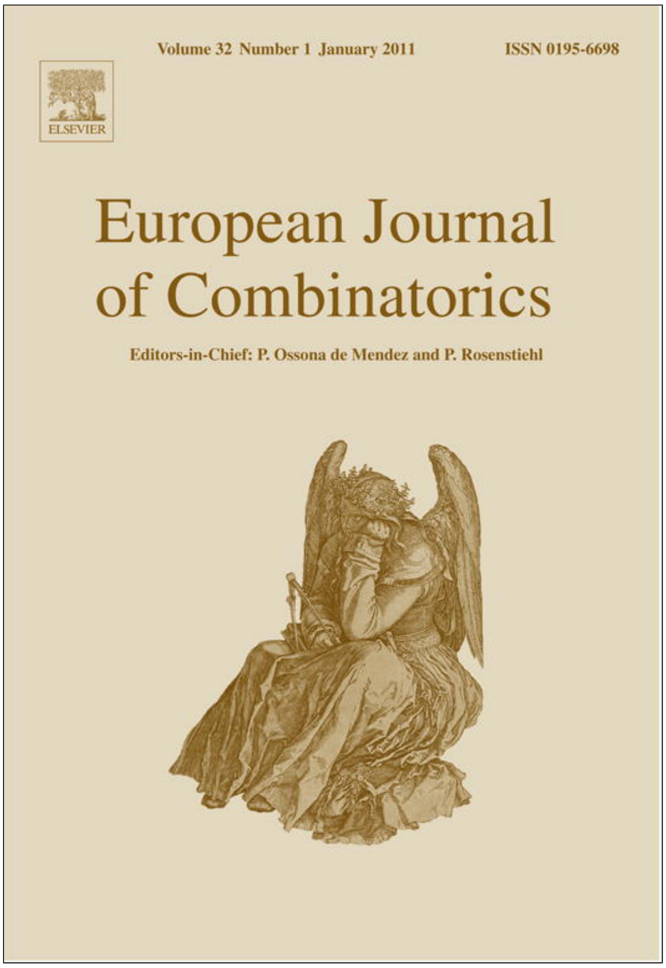
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On the growth of cocompact hyperbolic Coxeter groups

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ABSTRACT

For an arbitrary cocompact hyperbolic Coxeter group G with a finite generator set S and a complete growth function $f_S(x) = P(x)/Q(x)$, we provide a recursion formula for the coefficients of the denominator polynomial $Q(x)$. It allows us to determine recursively the Taylor coefficients and to study the arithmetic nature of the poles of the growth function $f_S(x)$ in terms of its subgroups and exponent variety. We illustrate this in the case of compact right-angled hyperbolic n-polytopes. Finally, we provide detailed insight into the case of Coxeter groups with at most 6 generators, acting cocompactly on hyperbolic 4-space, by considering the three combinatorially different families discovered and classified by Lannér, Kaplinskaya and Esselmann, respectively.

1. Overview and results

Let G be a discrete group generated by finitely many reflections in hyperplanes (mirrors) of hyperbolic space $\mathbb{H}^n$ such that the orbifold $\mathbb{H}^n/G$ is compact. We call G a cocompact hyperbolic Coxeter group and denote by S the (natural) set of generating reflections. For each generator $s \in S$, one has $s^2 = 1$ while two distinct elements $s, s' \in S$ satisfy either no relation if the corresponding mirrors admit a common perpendicular or provide the relation $(ss')^m = 1$ for an integer $m = m(s, s') > 1$ if the mirrors intersect. The images of the mirrors decompose $\mathbb{H}^n$ into connected components each of whose closures gives rise to a compact convex fundamental polytope $P \subset \mathbb{H}^n$ for G with dihedral angles of type π/p where $p \geq 2$ is an integer. Hence, P is a simple polytope so that each k-face is contained in exactly $n-k$ facets. We call P a Coxeter polytope and use the standard notation by means of the associated Coxeter graph simultaneously for G and P (cf. [6, Chapter 3] and [23, Chapter 5]). In particular, two nodes in the Coxeter graph Γ of G corresponding to mirrors intersecting under

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the angle of $\pi/3$ (respectively π/p) are connected by a simple edge (respectively by an edge with label p). If two mirrors are perpendicular (or admit a common perpendicular), their nodes are not joined at all (are joined by a dotted line).

The focus of this work is the growth series of G defined by

$$f_S(x) = \sum_{w \in G} x^{l_S(w)} = 1 + |S|x + \cdots = 1 + \sum_{i \geq 1} a_i x^i,$$

where $l_S(w)$ denotes the (minimal) word length of w with respect to S, and where a_i is the number of words w with $l_S(w) = i$. We investigate its explicit properties and the growth rate τ as given by the inverse of the radius of convergence of $f_S(x)$. Of special interest is the arithmetic nature of τ as formulated in the general conjecture below.

In this context, the following classical facts are of fundamental importance. By a result of Steinberg [21], $f_S(x)$ is the power series of a rational function. For a cocompact hyperbolic Coxeter group, a result of Milnor [13] implies that $\tau > 1$ and that τ coincides with the biggest (real) pole of $f_S(x)$ (see also [6, Section 17.1, p. 322]). Furthermore, in the same case, the rational function $f_S(x)$ is reciprocal (resp. anti-reciprocal) for n even (resp. n odd) (cf. [4, Corollary, p. 376] and, for G having only finite Coxeter subgroups, [19]). More precisely,

$$f_S(x^{-1}) = \begin{cases} f_S(x) & \text{for } n \equiv 0(2), \\ -f_S(x) & \text{for } n \equiv 1(2). \end{cases} \tag{1.1}$$

Very useful is Steinberg's formula [21]

$$\frac{1}{f_S(x^{-1})} = \sum_{\substack{G_T < G \\ \text{finite}}} \frac{(-1)^{|T|}}{f_T(x)}, \tag{1.2}$$

allowing to express $f_S(x^{-1})$ in terms of the growth series $f_T(x)$ of the finite Coxeter subgroups G_T, $T \subset S$, of G where $G_\varnothing = \{1\}$. Recall that any subset $T \subset S$ generates a Coxeter group G_T which may be finite or infinite, reducible or irreducible. A finite Coxeter subgroup $G_T < G$ arises as stabiliser of a certain face of P and has a growth function $f_T(x)$ which, by a result of Solomon [20], is a polynomial given by a product

$$f_T(x) = \prod_{i=1}^{t} [m_i + 1]. \tag{1.3}$$

Here we use the standard notations $[k] := 1 + x + \cdots + x^{k-1}$, $[k, l] = [k] \cdot [l]$ and so on, and denote by $m_1 = 1, m_2, \ldots, m_t$ the exponents of the Coxeter group G_T (cf. Table 1; for references, see [5, Section 9.7] or [6, Chapter 17], for example). In particular, a maximal finite Coxeter subgroup G_T of G acting on $\mathbb{H}^n$ is of rank $|T| = n$ and stabilises a vertex of P whose vertex neighborhood is a cone over a spherical $(n-1)$-simplex P_v due to the simplicity of P.

Table 1
Exponents and growth polynomials of irreducible finite Coxeter groups.

Graph	Exponents	Growth series $f_S(x)$
A_n	$1, 2, \ldots, n-1, n$	$[2, 3, \ldots, n, n+1]$
B_n	$1, 3, \ldots, 2n-3, 2n-1$	$[2, 4, \ldots, 2n-2, 2n]$
D_n	$1, 3, \ldots, 2n-5, 2n-3, n-1$	$[2, 4, \ldots, 2n-2] \cdot [n]$
$G_2^{(m)}$	$1, m-1$	$[2, m]$
F_4	$1, 5, 7, 11$	$[2, 6, 8, 12]$
E_6	$1, 4, 5, 7, 8, 11$	$[2, 5, 6, 8, 9, 12]$
E_7	$1, 5, 7, 9, 11, 13, 17$	$[2, 6, 8, 10, 12, 14, 18]$
E_8	$1, 7, 11, 13, 17, 19, 23, 29$	$[2, 8, 12, 14, 18, 20, 24, 30]$
H_3	$1, 5, 9$	$[2, 6, 10]$
H_4	$1, 11, 19, 29$	$[2, 12, 20, 30]$

Finally, the growth series $f_S(x)$ of a Coxeter group acting cocompactly on $\mathbb{H}^n$ is related to the Euler characteristic of G and the volume of P, and therefore of $\mathbb{H}^n/G$, as follows (see [8]).

$$\frac{1}{f_S(1)} = \chi(G) = \begin{cases} \dfrac{(-1)^{\frac{n}{2}} 2\mathrm{vol}_n(P)}{\mathrm{vol}_n(\mathbb{S}^n)}, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd.} \end{cases} \tag{1.4}$$

Of special interest is the arithmetic nature of the growth rate τ. By results of [1,14] (see also [9]), it is known that the growth rate τ of a Coxeter group G acting cocompactly on $\mathbb{H}^n$ is a *Salem number* if $n = 2, 3$. That is, τ is a real algebraic integer >1 all of whose conjugates have absolute value not greater than 1, and at least one has absolute value equal to 1. It follows that the minimal polynomial of τ is palindromic with roots coming in inversive pairs. For $n \geq 4$, the growth rate τ is not a Salem number anymore. This was first observed by Cannon [1] who considered $n = 4$ and Coxeter groups G with 5 generators. Based on substantial experimental data, we make the following claim concerning the *positive* poles (appearing in inversive pairs) of the growth function of G (cf. [15, Section 5]).

Conjecture. *Let G be a Coxeter group acting cocompactly on $\mathbb{H}^n$ with natural generating set S and growth series $f_S(x)$. Then,*

(a) *for n even, $f_S(x)$ has precisely $\frac{n}{2}$ poles $0 < x_1 < \cdots < x_{\frac{n}{2}} < 1$ in the open unit interval $]0, 1[$;*

(b) *for n odd, $f_S(x)$ has precisely the pole 1 and $\frac{n-1}{2}$ poles $0 < x_1 < \cdots < x_{\frac{n-1}{2}} < 1$ in the interval $]0, 1]$.*

In both cases, the poles are simple, and the non-real poles of $f_S(x)$ are contained in the annulus of radii $x_\star$ and $x_\star^{-1}$ for some $\star \in \left\{1, \ldots, \left[\frac{n}{2}\right]\right\}$.

In order to study such arithmetical properties of the growth series of an arbitrary cocompact Coxeter group G acting with generating set S in $\mathbb{H}^n$, we need to control the denominator polynomial of its growth series $f_S(x)$ very well. To this end, we associate to it a certain complete form,

$$f_S(x) = \frac{P(x)}{Q(x)} = \frac{\prod_{i=1}^{r}[n_i]}{\sum_{i=0}^{d} b_i x^i},$$

where $P(x), Q(x) \in \mathbb{Z}[x]$ are of equal degree, and where $[k]$ is as in (1.3). The integers r and $n_1, \ldots, n_r \geq 2$ are related to the finite Coxeter subgroups of G and their exponents. Inspired by an idea of Chapovalov et al. [3], we are able to derive a recursion formula for the coefficients b_i of the denominator polynomial Q. In the recursion appear beside $|S|$ and r certain counting functions such as $N_k = \mathrm{card}\,\{n_l > k \mid 1 \leq l \leq r\}$ for $2 \leq k \leq i$ related to the numerator P of $f_S(x)$ (cf. (2.9) and (2.13)). Let us point out that Steinberg's formula does not provide this sort of information about the growth denominator polynomial but helps in single cases to shed light on the above conjecture. By applying our recursion formula to a given group G, we exhibit an algorithm to determine the poles and the coefficients a_i in the growth series

$$f_S(x) = 1 + \sum_{i \geq 1} a_i x^i$$

in a completely explicit manner and to control the growth of words in the Cayley diagram of G with respect to the word metric induced by S. Notice that the cardinalities a_i are usually very difficult to determine since they depend on the number and the relations between the generators. Hence, it is not surprising that all our formulas depend heavily on the combinatorics of the subgroup structure of G as well.

Nevertheless, there are various applications of our recursion formulas. Firstly, we apply the recursion to the elementary family of compact right-angled hyperbolic Coxeter polytopes (see Proposition 3.2). For such a polytope P in $\mathbb{H}^4$, having f_0 vertices and f_3 facets, the associated growth series is given by

$$\frac{(1+x)^4}{1+(4-f_3)x+(f_0-2f_3+6)x^2+(4-f_3)x^3+x^4},$$

has precisely 2 inversive pairs of positive simple poles, and is, by (1.4), of covolume equal to

$$\mathrm{vol}_4(P) = \frac{f_0 - 4f_3 + 16}{12}\pi^2. \tag{1.5}$$

This result confirms our conjecture. Let us mention another result in dimension 4 presented in [24]. Therein, the growth rates of the infinite sequence of cocompact Coxeter groups acting on $\mathbb{H}^4$ are determined which are constructed as m-garlands based on the doubly truncated Coxeter orthoschemes [5, 3, 5, 3] and [4, 3, 5, 3]. The denominator of each of these growth functions is a palindromic polynomial of degree 18 with exactly two pairs of real (simple) roots $x_m^{-1} < y_m^{-1} < 1 < y_m < x_m$ while all the other conjugates lie on the unit circle.

At the end, we shall apply our results in order to confirm our conjecture about the growth behavior of cocompact Coxeter groups acting with at most 6 generating reflections on $\mathbb{H}^4$ (see Theorem 4.1). We shall discuss these aspects by outlining proofs, only (cf. [15]).

2. Recursion formulas for growth coefficients

2.1. The complete form

Let G be a Coxeter group acting cocompactly on hyperbolic space $\mathbb{H}^n$. Denote by S its natural set of generating reflections, and consider the growth series of G,

$$f_S(x) = 1 + |S|x + \cdots = 1 + \sum_{i\geq 1} a_i x^i, \tag{2.1}$$

which is an (anti-)reciprocal rational function for n even (odd) according to (1.1). It can be written as a quotient $f_S(x) = \frac{p(x)}{q(x)}$ of relatively prime polynomials $p, q \in \mathbb{Z}[x]$. By (1.2) and (1.3), the polynomials p, q are of equal degree over the integers. On the other hand, consider the denominator of the sum in Steinberg's formula (1.2)

$$\sum_{T\in\mathcal{F}} \frac{(-1)^{|T|}}{f_T(x)},$$

where $\mathcal{F} = \{T \subset S \mid G_T \text{ is finite}\}$. The least common multiple

$$\mathrm{Virg}(S) := \mathrm{LCM}\{f_T(x) \mid T \in \mathcal{F}\}$$

is called the *virgin form* of the numerator of $(-1)^n f_S(x)$, and $(-1)^n f_S(x)$ can be expressed as a rational function with numerator equal to $\mathrm{Virg}(S)$ (see [3, Corollary 5.2.2a]). Although each constituent $f_T(x) = \prod_{i=1}^{t}[m_i+1]$, $T \in \mathcal{F}$, is a product of polynomials of type $[k]$ according to (1.3), certain factorisation properties of $[k]$ prevent $\mathrm{Virg}(S)$ from being a product of $[k]$'s, only (cf. Example 1). More precisely, there is the factorisation (cf. [17, Section 3.3])

$$[k] = \prod_{\substack{d\mid k\\ d>1}} \Phi_d(x),$$

where $\Phi_d(x)$ denotes the dth cyclotomic polynomial of degree equal to Euler's function $\varphi(d)$. The polynomial $\Phi_d(x)$ is irreducible in $\mathbb{Z}[x]$ and, for $d > 2$, of even degree. If p is prime and $d = pm$, it satisfies the property

$$\Phi_{pm}(x) = \begin{cases} \Phi_m(x^p) & \text{if } p \mid m, \\ \dfrac{\Phi_m(x^p)}{\Phi_m(x)} & \text{else.} \end{cases}$$

Since, for later purposes, we are interested in having uniformly tractable numerators for $f_S(x)$, we modify $\mathrm{Virg}(S)$ in the following way. Denote by $\mathrm{Ext}(S) \in \mathbb{Z}[x]$ the monic polynomial arising as the unique common multiple of all $f_T(x)$, $T \in \mathcal{F}$, such that

$$\mathrm{Ext}(S) = \prod_{i=1}^{r} [n_i],$$

where the integers $r, n_1, \ldots, n_r \geq 2$ with $n_i = m_i + 1$ are minimal. Since $\mathrm{Ext}(S) = \mathrm{Virg}(S) \cdot R(x)$ for some polynomial $R(x) \in \mathbb{Z}[x]$, $\mathrm{Ext}(S)$ is called the *extended form* of $\mathrm{Virg}(S)$. Denote by

$$P(x) := \mathrm{Ext}(S) \quad \text{and} \quad Q(x) := (-1)^n q(x) \cdot R(x)$$

the extended form of the numerator $p(x)$ and of the denominator $q(x)$ of $f_S(x)$. Then, the growth series $f_S(x)$ can be written as a rational function $P(x)/Q(x)$ which is called its *complete form*, a notion going back to Chapovalov et al. (see [3, Paragraph 5.4.2]). Let us point out that $P(x)$ and $Q(x)$ are in general no more relatively prime. An important feature of putting a growth series into its complete form is that the numerator P is simply a product of polynomials $[k]$ which is of advantage when taking iterative derivatives and evaluating at 0. The passage to the complete form does not change the number of the real poles and their localisation in the complex plane. In fact, the extension of the denominator $q(x)$ arises by multiplying it with cyclotomic polynomials of degree bigger than 1. The next example illustrates the above procedure.

Example 1. Consider the cocompact hyperbolic simplex group G_L acting on $\mathbb{H}^4$, with set S of 5 reflections related by the graph

$$\Gamma_L : \bullet \overset{5}{\text{——}} \bullet \text{——} \bullet \text{——} \bullet \overset{4}{\text{——}} \bullet\,,$$

and with growth series $f_S(x) = p(x)/q(x)$. By means of (1.2) and the list of exponents of the subgroups involved (see Table 1), one computes $\mathrm{Virg}(S) = [2, 12, 20, 30]\Phi_8(x)$. Therefore, the complete form of $f_S(x)$ is given by the quotient of $P(x) = \mathrm{Ext}(S) = [2, 8, 12, 20, 30]$ divided by $Q(x) = [4]\,q(x)$, where we used the decomposition $[8] = [4]\Phi_8(x)$.

2.2. *The additive nature of* Ext(S)

Write $f_S(x) = P(x)/Q(x)$ with

$$P(x) = \prod_{i=1}^{r} [n_i] \quad \text{and} \quad Q(x) = \sum_{i=0}^{d} b_i x^i \in \mathbb{Z}[x] \tag{2.2}$$

according to Section 2.1. Since f_S is (anti-)reciprocal for n even (odd), and since each factor $[k]$ of P (and therefore P itself) is a palindromic polynomial, satisfying the property $F(x) = x^{\deg F} F(x^{-1})$, the numerator Q is an (anti-)palindromic polynomial for n even (odd). This means that, for n odd, Q satisfies the property $Q(x) = -x^{\deg Q} Q(x^{-1})$. Our aim is to derive a recursion formula for the coefficients b_i of $Q(x)$. Inspired by [3], we will differentiate iteratively $\pm 1/f_S(x) = 1/f_S(x^{-1})$ by means of Steinberg's formula (1.2) and compare it – after evaluation at $x = 0$ – with the corresponding expression for $\pm Q(x)/P(x)$. Since $f_S(0) = 1$ and $P(0) = 1$, one has $b_0 = \pm b_d = 1$. Furthermore, $Q^{(l)}(0) = l!\, b_l$. One also observes that $P'(0) = r$. However, by (2.2), $P(x)$ is a product of factors of type $[k] = 1 + x + \cdots + x^{k-1}$ so that higher derivatives of it become complicated expressions. The following lemma about the *additive* character of P is therefore very useful.

Lemma 2.1. *Let $r \geq 1$ and $n_1, \ldots, n_r \geq 2$ be integers. Then,*

$$\begin{aligned}(x-1)^{r-1} \prod_{i=1}^{r} [n_i] = {} & [n_1 + \cdots + n_r] - \sum_{1 \leq i \leq r} [n_1 + \cdots + \widehat{n_i} + \cdots + n_r] \\ & + \sum_{1 \leq i < j \leq r} [n_1 + \cdots + \widehat{n_i} + \cdots + \widehat{n_j} + \cdots + n_r] \\ & - \cdots + (-1)^{r-1} \sum_{i=1}^{r} [n_i].\end{aligned} \tag{2.3}$$

Proof. We proceed by induction. Since

$$[k] = 1 + x + \cdots + x^{k-1} = \frac{x^k - 1}{x - 1},$$

one immediately deduces that

$$\begin{aligned}[n_1][n_2] &= \frac{x^{n_1} - 1}{x - 1} \cdot \frac{x^{n_2} - 1}{x - 1} = \frac{1}{(x-1)^2}\left\{x^{n_1+n_2} - x^{n_1} - x^{n_2} + 1\right\} \\ &= \frac{1}{(x-1)^2}\left\{(x^{n_1+n_2} - 1) - (x^{n_1} - 1) - (x^{n_2} - 1)\right\} \\ &= \frac{1}{x-1}\left\{[n_1 + n_2] - [n_1] - [n_2]\right\}. \end{aligned} \tag{2.4}$$

By means of the induction hypothesis and by using (2.4), an easy rearrangement of the terms suffices to finish the proof. □

Remark 1. It is convenient to write Eq. (2.3) in a more efficient way by introducing the following notation. Let $X = \{x_1, \ldots, x_k\} \subseteq \{1, \ldots, r\}$ be a non-empty index subset, with $x_i < x_k$ if $i < k$, and write

$$n_X := n_{x_1} + \cdots + n_{x_k}.$$

Then, (2.3) can be expressed in the form

$$(x-1)^{r-1} \prod_{i=1}^{r} [n_i] = \sum_{\emptyset \neq X \subseteq \{1,\ldots,r\}} (-1)^{r-|X|} [n_X]. \tag{2.3$'$}$$

Now, by differentiating l-times a term $[n]$ and evaluating it at $x = 0$, one obtains

$$[n]^{(l)}(0) = l!\epsilon_l(n) \quad \text{where } \epsilon_l(n) := \begin{cases} 1 & \text{if } l < n, \\ 0 & \text{if } l \geq n. \end{cases} \tag{2.5}$$

The lth derivative of the factor $1/(x-1)^{r-1}$ at $x = 0$ yields, for $l \geq 1$,

$$\left(\frac{1}{(x-1)^{r-1}}\right)^{(l)}(0) = (-1)^{r-1} \prod_{i=0}^{l-1} (r + i - 1). \tag{2.6}$$

Corollary 2.2. *Let $n_1, \ldots, n_r \geq 2$ be integers. Then, for $l \geq 1$,*

$$\begin{aligned}\left(\prod_{i=1}^{r} [n_i]\right)^{(l)}(0) = {}& l! \sum_{\emptyset \neq X \subseteq \{1,\ldots,r\}} (-1)^{|X|+1} \epsilon_l(n_X) \\ & + \sum_{j=0}^{l-1} \left\{ \frac{l!}{(l-j)!} \prod_{k=1}^{l-j} (r - 2 + k) \cdot \sum_{\emptyset \neq X \subseteq \{1,\ldots,r\}} (-1)^{|X|+1} \epsilon_j(n_X) \right\}. \end{aligned} \tag{2.7}$$

Proof. By (2.3′), we can write $\prod_{i=1}^{r} [n_i] =: u(x) \cdot v(x)$ with

$$u(x) = \frac{1}{(x-1)^{r-1}} \quad \text{and} \quad v(x) = \sum_{\emptyset \neq X \subseteq \{1,\ldots,r\}} (-1)^{r-|X|} [n_X].$$

Then,

$$\left(\prod_{i=1}^{r} [n_i]\right)^{(l)}(0) = (-1)^{r-1} v^{(l)}(0) + \sum_{j=0}^{l-1} \binom{l}{j} u^{(l-j)}(0) v^{(j)}(0),$$

where, by (2.5) and (2.6), for $l-j\geq 1$,

$$u^{(l-j)}(0)=(-1)^{r-1}\prod_{k=1}^{l-j}(r-2+k),$$
$$v^{(j)}(0)=j!\sum_{\emptyset\neq X\subseteq\{1,\dots,r\}}(-1)^{r-|X|}\epsilon_j(n_X),$$

from which the corollary follows. □

Remark 2. For practical purposes, the following recursive version of Corollary 2.2 is useful (cf. [15, Proposition 4.5]). Denote by $g_r(x):=\prod_{i=1}^{r}[n_i]$. Then,

$$g_r^{(l)}(0)=(-1)^{r+1}l!\sum_{X\subsetneq\{1,\dots,r\}}(-1)^{|X|}\,\text{card}\{(r-|X|)\text{-tuples}Y\mid n_Y>r\}$$
$$+\sum_{j=1}^{l}\binom{l}{j}(-1)^{j+1}\prod_{k=1}^{j}(r-k)g_r^{(l-j)}(0). \tag{2.8}$$

As an application of Corollary 2.2, we describe the cases $l=1,2,3$ explicitly. To this end, consider the numbers

$$N_k=N_k(G):=\text{card}\{n_i>k\mid 1\leq i\leq r\}, \tag{2.9}$$

for $k\in\mathbb{N}$, which satisfy $N_0=N_1=r$.

Corollary 2.3. *Let $n_1,\dots,n_r\geq 2$ be integers, and let $g_r(x)=\prod_{i=1}^{r}[n_i]$. Then,*

$$\begin{aligned}&g_r'(0)=r\\&g_r''(0)=r(r-1)+2N_2\\&g_r^{(3)}(0)=r(r-1)(r-2)+6(r-1)N_2+6N_3.\end{aligned} \tag{2.10}$$

Proof. By taking once the derivative of the product $g_r(x)$ and evaluating it at $x=0$ yields the claim, since

$$\sum_{\emptyset\neq X\subseteq\{1,\dots,r\}}(-1)^{|X|}=\sum_{k=1}^{r}(-1)^k\binom{r}{k}=-1. \tag{2.11}$$

Consider the second derivative $g_r''(x)$. By means of (2.7), and since $\epsilon_0(n_X)=\epsilon_1(n_X)=1$, we obtain

$$g_r''(0)=\left\{r(r-1)+2(r-1)\right\}\cdot\sum_{\emptyset\neq X\subseteq\{1,\dots,r\}}(-1)^{|X|+1}+2N_2+2\cdot\sum_{\substack{X\subseteq\{1,\dots,r\}\\|X|\geq 2}}(-1)^{|X|+1}.$$

By (2.11), the last term can be transformed in order to yield the desired equality. As for $g_r^{(3)}(0)$, a similar consideration based on Remark 2 gives the desired result. □

Remark 3. We will apply Corollary 2.2 later in the following inductive way. The lth derivative of the inverse function $h_r(x)=1/g_r(x)$ evaluated at $x=0$ can be expressed in terms of the lower order derivatives of $h_r(x)$ and $g_r(x)$ at 0 as follows.

$$\left(\frac{1}{g_r(x)}\right)^{(l)}(0)=-\sum_{j=1}^{l}\binom{l}{j}g_r^{(j)}(0)\left(\frac{1}{g_r(x)}\right)^{(l-j)}(0). \tag{2.12}$$

This formula is a consequence of Leibniz' rule applied to $g_r(x)\cdot\frac{1}{g_r(x)}$ and $g_r(0)=1$.

2.3. The recursion formula

Let us return to a cocompact hyperbolic Coxeter group G with set of generating reflections S and growth series in complete form $f_S(x) = P(x)/Q(x)$, that is, $P(x) = \prod_{i=1}^{r}[n_i]$ and $Q(x) = 1 + \sum_{i=1}^{d} b_i x^i$ (see (2.2)). By Steinberg's formula (1.2),

$$\frac{1}{f_S(x)} = (-1)^n \sum_{T\in\mathcal{F}} \frac{(-1)^{|T|}}{f_T(x)}, \tag{1.2'}$$

where $\mathcal{F} = \{T \subset S \mid G_T < G \text{ is finite}\}$, as usually. Each finite Coxeter subgroup G_T of G has a growth polynomial of the form

$$f_T(x) = \prod_{i=1}^{|T|}[1+m_i] =: \prod_{i=1}^{|T|}[c_i], \tag{2.13}$$

where the exponents $m_i = m_i(T)$ depend on G_T as indicated in Table 1. Let

$$C_k = C_k(T) := \operatorname{card}\{c_i > k \mid 1 \le i \le |T|\},$$

and consider the set

$$\mathcal{F}' := \{T \subset S \mid |T| \ge 2 \text{ and } G_T \text{ is finite}\}. \tag{2.14}$$

We are now ready to present formulas for the coefficients b_1, b_2, b_3 of Q. Observe that the coefficient b_1 has first been described in [3, Theorem 5.4.3], but by a different method. In the proof of [3], there is furthermore a little flaw concerning the (non-)reciprocity of $f_S(x)$ when deriving and evaluating its inverse at $x = 0$.

Proposition 2.4. *Let G be a Coxeter group, with set S of generating reflections, which acts cocompactly on $\mathbb{H}^n$. Denote by $f_S(x) = P(x)/Q(x)$ its growth series in complete form with $P(x) = \prod_{i=1}^{r}[n_i]$ and $Q(x) = 1 + \sum_{i=1}^{d} b_i x^i$. Then,*

$$b_1 = r - |S|, \tag{2.15}$$

$$\begin{aligned} 2b_2 &= (-1)^{n+1}2|S| + (-1)^n\left(\sum_{T\in\mathcal{F}'}(-1)^{|T|}|T|(|T|+1)\right) \\ &\quad + (-1)^{n+1}2\left(\sum_{T\in\mathcal{F}'}(-1)^{|T|}C_2\right) - r(r+1) + 2N_2 + 2rb_1, \end{aligned} \tag{2.16}$$

$$\begin{aligned} 6b_3 &= (-1)^n 6|S| + (-1)^{n+1}\cdot\left(\sum_{T\in\mathcal{F}'}(-1)^{|T|}|T|(|T|+1)(|T|+2)\right) \\ &\quad + (-1)^n 6\left(\sum_{T\in\mathcal{F}'}(-1)^{|T|}\left(-C_3 + (|T|+1)C_2\right)\right) \\ &\quad + r(r+1)(r+2) + 6N_3 - 6(r+1)N_2 + 3(2N_2 - r(r+1))b_1 + 6rb_2. \end{aligned} \tag{2.17}$$

Proof. In order to determine b_1, recall that $f_S(x)$ is given by (2.1) in the form

$$f_S(x) = \sum_{i\ge 0} a_i x^i = 1 + |S|x + \sum_{i\ge 2} a_i x^i,$$

where $a_i > 0,\ i \ge 2$, are certain cardinalities. For example, $a_1 = |S|$. Since $\sum_{i\ge 0} a_i x^i = P(x)/Q(x)$, it follows that

$$\left(\sum_{i=0}^{d} b_i x^i\right)\left(1 + |S|x + a_2x^2 + \cdots\right) = \prod_{i=1}^{r}[n_i]. \tag{2.18}$$

A comparison of coefficients in (2.18) leads to $r = b_1 + |S|$.

As for b_2, one computes by means of (1.2) and (1.3) that

$$\left(\frac{1}{f_S}\right)''(0) = (-1)^n \left\{-2|S| + \sum_{T\in\mathcal{F}'}(-1)^{|T|}\left(-\frac{f_T''(0)(f_T(0))^2 - 2f_T(0)(f_T'(0))^2}{(f_T(0))^4}\right)\right\}$$
$$= (-1)^n \left\{-2|S| + \sum_{T\in\mathcal{F}'}(-1)^{|T|}\left(2(f_T'(0))^2 - f_T''(0)\right)\right\}.$$

By Corollary 2.3, applied to $f_T(x) = \prod_{i=1}^{|T|}[c_i]$,

$$\left(\frac{1}{f_S}\right)''(0) = (-1)^n \left(-2|S| + \sum_{T\in\mathcal{F}'}(-1)^{|T|}\{|T|(|T|+1) - 2C_2\}\right).$$

On the other hand side,

$$\left(\frac{1}{f_S}\right)''(0) = \left(\frac{Q}{P}\right)''(0).$$

Since $Q^{(l)}(0) = l!b_l$ and $b_1 = r - |S|$, we obtain by Corollary 2.3 that

$$\left(\frac{Q}{P}\right)''(0) = r(r+1) - 2N_2 + 2b_2 - 2rb_1.$$

It remains then to compare the two expressions for $(1/f_S)''(0)$ in order to obtain the desired formula. In a similar way one verifies the claim for b_3. □

Application. The proof of the identity (2.16) for b_2 above can be performed for b_1 as well. Combined with (2.15), it reveals then some information about the distribution of the *finite* and *infinite* subgroups of G which is very useful. Since we are dealing here only with cocompact groups, any infinite Coxeter subgroup of G is hyperbolic, and we deduce that

$$\sum_{\substack{T\subset S\\ |G_T|<\infty}}(-1)^{|T|}|T| = (-1)^n|S|; \qquad \sum_{\substack{T\subseteq S\\ |G_T|=\infty}}(-1)^{|T|}|T| = (-1)^{n+1}|S|. \tag{2.19}$$

Notice that the second identity in (2.19) follows from the first one by using the well-known combinatorial identity

$$\sum_{\emptyset\subseteq T\subseteq S}(-1)^{|T|}|T| = 0.$$

For illustration, consider the Coxeter group G given by the graph

$$\Gamma : \bullet \cdots \bullet \overset{p}{\text{———}} \bullet \overset{q}{\text{———}} \bullet \overset{r}{\text{———}} \bullet, \qquad p,q,r\geq 3, \quad \frac{1}{p}+\frac{1}{q} > \frac{1}{2}, \quad \frac{1}{q}+\frac{1}{r} < \frac{1}{2}, \tag{2.20}$$

which acts cocompactly on $\mathbb{H}^3$ and is generated by five reflections in the facets of a certain simplicial prism (more precisely, a *simply truncated orthoscheme*) of dihedral angles $\pi/p, \pi/q, \pi/r$. Each subgraph containing $\bullet\cdots\bullet$ in (2.20) is of infinite order. A little computation with respect to (2.20) confirms (2.19) as follows.

$$\sum_{\substack{T\subseteq S\\ |G_T|=\infty}}(-1)^{|T|}|T| = 2\cdot 1 - 3\cdot 4 + 4\cdot 5 - 5\cdot 1 = 5.$$

In general, the coefficient b_k, $k\geq 4$, of the denominator polynomial Q of $f_S(x)$ can be deduced from $b_1,\ldots,b_{k-1}$ as follows. By means of Steinberg's formula (1.2′),

$$\left(\frac{1}{f_S}\right)^{(k)}(0) = (-1)^{n+k+1}k!|S| + (-1)^n\sum_{T\in\mathcal{F}'}(-1)^{|T|}\left(\frac{1}{f_T}\right)^{(k)}(0), \tag{2.21}$$

where $\mathcal{F}'$ is given by (2.14). On the other hand, the complete form of $f_S(x) = P(x)/Q(x)$ as given by (2.2) leads to

$$\left(\frac{1}{f_S}\right)^{(k)}(0) = \left(\frac{1}{P}\right)^{(k)}(0) + \sum_{j=1}^{k-1}\binom{k}{j} j!b_j \left(\frac{1}{P}\right)^{(k-j)}(0) + k!b_k. \tag{2.22}$$

By comparing (2.21) with (2.22), one derives a first formula for the coefficient b_k as follows.

$$\begin{aligned} k!b_k &= (-1)^{n+k+1}k!|S| + (-1)^n P_k^\tau - P_k + B_k, \quad \text{where} \\ P_k &:= \left(\frac{1}{P}\right)^{(k)}(0), \\ P_k^\tau &:= \sum_{T\in\mathcal{F}'}(-1)^{|T|}\left(\frac{1}{f_T}\right)^{(k)}(0), \\ B_k &:= -\sum_{j=1}^{k-1}\binom{k}{j} j!b_j \left(\frac{1}{P}\right)^{(k-j)}(0) = -\sum_{j=1}^{k-1}\frac{k!}{(k-j)!}b_j P_{k-j}. \end{aligned} \tag{2.23}$$

The different terms in (2.23) can be determined as follows. By (2.12), we obtain the recursion

$$\begin{aligned} P_k &= -\sum_{j=1}^{k}\binom{k}{j} P^{(j)}(0)P_{k-j}, \\ P_k^\tau &= \sum_{T\in\mathcal{F}'}(-1)^{|T|+1}\left(\sum_{j=1}^{k}\binom{k}{j} f_T^{(j)}(0)\left(\frac{1}{f_T}\right)^{(k-j)}(0)\right). \end{aligned} \tag{2.24}$$

Corollary 2.2 together with (2.13) yields now similar recursion identities for both parts, that is,

$$\begin{aligned} P_k \;&=\; -\sum_{j=1}^{k}\frac{k!}{(k-j)!}\left(\sum_{\emptyset\neq X\subseteq\{1,\ldots,r\}}(-1)^{|X|+1}\epsilon_j(n_X)\right. \\ &\quad \left. + \sum_{i=0}^{j-1}\left\{\frac{j!}{(j-i)!}\prod_{l=1}^{j-i}(r-2+l)\cdot\sum_{\emptyset\neq X\subseteq\{1,\ldots,r\}}(-1)^{|X|+1}\epsilon_i(n_X)\right\}\right)P_{k-j}, \\ P_k^\tau \;&=\; \sum_{T\in\mathcal{F}'}(-1)^{|T|+1}\sum_{j=1}^{k}\frac{k!}{(k-j)!}\left(\sum_{\emptyset\neq Y\subseteq\{1,\ldots,|T|\}}(-1)^{|Y|+1}\,\epsilon_j(c_Y)\right. \\ &\quad \left. + \sum_{i=0}^{j-1}\left\{\frac{j!}{(j-i)!}\prod_{l=1}^{j-i}(|T|-2+l)\cdot\sum_{\emptyset\neq Y\subseteq\{1,\ldots,|T|\}}(-1)^{|X|+1}\,\epsilon_i(c_Y)\right\}\left(\frac{1}{f_T}\right)^{(k-j)}(0)\right). \end{aligned} \tag{2.25}$$

Finally, for B_k, we easily derive the relation

$$B_k = -kP_{k-1}b_1 - \sum_{j=2}^{k-2}\left(\frac{k!}{(k-j)!}P_{k-j}b_j\right) + k!rb_{k-1}. \tag{2.26}$$

Then, by plugging (2.24)–(2.26) into (2.23), the following recursion concept follows where we add for completeness the first values according to Proposition 2.4. Recall the notations $\mathcal{F}'$, N_k, C_k and $\epsilon_k(X)$ according to (2.5), (2.9) and (2.14).

Theorem 2.5 (*The Recursion Formula*)**.** *Let G be a Coxeter group with set S of generating reflections acting cocompactly on $\mathbb{H}^n$. Denote by $f_S(x) = P(x)/Q(x)$ its growth series in complete form with $P(x) = \prod_{i=1}^{r}[n_i]$*

and $Q(x) = 1 + \sum_{i=1}^{d} b_i x^i$. *Then, for* $k \geq 4$, *and with* $P_k = \left(\frac{1}{P}\right)^{(k)}(0)$,

$$
\begin{aligned}
& b_1 = r - |S|, \\
& 2b_2 = (-1)^{n+1} 2|S| + (-1)^n \left(\sum_{T \in \mathcal{F}'} (-1)^{|T|} |T|(|T|+1) \right) \\
& \quad + (-1)^{n+1} 2 \left(\sum_{T \in \mathcal{F}'} (-1)^{|T|} C_2 \right) - r(r+1) + 2N_2 + 2rb_1, \\
& 6b_3 = (-1)^n 6|S| + (-1)^{n+1} \cdot \left(\sum_{T \in \mathcal{F}'} (-1)^{|T|} |T|(|T|+1)(|T|+2) \right) \\
& \quad + (-1)^n 6 \left(\sum_{T \in \mathcal{F}'} (-1)^{|T|} \left(-C_3 + (|T|+1)C_2\right) \right) \\
& \quad + r(r+1)(r+2) + 6N_3 - 6(r+1)N_2 + 3(2N_2 - r(r+1))b_1 + 6rb_2, \\
& k!b_k = (-1)^{n+k+1} k!|S| + \sum_{j=1}^{k} \frac{k!}{(k-j)!j!} \left(j! \sum_{\emptyset \neq X \subseteq \{1,\dots,r\}} (-1)^{|X|+1} \epsilon_j(n_X) \right. \\
& \quad \left. + \sum_{i=0}^{j-1} \left\{ \frac{j!}{(j-i)!} \prod_{l=1}^{j-i} (r-2+l) \cdot \sum_{\emptyset \neq X \subseteq \{1,\dots,r\}} (-1)^{|X|+1} \epsilon_i(n_X) \right\} \right) P_{k-j} \\
& \quad + \sum_{T \in \mathcal{F}'} (-1)^{n+|T|+1} \sum_{j=1}^{k} \frac{k!}{(k-j)!j!} \left(j! \sum_{\emptyset \neq Y \subseteq \{1,\dots,|T|\}} (-1)^{|Y|+1} \epsilon_j(c_Y) \right. \\
& \quad \left. + \sum_{i=0}^{j-1} \left\{ \frac{j!}{(j-i)!} \prod_{l=1}^{j-i} (|T|-2+l) \cdot \sum_{\emptyset \neq Y \subseteq \{1,\dots,|T|\}} (-1)^{|X|+1} \epsilon_i(c_Y) \right\} \left(\frac{1}{f_T} \right)^{(k-j)} (0) \right) \\
& \quad - kP_{k-1}b_1 - \sum_{j=2}^{k-2} \left(\frac{k!}{(k-j)!} P_{k-j} b_j \right) + k!rb_{k-1}.
\end{aligned}
\tag{2.27}
$$

It is obvious that formula (2.27) of Theorem 2.5 depends strongly on the finite Coxeter subgroups of a given group, together with their exponents. For a family of hyperbolic Coxeter polytopes with fixed combinatorial structure, the algorithm of Theorem 2.5 can be implemented into a computer program by encoding the details about all finite irreducible Coxeter groups according to Table 1.

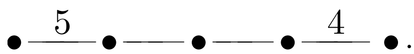
3. A first application

The recursion formula of Theorem 2.5 can be best adapted to families of groups whose subgroup structure is uniform. A first such example are right-angled hyperbolic Coxeter groups. More precisely, consider a hyperbolic Coxeter group G with presentation

$$G = \langle S = \{s_1, \dots, s_k\} \mid (s_i s_j)^{m_{ij}} = 1 \rangle.$$

Then, G is called *right-angled* if and only if $m_{ij} \in \{1, 2, \infty\}$. The terminology is justified by the fact that a fundamental polyhedron $P \subset \mathbb{H}^n$ has all dihedral angles equal to $\pi/2$ (see also [16]). Notice that each subgroup of G and all l-faces, $2 \leq l \leq n-1$, of P are right-angled. By results of Vinberg [23], there exist no cocompact right-angled Coxeter groups in $\mathbb{H}^n$ for $n \geq 5$. For $n = 2$, right-angled Coxeter polygons are realisable as long as they have at least five vertices. For $n = 3$, the (compact) right-angled dodecahedron is the one with the minimal number of facets (and vertices). A beautiful example in $\mathbb{H}^4$

is the compact (regular) 120-cell of dihedral angle $\pi/2$ whose symmetry group is generated by the reflections of the Coxeter group



Let G be a cocompact right-angled Coxeter group, with generating set S, and which acts on $\mathbb{H}^n$. Hence, $n \leq 4$. Consider the growth series $f_S(x) = P(x)/Q(x)$ of G in its complete form (see Section 2). Each finite Coxeter subgroup G_T of G is right-angled and, by (1.3), has a growth series equal to $[2]^{|T|}$. Hence, the numerator in its virgin form of $f_S(x)$ equals $[2]^n$, since the maximal (right-angled) subgroups in G are of rank n. We obtain

$$f_S(x) = \frac{[2]^n}{Q(x)} \quad \text{with } Q(x) = 1 + \sum_{i=1}^{n} b_i x^i. \tag{3.1}$$

Recall that $b_{n-i} = (-1)^n b_i$ for all $0 \leq i \leq [n/2]$, since $Q(x)$ is (anti-)palindromic. Furthermore $n \leq 4$, so that at most the coefficients b_0, b_1, b_2 are of pertinence in (3.1). As a consequence of Theorem 2.5, one deduces easily the following result.

Corollary 3.1. *Let G be a right-angled hyperbolic Coxeter group, with generating set S, which acts cocompactly on $\mathbb{H}^n$, $n \leq 4$. Then, the coefficients b_i, $1 \leq i \leq [n/2]$, of $Q(x)$ in (3.1) are given by*

$$b_1 = n - |S|$$
$$b_2 = \frac{n}{2}(n - 2|S| - 1) + \frac{(-1)^n}{2}\left(\sum_{T \in \mathcal{F}'} (-1)^{|T|}|T|(|T|+1) - 2|S|\right), \tag{3.2}$$

where $\mathcal{F}' = \{T \subset S \mid |T| \geq 2 \text{ and } G_T \text{ is finite}\}$.

Remark 4. Consider a group as in Corollary 3.1 together with its growth series $f_S(x) = \sum_{i \geq 0} a_k x^k = 1 + |S|x + \sum_{k \geq 2} a_k x^k$ where $a_k > 0$ is the number of S-words of length k in G. Formula (3.1) yields the following recursion for a_k with $a_0 = 1$ and $a_1 = |S|$.

$$a_k = \begin{cases} \binom{n}{k} - \sum_{j=1}^{k} a_{k-j} b_j & \text{for } 2 \leq k \leq n; \\ -\sum_{j=1}^{n} a_{k-j} b_j & \text{for } 2 \leq k \leq n, \end{cases}$$

where $b_0 = 1$, and the coefficients b_i, $i = 1, \ldots, n$, are given by Corollary 3.1, and $b_i = 0$ for $i > n$.

For example, a *hexagonal* right-angled Coxeter group G_H acting cocompactly on $\mathbb{H}^2$ has a Coxeter series $f_S(x) = 1 + 6x + 24x^2 + 90x^3 + 336x^4 + 1254x^5 + 4680x^6 + 17\,466x^7 + 65\,184x^8 + 243\,270x^9 + 907\,896x^{10} + \cdots$.

In what follows, we present a combinatorial formula for b_2 in Corollary 3.1. Consider an arbitrary convex n-polytope $P \subset \mathbb{H}^n$. Its *f-vector* $f = (f_0, f_1, \ldots, f_{n-1})$ has components f_i given by the numbers of i-faces of P. They are related by Euler's formula according to

$$\sum_{i=0}^{n-1} (-1)^i f_i = 1 - (-1)^n. \tag{3.3}$$

Proposition 3.2. *Let G be a right-angled Coxeter group, with generating set S, acting cocompactly on $\mathbb{H}^4$ with fundamental polytope P and f-vector (f_0, f_1, f_2, f_3). Let $f_S(x)$ denote the growth series of G in its complete form. Then,*

(a) $f_S(x) = \frac{[2]^4}{1+(4-f_3)x+(f_0-2f_3+6)x^2+(4-f_3)x^3+x^4}$;

(b) *$f_S(x)$ has four distinct (simple) poles given by*

$$x_1 = \frac{1}{4}\left(\alpha + \sqrt{\gamma} + \sqrt{\beta + 2\alpha\sqrt{\gamma}}\right); \qquad x_1^{-1} = \frac{1}{4}\left(\alpha + \sqrt{\gamma} - \sqrt{\beta + 2\alpha\sqrt{\gamma}}\right);$$
$$x_2 = \frac{1}{4}\left(\alpha - \sqrt{\gamma} + \sqrt{\beta - 2\alpha\sqrt{\gamma}}\right); \qquad x_2^{-1} = \frac{1}{4}\left(\alpha - \sqrt{\gamma} - \sqrt{\beta - 2\alpha\sqrt{\gamma}}\right),$$

where

$$\alpha = f_3 - 4, \qquad \beta = 2\alpha f_3 - 4f_0, \qquad \gamma = f_3^2 - 4f_0.$$

(c) -1 *is a root of multiplicity* 4 *of* $f_S(x)$.
(d) *The volume of* P *is given by*

$$\mathrm{vol}_4(P) = \frac{f_0 - 4f_3 + 16}{12}\pi^2.$$

Proof. As for (a), it is sufficient by Corollary 3.1 to show that $b_2 = f_0 - 2f_3 + 6$. Since P is simple, $2f_0 = f_1$, and the number of finite Coxeter subgroups G_T of G with $|T| = l$ equals f_{4-l}, for $l = 1, \ldots, 4$. Moreover, $f_3 = |S|$, and by Euler's formula (3.3), $f_2 = f_0 + |S|$. By a direct calculation, one gets

$$\sum_{T \in \mathcal{F}'} (-1)^{|T|}|T|(|T| + 1) = 6f_2 - 12f_1 + 20f_0 = 6|S| + 2f_0 = 6f_3 + 2f_0. \tag{3.4}$$

By plugging (3.4) into (3.2), we deduce that $b_2 = f_0 - 2f_3 + 6$. Property (c) follows easily from (a) since $f_S(-1) = \frac{[2]^4(-1)}{f_0} = 0$, and property (d) is a direct consequence of (a) and Heckman's formula (1.4).

As for (b), consider the denominator $Q(x) = 1 + (4 - f_3)x + (f_0 - 2f_3 + 6)x^2 + (4 - f_3)x^3 + x^4$ of $f_S(x)$ in (a). The polynomial $Q(x)$ is quartic over the integers with discriminant (see [18, Discriminants])

$$\Delta = f_0(16 + f_0 - 4f_3)(f_3^2 - 4f_0)^2.$$

Since P is simple with 2-faces being at least pentagonal, $f_0 \geq 5f_3$. Suppose that $f_3^2 > 4f_0$. Then, $\Delta > 0$, and $Q(x)$ has only simple and real roots whose explicit form (b) can be determined by a standard method. In fact, by applying the transformation $x = X + 1/X$ to the quartic polynomial $Q(x)$, which does not change the discriminant, one obtains a reduced cubic $\widetilde{Q}(x)$ with explicit formulas for its roots (see [18, Classical Formulas]). It remains to show that $f_3^2 > 4f_0$. Suppose on the contrary that $f_3^2 \leq 4f_0$ and consider a facet F^* of P with maximal number $N := f_0(F^*) = \max f_0(F)$ of vertices among all facets F of P, that is,

$$4f_0 = \sum_F f_0(F) \leq Nf_3 \quad \text{whence } f_3 \leq N.$$

We conclude the proof by showing that $N \geq f_3 \geq f_0(F^*) + 1 = N + 1$. Indeed, since P is simple, precisely one additional edge of P emanates from each of the $N = f_0(F^*)$ vertices to the outside of F^*. We show that these N edges give rise to N different (but not necessarily disjoint) facets F_* of P, beside F^*, and this by contraposition. Since all facets are convex and meet properly at 2-faces of P, the assumption of the opposite can hold only if two vertices v_1, v_2 belong to a common edge of F^* and if their edges leaving F^* arrive at vertices w_1, w_2 which may coincide in or lie on an edge of F_*. Hence, the convex hull of the vertices v_1, v_2, w_1, w_2 is a right-angled triangular or quadrilateral 2-face of P which is impossible. Therefore, $N \geq f_3 \geq f_0(F^*) + 1 = N + 1$. □

Remark 5. In [6, Example 17.4.3], a result analogous to Proposition 3.2(a) for the three-dimensional case is presented. More precisely, the growth series of a cocompact right-angled Coxeter group in $\mathbb{H}^3$ is given by

$$f_S(x) = \frac{[2]^3}{1 - (f_2 - 3)x + (f_2 - 3)x^2 - x^3}, \tag{3.5}$$

which has the three positive real poles 1, τ, τ^{-1} where

$$\tau = \frac{(f_2 - 4) + \sqrt{(f_2 - 4)^2 - 4}}{2}.$$

In fact, Euler's formula (3.3), $f_0 - f_1 + f_2 = 2$, together with the (vertex) simplicity $3f_0 = 2f_1$ and the evident inequality $f_2 \geq 4$, yields that $f_S(x)$ has only real simple roots.

Example 2. Let G_{120} be the Coxeter group generated by the 120 reflections with respect to the facets of a right-angled (compact) 120-cell $P \subset \mathbb{H}^4$. The polyhedron P has f-vector

$$f = (600, 1200, 720, 120)$$

and is the four-dimensional analogue of a right-angled dodecahedron D. In fact, all facets of P are isometric to D. The volume of P equals $34\pi^2/3$ by Proposition 3.2(d). This value can also be obtained by studying the symmetry group of P and by determining the covolume of its 14,000 index Coxeter simplex subgroup according to [11, Appendix], that is,

$$\mathrm{vol}(P) = 14{,}400 \cdot \mathrm{covol}_4(\bullet \overset{5}{\text{———}} \bullet \text{———} \bullet \text{———} \bullet \overset{4}{\text{———}} \bullet) = 14{,}400 \cdot \frac{17\pi^2}{21{,}600} = \frac{34\pi^2}{3}.$$

By means of (3.1) and Proposition 3.2, the growth series of G_{120} with respect to the set S of the above reflections is given by

$$f_S(x) = \frac{[2]^4}{1 - 116x + 366x^2 - 116x^3 + x^4}, \tag{3.6}$$

implying that $f_S(x)$ possesses exactly two pairs of real poles, which are positive and simple.

4. Growth of Coxeter groups with at most 6 generators in $\mathbb{H}^4$

Consider a hyperbolic cocompact Coxeter group G with generating set of reflections S acting in low dimensions $n \geq 2$. For $n = 2$, Cannon and Wagreich [2] showed that the growth series $f_S(x)$ is a quotient of relatively prime monic polynomials over the integers for which the denominator splits into exactly one Salem polynomial and (possibly none) distinct irreducible cyclotomic polynomials. Here, a Salem polynomial is a palindromic irreducible monic polynomial over the integers with exactly one (inversive) pair of real roots $\alpha^{-1}, \alpha > 1$ and with all other conjugates lying on the unit circle. The root α is called a Salem number. Hence, the growth rate τ of any (of the infinitely many) planar cocompact hyperbolic Coxeter groups is a Salem number. In [9], Hironaka showed that the smallest growth rate which arises in this way equals Lehmer's number given by the root $\alpha_L > 1$ of the Salem polynomial of smallest known degree

$$L(x) = 1 + x - x^3 - x^4 - x^5 - x^6 - x^7 + x^9 + x^{10}.$$

For $n = 3$, Parry [14] proved a result, analogous to the one in [2] above, for cocompact hyperbolic Coxeter groups providing a unified proof for $n = 2$ and $n = 3$ and extending Cannon's result [1] from the case of the nine Coxeter tetrahedra to arbitrary compact Coxeter polyhedra. Parry's proof is based on a special relationship between anti-reciprocal functions and Salem numbers. However, for $n \geq 4$, growth rates of cocompact hyperbolic Coxeter groups are not Salem numbers anymore as is illustrated by the example of the compact right-angled 120-cell in $\mathbb{H}^4$ according to (3.6).

In the following, we will describe in detail (see Theorem 4.1) the growth series of a cocompact hyperbolic Coxeter group, generated by at most six reflections in $\mathbb{H}^4$, and show that its positive poles arise always in precisely 2 inversive pairs $x_1^{-1} < x_2^{-1} < 1 < x_2 < x_1$, and each one is of multiplicity 1. The growth rate $\tau = x_1$ is a *Perron number*, that is, τ is a real algebraic integer all of whose conjugates are of strictly smaller absolute value. The non-real poles of the growth function come in quadruplets which do not all lie on the unit circle anymore (cf. also [3]). A rigorous proof of all our observations is very technical (and partially computer-based) and necessitates a closer analysis of the Coxeter groups under consideration with respect to their Coxeter subgroup structure (see [15] for all details).

Indeed, here lies the qualitative difference to the lower-dimensional cases $n = 2$ resp. $n = 3$ where the maximal finite Coxeter subgroups are dihedral groups resp. spherical triangle groups with a very limited, manageable variety of exponents, and this independently of the number of generators. The following exposition will document that similar growth questions in higher-dimensional hyperbolic spaces become nearly intractable.

Let G be a Coxeter group, with natural generating set S such that $|S| \leq 6$, and which acts cocompactly on $\mathbb{H}^4$. There is a complete classification which shows that they fall combinatorially into three (finite) families. For $|S| = 5$, G is a Coxeter simplex group, denoted by G_L. They were discovered and classified by Lannér (cf. [23, Chapter 3, Table 3]) and are nowadays called *Lannér groups*. If $|S| = 6$, then G is a *Kaplinskaya group* G_K or an *Esselmann group* G_E, which are characterised as follows. A fundamental polytope of G_K is a product of a 1-simplex with a 3-simplex and has eight vertices, while a fundamental polytope of G_E is a product of two triangles with nine vertices. The classification of the Kaplinskaya groups can be found in [10], and the list of all Esselmann groups is in [7].

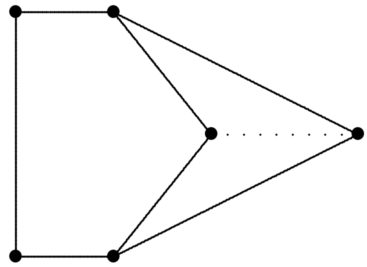


Fig. 1. The graph of the Kaplinskaya group G_{66}.

Theorem 4.1. *Let G be a Lannér group, an Esselmann group or a Kaplinskaya group, respectively, acting with natural generating set S on $\mathbb{H}^4$. Then,*

(1) *the growth series $f_S(x)$ of G is a quotient of relatively prime, monic and palindromic polynomials of equal degree over the integers.*
(2) *The growth series $f_S(x)$ of G possesses four distinct positive real poles appearing in pairs (x_1, x_1^{-1}) and (x_2, x_2^{-1}) with $x_1 < x_2 < 1 < x_2^{-1} < x_1^{-1}$; these poles are simple.*
(3) *The growth rate $\tau = x_1^{-1}$ is a Perron number.*
(4) *The non-real poles of $f_S(x)$ are contained in an annulus of radii x_2, x_2^{-1} around the unit circle.*
(5) *The growth series $f_S(x)$ of the Kaplinskaya group G_{66} with graph K_{66} (cf. Fig. 1) has four distinct negative and four distinct positive simple real poles; for $G \neq G_{66}$, $f_S(x)$ has no negative pole.*

Remark 6. The exceptional role (5) of the Kaplinskaya group G_{66}, having a growth series with 4 inversive pairs of distinct real poles, was first discovered by Zehrt (cf. [24]).

Sketch of the proof of Theorem 4.1. We will only discuss the ingredients of the proofs for (1)–(3), and this especially for the simplex case. At the end, we indicate how the proof extends for the families G_E and G_K (for more details, see [15]). Consider a Lannér group G_L with natural generator set S and denote by G_T, $T \subset S$, a maximal finite Coxeter subgroup of G_L. Associate to G_T the help function

$$h_T^L(x) := -\frac{1}{x+1} + \frac{1}{3}\sum_U \frac{1}{f_U(x)} - \frac{1}{2}\sum_V \frac{1}{f_V(x)} + \frac{1}{f_T(x)}, \tag{4.1}$$

where U varies over the six 2-element subsets and V varies over the four 3-element subsets of T. By means of Steinberg's formula (1.2′),

$$\frac{1}{f_S(x)} = 1 + \sum_{G_T \text{ maximal}} h_T^L(x). \tag{4.2}$$

Note that there exists only a very limited number of different maximal finite Coxeter subgroups in G_L as the weights of the Coxeter graph of G_L are at most equal to 5. By taking into account their reducibility properties, the following important auxiliary result can be shown by a case-by-case study (cf. [15, Lemma 3.8]).

Lemma 4.2. *The help function $h_T^L(x)$ (4.1) associated to a maximal finite Coxeter subgroup G_T of a Lannér group G_L can be written as the quotient*

$$h_T^L(x) = -x\frac{n(x)}{d(x)}, \tag{4.3}$$

where $n(x)$ and $d(x)$ are palindromic polynomials of even degrees over the integers. Moreover, $d(x)$ is cyclotomic with $\deg d = \deg n + 2$. Furthermore, $h_T^L(x)$ is negative for $x > 0$ and strictly decreasing on $(-1, 0)$.

By plugging (4.3) into (4.2), $f_S(x)$ becomes a quotient of palindromic integer polynomials of equal (even) degree. It is a well-known result that each palindromic integer polynomial of degree m can be factored into a product of a constant times linear (if m is odd), quadratic and quartic palindromic polynomials with real coefficients (see also [15, Proposition D.11]). It follows from this that $f_S(x)$ is a quotient of monic palindromic integer polynomials which are prime.

For the study of the real poles of $f_S(x)$, it suffices to consider the interval $[-1, 1]$ as $f_S(x)$ is reciprocal. Recall that $f_S(0) = 1$ and that $f_S(1) > 0$ by (1.4). Furthermore, we prove the following.

Lemma 4.3. *Let $f_S(x)$ be the growth series of a cocompact Coxeter group G acting on $\mathbb{H}^4$ with natural generator set S satisfying $|S| \le 6$. Then, $f_S(-1) = 0$.*

Proof. By Steinberg's formula (1.2), it is sufficient to show that the growth polynomial $f_T(x)$ of at least one maximal (or, equivalently, rank 4) finite Coxeter subgroup G_T of G factorises according to $f_T(x) = [2]^4 g_T(x)$ where $g_T \in \mathbb{Z}[x]$ with $g_T(-1) \neq 0$. Since the natural generator set S of G is of cardinality at most 6, G contains at least one subgroup G_T of type B_4, D_4, F_4 or H_4. This is due to its combinatorial structure (cf. also [23,7,10]). By Table 1, each of the groups B_4, D_4, F_4, H_4 has only *odd* exponents and therefore a growth polynomial $f_T(x)$ splitting into 4 factors of type $[2k]$. Now, observe that $[2k] = [2]\sum_{i=0}^{k-1} x^{2i}$ so that $[2k](-1) = 0$. □

All the above observations together with Lemmas 4.2 and 4.3 allow us to conclude that $f_S(x)$ is positive and strictly increasing on $(-1, 0]$. Since $f_S(x)$ is reciprocal, it is non-singular on $\mathbb{R}_{\le 0}$. For the study of the behavior of $f_S(x)$ on $I := [0, 1]$, we know that $f_S(x)$ is a rational function and has a real pole $0 < x_1 < 1$ given by the convergence radius. This follows since coefficients a_i of the series $f_S(x)$ are positive and real (cf. Section 1; [6, Section 17.1]). In particular, x_1 is a real algebraic integer whose inverse x_1^{-1} is the growth rate τ of G_L. Hence, τ is a Perron number.

For the proof of the remaining claims, a distinction of several cases and the help of a computer are needed to control the graphs of the help functions in the decomposition (4.2) on I (see [15, pp. 28–47]). By doing this, it turns out that their sum

$$H^L(x) := \sum_{G_T \text{ maximal}} h_T^L(x) = \frac{1}{f_S(x)} - 1 \tag{4.4}$$

is negative on I, and that $H^L(x)$ is either strictly decreasing on I or possesses exactly one negative minimum in I. Since x_1 is a pole of $f_S(x)$, and $1/f_S(1) > 0$, it follows that $H^L(x_1) = -1$ and $H^L(1) > -1$. Therefore, $H^L(x)$ cannot be strictly decreasing on I, but possesses exactly one negative minimum M. That is, there is a unique $x_M \in I$ such that $H^L(x_M) = M$. Obviously, $x_M \ge x_1$, since x_1 equals the radius of convergence of $f_S(x)$. Summarising, we can deduce that $f_S(x)$ possesses exactly two simple poles in I if $x_M > x_1$, or it has a pole of (positive) even order in I if $x_M = x_1$.

As for the simplicity of the poles x_1, x_2 of $f_S(x)$, there are no criteria known to us allowing to conclude it without precise knowledge of the denominator coefficients (cf. [12, for example]. By means of the recursion formula for these coefficients (see Theorem 2.5), the computer implementation of this algorithm helps to prove this last claim.

In the cases of Esselmann groups G_E and Kaplinskaya groups G_K, our strategy is essentially the same, apart from some particularities which have to be dealt with carefully. Furthermore, the help functions have to be adapted to the different combinatorial features of G_E and G_K. We finish this outline by providing their explicit shapes.

Recall that an Esselmann polytope has the combinatorial type of a direct product of two triangles and possesses therefore precisely nine vertices. The Coxeter graph Γ_E of an Esselmann group G_E contains two disjoint Lannér diagrams, called L_1 and L_2, each of them with three nodes. Let G_T be one of the nine maximal finite Coxeter subgroups of G_E where T denotes its natural generating set. The help function for h_T^E is defined by

$$h_T^E(x) := h_T^L(x) + \frac{1}{3(1+x)} - \frac{1}{12}\sum_W \frac{1}{f_W(x)}, \tag{4.5}$$

where $h_T^L(x)$ is the function (4.1) given by

$$h_T^L(x) = -\frac{1}{x+1} + \frac{1}{3}\sum_U \frac{1}{f_U(x)} - \frac{1}{2}\sum_V \frac{1}{f_V(x)} + \frac{1}{f_T(x)},$$

where U is a 2-element subset, V is a 3-element subset, and where W is a subset of T satisfying the following condition. The set W consists of four pairs of generators (s_p, s_q) such that the node in Γ_E corresponding to s_p belongs to L_1, while the node corresponding to s_q belongs to L_2.

A Kaplinskaya polytope has the combinatorial type of a simplicial prism and possesses therefore precisely eight vertices. The Coxeter graph Γ_K of a Kaplinskaya group G_K contains a Lannér diagram L with four nodes which represents a tetrahedron P, and two additional nodes which represent the reflections through the top respectively the bottom of the simplicial prism $P \times [0, 1]$. Let G_T be one of the eight maximal finite Coxeter subgroups of G_K where T denotes its natural generating set. The help function h_T^K is defined by

$$h_T^K(x) := h_T^L(x) + \frac{1}{4(1+x)} - \frac{1}{12}\sum_W \frac{1}{f_W(x)}, \tag{4.6}$$

where $h_T^L(x)$ is the function (4.1) and where W is a subset of T containing three pairs (s_{L_1}, s_b), (s_{L_2}, s_b) and (s_{L_3}, s_b) such that s_{L_j} belongs to L, for $j = 1, 2, 3$, while $s_b \notin L$. $\square$

Remark 7. For a cocompact hyperbolic Coxeter group, acting on $\mathbb{H}^4$ with a set S of generating reflections such that $|S| \leq 6$, the growth series $f_S(x)$ can be put into the form $R(x)/S(x)$ with monic palindromic polynomials $R, S \in \mathbb{Z}[x]$, $\deg R = \deg S$, and

$$R(x) = \begin{cases} [2, 8, 12, 20, 30] & \text{if } G \text{ is a Lannér group,} \\ [2, 6, 8, 12, 20, 30] & \text{if } G \text{ is an Esselmann or a Kaplinskaya group.} \end{cases} \tag{4.7}$$

For the coefficients β_k of the denominator $S(x) = 1 + \sum_{k\geq 1} \beta_k x^k$, the recursion of Theorem 2.5 applies as well. Observe that the numerator R and the denominator S are in general not prime. The result (4.7) follows easily by passing to the complete form P/Q and by extending P, Q simultaneously in a suitable way according to the Coxeter subgroup structure of all G of fixed type L, E or K as described in [23,7,10], and by using divisibility properties of the associated exponents.

Remark 8. The growth series of any hyperbolic cocompact Coxeter group G acting on $\mathbb{H}^n$, $n \geq 2$, vanishes at -1 *if at least one of its maximal finite Coxeter subgroups has a growth polynomial with odd exponents only*. Notice that this condition, for $n = 2$, excludes only the triangle groups $G = (p, q, r)$ with p, q, r odd. For $n = 3$, Parry's formulas [14, (0.5), (0.6)] for $f_S(x)$ imply that $f_S(-1) = 0$ without imposing any condition. However, for $n \geq 4$, the above condition is in general not true anymore. In fact, the Tumarkin group G_* acting cocompactly on $\mathbb{H}^6$ with Coxeter graph given in Fig. 2 (cf. [22]) does not have a maximal Coxeter subgroup all of whose exponents are odd. Nevertheless, its growth series splits the factor $[2]^4$ so that -1 is a root of multiplicity 4. As a byproduct, the computation shows also, by (1.4), that the covolume of G_* is given by $2,077\pi^3/17,010,000 \simeq 0.003786$.

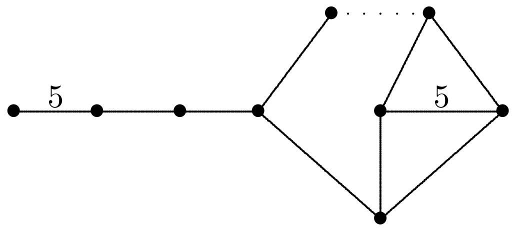


Fig. 2. A compact Tumarkin polytope with 9 facets in $\mathbb{H}^6$.

Finally, by analyzing *all known* examples of cocompact hyperbolic Coxeter groups acting in dimensions bigger than two (and less than nine), we see that -1 is always a root of the growth series.

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