4}+\frac{1}{2} v_{2,3,5}+v_{2,3,6}+v_{2,4,4}+\frac{3}{2} v_{3,3,3}+v_{2,2,2,2}+e_{5}+F-8 \\ \frac{1}{2} v_{2,2,3}+\frac{1}{2} v_{2,2,4}+\frac{1}{2} v_{2,2,5}+v_{2,3,3}+v_{2,3,4}+v_{2,3,5}+\frac{1}{2} v_{2,3,6}+v_{2,4,4}+\frac{3}{2} v_{3,3,3}+v_{2,2,2,2}+e_{6}+F-8 .\end{array}\right.$
By the equality (3.1),

$$
n_{2} \geq \max \left\{e_{3}, e_{4}, e_{5}, e_{6}\right\}+v_{2,2,2,2}+\frac{1}{2}\left(v_{2,3,6}+v_{2,4,4}+v_{3,3,3}\right)+F-8
$$

If $P$ has one of the cusps of type $(2,3,6),(2,4,4)$ and $(3,3,3)$, then max $\left\{e_{3}, e_{4}, e_{5}, e_{6}\right\}$ is strictly greater than 1 . Therefore, by the assumption that $P$ has at least two cusps, $n_{2}$ is non-negative.

Since the coefficients of the polynomial $H_{2,3,4,5,6}(t)$ are integers, the following result is proved analogously to Lemma 3.1.2.
Lemma 3.1.3. Suppose that $P$ has one of the cusps of type $(2,3,6),(2,4,4)$ and $(3,3,3)$. Then, the coefficients $n_{2}, n_{4}$ and $n_{6}$ are non-negative.

Proof of Theorem 3.1.1 By Proposition 3.1.1, if $P$ is a right-angled noncompact hyperbolic Coxeter polyhedron, then the growth rate is a Perron number. Therefore, we may assume that $P$ is not a right-angled polyhedron. Suppose that $P$ has at least two cusps or one of the cusps of type $(2,3,6),(2,4,4)$ and $(3,3,3)$. Then, by Lemma 3.1.1, Lemma 3.1.2 and Lemma 3.1.3, the denominator polynomial $H_{2,3,4,5,6}(t)$ of the growth function $f_{P}(t)$ satisfies the conditions of Proposition 3.0.1. Finally, the remaining case is that $P$ is not a right-angled polyhedron, and has the unique cusp which is furthermore of type (2, 2, 2, 2). Apply Theorem 2.3.3 (ii) and consider the unique hyperbolic polyhedron $\tilde{P}$ obtained by opening this cusp in $P$. By a result of Kolpakov [11, Theorem 5], the growth rate of $P$ is a Pisot number and therefore also a Perron number.

### 3.2 Non-compact Coxeter polyhedra some of whose dihedral angles are $\frac{\pi}{k}$ for $k \geq 7$

In this section, we calculate the growth function $f_{P}(t)$ of a non-compact hyperbolic Coxeter polyhedron $P$ some of whose dihedral angles are $\frac{\pi}{k}$ for $k \geq 7$ and prove the following theorem.
Theorem 3.2.1. [25] Let $P$ be a non-compact hyperbolic Coxeter polyhedron some of whose dihedral angles are $\frac{\pi}{k}$ for $k \geq 7$. Then the growth rate of $P$ is a Perron number.

Let us denote the number of $\frac{\pi}{k}$-edges with $k \geq 7$ by $\sigma$. By Theorem 2.3.2, the proof of Theorem 3.2.1 will be divided into two steps; one is the case of $\sigma=F-3$, and second is of $\sigma \leq F-4$.

### 3.2.1 The growth rates in the case of $\sigma=F-3$

By Theorem 2.3.2, $P$ has a unique cusp which is furthermore of type (2,2,2,2). Apply Theorem 2.3.3 (ii) and consider the unique hyperbolic polyhedron $\tilde{P}$ obtained by opening this cusp in $P$. Then, $\tilde{P}$ is a compact Coxeter polyhedron whose growth rate is a Salem number. By a result of Kolpakov (Theorem 5 [11]), the growth rate of $P$ is then a Pisot number and therefore also a Perron number.

### 3.2.2 The growth rates in the case of $\sigma \leq F-4$

In this subsection, let us first prove the following theorem.
Theorem 3.2.2. Suppose that $\sigma \leq F-4$ and $P$ satisfies the following inequality

$$
\begin{equation*}
v_{2,2,2,2}+e_{3}+e_{4}+e_{5}+e_{6}+F-8 \geq 0 \tag{3.2}
\end{equation*}
$$

Then the growth rate of $P$ is a Perron number.
In order to prove Theorem 3.2.2, we shall use the following notation and terminology in this subsection. For any hyperbolic Coxeter polyhedron $P$, the boundary $\partial P$ is homeomorphic to $S^{2}$. This implies that the 1-skeleton of $P$ provides an abstract Coxeter polyhedron $C$. We call $C$ the abstract Coxeter polyhedron associated to $P$. Suppose that $C$ is an abstract Coxeter polyhedron and that $v$ is a vertex with valency $i$ for $i=3$ or $i=4$. Let $c_{1}, \cdots, c_{i}$ be the edges of $C$ incident to $v$ and denote by $\frac{\pi}{k_{i}}$ the label of the edge $c_{i}$.

- If a vertex $v$ of $C$ with valency 3 satisfies the inequality $\frac{1}{k_{1}}+\frac{1}{k_{2}}+\frac{1}{k_{3}}>1$, we call $v$ a spherical vertex of type $\left(k_{1}, k_{2}, k_{3}\right)$.
- If a vertex $v$ of $C$ with valency 3 satisfies the equality $\frac{1}{k_{1}}+\frac{1}{k_{2}}+\frac{1}{k_{3}}=1$, we call $v$ a Euclidean vertex of type $\left(k_{1}, k_{2}, k_{3}\right)$.
- If a vertex $v$ of $C$ with valency 3 satisfies the inequality $\frac{1}{k_{1}}+\frac{1}{k_{2}}+\frac{1}{k_{3}}<1$, we call $v$ a hyperbolic vertex of type $\left(k_{1}, k_{2}, k_{3}\right)$.
- If a vertex $v$ of $C$ with valency 4 satisfies the equality $k_{1}=k_{2}=k_{3}=k_{4}=2$, we call $v$ a Euclidean vertex of type (2,2,2,2).
- A vertex $v$ of $C$ with valency 4 different from a Euclidean vertex is called a hyperbolic vertex of valency 4.
- $\mathcal{V}_{k_{1}, k_{2}, k_{3}}$ denotes the number of spherical vertices of type $\left(k_{1}, k_{2}, k_{3}\right)$ of $C$.
- $\mathcal{E}_{k}$ denotes the number of edges labeled by $\frac{\pi}{k}$ of $C$.
- $\mathcal{F}$ denotes the number of faces of $C$.

A spherical, Euclidean or hyperbolic vertex $v$ of type $\left(k_{1}, k_{2}, k_{3}\right)$ of $C$ corresponds to a spherical, Euclidean or hyperbolic Coxeter triangle $\Delta_{k_{1}, k_{2}, k_{3}}$ whose interior angles are $\frac{\pi}{k_{1}}, \frac{\pi}{k_{2}}$ and $\frac{\pi}{k_{3}}$, respectively. We denote by $f_{k_{1}, k_{2}, k_{3}}(t)$ the growth function of $\Delta_{k_{1}, k_{2}, k_{3}}$. Then the abstract growth function $f_{C}(t)$ of $C$ is defined by the following identity.

$$
\frac{1}{f_{C}\left(t^{-1}\right)}:=1-\frac{\mathcal{F}}{[2]}+\sum_{k \geq 2} \frac{\mathcal{E}_{k}}{[2 ; k]}-\sum_{\frac{1}{k_{1}}+\frac{1}{k_{2}}+\frac{1}{k_{3}}>1} \frac{\mathcal{V}_{k_{1}, k_{2}, k_{3}}}{f_{k_{1}, k_{2}, k_{3}}(t)}
$$

In the sequel, let $C$ be the abstract Coxeter polyhedron associated with $P$ and $C^{\prime}$ be the abstract Coxeter polyhedron obtained from $C$ by changing one of the labels of $C$ from $\frac{\pi}{k}$ to $\frac{\pi}{6}$ (see Fig 3.1). Then we can see that the abstract growth function $f_{C}(t)$ of $C$ is equal to the growth function $f_{P}(t)$ of $P$.


Figure 3.1:
By Andreev's theorem (see Theorem 2.3.1), the endpoints of a $\frac{\pi}{k}$-edge of $P$ are vertices of type $(2,2, k)$ for $k \geq 7$ so that the abstract polyhedron $C^{\prime}$ has at least one Euclidean vertex and no hyperbolic vertices of valency 4. Then the growth function $f_{P}(t)$ of $P$ differs from the abstract growth function $f_{C^{\prime}}(t)$ of $C^{\prime}$ in the terms related to changing the label. This implies the following identity by using the relation $\frac{1}{\left.[k] t^{-1}\right)}=\frac{t^{k-1}}{[k]}$.

$$
\begin{align*}
\frac{1}{f_{P}(t)} & =\frac{1}{f_{C^{\prime}}(t)}+\left\{\left(-\frac{t^{6}}{[2 ; 6]}+\frac{2 t^{7}}{[2 ; 2 ; 6]}\right)+\left(\frac{t^{k}}{[2 ; k]}-\frac{2 t^{k+1}}{[2 ; 2 ; k]}\right)\right\} \\
& =\frac{1}{f_{C^{\prime}}(t)}+\frac{(t-1)}{[2 ; 2 ; 6 ; k]} \sum_{n=6}^{k-1} t^{n} \tag{3.3}
\end{align*}
$$

Proof of Theorem 3.2.2. Let $P \subset \mathbb{H}^{3}$ be a non-compact Coxeter polyhedron with $F \geq 5$ faces. The proof of the theorem proceeds by induction on the number
$\sigma$ of $\frac{\pi}{k}$-edges with $k \geq 7$ of $P$. More specifically, denote by $P_{\sigma}$ such a polyhedron with dihedral angles $\frac{\pi}{k_{1}}, \cdots, \frac{\pi}{k_{\sigma}}$ where $k_{1}, \cdots, k_{\sigma} \geq 7$. In order to prove that the growth rate of $P_{\sigma}$ is a Perron number, we show that the growth function $f_{P_{\sigma}}(t)$ of $P_{\sigma}$ satisfies the following identity.

$$
\frac{1}{f_{P_{\sigma}}(t)}=\frac{(t-1) Q_{\sigma}(t)}{\left[2 ; 2 ; 6 ; k_{1} ; \cdots ; k_{\sigma}\right]\left(1+2 t^{2}+2 t^{4}+2 t^{6}+2 t^{8}+t^{10}\right)}
$$

where $Q_{\sigma}(t)$ is the integer polynomial of degree $k_{1}+\cdots+k_{\sigma}+16-\sigma$ whose constant term is equal to -1 and the coefficients of $Q_{\sigma}(t)$ except its constant term are non-negative.

Step 1 In the case $\sigma=1$, consider the abstract Coxeter polyhedron $C_{1}^{\prime}$ whose labels lie in the set $\left\{\left.\frac{\pi}{k} \right\rvert\, k=2,3,4,5,6\right\}$ by construction. By the calculation of subsection 3.1.5, $\frac{1}{f_{C_{1}^{\prime}}(t)}$ is written as

$$
\frac{1}{f_{C_{1}^{\prime}}(t)}=\frac{(t-1)}{[2 ; 4 ; 6 ; 10]} H_{2,3,4,5,6}(t)
$$

where $H_{2,3,4,5,6}(t)$ is the integer polynomial of at most degree 17 . Then, by using Mathematica, we see that the polynomial $H_{2,3,4,5,6}$ is divisible by the polynomial $[2]=t+1$.

$$
\frac{1}{f_{C_{1}^{\prime}}(t)}=\frac{(t-1)}{[4 ; 6 ; 10]} G_{2,3,4,5,6}(t),
$$

where $G_{2,3,4,5,6}(t):=\frac{H_{2,3,4,5,6}(t)}{[2]}$ is the integer polynomial of degree 16. By using Mathematica, $G_{2,3,4,5,6}(t)$ can be rewritten as follows

$$
\begin{aligned}
G_{2,3,4,5,6}(t) & =\left(v_{2,3,6}^{\prime}+v_{2,4,4}^{\prime}+v_{3,3,3}^{\prime}+v_{2,2,2,2}^{\prime}-1\right) t^{16} \\
& +\left(v_{2,3,6}^{\prime}+v_{2,4,4}^{\prime}+v_{3,3,3}^{\prime}+F^{\prime}-4\right) t^{15} \\
& +\left(\frac{1}{2} v_{2,2,3}^{\prime}+\frac{1}{2} v_{2,2,4}^{\prime}+\frac{1}{2} v_{2,2,5}^{\prime}+\frac{1}{2} v_{2,2,6}^{\prime}+v_{2,3,3}^{\prime}+v_{2,3,4}^{\prime}+v_{2,3,5}^{\prime}+3 v_{2,3,6}^{\prime}+3 v_{2,4,4}^{\prime}+\frac{5}{2} v_{3,3,3}^{\prime}+3 v_{2,2,2,2}^{\prime}-4\right) t^{14} \\
& +\left(\frac{1}{2} v_{2,2,2}^{\prime}+\frac{1}{2} v_{2,2,4}^{\prime}+\frac{1}{2} v_{2,2,5}^{\prime}+\frac{1}{2} v_{2,2,6}^{\prime}+\frac{1}{2} v_{2,3,3}^{\prime}+v_{2,3,4}^{\prime}+v_{2,3,5}^{\prime}+3 v_{2,3,6}^{\prime}+\frac{5}{2} v_{2,4,4}^{\prime}+3 v_{3,3,3}^{\prime}+v_{2,2,2,2}^{\prime}+2 F^{\prime}-10\right) t^{13} \\
& +\left(\frac{3}{2} v_{2,2,3}^{\prime}+v_{2,2,4}^{\prime}+\frac{3}{2} v_{2,2,5}^{\prime}+\frac{3}{2} v_{2,2,6}^{\prime}+2 v_{2,3,3}^{\prime}+\frac{5}{2} v_{2,3,4}^{\prime}+3 v_{2,3,5}^{\prime}+5 v_{2,3,6}^{\prime}+5 v_{2,4,4}^{\prime}+\frac{9}{2} v_{3,3,3}^{\prime}+5 v_{2,2,2,2}^{\prime}-8\right) t^{12} \\
& +\left(v_{2,2,2}^{\prime}+v_{2,2,4}^{\prime}+\frac{1}{2} v_{2,2,5}^{\prime}+v_{2,2,6}^{\prime}+v_{2,3,3}^{\prime}+2 v_{2,3,4}^{\prime}+\frac{5}{2} v_{2,3,5}^{\prime}+4 v_{2,3,6}^{\prime}+4 v_{2,4,4}^{\prime}+4 v_{3,3,3}^{\prime}+2 v_{2,2,2,2}^{\prime}+3 F^{\prime}-16\right) t^{11} \\
& +\left(2 v_{2,2,3}^{\prime}+\frac{3}{2} v_{2,2,4}^{\prime}+\frac{5}{2} v_{2,2,5}^{\prime}+2 v_{2,2,6}^{\prime}+3 v_{2,3,3}^{\prime}+\frac{7}{2} v_{2,3,4}^{\prime}+\frac{9}{2} v_{2,3,5}^{\prime}+6 v_{2,3,6}^{\prime}+6 v_{2,4,4}^{\prime}+6 v_{3,3,3}^{\prime}+6 v_{2,2,2,2}^{\prime}-11\right) t^{10} \\
& +\left(v_{2,2,2}^{\prime}+v_{2,2,4}^{\prime}+v_{2,2,6}^{\prime}+v_{2,3,3}^{\prime}+2 v_{2,3,4}^{\prime}+3 v_{2,3,5}^{\prime}+4 v_{2,3,6}^{\prime}+4 v_{2,4,4}^{\prime}+4 v_{3,3,3}^{\prime}+2 v_{2,2,2,2}^{\prime}+4 F^{\prime}-20\right) t^{9} \\
& +\left(2 v_{2,2,3}^{\prime}+\frac{3}{2} v_{2,2,4}^{\prime}+3 v_{2,2,5}^{\prime}+2 v_{2,2,6}^{\prime}+3 v_{2,3,3}^{\prime}+\frac{7}{2} v_{2,3,4}^{\prime}+5 v_{2,3,5}^{\prime}+6 v_{2,3,6}^{\prime}+6 v_{2,4,4}^{\prime}+6 v_{3,3,3}^{\prime}+6 v_{2,2,2,2}^{\prime}-12\right) t^{8} \\
& +\left(v_{2,2,2}^{\prime}+v_{2,2,4}^{\prime}+v_{2,2,6}^{\prime}+v_{2,3,3}^{\prime}+2 v_{2,3,4}^{\prime}+3 v_{2,3,5}^{\prime}+4 v_{2,3,6}^{\prime}+4 v_{2,4,4}^{\prime}+4 v_{3,3,3}^{\prime}+2 v_{2,2,2,2}^{\prime}+4 F^{\prime}-20\right) t^{7} \\
& +\left(2 v_{2,2,3}^{\prime}+\frac{3}{2} v_{2,2,4}^{\prime}+\frac{5}{2} v_{2,2,5}^{\prime}+2 v_{2,2,6}^{\prime}+3 v_{2,3,3}^{\prime}+\frac{7}{2} v_{2,3,4}^{\prime}+\frac{9}{2} v_{2,3,5}^{\prime}+5 v_{2,3,6}^{\prime}+5 v_{2,4,4}^{\prime}+5 v_{3,3,3}^{\prime}+5 v_{2,2,2,2}^{\prime}-11\right) t^{6}
\end{aligned}
$$

$$
\begin{aligned}
& +\quad\left(v_{2,2,2}^{\prime}+v_{2,2,4}^{\prime}+\frac{1}{2} v_{2,2,5}^{\prime}+v_{2,2,6}^{\prime}+v_{2,3,3}^{\prime}+2 v_{2,3,4}^{\prime}+\frac{5}{2} v_{2,3,5}^{\prime}+3 v_{2,3,6}^{\prime}+3 v_{2,4,4}^{\prime}+3 v_{3,3,3}^{\prime}+2 v_{2,2,2,2}^{\prime}+3 F^{\prime}-16\right) t^{5} \\
& +\quad\left(\frac{3}{2} v_{2,2,3}^{\prime}+v_{2,2,4}^{\prime} \frac{3}{2} v_{2,2,5}^{\prime}+\frac{3}{2} v_{2,2,6}^{\prime}+2 v_{2,3,3}^{\prime}+\frac{5}{2} v_{2,3,4}^{\prime}+3 v_{2,3,5}^{\prime}+3 v_{2,3,6}^{\prime}+3 v_{2,4,4}^{\prime}+\frac{7}{2} v_{3,3,3}^{\prime}+3 v_{2,2,2,2}^{\prime}-8\right) t^{4} \\
& +\quad\left(\frac{1}{2} v_{2,2,2}^{\prime}+\frac{1}{2} v_{2,2,4}^{\prime}+\frac{1}{2} v_{2,2,5}^{\prime}+\frac{1}{2} v_{2,2,6}^{\prime}+\frac{1}{2} v_{2,3,3}^{\prime}+v_{2,3,4}^{\prime}+v_{2,3,5}^{\prime}+v_{2,3,6}^{\prime}+\frac{3}{2} v_{2,4,4}^{\prime}+v_{3,3,3}^{\prime}+v_{2,2,2,2}^{\prime}+2 F^{\prime}-10\right) t^{3} \\
& +\quad\left(\frac{1}{2} v_{2,2,3}^{\prime}+\frac{1}{2} v_{2,2,4}^{\prime}+\frac{1}{2} v_{2,2,5}^{2}+\frac{1}{2} v_{2,2,6}^{\prime}+v_{2,3,3}^{\prime}+v_{2,3,4}^{\prime}+v_{2,3,5}^{\prime}+v_{2,3,6}^{\prime}+v_{2,4,4}^{\prime}+\frac{3}{2} v_{3,3,3}^{\prime}+v_{2,2,2,2}^{\prime}-4\right) t^{2} \\
& +\quad\left(F^{\prime}-4\right) t-1
\end{aligned}
$$

where $F^{\prime}, v_{2,2,2,2}^{\prime}$ and $v_{k_{1}, k_{2}, k_{3}}^{\prime}$ denote respectively the number of faces, Euclidean vertices of type ( $2,2,2,2$ ) and spherical vertices of type $\left(k_{1}, k_{2}, k_{3}\right)$ of $C_{1}^{\prime \prime}$. We denote $n_{i}$ by the $i$-th coefficient of the polynomial $G_{2,3,4,5,6}(t)$. By using the identities (2.3)-(2.10) and the inequality (2.12), we can see that the following inequalities.

$$
\begin{align*}
n_{i} \geq 0 & (i=1,3,5,7,9,11,13,15)  \tag{3.4}\\
n_{i}+n_{i+1} \geq 0 & (i=1, \cdots, 15)  \tag{3.5}\\
n_{i}+n_{i+1}+n_{i+2} \geq 0 & (i=1, \cdots, 14) \tag{3.6}
\end{align*}
$$

By using the identity (3.3), we can see that

$$
\begin{aligned}
\frac{1}{f_{P_{1}}(t)} & =\frac{1}{f_{C_{1}^{\prime}}(t)}+\frac{(t-1)}{\left[2 ; 2 ; 6 ; k_{1}\right]} \sum_{i=6}^{k_{1}-1} t^{i} \\
& =\frac{(t-1)}{[4 ; 6 ; 10]} G_{2,3,4,5,6}(t)+\frac{(t-1)}{\left[2 ; 2 ; 6 ; k_{1}\right]} \sum_{i=6}^{k_{1}-1} t^{i} \\
& =\frac{(t-1)}{[2 ; 2 ; 5 ; 6]\left(1+t^{2}\right)\left(1-t+t^{2}-t^{3}+t^{4}\right)} G_{2,3,4,5,6}(t)+\frac{(t-1)}{\left[2 ; 2 ; 6 ; k_{1}\right]} \sum_{i=6}^{k_{1}-1} t^{i} \\
& =\frac{(t-1)\left\{\left[k_{1}\right] G_{2,3,4,5,6}(t)+\left(1+2 t^{2}+2 t^{4}+2 t^{6}+2 t^{8}+t^{10}\right) \sum_{i=6}^{k_{1}-1} t^{i}\right\}}{\left[2 ; 2 ; 6 ; k_{1}\right]\left(1+2 t^{2}+2 t^{4}+2 t^{6}+2 t^{8}+t^{10}\right)}
\end{aligned}
$$

Let $Q_{1}(t):=\left[k_{1}\right] G_{2,3,4,5,6}(t)+\left(1+2 t^{2}+2 t^{4}+2 t^{6}+2 t^{8}+t^{10}\right) \sum_{i=6}^{k_{1}-1} t^{i}$.

$$
\begin{aligned}
& {\left[k_{1}\right] G_{2,3,4,5,6}(t)} \\
& \quad=\left(\sum_{j=0}^{k_{1}-1} t^{j}\right)\left(\sum_{i=1}^{16} n_{i} t^{i}-1\right) \\
& \quad=\sum_{j=0}^{k=1-1} \sum_{i=1}^{16} n_{i} t^{i+j}-\sum_{j=0}^{k_{1}-1} t^{j} \\
& \quad=\sum_{i=1}^{k_{1}+15}\left\{\chi_{\left[1, k_{1}\right]}(i) n_{1}+\cdots+\chi_{\left[16, k_{1}+15\right]}(i) n_{16}\right\} t^{i}-\sum_{j=0}^{k_{1}-1} t^{j} .
\end{aligned}
$$

$$
\begin{aligned}
(1 & \left.+2 t^{2}+2 t^{4}+2 t^{6}+2 t^{8}+t^{10}\right) \sum_{i=6}^{k_{1}-1} t^{i} \\
& =\sum_{i=8}^{k_{1}+9}\left\{2\left(\chi_{\left[8, k_{1}+1\right]}+\chi_{\left[10, k_{1}+3\right]}+\chi_{\left[12, k_{1}+5\right]}+\chi_{\left[14, k_{1}+7\right]}\right)(i)+\chi_{\left[16, k_{1}+9\right]}(i)\right\} t^{i}+\sum_{j=6}^{k_{1}-1} t^{j} .
\end{aligned}
$$

where $\chi_{[p, q]}$ is defined to be the simple function on the closed interval $[p, q]$. Then the degree of $Q_{1}(t)$ is $k_{1}+15$, so that we can represent $Q_{1}(t)$ as $\sum_{i=1}^{k_{1}+15} n_{i}^{(1)} t^{i}-1$ and $n_{i}^{(1)}$ is written as follows.
$n_{i}^{(1)}=\sum_{j=1}^{16} \chi_{\left[j, k_{1}+j-1\right]}(i) n_{j}+2\left(\chi_{\left[8, k_{1}+1\right]}+\chi_{\left[10, k_{1}+3\right]}+\chi_{\left[12, k_{1}+5\right]}+\chi_{\left[14, k_{1}+7\right]}\right)(i)+\chi_{\left[16, k_{1}+9\right]}(i)-\chi_{[1,5]}(i)$.
Therefore, by combining the inequalities (3.4), (3.5) and (3.6), we can obtain the following inequalities and identities.

$$
\begin{aligned}
n_{i}^{(1)} & \geq 0\left(6 \leq i \leq k_{1}+15\right) . \\
n_{5}^{(1)} & =n_{5}+n_{4}+n_{3}+n_{2}+n_{1}-1 \\
n_{4}^{(1)} & =n_{4}+n_{3}+n_{2}+n_{1}-1 \\
n_{3}^{(1)} & =n_{3}+n_{2}+n_{1}-1 \\
n_{2}^{(1)} & =n_{2}+n_{1}-1=v_{2,2,2,2}^{\prime}+e_{3}^{\prime}+e_{4}^{\prime}+e_{5}^{\prime}+e_{6}^{\prime}+F^{\prime}-9 \\
n_{1}^{(1)} & =n_{1}-1=F^{\prime}-5,
\end{aligned}
$$

Since $C_{1}^{\prime}$ is obtained from $P_{1}$ by changing one dihedral angle from $\frac{\pi}{k_{1}}$ to $\frac{\pi}{6}, n_{2}^{(1)}$ can be rewritten as

$$
\begin{equation*}
n_{2}^{(1)}=v_{2,2,2,2}+e_{3}+e_{4}+e_{5}+e_{6}+F-8 \tag{3.7}
\end{equation*}
$$

The equality (3.7) together with $F^{\prime}=F \geq 5$ mean that the coefficients of $Q_{1}(t)$ except its constant term are non-negative under the assumption of Theorem 3.2.2. Therefore, by Proposition 3.0.1, the growth rate of $P_{1}$ is a Perron number.

Step 2. We assume that the following identity holds for the growth function $f_{P_{\sigma-1}}(t)$ of $P_{\sigma-1}$ for $\sigma \geq 2$ as inductive hypothesis.

$$
\frac{1}{f_{P_{\sigma-1}}(t)}=\frac{(t-1) Q_{\sigma-1}(t)}{\left[2 ; 2 ; 5 ; 6 ; k_{1} ; \cdots ; k_{\sigma-1}\right]\left(1+2 t^{2}+2 t^{4}+2 t^{6}+2 t^{8}+t^{10}\right)}
$$

where $Q_{\sigma-1}(t)$ is a polynomial of degree $k_{1}+\cdots+k_{\sigma-1}+16-(\sigma-1)$ and the coefficients of $Q_{\sigma-1}(t)$ except its constant term are non-negative. By the identity (3.3) we deduce that the following identities.

$$
\begin{aligned}
\frac{1}{f_{P_{\sigma}}(t)} & =\frac{(t-1)}{[2 ; 2 ; 6]}\left\{\frac{Q_{\sigma-1}(t)}{\left[k_{1} ; \cdots ; k_{\sigma-1}\right]\left(1+2 t^{2}+2 t^{4}+2 t^{6}+2 t^{8}+t^{10}\right)}+\frac{\sum_{n=6}^{k_{\sigma}-1} t^{n}}{\left[k_{\sigma}\right]}\right\} \\
& =\frac{(t-1)\left\{\left[k_{\sigma}\right] Q_{\sigma-1}(t)+\left[k_{1} ; \cdots ; k_{\sigma-1}\right]\left(1+2 t^{2}+2 t^{4}+2 t^{6}+2 t^{8}+t^{10}\right) \sum_{n=6}^{k_{\sigma}-1} t^{n}\right\}}{\left[2 ; 2 ; 6 ; k_{1} ; \cdots ; k_{\sigma}\right]\left(1+2 t^{2}+2 t^{4}+2 t^{6}+2 t^{8}+t^{10}\right)}
\end{aligned}
$$

Let $Q_{\sigma}(t):=\left[k_{\sigma}\right] Q_{\sigma-1}(t)+\left[k_{1} ; \cdots ; k_{\sigma-1}\right]\left(1+2 t^{2}+2 t^{4}+2 t^{6}+2 t^{8}+t^{10}\right) \sum_{n=6}^{k_{\sigma}-1} t^{n}$ and $R(t):=\left[k_{1} ; \cdots ; k_{\sigma-1}\right]\left(1+2 t^{2}+2 t^{4}+2 t^{6}+2 t^{8}+t^{10}\right) \sum_{n=6}^{k_{\sigma}-1} t^{n}$. Note that the coefficients of $R(t)$ is non-negative. Moreover, the coefficients of $i$-th terms are positive for $6 \leq i \leq k_{\sigma}-1$.

$$
\begin{align*}
\operatorname{deg}\left[k_{\sigma}\right] Q_{\sigma}(t) & =\left(k_{\sigma}-1\right)+\operatorname{deg} Q_{\sigma-1} \\
& =k_{1}+\cdots+k_{\sigma}+16-\sigma .  \tag{3.8}\\
\operatorname{deg} R(t) & =\left(k_{1}-1\right)+\cdots+\left(k_{\sigma-1}-1\right)+10+\left(k_{\sigma}-1\right) \\
& =k_{1}+\cdots+k_{\sigma}+10-\sigma . \tag{3.9}
\end{align*}
$$

The equalities (3.8) and (3.9) imply that the degree of $Q_{\sigma}(t)$ is equal to $k_{1}+\cdots+$ $k_{\sigma}+16-\sigma$. We denote $n_{i}^{(\sigma-1)}$ by the $i$-th coefficient of the polynomial $Q_{\sigma-1}(t)$, so that $Q_{\sigma-1}(t)$ can be rewritten as $\sum_{i \geq 1} n_{i}^{(\sigma-1)} t^{i}-1$.

$$
\begin{aligned}
Q_{\sigma}(t)= & {\left[k_{\sigma}\right]\left(\sum_{i \geq 1} n_{i}^{(\sigma-1)} t^{i}\right)-\left[k_{\sigma}\right]+R(t) } \\
= & \left(\sum_{i=0}^{6} t^{i}+\sum_{i=7}^{k_{\sigma}-1} t^{i}\right)\left(\sum_{i \geq 1} n_{i}^{(\sigma-1)} t^{i}\right)-\left(1+\sum_{i=1}^{5} t^{i}+\sum_{i=6}^{k_{\sigma}-1} t^{i}\right)+R(t) \\
= & \left(\sum_{i \geq 1} n_{i}^{(\sigma-1)} t^{i}\right)+\left(\sum_{i \geq 2} n_{i-1}^{(\sigma-1)} t^{i}\right)+\left(\sum_{i \geq 3} n_{i-2}^{(\sigma-1)} t^{i}\right)+\left(\sum_{i \geq 4} n_{i-3}^{(\sigma-1)} t^{i}\right) \\
& +\left(\sum_{i \geq 5} n_{i-4}^{(\sigma-1)} t^{i}\right)+\left(\sum_{i \geq 6} n_{i-5}^{(\sigma-1)} t^{i}\right)+\left(\sum_{i \geq 7} n_{i-6}^{(\sigma-1)} t^{i}\right) \\
& \quad+\sum_{j=7}^{k_{\sigma}-1} \sum_{i \geq 1} n_{i}^{(\sigma-1)} t^{i+j}+\left\{R(t)-\sum_{i=6}^{k_{\sigma}-1} t^{i}\right\}-\sum_{i=1}^{5} t^{i}-1,
\end{aligned}
$$

and hence we obtain the following inequality and identities once we represent $Q_{k}(t)$ as $\sum n_{i}^{(\sigma)} t^{i}-1$.

$$
\begin{aligned}
& n_{i}^{(\sigma)} \geq 0(i \geq 6) \\
& n_{5}^{(\sigma)}=n_{5}^{(\sigma-1)}+n_{4}^{(\sigma-1)}+n_{3}^{(\sigma-1)}+n_{2}^{(\sigma-1)}+n_{1}^{(\sigma-1)}-1
\end{aligned}
$$

$$
\begin{aligned}
n_{4}^{(\sigma)} & =n_{4}^{(\sigma-1)}+n_{3}^{(\sigma-1)}+n_{2}^{(\sigma-1)}+n_{1}^{(\sigma-1)}-1 . \\
n_{3}^{(\sigma)} & =n_{3}^{(\sigma-1)}+n_{2}^{(\sigma-1)}+n_{1}^{(\sigma-1)}-1 . \\
n_{2}^{(\sigma)} & =n_{2}^{(\sigma-1)}+n_{1}^{(\sigma-1)}-1 . \\
n_{1}^{(\sigma)} & =n_{1}^{(\sigma-1)}-1=n_{1}^{(1)}-(\sigma-1) .
\end{aligned}
$$

By the result of Step 1,

$$
n_{1}^{(\sigma)}=n_{1}^{(1)}-(\sigma-1)=F-4-\sigma .
$$

Therefore the coefficients of $Q_{\sigma}(t)$ except its constant term are non-negative and the constant term of $Q_{\sigma}(t)$ is equal to -1 if $P$ satisfies the inequality $F-4 \geq k$. Therefore, by Proposition 3.0.1, the growth rate of $P_{\sigma}$ is a Perron number.

### 3.2.3 The proof of Theorem 3.2.1

By Theorem 3.2.2, the condition (3.2) is sufficient in order to deduce that the growth rate of $P$ is a Perron number when $F-4 \geq \sigma$. First, suppose that $P$ is a non-compact hyperbolic Coxeter polyhedron with $F \geq 7$. Since $P$ has at least 1 cusp, we get the following inequality

$$
v_{2,2,2,2}+e_{3}+e_{4}+e_{5}+e_{6}+F-8 \geq 1+7-8=0
$$

which allow us to conclude. Therefore, it remains to consider non-compact Coxeter polyhedra with $F=5$ or $F=6$ faces and which do not satisfy the inequality (3.2) of Theorem 3.2.2. Figure 3.2 shows all possible combinatorial structures of acuteangled convex polyhedra with 4,5 or 6 faces [5].

We use Andreev's Theorem (see Theorem 2.3.1) in order to describe a noncompact hyperbolic Coxeter polyhedron with 5 or 6 faces which does not satisfy inequality (3.2).

By Theorem 2.3.3 and Andreev's Theorem, it is not difficult to see that a noncompact finite volume hyperbolic Coxeter polyhedron $P$ with 5 or 6 faces and with at least one $\frac{\pi}{k}$-edge for $k \geq 7$ has to be of combinatorial type (ii), (iv), (v), (viii), (ix), (x). If the combinatorial structure of $P$ is (viii), $P$ has 2 cusps of type $(2,2,2,2)$ and if the combinatorial structure is (ix) or (x), $P$ has at least one of cusps of type $(2,3,6)$ or $(2,4,4)$ or $(3,3,3)$. Hence, the inequality (3.2) holds for polyhedra $P$ of type (viii), (iv) or (x), and by Theorem 3.2.2 their growth rates are Perron numbers.

Consider finally Coxeter polyhedra $P$ of type (ii), (iv) or (v). First and by means of Theorem 2.3.3, we determine which edges of $P$ subject to (ii), (iv) or (v) can be of the form $\frac{\pi}{k}$ for $k \geq 7$. In this way, we can deduce that each such polyhedron $P$ results from opening cusps of type (2,2,2,2) as shown in Figure 3.3.


Figure 3.2:


Figure 3.3:

In Figure 3.3, labels on edges mean the dihedral angles and $k, k_{1}, k_{2} \geq 7$. If the inequality (3.2) does not hold for the case of (iv) or (v), all of the dihedral angles other than $\frac{\pi}{k_{1}}, \frac{\pi}{k_{2}}$ are $\frac{\pi}{2}$, since $v_{2,2,2,2}=1$.
Proposition 3.2.1. Suppose that the combinatorial structure of $P$ is (iv) or (v). Then the growth rate of $P$ is a Perron number.

Proof. By means of Steinberg's formula (see Theorem 2.1.3), we can calculate the growth function $f_{P}(t)$ of $P$ as follows.

$$
\begin{aligned}
\frac{1}{f_{P}(t)} & =1-\frac{6 t}{[2]}+\frac{9 t^{2}}{[2 ; 2]}+\frac{t^{k_{1}}}{\left[2 ; m_{1}\right]}+\frac{t^{k_{2}}}{\left[2 ; k_{2}\right]}-\frac{2 t^{3}}{[2 ; 2 ; 2]}-\frac{2 t^{k_{1}+1}}{\left[2 ; 2 ; k_{1}\right]}-\frac{2 t^{k_{2}+1}}{\left[2 ; 2 ; k_{2}\right]} \\
& =\frac{(t-1)\left\{(2 t+1)\left[k_{1} ; k_{2}\right]-(t+1)\left(\left[k_{1}\right]+\left[k_{2}\right]\right)\right\}}{\left[2 ; 2 ; 2 ; k_{1} ; k_{2}\right]}
\end{aligned}
$$

Let $Q(t):=(2 t+1)\left[k_{1} ; k_{2}\right]-(t+1)\left(\left[k_{1}\right]+\left[k_{2}\right]\right)$. We may assume that $k_{1} \geq k_{2}$, without loss in generality.

If $k_{1}=k_{2}, Q(t)$ can be rewritten as,

$$
\begin{aligned}
Q(t) & =\left[k_{1}\right]\left\{(2 t+1)\left[k_{1}\right]-(2 t+2)\right\} \\
& =\left[k_{1}\right]\left(2 \sum_{i=0}^{k_{1}-1} t^{i+1}+\sum_{i=0}^{k_{1}-1} t^{i}-2 t-2\right) \\
& =\left[k_{1}\right]\left(2 t^{k_{1}}+3 t^{k_{1}-1}+3 t^{k_{1}-2}+\cdots+3 t^{2}+t-1\right)
\end{aligned}
$$

If $k_{1}>k_{2}, Q(t)$ can be rewritten as,

$$
\begin{aligned}
Q(t) & =(2 t+1)\left\{\left(t^{k_{1}-1}+\cdots+t^{k_{2}}\right)\left[k_{2}\right]+\left[k_{2}\right]^{2}\right\}-(t+1)\left\{\left(t^{k_{1}-1}+\cdots+t^{k_{2}}\right)+2\left[k_{2}\right]\right\} \\
& =(2 t+1)\left(t^{k_{1}-1}+\cdots+t^{k_{2}}\right)\left[k_{2}\right]-(t+1)\left(t^{k_{1}-1}+\cdots+t^{k_{2}}\right)+\left[k_{2}\right]\left\{(2 t+1)\left[k_{2}\right]-(2 t+2)\right\} \\
& =\left[k_{1}\right]\left(2 t^{k_{2}}+3 t^{k_{2}-1}+\cdots+3 t^{2}+t\right)+t\left(t^{k_{1}-1}+\cdots+t^{k_{2}}\right)-\left[k_{2}\right]
\end{aligned}
$$

By the above calculation, the coefficients of $Q(t)$ except its constant term are non-negative.

Therefore we can apply Proposition 3.0.1 to conclude that the growth rate is a Perron number.

It remains to study the growth rates of non-compact Coxeter triangular prisms $P$ (see Fig 3.3). Since $P$ has at least one vertex at infinity, $P$ has precisely one $\frac{\pi}{k}$-edge for $k \geq 7$. By contraction of this edge to a vertex of type $(2,2,2,2)$ (see Theorem 2.3.3), $P$ deforms into exactly one among the hyperbolic Coxeter pyramid $\hat{P}$ which have been entirely classified by Tumarkin [21]. In this way, we can deduce a precise configuration for $P$ (see Fig 3.4) and prove the following result.

Proposition 3.2.2. Suppose that $P$ is a Coxeter triangular prism and $P$ does not satisfy the inequality (3.2). Then $P$ has the dihedral angles as in Figure 3.4 and the growth rate of $P$ is a Perron number.


Figure 3.4:

Proof. Case.(I) By means of Steinberg's formula, we can calculate the growth function $f_{P}(t)$ of $P$, and hence the growth function is written as,

$$
\frac{1}{f_{P}(t)}=\frac{(t-1)\left(2 t^{k+2}+3 t^{k+1}+4 t^{k}+\cdots+4 t^{4}+3 t^{3}+t^{2}-1\right)}{[2 ; 2 ; 4 ; k]}
$$

Case.(II) The growth function is calculated in the same manner:

$$
\frac{1}{f_{P}(t)}=\frac{R(t)}{[2 ; 2 ; 2 ; 3 ; 6 ; k]}
$$

where

$$
\begin{gathered}
R(t)=2 t^{k+8}+5 t^{k+7}+7 t^{k+6}+7 t^{k+5}+6 t^{k+4}+5 t^{k+3}+3 t^{k+2}+t^{k+1} \\
-t^{9}-4 t^{8}-7 t^{7}-8 t^{6}-7 t^{5}-6 t^{4}-4 t^{3}-t^{2}+t+1
\end{gathered}
$$

Let us notice that $R(t)$ is divisible by $[2 ; 3]$ and $(t-1)$. Therefore $f_{P}(t)$ can be rewritten as,
$\frac{1}{f_{P}(t)}=\frac{(t-1)\left(2 t^{k+4}+3 t^{k+3}+4 t^{k+2}+5 t^{k+1}+6 t^{k}+\cdots+6 t^{6}+5 t^{5}+3 t^{4}+2 t^{3}+t^{2}-1\right)}{[2 ; 2 ; 6 ; k]}$
Hence, we can apply Proposition 3.0.1 to conclude that the growth rate is a Perron number.

Proof of Theorem 3.2.1 Let $P$ be a non-compact hyperbolic Coxeter polyhedron having at least one dihedral angle of the form $\frac{\pi}{k}$ for some integer $k \geq 7$ and $\sigma$ be the number of $\frac{\pi}{k}$-edges of $P$ with $k \geq 7$. By Theorem 2.3.2, $P$ satisfies the inequality $\sigma \leq F-3$. If the equality $\sigma=F-3$ holds for $P$, by combining with
the observation in subsection 3.2.1, the growth rate of $P$ is a Perron number. If the inequality $\sigma \leq F-4$ holds for $P$, there are two cases that can be considered. First, the case that $P$ satisfies the inequality (3.2). In this case, by Theorem 3.2 .2 , the growth rate of $P$ is a Perron number. Second, the case that $P$ does not satisfy the inequality (3.2). In this case, $P$ has to be of combinatorial type (ii), (iv) or (v) (see Fig 3.2). By Proposition 3.2.1 (resp. Proposition 3.2.2), if the combinatorial structure of $P$ is (iv) or (v) (resp. (ii)), the growth rate of $P$ is a Perron number.

## Chapter 4

## An infinite sequence of ideal hyperbolic Coxeter 4-polytopes whose growth rates are Perron numbers

### 4.1 Construction of infinite sequence of ideal nonsimple hyperbolic Coxeter polytopes

In this Section, we construct an infinite sequence $\left\{P_{n}\right\}_{n \in \mathbb{N}}$ of non-simple ideal hyperbolic Coxeter 4-polytopes by glueing ideal hyperbolic Coxeter 4-pyramids along their isometric facets. First, we introduce the vertical projection $p_{\infty}$ from $\infty$ to $\mathbb{R}^{3}$ and describe how to see hyperbolic 4-polytopes in terms of the projection. Second, we review hyperbolic Coxeter 4 -pyramids $P_{1}$ over the product of three simplexes which are completely classified by Tumarkin [21] and then construct the infinite sequence $\left\{P_{n}\right\}_{n \in \mathbb{N}}$. Finally, we determine the combinatorial structure of $P_{n}$ in order to calculate the growth rate $\tau_{P_{n}}$. In the sequel, we call 2-faces of 4 -polytope faces.

### 4.1.1 The vertical projection from $\infty$

A horosphere $\Sigma=\Sigma_{u}$ based at a point at infinity $u \in \partial \mathbb{H}^{d}$ is defined to be a 3-dimensional Euclidean sphere in $\mathbb{H}^{4}$ tangent to $\mathbb{R}^{3}$ at $u$ (resp. a Euclidean hyperplane parallel to $\mathbb{R}^{3}$ ) if $u$ is situated on $\mathbb{R}^{3}$ (resp. $u=\infty$ ). The restriction of the hyperbolic metric to the horosphere $\Sigma$ turns $\Sigma$ into a Euclidean 3-space.
Lemma 4.1.1. [18, Theorem 6.4.5] Suppose that $P=\cap_{i=1}^{m} H_{i}^{-}$is a non-compact hyperbolic 4-polytope of finite volume and $u$ is a vertex at infinity of $P$. Let $\Sigma$
be a horosphere based at $u$ such that $\Sigma$ intersects with $P$ only at the bounding hyperplanes incident to $u$. Then, $L(u):=P \cap \Sigma$ has the following properties.

- $L(u)$ is a 3-dimensional Euclidean polytope in $\Sigma$.
- For any bounding hyperplane $H_{i}$ incident to $u, H_{i} \cap L(u)$ is a bounding hyperplane of $L(u)$ in $\Sigma$.
- If 2 facets $F_{i}:=H_{i} \cap P$ and $F_{j}:=H_{j} \cap P$ make the face of $P$, then the intersection of $F_{i} \cap L(u)$ and $F_{j} \cap L(u)$ is an edge of $L(u)$ and the dihedral angle $\angle F_{i} \cap F_{j}$ is equal to the dihedral angle $\angle\left(F_{i} \cap L(u)\right) \cap\left(F_{j} \cap L(u)\right)$

Consider the vertical projection from $\infty$ denoted by

$$
p_{\infty}: \mathbb{H}^{4} \rightarrow \mathbb{R}^{3} ;(x, y, z, t) \mapsto(x, y, z)
$$

Let $P=\cap_{i=1}^{m} H_{i}^{-}$be a non-compact hyperbolic 4-polytope of finite volume and $u$ be a vertex at infinity of $P$. By using the translation of $\mathbb{R}^{3}$ which maps $u$ to 0 and the inversion with respect to the unit sphere in $\mathbb{R}^{4}$, we may assume that $u$ is $\infty$. If a hyperplane $H_{i}$ is incident to (resp. not incident to) $\infty$, then $H_{i}$ is a Euclidean hyperplane (resp. hemisphere) in $\mathbb{H}^{4}$ orthogonal to $\mathbb{R}^{3}$. Note that in our setting any closed half-space $H_{i}^{-}$contains $\infty$. Since the vertical projection $p_{\infty}$ maps any horosphere $\Sigma$ based at $\infty$ conformally onto $\mathbb{R}^{3}$, by using Lemma 4.1.1, we can treat dihedral angles between 2 bounding hyperplanes of $P$ incident to $\infty$ as the corresponding dihedral angles in the 3 -dimensional Euclidean polytope $p_{\infty}(L(\infty))$. Suppose that the bounding hyperplanes $H_{i}$ and $H_{j}$ of $P$ are not incident to $\infty$. By choosing a point in $H_{i} \cap H_{j} \cap \mathbb{R}^{3}$ and considering the outer normal vectors $u_{i}$ and $u_{j}$, the dihedral angle $\angle H_{i} \cap H_{j}$ in $P$ is given by $\arccos -\left(u_{i}, u_{j}\right)$.


Figure 4.1: The dihedral angle in $\mathbb{R}^{3}$

### 4.1.2 The ideal hyperbolic Coxeter pyramid $P_{1}$.

In [21], Tuamrkin classified all hyperbolic Coxeter 4-pyramids whose apex at infinity has a cubical structure. In particular, there exists an ideal hyperbolic Coxeter 4 -pyramid $P_{1}$ with Coxeter scheme shown in Figure 4.2.


Figure 4.2: The Coxeter scheme $X\left(P_{1}\right)$
In the sequel, we use the following notations.

- The non-simple vertex of $P_{1}$ is denoted by $u$.
- $F_{0}$ denotes the unique cubical facet of $P_{1}$.
- The pyramidal facets of $P_{1}$ are denoted by $F_{1}, \cdots, F_{6}$. The facets have property that $F_{i}$ and $F_{i+1}(i=1,3,5)$ meet at the non-simple vertex $u$ of $P_{1}$ and the dihedral angle formed by $F_{i}$ and $F_{0}$ is equal to $\frac{\pi}{4}$ for $i=1,2$.
- If the intersection of facets $F_{i}$ and $F_{j}$ is a (polygonal) face of $P_{1}$, we denote it by $f_{i j}$. $f_{i j}$ is the ridge of the dihedral angle $\angle F_{i} \cap F_{j}$.
- The hyperplane carrying $F_{i}$ is denoted by $H_{i}$.

Since the vertex link of $u$ is a Euclidean right-angled cube given by $\widetilde{A}_{1} \times \widetilde{A}_{1} \times \widetilde{A}_{1}$, by using suitable isometries of $\mathbb{H}^{4}, P_{1}$ can be normalized as follows:

- The vertex $u$ is $\infty$.
- The hyperplane $H_{0}$ is the unit hemisphere centered at origin.
- The hyperplanes $H_{1}$ and $H_{2}$ are orthogonal to the $x$-axis.
- The hyperplanes $H_{3}$ and $H_{4}$ are orthogonal to the $y$-axis.
- The hyperplanes $H_{5}$ and $H_{6}$ are orthogonal to the $z$-axis.

Under this normalization of $P_{1}$, we can depict $p_{\infty}\left(P_{1}\right)$ according to Figure 4.3. The coordinates of the eight vertices $A, B, C, D, E, F, G$ and $H$ are

$$
A=\left(\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2}\right) \quad B=\left(-\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2}\right) \quad C=\left(-\frac{1}{\sqrt{2}},-\frac{1}{2}, \frac{1}{2}\right)
$$

$$
\begin{array}{ll}
D=\left(\frac{1}{\sqrt{2}},-\frac{1}{2}, \frac{1}{2}\right) & E=\left(\frac{1}{\sqrt{2}}, \frac{1}{2},-\frac{1}{2}\right) \quad F=\left(-\frac{1}{\sqrt{2}}, \frac{1}{2},-\frac{1}{2}\right) \\
G=\left(-\frac{1}{\sqrt{2}},-\frac{1}{2},-\frac{1}{2}\right) & H=\left(\frac{1}{\sqrt{2}},-\frac{1}{2},-\frac{1}{2}\right) .
\end{array}
$$



Figure 4.3: $p_{\infty}\left(P_{1}\right)$
In Figure 4.3, the hyperplanes carrying the quadrangular faces $\mathrm{ADHE}, \mathrm{ABFE}$ and ABCD are $p_{\infty}\left(H_{1}\right), p_{\infty}\left(H_{3}\right)$ and $p_{\infty}\left(H_{5}\right)$. We take a copy of $P_{1}$, denoted by $P_{1}^{\prime}$. The facets $F_{k}^{\prime}$ of $P_{1}^{\prime}$ is isometric to the facet $F_{k}$ of $P_{1}$ for $k=0, \cdots, 6$. Then, glue two isometric 4-pyramids $P_{1}$ and $P_{1}^{\prime}$ along the facet $F_{1}$ of $P_{1}$ and the facet $F_{2}^{\prime}$ of $P_{1}^{\prime}$ to obtain a new polytope $P_{2}$.


Figure 4.4: The projective images of $P_{1}$ and $P_{1}^{\prime}$
The projective image of the polytope $P_{2}$ is depicted in Figure 4.5. By the glueing procedure, the facets $F_{1}$ of $P_{1}$ and $F_{2}^{\prime}$ of $P_{1}^{\prime}$ do not appear in $P_{2}$. Since the hyperplanes $p_{\infty}\left(H_{3}\right), p_{\infty}\left(H_{4}\right), p_{\infty}\left(H_{5}\right)$ and $p_{\infty}\left(H_{6}\right)$ of $P_{1}$ and $P_{1}^{\prime}$ coincide with each other, the faces $f_{13}, f_{14}, f_{15}, f_{16}$ in $P_{1}$ and $f_{23}, f_{24}, f_{25}, f_{26}$ in $P_{1}^{\prime}$ do not appear in $P_{2}$ as well. On the other hand, $P_{2}$ has some new faces; one is the quadrangular face coming from the cubical facet $F_{0}$ in $P_{1}$ and $P_{1}^{\prime}$, and the other new faces are


Figure 4.5: The projective image of the resulting 4-polytope $P_{2}$.
composed by the unions of $f_{34}, f_{45}, f_{56}$ and $f_{63}$ in $P_{1} \cup P_{1}^{\prime}$. Since the pyramidal facets $F_{2}$ in $P_{1}$ and $F_{1}^{\prime}$ in $P_{1}^{\prime}$ do not contribute to the glueing procedure, $P_{2}$ has the two facets $F_{1}$ and $F_{2}$ in its boundary.

In summary, we obtain the following combinatorial data for $P_{2}$.

- $P_{2}$ has 8 facets; 2 cubical facets, 2 pyramidal facets and 4 facets with 6 faces.
- $P_{2}$ has 23 faces; (i) 8 triangular faces come from $F_{2}$ of $P_{1}$ and $F_{1}^{\prime}$ of $P_{1}^{\prime}$, (ii) 10 quadrangular faces come from $F_{0}$ in $P_{1}$ and $P_{1}^{\prime}$, (iii) only one quadrangular face comes from the intersection of $F_{1}$ in $P_{1}$ and $F_{2}^{\prime}$ in $P_{1}^{\prime}$, (iv) 4 quadrangular faces come from the union of $f_{34}, f_{45}, f_{56}$ and $f_{63}$ of $P_{1}$ and $P_{1}^{\prime}$.
- $P_{2}$ has 28 edges.
- $P_{2}$ has 13 ideal vertices; only the vertex $\infty$ is non-simple.

Since the two pyramidal facets of $P_{2}$ are isometric to the pyramidal facets $F_{1}$ and $F_{2}$ of $P_{1}$, we can repeat this procedure by glueing $P_{1}$ and $P_{2}$ along their pyramidal facets, and the resulting 4-polytope is denoted by $P_{3}$. By induction, glueing a copy of $P_{1}$ to $P_{n-1}$ gives rise to a new polytope denoted by $P_{n}$. In fact, the ideal hyperbolic 4-polytope $P_{n}$ is obtained by glueing $n$ copies of $P_{1}$ along the isometric facets $F_{1}$ and $F_{2}$.

### 4.1.3 The combinatorial structure of $P_{n}$.

Lemma 4.1.2. $P_{n}$ has the following combinatorial data.
(Facets) $(n+6)$ facets; $n$ cubical facets, 2 pyramidal facets and the other 4 facets have $(n+4)$ faces.
(Faces) $(5 n+13)$ faces; 8 triangular faces, $5 n+1$ quadrilateral faces and $4(n+2)$ gonal faces.
(Edges) $(8 n+12)$ edges.
(Vertices) $(4 n+5)$ vertices; $4 n+4$ simple vertices and only one non-simple vertex.
Proof. It suffices to consider $p_{\infty}\left(P_{n}\right)$. Indeed, the projective image $P_{n}$ consists of $n$ right quadrangular prisms inscribed in closed balls of radius 1 (see Fig 4.6).


Figure 4.6: The projective image of $P_{n}$
We use the following notation and terminology to describe $P_{n}$.

- The 2 pyramidal facets of $P_{n}$ are denoted by $F_{1}$ and $F_{2}$.
- The $n$ cubical facets of $P_{n}$ are denoted by $C_{1}, \cdots, C_{n}$. Moreover, we suppose that $C_{1} \cap F_{1}, C_{n} \cap F_{2}$ and $C_{i} \cap C_{i+1}$ are the quadrilateral faces.
- The remaining facets of $P_{n}$ are denoted by $G_{1}, G_{2}, G_{3}, G_{4}$. Moreover, we suppose that $G_{i} \cap G_{i+1}(i \bmod 4)$ is a $(n+2)$-gonal face.
- $X_{n}$ denotes the Coxeter scheme of $P_{n}$.
- If a face of $P_{n}$ has the dihedral angle $\frac{\pi}{m}$, we call it a $\frac{\pi}{m}$-face.


Figure 4.7: The front, top, back, and bottom planes are labeled by $G_{1}, G_{2}, G_{3}$, and $G_{4}$, respectively, following the notations for $P_{n}$.

Let us determine the elliptic and parabolic subschemes of $X_{n}$.
(1) By Lemma 4.1.2, $X_{n}$ has $n+6$ vertices.
(2) Since each quadrilateral face $C_{i} \cap C_{i+1}$ is the intersection of glueing facets, its dihedral angle $\angle C_{i} \cap C_{i+1}$ is equal to $\frac{\pi}{2}$. If we glue $P_{n-1}$ and $P_{1}$ along their isometric pyramidal facets, then all faces of $P_{n-1}$ and $P_{1}$ which are not incident to the glueing facets are invariant. Therefore, we have the following situation.

- The triangular faces $F_{i} \cap G_{j}$ are $\frac{\pi}{2}$-faces.
- The $(n+2)$-gonal faces $G_{i} \cap G_{i+1}$ are $\frac{\pi}{2}$-faces.
- The quadrilateral faces $G_{i} \cap C_{j}$ are $\frac{\pi}{3}$-faces.
- The quadrilateral faces $C_{1} \cap F_{1}$ and $C_{n} \cap F_{2}$ are $\frac{\pi}{4}$-faces.
(3) Each edge of $P_{n}$ is expressed as the intersection of precisely three facets.
- If an edge is the intersection $F_{i} \cap G_{j} \cap G_{j+1}$, it corresponds to the elliptic subscheme $A_{1} \times A_{1} \times A_{1}$ of $X_{n}$.
- If an edge is the intersection $F_{1} \cap G_{i} \cap C_{1}$ or $F_{2} \cap G_{i} \cap C_{n}$, it corresponds to the elliptic subscheme $B_{3}$ of $X_{n}$.
- If an edge is the intersection $G_{i} \cap G_{i+1} \cap C_{j}$, it corresponds to the elliptic subscheme $A_{3}$ of $X_{n}$.
- If an edge is the intersection $G_{i} \cap C_{j} \cap C_{j+1}$, it corresponds to the elliptic subscheme $A_{3}$ of $X_{n}$.
(4) Each vertex corresponds to a parabolic subscheme of $X_{n}$.
- If a vertex is the intersection $F_{1} \cap G_{i} \cap G_{i+1} \cap C_{1}$ or $F_{2} \cap G_{i} \cap G_{i+1} \cap C_{n}$, then it corresponds to the parabolic subscheme $\tilde{B}_{3}$ of $X_{n}$.
- If a vertex is the intersection $G_{i} \cap G_{i+1} \cap C_{j} \cap C_{j+1}$, then it corresponds to the parabolic subscheme $\tilde{A}_{3}$ of $X_{n}$.
- The non-simple vertex corresponds to the parabolic subscheme $\tilde{A}_{1} \times \tilde{A}_{1} \times \tilde{A}_{1}$ of $X_{n}$.


### 4.2 The growth function of $P_{n}$

By implementing the combinatorial data of $P_{n}$ into Steinberg's formula (see Theorem 2.1.3), the growth function $f_{n}(t)$ of $P_{n}$ can be calculated as follows.

$$
\frac{1}{f_{n}\left(t^{-1}\right)}=1-\frac{n+6}{[2]}+\frac{n+11}{[2,2]}+\frac{4 n}{[2,3]}+\frac{2}{[2,4]}-\frac{8}{[2,2,2]}-\frac{8}{[2,4,6]}-\frac{8 n-4}{[2,3,4]} .
$$

By using Mathematica, the growth function $f_{n}(t)$ can be expressed as

$$
f_{n}\left(t^{-1}\right):=\frac{N_{n}(t)}{D_{n}(t)}
$$

where

$$
\begin{aligned}
N_{n}(t)= & (t+1)^{3}\left(t^{2}+1\right)\left(t^{2}-t+1\right)\left(t^{2}+t+1\right) \\
D_{n}(t)= & t^{9}-(n+3) t^{8}-(n-4) t^{7}+(2 n-8) t^{6} \\
& \quad+(2 n+8) t^{5}+(2 n-8) t^{4}-(2 n-11) t^{3}+(3 n-5) t^{2}+(3 n+4) t-4(n+1) .
\end{aligned}
$$

Lemma 4.2.1. All the roots of $D_{n}(t)$ are simple.
Proof. We show that the resultant $R\left(D_{n}(t), D_{n}^{\prime}(t)\right)$ of $D_{n}(t)$ and $D_{n}^{\prime}(t)$ is not equal to 0 for any $n \in \mathbb{N}$. By using Mathematica, we can calculate it as follows:

$$
\begin{aligned}
R\left(D_{n}(t), D_{n}^{\prime}(t)\right) & =9367548196608 n^{16}-84315693201408 n^{15}-3211145218356480 n^{14} \\
& -13452086684085248 n^{13}-76883986729280512 n^{12}-221310749589989376 n^{11} \\
& -369276695931527424 n^{10}-436823682353681408 n^{9}-375744535536699392 n^{8} \\
& -227155659791212544 n^{7}-100271146222672128 n^{6}-28147372028425216 n^{5} \\
& -2791806794781440 n^{4}-1194005028478976 n^{3}-23952968404992 n^{2} \\
& -2787725279232 n .
\end{aligned}
$$

By using Descartes' rule [17, Corollary 1, p.28], $R\left(D_{n}(t), D_{n}^{\prime}(t)\right)$ has at most one positive real root as a real polynomial with related to the index $n$. We can check the following equalities by using Mathematica.

$$
\begin{aligned}
& R\left(D_{25}(t), D_{25}^{\prime}(t)\right)=-5236764089528548306162419869100800 \\
& R\left(D_{26}(t), D_{26}^{\prime}(t)\right)=18356309345841539117459400503775232
\end{aligned}
$$

Hence, $R\left(D_{n}(t), D_{n}^{\prime}(t)\right) \neq 0$ for any $n \in \mathbb{N}$.

### 4.2.1 The distribution of the real roots of $D_{n}(t)$

Lemma 4.2.2. Let $w(t)$ be the number of sign changes in the Sturm sequence of $D_{n}(t)$ and $D_{n}^{\prime}(t)$. Then, $w(0)=\left\{\begin{array}{ll}6 & (1 \leq n \leq 25) \\ 5 & (26 \leq n)\end{array}\right.$ and $w(\infty)=\left\{\begin{array}{ll}3 & (1 \leq n \leq 25) \\ 2 & (26 \leq n)\end{array}\right.$. Moreover, by using Sturm's theorem, the number of positive real roots of $D_{n}(t)$ is equal to 3 for any $n \in \mathbb{N}$.

Proof. The equality $D_{n}(0)=-4(n+1)$ implies that 0 is not a root of $D_{n}(t)$ for any $n \in \mathbb{N}$. By using Mathematica, the Sturm sequence $S\left(D_{n}, D_{n}^{\prime}\right)$ can be calculated (see Appendix). Let us write $S\left(D_{n}, D_{n}^{\prime}\right)=\left\{d_{0}, \cdots, d_{9}\right\}$, and denote the $i$-th coefficient of $d_{k}(t)$ as $a_{i}^{(k)}$, that is,

$$
\begin{equation*}
d_{k}(t)=\sum_{i=0}^{9-k} a_{i}^{(k)} t^{i} \tag{*}
\end{equation*}
$$

Then, $w(0)$ (resp. $w(\infty)$ ) is equal to the number of sign changes in the sequence $a_{0}^{(0)}, \cdots, a_{0}^{(9)}$ (resp. $\left.a_{9}^{(0)}, a_{8}^{(1)}, \cdots, a_{1}^{(8)}, a_{0}^{(9)}\right)$. The sign of each coefficient $a_{i}^{(k)}$ depends on $n \in \mathbb{N}$. Let us determine their signs. For example, we consider the sign of $a_{0}^{(5)}$. The sign of $a_{0}^{(5)}$ depends on the following factor polynomial $p(n)$ (see Appendix);

$$
\begin{aligned}
p(n)=13008 n^{8} & +20600 n^{7}-1607896 n^{6}+2420092 n^{5}+2017855 n^{4} \\
& +899112 n^{3}+1122697 n^{2}-1476508 n-45088
\end{aligned}
$$

The difference of $p(n+1)$ and $p(n)$ equals

$$
\begin{aligned}
p(n+1)-p(n)=52032 n^{7} & +254212 n^{6}-4243164 n^{5}-5193210 n^{4} \\
& +781934 n^{3}+7841885 n^{2}+7857749 n+1704480 .
\end{aligned}
$$

By Descartes' rule, the number of positive real zeroes of $p(n+1)-p(n)$ is at most 2. Consider

$$
\begin{aligned}
& p(2)-p(1)=9055918>0 \\
& p(3)-p(2)=-140899954<0 \\
& p(8)-p(7)=-10316213144<0 \\
& p(9)-p(8)=16414574600>0
\end{aligned}
$$

This observation shows that

$$
\left\{\begin{array}{l}
p(2)>p(1) \\
p(2)>p(3)>\cdots>p(7)>p(8) \\
p(8)<p(9)<\cdots<p(n)<p(n+1)<\cdots
\end{array}\right.
$$

Moreover,

$$
\begin{aligned}
p(1) & =3363872 \\
p(3) & =-260324200 \\
p(9) & =-39144733360
\end{aligned}
$$

$$
p(10)=162088321532
$$

Therefore, we can determine the sign of $a_{0}^{(5)}$ as follows.

$$
a_{0}^{(5)} \begin{cases}>0 & (n=1,2) \\ <0 & (3 \leq n \leq 9) \\ >0 & (n \geq 10)\end{cases}
$$

The remaining cases concerning $a_{i}^{(k)}$ follow by analogy.
We can calculate $w(-\infty)$ analogously to the proof of Lemma 4.2.2, in such a way that

$$
w(-\infty)= \begin{cases}6 & (1 \leq n \leq 25) \\ 7 & (26 \leq n)\end{cases}
$$

Therefore, by combining Lemma 4.2.2 and Sturm's theorem, we obtain the following result.
Proposition 4.2.1. The denominator polynomial $D_{n}(t)$ has the following real roots:

$$
\begin{cases}\text { three positive roots and no negative roots } & (1 \leq n \leq 25) \\ \text { three positive roots and two negative roots } & (n \geq 26)\end{cases}
$$

### 4.2.2 The distribution of the complex roots of $D_{n}(t)$

By applying the method prescribed in section 2.4.3, we can deduce an upper bound for the absolute values of all complex roots of $D_{n}(t)$.

1. Calculate the two real polynomials $\Phi(t)$ and $\Psi(t)$ which are given by

$$
D_{n}(z(t))=\frac{\Phi(t)+i \Psi(t)}{(t+i)^{\operatorname{deg} D_{n}}}
$$

where $z(t)=2 \frac{t-i}{t+i}$. By using Mathematica, $\Phi(t)$ and $\Psi(t)$ can be written as follows:

$$
\begin{gathered}
\Phi(t)=-(162 n+56) t^{9}+(6456 n-6512) t^{7}-(2476 n-49792) t^{5} \\
-(7176 n+60048) t^{3}+(894 n+13752) t \\
\Psi(t)=(2034 n-456) t^{8}-(8280 n-24880) t^{6}-(7188 n+67136) t^{4} \\
+(4136 n+36816) t^{2}-(14 n+2808)
\end{gathered}
$$

2. By using Mathematica, we can show that the resultant of $\Phi(t)$ and $\Psi(t)$ is not equal to 0 for any $n \in N$. Therefore $D_{n}(t)$ has no roots on the circle $S_{2}$ of radius 2 centered at the origin.
3. By using Mathematica, the Sturm sequence $S(\Phi, \Psi)$ can be calculated.
4. In a manner similar to the argument in section 4.2.1, we can calculate the numbers of sign changes $w(\infty)$ and $w(-\infty)$ in $S(\Phi, \Psi)$.
Lemma 4.2.3. For any $n \in \mathbb{N}, w(\infty)=8$ and $w(-\infty)=1$. By Corollary 2.4.1, the number of roots of $D_{n}(t)$ contained in the closed disk of radius 2 centered at the origin in the complex plane $\mathbb{C}$ is equal to 8.

Theorem 4.2.1. The growth rate of the polytope $P_{n}$ is a Perron number for any $n \in \mathbb{N}$.

Proof. By Lemma 4.2.3, the absolute values of the 8 roots of $D_{n}(t)$ are strictly less than 2. Since $\operatorname{deg} D_{n}(t)=9$, it is sufficient to prove that $D_{n}(t)$ has a positive real root which is greater than 2. In order to prove that, we consider $w(2)$. By Section 2.4.1, we obtain

$$
w(2)= \begin{cases}4 & (1 \leq n \leq 25) \\ 3 & (26 \leq n)\end{cases}
$$

Therefore, by Sturm's theorem, the polynomial $D_{n}(t)$ has the unique positive real root which is strictly greater than 2 for any $n \in \mathbb{N}$.

### 4.3 Appendix: the Sturm sequence of $D_{n}(t)$ and $D_{n}^{\prime}(t)$

In this section, we provide the Sturm sequence $S\left(D_{n}, D_{n}^{\prime}\right)=\left\{d_{0}, \cdots, d_{k}\right\}$ considered in Section 4.2.1, ((*)).

$$
\begin{aligned}
& d_{0}(t)= t^{9}-(n+3) t^{8}-(n-4) t^{7}+(2 n-8) t^{6} \\
&+(2 n+8) t^{5}+(2 n-8) t^{4}-(2 n-11) t^{3}+(3 n-5) t^{2}+(3 n+4) t-4(n+1) \\
& d_{1}(t)=9 t^{8}-8(n+3) t^{7}-7(n-4) t^{6}+6(2 n+8) t^{5} \\
&+5(2 n+8) t^{4}+4(2 n-8) t^{3}-3(2 n-11) t^{2}+2(3 n-5) t+(3 n+4) \\
& d_{2}(t)=\frac{1}{81}\left\{\left(8 n^{2}+66 n\right) t^{7}+\left(7 n^{2}-61 n+132\right) t^{6}+\left(-12 n^{2}-60 n-144\right) t^{5}\right. \\
&+\left(-10 n^{2}-160 n+240\right) t^{4}+\left(-8 n^{2}+116 n-498\right) t^{3}+\left(6 n^{2}-204 n+216\right) t^{2} \\
&\left.\quad+\left(-6 n^{2}-224 n-258\right) t-3 n^{2}+311 n+312\right\}
\end{aligned}
$$

$$
\begin{aligned}
d_{3}(t)=\frac{81}{4 n^{2}(4 n+33)^{2}} & \left\{\left(39 n^{4}+266 n^{3}-2673 n^{2}-1848 n-1936\right) t^{6}\right. \\
+ & \left(36 n^{4}+612 n^{3}+3956 n^{2}+4480 n+2112\right) t^{5} \\
+ & \left(54 n^{4}+470 n^{3}-1872 n^{2}-4372 n-3520\right) t^{4} \\
+ & \left(-88 n^{4}-776 n^{3}+3866 n^{2}+6246 n+7304\right) t^{3} \\
+ & \left(150 n^{4}+1374 n^{3}-3216 n^{2}-1660 n-3168\right) t^{2} \\
+ & \left(162 n^{4}+2508 n^{3}+8540 n^{2}+8870 n+3784\right) t \\
& \left.-259 n^{4}-3428 n^{3}-7161 n^{2}-8548 n-4576\right\}
\end{aligned}
$$

Next, we list the coefficients of the polynomials $d_{4}(t), \cdots, d_{8}(t)$.
The denominator of $d_{4}(t)=81(1936+n(1848+n(2673-n(266+39 n))))^{2}$

$$
\begin{aligned}
& a_{5}^{(4)}=8 n^{2}(4 n+33)^{2}\left(270 n^{6}-930 n^{5}-59765 n^{4}-72316 n^{3}-51247 n^{2}-34920 n+11920\right) \\
& a_{4}^{(4)}=-16 n^{2}(4 n+33)^{2}\left(51 n^{6}+1630 n^{5}+7368 n^{4}-68445 n^{3}-3176 n^{2}-41152 n+16768\right) \\
& a_{3}^{(4)}=8 n^{2}(4 n+33)^{2}\left(471 n^{6}+6452 n^{5}-5086 n^{4}-176746 n^{3}-54403 n^{2}-120344 n-8944\right) \\
& a_{2}^{(4)}=16 n^{2}(4 n+33)^{2}\left(153 n^{6}-411 n^{5}-32385 n^{4}-33106 n^{3}-44007 n^{2}-20216 n-7664\right) \\
& a_{1}^{(4)}=-8 n^{2}(4 n+33)^{2}\left(579 n^{6}+14834 n^{5}+101041 n^{4}+47610 n^{3}+25760 n^{2}+3472 n-25280\right) \\
& a_{0}^{(4)}=16 n^{2}(33+4 n)^{2}\left(10304+60992 n+92088 n^{2}+112317 n^{3}+78944 n^{4}+5932 n^{5}+33 n^{6}\right)
\end{aligned}
$$

The denominator of $d_{5}(t)=4 n^{2}(33+4 n)^{2}\left(11920-34920 n-51247 n^{2}-72316 n^{3}\right.$

$$
\left.-59765 n^{4}-930 n^{5}+270 n^{6}\right)^{2}
$$

$$
\begin{aligned}
a_{4}^{(5)} & =-81\left(39 n^{4}+266 n^{3}-2673 n^{2}-1848 n-1936\right)^{2}\left(246 n^{8}-5794 n^{7}+360959 n^{6}\right. \\
& \left.+5606880 n^{5}-3313218 n^{4}+6140122 n^{3}-3491843 n^{2}+2584756 n-544176\right) \\
a_{3}^{(5)} & =162\left(39 n^{4}+266 n^{3}-2673 n^{2}-1848 n-1936\right)^{2}\left(5289 n^{8}+5992 n^{7}-788952 n^{6}\right. \\
& \left.-810030 n^{5}-5107313 n^{4}+118907 n^{3}-2823408 n^{2}+1353973 n-43828\right) \\
a_{2}^{(5)} & =-81\left(39 n^{4}+266 n^{3}-2673 n^{2}-1848 n-1936\right)^{2}\left(8442 n^{8}-32742 n^{7}-1868957 n^{6}\right. \\
& \left.-1946748 n^{5}-4253223 n^{4}-1203496 n^{3}-1818280 n^{2}+440564 n-127008\right) \\
a_{1}^{(5)} & =-162\left(39 n^{4}+266 n^{3}-2673 n^{2}-1848 n-1936\right)^{2}\left(6261 n^{8}+27352 n^{7}-543939 n^{6}\right. \\
& \left.+1168425 n^{5}-740209 n^{4}-333809 n^{3}-454006 n^{2}-793981 n+269220\right) \\
a_{0}^{(5)} & =81\left(39 n^{4}+266 n^{3}-2673 n^{2}-1848 n-1936\right)^{2}\left(13008 n^{8}+20600 n^{7}-1607896 n^{6}\right.
\end{aligned}
$$

$$
\left.+2420092 n^{5}+2017855 n^{4}+899112 n^{3}+1122697 n^{2}-1476508 n-45088\right)
$$

The denominator of $d_{6}(t)=81\left(-1936-1848 n-2673 n^{2}+t h e 266 n^{3}+39 n^{4}\right)^{2}$

$$
\begin{aligned}
& \left(-544176+2584756 n-3491843 n^{2}+6140122 n^{3}-3313218 n^{4}\right. \\
& \left.+5606880 n^{5}+360959 n^{6}-5794 n^{7}+246 n^{8}\right)^{2}
\end{aligned}
$$

$$
\begin{aligned}
& a_{3}^{(6)}=-8 n^{2}(4 n+33)^{2}\left(270 n^{6}-930 n^{5}-59765 n^{4}-72316 n^{3}-51247 n^{2}-34920 n+11920\right)^{2} \\
& \quad\left(403481 n^{10}+2480778 n^{9}-37969219 n^{8}-158119702 n^{7}-1100390746 n^{6}-216055166 n^{5}\right. \\
& \left.\quad-1160964773 n^{4}+282443786 n^{3}-329580155 n^{2}+172728524 n-35052620\right) \\
& a_{2}^{(6)}=16 n^{2}(4 n+33)^{2}\left(270 n^{6}-930 n^{5}-59765 n^{4}-72316 n^{3}-51247 n^{2}-34920 n+11920\right)^{2} \\
& \quad\left(169494 n^{10}+14649 n^{9}-18830064 n^{8}+62828800 n^{7}-387398843 n^{6}+226406803 n^{5}\right. \\
& \left.\quad-413299018 n^{4}+245275527 n^{3}-138927361 n^{2}+67186063 n-4007124\right) \\
& a_{1}^{(6)}=8 n^{2}(4 n+33)^{2}\left(270 n^{6}-930 n^{5}-59765 n^{4}-72316 n^{3}-51247 n^{2}-34920 n+11920\right)^{2} \\
& \quad\left(474903 n^{10}+4516538 n^{9}-11601465 n^{8}+104831670 n^{7}+294141284 n^{6}-180768204 n^{5}\right. \\
& \left.\quad+111338775 n^{4}-296355112 n^{3}+31452859 n^{2}-39181768 n+10452012\right) \\
& a_{0}^{(6)}=-16 n^{2}(4 n+33)^{2}\left(270 n^{6}-930 n^{5}-59765 n^{4}-72316 n^{3}-51247 n^{2}-34920 n+11920\right)^{2} \\
& \quad\left(252601 n^{10}+1535932 n^{9}-10172760 n^{8}+137682333 n^{7}+130244020 n^{6}+208421539 n^{5}\right. \\
& \left.\quad+143139607 n^{4}+2115857 n^{3}+44003972 n^{2}-41200307 n+18745192\right)
\end{aligned}
$$

The denominator of $d_{7}(t)=4 n^{2}(33+4 n)^{2}\left(11920-34920 n-51247 n^{2}-72316 n^{3}\right.$

$$
\left.-59765 n^{4}-930 n^{5}+270 n^{6}\right)^{2}(-35052620+172728524 n
$$

$$
-329580155 n^{2}+282443786 n^{3}-1160964773 n^{4}-216055166 n^{5}
$$

$$
\left.-1100390746 n^{6}-158119702 n^{7}-37969219 n^{8}+2480778 n^{9}+403481 n^{10}\right)^{2}
$$

$$
\begin{aligned}
a_{2}^{(7)} & =81\left(39 n^{4}+266 n^{3}-2673 n^{2}-1848 n-1936\right)^{2}\left(246 n^{8}-5794 n^{7}+360959 n^{6}+5606880 n^{5}\right. \\
& \left.-3313218 n^{4}+6140122 n^{3}-3491843 n^{2}+2584756 n-544176\right)^{2}\left(48400755 n^{12}\right. \\
& +245803454 n^{11}-4721345357 n^{10}-11572421870 n^{9}-124324436353 n^{8}-146160412422 n^{7} \\
& -206861074257 n^{6}-134297550268 n^{5}-66775078001 n^{4}-24225751096 n^{3}+3620403819 n^{2} \\
& -813838328 n+111404496) \\
a_{1}^{(7)} & =162\left(39 n^{4}+266 n^{3}-2673 n^{2}-1848 n-1936\right)^{2}\left(246 n^{8}-5794 n^{7}+360959 n^{6}+5606880 n^{5}\right. \\
& \left.-3313218 n^{4}+6140122 n^{3}-3491843 n^{2}+2584756 n-544176\right)^{2}\left(9127365 n^{12}\right. \\
& +43738914 n^{11}-1050600669 n^{10}-2134594907 n^{9}-221052668 n^{8}+8764159647 n^{7} \\
& +1193739782 n^{6}+16709700491 n^{5}+4028829086 n^{4}+2954840024 n^{3}-2598459169 n^{2}
\end{aligned}
$$

$-405956928 n-67672272)$

$$
\begin{aligned}
a_{0}^{(7)} & =-81\left(39 n^{4}+266 n^{3}-2673 n^{2}-1848 n-1936\right)^{2}\left(246 n^{8}-5794 n^{7}+360959 n^{6}+5606880 n^{5}\right. \\
& \left.-3313218 n^{4}+6140122 n^{3}-3491843 n^{2}+2584756 n-544176\right)^{2}\left(59130903 n^{12}\right. \\
& +320783028 n^{11}-5921870437 n^{10}-16668405100 n^{9}-117418503841 n^{8}-151967821848 n^{7} \\
& -180213457131 n^{6}-140644288440 n^{5}-51131969275 n^{4}-32331152680 n^{3}+5676560341 n^{2} \\
& -2814520288 n-23940048)
\end{aligned}
$$

The denominator of $d_{8}(t)=81\left(-1936-1848 n-2673 n^{2}+t h e 266 n^{3}+39 n^{4}\right)^{2}$

$$
\begin{aligned}
& \left(-544176+2584756 n-3491843 n^{2}+6140122 n^{3}-3313218 n^{4}\right. \\
& \left.+5606880 n^{5}+360959 n^{6}-5794 n^{7}+246 n^{8}\right)^{2}(111404496-813838328 n \\
& +3620403819 n^{2}-24225751096 n^{3}-66775078001 n^{4}-134297550268 n^{5} \\
& -206861074257 n^{6}-146160412422 n^{7}-124324436353 n^{8} \\
& \left.-11572421870 n^{9}-4721345357 n^{10}+245803454 n^{11}+48400755 n^{12}\right)^{2}
\end{aligned}
$$

$$
\begin{aligned}
& a_{1}^{(8)}=16 n^{2}(4 n+33)^{2}\left(270 n^{6}-930 n^{5}-59765 n^{4}-72316 n^{3}-51247 n^{2}-34920 n+11920\right)^{2} \\
& \quad\left(403481 n^{10}+2480778 n^{9}-37969219 n^{8}-158119702 n^{7}-1100390746 n^{6}-216055166 n^{5}\right. \\
& \left.\quad-1160964773 n^{4}+282443786 n^{3}-329580155 n^{2}+172728524 n-35052620\right)^{2} \\
& \left(1462545045 n^{14}-10472627469 n^{13}-402243294759 n^{12}-1104112693071 n^{11}\right. \\
& \quad-8571517376059 n^{10}-16797900884717 n^{9}-22904507347277 n^{8}-22168784110521 n^{7} \\
& \quad-14235620251809 n^{6}-6907194126551 n^{5}-2062300172501 n^{4}-196719185377 n^{3} \\
& \left.\quad-72614586920 n^{2}+4391952 n-226865664\right)
\end{aligned}
$$

$$
\begin{aligned}
& a_{0}^{(8)}=-16 n^{2}(4 n+33)^{2}\left(270 n^{6}-930 n^{5}-59765 n^{4}-72316 n^{3}-51247 n^{2}-34920 n+11920\right)^{2} \\
& \quad\left(403481 n^{10}+2480778 n^{9}-37969219 n^{8}-158119702 n^{7}-1100390746 n^{6}-216055166 n^{5}\right. \\
& \left.\quad-1160964773 n^{4}+282443786 n^{3}-329580155 n^{2}+172728524 n-35052620\right)^{2} \\
& \quad\left(682442280 n^{14}-13967744415 n^{13}-318617986273 n^{12}-866028050552 n^{11}\right. \\
& \quad-5973136686946 n^{10}-11470936502501 n^{9}-15278417145211 n^{8}-15018314214172 n^{7} \\
& \quad-9591556809634 n^{6}-5038052836203 n^{5}-1582742665577 n^{4}-286371055374 n^{3} \\
& \left.\quad-76587929392 n^{2}-3723242592 n-226865664\right)
\end{aligned}
$$

The numerator of $d_{0}=81\left(39 n^{4}+266 n^{3}-2673 n^{2}-1848 n-1936\right)^{2}\left(246 n^{8}-5794 n^{7}\right.$

$$
\begin{aligned}
& +360959 n^{6}+5606880 n^{5}-3313218 n^{4}+6140122 n^{3}-3491843 n^{2} \\
& +2584756 n-544176)^{2}\left(48400755 n^{12}+245803454 n^{11}-4721345357 n^{10}\right. \\
& -11572421870 n^{9}-124324436353 n^{8}-146160412422 n^{7}-206861074257 n^{6} \\
& -134297550268 n^{5}-66775078001 n^{4}-24225751096 n^{3}+3620403819 n^{2} \\
& -813838328 n+111404496)^{2}\left(36591985143 n^{15}-329358176568 n^{14}\right. \\
& -12543536009205 n^{13}-52547213609708 n^{12}-300328073161252 n^{11} \\
& -864495115585896 n^{10}-1442487093482529 n^{9}-1706342509194068 n^{8} \\
& -1467752091940232 n^{7}-887326796059424 n^{6}-391684164932313 n^{5} \\
& -109950671986036 n^{4}-10905495292115 n^{3}-4664082142496 n^{2} \\
& -93566282832 n-10889551872)
\end{aligned}
$$

The denominator of $d_{0}=4 n(4 n+33)^{2}\left(270 n^{6}-930 n^{5}-59765 n^{4}-72316 n^{3}-51247 n^{2}-34920 n\right.$

$$
+11920)^{2}\left(403481 n^{10}+2480778 n^{9}-37969219 n^{8}-158119702 n^{7}\right.
$$

$$
-1100390746 n^{6}-216055166 n^{5}-1160964773 n^{4}+282443786 n^{3}
$$

$$
\left.-329580155 n^{2}+172728524 n-35052620\right)^{2}\left(1462545045 n^{14}\right.
$$

$$
-10472627469 n^{13}-402243294759 n^{12}-1104112693071 n^{11}
$$

$$
-8571517376059 n^{10}-16797900884717 n^{9}-22904507347277 n^{8}
$$

$$
-22168784110521 n^{7}-14235620251809 n^{6}-6907194126551 n^{5}
$$

$$
-2062300172501 n^{4}-196719185377 n^{3}-72614586920 n^{2}+4391952 n
$$

$$
-226865664)^{2}
$$

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