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Cusped hyperbolic orbifolds of minimal volume in dimensions less than 11

THESIS

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Fir de Nicki

Abstract

Cusped hyperbolic orbifolds are quotients of hyperbolic n-space by discrete subgroups of hyperbolic isometries having finite volume. By a theorem of Kazhdan and Margulis, the volume spectrum of these spaces possesses a minimal value $\mu_n > 0$ in every dimension $n \geq 2$. In our work, we generalize the results of Siegel and Meyerhoff in dimensions 2 and 3 by determining the values μ_n for $4 \leq n \leq 9$ and by constructing the (unique) cusped orbifold Q^n of volume μ_n. The method we develop is based on results due to Adams in dimension 3 and combines geometric and algebraic group theory. It is universally applicable but its success in dimension $n \leq 9$ relies on the existence of hyperbolic Coxeter n-simplices with one vertex at infinity and of (uniquely determined) densest lattice packings in euclidean $(n-1)$-space. In particular, the face-centered-cubic packing from crystallography respectively the packing with inballs of the tessellation with regular 24-cells in dimension 4 enter into the construction of Q^4 respectively Q^5.

This work is a self-contained version of our articles [HK] and [H],

[H] T. Hild, The cusped hyperbolic orbifolds of minimal volume in dimensions less than ten, to appear in J. of Algebra.

[HK] T. Hild, R. Kellerhals, The fcc-lattice and the cusped hyperbolic 4-orbifold of minimal volume, to appear in J. London Math. Soc.

Zusammenfassung

Eine hyperbolische Orbifold mit Spitzen ist ein Quotient endlichen Volumens des hyperbolischen n-Raums modulo einer diskreten Untergruppe der hyperbolischen Isometriegruppe. Nach einem Satz von Kazhdan und Margulis besitzt das Volumenspektrum dieser Räume in jeder Dimension $n \geq 2$ einen minimalen Wert $\mu_n > 0$. In unserer Arbeit bestimmen wir diese Werte μ_n für $4 \leq n \leq 9$ und konstruieren die (eindeutig bestimmte) Orbifold Q^n mit Spitzen, deren Volumen μ_n beträgt. Hiermit verallgemeinern wir die entsprechenden Resultate von Siegel und Meyerhoff in den Dimensionen 2 und 3. Die von uns entwickelte Methode stützt sich auf Resultate, welche in Dimension 3 auf Adams zurückgehen und kombiniert geometrische und algebraische Gruppentheorie. Sie ist universell einsetzbar doch ihr Erfolg in den Dimensionen $n \leq 9$ beruht auf der Existenz sowohl n-dimensionaler hyperbolischer Coxetersimplizes mit einer Ecke im Unendlichen als auch (eindeutig bestimmter) dichtester Gitterkugelpackungen im euklidischen $(n-1)$-Raum. Insbesondere benutzt die Konstruktion von Q^4 beziehungsweise Q^5 das flächenzentrierte kubische Gitter der Kristallographie beziehungsweise die Packung bestehend aus Inkugeln der 4-dimensionalen Pflasterung mit regulären 24-Zells.

Diese Arbeit ist eine in sich abgeschlossene Version unserer beiden Artikel [HK] und [H],

[H] T. Hild, The cusped hyperbolic orbifolds of minimal volume in dimensions less than ten, to appear in J. of Algebra.

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Résumé

Une orbifold hyperbolique avec pointes est un quotient de l'espace hyperbolique de dimension n par un groupe discret d'isométries hyperboliques, ayant un volume fini. D'après un théorème de Kazhdan et Margulis, le spectre du volume de ces espaces possède une valeur minimale $\mu_n > 0$ dans chaque dimension $n \geq 2$. Dans notre travail, nous déterminons ces valeurs μ_n pour $4 \leq n \leq 9$ et construisons l'(unique) orbifold avec pointe Q^n dont le volume vaut μ_n. Par ce fait, nous généralisons les résultats analogues de Siegel et Meyerhoff en dimensions 2 et 3. La méthode que nous développons est basée sur les résultats obtenus par Adams en dimension 3 et combine les théories géométrique et algébrique des groupes. Elle est universellement applicable mais son succès en dimensions $n \leq 9$ est dû aussi bien à l'existence de simplexes hyperboliques de Coxeter avec un sommet à l'infini en dimensions $n \leq 9$ qu'au fait que les réseaux euclidiens les plus denses sont connus (et uniquement déterminés) en dimensions $n \leq 8$. En particulier, le réseau cubique fcc de la cristallographie respectivement le pavage avec des 24-cells réguliers en dimension 4 sont utilisés pour la construction de Q^4 respectivement Q^5.

Cette thèse est une version autonome de nos articles [HK] et [H],

[H] T. Hild, The cusped hyperbolic orbifolds of minimal volume in dimensions less than ten, to appear in J. of Algebra.

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I also want to thank both co-advisors, Colin Adams and Anand Dessai, for their interest in my work and their suggestions.

I am especially grateful to Marston Conder for inviting me to New Zealand. My three months research stay in Auckland was from a mathematical but also non-mathematical point of view an unforgettable experience. I thank all the members of the Math Department of Auckland for their hospitality.

I thank the members of the Math Department of Fribourg for the pleasant working environment, in particular my office neighbors Gilles, Thomas and Christine. I truly enjoyed being a member of the constantly increasing dynamic geometry group. I owe special thanks to Patrick Ghanaat, Ralph Strebel and Andreas Bernig for their interest in my work and some inspiring discussions and remarks.

Financial support from the Swiss National Science Foundation is gratefully acknowledged.

Finally I thank my family for the constant support and my grandfather who draw the tree of the introduction below. Thank you Michèle for your encouragement and the beautiful moments we shared.

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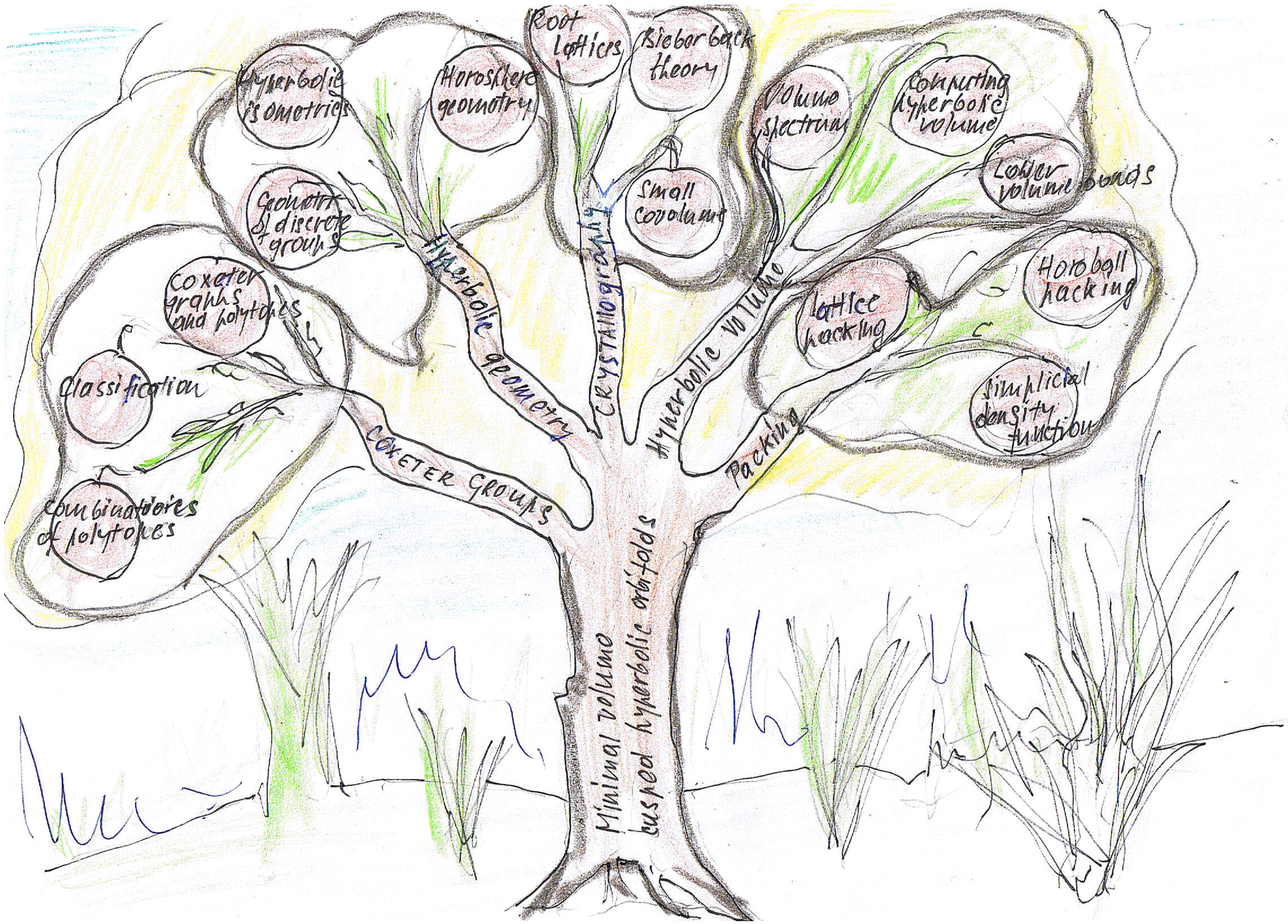
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1 Introduction

Let H^n denote the n-dimensional hyperbolic space and $I(H^n)$ the group of hyperbolic isometries.

In this thesis, we study non-compact, complete, hyperbolic orbifolds of finite volume, that is, quotients of H^n by non-cocompact, discrete subgroups of $I(H^n)$ of finite covolume. By the finite volume condition, H^n/Γ possesses a finite number of unbounded ends of finite volume each. These parts are called cusps of the orbifold and are associated to conjugacy classes of parabolic subgroups of Γ. A cusp boundary is a compact euclidean orbifold E^{n-1}/Γ_q, where Γ_q is the stabilizer in Γ of a parabolic fixed point $q \in \partial H^n$.

While hyperbolic manifolds are difficult to construct in dimensions $n > 3$, we have the nice class of hyperbolic orbifolds modelled upon discrete Coxeter groups of H^n at hand. A hyperbolic Coxeter group Γ_C is generated by the reflections in the sides of a Coxeter polytope P_C characterized by dihedral angles of the form $\frac{\pi}{k}$, $k \geq 2$ an integer. Among Coxeter polytopes, Coxeter simplices T_C are distinguished by their simple combinatorics. In particular, the mirror images of T_C under the action of Γ_C provide a very symmetric tessellation of H^n. However, hyperbolic Coxeter simplices do not exist anymore in dimensions $n > 9$. The start of this thesis was a strong believe of Ruth Kellerhals that a connection exists between Coxeter simplices with one vertex at infinity and cusped orbifolds of small or minimal volume.

Understanding the volume spectrum

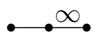
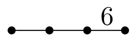
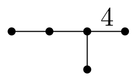
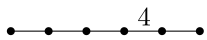
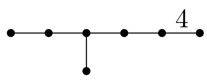
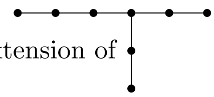
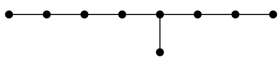
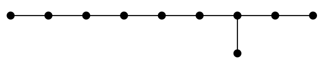
$$V_\infty^n := \{\mathrm{vol}_n(H^n/\Gamma) : H^n/\Gamma \text{ a cusped hyperbolic orbifold}\}$$

for cusped hyperbolic n-orbifolds would constitute a first step towards a complete classification by controlling one of its most important topological invariants. It is known that V_∞^n is well-ordered for $n = 2, 3$, and discrete for $n \geq 4$. By a theorem of Kazhdan-Margulis [KM], there is a minimal element $\mu_n > 0$ in V_∞^n for every dimension $n \geq 2$. We address the problem to find all hyperbolic n-orbifolds of volume μ_n and to determine the value μ_n explicitly. For $n \leq 9$, we achieve a complete solution which can be stated as follows.

Main Theorem. *For $4 \leq n \leq 9$, let H^n/Γ_* be a cusped hyperbolic orbifold of minimal volume μ_n. Then, up to isomorphism, Γ_* is related to a hyperbolic Coxeter simplex group according to the table below, and as such uniquely determined.*

It generalizes the results in dimensions 2 and 3 due to C. L. Siegel [Sie, 1945] and R. Meyerhoff [Me2, 1985] respectively. Our proof is based on a generalization of methods developed by C. Adams [Ad1] in dimension 3. It is an interplay between algebraic and geometric group theory. For the convenience of the reader, we included an illustrative survey in form of a tree on the previous page. The five branches indicate the mathematical areas that enter the construction of H^n/Γ_* for $n \leq 9$. Every apple on a branch gives an important keyword associated with the theory symbolized by the branch.

The method we developped is universal. However, the lack of finite volume hyperbolic Coxeter simplices increases considerably the complexity of the arguments in higher dimensions. At the end of this thesis, we discuss the 10-dimensional case, only. In particular, we prove that H^{10}/Γ_* has at most two cusps and determine their shape. In this way, we can at least pinch the value μ_{10}.

Dim n	Γ_*	μ_n	
2		$\frac{\pi}{6}$	$\approx 5.23 \cdot 10^{-1}$
3		$\frac{1}{8}\mathbf{Л}\left(\frac{\pi}{3}\right)$	$\approx 4.23 \cdot 10^{-2}$
4		$\frac{\pi^2}{1\ 440}$	$\approx 6.85 \cdot 10^{-3}$
5		$\frac{7\ \zeta(3)}{46\ 080}$	$\approx 1.83 \cdot 10^{-4}$
6		$\frac{\pi^3}{777\ 600}$	$\approx 3.98 \cdot 10^{-5}$
7	$\mathbb{Z}_2$-extension of 	$\frac{\sqrt{3}\ L(4,3)}{1\ 720\ 320}$	$\approx 9.46 \cdot 10^{-7}$
8		$\frac{\pi^4}{4\ 572\ 288\ 000}$	$\approx 2.13 \cdot 10^{-8}$
9		$\frac{\zeta(5)}{22\ 295\ 347\ 200}$	$\approx 4.65 \cdot 10^{-11}$

The present work is a self-contained version of our two articles [HK] and [H] accepted for publication in Journal of the London mathematical society and the Journal of Algebra respectively. We tried to include all information a non-expert needs to understand the methods and the proof of our theorem.

[H] T. Hild, The cusped hyperbolic orbifolds of minimal volume in dimensions less than ten, to appear in J. of Algebra.

[HK] T. Hild, R. Kellerhals, The fcc-lattice and the cusped hyperbolic 4-orbifold of minimal volume, to appear in J. London Math. Soc.

2 Hyperbolic geometry

2.1 Preliminaries

As a reference for the material in this chapter, we recommend [Rat] and [Thu].

Hyperbolic n-space. Up to isometry, hyperbolic n-space $\mathbb{H}^n$ is the only simply connected, complete, n-dimensional, Riemannian manifold of constant sectional curvature $\kappa = -1$. We use the conformal Poincaré model $H^n = (U^n, d)$ consisting of the upper half-space

$$U^n = \{x \in \mathbb{R}^n : x_n > 0\} \subset \mathbb{R}^n$$

equipped with the hyperbolic distance function

$$\cosh\left(d\left(x,y\right)\right) = 1 + \frac{\|x-y\|^2}{2x_n y_n}; x, y \in U^n.$$

Here and in what follows, $\|.\|$ denotes the euclidean norm in $\mathbb{R}^n$ induced by the standard scalar product $\langle .,. \rangle$. We often write E^n instead of $\mathbb{R}^n$ to indicate that we consider the euclidean metric space $(\mathbb{R}^n, \langle .,. \rangle)$. In particular, we have

$$d\left(\left(0,\ldots,0,x_n\right),\left(0,\ldots,0,y_n\right)\right) = \left|\ln \frac{x_n}{y_n}\right|, \text{ for all } x_n, y_n > 0. \quad (1)$$

The Riemannian metric of the geometric model H^n is

$$ds^2 = \frac{\sum_{i=1}^n dx_i^2}{x_n^2} \quad (2)$$

and yields the volume element

$$dv = \frac{dx_1 \cdots dx_n}{x_n^n}. \quad (3)$$

The boundary at infinity of H^n is the one-point compactification $\partial H^n = \partial U^n \cup \{\infty\} = \overline{\mathbb{R}}^{n-1}$. Elements of ∂H^n will be called *points at infinity*, those of H^n *ordinary points*.

Spheres and horospheres. In the upper half-space model, a hyperbolic sphere

$$S\left(a,r\right) := \{x \in H^n : d\left(a,x\right) = r\}$$

of center $a \in H^n$ and radius $r > 0$ equals, as a set, the euclidean sphere contained in U^n with center $(a_1, \ldots, a_{n-1}, a_n \cosh r)$ and radius $a_n \sinh r$. Fix a point $c \in S\left(a,r\right) = S$ and let l be the geodesic half-line through a with endpoints c and $p \in \partial H^n$. Expand S by moving the center a away from c on l, while keeping c on S. The resulting hypersurface S_p is called a *horosphere based at* $p \in \partial H^n$. *Geodesics* and, more generally, totally geodesic *k-planes* in H^n are intersections of U^n with a euclidean k-plane or k-sphere orthogonal to $\mathbb{R}^{n-1} = \partial U^n$. Note that horosphere S_p is orthogonal to every geodesic with endpoint p. Therefore, S_p is a euclidean hyperplane parallel to $\mathbb{R}^{n-1}$ if $p = \infty$ and the intersection of U^n with a euclidean sphere in $\overline{U}^n$ tangent to $\mathbb{R}^{n-1}$ in p if $p \neq \infty$. In particular, the horosphere

$$S_\infty\left(\rho\right) := \{x \in U^n : x_n = \rho\}$$

is said to be *at euclidean height* $\rho > 0$. By means of (2), $S_\infty(\rho)$ inherits a positive multiple of the euclidean metric

$$ds^2 \mid_{S_\infty(\rho)} = \frac{1}{\rho^2} \sum_{i=1}^{n-1} dx_i^2. \tag{4}$$

All horospheres being congruent to $S_\infty(\rho)$, they are all endowed with a flat Riemannian structure. The connected region bounded by S_p and meeting ∂H^n only at p is termed *horoball* B_p. In particular, the horoball

$$B_\infty(\rho) := \{x \in U^n : x_n > \rho\}$$

is said to be *at euclidean height* $\rho > 0$.

Hyperbolic isometries. The set of all hyperbolic isometries is denoted by $I(H^n)$. Every $\gamma \in I(H^n)$ is a finite composition of reflections with respect to hyperplanes. It naturally extends to a homeomorphism of $\overline{H}^n = H^n \cup \partial H^n$ and is completely determined by its behaviour on ∂H^n. To see this, we endow ∂H^n with the chordal metric. A *generalized sphere* $\hat{S}(a, r)$ of $\overline{\mathbb{R}}^{n-1}$ is either a euclidean sphere

$$S(a, r) := \{\|x - a\| = r\}$$

or an extended euclidean hyperplane

$$P(a, r) := \{\langle x, a\rangle = r\} \cup \{\infty\}$$

with unit normal vector $a \in \mathbb{R}^{n-1}$ and passing through ra, $r \in \mathbb{R}$. Note that topologically, $P(a, r)$ is a sphere. In the same spirit, a reflection $\hat{\sigma}_{a,r}$ of $\overline{\mathbb{R}}^{n-1}$ with respect to $\hat{S}(a, r)$ is either a reflection $\rho_{a,r}$ with respect to $P(a, r)$ or an inversion $\sigma_{a,r}$ in $S(a, r)$:

$$\rho_{a,r}(x) = x + 2(r - \langle a, x\rangle)\, a \qquad , \text{ if } x \in \mathbb{R}^n \ ; \ \rho_{a,r}(\infty) = \infty,$$
$$\sigma_{a,r}(x) = a + \left(\tfrac{r}{\|x-a\|}\right)^2 (x - a) \quad , \text{ if } x \in \mathbb{R}^n \ ; \ \sigma_{a,r}(\infty) = a \ ; \ \sigma_{a,r}(a) = \infty.$$

Finite compositions of $\hat{\sigma}_{a,r}$ are called *Möbius transformations* of $\overline{\mathbb{R}}^{n-1}$. The group Möb$(n-1)$ of all $(n-1)$-dimensional Möbius transformations is naturally isomorphic to $I(H^n)$. Indeed, if we interpret $\mathbb{R}^{n-1}$ as the boundary of U^n and write $\tilde{a} = (a, 0)$, then the isomorphism is given via the *Poincaré extension*

$$\hat{\sigma}_{a,r} \longmapsto \hat{\sigma}_{\tilde{a},r} \mid_{U^n} .$$

For more details, we refer to [Rat, §4.4 and §4.6].

Classification of hyperbolic isometries. The Brouwer fixed point theorem forces $\gamma \in I(H^n)$ to have a fixed point in $\overline{H}^n$. For $\gamma \neq id$, there are three mutually excluding situations depending only on its conjugacy class in $I(H^n)$. We call γ *elliptic* if γ fixes a point $p \in H^n$. It leaves every hyperbolic sphere $S(p, r)$ setwise invariant and acts on this sphere as an isometry of the induced spherical metric. A *parabolic* isometry γ has no fixed point in H^n and exactly one fixed point $p \in \partial H^n$. Every horosphere S_p is left invariant by γ which acts on it as an isometry of the induced flat metric. In particular, a parabolic element fixing ∞ acts on $S_\infty(1)$ as a euclidean isometry. Finally, a *loxodromic* isometry has no fixed point in H^n and exactly two fixed points $p, q \in \partial H^n$. Denote by l the geodesic line joining p and q. Then γ leaves invariant every banana tube or cylinder $\{x \in H^n : d(l, x) = r\}$ of radius $r > 0$ around l and is the composition of a non-trivial hyperbolic translation along l with a possibly trivial elliptic isometry fixing l. We note the following useful result (see [Rat, Thm. 4.7.3], for example).

Lemma 1 *Let S_a and S_b be two horospheres and $\gamma \in I(H^n)$ such that $\gamma(S_a) = S_b$. Then, γ is an isometry with respect to the induced flat metrics of S_a and S_b.*

Hyperbolic polytopes. The terminology used in the context of polytopes is not unified. This paragraph should help to avoid any confusion later. A hyperbolic *polytope* $P^n = P$ in dimension n, or simply *n-polytope*, is the intersection of the convex hull of finitely many points in $\overline{H}^n$ with H^n. We use the name *polygon* for P^2 and *polyhedron* for P^3. A *side* S of P is a non-empty subset of the boundary ∂P contained in a unique geodesic hyperplane $\langle S \rangle$ of H^n. Note that P has *finite volume* and only *finitely* many sides.

If $\langle S \rangle^-$ denotes the half-space bounded by $\langle S \rangle$ and containing P, then

$$P = \bigcap_{S \text{ a side of } P} \langle S \rangle^- . \tag{5}$$

The definition of P as an intersection of finitely many half-spaces as in (5) would have been equivalent to our definition. We often consider *k-faces* of P which are defined inductively as follows: the only n-face is P itself and a k-face is the side of a $(k+1)$-face. For our work, the $(n-2)$-faces or *ridges* and the 0-faces or *vertices* are of special interest. We distinguish between *ordinary* and *ideal* vertices. The latter ones lie at infinity and do not belong to P. Their number is at most finite. A polytope all of whose vertices are ideal is called *ideal polytope.* For completeness, we mention that a 1-face of P is called an *edge.*

Analogously, a spherical or euclidean polytope will always be the convex hull of finitely many points in S^n respectively E^n.

Vertex figures and solid angles. Vertex figures and solid angles generalize the angle notion for polygons to higher dimensions. Every ordinary boundary point x of a polytope $P^n = P$ lives in the interior of a d-face $F^d = F$, with $0 \le d \le n-1$. Choose a radius $r > 0$ so small that the hyperbolic n-sphere $S(x, r)$ centered in x only intersects the sides of P incident with F. Such a radius exists because P has only finitely many sides. Denote by $F^\perp$ the $(n-d)$-plane passing through x and orthogonal to F. The intersection of $F^\perp$ with the *sufficiently small* sphere $S(x, r)$ is an $(n-d-1)$-sphere $S^{n-d-1}(x, r)$. We define the *solid angle* of P with apex F to be the spherical $(n-d-1)$-polytope

$$V_F := \frac{1}{r}\left(P \cap S^{n-d-1}(x, r)\right).$$

Up to isometry, V_F is independent of $x \in F$. By abuse of language, the name of solid angle is also used for the spherical volume of V_F. Often, a formula involving solid angles is simplified when using the *normalized volume* obtained by dividing the solid angle by the volume

$$\Omega^{n-1} := \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \tag{6}$$

of the unit $(n-1)$-sphere. We write

$$\omega_{n-d-1}\left(F \mid P\right) := \frac{\mathrm{vol}_n\left(V_F\right)}{\Omega^{n-d-1}} \tag{7}$$

for the normalized volume of the solid angle with apex F in P. In particular, we set $\omega_{-1}\left(P \mid P\right) = 1$ and $\omega_0\left(F^{n-1} \mid P\right) := 1/2$.

Some of the solid angles in P have special names. The solid angle with apex a ridge R of P is called *dihedral angle* α_R of P. We have $\alpha_R = 2\pi\ \omega_1\left(R \mid P\right)$. The solid angle with apex a vertex $v \in D$ is termed *vertex figure* in v. In the case of a hyperbolic polytope P, the vertex figure in an ideal vertex w is defined to be the euclidean $(n-1)$-polytope

$$V_w := P \cap S_w$$

in the normalized flat metric of a sufficiently small horosphere S_w based at w.

2.2 Geometry of discrete groups

Complete hyperbolic orbifolds. A classification theorem of Riemannian geometry tells us that every connected, complete, n-dimensional real Riemannian manifold of constant sectional curvature $\kappa = -1$ is isometric to a quotient H^n/Γ of H^n by a discrete, torsionfree subgroup of $I(H^n)$. Since torsionfree subgroups of $I(H^n)$ act without fixed points in H^n, only parabolic and loxodromic elements occur in Γ. If we enlarge this class of *hyperbolic manifolds* by allowing elliptic isometries in Γ, we obtain the class of (complete) *hyperbolic n-orbifolds*. Every fixed point of an elliptic element projects to a singularity of the quotient space H^n/Γ. A proof showing the completeness of these quotients can be found in chapter 13 of [Thu] or [Rat].

Two orbifolds H^n/Γ_1 and H^n/Γ_2 are isometric if and only if Γ_1 and Γ_2 are conjugate in $I(H^n)$ [Rat, Thm 8.1.5]. Therefore, we are interested in properties and objects invariant under conjugation. For instance, the stabilizers of two Γ-equivalent points $x, \gamma x \in \overline{H}^n$ are conjugated, i.e.

$$\Gamma_{\gamma x} = \gamma \Gamma_x \gamma^{-1}.$$

First, consider the stabilizer $\Gamma_x < \Gamma$ of an ordinary point $x \in H^n$. Containing elliptic elements only, Γ_x is isomorphic to a finite subgroup of the orthogonal group $O(n)$. This forces the fixed point set

$$\mathfrak{S}(\gamma) := \{x \in H^n : \gamma x = x\}$$

of a non-trivial isometry $1 \neq \gamma \in \Gamma$ to be empty or a hyperbolic k-plane different from H^n. We deduce that the set

$$\mathfrak{S} := \bigcup_{1 \neq \gamma \in \Gamma} \mathfrak{S}(\gamma)$$

of all points in H^n fixed by an element in Γ is a closed, nowhere dense subset. Under the canonical projection $\pi : H^n \to H^n/\Gamma$, $\mathfrak{S}$ projects to the so-called *singular set* of H^n/Γ.

Now, let $x \in \partial H^n$ be the fixed point of a parabolic isometry. A discrete subgroup of $I(H^n)$ cannot contain a loxodromic and parabolic isometry with common fixed point [Rat, Thm 5.5.4]. The stabiliser Γ_x therefore contains only parabolic and elliptic elements. It is called *elementary of parabolic type*. After conjugation, assume $x = \infty$ to see that Γ_∞ is the Poincaré extension of an infinite discrete subgroup of euclidean isometries acting on $\mathbb{R}^{n-1}$ [Rat, Thm 5.5.5] and leaves invariant every horosphere S_∞. By means of Lemma 1, Γ_∞ acts as an infinite discrete subgroup of isometries in the flat metric of S_∞.

Consider a subset $V \subset H^n/\Gamma$ pierced by the singular set. The restriction of the canonical projection π to $\pi^{-1}(V)$ is clearly not injective. Nevertheless, we call V *embedded* if there is a point x in the closure of $\pi^{-1}(V)$ such that π induces an isometry from $\pi^{-1}(V)/\Gamma_x$ onto V. By abuse of language, we also say that $\pi^{-1}(V)$ embeds in H^n/Γ. In particular, for r satisfying

$$0 < r < \frac{1}{2} d(x, \Gamma x \setminus \{x\}),$$

a hyperbolic ball $B(x,r)$ of radius r centered in an ordinary point $x \in H^n$ embeds in H^n/Γ since $B(\pi(x), r)$ is isometric to $B(x,r)/\Gamma_x$ [Rat, Thm 13.1.1]. Later, we will see that each horoball B_x based at the fixed point $x \in \partial H^n$ of a parabolic isometry $\gamma \in \Gamma$, and of sufficiently small euclidean radius, projects to an embedded subset B_x/Γ_x of H^n/Γ.

Fundamental domains. The link between algebraic and geometric group theory is made by the notion of fundamental domain. The terminology used in connection with fundamental domains is not unified. In this work, a *fundamental domain* for a discrete group $\Gamma < I(H^n)$ is a *closed* set $D \subset H^n$, such that the Γ-images of its interior $\mathring{D}$ are all disjoint and the family

$$\mathcal{T} = \{\gamma D : \gamma \in \Gamma\}, \tag{8}$$

which we force to be *locally finite*, partitions H^n. In general, D is not uniquely determined. The Dirichlet construction yields a particularly useful fundamental domain (see below).

Different elements in (8) intersect only on their boundaries, that is

$$D \cap \gamma D \subset \partial D$$

for all non-trivial $\gamma \in \Gamma$. Therefore each interior point of D has trivial stabiliser in Γ whereas the set of fixed points $\mathfrak{S}$, modulo the action of Γ, lies on the boundary ∂D. The inclusion $\iota : D \hookrightarrow H^n$ induces a homeomorphism $\kappa : D/\Gamma \to H^n/\Gamma$ [Rat, Thm 6.5.8]. Up to some identifications of boundary points, D can thus be used as a *topological model* for the orbifold H^n/Γ.

If $\mathrm{vol}(\partial D) = 0$, then D is said to be *proper*. Since all proper fundamental domains of Γ have the same volume [Rat, Thm 6.5.5], the *covolume* of Γ is well-defined by

$$\mathrm{covol}(\Gamma) := \mathrm{vol}(D), \ D \text{ a proper fundamental domain of } \Gamma.$$

Outside its singular set, H^n/Γ has the local structure of a manifold. The canonical projection π maps D onto H^n/Γ so that $\mathrm{vol}(H^n/\Gamma) \leq \mathrm{vol}(D)$. But, since π is injective on $\mathring{D}$, we also have $\mathrm{vol}(H^n/\Gamma) \geq \mathrm{vol}\left(\mathring{D}\right)$. Combining both inequalities, we conclude that the volume of a complete orbifold H^n/Γ equals the covolume of Γ, that is the volume of a proper fundamental domain of Γ.

Dirichlet fundamental domains. Dirichlet's construction of a fundamental domain for a discrete group $\Gamma < I(H^n)$ starts, as usually, with the choice of an ordinary point $u \in H^n$ with trivial stabilizer [Rat, Thm 6.5.14]. To every $1 \neq \gamma \in \Gamma$, we associate the closed half-space

$$H_\gamma(u) := \{x \in H^n : d(x,u) \leq d(x, \gamma u)\}$$

containing u and bounded by the perpendicular bisector

$$P_\gamma(u) := \{x \in H^n : d(x,u) = d(x, \gamma u)\}$$

of the geodesic joining u to γu. The *Dirichlet fundamental domain* $D(u)$ with center u is then given by

$$D(u) := \begin{cases} H^n, \text{ if } \Gamma = \{1\} \\ \bigcap\limits_{1 \neq \gamma \in \Gamma} H_\gamma(u) \end{cases}.$$

A parabolic isometry in Γ forces $D(u)$ to be unbounded [Rat, Thm 6.5.7]. If Γ has finite covolume, $D(u)$ is the intersection of only finitely many half-spaces $H_\gamma(u)$, hence a hyperbolic n-polytope. In what follows, we restrict ourselves to the finite volume case and put $D = D(u)$. We will see that modulo the action of Γ, every ideal vertex corresponds to the conjugacy class in $I(H^n)$ of a parabolic subgroup in Γ.

Side-pairing and Poincaré relations. Besides being proper, D is *exact.* This means that with every side S_i of D comes an isometry $1 \neq s_i \in \Gamma$ satisfying

$$S_i = D \cap s_i D.$$

This *adjacency transformation* s_i is uniquely determined and satisfies

$$S_i' := s_i^{-1} S_i \text{ is a side of } D.$$

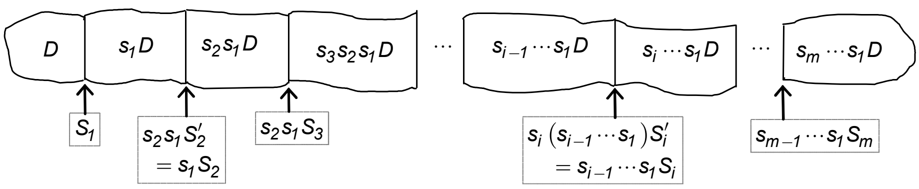
Since $S_i' = s_i^{-1} D \cap D$, we deduce the *side-pairing relation*

$$s_i s_i' = 1. \tag{9}$$

Consider again the tessellation $\mathcal{T}$ defined in (8) and call *chain* in $\mathcal{T}$ a finite sequence of adjacent polytopes. The set of all chains in $\mathcal{T}$ is denoted by $\mathcal{K}$. In $\mathcal{T}$, the passage from the polytope $s_{i-1} \cdots s_2 s_1 D$ to its adjacent polytope $s_i \cdots s_2 s_1 D$ with shared side $s_{i-1} \cdots s_2 s_1 S_i$ is described by the adjacency transformation s_i. The uniqueness of s_i implies a one-to-one correspondence

$$\begin{array}{rccc} \eta: & G & \longrightarrow & \mathcal{K} \\ & s_m \cdots s_2 s_1 & \longmapsto & \{D, s_1 D, s_2 s_1 D, \ldots, s_m \cdots s_2 s_1 D\} \end{array}$$

between the group G generated by the set of adjacency transformations $\Phi := \{s_i : S_i \text{ a side of } D\}$ and $\mathcal{K}$. Any member of $\mathcal{T}$ being joined to D by a chain, we conclude that $G = \Gamma$.



A *cycle* in $\mathcal{K}$ is a chain whose first and last members coincide. Every cycle in $\mathcal{K}$ corresponds, under η, to a relation in Γ. If the polytopes of a cycle all share a common ridge in D, the corresponding relation

$$s_f \cdots s_2 s_1 = 1 \tag{10}$$

of elements $s_i \in \Phi$ is termed *Poincaré relation.* One can show that

$$\langle \Phi : \text{Side-pairing relations, Poincaré relations} \rangle \tag{11}$$

yields a group presentation for Γ [Rat, Thm 13.5.3].

The next paragraph provides necessary conditions for the solid angles of fundamental polytopes. These conditions are hidden in every construction of orbifolds and indirectly enter the computation of orbifold volumes.

Cycle conditions. The side-pairing relations (9) imply that any class of Γ-equivalent sides of D consists of one or two elements depending on whether the associated adjacency transformation leaves the side invariant or not. On the other hand, a Poincaré relation $s_f \cdots s_2 s_1 = 1$ corresponds to the cycle around a ridge F^{n-2} of D. Clearly,

$$F^{n-2} \subset s_i \cdots s_1 D \Longleftrightarrow (s_i \cdots s_1)^{-1} F^{n-2} \subset D, \forall i = 1, \ldots, f.$$

In what follows, we use the notation $\gamma_i := (s_i \cdots s_1)^{-1}$ and $F_i^{n-2} := \gamma_i F^{n-2}$. By means of (11), the family

$$\mathcal{F}^{n-2} := \left\{ F_f^{n-2} = F^{n-2} = F_0^{n-2}, F_1^{n-2}, F_2^{n-2}, \ldots, F_{f-1}^{n-2} \right\}$$

contains all, possibly repeated, ridges of D equivalent to F^{n-2} under Γ. Every boundary point $x \in \partial D$ lies either in the the interior of a side or in a ridge. Consider the equivalence class

$$[x] := \{x_f = x = x_0, x_1, x_2, \ldots, x_{f-1}\} = \Gamma x \cap D \tag{12}$$

containing all boundary points $x_i = \gamma_i(x)$ equivalent to x. Fix $0 \leq d \leq n-1$. Since two d-faces $F_1, F_2 \subset D$ are Γ-equivalent if and only if their respective interiors contain points $x_1 \in \mathring{F}_1$and $x_2 \in \mathring{F}_2$ of the same equivalence class $[x_1] = [x_1]$, Poincaré and side-pairing relations intrinsically define an equivalence relation on the set $\mathcal{F}^d(D)$ of all d-faces in D. In the sequel, we use the following notations:

> v^d denotes the number of equivalence classes $\mathcal{F}^d_{(1)}, \mathcal{F}^d_{(2)}, \ldots, \mathcal{F}^d_{(v^d)}$ in $\mathcal{F}^d(D)$. Every class $\mathcal{F}^d_{(i)}$ contains $f^d_{(i)}$ faces $F^d_{(i)j}$.

Similarly, we obtain equivalence classes on the set $\mathcal{I}(D)$ of ideal vertices.

> $\mathcal{I}(D)$ splits into m equivalence classes $\mathcal{I}_{(1)}, \mathcal{I}_{(2)}, \ldots, \mathcal{I}_{(m)}$, every class $\mathcal{I}_{(i)}$ containing $l_{(i)}$ ideal vertices $w_{(i)j}$.

For fixed $0 \leq d \leq n-1$ and $1 \leq i \leq v^d$, we denote the order of a setwise stabilizer of a face in $\mathcal{F}^d_{(i)}$ by $g^d_{(i)}$. On can show that the normalized angles with apex $F^d_{(i)j}$ satisfy the cycle condition

$$\sum_{F^d_{(i)} \in \mathcal{F}^d_{(i)}} \omega_{n-d-1}\left(F^d_{(i)} \mid D\right) = \frac{1}{g^d_{(i)}}. \tag{13}$$

In particular, the dihedral angle sum in Γ-equivalent ridges F^{n-2} of D equals a submultiple of 2π:

$$\sum_{F^{n-2}_{(i)} \in \mathcal{F}^{n-2}_{(i)}} \delta_{F^{n-2}_{(i)}}(D) = \frac{2\pi}{g^{n-2}_{(i)}}. \tag{14}$$

More details can be found in [Zeh, Thm. 13.1.1]. As explained by Poincaré's fundamental polyhedron theorem [Rat, §13.5], the dihedral angle sum condition (14) is, in some sense, already a sufficient condition for D to be the fundamental domain for a discrete subgroup of $I\left(H^n\right)$.

Cusp of an orbifold. The present paragraph defines the most important objects of our work: the cusps of an orbifold. We use the notation introduced above. Consider an equivalence class $\mathcal{I}=[w]=\{w_1,\dots w_{l-1},w_l=w\}$ of ideal vertices in D. Let $\gamma_j\in\Gamma$ be the isometry satisfying $w_j=\gamma_j w$. Every ideal vertex w_j of D is fixed by a parabolic isometry in Γ [Rat, Corollary to Thm 12.2.4]. Since it cannot be at the same time fixed point of a loxodromic element [Rat, Thm 5.5.4], we can associate with each w_j, a horosphere S_{w_j} based at w_j such that $S_{w_j}=\gamma_j S_w$. The union of vertex figures

$$V_{[w]}=\bigcup_{j=1}^{l}\gamma_j^{-1}\left(S_{w_j}\cap D\right)\subset S_w$$

is formed by adjacent, bounded euclidean $(n-1)$-polytopes, no two of which have common interior points. The set $V_{[w]}$ represents a fundamental domain for the action of the stabiliser Γ_w of elementary parabolic type on S_w. By conjugation, we may assume $w=\infty$ and the sufficiently small horosphere $S_\infty(r)$ to be at euclidean height $r>0$, i.e

$$S_\infty(r)=\{x\in U^n : x_n=r\}\,.$$

The finite volume condition forces Γ_∞ to be the Poincaré extension of a cocompact discrete subgroup of euclidean isometries. It leaves every horosphere $S_\infty(\rho)$, $\rho<r$ and therefore every horoball $B_\infty(r)$ setwise invariant. The quotient set $C_\infty(r):=B_\infty(r)/\Gamma_\infty$ is an unbounded hyperbolic n-orbifold embedded in H^n/Γ. It is called a *cusp* of H^n/Γ with *cusped point* $[w]$. We conclude that there is a one-to-one correspondence between the conjugacy classes of parabolic isometries in Γ and the cusps of H^n/Γ.

Maximal and canonical cusps. The euclidean height $r>0$ of $S_\infty(r)$ is not uniquely defined. More precisely, the sequence of all sufficiently small horoballs $\left(B_\infty(r)\right)_{r\in(0,d)}$ defines a sequence of nested cusps $\left(C_\infty(r)\right)_{r\in(0,d)}$ in H^n/Γ. An answer to the question about the best choice for r depends on the situation and the results one is interested in. For instance, a lower volume bound for a unbounded orbifold is given by the sum of cusp volumes as long as their interiors are disjoint. Furthermore, we want the cusps in H^n/Γ to be maximal in the sense that no cusp can be enlarged without failing to be embedded. In practice, such a set of *maximal disjoint cusps* is not easy to specify.

Another set of disjoint cusps is the one formed by *canonical* cusps (compare for example [Bea, §7.37], [Shi, Lemma 4 & 5], [Me1], for $n=3$ and [He1], [K4, Lemma 2.7 & 2.8] for arbitrary n). Albeit not necessarily maximal, the canonical cusp has the advantage of an easy construction. By a hyperbolic isometry, we map the fixed point of an associated parabolic element to ∞. We already know that Γ_∞ acts on ∂U^n as a cocompact discrete subgroup of euclidean isometries. Such groups are called crystallographic and are the subject of Chapter 4. There, we explain that Γ_∞ contains non-trivial translations giving rise to the notion of *minimal translation length* $\omega>0$. The particular horoball $B_\infty(\omega)$ at euclidean height ω is called *canonical*, and the *canonical cusp* $C_\infty^{\mathrm{can}}(\omega)=C_\infty(\omega)$ is defined to be its image under projection to H^n/Γ.

2.3 Volume of non-euclidean polytopes

By definition, the volume of an orbifold H^n/Γ equals the volume of a fundamental polytope D for Γ. A good understanding of the volume spectrum for hyperbolic n-orbifolds leads unavoidably to the non-trivial problem of computing the volume of hyperbolic n-polytopes. All results known so far about non-euclidean volume determination are based on ideas developed by L. Schläfli and N. I. Lobachevsky (compare [Sch] and [Lo1], [Lo2]). Good references for the following paragraphs are [Vin1, Part I, Chapter 7], the PhD thesis of Zehrt [Zeh] and the survey article [K3].

Poincaré's angle sum. A volume formula which goes back to Poincaré [Poi, for $n = 2$] links the volume of an even dimensional hyperbolic polytope $P^{2m} = P$ with its *generalized alternating angle sum*

$$W(P) := \sum_{d=0}^{2m} (-1)^d \sum_{F^d \in \mathcal{F}^d(P)} \omega_{n-d-1}\left(F^d \mid P\right) \tag{15}$$

by means of

$$W(P) = (-1)^m \frac{2}{\Omega^{2m}} \operatorname{vol}_{2m}(P). \tag{16}$$

It can be viewed as an angle analog of Euler's Polyhedron Theorem. If D is a fundamental polytope for a discrete group Γ, then the cycle condition (13) transforms $W(D)$ into an alternating sum of stabilizer orders

$$W(D) = \sum_{d=0}^{2m} (-1)^d \sum_{i=1}^{v^d} \frac{1}{g^d_{(i)}}.$$

Poincaré's formula (16) reflects a fundamental difference between even and odd dimensions in connection with non-euclidean volume. Indeed, the alternating angle sum of a polytope always vanishes in odd dimensions and yields no information about volume.

It is interesting that for hyperbolic *manifolds* $M^n = H^n/\Gamma$ of finite volume, the alternating angle sum of a fundamental polytope for Γ equals the *Euler characteristic* $\chi(M^n)$. By following the elementary approach of Hopf in the compact case [Hop], Kellerhals and Zehrt [KZe, Thm 3.3] gave an elementary combinatorial-metrical proof of the Gauß-Bonnet formula

$$\chi(M^n) = \begin{cases} (-1)^m \frac{2}{\Omega^n} \operatorname{vol}_n(M^n) & , n = 2m \\ 0 & , n = 2m+1 \end{cases}. \tag{17}$$

Schläfli's volume differential. The qualitative difference between odd and even dimensions is also reflected by *Schläfli's differential formula*

$$d\operatorname{vol}_n(P) = \kappa \frac{1}{n-1} \sum_{F^{n-2} \in \mathcal{F}^{n-2}(P)} \operatorname{vol}_{n-2}\left(F^{n-2}\right) d\alpha_{F^{n-2}}. \tag{18}$$

Here, the constant κ is -1 or 1 depending on whether P is hyperbolic or spherical. The inductive character of (18) allows to deduce volume formulae for a fixed parity of the dimension.

In the case of **even dimension**, we start with the assumption

$$\mathrm{vol}_0\left(\{\mathrm{point}\}\right) = 1$$

to obtain the well-known hyperbolic defect formula which expresses hyperbolic area as a function of its angles. This procedure was generalized to higher dimensions by Schläfli, the main problem being the determination of the integration constant. The general *reduction formula* expresses even dimensional hyperbolic volumes as functions of odd-dimensional spherical volumes by means of

$$\mathrm{vol}_{2m}\left(P\right) = \frac{\Omega^{2m}}{2}\sum_{j=0}^{m}\sum_{F^{2j}\in\mathcal{F}^{2j}(P)}\sigma^{2j}\left(F^{2j}\right)\omega_{2m-2j-1}\left(F^{2j}\mid P\right). \tag{19}$$

The *Schläfli invariants* $\sigma^{2j}\left(F^{2j}\right)$ are rational constants depending only on the combinatorics of F^{2j}. Zehrt computed the Schläfli invariants for certain even dimensional hyperbolic and spherical polytopes [Zeh, Thm 12.3.1].

In **odd dimensions,** the integration problem is considerably harder. First difficulties already arise in dimension 3. Indeed, even the volume formulae for the combinatorially simplest polyhedra require very complicated functions related to the *classical polylogarithm*

$$\mathrm{Li}_k\left(z\right) = \sum_{r=1}^{\infty}\frac{z^r}{r^k} = \int_0^{\infty}\frac{\mathrm{Li}_{k-1}\left(t\right)}{t}dt, \quad z\in\mathbb{C} \text{ and } |z|\leq 1.$$

As examples we cite the *Lobachevsky function*

$$\text{Л}\left(\alpha\right) = \frac{1}{2}\sum_{r=1}^{\infty}\frac{\sin\left(2r\alpha\right)}{r^2} = -\int_0^{\alpha}\frac{\log\left(1-t\right)}{t}dt = \frac{1}{2}\,\mathrm{Im}\left(\mathrm{Li}_2\left(e^{2i\alpha}\right)\right),$$

the *Riemann zeta function*

$$\boldsymbol{\zeta}\left(k\right) = \mathrm{Li}_k\left(1\right) = \sum_{r=1}^{\infty}\frac{1}{r^k}$$

and the *Dirichlet L-function*

$$L\left(k,d\right) = \sum_{r=1}^{\infty}\binom{r}{d}\frac{1}{r^k},$$

where $\binom{r}{d}$ is the Legendre symbol. While Kellerhals [K2] could at least express the volumes of some combinatorially simple hyperbolic 5-polytopes in terms of the *Trilobachevsky function*

$$\text{Л}_3\left(\alpha\right) = \frac{1}{4}\,\mathrm{Re}\left(\mathrm{Li}_3\left(e^{2i\alpha}\right)\right),$$

the volume problem for $n\geq 7$ is, up to now, completely open.

2.4 Volume of hyperbolic quotients

Volume spectrum. The volume of a hyperbolic n-orbifold is one of its most important topological invariants. The *topological invariance* is a consequence of Gauß-Bonnet in dimension 2, of Gromov-Thurston in dimension $n \geq 3$ ([Thu, Thm 6.2], for example). Understanding the set of possible volumes would constitute a first step on the long way to a complete classification. This chapter tries to give an overview of known results and an idea of the difficulties related to the study of the volume spectrum

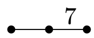
$$V^n := \{\mathrm{vol}_n\,(Q^n) : Q^n \text{ a hyperbolic } n\text{-orbifold}\}\,.$$

In what follows, we restrict ourselves to the finite volume case and often omit the adjective hyperbolic since no confusion is possible.

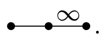
The **2-dimensional** spectrum V^2 is well understood. In 1945, Siegel found a formula which relates the volume of a hyperbolic orbifold H^2/Γ to the orders of stabilizer subgroups of Γ and the number $|\mathcal{I}\,(P)|$ of ideal vertices of a fundamental polygon P for Γ [Sie] (see also [Bea, §10.4 & 10.6]) :

$$\mathrm{vol}_2\left(H^2/\Gamma\right) = -2\pi\left[\sum_{i=1}^{v^0}\left(\frac{1}{g^0_{(i)}} - \frac{1}{2}f^0_{(i)}\right) + 1 - \frac{1}{2}\,|\mathcal{I}\,(P)|\right]. \qquad (20)$$

Substituting integers for the parameters v^0, $g^0_{(i)}$, $f^0_{(i)}$ and $|\mathcal{I}\,(P)|$, he could conclude that the volume of a 2-orbifold is bounded from below by $\frac{\pi}{42}$ and that, up to isometry, the orbifold corresponding to the Coxeter group (see Chapter 3) with graph



is unique to have area $\frac{\pi}{42}$. Moreover, the only non-compact orbifold of smallest possible volume $\frac{\pi}{6}$ is isometric to the quotient H^2/Γ, with Γ the Coxeter group represented by



Siegel's formula (20) generalizes the well-known Gauß-Bonnet formula which gives the volume of oriented compact manifolds M^2 according to

$$\mathrm{vol}_2\left(M^2\right) = 4\pi\,(g-1)\,, \qquad (21)$$

with $g > 1$ the genus of M^2. In contrast to the discrete volume spectrum for manifolds, one can show that V^2 is well-ordered of order type ω^ω.

The **3-dimensional** structure theory for the volume spectrum W^3_o of oriented *manifolds* is developed by Jørgensen and Thurston [Thu, Chapter 5 & 6]. By proving that volumes decrease when Dehn surgery is performed on a cusp of the manifold, they showed that W^3_o is well-ordered of order type ω^ω. The limit points in W^3_o are exactly the volumes of non-compact manifolds. More precisely, the volume of an $(m+1)$-cusped manifold is approximated from below by volumes of m-cusped manifolds. Since any non-orientable manifold is doubly covered by an oriented manifold, we know that W^3 is also well-ordered.

In [DM], Dunbar and Meyerhoff use results of Adams [Ad1, Section 2] to generalize this structure theory to the *orbifold* case. The volume spectrum V^3 is still of order type ω^ω and limit points of V^3 correspond to cusped orbifolds. But this time, the converse is false since there is a second type of so-called rigid cusps which do not allow Dehn-filling. The volumes of non-compact orbifolds all of whose cusps are rigid, represent isolated points in V^3.

The passage from **dimension** 3 to 4 reveals a big surprise. Indeed, by a result of Wang [Wan], V^n is discrete for all $\boldsymbol{n \geq 4}$.

Small volumes. The volume spectrum gives a good idea of the variety of hyperbolic orbifolds. We are are especially interested in the small values of V^n. In dimension 2, Siegel's formula implies a lower bound for the volume of an orbifold and can be used to determine the singleton of minimal volume. The existence of a positive lower bound for arbitrary $n \geq 2$ is a consequence of a result due to Kazhdan and Margulis [KM], who showed that every hyperbolic orbifold contains a ball whose radius depends only on the dimension. This geometric fact implies the following fundamental

Theorem 2 *The volume spectrum V^n contains a positive minimal value for all $n \geq 2$.*

The natural question now asks for this minimal value. Closely related is the problem of constructing all hyperbolic n-orbifolds whose volume equals this minimal value. Is there only one such orbifold up to isometry? In dimension 2, we already know that the answer is yes. We will see that the answer remains yes at least in dimensions $n \leq 9$.

Minimal volume cusped manifolds. The Gauß-Bonnet formula (17) shows that, in particular, the discrete volume spectrum $W^n \subset V^n$ of hyperbolic *manifolds* in even dimensions $2m$ is of the form

$$W^{2m} \subset \frac{\Omega^{2m}}{2}\mathbb{N}. \tag{22}$$

In **dimension 2**, the alternative version (21) of the Gauß-Bonnet formula implies that

$$W^2 = 2\pi\mathbb{N}.$$

Exactly four non-homeomorphic hyperbolic Riemann surfaces have minimal volume 2π, namely the 3-punctured sphere, the 1-punctured torus, the 1-punctured Klein bottle and the projective plane with one handle.

By gluing together the sides of an ideal, regular 24-cell in H^4, Ratcliffe and Tschantz constructed several cusped hyperbolic **4-manifolds** with Euler characteristic 1 and hence of minimal volume $4\pi^2/3$ [RT1]. In addition, they showed

$$W^4 = \frac{4\pi^2}{3}\mathbb{N}.$$

In a joint work with Everitt [ERT], they found cusped hyperbolic **6-manifolds** with Euler characteristic 1, hence minimal volume $8\pi^3/15$. All these manifolds are constructed by gluing copies of an all right-angled Coxeter polytope. In higher even dimensions however, no similar constructions yielding Euler characteristic 1 and hence minimal volume have been realized until now. In fact, we do not even know if $(2m)$-manifolds of Euler characteristic 1 still exist.

Selberg lemma. We come back to the orbifold case. The subset relation (22) allows some conclusions about the even dimensional volume spectrum of hyperbolic orbifolds. Indeed, by the Selberg lemma [Sel] (see also [Rat, Corollary 4 to Thm 7.5.7]), every discrete subgroup $\Gamma < I(H^n)$ has a fixed point free subgroup Γ' of finite index $[\Gamma : \Gamma'] = i$. In other words, every hyperbolic orbifold H^n/Γ is finitely covered by a hyperbolic manifold H^n/Γ' and

$$\mathrm{vol}_n\left(H^n/\Gamma\right) = \frac{\mathrm{vol}_n\left(H^n/\Gamma'\right)}{i}.$$

We deduce

$$V^{2m} \subset \frac{\Omega^{2m}}{2}\mathbb{Q}.$$

Unfortunately the proof of Selberg's lemma is not constructive so that the question of how to determine a covering manifold for a given orbifold remains open and is in general a very hard problem. Likewise, the related problem of finding all possible indices i of fixed point free subgroups of Γ is far from being solved. It is not even known if a manifold of smallest volume covers an orbifold of smallest volume. The volume spectrum in dimension $n \geq 4$ being discrete, we only conclude that in some sense, most indices i are forbidden.

Zehrt's formula. Combining the Schläfli reduction formula (19) and the cycle condition (13), Zehrt generalized Siegel's formula (20) to higher even dimensions [Zeh, Chapter 13]

$$\begin{aligned}\mathrm{vol}_{2m}\left(P^{2m}\right) &= \tfrac{\Omega^{2m}}{2}\sum_{d=0}^{m}\sum_{i=1}^{\upsilon^{2d}}\sum_{F^{2d}_{(i)}\in\mathcal{F}^{2d}_{(i)}(P^{2m})}\sigma^{2d}\left(F^{2d}_{(i)}\right)\omega_{2m-2d-1}\left(F^{2d}_{(i)} \mid P^{2m}\right)\\ &= \tfrac{\Omega^{2m}}{2}\sum_{d=0}^{m}\sum_{i=1}^{\upsilon^{2d}}\sigma^{2d}\left(\mathcal{F}^{2d}_{(i)}\right)\tfrac{1}{g^{d}_{(i)}}.\end{aligned} \tag{23}$$

The last equality uses the fact that the Schläfli invariants $\sigma^{2d}\left(F^{2d}_{(i)}\right)$, depending on the combinatorics of $F^{2d}_{(i)}$ only, are the same for all faces in a given class $\mathcal{F}^{2d}_{(i)}$. By substituting the expressions he found for $\sigma^{2d}\left(\mathcal{F}^{2d}_{(i)}\right)$, Zehrt proved that (23) equals Siegel's formula in dimension 2. In contrast to the 2-dimensional case, it seems that Zehrt's formula (23) does not help to determine hyperbolic orbifolds of minimal volume. As an example, just take a look at the formula giving the volume of a fundamental polytope P for a discrete group Γ in dimension 4 [Zeh, Corollary 13.2.3]:

$$\begin{aligned}\mathrm{vol}_4(P) = \tfrac{\Omega^4}{2}\Bigg[&\sum_{i=1}^{\upsilon^0}\left(\tfrac{1}{g^0_{(i)}} - \tfrac{1}{2}f^0_{(i)}\right) + \\ &+\sum_{i=1}^{\upsilon^2}\left(1 - \tfrac{1}{2}a^0\left(F^2_{(i)}\right)\right)\left(\tfrac{1}{g^2_{(i)}} - \tfrac{1}{2}f^2_{(i)}\right) + 1 - \tfrac{1}{2}\left|\mathcal{I}(P)\right|\Bigg],\end{aligned}$$

where $a^0\left(F^2_{(i)}\right)$ represents the number of ordinary vertices of P.

Volumes of Coxeter simplices. Four our work, the most important result is the list of **all** Coxeter simplex volumes which is due to Johnson, Kellerhals, Ratcliffe and Tschantz [JKRT]. Their results are based on various considerations some of which are described above.

Lower volume bounds. Formulas like (23) are not suited in our quest of the minimal value in V^n. Another approach is by searching lower volume bounds and then improving these bounds step by step. Since unbounded hyperbolic quotients are more frequent in some sense and easier to handle, we restrict our investigations to this case.

A lower volume bound for an unbounded orbifold is given by the volume of the canonical cusp $C_\infty(\omega)$. By means of (3), the volume of $C_\infty(\omega)$ easily computes to

$$\mathrm{vol}_n\left(C_\infty(\omega)\right) = \int_{C_\infty(\omega)} dv = \int_{F\times(\omega,\infty)} \frac{dx_1\cdots dx_n}{x_n^n} = \frac{\mathrm{vol}_{n-1}(F)}{(n-1)\ \omega^{n-1}}, \qquad (24)$$

where $\mathrm{vol}_{n-1}(F)$ is the euclidean volume of a fundamental domain F for Γ_∞ acting on ∂U^n.

3 Coxeter groups

Throughout this chapter, X^n denotes the hyperbolic space H^n, the sphere S^n or the euclidean space E^n of curvature $\kappa = -1, 1$ or 0 respectively. The difficulty of finding fixed point free discrete subgroups of $I(H^n)$ makes the construction of hyperbolic manifolds in dimensions $n > 3$ a very complicated task. Clearly, the situation becomes easier when we omit the condition that the group has to be fixed point free. This chapter describes a special class of discrete subgroups providing easy examples of hyperbolic orbifolds in low dimensions. We restrict ourselves to the finite volume case. A general reference for this chapter is [Vin2] or the more condensed version [Vin1, Part I, Chapter 6 and Part II, Chapter 5].

Coxeter groups, polytopes and graphs. We call *discrete reflection group* every discrete subgroup of $I(X^n)$ generated by reflections with respect to hyperplanes in X^n. In honour of H. S. M. Coxeter who first classified these groups in euclidean and spherical space, a discrete reflection group Γ is said to be a hyperbolic, elliptic or parabolic *Coxeter group* depending on whether κ is -1, 1 or 0. Distinguished among all discrete subgroups by the simplicity of their geometric description, they provide the simplest examples of orbifolds. The family of all mirror hyperplanes associated to the elements of Γ decomposes X^n into polytopes that are permuted under the action of Γ. Any polytope $P \subset X^n$ of the resulting tessellation is a fundamental domain for Γ and Γ is generated by the reflections with respect to the bounding hyperplanes of P [Vin1, Prop 1.4.]. By abuse of language, we say that Γ is generated by the reflections in the sides of P. The polytope P is called *Coxeter polytope associated to* Γ. One can show that a polytope is a Coxeter polytope for some Coxeter group Γ if and only if all of its dihedral angles are submultiples of π, i.e. of the form $\frac{\pi}{k}$, $k \in \mathbb{N}$, $k \geq 2$.

Being a discrete subgroup of $I(X^n)$, the generating reflections of a Coxeter group are adjacency transformations and verify side-pairing and Poincaré relations. We denote the reflection in the side S_i of P by s_i. The side-pairing relations are of the form $s_i^2 = 1$. If π/k_{ij} is the dihedral angle in the ridge $R_{ij} = S_i \cap S_j$, then the cycle of adjacent Γ-images of P around R_{ij} yields the Poincaré relation $(s_i s_j)^{k_{ij}} = 1$. Two parallel sides S_i and S_j are indicated by $(s_i s_j)^{\infty}$ and $k_{ij} = \infty$. Note that s_i and s_j then generate an infinite cyclic group.

Coxeter groups and polytopes are most conveniently described by *Coxeter graphs*. Depending on the situation, a vertex i represents the generating reflection $s_i \in \Gamma$ or the corresponding side $S_i \subset P$. An edge joining two vertices i and j is marked by the integer $2 \leq k_{ij} \leq \infty$ associated to the relation $(s_i s_j)^{k_{ij}}$. To simplify matters, we omit the edges marked 2 and leave an edge unmarked if $k_{ij} = 3$. For completeness, we add that the vertices representing two divergent sides in H^n with a common perpendicular of length l are joined by a dotted edge marked l. However, this case will not occur in our work. In the sequel, we shall use the terms *node* and *branch* for the vertices and edges of graphs in order to avoid confusion with 0- and 1-faces of polytopes.

Classification. Using linear algebra, it is possible to check if the polytope represented by a given Coxeter graph is realizable in X^n. Modulo congruence, this polytope is then uniquely determined. A Coxeter graph is termed spherical, euclidean of hyperbolic depending on whether the polytope it represents is realized in S^n, E^n or H^n.

It turns out that every **spherical** Coxeter polytope is a simplex while a **euclidean** Coxeter polytope of finite volume is a direct product of Coxeter simplices (compare [Cox, § 11.2], for example). Euclidean and spherical Coxeter simplices exist in all dimensions and have been completely classified by Coxeter. Their graphs can be found in [Vin1, Table 1 & 2 p.202], for example.

Note that the Coxeter graph of a spherical simplex $T \subset S^{n-1}$ also represents an unbounded simplicial cone $C \subset E^n$ with apex 0, such that

$$C \cap S^{n-1} = T.$$

Vice versa, the spherical vertex figure T in 0 of a euclidean Coxeter simplex arises by omitting its bounding hyperplane opposite to C. Since each vertex figure of a euclidean or spherical Coxeter simplex is a spherical Coxeter simplex, we conclude that every subgraph obtained from a spherical of euclidean Coxeter simplex graph by removing one of the nodes together with every incident branch, is spherical.

The **hyperbolic** case is much more complicated (see [Vin2], for example). While, up to isometry, a hyperbolic Coxeter polytope of finite volume is uniquely determined by its connected graph, the number of nodes of this graph can be arbitrarily large. Therefore, we cannot hope for a simple classification of these polytopes.

By the same argument as above, the vertex figure in every ordinary vertex is represented by a spherical Coxeter simplex graph. Analogously, the euclidean vertex figure associated to an ideal vertex is determined by a euclidean Coxeter graph. In the neighborhood of every ideal vertex, a hyperbolic Coxeter polytope has therefore the combinatorial structure of a cone over a direct product of euclidean Coxeter simplices.

In low dimensions, hyperbolic Coxeter groups give nice examples of hyperbolic orbifolds. Unfortunately, for n sufficiently large, there are no more Coxeter polytopes and thus no discrete reflection groups of finite covolume in H^n. The bounds $n < 30$ in the compact case and $n < 996$ in the finite volume case are due to Vinberg, Prokhorov and Khovanskij, respectively. Concrete examples however are only known in dimensions $n \leq 8$ and $n \leq 21$ respectively.

It is worth while to note that the classification problem for hyperbolic Coxeter n-simplices is completely resolved. They are described by hyperbolic Coxeter graphs, all of whose subgraphs with n nodes are elliptic or parabolic. In the compact case, hyperbolic Coxeter simplices exist only in dimensions $n \leq 4$. In the finite volume case Coxeter simplices live in dimensions $n \leq 9$, only. A complete list can be found in [Vin1, Table 3 & 4], for example. One has the impression that in some sense, these simplices yield the most symmetric tessellations of H^n. This feeling will be confirmed by the statement of our main theorem in Chapter 7.

Important examples. For $n \geq 2$, the linear Coxeter graph A_n defined by

, n nodes,

represents a spherical simplex in dimension $n-1$. It can be interpreted as the euclidean fundamental cone for the symmetry group of the regular simplex spanned by the centers of $n+1$ mutually touching congruent balls in E^n.

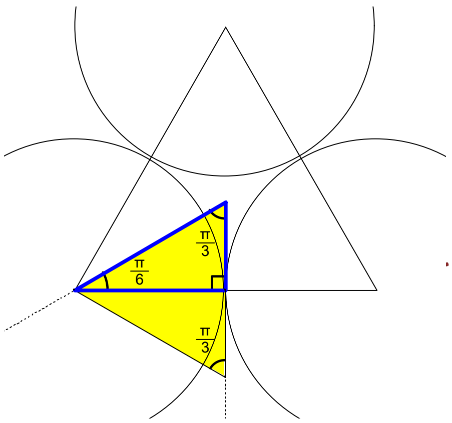


Figure 1

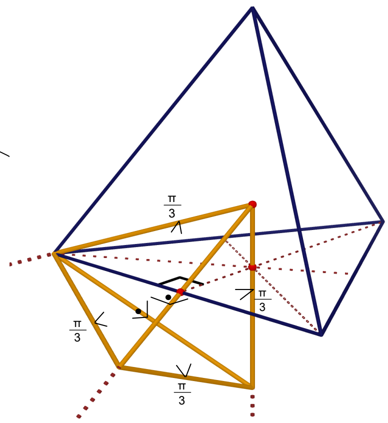
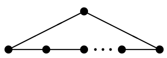


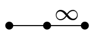
Figure 2

Chopping this cone by a suitable hyperplane and taking the bounded part yields a euclidean simplex with Coxeter graph

, $n+1$ nodes,

as indicated by the yellow color in the Figures 1 and 2 above.

The linear graph

 (25)

describes a right angled hyperbolic triangle T with one ideal vertex symbolized by the euclidean subgraph •$\overset{\infty}{—}$•. Note that the associated Coxeter group induces the symmetry group of the $\mathbb{Z}$-lattice. The spherical subgraphs •—• and • • in (25) represent the dihedral angles $\frac{\pi}{3}$ and $\frac{\pi}{2}$ at the ordinary vertices of T.

We will see in Chapter 7, that the vertex with vertex figure •—• can be interpreted as the center of an (ideal) regular hyperbolic triangle spanned by the (base or) center points of three mutually touching (horo)discs (see Figure 3).

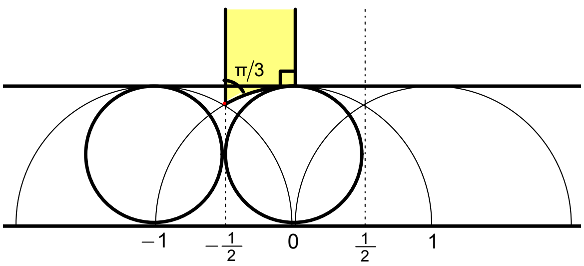
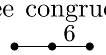
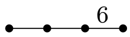


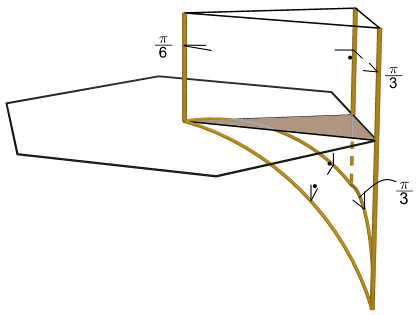
Figure 3

Among all non-cocompact discrete subgroups of $I\left(H^2\right)$, the Coxeter group defined by (25) has minimal covolume (see Chapter 6).

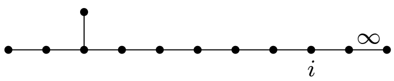
Consider again the euclidean cone defined by A_2. This time we chop it by the edge of the simplex spanned by the centers of the three congruent, mutually touching balls to obtain the euclidean Coxeter triangle  indicated by the blue color in Figure 1. This triangle represents the euclidean vertex figure in the unique ideal vertex of the hyperbolic tetrahedron defined by the Coxeter graph

 . (26)

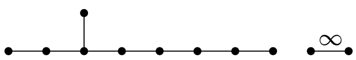
Note that it is also a characteristic simplex for the symmetry group of the hexagonal packing of euclidean discs. Again, one of the spherical subgraphs with three nodes belongs to the A_n-family. We will see in Chapter 6 that the vertex with vertex figure can be interpreted as the center of an (ideal) regular hyperbolic tetrahedron spanned by the base points of four mutually touching (horo)balls. The Coxeter group defined by (26) has minimal covolume among all non-cocompact discrete subgroups of $I\left(H^3\right)$.



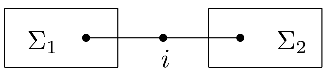
In dimension 10, consider the hyperbolic Coxeter polytope P_1^{10} determined by

 . (27)

As such, it is of simplest combinatorial type: it has twelve sides and exactly one ideal vertex whose 9-dimensional vertex figure is determined by the euclidean subgraph



obtained by discarding the node i. Being disconnected, this vertex figure is a direct product of two euclidean simplices. Both simplices are well known to crystallographers since they are fundamental domains for the affine Weyl group of the E_8-lattice and the $\mathbb{Z}$-lattice respectively. Root lattices will be studied in Chapter 4. The side represented by the node i is the only one which is not incident with the ideal vertex. Using the self-defining abbreviation

 ,

all 9-dimensional spherical vertex figures of P_1^{10} can be found by removing one node from Σ_1 and one node from Σ_2. They are listed in the Appendix A3 where we compute the volume of P_1^{10}. Its volume yields an upper bound for the minimal volume μ_{10} of cusped hyperbolic orbifolds (cf. Chapter 8).

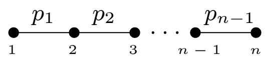
Wythoff's construction. We now describe the so-called *Wythoff construction* which is used to define an important class of polytopes in X^n. In particular, regular polytopes belong to this class.

Consider a spherical Coxeter n-simplex T with associated graph Σ and group Γ. To every side S_i of T corresponds a unique vertex v_i opposite to S_i. We interpret Γ as a subgroup of $I\left(E^{n+1}\right)$ with fixed point 0 and consider the Γ-orbit of v_i. Their convex hull in E^{n+1} defines a euclidean polytope $P_{v_i} = P$. In what follows, we symbolize this procedure by drawing a ring around the node i of Σ. We speak about the *ringed graph* of P.

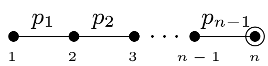
The dimension of P equals the number of nodes in the connected component Σ_i of i in Σ. Moreover, P is contained in the mirror hyperplanes of the reflections represented by the nodes of $\Sigma \setminus \Sigma_i = \Sigma_*$. Therefore, the symmetry group of P is induced by the Coxeter group with graph Σ_i while the Coxeter group with graph Σ_* leaves P pointwise invariant. Finally, the Coxeter graph of the setwise stabilizer $\mathrm{Stab}\,(P)$ of P in Γ is obtained when removing from Σ the node i together with all incident branches.

Throughout this paragraph, let Σ be a ringed, connected, spherical Coxeter graph with n nodes. Then, Σ yields an n-dimensional polytope P. Consider the center c of a side F^{n-1} of P. By construction it is the image under Γ of a vertex $v_j \neq v_i$ of T. For simplicity, we assume $c = v_j$. The side S_j opposite v_j is the only side of T which is not incident with v_j. Hence, the ringed graph σ representing F^{n-1} is derived from the graph Σ by removing the node j together with all incident branches. But the ringed graph σ has to be connected since F^{n-1} has dimension $n-1$. We conclude that the node j is at a free end of Σ. Analogously, the ringed graph representing an $(n-k)$-face is obtained by discarding the node representing its center from the ringed graph of an $(n-k+1)$-face, for $1 \leq k \leq n$.

Regular polytopes. Until now, a ringed Coxeter graph represents a euclidean polytope. This paragraph explains how the same graph can be interpreted to yield a spherical or a hyperbolic polytope as well. First, we apply Wythoff's construction to a spherical Coxeter simplex T with *linear* graph Σ



and corresponding group Γ. The vertex of T which is symbolized by the node i is termed v_i. The euclidean polytope $P_e \subset E^n$ defined by putting a ring around the node n,



is called a *regular* polytope since all its sides and all its dihedral angles are the same. We write α_e for its dihedral angle. Let T_e denote the bounded euclidean simplex obtained when cutting the cone determined by Σ with a euclidean hyperplane orthogonal to the edge $(0, v_n)$. Then

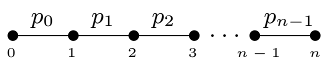
$$P_e = \bigcup_{\gamma \in \Gamma} \gamma T_e,$$

and Σ determines P_e uniquely up to similarity.

Now, by using the natural embedding $S^n \subset S^{n+1}$, we can equally well interpret Σ as a simplicial cone with apex $a \in S^{n+1} \setminus S^n$ and base $T \subset S^n$. Chopping this cone with a spherical hyperplane H_s orthogonal to $(0, v_n)$ yields a bounded spherical simplex T_s. The disjoint union of all Γ-images of T_s builds up a regular spherical polytope P_s. Depending on the distance from H_s to a, the dihedral angle α_s of P_s satisfies $\alpha_s \in (\alpha_e, \pi)$.

Finally, Σ describes also a hyperbolic cone in H^{n+1} with apex $a \in H^{n+1}$ and base $T \subset S^n(a, r)$, $r > 0$. Chop the cone with a hyperbolic hyperplane H_h orthogonal to $(0, v_n)$ to obtain a bounded hyperbolic simplex T_h. The Γ-images of T_h add up to a regular hyperbolic polytope P_h. Depending on the distance from H_h to a, the dihedral angle α_h of P_h satisfies $\alpha_h \in (\alpha_0, \alpha_e)$. Up to isomorphism, all regular polytopes in S^{n+1} and H^{n+1} are uniquely determined by Σ and α_s or α_h. In particular, the minimal dihedral angle α_0 corresponds to the ideal regular polytope defined by Σ. The angles α_e and α_0 are listed in [Vin1, Part II, Chapter 5, Table 2], for example.

Regular tessellations. A regular polytope generates a tessellation $\mathcal{T}$ of X^n if its dihedral angle is of the form $\frac{2\pi}{p_0}$, with p_0 an integer. In that case, the Coxeter group with graph Σ'



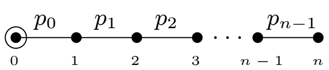
induces the symmetry group of $\mathcal{T}$. Note that Σ' is obtained by adding a node to the graph Σ defined above and joining it to the node 1 by a branch marked p_0. Up to congruence, a regular tessellation of E^n, S^n or H^n is therefore uniquely determined by a linear euclidean, spherical or hyperbolic Coxeter graph with the following property

> deleting any node of the graph Σ' yields an elliptic graph. Only in the case of an hyperbolic tessellation with ideal regular polytopes, the graph with the node n removed is parabolic.

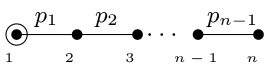
Regular hyperbolic tessellations are classified and listed in [Vin1, Part II, Chapter 5, Table 6]. For compact polytopes, they only exist in dimensions $n \leq 4$. Tessellations with ideal regular polytopes are realizable in dimensions $n \leq 5$ and represented in the following table.

n	2	3	4	5
σ	α ∞ , $3 \leq \alpha < \infty$	β 3 6 , $3 \leq \beta \leq 5$ 3 4 4	3 4 3 4	3 3 3 4 3

Vertex figure of a lattice. In the spirit of Wythoff's construction, ringing the node 0 of the linear *parabolic* Coxeter graph Σ' provides a lattice Λ in E^n:



The lattice points are precisely the centers of the copies of P (determined by Σ) in the tessellation $\mathcal{T}$. For simplicity, assume that the center of P is $0 \in \Lambda$. A vector of minimal norm in Λ is the center of a copy adjacent to P in $\mathcal{T}$. Their convex hull $V(0)$ is a polytope called *vertex figure of* Λ *in* 0. Remember that the symmetry group Γ of P is given by the Coxeter group with graph Σ. Up to magnification by a factor 2, $V(0)$ is the polytope with vertices Γv_1. The ringed graph of $V(0)$ is thus obtained by putting a ring around the node 1 of Σ.



Forcing Σ to be linear is very restrictive. Indeed, there is just one regular tessellation in E^n for $n > 4$, namely the one with hypercubes. Fortunately all the statements of this paragraph still hold for *Coxeter trees* under the condition that the node 0 is joined to only *one* other node 1. In particular, all root lattices can be represented by a ringed graph. This simplifies the notation in Chapter 4. In fact, the polytopes whose combinatorics we determine in Appendix A1 are all vertex figures of root lattices.

Combinatorics. In this paragraph, we explain how the representation by a ringed Coxeter tree can be used to determine the combinatorics of the vertex figure $V(0)$ (compare [Cox, §11]). More precisely, we are interested in the number N_m^{n-k} of $(n-k)$-faces in $V(0)$ which are equivalent to $F_m^{n-k} \subset V(0)$ under the action of Γ. Elementary group theory yields

$$N_m^{n-k} = \frac{|\Gamma|}{\left|\mathrm{Stab}\left(F_m^{n-k}\right)\right|},$$

where $\mathrm{Stab}\left(F_m^{n-k}\right)$ is the setwise stabilizer of F_m^{n-k} in Γ. As specified above, the graph of $\mathrm{Stab}\left(F_m^{n-k}\right)$ is obtained by removing the node representing the center of F_m^{n-k} from the original graph Σ. The orders of elliptic Coxeter groups are listed in [Vin1, Part II, Chapter 5, Table 1], for example. In Appendix A1, we present six examples of polytopes whose combinatorics are needed in the proof of our main theorem.

4 Crystallography

The shape of cusp boundaries is related to properties of crystallographic groups. For our work, a good understanding of the geometric properties in crystallography is therefore crucial. Good references are [Bou], [CS] and [BrB].

Euclidean lattices. An n-dimensional euclidean *lattice* is the set of all integral linear combinations of n linearly independent vectors $\varepsilon_1, \ldots, \varepsilon_n$ in euclidean m-space E^m, $m \geq n$, i.e.

$$\Lambda = \mathbb{Z}\varepsilon_1 \oplus \ldots \oplus \mathbb{Z}\varepsilon_n.$$

In what follows, we often identify Λ with the discrete group of translations along its lattice vectors, that is with a subgroup of $I(\text{span}(\varepsilon_1, \ldots, \varepsilon_n)) \simeq I(E^n)$. A fundamental domain for Λ is given by the parallelotope

$$P_\Lambda := \left\{ \sum\nolimits_{i=1}^{n} \lambda_i \varepsilon_i,\ 0 \leq \lambda_i \leq 1 \right\}.$$

Its volume may be expressed in terms of the Gram matrix $G_\Lambda = (\langle \varepsilon_i, \varepsilon_j \rangle)_{i,j}$ according to

$$\text{vol}_n(P_\Lambda) = \sqrt{\det(G_\Lambda)}. \tag{28}$$

Note that a basis change matrix A for Λ has integral entries only and $\det(A) = \pm 1$. Such a matrix $A \in GL(n, \mathbb{Z})$ is called *unimodular.* Since $\det(G_\Lambda) = \det\left(AG_\Lambda A^{-1}\right)$ is independent of the lattice basis, the *volume* of Λ is well defined by

$$\text{vol}_n(\Lambda) := \text{vol}_n(P_\Lambda).$$

Dual lattices. The lattice generated by the dual vectors $\varepsilon_1^*, \ldots, \varepsilon_n^*$ is called *dual lattice* and denoted by

$$\Lambda^* = \{x \in E^n : \langle x, y \rangle \in \mathbb{Z},\ \forall y \in \Lambda\}.$$

Plugging the well-known identity $G_\Lambda G_{\Lambda^*} = I$ from linear algebra into (28) yields

$$\text{vol}_n(\Lambda^*) = \frac{1}{\text{vol}_n(\Lambda)}. \tag{29}$$

A lattice whose Gram matrix has integral entries only is termed *integral* lattice. Integral lattices can be interpreted as sublattices of their duals and, according to (29), satisfy

$$\text{vol}_n(\Lambda) = \sqrt{[\Lambda^* : \Lambda]}. \tag{30}$$

Note that integral self-dual lattices are unimodular by means of (30). Prominent examples of unimodular lattices are the cubic lattice $\mathbb{Z}^n$ and the root lattices D_4 and E_8.

Automorphism groups. Every lattice Λ comes with its *automorphism group* $\text{Aut}(\Lambda)$ that consists of all euclidean isometries fixing the origin and mapping Λ onto itself. Since Λ is discrete, $\text{Aut}(\Lambda)$ is a finite subgroup of $O(n)$. One can show that dual lattices have the same automorphism group, i.e.

$$\text{Aut}(\Lambda^*) = \text{Aut}(\Lambda).$$

Adjoining the translations along lattice vectors to $\mathrm{Aut}(\Lambda)$ yields the *affine automorphism group* $\mathrm{Aut}_a(\Lambda)$ of Λ. We remark that $\mathrm{Aut}_a(\Lambda)$ is the group of all distance preserving euclidean transformations mapping Λ onto itself, that is, it equals the symmetry group of Λ.

Root lattices. An integral lattice Λ is said to be *even* if all its vectors have even squared norm, that is

$$\langle x, x\rangle \equiv 0 \pmod 2, \ \forall x \in \Lambda.$$

The *minimal vectors* of norm $\omega = \sqrt{2}$ are the *roots* of Λ. An even integral lattice generated by its roots is called a *root lattice.* Chapter VI of [Bou] presents an introduction to their theory. It turns out that any root lattice Λ admits a *fundamental basis* $\alpha_1, \dots, \alpha_n$ verifying

$$\langle \alpha_i, \alpha_i\rangle = 2, \forall i \in \{1, \dots, n\} \qquad \text{and} \qquad \langle \alpha_i, \alpha_j\rangle \in \{0, -1\}, \forall i \neq j$$

and is a direct sum of irreducible ones:

$$\Lambda = \Lambda_1 \oplus \dots \oplus \Lambda_k.$$

Among the euclidean lattices, root lattices are distinguished by extremal properties. In this context, it is preferable to normalize the situation in order to have *minimal norm* $\omega = 1$.

Therefore, roots have minimal norm 1 in what follows and the elements of a fundamental root basis $\alpha_1, \dots, \alpha_n$ satisfy

$$\langle \alpha_i, \alpha_i\rangle = 1, \forall i \in \{1, \dots, n\} \qquad \text{and} \qquad \langle \alpha_i, \alpha_j\rangle \in \left\{0, -\frac{1}{2}\right\}, \forall i \neq j. \tag{31}$$

All vectors x of a root lattice then satisfy

$$\langle x, x\rangle \in \mathbb{N}. \tag{32}$$

Weyl groups. To every root α, we associate the family of parallel hyperplanes

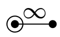
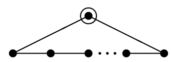
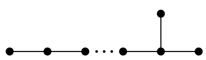
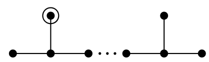
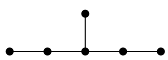
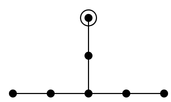
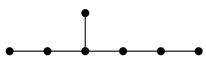
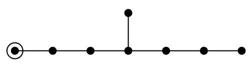
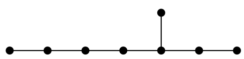
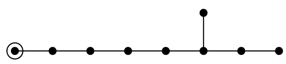
$$P_{\alpha,k} := \left\{\langle x, \alpha\rangle = \frac{k}{2}\right\}, \ k \in \mathbb{Z},$$

orthogonal to α and passing through $\frac{k}{2}\alpha$. The reflection $\rho_{\alpha,k}$ with respect to $P_{\alpha,k}$ equals the reflection in $P_{\alpha,0}$ followed by k translations along the root α :

$$\rho_{\alpha,k}(x) = x - 2\left[\langle x, \alpha\rangle - \frac{k}{2}\right]\alpha = \rho_{\alpha,0} + k\alpha. \tag{33}$$

From now on, let $\mathcal{R}$ denote the set of roots associated to an *irreducible* root lattice Λ. The finite subgroup of $\mathrm{Aut}(\Lambda)$ generated by the reflections $\rho_{\alpha,0}$, $\alpha \in \mathcal{R}$, is named the *Weyl group* $W(\Lambda)$ of Λ. The union of all hyperplanes $P_{\alpha,0}$, $\alpha \in \mathcal{R}$, partitions E^n into simplicial cones with apex 0. Every such cone C is a fundamental domain for $W(\Lambda)$. Moreover, the roots associated to the bounding hyperplanes of C form a fundamental root basis for Λ [Bou, Chapter VI, n°1.5]. By means of property (31), the vertex figure in 0 of C is a simplex described by a connected spherical Coxeter graph all of whose branches are unmarked.

Affine Weyl groups. Adjoining the translations along root vectors to $W(\Lambda)$ yields the infinite *affine Weyl group* $W_a(\Lambda)$. Formula (33) shows that $W_a(\Lambda)$ is generated by the reflections with respect to the hyperplanes $P_{\alpha,k}$, $\alpha \in \mathcal{R}$, $k \in \mathbb{Z}$. The union of all $P_{\alpha,k}$ provides a tiling of E^n with simplices. Every simplex T defines a fundamental domain for $W_a(\Lambda)$. Those with vertex 0 are obtained by chopping one of the fundamental cones C for $W(\Lambda)$ with the hyperplane $P_{\hat{\alpha},1}$ associated to the highest root $\hat{\alpha}$ corresponding to C. We will not need a precise definition of the highest root in what follows. The interested reader is referred to [Bou, Chapter VI, n°1.8 & 2.1]. Much more important for us is the fact that T is a euclidean Coxeter simplex whose graph is obtained by adding a node representing $P_{\hat{\alpha},1}$ and one or two unmarked branches to the graph of C. As explained at the end of Chapter 3, this euclidean graph with a ring around the additional node describes Λ. The following list gives all connected spherical Coxeter graphs defining a Weyl group and the ringed graphs for the associated root lattices (cf. [CS, Table 4.1], for example).

Root lattice	Weyl group	ringed graph
A_1		
A_n $n \geq 2$ nodes		
D_n $n \geq 4$ nodes		
E_6		
E_7		
E_8		

Crystallographic groups. By definition, *crystallographic groups* are cocompact discrete groups of euclidean isometries. They describe the symmetries of lattices and have been characterized algebraically by Ludwig Bieberbach in [Bie]. In particular, affine Weyl and automorphism groups are crystallographic groups. The following paragraph summarizes some of Bieberbach's results.

A (linear) isometry of E^n is of the form $(a, A) := (x \mapsto Ax + a)$, with $a \in \mathbb{R}^n$ and $A \in O(n)$. Every crystallographic group $\Gamma < I(E^n)$ therefore comes with two important groups. The first one is its normal subgroup of translations $\Lambda \lhd \Gamma$ identifed with the corresponding *euclidean lattice*. It is the kernel of the canonical homomorphic projection

$$\begin{array}{cccc} p: & \Gamma & \longrightarrow & O(n) \\ & (a, A) & \longmapsto & A \end{array}$$

mapping each element $(a, A) \in \mathbb{R}^n \rtimes O(n)$ to its differential. The second group, called the *point group* ϕ of Γ, is defined as the image of p. Let $\iota : \Lambda \hookrightarrow \Gamma$ denote the natural injection $a \mapsto (a, I)$, I the identity matrix as usually. Then we have an exact sequence of groups

$$1 \to \Lambda \overset{\iota}{\hookrightarrow} \Gamma \overset{p}{\to} \phi \to 1. \tag{34}$$

Characterized as the unique maximal free abelian subgroup of Γ, the lattice Λ must have rank n and finite index $[\Gamma : \Lambda] = |\phi|$ in Γ [Rat, Thm. 7.4.2]. On the other hand, ϕ may be identified with a subgroup of $\mathrm{Aut}\,(\Lambda)$ by means of $A \mapsto (0, A)$.

Two crystallographic groups Γ_1 and Γ_2 are isomorphic if and only if there is an affine transformation $\alpha \in \mathbb{R}^n \rtimes O\,(n)$ such that

$$\Gamma_2 = \alpha^{-1}\Gamma_1\alpha.$$

In what follows, we always suppose that 0 belongs to Λ and use a lattice basis rather than the standard basis to express the elements of Γ. With respect to this basis, Λ is just the lattice of integers $\mathbb{Z}^n$. Every element of the point group ϕ becomes a unimodular matrix $A \in GL\,(n, \mathbb{Z})$. Therefore, two crystallographic groups Γ_1 and Γ_2 of the same order are isomorphic if and only if there is a vector $u \in \mathbb{R}^n$ and a unimodular matrix $U \in GL\,(n, \mathbb{Z})$ such that the representation $(g_1, G_1) \in \mathbb{Z}^n \rtimes GL\,(n, \mathbb{Z})$ of every element in Γ_1 is conjugate to the representation $(g_2, G_2) \in \mathbb{Z} \rtimes GL\,(n, \mathbb{Z})$ of an element in Γ_2 according to

$$(u, U)^{-1}\,(g_1, G_1)\,(u, U) = (g_2, G_2)\,.$$

Classification of crystallographic groups. In 1911, Bieberbach answered affirmatively Hilbert's 18th problem by proving that in every dimension n, the number of isomorphism classes of crystallographic groups is finite. For an elementary proof of Bieberbach's theorem, see [Bus]. A complete classification exists only in dimensions $n \leq 4$. It is due to Fedorov and Schönflies in dimensions 2 and 3, to Brown, Bülow, Neubüser, Wondratschek and Zassenhaus in dimensions 4 (compare e.g. [Bur] for $n = 2$ or 3 and [BrB], for $n = 4$). Fortunately, the classification in isomorphism classes is more than sufficient for our needs. In fact, a lot of information about a crystallographic group is encoded in its finite point group. The classification into arithmetic and geometric crystal classes plays on this assertion.

Two crystallographic groups belong to the same *arithmetic crystal class* or simply $\mathbb{Z}$-*class* if their point groups ϕ_1 and ϕ_2 are conjugate according to

$$\exists U \in GL\,(n, \mathbb{Z}) \text{ such that } \phi_2 = U\phi_1 U^{-1}. \tag{35}$$

It is important to note that every $\mathbb{Z}$-class consists of full isomorphism classes. If we reduce even more the importance of the lattice vectors by allowing the basis change matrices to have real entries, then the set of all $\mathbb{Z}$-classes splits into *geometric crystal classes* or simply $\mathbb{Q}$-*classes*. More precisely, two crystallographic groups belong to the same $\mathbb{Q}$-class if their point groups ϕ_1 and ϕ_2 satisfy

$$\exists U \in GL\,(n, \mathbb{R}) \text{ such that } \phi_2 = U\phi_1 U^{-1}.$$

The name of $\mathbb{Q}$-class is justified because the existence of a real such matrix U implies automatically the existence of a rational matrix with the same properties.

Symmorphic groups. Remember that the affine automorphism group $\mathrm{Aut}_a(\Lambda)$ of an n-dimensional euclidean lattice Λ is the crystallographic group that contains all euclidean isometries mapping Λ onto itself. In other words, $\mathrm{Aut}_a(\Lambda)$ is the semi-direct product of $\mathbb{Z}^n$ with $\mathrm{Aut}(\Lambda)$:

$$\mathrm{Aut}_a(\Lambda) = \mathbb{Z}^n \rtimes \mathrm{Aut}(\Lambda)$$
$$(a, A) \circ (b, B) = (a + Ab, AB)\,.$$

A crystallographic group Γ defined as the semi-direct product of $\mathbb{Z}^n$ with a subgroup of $\mathrm{Aut}(\Lambda)$ is called *symmorphic*. Some authors prefer the name of *split group* instead of symmorphic group in order to point out that the exact sequence (34) splits for Γ, meaning that a homomorphism $q : \mathrm{Aut}(\Lambda) \to \Gamma$ exists, with $p \circ q = id$. A symmorphic group is characterized by the property to contain for each motion also its linear and translational parts. Let us point out, that in an equivalent but more geometric way, a crystallographic group Γ is symmorphic if and only if the Γ-orbit of 0 equals the lattice Λ.

It is important to note that not every crystallographic group is symmorphic. We cite the following example in dimension 2 : consider the dihedral group of order 8 generated by the point inversion $-I$, the rotation R of $\pi/2$ around 0 and the reflection C in the x-axis. Then the crystallographic group Γ generated by $(-I, 0)\,,\ (R, 0)\,,\ (C, (1/2, 1/2))$ and $(I, u)\,,\ u \in \mathbb{Z}^2$, is not symmorphic. In fact, the underlying lattice is the square lattice $\mathbb{Z}^2$ but the Γ-orbit of 0 contains besides the lattice points, also the midpoints $\Lambda\,(1/2, 1/2)$ of each square.

Bravais types. The elements of an isomorphism class are either all symmorphic or all non-symmorphic. Among the isomorphism classes belonging to a given $\mathbb{Z}$-class, there is exactly one class of symmorphic elements. In particular, given a lattice Λ, there is one $\mathbb{Z}$-class that contains the isomorphism class of $\mathrm{Aut}_a(\Lambda)$. This class is called *Bravais $\mathbb{Z}$-class* for Λ. Two lattices are said to be of the same *Bravais type* if they both determine the same Bravais $\mathbb{Z}$-class. One thus obtains a classification of the infinitely many lattices of fixed dimension into a finite number of Bravais types. In dimension 3, this definition yields the famous 14 Bravais lattices first enumerated by Auguste Bravais in 1849.

Normalized crystallographic groups. The covolume of a crystallographic group Γ equals the volume of the associated lattice Λ divided by the order of its point group ϕ, i.e.

$$\mathrm{covol}(\Gamma) = \frac{\mathrm{vol}_n(\Lambda)}{|\phi|}.$$

Since similar lattices Λ and $\beta\Lambda + b$, $\beta \in \mathbb{R}^*$, $b \in \mathbb{R}^n$ have the same point group ϕ, a normalization to minimal norm $\omega = 1$ in Λ justifies to speak about the crystallographic group of *smallest covolume* in dimension n. Indeed, having a finite number of Bravais types and $\mathbb{Z}$-classes only, and every point group having finite order, a normalized crystallographic group of smallest covolume exists. We will see that, at least in dimensions $n \leq 8$, it is uniquely determined if we restrict ourselves to symmorphic groups.

Biggest and second biggest order of a $\mathbb{Q}$-class. Denote by φ_n, ψ_n the biggest, respectively second biggest order of a point group in dimension n. For $n \leq 4$, the classification of crystallographic groups yields φ_n and ψ_n in an explicit way(see [BrB, Fig 5, Fig 6, Fig 7], for example).

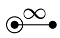
For $5 \leq n \leq 9$, we lack of a complete classification but Plesken and Pohst listed all $\mathbb{Q}$-classes of maximal finite irreducible subgroups of $GL(n, \mathbb{Z})$ together with the $\mathbb{Z}$-classes they contain (see [PP1] for $n = 5, 7$, [PP2] for $n = 6$, [PP3] for $n = 9$ and [PP4] for $n = 8$). Reducible matrix groups are direct products of irreducible ones. Their order is obtained by multiplying the orders of their irreducible components. Starting with the maximal order of a representative of an irreducible $\mathbb{Q}$-class taken from the work of Plesken and Pohst and comparing with the orders of reducible matrix groups, we compute φ_n for $5 \leq n \leq 9$.

The representing element of the $\mathbb{Q}$-class of maximal order φ_n is denoted by G_n. It turns out that for $n \leq 8$, G_n is actually irreducible. However, φ_9 is the order of the reducible automorphism group $W(\mathbb{Z}) \times W(E_8)$ of the root lattice $\mathbb{Z} \oplus E_8$.

The determination of ψ_n is analogous but slightly more difficult. Not only the representatives of irreducible $\mathbb{Q}$-classes with second biggest order ψ_n' but also the subgroups of G_n have to be considered. Since all G_n are reflection groups, they contain a subgroup of index 2 generated by the elements of determinant 1. The computation of ψ_n thus involves ψ_n' and $\varphi_n/2$. For $n = 1, 3, 4, 6, 7, 8$ or 9, one shows that $\psi_n = \varphi_n/2$. In dimension 2, the irreducible dihedral group of order 8, which is the automorphism group of the square lattice $\mathbb{Z}^2$ has second biggest order. For $n = 5$, the reducible automorphism group of the root lattice $\mathbb{Z} \oplus D_4$ has order $\psi_5 = 2 \cdot 1152 = 2304$. The results are listed in the tables of Appendix A2.

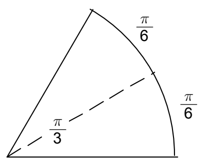
Lattices of maximal symmetry. In this paragraph, we determine all lattices L_i^n of fixed dimension having maximal symmetry. Note that a lattice in dimension n has *maximal symmetry* if its automorphism group equals G_n. In some dimensions, only one such lattice exists up to Bravais type. In that case, we omit the index i which, in general, enumerates distinct Bravais types.

In **dimension 1**, there is only one Bravais type L^1, namely the $\mathbb{Z}$-lattice. It has automorphism group $G_1 = \mathbb{Z}_2$ and can be described by the ringed graph

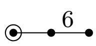


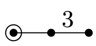
.

In **dimension 2**, the $\mathbb{Q}$-class of maximal order contains one $\mathbb{Z}$-class (cf. [BrB, Table 1A], for example). We conclude that there is only one Bravais lattice L^2 of maximal symmetry G_2, namely the selfdual hexagonal lattice A_2. The automorphism group of A_2 is the dihedral group of order 12 which contains the Weyl group $W(A_2)$ as a subgroup of index 2. Geometrically, this is interpreted by the following bisection of the spherical vertex figure of the fundamental cone for $W(A_2)$:



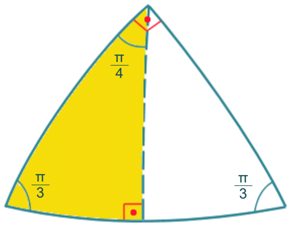
The ringed Coxeter graphs

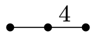


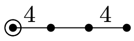
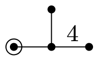
and


both describe the hexagonal lattice L^2.

In **dimension 3**, the $\mathbb{Q}$-class of maximal order contains three $\mathbb{Z}$-classes (cf. [BrB, Table 1B], for example). We conclude that there are three Bravais types of maximal symmetry G_3, the so-called cubic lattices. The Weyl group of the root lattice $D_3 = A_3$ has order $24 = \varphi_3/2$. However, the spherical vertex figure in 0 of its fundamental cone



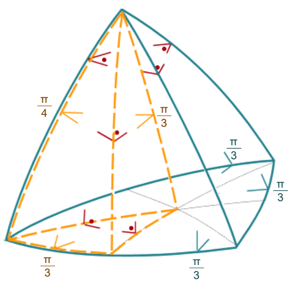
can be bisected into two congruent triangles with Coxeter graph  [Cox, §11.7]. This Coxeter graph thus represents G_3. There exist two different ringed euclidean Coxeter graphs describing euclidean 3-lattices with automorphism group G_3:

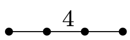
 and  .

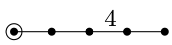
The left graph determines the *simple cubic lattice* $\mathbb{Z}^3$ (*sc-lattice*) while the right graph stands for the *face-centered cubic lattice* (*fcc-lattice*). Both lattices can be seen as generalized root lattice (cf. [Bou, Chapter VI], for example). The dual of the fcc-lattice is the *body-centered cubic lattice* (*bcc-lattice*). It is well-known that the three cubic lattices represent distinct Bravais types ([Max, Table 1], for example). Hence,

$$L_1^3 = \text{fcc-lattice},\ L_2^3 = \text{bcc-lattice},\ L_3^3 = \text{sc-lattice}.$$

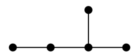
In **dimension 4**, the $\mathbb{Q}$-class of maximal order consists of one $\mathbb{Z}$-class only (cf. [BrB, Table 1C], for example). The Bravais type L^4 of maximal symmetry is represented by the self-dual root lattice D_4. Note that $[\mathrm{Aut}\,(D_4) : W\,(D_4)] = 6$. The geometric reason is given by the following dissection of the spherical vertex figure in 0 of a fundamental cone for $W\,(D_4)$



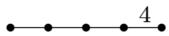
into six congruent orthoschemes with Coxeter graph  . We conclude that the ringed graph for L^4 is given by



Dimension 5 is similar to dimension 3. The $\mathbb{Q}$-class of maximal order splits into three $\mathbb{Z}$-classes yielding three Bravais types of lattices [PP1]. The Weyl group of the D_5-lattice has order $1920 = \varphi_5/2$. The spherical vertex figure in 0 of its fundamental cone is represented by the Coxeter graph

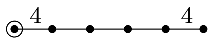


and bisects into two congruent tetrahedra with graph

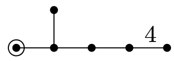


(36)

(cf. [Cox, §11.7], for example). Hence, (36) determines G_5. Again, there are two ringed euclidean graphs describing a lattice with vertex figure (36) in 0 :



and

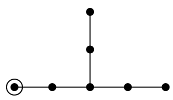


.

The left graph stands for the simple cubic lattice $\mathbb{Z}^5$ while the right graph represents the generalized fcc-lattice whose dual is the generalized bcc-lattice. By means of Table 1 in [Max], the three cubic lattices correspond to different Bravais types. We conclude that

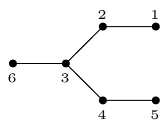
$$L_1^5 = \text{fcc-lattice},\ L_2^5 = \text{bcc-lattice},\ L_3^5 = \mathbb{Z}^5.$$

In **dimension 6**, the $\mathbb{Q}$-class of maximal order contains two $\mathbb{Z}$-classes [PP2]. Every element of this $\mathbb{Q}$-class is isomorphic to the $\mathbb{Z}_2$-extension of $W(E_6)$. By Table 1 of [Max], the two Bravais types of symmetry $W(E_6)$ correspond to the E_6-lattice described by the ringed graph



(37)

and its dual lattice. The theory of root lattices tells us that the quotient group $\mathrm{Aut}(E_6)/W(E_6)$ is isomorphic to the automorphism group of the Coxeter graph



(38)

which describes $W(E_6)$ [Bou, Chapter VI, n°4.2, Corollary to Prop 1]. Beside the identity, (38) admits the involutory automorphism ι permuting the nodes according to

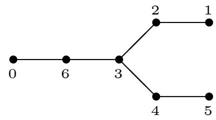
$$1 \longleftrightarrow 5,\ 2 \longleftrightarrow 4,\ 3 \longleftrightarrow 3 \text{ and } 6 \longleftrightarrow 6. \tag{39}$$

Hence, $\mathrm{Aut}(E_6)/W(E_6)$ is isomorphic to $\mathbb{Z}_2$ and $\mathrm{Aut}(E_6)$ equals G_6. We conclude that

$$L_1^6 = E_6\text{-lattice},\ L_2^6 = E_6^*\text{-lattice}.$$

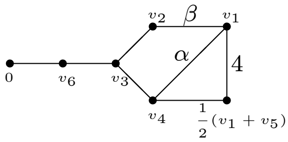
To describe a fundamental simplex for $\mathrm{Aut}_a(E_6)$, we need to understand the action of the isometry ε induced by ι in E^6. First, we point out that $\varepsilon \in \mathrm{Aut}(E_6)$. Indeed, beeing a lattice, L_1^6 admits the symmetry -1. Moreover, we know that the Weyl group of a root lattice contains an element w_0 mapping a fundamental cone C for its action onto its negative $-C$ [Bou, Chapter VI, n°1.6, Corollary 3 to Prop 17]. Since $-1 \notin W(E_6)$ [Bou, Chapter V, n°6.2, Corollary 3 to Prop 3 & Chapter VI, n°4.12 (X)], we deduce that $w_0 = -\varepsilon$ and the assertion follows.

Every node i in (38) represents a hyperplane $P_{\alpha_i,0}$ bounding C. Therefore, ε is a symmetry of the E_6-lattice which permutes the fundamental roots $\alpha_1, \ldots, \alpha_6$ and the associated hyperplanes $P_{\alpha_i,0}$ according to (39). The unringed graph

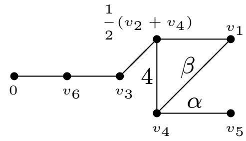


(40)

determines a fundamental simplex T for $W_a(E_6)$ obtained by chopping C with an additional hyperplane. Let v_i be the vertex opposite the side $P_{\alpha_i,0}$, for $i = 1, \ldots, 6$. Then, ε fixes the vertex 0 and permutes the vertices $v_1, \ldots, v_6$ according to (39). Denote by s_1 the reflection with respect to the hyperplane S_1 passing through the points $0, v_1, v_3, v_4, v_5, v_6$ and $\frac{1}{2}(v_2 + v_4)$, by s_2 the reflection with respect to S_2 through $0, v_1, v_2, v_3, v_4, v_6$ and $\frac{1}{2}(v_1 + v_5)$. Since $\varepsilon = s_1 \circ s_2$, a fundamental simplex T_0 for $\mathrm{Aut}_a(E_6)$ is obtained when bisecting T by means of S_1 or S_2. It can be represented by one of the following *weighted graphs*:



or

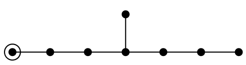


where $\alpha = \arccos\left(\frac{1}{2\sqrt{2}}\right)$ and $\beta = \pi - \alpha$. Here, the nodes represent the vertices or the sides of T_0 but not the corresponding reflections. A non-integral weight symbolizes a dihedral angle formed by two sides. Note that $\mathrm{Aut}_a(E_6)$ is no more generated by reflections. In particular, it does not contain the reflections s_i, $i = 1, 2$. Clearly, $\mathrm{Aut}_a(E_6)$ equals the $\mathbb{Z}_2$-extension of $W_a(E_6)$.

In **dimension 7**, the $\mathbb{Q}$-class of maximal order splits into two $\mathbb{Z}$-classes yielding two Bravais types [PP1]. The automorphism group of both lattices is $W(E_7)$. Using Table 1 of [Max], we have

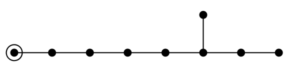
$$L_1^7 = E_7\text{-lattice}, \; L_2^7 = E_7^*\text{-lattice}.$$

The ringed Coxeter graph for L_1^7 is given by



.

In **dimension 8**, the $\mathbb{Q}$-class of maximal order contains only the $\mathbb{Z}$-class of the Weyl group $W(E_8)$. The unimodular E_8-lattice is represented by the ringed Coxeter graph



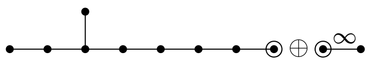
(compare [PP4]). As indicated in Table 1 of [Max],

$$L^8 = E_8\text{-lattice}.$$

Dimension 9 uses the results of dimension 1 and 8. The $\mathbb{Q}$-class of maximal order is the automorphism group of the reducible root lattice $\mathbb{Z} \oplus E_8$. Since $\text{Aut}\,(\mathbb{Z} \oplus E_8) = W\,(\mathbb{Z}) \times W\,(E_8)$, we have

$$L^9 = \mathbb{Z} \oplus E_8\text{-lattice.}$$

The lattice of maximal symmetry L^9 is described by the Coxeter graph



Here, be aware of the unusual interpretation of the non-connected Coxeter graph. In fact, it represents the direct sum of the *lattices* $\mathbb{Z}$ and E_8. However the non-connectedness does not imply that the space containing E_8 is orthogonal to the line containing $\mathbb{Z}$.

Geometry of the most symmetric lattices. Irreducible root lattices and their duals are well understood. References are [CS, Chapter 4 and 21] and [Bou]. An overview can be found in the tables of Appendix A2. The rest of this paragraph explains how the motivated reader can verify these data.

Fundamental roots $\alpha_1, \ldots \alpha_n$ for L_i^n and the corresponding highest root $\hat{\alpha}$ can be found in the references mentioned above. We normalize the situation to have minimal norm 1 in L_i^n and use the so-called even coordinate system for E_8. A fundamental simplex T_0 for $\text{Aut}_a\,(L_i^n)$ is bounded by the hyperplanes $P_{a_i,0}$, $i = 1, \ldots, n$, and $P_{\hat{\alpha},1}$. The fundamental simplices given in chapter 21 of [CS] are further normalized and dissected as described in the previous paragraph. Finally, the intersection of all but one hyperplane equals the vertex opposite the omitted plane. The Gram matrix $G = (\langle \alpha_i, \alpha_j \rangle)_{i,j}$ yields the volume of the normalized lattice L_i^n by means of

$$\text{vol}_n\,(L_i^n) = \sqrt{\det\,(G)}.$$

Covering radius and deep holes. Important quantitative invariants which come with a lattice Λ are its *packing radius* r_Λ and *covering radius* R_Λ. The packing radius is the maximal radius for congruent but non-overlapping balls centered in the lattice points. It equals half the minimal norm ω of the lattice. The covering radius stands for the minimal radius that congruent balls centered in the lattice points must have in order to cover the whole space E^n. While the packing radius in our normalized setting $\omega = 1$ is always $1/2$, the covering radius changes from lattice to lattice. Every point of E^n whose distance from the nearest lattice point is a local maximum is called a *hole* of L_i^n. The holes for which this distance is a global maximum and equals the covering radius are termed *deep holes.* Among the vertices of T_0, the one of maximal norm is a deep hole. In Chapter 5, we define the Dirichlet-Voronoĭ-cell of a ball in a sphere packing. For lattice packings, one can show that every vertex of this Dirichlet-Voronoĭ-cell is a hole. For more information, see [CS, Chapter 2].

5 Packings

A *sphere packing* $\mathcal{B}$ in $X^n = E^n$, S^n or H^n is a set of non-overlapping congruent n-balls. More appropriate would be the name of ball packing. The interested reader should consult [FTK] or [Rog]. A quantitative value associated with a packing is its density $\delta(\mathcal{B})$, the proportion of space occupied by the balls in $\mathcal{B}$. In this chapter, we focus on some aspects of euclidean sphere packings and hyperbolic horoball packings. In particular, we will see how upper density bounds can be used to determine lower volume bounds for lattices, cusps and hyperbolic orbifolds.

Euclidean sphere packings. The density $\delta(\mathcal{B} \mid D)$ of a euclidean sphere packing $\mathcal{B}$ with respect to a bounded region $D \subset E^n$ is just the fraction of space in D filled by balls of the packing. A suitable definition for the *density* $\delta(\mathcal{B})$ of $\mathcal{B}$ with respect to the whole space E^n starts with the density of $\mathcal{B}$ with respect to an arbitrary ball $\mathbb{B}(r)$ of radius $r > 0$ and then takes the lim sup over r. Indeed, the real number

$$\delta(\mathcal{B}) := \limsup_{r \longrightarrow \infty} \left[\frac{1}{\mathrm{vol}_n(\mathbb{B}(r))} \sum_{B \subset \mathcal{B} \cap \mathbb{B}(r)} \mathrm{vol}_n(B) \right] < 1$$

is well-defined and independent of the choice for $\mathbb{B}(r)$. An extremely difficult problem is the determination of the densest packings in dimension n. The solution is only known for $n = 2$ and 3. In 1611, Kepler asserted that the canonball packing associated to the fcc-lattice D_3 has maximal density $\pi/\sqrt{18}$ among all packings in dimension 3. Thue affirmed in 1890 that the hexagonal lattice packing A_2 is unique to have maximal density $\pi/12$ among all 2-dimensional packing. While the verification of Thue's theorem is elementary, Kepler's conjecture remained open for almost 400 years until Hales published his proof in 1998. We recommend his survey article [Hal].

Upper density bounds. In higher dimensions there are still upper density bounds. Chapter 2.3 of [FTK] gives an overview. In this paragraph we concentrate on the simplicial density bound due to Rogers. By definition, the *simplicial density* d_n is the ratio

$$d_n := (n+1) \frac{\mathrm{vol}_n\left(B(r) \cap S^n_{\mathrm{reg}}(2r)\right)}{\mathrm{vol}_n\left(S^n_{\mathrm{reg}}(2r)\right)}$$

between the total volume of the sectors of $(n+1)$ mutually touching balls $B(r)$ of radius $r > 0$ lying in the euclidean regular simplex $S^n_{\mathrm{reg}}(2r)$ of edge length $2r$ spanned by their centers and the volume of the whole simplex $S^n_{\mathrm{reg}}(2r)$. Note that d_n is independent of the radius r. In 1958, Rogers proved the following theorem in [Rog, Thm 7.1].

Theorem 3 *The density of an arbitrary sphere packing $\mathcal{B}$ in E^n satisfies*

$$\delta(\mathcal{B}) \leq d_n. \tag{41}$$

Compared to other density bounds, the simplicial bound has the significant advantage to generalize to sphere packings in spherical and hyperbolic space. In particular, it is sharp in dimension 2. The first few values for d_n are listed below (cf. [CS, Table 1.2], for example).

Dim n	1	2	3	4
$d_n \simeq$	1	0.90691	0.77367	0.64780
Dim n	5	6	7	8
$d_n \simeq$	0.52570	0.41921	0.32984	0.25676

Euclidean lattice packings. A sphere packing $\mathcal{B}_\Lambda$ is said to be a *lattice packing* if the centers of its balls form a euclidean lattice Λ. As usually, the minimal norm of Λ is assumed to be 1. Moreover, we always suppose that the radius of the balls in $\mathcal{B}_\Lambda$ equals the packing radius $1/2$. The periodicity of Λ reduces the study of density to a local problem. More precisely, $\delta(\mathcal{B}_\Lambda)$ can be proven to be the ratio

$$\delta(\mathcal{B}_\Lambda) = \frac{\mathrm{vol}_n\left(B\left(\frac{1}{2}\right)\right)}{\mathrm{vol}_n(\Lambda)} = \frac{\Omega^{n-1}}{2^n n\,\mathrm{vol}_n(\Lambda)}, \tag{42}$$

where Ω^{n-1} denotes the volume (6) of the unit $(n-1)$-sphere and $B(1/2)$ the ball of radius $1/2$. The right term in formula (42) proves the scaling invariance of $\delta(\mathcal{B}_\Lambda)$. Under these regularity conditions, the question about the densest lattice packing is answered in dimensions $n \leq 8$ due to results of Lagrange [Lag], Gauß [Gau], Korkine and Zolotareff [KZ1], [KZ2] as well as Blichfeldt [Bli].

Remember the root lattice L_1^n defined in Chapter 3. For this work, it is crucial that beside having maximal symmetry, L_1^n is the unique lattice that generates the densest euclidean lattice packing in dimension n. The corresponding density δ_n is as indicated in the following table.

n	L_1^n	δ_n		n	L_1^n	δ_n	
1	$\mathbb{Z}$	1		5	D_5	$\frac{\pi^2}{15\sqrt{2}}$	Korkine, Zolotareff 1877
2	A_2	$\frac{\pi}{\sqrt{12}}$	Lagrange 1773	6	E_6	$\frac{\pi^3}{48\sqrt{3}}$	Blichfeldt 1925
3	A_3	$\frac{\pi}{\sqrt{18}}$	Gauß 1831	7	E_7	$\frac{\pi^3}{105}$	Blichfeldt 1926
4	D_4	$\frac{\pi^2}{16}$	Korkine, Zolotareff 1872	8	E_8	$\frac{\pi^4}{384}$	Blichfeldt 1934

Smallest volume lattices. By means of formula (42), an upper bound for the lattice density yields a lower volume bound for normalized euclidean lattices.

As an example, Rogers' simplicial density bound (41) forces the volume of every normalized lattice Λ in E^n to verify

$$\mathrm{vol}_n(\Lambda) \geq \frac{\mathrm{vol}_n\left(B\left(\frac{1}{2}\right)\right)}{d_n}.$$

For $n \leq 8$, we can optimize this lower bound by using the maximal lattice density δ_n instead of d_n. We conclude that every normalized lattice volume exceeds the volume of L_1^n, i.e.

$$\mathrm{vol}_n(\Lambda) \geq \mathrm{vol}_n(L_1^n), \tag{43}$$

and equality holds if and only if $\Lambda = L_1^n$.

Hyperbolic packings. Questions about the density of a sphere packing in hyperbolic space are much more complicated than in euclidean space. The reason for that is the lack of similarities in H^n. In particular, K. Böröczky sen. observed that H^n admits no reasonable global notion of density. He published his arguments in a hungarian journal but one of his examples can be found in chapter 3.2 of [FTK]. Nevertheless, one can generalize the notion of local density (42) from the lattice packing case.

Consider an arbitrary sphere packing $\mathcal{B}$ in X^n. Every ball B in $\mathcal{B}$ lives in its *Dirichlet-Voronoĭ-cell* $D(B,\mathcal{B})$ consisting of all points in X^n closer to B than to any other ball in $\mathcal{B}$, i.e.

$$D(B,\mathcal{B}) = \{x \in X^n : d(x,B) \leq d(x,B'),\ \forall B' \in \mathcal{B}\} \supset B.$$

The *local density* of B in $\mathcal{B}$ is then defined by

$$ld(B,\mathcal{B}) := \frac{\mathrm{vol}_n(B)}{\mathrm{vol}_n(D(B,\mathcal{B}))}.$$

It is the right object to use when searching for lower volume bounds in the context of discrete group quotients in H^n. First, suppose a hyperbolic **manifold** H^n/Γ contains an embedded ball $b(r) = b$ of radius $r > 0$. Let B be one of the balls in H^n covering b. Altogether, the Γ-copies of B form a *periodic* hyperbolic sphere packing $\mathcal{B}_\Gamma := \pi^{-1}(b)$ which is *precisely invariant* under Γ, that is,

$$\gamma B \cap B = \emptyset,\ \forall \gamma \in \Gamma.$$

If c is the center of B, then the Dirichlet-Voronoĭ-cell of B with respect to $\mathcal{B}_\Gamma$ equals the Dirichlet fundamental domain for Γ with center c. In particular, the balls in $\mathcal{B}_\Gamma$ have congruent Dirichlet-Voronoĭ-cells. Hence

$$\mathrm{vol}_n(H^n/\Gamma) = \mathrm{vol}_n(D(B,\mathcal{B})) = \frac{\mathrm{vol}_n(B)}{ld(B,\mathcal{B}_\Gamma)}. \tag{44}$$

In the **orbifold** case, the center c of B may project to the singular set of H^n/Γ. The stabilizer $\Gamma_c < \Gamma$ is then isomorphic to a non-trivial, finite subgroup of $O(n)$. Nevertheless, B is still precisely Γ-invariant:

$$\gamma B = B,\ \forall \gamma \in \Gamma_c \qquad \text{and} \qquad \gamma B \cap B = \emptyset,\ \forall \gamma \in \Gamma \setminus \Gamma_c\ .$$

Since $D(B, \mathcal{B}_\Gamma)/\Gamma_c$ is a fundamental domain for the action of Γ_c on $D(B, \mathcal{B}_\Gamma)$, the quotient $D(B, \mathcal{B}_\Gamma)/\Gamma_c$ is a fundamental domain for Γ acting in H^n (compare [Bea, Section 9.6], for $n = 2$). Finally

$$ld(B, \mathcal{B}_\Gamma) = \frac{\mathrm{vol}_n(B)}{\mathrm{vol}_n(D(B, \mathcal{B}_\Gamma))} = \frac{\mathrm{vol}_n(B/\Gamma_c)}{\mathrm{vol}_n(D(B, \mathcal{B}_\Gamma)/\Gamma_c)},$$

and equality (44) holds for orbifolds too.

Simplicial density function. In 1978, Böröczky sen. confirmed a conjecture of Fejes Tóth and Coxeter by proving, for packings in X^n, the validity of Rogers' simplicial bound formulated in terms of local density [Bör, Thm 1]. Here we discuss the hyperbolic case only. The *simplicial density function* in H^n is defined by

$$d_n(r) := (n+1)\frac{\mathrm{vol}_n\left(B(r) \cap S^n_{\mathrm{reg}}(2r)\right)}{\mathrm{vol}_n\left(S^n_{\mathrm{reg}}(2r)\right)},$$

where $S^n_{\mathrm{reg}}(2r)$ denotes the regular simplex of edge length $2r$ spanned by the centers of $(n+1)$ mutually touching balls $B(r)$ of radius $r > 0$. The big difference with the euclidean case is the dependence of $d_n(r)$ on the radius r. By looking at the curvature dependence of the volume element in H^n, we can interpret the euclidean d_n as a limiting density $d_n = \lim_{r \to 0} d_n(r)$ on H^n. This justifies the use of the same letter d_n as in the euclidean case.

Böröczky showed that the local density of a sphere packing with balls of radius r is always bounded from above by the simplicial density $d_n(r)$. Together with (44), we deduce the following lower volume bound for hyperbolic n-orbifolds containing an embedded ball of radius r (compare [Me3])

$$\mathrm{vol}_n(H^n/\Gamma) = \frac{\mathrm{vol}_n(B(r))}{ld(B(r), \mathcal{B}_\Gamma)} \geq \frac{\mathrm{vol}_n(B(r))}{d_n(r)}. \tag{45}$$

Horoball packings. An even more general version of Rogers' bound treats the case of horoball packings in $\overline{H}^n$. By definition, a *horoball packing* is a set $\mathcal{B}_\infty$ of non-overlapping horoballs in $\overline{H}^n$. Let $B_q = B$, $q \in \partial H^n$ be a horoball in $\mathcal{B}_\infty$. We write $\mathrm{dist}(x, B)$ for the length of the unique perpendicular from a point $x \in H^n$ to the bounding horosphere of B. Here, $\mathrm{dist}(x, B)$ is taken to be negative if $x \in B$. Roughly speaking, the local density of B with respect to $\mathcal{B}_\infty$ is the proportion of space occupied by B in its Dirichlet-Voronoĭ-cell

$$DV(B, \mathcal{B}_\infty) := \{x \in H^n : \mathrm{dist}(x, B) \leq \mathrm{dist}(x, B'), \forall B' \in \mathcal{B}_\infty\}.$$

Since both, B and $DV(B, \mathcal{B}_\infty)$ have infinite volume, we need to adapt the concept of local density. Consider the cone $C_q(r)$ in $\overline{H}^n$ with apex q and consisting of all geodesics with limit point q passing through the euclidean ball $B^{n-1}(c, r)$ of center c and radius $r > 0$ living on the horosphere ∂B_q. Then, the well-defined real number

$$ld_n(B, \mathcal{B}_\infty) := \limsup_{r \longrightarrow \infty} \frac{\mathrm{vol}_n(B \cap C_q(r))}{\mathrm{vol}_n(DV(B, \mathcal{B}_\infty) \cap C_q(r))} < 1$$

is independent of the choice for c and measures the *local density* of B with respect to $\mathcal{B}_\infty$.

This notion of local density is best suited for searching lower volume bounds of cusped hyperbolic orbifolds H^n/Γ. A cusp C in H^n/Γ is covered by the periodic packing $\mathcal{B}_\infty = \pi^{-1}(C)$ of horoballs. Let B_q be one of the horoballs in $\mathcal{B}_\infty$. Then B_q is precisely invariant under the action of Γ, i.e.

$$\gamma B_q = B_q, \forall \gamma \in \Gamma_q \qquad \text{and} \qquad \gamma B_q \cap B_q = \emptyset, \forall \gamma \notin \Gamma \setminus \Gamma_q \ . \tag{46}$$

For such a periodic packing, the local density is the same for all horoballs. By means of (46), a fundamental domain D for the action of Γ_q on $DV(B_q, \mathcal{B}_\infty)$ is automatically a fundamental domain for Γ (compare [Bea, Section 9.6], for $n = 2$). Therefore, we deduce the following simple expression for the local density

$$ld_n(B_q, \mathcal{B}_\infty) = \frac{\mathrm{vol}_n(B_q \cap D)}{\mathrm{vol}_n(D)} = \frac{\mathrm{vol}_n(C)}{\mathrm{vol}_n(H^n/\Gamma)} \ . \tag{47}$$

Simplicial horoball bound. The simplicial horoball density $d_n(\infty)$ in H^n is defined to be the density of $(n+1)$ horoballs B mutually touching one another with respect to the ideal regular n-simplex $S^n_{\mathrm{reg}}(\infty)$ spanned by the base points of these horospheres, i.e.

$$d_n(\infty) = (n+1)\,\frac{\mathrm{vol}_n\left(B \cap S^n_{\mathrm{reg}}(\infty)\right)}{\mathrm{vol}_n\left(S^n_{\mathrm{reg}}(\infty)\right)} \ .$$

Alternatively, we can interpret $d_n(\infty)$ as the limiting density $\lim_{r\to\infty} d_n(r)$. In the last paragraph of his paper [Bör, Thm. 4], Böröczky formulates the generalization of Rogers' simplicial bound to horoball packings:

Theorem 4 *In H^n, consider a horoball packing $\mathcal{B}_\infty$. Then the local density ld_n of each horoball with respect to its Dirichlet-Voronoĭ-cell cannot be greater than the simplicial horoball density $d_n(\infty)$:*

$$ld_n \leq d_n(\infty) \, . \tag{48}$$

In combination with (47), Theorem 4 yields the following lower volume bound for a non-compact hyperbolic n-orbifold with cusp C (compare [Me3] and [K1, Lemma 3.2], for example)

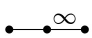
$$\mathrm{vol}_n(H^n/\Gamma) = \frac{\mathrm{vol}_n(C)}{ld(B, \mathcal{B}_\Gamma)} \geq \frac{\mathrm{vol}_n(C)}{d_n(\infty)} \ . \tag{49}$$

The volume bound (49) is crucial in order to determine the non-compact orbifolds of minimal volume. In fact, it is one of the most important ingredients in the proof of almost all results known so far about small volume cusped hyperbolic orbifolds.

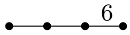
In Chapter 3, we expained how Coxeter graphs can be used to describe regular tessellations. Böröczky's proof shows that equality in the simplicial horoball bound (48) holds if and only if the horoballs are inscribed in the unbounded cells of a regular tessellation with Coxeter graph

$, m \geq 3.$

The table on page 23 reveals precisely two such tilings. The first one lives in dimension 2 and defines the packing with horodiscs inscribed in the tessellation



with regular convex tiles having infinitely many sides and dihedral angle $\pi/3$. Its dual tessellation is the famous tiling of H^2 with ideal triangles all of whose vertices are surrounded by infinitely many triangles. The second one describes the packing in H^3 with horoballs representing the inballs of the regular tessellation

 .

Its dual tiling consists of ideal regular tetrahedra with dihedral angle $\pi/3$ building up a 6-cycle around each edge.

For all $n \geq 2$, R. Kellerhals expressed $d_n(\infty)$ in terms of the volume v_n of $S^n_{\text{reg}}(\infty)$ [K1, Thm 2.1]:

$$d_n(\infty) = \frac{n+1}{n-1}\frac{n}{2^{n-1}}\prod_{k=2}^{n-1}\left(\frac{k-1}{k+1}\right)^{\frac{n-k}{2}}\frac{1}{v_n} . \tag{50}$$

A representation of v_n as a power series is due to J. Milnor [Mil, How to compute volume in hyperbolic space, Section 4]:

$$v_n = \sqrt{n}\sum_{k=0}^{\infty}\frac{\beta(\beta+1)\cdots(\beta+k+1)}{(n+2k)!}A_{n,k} ,$$

with

$$\beta = \frac{n+1}{2} \text{ and } A_{n,k} = \sum_{i_0+\cdots i_n=k,\, i_u\geq 0}\frac{(2i_0)!\cdots(2i_n)!}{i_0!\cdots i_n!} .$$

The first few values for $d_n(\infty)$ are listed below (see [K5, Table 1.4.4]).

Dim n	1	2	3	4	5
$d_n(\infty) \simeq$	–	0.95493	0.85328	0.73046	0.60695
Dim n	6	7	8	9	10
$d_n(\infty) \simeq$	0.49339	0.39441	0.31114	0.24285	0.18789

6 Minimal volume cusped hyperbolic orbifolds

Throughout the rest of this work, we restrict ourselves to the non-compact, finite volume, hyperbolic orbifold case. We write $\mu_n > 0$ for the smallest value of the volume spectrum

$$V_\infty^n := \{\mathrm{vol}_n(Q^n) : Q^n \text{ a cusped hyperbolic } n\text{-orbifold}\}.$$

The problem we are interested in is the determination of all cusped orbifolds of minimal volume μ_n. The values μ_2 and μ_3 and the corresponding orbifolds are due to Siegel [Sie, 1945] and Meyerhoff [Me2, 1985], respectively. Our main theorem gives all minimal volume cusped hyperbolic orbifolds in dimensions less than 10. This chapter is the content of our two papers [HK] and [H], accepted for publication in the Journal of the London Mathematical Society and the Journal of Algebra.

6.1 Lower bounds

A first lower bound. Our strategy to determine cusped hyperbolic orbifolds of minimal volume is as follows: by considering particular cusps, we obtain lower volume bounds which are improved in successive steps until they realize the volume of a minimizer. This turns out to be successful in dimensions less than 10. The fundamental difference with the compact case comes from the euclidean structure of the cusp boundary which allows us to use our knowledge about crystollographic groups. Cornerstone of the general method is the lower volume bound (49) for an orbifold H^n/Γ with disjoint embedded cusps $C_1, \ldots, C_m$:

$$\mathrm{vol}_n(H^n/\Gamma) \geq \frac{\sum_{i=1}^m \mathrm{vol}_n(C_i)}{d_n(\infty)}. \tag{51}$$

Canonical cusp. At the end of Chapter 2.2, we already mentioned two types of cusps, the canonical and the maximal one. Because of their easy construction, we first consider *canonical* cusps. It is well known that the canonical cusps associated to inequivalent parabolic elements in Γ are *embedded* and *disjoint* (compare for example [Bea, §7.37], [Shi, Lemma 4 & 5], [Me1], for $n = 3$, and [He1], [K4, Lemma 2.7 & 2.8] for arbitrary n). The proof of this property exploits the *Leutbecher-Shimizu inequality* (see [Leu] and [Shi, Lemma 4], for $n = 3$, [Wat, Thm 8], [K4, Thm 1.2] and [He2], for arbitrary n).

Improving the lower bound. After conjugation, we always assume that ∞ is a parabolic fixed point of Γ. By the finite volume condition, the stabilizer Γ_∞ is the Poincaré extension of an $(n-1)$-dimensional crystallographic group. In what follows, we use the same symbols for elements acting on the ground space $\partial U^n = E^{n-1}$ and their Poincaré extensions. In Chapter 4, we saw that every crystallographic group comes with a lattice Λ_∞ and a point group ϕ_∞. As usually, let ω denote the minimal norm of Λ_∞. By definition, the canonical horoball based at ∞ equals

$$B_\infty(\omega) = \{x \in H^n : x_n > \omega\}.$$

The horoball packing $\mathcal{B}_\infty = \Gamma B_\infty$ yields, after projection, the canonical cusp $C_\infty(\omega)$ with cusped point ∞. Here, we use ∞ as a representative of the class

$[\infty]$ of parabolic fixed points equivalent to ∞ under Γ.

Formula (24) yields the volume of $C_\infty(\omega)$ in terms of ω and the euclidean volume of a proper fundamental domain F for the crystallographic group $\Gamma_\infty < I\left(E^{n-1}\right)$. The volume of F equals the volume of Λ_∞ devided by the index of Λ_∞ in Γ_∞. Hence,

$$\mathrm{vol}_n\left(C_\infty(\omega)\right) = \frac{\mathrm{vol}_{n-1}\left(\Lambda_\infty\right)}{(n-1)\ \left[\Gamma_\infty : \Lambda_\infty\right]\ \omega^{n-1}}\ . \tag{52}$$

Since the quotient $\mathrm{vol}_{n-1}\left(\Lambda_\infty\right)/\omega^{n-1}$ is invariant under rescaling, we may always suppose the lattice to be normalized, i.e. $\omega = 1$. In the paragraph about smallest lattice volumes in Chapter 5, we presented the values

$$V_n := \frac{\mathrm{vol}_n\left(B\left(\frac{1}{2}\right)\right)}{\delta_n}\ ,\ 1 \leq n \leq 8, \tag{53}$$

which are in fact the *minimal volumes* $\mathrm{vol}_n\left(L_1^n\right)$ *for normalized lattices in* E^n. Substituting (52) in (51), we obtain the refined bound

$$\mathrm{vol}_n\left(H^n/\Gamma\right) \geq m \frac{V_{n-1}}{(n-1)\,\varphi_{n-1} d_n(\infty)} \tag{54}$$

for m-cusped n-orbifolds. Remember that φ_{n-1} symbolizes the maximal point group order in dimension $n-1$. The beauty of the volume bound (54) lies in its validity for arbitrary dimensions. But is there a dimension $n \geq 2$, such that the right hand side of (54) with $m = 1$ represents the volume of a cusped orbifold?

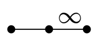
Known minimizers. A necessary condition for equality in (54) is the existence of a periodic horoball packing in dimension n with local density $d_n(\infty)$. The last paragraph of Chapter 5 excludes the existence of such a packing for $n \geq 4$. We immediately conclude that the volume of a cusped orbifold satisfies

$$\mathrm{vol}_n\left(H^n/\Gamma\right) > \frac{V_{n-1}}{(n-1)\,\varphi_{n-1} d_n(\infty)}\ ,\ \text{for } n \geq 4.$$

In dimension 2 and 3 however, the equality sign in (54) is possible. For $n = 2$, we find again Siegel's result. Indeed, inspection of the tables in Appendix A2 reveals $V_1 = 1$ and $\varphi_1 = 2$. Together with $d_2(\infty) = \frac{3}{\pi}$, we have

$$\mathrm{vol}_2\left(H^2/\Gamma\right) \geq \frac{\pi}{6} = \mu_2.$$

Equality can only occur if the canonical horoballs covering the cusp are the inscribed discs of the regular tessellation represented by the Coxeter graph

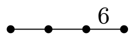


(55)

with density $\frac{3}{\pi}$. We write Γ_0^2 for the hyperbolic Coxeter group represented by (55). For $n = 3$, the table yields $V_2 = \frac{\sqrt{3}}{2}$ and $\varphi_2 = 12$. Together with $d_3(\infty) = \frac{\sqrt{3}}{2v_3}$, where $v_3 = \mathrm{vol}_3\left(S^3_{\mathrm{reg}}(\infty)\right) = 3\ \text{Л}\left(\frac{\pi}{3}\right)$, we obtain

$$\mathrm{vol}_3\left(H^3/\Gamma\right) \geq \frac{\text{Л}\left(\frac{\pi}{3}\right)}{8} = \mu_3.$$

Here, equality only occurs if the canonical horoballs covering the canonical cusp are the inscribed balls of the regular tessellation given by the Coxeter graph

 . (56)

In 1985, Meyerhoff was the first one to develop this strategy in order to determine the minimal value $\mu_3 \in V_\infty^3$. More precisely, he showed in [Me2] and [Me3] that the orbifold $H^3/PGL\left(\mathcal{O}_3\right)$, with $\mathcal{O}_3$ the ring of integers in $\mathbb{Q}\left(\sqrt{-3}\right)$, has minimum volume among all orientable cusped hyperbolic 3-orbifolds. It is the oriented double cover of the quotient of H^3 by the group Γ_0^3 determined by the graph (56). Since every non-oriented orbifold is doubly covered by an oriented one, it follows that H^3/Γ_0^3 is the only non-compact 3-orbifold of smallest volume $\mu_3 = \frac{1}{8}\mathbf{Л}\left(\frac{\pi}{3}\right)$ (compare [Ad4, Conclusion]).

Maximal cusp. It is important to note the *maximality* of the canonical cusp $C_\infty(1)$ in H^2/Γ_0^2 and H^3/Γ_0^3. Being the inballs of a regular tessellation, the canonical horoballs projecting to $C_\infty(1)$ have common *touching points*. By touching point between two horoballs, we mean a point common to their bounding horospheres. Enlarging the canonical horoballs would thus lead to self-intersections and the resulting cusp in the quotient would no longer be embedded. Hence, $C_\infty(1)$ satisfies the definition of a maximal cusp as given in Chapter 2.2

Colin Adams studied *maximal* cusps instead of canonical ones (compare [Ad1], [Ad2], [Ad3], [Ad4] and [Ad5]). The idea of his strategy in the one-cusped case can be described as follows. Consider the packing $\mathcal{B}_\infty$ of horoballs covering the maximal cusp. After conjugation, one of the horoballs in $\mathcal{B}_\infty$ is assumed to be based at ∞ and at euclidean height 1. We put $B_\infty = B_\infty(1)$. Note that in this setting, the minimal norm of the lattice Λ_∞ is not necessarily 1. By maximality, there are horoballs in $\mathcal{B}_\infty$ with euclidean diameter 1. These horoballs are called *fullsized*, while the remaining horoballs in $\mathcal{B}_\infty$ are termed *non-fullsized*. Conjugating again, we may suppose that one of the fullsized horoballs, B_0 say, is based at 0. Then, there is at least a fullsized horoball based at every point of the orbit $\Gamma_\infty(0) \subset \partial H^n$. In particular, every lattice point $\Lambda_\infty(0)$ is the base point of a fullsized horoball. Orthogonal projection of the elements of the packing $\mathcal{B}_\infty \setminus \{B_\infty\}$ to the boundary ∂B_∞ yields a euclidean arrangement of, in general, overlapping balls. This arrangement is called *horoball diagram*. Combining the theory of crystallographic groups with some horoball geometry, it is possible to obtain conclusions about realizable horoball diagrams. The corresponding cusp volume is then computed with formula (24). Plugging it into (49), a lower bound for the resulting orbifold is obtained. In the best possible case, this bound equals μ_n.

Results due to Adams. In the way described above, Adams first determined the non-compact hyperbolic 3-manifold of minimal volume [Ad2]. The main argument in his proof is the existence of fullsized horoballs based at points outside the orbit $\Gamma_\infty(0)$ [Ad2, Thm. 2]. In fact, suppose that Γ_∞ identifies all fullsized horoballs. Since $\mathcal{B}_\infty$ consists of Γ-images of B_∞, there is a $\gamma \in \Gamma$ with $\gamma B_0 = B_\infty$. Denote by a the touching point of B_0 with B_∞. Then γB_∞ is a fullsized horoball touching B_∞ in $\gamma(a)$. By our assumption, there is a $\lambda \in \Gamma_\infty$ with $\lambda\gamma B_\infty = B_0$. The element $\lambda\gamma \in \Gamma$ fixes the point a what is forbidden in the manifold case. A fundamental domain for Γ_∞ thus contains at least two projections of fullsized horoballs. Knowing that the maximal order of a point group associated with a fixed point free crystallographic group in dimension 2 equals 1 ([K1, (3.3)], for example), the lower bound (54) becomes

$$\mathrm{vol}_3\left(M^3\left(\Gamma\right)\right) \geq v_3 = 3\mathbf{JI}\left(\frac{\pi}{3}\right).$$

Adams showed that the Gieseking manifold is the unique cusped 3-manifold with volume $v_3 = \mathrm{vol}_3\left(S^{\infty}_{\mathrm{reg}}\right)$.

The cusp of the minimal volume 3-orbifold H^3/Γ^3_0 turns out to be rigid. Therefore, μ_3 is an isolated point of V^3_∞. The study of realizable horoball diagrams allowed Adams to find the three smallest limit points of V^3_∞ (see [Ad1]). In particular, the smallest limit volume is uniquely attained by a quotient of H^3 by $PSL\left(2, \mathcal{O}_1\right)$ and its volume equals $1/12$ that of the regular ideal octahedron, that is $0.30532\ldots$ The main idea of his proof is to show that the only singular axes intersecting a non-rigid cusp are of order two and go directly out the cusp. This property clearly restricts the possible horoball diagrams.

6.2 Horoball geometry

Consider the horoball packing $\mathcal{B}_\infty$ covering a maximal cusp of H^n/Γ. After conjugation, we assume that $\mathcal{B}_\infty$ is the set of all Γ-images of $B_\infty(\rho)$ based at ∞ and at euclidean height $\rho > 0$. In our notation, Λ_∞ always denotes the lattice of Γ_∞ interpreted, via Poincaré extension, as a crystallographic group acting in the euclidean metric of $\partial U^n = E^{n-1}$. When studying the acting of Γ_∞ on the horosphere $\partial B_\infty(\rho) = S_\infty(\rho)$, one has to be careful. To avoid any confusion later, we give a detailed description of the situation.

Consider the point $(0, \rho) \in S_\infty(\rho)$ lying above 0. The orbit of $(0, \rho)$ under the action of $\Lambda_\infty < \Gamma_\infty$ in H^n consists of the points $(x, \rho) \in S_\infty(\rho)$, $x \in \Lambda_\infty(0)$ lying above lattice points of $E^{n-1} = \{x_n = 0\}$. Every point in this orbit is a touching point of $B_\infty(\rho)$ with a fullsized horoball of $\mathcal{B}_\infty$. Two points (a, ρ) and (b, ρ) of the orbit $\Lambda_\infty(0, \rho)$ are related by the translation $x \mapsto x + (b - a)$ in Λ_∞. In the induced metric of $S_\infty(\rho)$, however, this translation reads $x \mapsto x + \frac{1}{\rho}(b - a)$ by means of (2). Hence, when Γ_∞ is interpreted as a crystallographic group acting in the induced metric of $S_\infty(\rho)$, the translational part of all its motions has to be multiplied by $\frac{1}{\rho}$. In the following, we write $\frac{1}{\rho}\Lambda_\infty$ for the lattice group acting on $S_\infty(\rho)$. This notation is justified since the orbit of $(0, \rho)$ in the induced metric equals the set $\frac{1}{\rho}\Lambda_\infty(0, \rho)$.

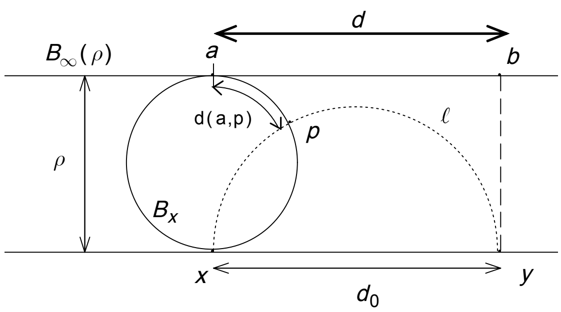
For every $\gamma \in \Gamma \setminus \Gamma_\infty$, the horoball $\gamma B_\infty(\rho) = B_{\gamma(\infty)}(\rho)$ based at $\gamma(\infty) \in \partial U^n$ belongs to $\mathcal{B}_\infty \setminus B_\infty(\rho)$. The conjugate group $\gamma\frac{1}{\rho}\Lambda_\infty\gamma^{-1}$ of $\frac{1}{\rho}\Lambda_\infty$ is a subgroup of the stabilizer $\Gamma_{\gamma(\infty)}$ and, by Lemma 1, acts as a crystallographic group in the induced metric of $\partial B_{\gamma(\infty)} = S_{\gamma(\infty)}$. It is associated to the lattice

$$\gamma\frac{1}{\rho}\Lambda_\infty\gamma^{-1}(\gamma(0, \rho)) = \gamma\left(\frac{1}{\rho}\Lambda_\infty\right) \subset S_{\gamma(\infty)}.$$

For our work, it is important to realize that every point of $\gamma\left(\frac{1}{\rho}\Lambda_\infty\right)$ is the touching point of $B_{\gamma(\infty)}$ with another horoball in $\mathcal{B}_\infty$.

For a detailed investigation of $\mathcal{B}_\infty$, we need formulae comparing distances on touching horospheres. We present a lemma and two corollaries which have been extensively used by Adams, when studying horoball diagrams, and by ourselves in [HK] and [H].

Lemma 5 *Consider $B_\infty(\rho)$ and a fullsized horoball B_x based at x. Let l be a geodesic with endpoints x and $y \in \mathbb{R}^{n-1} \setminus \{x\}$. Denote by d the distance from $a := (x, \rho)$ to (y, ρ) on $\partial B_\infty(\rho)$, by d_0 the euclidean distance from x to y. Call p the intersection point of l with $S_x = \partial B_x$. Then the distance $d(a, p)$ from a to p on S_x is given by*



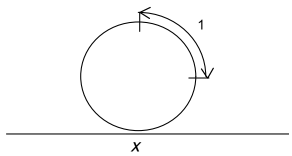
$$d(a, p) = \frac{1}{d} = \frac{\rho}{d_0}.$$

Proof. (cf. [Ad1, Lemma 4.3] and [HK, Lemma 1]) Assume $x = 0$ and let $\gamma \in I(H^n)$ be the half-turn around the geodesics with endpoints $-y$ and y. Then $\gamma(B_0)$ is a horoball $B_\infty(\tau)$ centered at ∞ and at euclidean height $\tau > 0$. Formula (1) implies $\tau = d_0^2/\rho$. Since γ maps l onto the geodesic with endpoints y and ∞, we have

$$d(a,p) = d(a,\gamma(p)) \overset{(2)}{=} \frac{d_0}{\tau} = \frac{\rho}{d_0} \overset{(2)}{=} \frac{1}{d} .$$

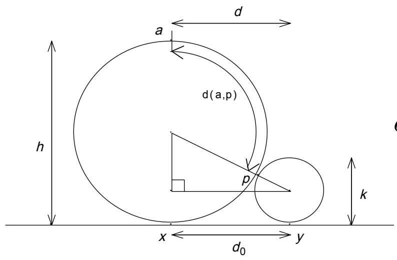
□

Corollary 6 *Let $B_x(h)$ be a horoball of euclidean diameter $h > 0$ based at $x \neq \infty$. Then the interior of its upper hemisphere is an open ball of radius 1 in the induced metric of its boundary $S_x(h)$.*



Proof. The euclidean distance from x to the second endpoint $y \in \mathbb{R}^{n-1} \setminus \{x\}$ of a geodesic emanating from x and passing through an equator point of $B_x(h)$ equals h. □

Corollary 7 *Let $B_x(h)$ and $B_y(k)$ be two horoballs with common touching point p and of euclidean diameter h and k respectively. Call d_0 the euclidean distance between their base points x and y. Denote by $d = d_0/h$ the distance from (x,h) to (y,h) on the horosphere based at ∞ and at euclidean height h. Then the distance $d(a,p)$ from a to p on $\partial B_x(h)$ is given by*



$$d(a,p) = \frac{\sqrt{h}}{\sqrt{k}} = \frac{d_0}{k} = \frac{h}{d_0} = \frac{hd}{k} = \frac{1}{d} .$$

Proof. Assume $h > k$. The euclidean centers of $B_x(h)$ and $B_y(k)$ together with $\left(x, \frac{h}{2} - \frac{k}{2}\right)$ span a right angled triangle. By Pythagoras,

$$d_0^2 + \left(\frac{h}{2} - \frac{k}{2}\right)^2 = \left(\frac{h}{2} + \frac{k}{2}\right)^2 \Longrightarrow d_0 = \sqrt{hk} .$$

□

6.3 Small volume cusped hyperbolic orbifolds

In this paragraph, let $2 \leq n \leq 9$. By definition, a cusped hyperbolic orbifold H^n/Γ is of *small volume* if

$$\mathrm{vol}_n\left(H^n/\Gamma\right) < \frac{3}{2}\,\frac{\mathrm{vol}_{n-1}\left(B^{n-1}\left(\frac{1}{2}\right)\right)}{(n-1)\ \varphi_{n-1}\ \boldsymbol{d_{n-1}}\ d_n\left(\infty\right)} =: \theta_n. \tag{57}$$

Notice that this definition differs from that as given in [HK] and [H]. We come back to this point at the end of this paragrah.

In order to get information about the cusps in the small volume case, we exploit the lower volume bound (54) by using (53). We obtain

$$\mathrm{vol}_n\left(H^n/\Gamma\right) \geq m\frac{V_{n-1}}{(n-1)\,\varphi_{n-1} d_n\left(\infty\right)} = m\frac{\mathrm{vol}_{n-1}\left(B^{n-1}\left(\frac{1}{2}\right)\right)}{(n-1)\ \varphi_{n-1}\ \boldsymbol{\delta_{n-1}}\ d_n\left(\infty\right)}$$

for m-cusped hyperbolic orbifolds H^n/Γ. The quotients H^n/Γ investigated in the following propositions are all assumed to be of small volume. Recall that by (43), the root lattice L_1^{n-1} is the only lattice of minimal volume V_{n-1} in dimension $n-1 \leq 8$. Furthermore, the symmetry group G_{n-1} of L_1^{n-1} has maximal order φ_{n-1} among all point groups.

The first proposition is an immediate corollary of (57).

Proposition 8 *A small volume non-compact hyperbolic orbifold has exactly one cusp.*

Proof. The lower volume bound (54) in the case $m \geq 2$ in combination with $\delta_{n-1} \leq d_{n-1}$ yields the contradiction $\mathrm{vol}_n\left(H^n/\Gamma\right) \geq \theta_n$. □

As usually, suppose $\Gamma_\infty \neq 1$ and let $\mathcal{B}_\infty$ be the horoball packing covering the unique maximal cusp of H^n/Γ. We write $B_\infty(\rho) \in \mathcal{B}_\infty$ for the horoball at euclidean height ρ based at ∞ and $C_\infty(\rho)$ for the associated cusp. The next four propositions explain how the small volume condition implies strong constraints for the algebraic definition of Γ as well as for the geometry of $\mathcal{B}_\infty$ and the cusp boundary (cf. [Ad1, Lemma 4.7 & 4.8], [HK, §2.3] and [H, Lemma 2 & 3]).

Proposition 9 *The point group ϕ_∞ of Γ_∞ has maximal order φ_{n-1}.*

Proof. Assume $|\phi_\infty| < \varphi_{n-1}$. Then $|\phi_\infty|$ is bounded from above by the second biggest point group order ψ_{n-1}. We consult the tables in Appendix A2 to find

$$\psi_{n-1} \leq \frac{2}{3}\varphi_{n-1}. \tag{58}$$

Plugging (58) into (54) leads to the contradiction $\mathrm{vol}_n\left(H^n/\Gamma\right) \geq \theta_n$. □

Corollary 10 *The lattice subgroup Λ_∞ of Γ_∞ is isometric to one of the lattices L_i^{n-1} having maximal symmetry.*

All the lattices L_i^{n-1} and their volumes are listed in the tables of Appendix A2. In dimension $n-1 = 1, 2, 4$ or 8, only one such lattice $L_1^{n-1} = L^{n-1}$ exists and we conclude that for these dimensions, Λ_∞ is isometric to L^{n-1}.

Proposition 11 *The lattice Λ_∞ identifies all fullsized horoballs in $\mathcal{B}_\infty$.*

Proof. Orthogonal projection of all fullsized horoballs in $\mathcal{B}_\infty$ to the ground space $\partial U^n = E^{n-1}$ yields a euclidean sphere packing $\mathcal{B}_0$ with balls of radius $\rho/2$. Let K_1 and K_2 be the projections of fullsized horoballs based at a point of $\Lambda_\infty(0) \subset \partial U^n$ and a non-lattice point, respectively. Because $\mathcal{B}_\infty$ is precisely invariant under Λ_∞, the same is true for $\mathcal{B}_0$. Modulo the action of Λ_∞, a fundamental domain P_{Λ_∞} for Λ_∞ acting on E^{n-1} contains therefore at least the Dirichlet-Voronoĭ-cells of K_1 and K_2 with respect to $\mathcal{B}_0$. Using Roger's simplicial bound (41) for the local density of a ball in its Dirichlet-Voronoĭ-cell, we deduce

$$\begin{aligned}\operatorname{vol}_{n-1}(P_{\Lambda_\infty}) &\geq \operatorname{vol}_{n-1}(DV(K_1) \cup DV(K_2)) \\ &= \operatorname{vol}_{n-1}(DV(K_1)) + \operatorname{vol}_{n-1}(DV(K_2)) \\ &\geq 2\tfrac{\operatorname{vol}_{n-1}(B(\rho/2))}{d_{n-1}}.\end{aligned}$$

Plugging this bound into (52), we obtain the following volume estimate for the canonical cusp in H^n/Γ :

$$\operatorname{vol}_{n-1}(C_\infty(\rho)) \geq 2\frac{\operatorname{vol}_{n-1}\left(B\left(\frac{\rho}{2}\right)\right)}{(n-1)\ \varphi_{n-1}\ \rho^{n-1}\ d_{n-1}} = 2\frac{\operatorname{vol}_{n-1}\left(B\left(\frac{1}{2}\right)\right)}{(n-1)\ \varphi_{n-1}\ d_{n-1}}\ . \quad (59)$$

Substitution of (59) in (51) contradicts the small volume assumption by means of

$$\operatorname{vol}_n(H^n/\Gamma) \geq \theta_n.$$

□

Note the fundamental difference with the manifold case, where Γ_∞ never identifies all fullsized horoballs (see results of Adams in §6.1).

Corollary 12 *The parabolic subgroup Γ_∞ is symmorphic.*

Proof. Suppose Γ_∞ is not symmorphic. Then the orbit $\Gamma_\infty(0)$ contains non-lattice points. Hence, Λ_∞ cannot identify the fullsized horoballs based at these points with those horoballs based at lattice points. By Proposition 11, H^n/Γ cannot be of small volume. □

In Chapter 4, we learned that the affine automorphism groups $\operatorname{Aut}_a\left(L_i^{n-1}\right)$ of lattices with maximal symmetry are the only symmorphic crystallographic groups associated with a point group of maximal order φ_{n-1} and a lattice L_i^{n-1} having maximal symmetry. Propositions 9 and 11 thus imply the following.

Corollary 13 *The parabolic subgroup Γ_∞ is isomorphic to* $\operatorname{Aut}_a\left(L_i^{n-1}\right)$.

Proposition 14 *Every non-fullsized horoball in $\mathcal{B}_\infty$ is tangent to a larger one.*

Proof. Suppose that $\mathcal{B}_\infty$ contains a non-fullsized horoball B_1 not tangent to any larger one. By Corollary 6, the upper hemisphere of B_1 is an open ball of radius 1. Since B_1 is a Γ-image of $B_\infty(\rho)$, there exists an isometry $\gamma \in \Gamma$ such that $\gamma B_1 = B_\infty(\rho)$. By definition, $\mathcal{B}_\infty$ is left setwise invariant by γ. By Proposition 11, orthogonal projection of all fullsized horoballs in $\mathcal{B}_\infty$ to the horosphere $\partial B_\infty(\rho)$ yields a lattice packing $\mathcal{B}_0$ with balls of radius $1/2$ with respect to the induced metric. But the centers of all these balls lie outside a ball of radius 1. Indeed, the isometry γ maps the upper hemisphere of B_1 to a ball of radius 1 on $\partial B_\infty(\rho)$ which does not contain a touching point of $B_\infty(\rho)$ with a fullsized horoball. This *ball of no tangency* thus has the same effect than an additional radius 1/2 ball with the same center.

Now, interpret $\mathcal{B}_0$ as a lattice packing with balls of radius ρ with respect to the standard euclidean metric of U^n. By the same argument as in the proof of Proposition 11, a fundamental domain of the lattice associated with $\mathcal{B}_0$ contains at least the Dirichlet-Voronoĭ-cells of two balls of radius $\rho/2$, which contradicts the small volume condition. □

Proposition 15 *There are largest non-fullsized horoballs in $\mathcal{B}_\infty$ and they are tangent to fullsized ones.*

Proof. Proposition 14 implies that every non-fullsized horoball is tangent to one of strictly larger euclidean diameter. All horoballs in a chain of strictly increasing horoballs with euclidean diameter $d \in [\delta - \varepsilon, \delta]$ for some $\delta < 1$ and $\varepsilon > 0$ are inequivalent under Λ_∞. However, a fundamental domain P_{Λ_∞} for Λ_∞ acting on ∂U^n has finite diameter and cannot contain the base points of such a chain. □

Remark. In [HK] and [H], a cusped hyperbolic orbifold H^n/Γ, with $2 \le n \le 9$, was said to be of *small volume* when

$$\mathrm{vol}_n\left(H^n/\Gamma\right) < \alpha \frac{\mathrm{vol}_{n-1}\left(B^{n-1}\left(\frac{1}{2}\right)\right)}{(n-1)\ \varphi_{n-1}\ \boldsymbol{\delta_{n-1}}\ d_n(\infty)} =: \tilde{\theta}_n, \tag{60}$$

with

$$\alpha = \begin{cases} \frac{5}{3} & \text{, if } n-1 = 5 \\ 2 & \text{, elsewhere} \end{cases}.$$

At this point, we inform the reader of a misprint in [H], where we used $\alpha = 2$ for $n = 6$, too, which causes a problem with the assertion of Proposition 9. Clearly, $\tilde{\theta}_n > \theta_n$ increases the number of cusped orbifolds with small volume. The lemmata in [HK], [H], which correspond to the Propositions 8, 11, 14 and 15 above still hold for $\tilde{\theta}_n$. In fact, it suffices to prove that Proposition 11 holds.

Proof of Proposition 11 with $\tilde{\theta}_n$. Suppose that Γ_∞ does not identify all fullsized horoballs, and that B_∞ is at euclidean height 1. As in the proof above of Proposition 11, a fundamental domain for Λ_∞ in ∂B_∞ contains the orthogonal projections K_1 and K_2 of at least two disjoint fullsized horoballs in $\mathcal{B}_\infty$. For $n-1=2$, the claim follows immediately since $\delta_2 = d_2$ (compare with [Ad3, Lemma 2.2]). In dimensions $4 \le n \le 9$ we need to study more carefully the Γ_∞-orbits of K_1 and K_2. Under Γ_∞, these two balls of radius 1/2 give rise to two lattice packings

$$\mathcal{P}_{K_1} := \left\{\lambda K_1 : \lambda \in L_i^{n-1}\right\} \text{ and } \mathcal{P}_{K_2} := \left\{\lambda K_2 : \lambda \in L_i^{n-1}\right\}.$$

Here, we used the fact that the lattice subgroup of Γ_∞ is isomorphic to L_i^{n-1} as guaranteed by the Corollary to the Proposition 9. The most efficient way to combine both packings to the periodic packing $\mathcal{P}_{K_1} \cup \mathcal{P}_{K_2}$ is by putting the centers of the balls in $\mathcal{P}_{K_2}$ into the deep holes of $\mathcal{P}_{K_1}$. The distance from a deep hole to a nearest lattice point in L_i^{n-1} equals the covering radius R of L_i^{n-1}. In order to avoid any overlapping of balls in $\mathcal{P}_{K_1} \cup \mathcal{P}_{K_2}$, the lattice L_i^{n-1} has to be rescaled by a factor $\frac{1}{R}$. This rescaling increases the volume of L_i^{n-1} by a factor $\left(\frac{1}{R}\right)^{n-1}$. Using the values of R given in the tables of Appendix A2, we verify that

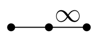
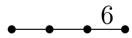
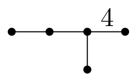
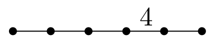
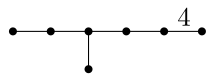
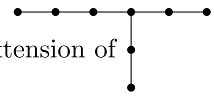
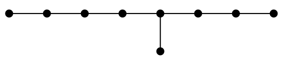
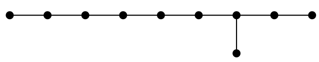
$$\mathrm{vol}_{n-1}\left(P_{\Lambda_\infty}\right) \geq \left(\frac{1}{R}\right)^{n-1} \mathrm{vol}_{n-1}\left(L_i^{n-1}\right) \geq \alpha V_{n-1},$$

which, together with (24) and (51), contradicts (60). □

7 Main theorem

We are now ready to formulate and prove our main theorem.

Main Theorem. *For $4 \leq n \leq 9$, let H^n/Γ_* be a cusped hyperbolic orbifold of minimal volume μ_n. Then, up to isomorphism, Γ_* is related to a hyperbolic Coxeter simplex group according to the following table, and as such uniquely determined.*

Dim n	Γ_*	μ_n	
2		$\frac{\pi}{6}$	$\approx 5.23 \cdot 10^{-1}$
3		$\frac{1}{8}\mathbf{JI}\left(\frac{\pi}{3}\right)$	$\approx 4.23 \cdot 10^{-2}$
4		$\frac{\pi^2}{1'440}$	$\approx 6.85 \cdot 10^{-3}$
5		$\frac{7\,\zeta(3)}{46'080}$	$\approx 1.83 \cdot 10^{-4}$
6		$\frac{\pi^3}{777'600}$	$\approx 3.98 \cdot 10^{-5}$
7	$\mathbb{Z}_2$-extension of 	$\frac{\sqrt{3}\ L(4,3)}{1'720'320}$	$\approx 9.46 \cdot 10^{-7}$
8		$\frac{\pi^4}{4'572'288'000}$	$\approx 2.13 \cdot 10^{-8}$
9		$\frac{\zeta(5)}{22'295'347'200}$	$\approx 4.65 \cdot 10^{-11}$

Proof. Let $4 \leq n \leq 9$ and H^n/Γ_* be a cusped hyperbolic orbifold of minimal volume μ_n. A look at the list of Coxeter simplex volumes in [JKRT] reveals that H^n/Γ_* is of small volume. To see this, we denote by T^n the unbounded hyperbolic Coxeter simplex of smallest volume ξ_n in dimension $n \neq 7$. In dimension 7, the smallest volume unbounded Coxeter simplex bisects in an obvious way into two congruent copies of the unbounded hyperbolic simplex T^7 of volume ξ_7. In each case

$$\mu_n \leq \xi_n < \theta_n .$$

It is important to note that the volumes ξ_n equal the values given in the right column of the table above, and that the simplices T^n are related to the Coxeter graphs in the middle column. To prove the theorem, it is therefore sufficient to show that a fundamental polytope of a discrete subgroup $\Gamma_* < I\,(H^n)$ with covolume μ_n is isometric to T^n.

Since H^n/Γ_* has small volume, it verifies all conditions given by the propositions and corollaries in §6.3. In particular, the parabolic subgroup associated with the unique cusp of H^n/Γ_* is isomorphic to the symmorphic crystallographic group $\mathrm{Aut}_a\left(L_i^{n-1}\right)$ whose point group $G_{n-1} = \mathrm{Aut}\left(L_i^{n-1}\right)$ has maximal order φ_{n-1}.

Modulo conjugation, assume $\Gamma_\infty \neq 1$. In what follows, $\mathcal{B}_\infty$ denotes the horoball packing covering the maximal cusp. In a similar way, the maximal horoball $B_\infty(1) = B_\infty \in \mathcal{B}_\infty$ based at ∞ is supposed to be at euclidean height 1. Write $S_\infty = \partial B_\infty$. In this setting, the induced metric on S_∞ is the standard euclidean metric. Finally, modulo conjugation, suppose the fullsized horoballs in $\mathcal{B}_\infty$ to be based at the lattice points $L_1^{n-1}(0) \subset \partial U^n$. As usually, the one based at 0 is termed B_0.

At this point it is worth while to inform the reader of a little difference with the setting in our proofs of [H] and [HK]. Instead of fixing the minimal translation length of L_i^{n-1} to 1 we now fix to 1 the euclidean height of B_∞. The reason is that this alternative setting is the one used by Adams in his work. Since the euclidean height of the canonical horoball based at ∞ equals the minimal translation length of L_i^{n-1}, Step 2a of the proof will show that both settings are in fact equivalent.

For the proof, we proceed in four steps:

Step 1. Γ_∞ is isomorphic to $\mathrm{Aut}_a\left(L_1^{n-1}\right)$.

Since H^n/Γ_* has small volume, Γ_∞ is isomorphic to $\mathrm{Aut}_a\left(L_i^{n-1}\right)$. It is thus sufficient to exclude the possibilities $i \neq 1$. To this end, we recall that the volume of the canonical cusp is given by formula (52). Note that the lattice volume used in this formula corresponds to the normalized lattice. Substitute the volumes of the normalized lattices L_i^{n-1} given in the tables of Appendix A2. Together with the lower bound (51), we have the following inequality

$$\frac{\mathrm{vol}_{n-1}\left(L_i^{n-1}\right)}{(n-1)\ \varphi_{n-1}\ d_n(\infty)} \leq \mathrm{vol}_{n-1}\left(H^n/\Gamma_*\right) \leq \xi_n\ ,$$

which is only satisfied if $i = 1$.

Step 2a. The maximal cusp is canonical.

By assumption, the maximal horoball B_∞ is at euclidean height 1. The euclidean height of the canonical horoball equals the minimal norm in L_1^{n-1}. Therefore we have to show that the minimal norm ω in L_1^{n-1} equals 1.

Suppose $\omega > 1$. Then the fullsized horoballs in $\mathcal{B}_\infty$ are not touching one another. An element $\Phi \in \Gamma_*$ with $\Phi(B_0) = B_\infty$ maps B_∞ onto a fullsized horoball which touches B_∞ in p, say. By Proposition 11, L_1^{n-1} identifies all fullsized horoballs. Hence, there is a $\lambda \in L_1^{n-1}$ such that $\lambda \circ \Phi \in \Gamma_*$ permutes B_0 and B_∞.

Let $\mathcal{F}$ be the set of all fullsized horoballs based at *minimal* elements $x_i \in L_1^{n-1}$, that is $\|x_i\| = \omega$. Lemma 1 implies that the isometry Φ maps $\mathcal{F}$ onto the set of all non-fullsized horoballs touching the fullsized horoball $\Phi(B_\infty)$ at points which lie at a distance ω from p on ∂B_∞. By Lemma 5, $\lambda \circ \Phi$ maps $\mathcal{F}$ onto the set $\mathcal{N}$ of all non-fullsized horoballs touching B_0 with base points y_i of norm $\frac{1}{\omega}$.

We now analyze the relative position of the points x_i and y_i. Consider the vertex figure $V(0) = V$ of L_1^{n-1} in 0. By means of Wythoff's construction explained in Chapter 3, the ringed graphs in Appendix A1 define V up to similarity. Being minimal, the points x_i lie on the lines passing through 0 and the vertices of V.

Since $\mathcal{B}_\infty$ is precisely Γ_*-invariant, both sets $\mathcal{F}$ and $\mathcal{N}$ are setwise invariant under the point group $\mathrm{Aut}\left(L_1^{n-1}\right)$. By definition, $\mathrm{Aut}\left(L_1^{n-1}\right)$ is the symmetry group of V. Therefore, the $\mathrm{Aut}\left(L_1^{n-1}\right)$-orbits of x_i and y_i have equal length N^0, where N^0 denotes the number of vertices of V. In Appendix A1, we use the method described in last paragraph of Chapter 3 in order to determine the number N_j^d of d-faces in V equivalent to a given face F_j^d under the action of $\mathrm{Aut}\left(L_1^{n-1}\right)$.

If $n \neq 5$, then $N_j^d \neq N^0$ for all $0 < d \leq n-1$ and j. Hence, the points y_i are colinear with 0 and x_i. In dimension 5, the vertex figure of L^4 is a self-dual 24-cell and the point y_i could also be colinear with 0 and the midpoints m_i of the 3-sides. In that case, the rays $(0, x_i)$ and $(0, y_i)$ would subtend an angle $\frac{\pi}{4}$. All horoballs are disjoint so that we conclude in both cases that

$$\begin{aligned} 2\tfrac{1}{\omega} \leq \omega \quad &\Longleftrightarrow \quad \omega \geq \sqrt{2} \quad , \text{ if } n \neq 5 \\ 2\tfrac{1}{\omega}\cos\left(\tfrac{\pi}{4}\right) \leq \omega \quad &\Longleftrightarrow \quad \omega \geq \sqrt[4]{2} \quad , \text{ if } n = 5. \end{aligned}$$

Together with (52) and (51), we obtain the following contradiction

$$\mu_n = \mathrm{vol}_{n-1}\left(H^n/\Gamma_*\right) \geq \frac{\omega^{n-1} V_{n-1}}{(n-1)\ \varphi_{n-1}\ d_n(\infty)} > \xi_n \geq \mu_n \ .$$

Therefore, $\omega = 1$ and the maximal horoball is canonical.

Step 2b. A special element in Γ_* .

We come back to the element $\lambda \circ \Phi \in \Gamma_*$. It fixes the touching point $(0,1)$ of B_∞ with B_0 and transposes 0 and ∞. Therefore $\lambda \circ \Phi$ is an elliptic element which leaves the geodesic $(0, \infty)$ setwise invariant. If σ denotes the reflection with respect to the bisector H_0 of B_∞ and B_0, we can write

$$\lambda \circ \Phi = \psi \circ \sigma,$$

where ψ is a hyperbolic isometry fixing the line $(0, \infty)$ pointwise. Because of $\omega = 1$, we have $\mathcal{F} = \mathcal{N}$. By definition, σ fixes the base points x_i of elements in $\mathcal{F}$ whereas ψ permutes the x_i and fixes their centroid 0. Consequently, ψ belongs to the symmetry group of the vertex figure V. Hence, $\sigma \in \Gamma_*$.

Step 3. Construction of a fundamental domain for Γ_*.

In what follows denote by T_0 a fundamental simplex of $\mathrm{Aut}_a\left(L_1^{n-1}\right)$. Then, a fundamental domain F^n for Γ_* is given by the intersection of the cylinder $Z^n = T_0 \times (0, \infty)$ with the Dirichlet-Voronoĭ-cell of B_∞, that is,

$$F^n = Z^n \cap DV\left(B_\infty\right).$$

By definition, $DV\left(B_\infty\right)$ is the set of all hyperbolic points lying above the bisectors of B_∞ with its Γ_*-images in $\mathcal{B}_\infty$. When intersecting with $DV\left(B_\infty\right)$, the cylinder Z^n is at least chopped by H_0, where H_0 is the mirror hyperplane of σ as above. Let Δ be the simplex of finite volume and vertex ∞ arising from Z^n by a cut along H_0. In the tables of Appendix A2, we listed the vertices v_i of T_0. By construction, the ordinary vertices of Δ are of the form

$$\tilde{v}_i := \left(v_i, \sqrt{1 - \|v_i\|^2}\right) \ , \ 1 \leq i \leq n. \tag{61}$$

A detailed study of $\mathcal{B}_\infty$ will allow to prove the following

$$\textbf{Assertion:} \quad \Delta = F^n. \tag{62}$$

The verification of this equality is quite technical and needs a case-by-case analysis with respect to the dimension n.

Given a non-fullsized horoball $B_q \in \mathcal{B}_\infty$ based at $q \in \partial H^n$ and of euclidean diameter $0 < h < 1$, we have to exclude a non-empty intersection of Δ with the bisector H_q of B_∞ with B_q. Note that the euclidean *radius* of H_q equals $\sqrt{h}$ by formula (1). We call *first generation* of horoballs the set $\mathcal{G}_1 = \mathcal{G}_1(B_0)$ of horoballs in $\mathcal{B}_\infty$ touching B_0. The base point q of an element in $\mathcal{G}_1$ is the σ-image of a point in $L_1^{n-1}(0)$, that is,

$$q = \sigma(x) = \frac{x}{\|x\|^2}, \text{ for some } x \in L_1^{n-1}(0).$$

Since L_1^{n-1} is a root lattice, we know by (32) that $\|x\|^2 \in \mathbb{N}$. By means of Lemma 5, the euclidean diameter h of an element B_q in $\mathcal{G}_1$ verifies

$$\sqrt{h} = \|q\| \Longleftrightarrow h = \frac{1}{j},\ j \in \mathbb{N}. \tag{63}$$

The bisector H_q thus has a euclidean radius of the form $\frac{1}{\sqrt{j}}$, $j \in \mathbb{N}$. More precisely, H_q is the hemisphere parametrized by

$$H_q = \left\{ y \in U^n : \|y - (q,0)\| = \frac{1}{\sqrt{j}} \right\}.$$

In the next lines, we say that H_q is *generated* by B_q. We remark that the intersection of H_q with $H_0 = \{y \in U^n : \|y\| = 1\}$ is given by

$$H_0 \cap H_q = \left\{ y \in U^n : \langle y, (q,0) \rangle = \frac{1}{2} \right\}. \tag{64}$$

Dimensions $n = 4, 5$ and 9 represent the warm-up case. Indeed, for these n, the diameter d of T_0 satisfies

$$\max_i \|v_i\| = d \le \frac{1}{\sqrt{2}}.$$

Therefore, by (61), the n^{th} coordinate of a vertex $\tilde{v}_i$ of Δ is at least $\frac{1}{\sqrt{2}}$, and no bisector intersecting Δ can be generated by an element of the $\mathcal{G}_1$-family. Proposition 15 implies, in particular, that the largest non-fullsized horoballs belong to $\mathcal{G}_1$ and the assertion follows.

Dimension $n = 6$ or 7. The diameter d of T_0 satisfies

$$\frac{1}{\sqrt{2}} < d \le \sqrt{\frac{2}{3}},$$

forcing the n^{th} coordinate of a vertex $\tilde{v}_i$ of Δ to exceed $\frac{1}{\sqrt{3}}$. In other words, only the bisectors generated by horoballs of euclidean diameter $\le \frac{1}{3}$ may be excluded immediately.

We call *second generation* $\mathcal{G}_2 = \mathcal{G}_2(B_0)$ the set of all horoballs touching a largest non-fullsized horoball $B_u \in \mathcal{G}_1$ which has non-empty intersection with Z^n. By (63), the euclidean diameter of B_u equals $\frac{1}{2}$. For its base point u, we deduce that $u = \frac{1}{2}\lambda$, $\lambda \in L_1^{n-1}$ with $\|\lambda\| = \sqrt{2}$. In other words, u is the vertex of T_0 of norm $\frac{1}{\sqrt{2}}$.

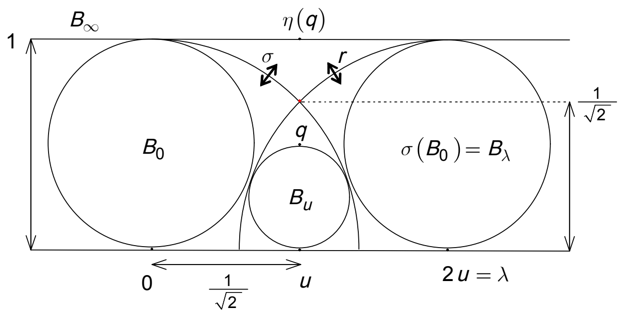
Proposition 14 implies that it is sufficient to exclude the chopping of Δ by the bisector H_u of B_u with B_∞ as well as the existence of a horoball of euclidean diameter $\frac{1}{3} < h < \frac{1}{2}$ in $\mathcal{G}_2$.

First we show the impossibility of the chopping. In the tables of Appendix A2, all vertices v_i are listed. There, the only vertex of norm exceeding $\frac{1}{\sqrt{2}}$ is symbolized by the letter t. We plug the coordinates of u and t into (61) and, by means of (64), conclude that the vertices $\tilde{u}$ and $\tilde{t}$ (cf. 61) of Δ belong to $H_0 \cap H_u$. Hence, $H_0 \cap H_u$ contains a 5-side of Δ, and non-trivial chopping of Δ by H_u does not happen.

Now, we use the properties of root lattices to show that the set of possible diameters h for horoballs in $\mathcal{G}_2$ is a subset of $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$. To this end, we consider the element $\eta := \sigma \circ r$, where r is the reflection with respect to the bisector H_λ of B_∞ and $\sigma(B_u) = B_\lambda$. Note that η is a half-turn around $\tilde{u}$ which permutes B_∞ and B_u. By Step 2b, $\eta \in \Gamma_*$. Since the point $(u, 1) \in S_\infty$ is transposed with the north pole $q = (u, 1/2) \in \partial B_u$ under η, the distances δ on ∂B_u from q to the touching points of B_u with $\mathcal{G}_2$-horoballs equal the distances in S_∞ from $(u, 1)$, or equivalently, from $u = \frac{1}{2}\lambda$ to lattice points l in $L_1^{n-1}(0)$. Hence

$$\delta = \|u - l\| = \sqrt{\frac{1}{4}\|\lambda\|^2 + \|l\|^2 - \langle \lambda, l \rangle} \; .$$

The properties of root lattices then imply that $\|\lambda\|^2 \in 2\mathbb{N}$, $\|l\|^2 \in \mathbb{N}$, and that $\langle \lambda, l \rangle$ is a linear combination of elements in $\left\{0, -\frac{1}{2}, 1\right\}$. We conclude that $\delta \in \sqrt{\frac{j}{2}}$, $j \in \mathbb{N}$, and by Corollary 7, $h \in \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$.



Dimension 8. Since the diameter verifies

$$d = \frac{\sqrt{3}}{2} \; ,$$

only horoballs of euclidean diameter $\leq 1/4$ may directly be excluded. We use again the coordinates given in the table of Appendix A2.

First, we show, by the same argument as in dimensions 6 and 7, that no horoball of the $\mathcal{G}_1$-family generates a bisector chopping Δ.

Then, again by the same argument as in dimensions 6 and 7, we show that the spectrum of possible euclidean diameters for elements in $\mathcal{G}_2$ is a subset of $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$. By Proposition 15, those horoballs of diameter $\frac{1}{2}$ belong to $\mathcal{G}_1$. Therefore, we only need to exclude the chopping of Δ by the bisector generated by a $\frac{1}{3}$-ball B_v in $\mathcal{G}_2$:

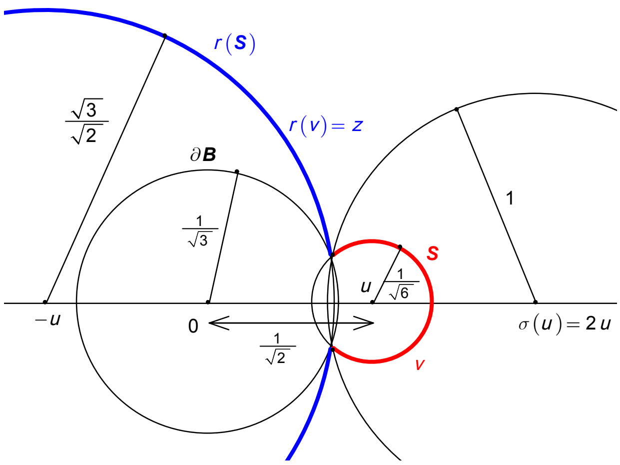
Since B_v is touching B_u, Corollary 7 shows that v lies on the sphere $\mathbb{S}$ of radius $\frac{1}{\sqrt{6}}$ centered in u. The reflection $r \in \Gamma_*$ defined as above interchanges B_v with a horoball $B' \in \mathcal{G}_1$. Moreover, v must lie outside the ball $\mathbb{B}$ of radius $\frac{1}{\sqrt{3}}$ centered at 0 since it has no overlap with B_0. The intersection $\partial\mathbb{B} \cap \mathbb{S}$ is a subset of the unit sphere centered at $\sigma(u)$. In fact, this sphere is the equator of the mirror hyperplane of r. We deduce that B' is of euclidean diameter $h \geq \frac{1}{3}$.

If $h = \frac{1}{3}$, then $B' = B_v \in \mathcal{G}_1$ and hence does not effect the construction of F^n. If $h = 1$, then $B_v \in \mathcal{G}_1(\sigma(B_u))$. Therefore, the bisector of B_v and B does not chop Δ.

We are left with the case $h = \frac{1}{2}$. The base points of largest non-fullsized horoballs in H^8 have coordinates of the form

$$\frac{1}{2\sqrt{2}}\left(2, 0^7\right), \frac{1}{2\sqrt{2}}\left(1^4, 0^4\right), \frac{1}{2\sqrt{2}}\left(1^2, 0^4, 1^2\right), \text{ or } \varepsilon\frac{1}{2\sqrt{2}}\left(\frac{3}{2}, \frac{1}{2}^7\right) \qquad (65)$$

from which the full list can be deduced by applying arbitrary permutations and signs to the first 6 vector entries, except for the vector with prefix ε, where an even number of minus signs is required for the first 6 vector entries (compare with [CS, Table 4.10]).



Consider the base point z of B'. It belongs to the list (65) and is a point of the sphere $r(\mathbb{S})$ of radius $\sqrt{3}/\sqrt{2}$ centered at $-u$. The bisector H_v of B_v and B_∞, whose radius equals $\frac{1}{\sqrt{3}}$, chops Δ if the following inequality holds:

$$\|r(z) - t\| < \frac{1}{2\sqrt{3}}, \qquad (66)$$

with t the only vertex of T_0 whose norm exceeds $\frac{1}{\sqrt{2}}$. However, there is no solution z with $v = r(z)$ satisfying condition (66). Therefore, the H_v does not chop Δ.

Finally, we have to consider the *third generation* $\mathcal{G}_3 = \mathcal{G}_3(B_0)$ consisting of all horoballs touching a non-fullsized horoball B_w intersecting Z^n and being of euclidean diameter $\frac{1}{3}$. The family $\mathcal{G}_3$ splits into two subsets $\mathcal{G}_3^1$ and $\mathcal{G}_3^2$, where $\mathcal{G}_3^1$ represents the case $B_w \in \mathcal{G}_1$, and $\mathcal{G}_3^2$ the case $B_w \in \mathcal{G}_2$. By Proposition 14 and 15, the assertion (62) follows if we can exclude the existence of horoballs in $\mathcal{G}_3$ having euclidean diameter $\frac{1}{4} < h < \frac{1}{3}$. We treat both cases $\mathcal{G}_3^1$ and $\mathcal{G}_3^2$ separately to show that the spectrum of possible euclidean diameters for horoballs in $\mathcal{G}_3$ is a subset of $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$.

For the subset $\mathcal{G}_3^1$, we find the base point w of B_w among the vertices of T_0 (compare the table in Appendix A2). Note that $\|w\| = \frac{1}{\sqrt{3}}$ and $\sigma(w) = 3w$. Denote by f the reflection with respect to the bisector of B_∞ and the fullsized horoball $\sigma(B_w)$. Then, $\sigma \circ f \in \Gamma_*$ conjugates the lattice L_1^7 acting on S_∞ to the lattice Λ' acting on ∂B_w. In particular, the touching point p of B_w and B_0 is the image of $(\sigma(w), 1)$ under $\sigma \circ f$. By Lemma 5, the distance from the north pole $q \in B_w$ to p equals $\frac{1}{\sqrt{3}}$ on ∂B_w. We deduce that q is the $(\sigma \circ f)$-image of the point

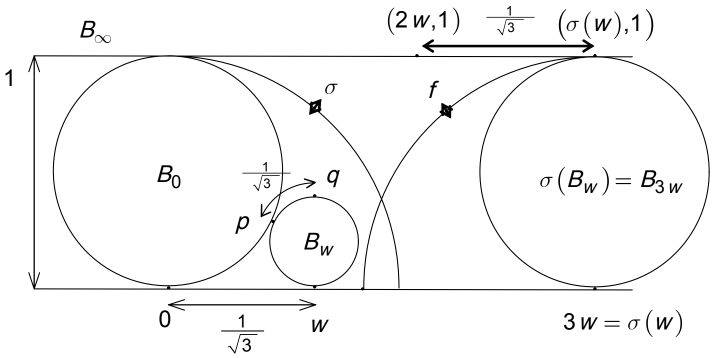
$$(a, 1) := \left(\frac{\|\sigma(w)\| - \frac{1}{\sqrt{3}}}{\|\sigma(w)\|} \sigma(w), 1 \right) = \left(\frac{2}{3}\sigma(w), 1 \right) = (2w, 1). \qquad (67)$$

Now, each $B' \in \mathcal{G}_3$ touches B_w at a point b of Λ'. The distance δ on ∂B_w from b to the north pole q of B_w equals the distance on S_∞ from $(c, 1) := \sigma \circ f(b) \in L_1^7(0) \subset S_\infty$ to $(a, 1)$, or equivalently, from c to a. We compute

$$\begin{aligned} \delta = \|a - c\| &= \sqrt{\|a\|^2 + \|c\|^2 - 2\langle a, c\rangle} \\ &\overset{(67)}{=} \sqrt{4\|w\|^2 + \|c\|^2 - 2\langle w, c\rangle} \\ &= \sqrt{\tfrac{4}{3}\|\sigma(w)\|^2 + \|c\|^2 - \tfrac{2}{3}\langle \sigma(w), c\rangle}. \end{aligned}$$

Since $\sigma(w)$ and c are points of the root lattice L_1^7, we conclude that

$$\delta \in \left\{ \frac{1}{\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3}}, 1, \frac{2}{\sqrt{3}}, \ldots \right\}.$$



By Corollary 7, the euclidean diameter of B' thus equals $\frac{1}{3\delta^2} \in \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$ as pretended.

For the subset $\mathcal{G}_3^2$, the horoball B_w touches B_u. Therefore, its base point w belongs to the points defined as $v = r(z)$, where z is in the list (65). Denote by g the reflection with respect to. the bisector of $B_w = B_{r(z)}$ and B_u. Observe that $\tau := \eta \circ g = \sigma \circ r \circ g \in \Gamma_*$.

By means of τ, the lattice L_1^7 on S_∞ conjugates to the lattice $\tilde{\Lambda}$ acting on $\partial B_{r(z)}$. In particular, τ sends the touching point $\tilde{p}$ of $B_{r(z)}$ and B_u to a lattice point $\tau(\tilde{p})$ of L_1^7. Let $\tilde{q}$ and q denote the north poles of $B_{r(z)}$ and B_u. Using $g\left(B_{r(z)}\right) = B_u$ and $\eta\left(B_u\right) = B_\infty$ together with Lemma 5, we deduce successively

$$d(\tilde{p}, \tilde{q}) = d(\tilde{p}, g(\tilde{q})) = \tfrac{\sqrt{2}}{\sqrt{3}} \; ; \; d(q, \tilde{p}) = \tfrac{\sqrt{3}}{\sqrt{2}} \; ,$$
and
$$d(q, g(\tilde{q})) = d(\eta(q), \tau(\tilde{q})) = \tfrac{\sqrt{3}}{\sqrt{2}} - \tfrac{\sqrt{2}}{\sqrt{3}} = \tfrac{1}{\sqrt{6}} \; .$$

and determine $\tau(\tilde{q})$ according to

$$\tau(\tilde{q}) = \eta(q) + \frac{u - r(z)}{\|u - r(z)\|} \; d(\tau(\tilde{q}), \eta(q)) = (2u - r(z), 1) \; . \tag{68}$$

The distance δ from $\tilde{q}$ to a lattice point $\tilde{c}$ of $\tilde{\Lambda}$ equals the distance from $\tau(\tilde{q})$ to $\tau(\tilde{c}) = c \in L_1^7$. Hence,

$$\begin{aligned} \delta = & \;\; \|\tau(\tilde{q}) - c\|^2 = \|\tau(\tilde{q}) - \tau(\tilde{p}) + \tau(\tilde{p}) - c\|^2 \\ = & \;\; \|\tau(\tilde{q}) - \tau(\tilde{p})\|^2 + \|\tau(\tilde{p}) - c\|^2 - 2\|\tau(\tilde{p})\|^2 + 2\langle\tau(\tilde{p}), c\rangle + 2\langle\tau(\tilde{q}), \tau(\tilde{p}) - c\rangle \; . \end{aligned}$$

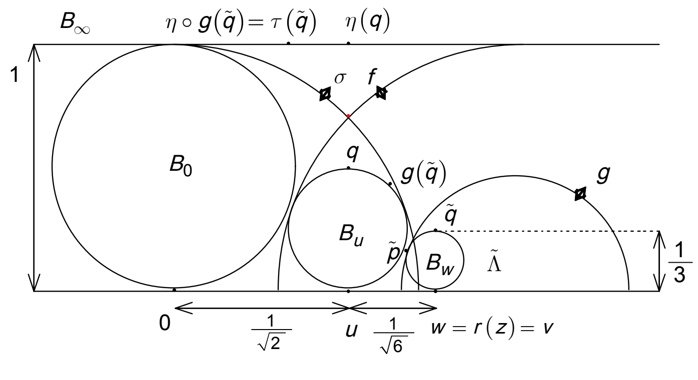
Corollary 7 yields $\|\tau(\tilde{q}) - \tau(\tilde{p})\|^2 = \frac{2}{3}$. Moreover,

$$\|\tau(\tilde{p}) - c\|^2 - 2\|\tau(\tilde{p})\|^2 + 2\langle\tau(\tilde{p}), c\rangle$$

is an integer since $\tau(\tilde{p}), c \in L_1^7$ are integral linear combinations of the basis vectors $\alpha_1, \ldots, \alpha_{n-1}$ satisfying $\langle\alpha_i, \alpha_k\rangle \in \{0, -\frac{1}{2}, 1\}$. Furthermore, one checks, with (68) and (65), that every $\tau(\tilde{q}) \in L_1^7$ is of the form

$$\tau(\tilde{q}) = \frac{1}{3} \; l \; , \; l \in L_1^7,$$

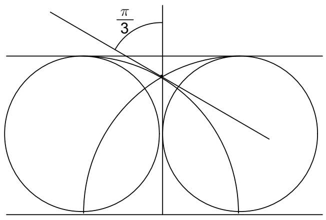
so that $2\langle\tau(\tilde{q}), \tau(\tilde{p}) - c\rangle \in \frac{1}{3}\mathbb{Z}$. We conclude that $\delta \in \frac{1}{3}\mathbb{Z}$. In other words, the possible euclidean diameters $\frac{1}{3\delta^2}$ for elements in $\mathcal{G}_3^2(B_w)$ belong to a subset of $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$.



Step 4. The geometric presentation of Γ_* .

By Step 3, a fundamental domain F^n for Γ_* is constructed by cutting the cylinder Z^n along the mirror hyperplane H_0 of σ. All $(n-1)$ bounding hyperplanes of Z^n which pass through 0 are perpendicular to H_0. The remaining face of Z^n is the bisector H_1 of B_0 with a fullsized horoball touching B_0. As indicated in the figure below, the dihedral angle subtended by H_0 and H_1 equals $\frac{\pi}{3}$.

If $n \neq 7$ $(n = 7)$, the simplex F^n is given by the Coxeter (weighted) graph obtained when adding to the Coxeter (weighted) graph of $\mathrm{Aut}_a\left(L_1^{n-1}\right)$, given in the tables of Appendix A2, one node together with an unmarked branch incident with the node 0. We conclude that F^n is isometric to the simplex T^n. Moreover, Step 1 and 2b show that for $n \neq 7$, Γ_* is generated by the reflections with respect to the sides of F^n. Therefore, Γ_* is isomorphic to the Coxeter group generated by the reflections in the side hyperplanes of F^n. In dimension 7, we should not forget to add the Poincaré extension of the involution ε defined in Chapter 4.



This finishes the proof of our main theorem. □

Alternative version of Step 4. In dimensions $n = 5, 6, 7, 8$, we can determine the geometric presentation of Γ_* in an alternative way. We decided to include this version here since it gives a better insight into the symmetry properties of the horoball packing $\mathcal{B}_\infty$.

Step 1 says that the parabolic subgroup of Γ_* is isomorphic to $\mathrm{Aut}_a\left(L_1^{n-1}\right)$. The Coxeter graph associated with $\mathrm{Aut}_a\left(L_1^{n-1}\right)$ reveals the existence of a very special subgroup, the group A_{n-1}, whose graph is obtained by omitting the node i. The node i represents the vertex v_i of the fundamental Coxeter simplex for $\mathrm{Aut}_a\left(L_1^{n-1}\right)$ characterized by $\|v_i\| = 1/\sqrt{2} = d$. This vertex is a deep hole equidistant from n mutually touching balls in the L_1^{n-1}-lattice packing $\mathcal{B}_0$ which, together with their centers, are permuted by means of the subgroup A_{n-1} of $\mathrm{Aut}_a\left(L_1^{n-1}\right)$.

After Poincaré extension, these balls become mutually tangent fullsized horoballs $B_0, \ldots, B_n$ in H^n. Here, B_0 is the horoball based at 0 as before. The reflection $\sigma \in \Gamma_*$ defined in Step 2b transposes B_0 and $B_\infty =: B_{n+1}$ and fixes $B_1, \ldots, B_{n-1}$ since the mirror hyperplane H_0 of σ touches ∂H^n at their base points. Their radical point (intersection of the $n+1$ bisector hyperplanes) equals the vertex $\tilde{v}_i$ of F^n lying above v_i. The base points of the horoballs $B_0, \ldots, B_{n+1}$ span an ideal, regular simplex $S^n_{\mathrm{reg}}(\infty)$ with center $\tilde{v}_i$ whose symmetry group A_n is generated by A_{n-1} and σ. We deduce that A_n is an elliptic Coxeter subgroup of Γ_*.

By Step 3, we know that a fundamental domain F^n for Γ_* is a hyperbolic simplex with precisely one ideal vertex. Moreover, Γ_* contains the elliptic subgroup A_n

and the parabolic subgroup $\mathrm{Aut}_a\left(L_1^{n-1}\right)$ represented by Coxeter graphs with n nodes representing n of the $n+1$ bounding hyperplanes of the simplex F^n. In Chapter 3, we explained that the passage from F^n to one of its vertex figures corresponds to the omission of one node together with its connecting edges in its weighted graph. We conclude that the only simplex with the desired vertex figures is the Coxeter simplex T^n. Finally, Γ_* is isomorphic to the Coxeter group represented by the graph of T^n.

8 About dimension 10

As shows the proof of the main theorem, our method for constructing minimal volume hyperbolic orbifolds holds for arbitrary n. The success of our method in dimensions $n \leq 9$ is due to the existence of hyperbolic Coxeter simplices serving as building blocks and the fact that the densest lattice packings are related to irreducible root lattices. In higher dimensions, however, densest lattice packings are not as yet identified and the number of familiar hyperbolic orbifolds is very small.

We decided to conclude this work with some remarks on dimension 10. In particular the following results are proven:

Proposition 16 *The value μ_n satisfies*

$$1.87 \cdot 10^{-11} \leq \mu_n \leq \frac{\pi^5}{5'431'878'144'000} \simeq 5.64 \cdot 10^{-11}.$$

Proposition 17 *The minimal volume cusped hyperbolic 10-orbifold has at most two cusps.*

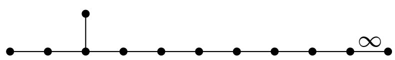
Proposition 18 *The point group ϕ_∞ is isomorphic to a subgroup of the direct product $W(E_8) \times \{\pm 1\}$, and its lattice subgroup is isomorphic to $E_8 \oplus \mathbb{Z}$.*

Several cusps. By taking into account the touching points between maximal cusps, Adams could improve, in dimension 3, the lower volume bound (54) for m-cusped hyperbolic n-orbifolds (compare [Ad3, Lemma 2.3 & 4.2] and [Ad1, Lemma 3.3]). More precisely, he could replace m by $2m$ in the manifold case and by $2m-1$ in the orbifold case, respectively. The bigger constant $2m$ in the manifold case is due to the presence of two non-equivalent fullsized horoball under the action of Γ_∞ (see results of Adams in Chapter 6.1 and Proposition 11). His arguments generalize to higher dimensions as in the manifold case [K1].

Lemma 19 *Let H^n/Γ be an m-cusped hyperbolic n-orbifold. Then,*

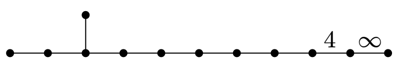
$$\mathrm{vol}_n\left(H^n/\Gamma\right) \geq (2m-1)\,\frac{\mathrm{vol}_{n-1}\left(B\left(1/2\right)\right)}{(n-1)\ \varphi_{n-1}\ d_{n-1}\ d_n\left(\infty\right)}\ .$$

Volume computation. Candidates for small or minimal volume in dimension 10 are the Coxeter polytopes P_1^{10} and P_2^{10} determined by the Coxeter graphs



(69)

and



(70)

respectively. In [Vin3], Vinberg studies reflective arithmetic groups associated to the unique, up to integral equivalence, odd unimodular integral quadratic form of type

$$f_n(x) = -x_0^2 + x_1^2 + \cdots + x_n^2$$

with signature $(n,1)$. In dimension 10, the fundamental polytope of the corresponding group is P_2^{10}.

Poincaré's formula (16) allows to compute the volume or the Euler characteristic of even dimensional Coxeter polytopes $P = P^{2m}$. The method goes as follows. Let Σ be the Coxeter graph of P. First, list all spherical subgraphs σ_q of Σ having q nodes, with q varying from 2 to $2m$. Then substitute the normalized spherical volumes of σ_q in the generalized angle sum $W\left(P^{2m}\right)$. Remember that all spherical Coxeter groups are classified, and their corresponding orders are listed in [Vin1, Part II, Chapter 5, Table 1], for example. Since the normalized volume of Σ_q equals the inverse of its order, the volume computation is just a matter of assiduity.

The generalized angle sum or Euler characteristic of P_2^{10} has already been determined by Ratcliffe and Tschantz in [RT2]:

$$W\left(P_2^{10}\right) = \chi\left(P_2^{10}\right) = -\frac{1}{11'147'673'600}.$$

The volume of P_2^{10} therefore computes to

$$\mathrm{vol}_{10}\left(P_2^{10}\right) = \frac{\pi^5}{329'204'736'000}.$$

Note that this is the smallest volume of a hyperbolic orbifold we found in the literature. Now, Schläfli's differential formula (18) implies that increasing one of the dihedral angles while leaving all other angles invariant, yields a polytope of smaller hyperbolic volume. Now, the polytope P_1^{10} is obtained by changing the dihedral angle $\frac{\pi}{4}$ of P_2^{10} to $\frac{\pi}{3}$. As a matter of fact, this change decreases the volume by a factor $\frac{33}{2}$. Indeed, using Poincaré's formula, we find

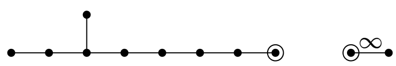
$$\mathrm{vol}_{10}\left(P_1^{10}\right) = \frac{\pi^5}{5'431'878'144'000} = \frac{2}{33}\,\mathrm{vol}_{10}\left(P_2^{10}\right). \tag{71}$$

The list of all spherical subgraphs of P_1^{10} and the calculation of the volume have been included in Appendix A3.

Some results in dimension 10. The Coxeter graph of P_1^{10} shows that the direct product $W(E_8) \times \{\pm 1\}$, which has maximal order φ_9 among all point groups in dimension 9, is a subgroup of (69). This justifies in some sense the use of

$$\theta_{10} := \frac{\pi^5}{5'431'878'144'000}$$

as a good upper bound for μ_{10} although we doubt that $\theta_{10} = \mu_{10}$. Up to isomorphism, there is only one lattice which can be associated to $W(E_8) \times \{\pm 1\}$, namely the root lattice $E_8 \oplus \mathbb{Z}$. For our quantitative purpose, we need information about the angle subtended by the spaces containing $\mathbb{Z}$ and E_8 respectively. If these spaces are mutually orthogonal, then the lattice can be described by a ringed Coxeter graph

$E_8 \perp \mathbb{Z} =$ 

.

However, there is another possibility when the basis vector in $\mathbb{Z}$-direction is of the form $\left(\lambda/2, 1/\sqrt{2}\right)$, with $\lambda \in E_8$ of second smallest norm $\sqrt{2}$. In this case, the angle equals $\frac{\pi}{4}$. The reader should note that $\lambda/2$ is a deep hole of E_8.

The lattice obtained is called *laminated lattice* Λ_9 (compare [CS, Chapter 6]). This second possibility makes the geometric description of possible hyperbolic orbifolds complicated.

Another problem we have to deal with is the fact that a densest lattice packing in dimension 9 is unknown even though Conway and Sloane write in [CS] that it is reasonable to guess that Λ_9 is the densest lattice. The best bound known today is Roger's bound d_9. The values of $\delta(\Lambda_9)$ and the simplicial density d_9 are given in [CS, Table 1.2] :

$$\delta(\Lambda_9) \simeq 0.14577, \; d_9 \simeq 0.19815.$$

In what follows, H^{10}/Γ denotes a cusped hyperbolic 10-orbifold of minimal volume μ_{10}. As usually, we suppose ∞ to be a parabolic fixed point of Γ. Let Γ_∞ be the corresponding parabolic subgroup with lattice subgroup Λ_∞ and point group ϕ_∞.

Proof of Proposition 18. We have seen at the end of Chapter 4 that the reducible direct product $W(E_8) \times \{\pm 1\}$ has *biggest* order $\varphi_9 = 2 \cdot \varphi_8 = 2 \cdot 192 \cdot 10!$ among all point groups in dimension 9. In [PP3], Plesken and Pohst show that the wreath product $C_2 \sim S_9$ is the *irreducible* point group of *maximal* order $2^9 9! = \frac{2}{15}\varphi_9$ in dimension 9. By using our results listed in the tables of Appendix A2, one easily verifies that the order of a *reducible* point group in dimension 9 which is not a subgroup of $W(E_8) \times \{\pm 1\}$ exceeds $\frac{1}{40}\varphi_9$.

Now, suppose ϕ_∞ is not a subgroup of $W(E_8) \times \{\pm 1\}$. Plugging $|\phi_\infty| \leq \frac{2}{15}\varphi_9$ and $V_9 \geq \frac{\mathrm{vol}_9(B(1/2))}{d_9}$ into (54) contradicts the minimal volume assumption by means of

$$\mathrm{vol}_{10}\left(H^{10}/\Gamma\right) \geq \frac{\mathrm{vol}_9\left(B\left(1/2\right)\right)}{9 \frac{2}{15}\; \varphi_9 \; d_9 \; d_{10}(\infty)} > \theta_{10.}$$

□

Proof of Proposition 17. Suppose H^{10}/Γ has more than two cusps. Then Lemma 19 implies the contradiction

$$\mathrm{vol}_{10}\left(H^{10}/\Gamma\right) \geq 5 \frac{\mathrm{vol}_9\left(B\left(1/2\right)\right)}{9 \; \varphi_9 \; d_9 \; d_{10}(\infty)} > \theta_{10.}$$

□

Proof of Proposition 16. The volume of H^{10}/Γ is pinched between the lower bound (54) and $\theta_{10} = \mathrm{vol}_{10}\left(P_1^{10}\right)$. Proposition 18 shows that the lattice acting on the horospheres covering the boundary of a cusp is isomorphic to $E_8 \oplus \mathbb{Z}$. We may therefore replace the simplicial density d_9 used in (54) by the density $\delta(\Lambda_9)$ of the laminated lattice Λ_9. By doing so, we obtain

$$1.87 \cdot 10^{-11} = \frac{\mathrm{vol}_9\left(B\left(1/2\right)\right)}{9 \; \varphi_9 \; \delta_9 \; d_{10}(\infty)} < \mu_{10} < \theta_{10} = \frac{\pi^5}{5'431'878'144'000} \; .$$

□

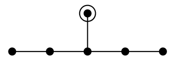
9 Appendix

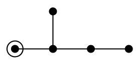
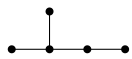
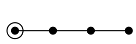
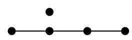
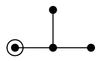
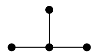
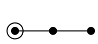
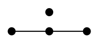
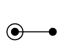
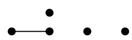
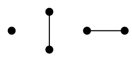
A1. We compute the combinatorics N_i^k of six polytopes defined by ringed Coxeter graphs. Remember that N_i^k equals the number of k-faces of the polytope which are equivalent to a given k-face F_i^k under the action of its symmetry group. Since the vertices of the polytope defined by a ringed Coxeter graph are by definition equivalent, we write N^0 instead of N_1^0 for its number.

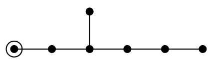
The 3-dimensional cuboctahedron •—⊙—4—•			
k	type of F_m^k	stabiliser of F_m^k	N_m^k
2	⊙—4—•	•—4—•	$N_1^2 = \frac{48}{8} = 6$
	⊙—•	•—•	$N_2^2 = \frac{48}{6} = 8$
1	⊙	•	$N_1^1 = \frac{48}{2} = 24$
0		• •	$N^0 = \frac{48}{2\cdot 2} = 12$

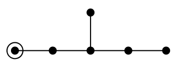
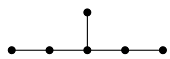
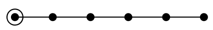
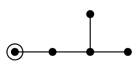
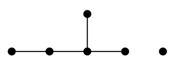
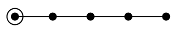
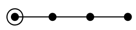
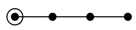
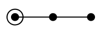
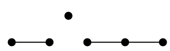
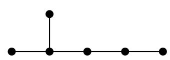
The 4-dimensional 24-cell ⊙—•—4—•—•			
3	⊙—•—4—•	•—•—4—•	$N_1^3 = \frac{1152}{2^3\cdot 3!} = 24$
2	⊙—•	•—• •	$N_1^2 = \frac{1152}{3!\cdot 2} = 96$
1	⊙	• •—•	$N_1^1 = \frac{1152}{2\cdot 3!} = 96$
0		•—•—4—•	$N^0 = \frac{1152}{2^4\cdot 4!} = 24$

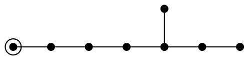
The 5-dimensional polytope •—⊙—•—•—4—•			
4	•—⊙—•—•	•—•—•—•	$N_1^4 = \frac{3840}{5!} = 32$
	⊙—•—•—4—•	•—•—•—4—•	$N_2^4 = \frac{3840}{2^4\cdot 4!} = 10$
3	•—⊙—•	•—• •—•	$N_1^3 = \frac{3840}{2\cdot 4!} = 80$
	⊙—•—•	•—•—•	$N_2^3 = \frac{3840}{4!} = 160$
2	⊙—•	•—• •	$N_1^2 = \frac{3840}{2\cdot 6} = 320$
	⊙—•	•—• •—4—•	$N_2^2 = \frac{3840}{6\cdot 8} = 80$
1	⊙	• •—4—•	$N_1^1 = \frac{3840}{2\cdot 8} = 240$
0		• •—•—4—•	$N^0 = \frac{8\cdot 9!}{2^5\cdot 6!} = 126$

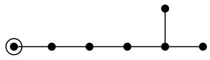
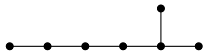
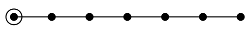
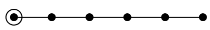
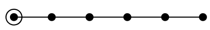
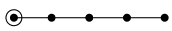
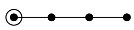
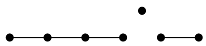
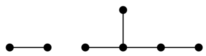
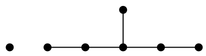
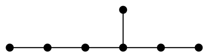
The 6-dimensional polytope 

5			$N_1^5 = N_2^5 = \frac{72 \cdot 6!}{2^4 5!} = 27$
4			$N_1^4 = N_2^4 = \frac{72 \cdot 6!}{2 \cdot 5!} = 216$
			$N_3^4 = \frac{72 \cdot 6!}{2^3 4!} = 270$
3			$N_1^3 = N_2^3 = \frac{72 \cdot 6!}{2 \cdot 4!} = 1080$
2			$N_1^2 = \frac{72 \cdot 6!}{6 \cdot 2 \cdot 2} = 2160$
1			$N_1^1 = \frac{72 \cdot 6!}{2 \cdot 6 \cdot 6} = 720$
0			$N^0 = \frac{72 \cdot 6!}{6!} = 72$

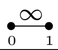
The 7-dimensional polytope 

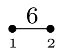
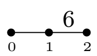
6			$N_1^6 = \frac{8 \cdot 9!}{72 \cdot 6!} = 56$
			$N_2^6 = \frac{8 \cdot 9!}{7!} = 576$
5			$N_1^5 = \frac{8 \cdot 9!}{2 \cdot 2^4 \cdot 5!} = 756$
			$N_2^5 = \frac{8 \cdot 9!}{6!} = 4032$
4			$N_1^4 = \frac{8 \cdot 9!}{5! \cdot 3!} = 4032$
			$N_2^4 = \frac{8 \cdot 9!}{2 \cdot 5!} = 12096$
3			$N_1^3 = \frac{8 \cdot 9!}{4! \cdot 3!} = 20160$
2			$N_1^2 = \frac{8 \cdot 9!}{4! \cdot 3! \cdot 2} = 10080$
1			$N_1^1 = \frac{8 \cdot 9!}{2 \cdot 6!} = 2016$
0			$N^0 = \frac{8 \cdot 9!}{2^5 \cdot 6!} = 126$

The 8-dimensional polytope 

7			$N_1^7 = \frac{192 \cdot 10!}{2^6 \cdot 7!} = 2160$
			$N_2^7 = \frac{192 \cdot 10!}{8!} = 17280$
6			$N_1^6 = \frac{192 \cdot 10!}{2 \cdot 7!} = 69120$
			$N_2^6 = \frac{192 \cdot 10!}{7!} = 138240$
5			$N_1^5 = \frac{192 \cdot 10!}{6! \cdot 2} = 483840$
4			$N_1^4 = \frac{192 \cdot 10!}{5! \cdot 3! \cdot 2} = 483840$
3			$N_1^3 = \frac{192 \cdot 10!}{4! \cdot 5!} = 241920$
2			$N_1^2 = \frac{192 \cdot 10!}{3! \cdot 2^4 \cdot 5!} = 60480$
1			$N_1^1 = \frac{192 \cdot 10!}{2 \cdot 72 \cdot 6!} = 6720$
0			$N^0 = \frac{172 \cdot 10!}{8 \cdot 9!} = 240$

A2. Crystallographic groups of minimal covolume and maximal symmetry.

Dimension 1	
Biggest and second biggest point group order:	
$\varphi_1 = 2$	$\psi_1 = 1$
Point group of maximal order:	
$G_1 = \mathbb{Z}_2$	
Lattice of maximal symmetry:	
$L^1 = \mathbb{Z}$-lattice	
Base vector:	
$\alpha_1 = (1)$	
Volume of the most symmetric lattice:	
$\mathrm{vol}_1(L^1) = 1$	
Fundamental simplex of $\mathrm{Aut}_a(L^1)$ **and vertices different from** 0**:**	
	$v_1 = (1)$

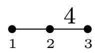
Dimension 2	
Biggest and second biggest point group order:	
$\varphi_2 = 12$	$\psi_2 = 8$
Point group of maximal order:	
$G_2 = \mathrm{Aut}(A_2)$, dihedral group of order 6	
Lattice of maximal symmetry:	
$L^2 = A_2$, hexagonal lattice	
Description of L^2**, modulo normalization:**	
$A_2 = \{(x_0, x_1, x_2) \in \mathbb{Z}^3 : x_0 + x_1 + x_2 = 0\}$	
Base vectors:	
$\alpha_1 = \frac{1}{\sqrt{2}}(-1, 1, 0)$	$\alpha_2 = \frac{1}{\sqrt{2}}(0, -1, 1)$
Volume of the most symmetric lattice:	
$\mathrm{vol}_2(L^2) = \frac{\sqrt{3}}{2}$	
Covering radius:	
$\frac{1}{\sqrt{3}}$	
Fundamental simplex of $\mathrm{Aut}_a(L^2)$ **and vertices different from** 0**:**	
	$v_1 = \frac{1}{\sqrt{2}}(-\frac{2}{3}, (\frac{1}{3})^2)$ $v_2 = \frac{1}{\sqrt{2}}((-\frac{1}{3})^2, \frac{2}{3})$

Dimension 3

Biggest and second biggest point group order:

$\varphi_3 = 48$	$\psi_3 = 24$

Point group of maximal order:

$G_3 = \mathrm{Aut}(A_3)$	

Lattices of maximal symmetry:

$L_1^3 = B_3$, fcc-lattice	$L_2^3 = B_3^*$, bcc-lattice	$L_3^3 = C_3$, cubic lattice

Description of L_1^3, modulo normalization:

$C_3 = A_3 = D_3 = \{(x_1, x_2, x_3) \in \mathbb{Z}^3 : \sum_{i=1}^{3} x_i \equiv 0, \mathrm{mod} 2\}$

Description of L_2^3, modulo normalization:

$C_3^* = A_3^* = D_3^* = (2\mathbb{Z})^3 \cup ((2\mathbb{Z})^3 + (1, 1, 1))$

Description of L_1^3:

$B_3 = \mathbb{Z}^3$

Base vectors:

$\alpha_1 = \frac{1}{\sqrt{2}}(0, 1, 1)$	$\omega_1 = \frac{1}{\sqrt{3}}(2, 0, 0)$	$\beta_1 = (1, 0, 0)$
$\alpha_2 = \frac{1}{\sqrt{2}}(1, 0, 1)$	$\omega_2 = \frac{1}{\sqrt{2}}(0, 2, 0)$	$\beta_2 = (0, 1, 0)$
$\alpha_3 = \frac{1}{\sqrt{2}}(1, 1, 0)$	$\omega_3 = \frac{1}{\sqrt{2}}(1, 1, 1)$	$\beta_3 = (0, 0, 1)$

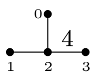
Volume of the most symmetric lattices:

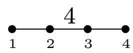
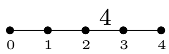
$\mathrm{vol}_3(L_1^3) = \frac{1}{\sqrt{2}}$	$\mathrm{vol}_3(L_2^3) = \frac{4}{3\sqrt{3}}$	$\mathrm{vol}_3(L_3^3) = 1$

Covering radius:

$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{5}}{2\sqrt{3}}$	$\frac{\sqrt{3}}{2}$

Fundamental simplex of $\mathrm{Aut}_a(L_1^3)$ and vertices different from 0:

	$v_1 = \frac{1}{\sqrt{2}}(0, 0, 1)$ $v_2 = \frac{1}{\sqrt{2}}(0, \frac{1}{2}, \frac{1}{2})$ $v_3 = \frac{1}{\sqrt{2}}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$

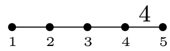
Dimension 4	
Biggest and second biggest point group order:	
$\varphi_4 = 1152$	$\psi_4 = 576$
Point group of maximal order:	
$G_4 = \mathrm{Aut}(D_4)$	
Lattice of maximal symmetry:	
$L^4 = D_4 = D_4^*$, checkerboard lattice	
Description of L^4, modulo normalization:	
$D_4 = \{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 : \sum_{i=1}^{4} x_i \equiv 0, \mathrm{mod} 2\}$	
Base vectors:	
$\alpha_1 = \frac{1}{\sqrt{2}}(-1, -1, 0^2)$	$\alpha_3 = \frac{1}{\sqrt{2}}(0, 1, -1, 0)$
$\alpha_2 = \frac{1}{\sqrt{2}}(1, -1, 0^2)$	$\alpha_4 = \frac{1}{\sqrt{2}}(0^2, 1, -1)$
Volume of the most symmetric lattice:	
$\mathrm{vol}_4(L^4) = \frac{1}{2}$	
Covering radius:	
$\frac{1}{\sqrt{2}}$	
Fundamental simplex of $\mathrm{Aut}_a(L^4)$ and vertices different from 0:	
	$v_1 = \frac{1}{\sqrt{2}}((\frac{1}{2})^2, 0^2)$
	$v_2 = \frac{1}{\sqrt{2}}(\frac{2}{3}, (\frac{1}{3})^2, 0)$
	$v_3 = \frac{1}{\sqrt{2}}((\frac{1}{3})^3, 0)$
	$v_4 = \frac{1}{\sqrt{2}}((\frac{1}{2})^4)$

Dimension 5

Biggest and second biggest point group order:

$\varphi_5 = 3840$	$\psi_5 = 2304$

Point group of maximal order:

$G_5 = \mathrm{Aut}(D_5)$	

Lattices of maximal symmetry:

$L_1^5 = B_5$, fcc-lattice	$L_2^5 = B_5^*$, bcc-lattice	$L_3^5 = C_5$, cubic lattice

Description of L_1^5, modulo normalization:

$C_5 = D_5 = \{(x_1, \dots, x_5) \in \mathbb{Z}^5 : \sum_{i=1}^5 x_i \equiv 0, \mathrm{mod} 2\}$

Description of L_2^5, modulo normalization:

$C_5^* = D_5^* = 2\mathbb{Z}^5 \cup ((2\mathbb{Z})^5 + (1^5))$

Description of L_1^5:

$B_5 = \mathbb{Z}^5$

Base vectors:

$\alpha_1 = \frac{1}{\sqrt{2}}(-1, -1, 0^3)$	$\omega_1 = \frac{1}{2}(2, 0^4)$	$\beta_1 = (1, 0^4)$
$\alpha_2 = \frac{1}{\sqrt{2}}(1, -1, 0^3)$	$\omega_2 = \frac{1}{2}(0, 2, 0^3)$	$\beta_2 = (0, 1, 0^3)$
$\alpha_3 = \frac{1}{\sqrt{2}}(0, 1, -1, 0^2)$	$\omega_3 = \frac{1}{2}(0^2, 2, 0^2)$	$\beta_3 = (0^2, 1, 0^2)$
$\alpha_4 = \frac{1}{\sqrt{2}}(0^2, 1, -1, 0)$	$\omega_4 = \frac{1}{2}(0^3, 2, 0)$	$\beta_4 = (0^3, 1, 0)$
$\alpha_5 = \frac{1}{\sqrt{2}}(0^3, 1, -1)$	$\omega_5 = \frac{1}{2}(1^5)$	$\beta_5 = (0^4, 1)$

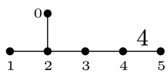
Volume of the most symmetric lattices:

$\mathrm{vol}_5(L_1^3) = \frac{1}{2\sqrt{2}}$	$\mathrm{vol}_5(L_2^5) = \frac{1}{2}$	$\mathrm{vol}_5(L_3^5) = 1$

Covering radius:

$\frac{\sqrt{5}}{2\sqrt{2}}$	$\frac{3}{4}$	$\frac{\sqrt{5}}{2}$

Fundamental simplex of $\mathrm{Aut}_a(L_1^5)$ and vertices different from 0:



$v_1 = \frac{1}{\sqrt{2}}(0^4, 1) = u$

$v_2 = \frac{1}{\sqrt{2}}(0^3, (\frac{1}{2})^2)$

$v_3 = \frac{1}{\sqrt{2}}(0^2, (\frac{1}{2})^3)$

$v_4 = \frac{1}{\sqrt{2}}(0, (\frac{1}{2})^4) = u$

$v_5 = \frac{1}{\sqrt{2}}((\frac{1}{2})^5) = t$

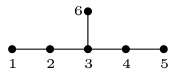
Dimension 6

Biggest and second biggest point group order:

$\varphi_6 = 2 \cdot 72 \cdot 6!$ | $\psi_6 = 72 \cdot 6!$

Point group of maximal order:

$G_6 = \mathrm{Aut}(E_6)$, that is
$G_6 = \mathbb{Z}_2$-extension of $W(E_6)$



Lattices of maximal symmetry:

$L_1^6 = E_6$-lattice | $L_2^6 = E_6^*$-lattice

Description of L_1^6, modulo normalization:

E_6 is the set of vectors in E_8 orthogonal to any A_2-lattice V of E_8.
If we choose $V = \mathrm{span}((0^6, 1, -1), (0^5, 1, -1))$, then

$E_6 = \{(x_1, \ldots, x_8) \in E_8 : x_1 = -x_2 = -x_3\}$

Description of L_2^6, modulo normalization:

Using the alternative root basis $\{\beta_i\}$ for E_6, we have

$E_6^* = E_6 \cup (E_6 + (0, (-\frac{2}{3})^2, (\frac{1}{3})^4)) \cup (E_6 + (0, (\frac{2}{3})^2, (-\frac{1}{3})^4))$.

Base vectors:

$\alpha_1 = \frac{1}{\sqrt{2}}(0^3, 1, -1, 0^3)$	$\beta_1 = \frac{1}{\sqrt{2}}(0, -1, 1, 0^5)$	$\omega_1 = \frac{2}{\sqrt{3}}(0, -1, 1, 0^5)$
$\alpha_2 = \frac{1}{\sqrt{2}}(0^4, 1, -1, 0^2)$	$\beta_2 = \frac{1}{\sqrt{2}}(0^2, -1, 1, 0^4)$	$\omega_2 = \frac{2}{\sqrt{3}}(0^2, -1, 1, 0^4)$
$\alpha_3 = \frac{1}{\sqrt{2}}(0^5, 1, -1, 0)$	$\beta_3 = \frac{1}{\sqrt{2}}(0^3, -1, 1, 0^3)$	$\omega_3 = \frac{2}{\sqrt{3}}(0^3, -1, 1, 0^3)$
$\alpha_4 = \frac{1}{\sqrt{2}}(0^6, 1, -1)$	$\beta_4 = \frac{1}{\sqrt{2}}(0^4, -1, 1, 0^2)$	$\omega_4 = \frac{2}{\sqrt{3}}(0^4, -1, 1, 0^2)$
$\alpha_5 = \frac{1}{\sqrt{2}}(\frac{1}{2}, (-\frac{1}{2})^6, \frac{1}{2})$	$\beta_5 = \frac{1}{\sqrt{2}}(0^5, -1, 1, 0)$	$\omega_5 = \frac{2}{\sqrt{3}}(0, (\frac{2}{3})^2, (\frac{-1}{3})^4, 0)$
$\alpha_1 = \frac{1}{\sqrt{2}}(0^6, 1^2)$	$\beta_6 = \frac{1}{\sqrt{2}}((\frac{1}{2})^4, (-\frac{1}{2})^4)$	$\omega_6 = \frac{2}{\sqrt{3}}((\frac{1}{2})^4, (-\frac{1}{2})^4)$

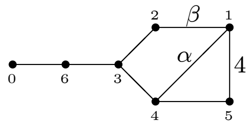
Volume of the most symmetric lattices:

$\mathrm{vol}_6(L_1^6) = \frac{\sqrt{3}}{8}$ | $\mathrm{vol}_6(L_2^6) = \frac{9\sqrt{3}}{64}$

Covering radius:

$\frac{\sqrt{2}}{\sqrt{3}}$ | $\frac{1}{\sqrt{2}}$

Fundamental simplex of $\mathrm{Aut}_a(L_1^6)$ and vertices different from 0:



$v_1 = \frac{1}{\sqrt{2}}(0^5, -(\frac{2}{3})^2, \frac{2}{3}) = t$

$v_2 = \frac{1}{\sqrt{2}}(-\frac{1}{4}, (\frac{1}{4})^4, -(\frac{5}{12})^2, \frac{5}{12})$

$v_3 = \frac{1}{\sqrt{2}}(0^2, (\frac{1}{3})^3, (-\frac{1}{3})^2, \frac{1}{3})$

$v_4 = \frac{1}{\sqrt{2}}(0^3, (\frac{1}{2})^2, (-\frac{1}{3})^2, \frac{1}{3})$

$v_5 = \frac{1}{\sqrt{2}}(0^4, \frac{1}{2}, (-\frac{1}{2})^2, \frac{1}{2})$=u

$v_6 = \frac{1}{\sqrt{2}}((\frac{1}{4})^5, -(\frac{1}{4})^2, \frac{1}{4})$

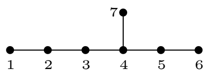
Dimension 7

Biggest and second biggest point group order:

$\varphi_7 = 8 \cdot 9!$ | $\psi_7 = 4 \cdot 9!$

Point group of maximal order:

$G_7 = W(E_7)$



Lattices of maximal symmetry:

$L_1^7 = E_7$-lattice | $L_2^7 = E_7^*$-lattice

Description of L_1^7, modulo normalization:

E_7 is the set of vectors in E_8 perpendicular to any minimal vector v of E_8. If we choose $v = (1^2, 0^6)$, then

$E_7 = \{(x_1, \ldots, x_8) \in E_8 : x_1 = -x_2\}$.

Description of L_2^7, modulo normalization:

Using the alternative root basis $\{\beta_i\}$ for E_7, we have
$E_7^* = E_7 \cup (E_7 + ((\frac{1}{4})^6, (-\frac{3}{4})^2))$.

Base vectors:

$\alpha_1 = \frac{1}{\sqrt{2}}(0^2, 1, -1, 0^4)$	$\beta_1 = \frac{1}{\sqrt{2}}(-1, 1, 0^6)$	$\omega_1 = \frac{3}{\sqrt{2}}(-1, 1, 0^6)$
$\alpha_2 = \frac{1}{\sqrt{2}}(0^3, 1, -1, 0^3)$	$\beta_2 = \frac{1}{\sqrt{2}}(0, -1, 1, 0^5)$	$\omega_2 = \frac{3}{\sqrt{2}}(0, -1, 1, 0^5)$
$\alpha_3 = \frac{1}{\sqrt{2}}(0^4, 1, -1, 0^2)$	$\beta_3 = \frac{1}{\sqrt{2}}(0^2, -1, 1, 0^4)$	$\omega_3 = \frac{3}{\sqrt{2}}(0^2, -1, 1, 0^4)$
$\alpha_4 = \frac{1}{\sqrt{2}}(0^5, 1, -1, 0)$	$\beta_4 = \frac{1}{\sqrt{2}}(0^3, -1, 1, 0^3)$	$\omega_4 = \frac{3}{\sqrt{2}}(0^3, -1, 1, 0^3)$
$\alpha_5 = \frac{1}{\sqrt{2}}(0^6, 1, -1)$	$\beta_5 = \frac{1}{\sqrt{2}}(0^4, -1, 1, 0^2)$	$\omega_5 = \frac{3}{\sqrt{2}}(0^4, -1, 1, 0^2)$
$\alpha_6 = \frac{1}{\sqrt{2}}(\frac{1}{2}, (-\frac{1}{2})^6, \frac{1}{2})$	$\beta_6 = \frac{1}{\sqrt{2}}(0^5, -1, 1, 0)$	$\omega_6 = \frac{3}{\sqrt{2}}(0^5, -1, 1, 0)$
$\alpha_7 = \frac{1}{\sqrt{2}}(0^6, 1^2)$	$\beta_7 = \frac{1}{\sqrt{2}}((\frac{1}{2})^4, (-\frac{1}{2})^4)$	$\omega_7 = \frac{3}{\sqrt{2}}((-\frac{3}{4})^2, (\frac{1}{4})^6)$

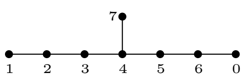
Volume of the most symmetric lattices:

$\text{vol}_7(L_1^7) = \frac{1}{8}$ | $\text{vol}_7(L_2^7) = \frac{8}{27\sqrt{3}}$

Covering radius:

$\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{7}}{2\sqrt{3}}$

Fundamental simplex of $\text{Aut}_a(L_1^7)$ and vertices different from 0:



$v_1 = \frac{1}{\sqrt{2}}(0^5, -1, \frac{1}{2}, -\frac{1}{2}) = t$

$v_2 = \frac{1}{\sqrt{2}}(0^4, (-\frac{1}{2})^2, \frac{1}{2}, -\frac{1}{2}) = u$

$v_3 = \frac{1}{\sqrt{2}}(0^3, (-\frac{1}{3})^3, \frac{1}{2}, -\frac{1}{2})$

$v_4 = \frac{1}{\sqrt{2}}(0^2, (-\frac{1}{4})^4, \frac{1}{2}, -\frac{1}{2})$

$v_5 = \frac{1}{\sqrt{2}}(\frac{1}{6}, -(\frac{1}{6})^5, \frac{1}{2}, -\frac{1}{2}) = w$

$v_6 = \frac{1}{\sqrt{2}}(0^6, \frac{1}{2}, -\frac{1}{2})$

$v_7 = \frac{1}{\sqrt{2}}((-\frac{1}{4})^6, \frac{1}{2}, -\frac{1}{2})$

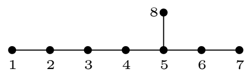
Dimension 8

Biggest and second biggest point group order:

$\varphi_8 = 192 \cdot 10!$ | $\psi_8 = 96 \cdot 10!$

Point group of maximal order:

$G_8 = W(E_8)$



Lattice of maximal symmetry:

$L^8 = E_8 = E_8^*$-lattice

Description of L^8, modulo normalization:

$E_8 = D_8 \cup (D_8 + ((\frac{1}{2})^8))$, that is

$E_8 = \{(x_1, \ldots, x_8) \in \mathbb{Z}^8 \cup (\mathbb{Z}^8 + ((\frac{1}{2})^8)) : \sum_{i=1}^{8} x_i \equiv 0, \mathrm{mod} 2\}$

Base vectors:

$\alpha_1 = \frac{1}{\sqrt{2}}(0, 1, -1, 0^5)$	$\alpha_5 = \frac{1}{\sqrt{2}}(0^5, 1, -1, 0)$
$\alpha_2 = \frac{1}{\sqrt{2}}(0^2, 1, -1, 0^4)$	$\alpha_6 = \frac{1}{\sqrt{2}}(0^6, 1, -1)$
$\alpha_3 = \frac{1}{\sqrt{2}}(0^3, 1, -1, 0^3)$	$\alpha_7 = \frac{1}{\sqrt{2}}(\frac{1}{2}, (-\frac{1}{2})^6, \frac{1}{2})$
$\alpha_4 = \frac{1}{\sqrt{2}}(0^4, 1, -1, 0^2)$	$\alpha_8 = \frac{1}{\sqrt{2}}(0^6, 1^2)$

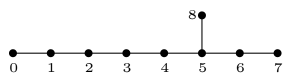
Volume of the most symmetric lattice:

$\mathrm{vol}_8(L^8) = \frac{1}{16}$

Covering radius:

$\frac{1}{\sqrt{2}}$

Fundamental simplex of $\mathrm{Aut}_a(L^8)$ and vertices different from 0:



$v_1 = \frac{1}{\sqrt{2}}(0^6, (\frac{1}{2})^2)$

$v_2 = \frac{1}{\sqrt{2}}(0^5, (\frac{1}{3})^2, \frac{2}{3})$

$v_3 = \frac{1}{\sqrt{2}}(0^4, (\frac{1}{4})^3, \frac{3}{4})$

$v_4 = \frac{1}{\sqrt{2}}(0^3, (-\frac{1}{5})^4, \frac{4}{5})$

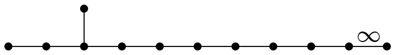
$v_5 = \frac{1}{\sqrt{2}}(0^2, (\frac{1}{6})^5, \frac{5}{6})$

$v_6 = \frac{1}{\sqrt{2}}(-\frac{1}{8}, (\frac{1}{8})^6, \frac{7}{8})$

$v_7 = \frac{1}{\sqrt{2}}(0^7, 1)$

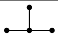
$v_8 = \frac{1}{\sqrt{2}}((\frac{1}{6})^7, \frac{5}{6})$

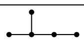
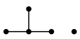
A3. We compute the volume of unbounded hyperbolic polytope P_1^{10} with Coxeter graph

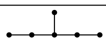
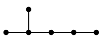
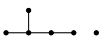
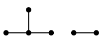
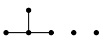


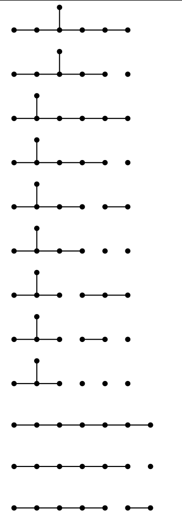
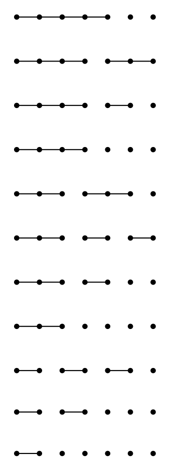
First, we list all n generator spherical subgraphs with multiplicity N :

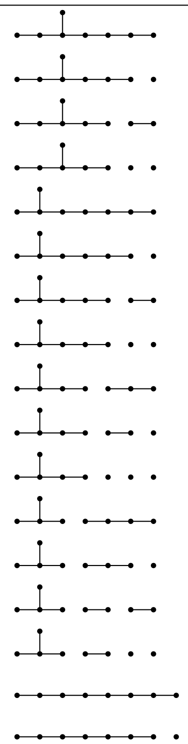
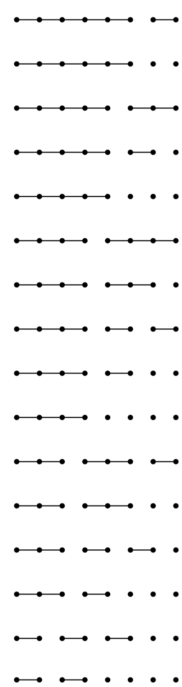
2 generators			**3 generators**		
N	σ	order	N	σ	order
10		$3!$	10		$4!$
55		2^2	79		$3! \cdot 2$
			121		2^3

4 generators					
N	σ	order	N	σ	order
1		$2^3 \cdot 4!$	26		$(3!)^2$
9		$5!$	214		$3! \cdot 2^2$
68		$4! \cdot 2$	132		2^4

5 generators					
N	σ	order	N	σ	order
2		$2^4 \cdot 5!$	147		$4! \cdot 2^2$
6		$2^4 \cdot 4!$	108		$(3!)^2 \cdot 2$
7		$6!$	240		$3! \cdot 2^3$
53		$5! \cdot 2$	66		2^5
43		$4! \cdot 3!$			

6 generators					
N	σ	order	N	σ	order
1		$2^7 \cdot 3^4 \cdot 5$	93		$5! \cdot 2^2$
1		$2^5 \cdot 6!$	17		$(4!)^2$
11		$2^4 \cdot 5! \cdot 2$	133		$4! \cdot 3! \cdot 2$
4		$2^3 \cdot 4! \cdot 3!$	115		$4! \cdot 2^3$
10		$2^3 \cdot 4! \cdot 2^2$	13		$(3!)^3$
6		$7!$	128		$(3!)^2 \cdot 2^2$
33		$6! \cdot 2$	107		$3! \cdot 2^4$
31		$5! \cdot 3!$	11		2^6

7 generators					
N	σ	order	N	σ	order
1		$2^{10} \cdot 3^4 \cdot 5 \cdot 7$	41		$6! \cdot 2^2$
5		$2^8 \cdot 3^4 \cdot 5$	23		$5! \cdot 4!$
1		$2^6 \cdot 7!$	72		$5! \cdot 3! \cdot 2$
4		$2^6 \cdot 6!$	53		$5! \cdot 2^3$
7		$2^4 \cdot 5! \cdot 3!$	35		$(4!)^2 \cdot 2$
16		$2^6 \cdot 5!$	18		$4! \cdot (3!)^2$
3		$2^3 \cdot (4!)^2$	97		$4! \cdot 3! \cdot 2^2$
9		$2^4 \cdot 4! \cdot 3!$	27		$4! \cdot 2^4$
4		$2^6 \cdot 4!$	18		$(3!)^3 \cdot 2$
5		$8!$	51		$(3!)^2 \cdot 2^3$
22		$7! \cdot 2$	12		$3! \cdot 2^5$
16		$6! \cdot 3!$			

8 generators					
N	σ	order	N	σ	order
1		$2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$	8		$7! \cdot 3!$
4		$2^{11} \cdot 3^4 \cdot 5 \cdot 7$	18		$7! \cdot 2^2$
3		$2^7 \cdot 3^4 \cdot 5 \cdot 3!$	10		$6! \cdot 4!$
6		$2^9 \cdot 3^4 \cdot 5$	21		$6! \cdot 3! \cdot 2$
1		$2^7 \cdot 8!$	14		$6! \cdot 2^3$
3		$2^7 \cdot 7!$	7		$(5!)^2$
2		$2^5 \cdot 6! \cdot 3!$	31		$5! \cdot 4! \cdot 2$
3		$2^7 \cdot 6!$	6		$5! \cdot (3!)^2$
5		$2^4 \cdot 5! \cdot 4!$	36		$5! \cdot 3! \cdot 2^2$
13		$2^5 \cdot 5! \cdot 3!$	6		$5! \cdot 2^4$
5		$2^7 \cdot 5!$	5		$(4!)^2 \cdot 3!$
2		$2^3 \cdot 4! \cdot 5!$	11		$(4!)^2 \cdot 2^2$
4		$2^4 \cdot (4!)^2$	13		$4! \cdot (3!)^2 \cdot 2$
1		$2^3 \cdot 4! \cdot (3!)^2$	17		$4! \cdot 3! \cdot 2^3$
2		$2^5 \cdot 4! \cdot 3!$	7		$(3!)^3 \cdot 2^2$
4		$9!$	3		$(3!)^2 \cdot 2^4$
13		$8! \cdot 2$			

9 generators					
N	σ	order	N	σ	order
3		$2^{15}\cdot 3^5\cdot 5^2\cdot 7$	3		$10!$
2		$2^{10}\cdot 3^4\cdot 5\cdot 7\cdot 3!$	6		$9!\cdot 2$
3		$2^{12}\cdot 3^4\cdot 5\cdot 7$	2		$8!\cdot 3!$
2		$2^7\cdot 3^4\cdot 5\cdot 4!$	6		$8!\cdot 2^2$
4		$2^8\cdot 3^4\cdot 5\cdot 3!$	3		$7!\cdot 4!$
1		$2^{10}\cdot 3^4\cdot 5$	6		$7!\cdot 3!\cdot 2$
1		$2^8\cdot 9!$	3		$7!\cdot 2^3$
2		$2^8\cdot 8!$	4		$6!\cdot 5!$
1		$2^6\cdot 7!\cdot 3!$	5		$6!\cdot 4!\cdot 2$
1		$2^8\cdot 7!$	6		$6!\cdot 3!\cdot 2^2$
1		$2^5\cdot 6!\cdot 4!$	6		$(5!)^2\cdot 2$
1		$2^6\cdot 6!\cdot 3!$	2		$5!\cdot 4!\cdot 3!$
3		$2^4\cdot (5!)^2$	4		$5!\cdot 4!\cdot 2^2$
5		$2^5\cdot 5!\cdot 4!$	3		$5!\cdot (3!)^2\cdot 2$
1		$2^4\cdot 5!\cdot (3!)^2$	2		$5!\cdot 3!\cdot 2^3$
2		$2^6\cdot 5!\cdot 3!$	1		$(4!)^2\cdot 3!\cdot 2$
1		$2^3\cdot 4!\cdot 6!$	2		$4!\cdot (3!)^2\cdot 2^2$
1		$2^4\cdot 4!\cdot 5!$			

10 generators					
N	σ	order	N	σ	order
1		$2^{14}\cdot 3^5\cdot 5^2\cdot 7\cdot 3!$	1		$2^5\cdot (5!)^2$
1		$2^{16}\cdot 3^5\cdot 5^2\cdot 7$	1		$11!$
1		$2^{10}\cdot 3^4\cdot 5\cdot 7\cdot 4!$	2		$10!\cdot 2$
1		$2^{11}\cdot 3^4\cdot 5\cdot 7\cdot 3!$	1		$9!\cdot 2^2$
1		$2^7\cdot 3^4\cdot 5\cdot 5!$	1		$8!\cdot 3!\cdot 2$
1		$2^8\cdot 3^4\cdot 5\cdot 4!$	1		$7!\cdot 5!$
1		$2^9\cdot 10!$	1		$7!\cdot 3!\cdot 2^2$
1		$2^9\cdot 9!$	1		$6!\cdot 5!\cdot 2$
1		$2^4\cdot 5!\cdot 6!$			

Then, we compute in each dimension, the sum of all normalized angles (inverse of orders) counted with multiplicity :

$$
\begin{aligned}
w_{-1} = \quad & 1 && = 1 \\
w_0 = \quad & 12\tfrac{1}{2} && = 6 \\
w_1 = \quad & 10\tfrac{1}{3!} + 55\tfrac{1}{2^2} && = \tfrac{185}{12} \\
w_2 = \quad & 10\tfrac{1}{4!} + 79\tfrac{1}{3!\cdot 2} + 121\tfrac{1}{2^3} && = \tfrac{177}{8} \\
w_3 = \quad & \tfrac{1}{2^3\cdot 4!} + 9\tfrac{1}{5!} + 68\tfrac{1}{4!\cdot 2} + 26\tfrac{1}{3!^2} + 214\tfrac{1}{3!\cdot 2^2} + 132\tfrac{1}{2^4} && = \tfrac{55\,831}{2880} \\
w_4 = \quad & 2\tfrac{1}{2^4\cdot 5!} + 6\tfrac{1}{2^4\cdot 4!} + 7\tfrac{1}{6!} + 53\tfrac{1}{5!\cdot 2} + 43\tfrac{1}{4!\cdot 3!} + \\
& +147\tfrac{1}{4!\cdot 2^2} + 108\tfrac{1}{3!^2\cdot 2} + 240\tfrac{1}{3!\cdot 2^3} + 66\tfrac{1}{2^5} && = \tfrac{5107}{480} \\
w_5 = \quad & \tfrac{1}{2^7\cdot 3^4\cdot 5} + \tfrac{1}{2^5\cdot 6!} + 11\tfrac{1}{2^4\cdot 5!\cdot 2} + 4\tfrac{1}{2^3\cdot 4!\cdot 3!} + 10\tfrac{1}{2^3\cdot 4!\cdot 2^2} + \\
& +6\tfrac{1}{7!} + 33\tfrac{1}{6!\cdot 2} + 31\tfrac{1}{5!\cdot 3!} + 93\tfrac{1}{5!\cdot 2^2} + 17\tfrac{1}{4!^2} + 133\tfrac{1}{4!\cdot 3!\cdot 2} + \\
& +115\tfrac{1}{4!\cdot 2^3} + 13\tfrac{1}{3!^3} + 128\tfrac{1}{3!^2\cdot 2^2} + 107\tfrac{1}{3!\cdot 2^4} + 11\tfrac{1}{2^6} && = \tfrac{5234\,389}{1451\,520} \\
w_6 = \quad & \tfrac{1}{2^{10}\cdot 3^4\cdot 5\cdot 7} + 5\tfrac{1}{2^8\cdot 3^4\cdot 5} + \tfrac{1}{2^6\cdot 7!} + 4\tfrac{1}{2^6\cdot 6!} + 7\tfrac{1}{2^4\cdot 5!\cdot 3!} + 16\tfrac{1}{2^6\cdot 5!} + \\
& +3\tfrac{1}{2^3\cdot 4!^2} + 9\tfrac{1}{2^4\cdot 4!\cdot 3!} + 4\tfrac{1}{2^6\cdot 4!} + 5\tfrac{1}{8!} + 22\tfrac{1}{7!\cdot 2} + 16\tfrac{1}{6!\cdot 3!} + \\
& +41\tfrac{1}{6!\cdot 2^2} + 23\tfrac{1}{5!\cdot 4!} + 72\tfrac{1}{5!\cdot 3!\cdot 2} + 53\tfrac{1}{5!\cdot 2^3} + 35\tfrac{1}{4!^2\cdot 2} + \\
& +18\tfrac{1}{4!\cdot 3!^2} + 97\tfrac{1}{4!\cdot 3!\cdot 2^2} + 27\tfrac{1}{4!\cdot 2^4} + 18\tfrac{1}{3!^3\cdot 2} + 51\tfrac{1}{3!^2\cdot 2^3} + && = \tfrac{24\,697}{34\,560} \\
w_7 = \quad & \tfrac{1}{2^{14}\cdot 3^5\cdot 5^2\cdot 7} + 4\tfrac{1}{2^{11}\cdot 3^4\cdot 5\cdot 7} + 3\tfrac{1}{2^7\cdot 3^4\cdot 5\cdot 3!} + 6\tfrac{1}{2^9\cdot 3^4\cdot 5} + \tfrac{1}{2^7\cdot 8!} + \\
& +3\tfrac{1}{2^7\cdot 7!} + 12\tfrac{1}{3!\cdot 2^5} + 2\tfrac{1}{2^5\cdot 6!\cdot 3!} + 3\tfrac{1}{2^7\cdot 6!} + 5\tfrac{1}{2^4\cdot 5!\cdot 4!} + 13\tfrac{1}{2^5\cdot 5!\cdot 3!} + \\
& +5\tfrac{1}{2^7\cdot 5!} + 2\tfrac{1}{2^3\cdot 4!\cdot 5!} + 4\tfrac{1}{2^4\cdot 4!^2} + \tfrac{1}{2^3\cdot 4!\cdot 3!^2} + 2\tfrac{1}{2^5\cdot 4!\cdot 3!} + 4\tfrac{1}{9!} + \\
& +13\tfrac{1}{8!\cdot 2} + 8\tfrac{1}{7!\cdot 3!} + 18\tfrac{1}{7!\cdot 2^2} + 10\tfrac{1}{6!\cdot 4!} 21\tfrac{1}{6!\cdot 3!\cdot 2} + 14\tfrac{1}{6!\cdot 2^3} + \\
& +7\tfrac{1}{5!^2} + 31\tfrac{1}{5!\cdot 4!\cdot 2} + 6\tfrac{1}{5!\cdot 3!^2} + 36\tfrac{1}{5!\cdot 3!\cdot 2^2} + 6\tfrac{1}{5!\cdot 2^4} + 5\tfrac{1}{4!^2\cdot 3!} + \\
& +11\tfrac{1}{4!^2\cdot 2^2} + 13\tfrac{1}{4!\cdot 3!^2\cdot 2} + 17\tfrac{1}{4!\cdot 3!\cdot 2^3} + 7\tfrac{1}{3!^3\cdot 2^2} + 3\tfrac{1}{3!^2\cdot 2^4} && = \tfrac{6414\,413}{87\,091\,200} \\
w_8 = \quad & 3\tfrac{1}{2^{15}\cdot 3^5\cdot 5^2\cdot 7} + 2\tfrac{1}{2^{10}\cdot 3^4\cdot 5\cdot 7\cdot 3!} + 3\tfrac{1}{2^{12}\cdot 3^4\cdot 5\cdot 7} + 2\tfrac{1}{2^7\cdot 3^4\cdot 5\cdot 4!} + \\
& +4\tfrac{1}{2^8\cdot 3^4\cdot 5\cdot 3!} + \tfrac{1}{2^{10}\cdot 3^4\cdot 5} + \tfrac{1}{2^8\cdot 9!} + 2\tfrac{1}{2^8\cdot 8!} + \tfrac{1}{2^6\cdot 7!\cdot 3!} + \tfrac{1}{2^8\cdot 7!} + \\
& +\tfrac{1}{2^5\cdot 6!\cdot 4!} + \tfrac{1}{2^6\cdot 6!\cdot 3!} + 3\tfrac{1}{2^4\cdot 5!^2} + 5\tfrac{1}{2^5\cdot 5!\cdot 4!} + \tfrac{1}{2^4\cdot 5!\cdot 3!^2} \\
& +2\tfrac{1}{2^6\cdot 5!\cdot 3!} + +\tfrac{1}{2^3\cdot 4!\cdot 6!} + +\tfrac{1}{2^4\cdot 4!\cdot 5!} + 3\tfrac{1}{10!} + 6\tfrac{1}{9!\cdot 2} + 2\tfrac{1}{8!\cdot 3!} + \\
& +6\tfrac{1}{8!\cdot 2^2} + 3\tfrac{1}{7!\cdot 4!} + 6\tfrac{1}{7!\cdot 3!\cdot 2} + 3\tfrac{1}{7!\cdot 2^3} + 4\tfrac{1}{6!\cdot 5!} + 5\tfrac{1}{6!\cdot 4!\cdot 2} + \\
& +6\tfrac{1}{6!\cdot 3!\cdot 2^2} + 6\tfrac{1}{5!^2\cdot 2} + 2\tfrac{1}{5!\cdot 4!\cdot 3!} + 4\tfrac{1}{5!\cdot 4!\cdot 2^2} + 3\tfrac{1}{5!\cdot 3!^2\cdot 2} + \\
& +2\tfrac{1}{5!\cdot 3!\cdot 2^3} + \tfrac{1}{4!^2\cdot 3!\cdot 2} + 2\tfrac{1}{4!\cdot 3!^2\cdot 2^2} && = \tfrac{21\,265}{6967\,296} \\
w_9 = \quad & \tfrac{1}{2^{14}\cdot 3^5\cdot 5^2\cdot 7\cdot 3!} + \tfrac{1}{2^{16}\cdot 3^5\cdot 5^2\cdot 7} + \tfrac{1}{2^{10}\cdot 3^4\cdot 5\cdot 7\cdot 4!} + \tfrac{1}{2^{11}\cdot 3^4\cdot 5\cdot 7\cdot 3!} + \\
& +\tfrac{1}{2^7\cdot 3^4\cdot 5\cdot 5!} + \tfrac{1}{2^8\cdot 3^4\cdot 5\cdot 4!} + \tfrac{1}{2^9\cdot 10!} + \tfrac{1}{2^9\cdot 9!} + \tfrac{1}{2^4\cdot 5!\cdot 6!} + \tfrac{1}{2^5\cdot 5!^2} + \\
& +\tfrac{1}{11!} + 2\tfrac{1}{10!\cdot 2} + \tfrac{1}{9!\cdot 2^2} + \tfrac{1}{8!\cdot 3!\cdot 2} + \tfrac{1}{7!\cdot 5!} + \tfrac{1}{7!\cdot 3!\cdot 2^2} + \tfrac{1}{6!\cdot 5!\cdot 2} && = \tfrac{4096\,223}{183\,936\,614\,400}
\end{aligned}
$$

According to (15), the alternating angle sum now computes to

$$W\left(P_1^{10}\right) = \begin{array}{l} 1 - 6 + \frac{185}{12} - \frac{177}{8} + \frac{55\,831}{2880} - \frac{5107}{480} + \frac{5234\,389}{1451\,520} - \\ - \frac{24\,697}{34\,560} + \frac{6414\,413}{87\,091\,200} - \frac{21\,265}{6967\,296} + \frac{4096\,223}{183\,936\,614\,400} \quad = -\frac{1}{183\,936\,614\,400}. \end{array}$$

Finally, by means of (16), the volume of P_1^{10} is given by

$$\mathrm{vol}_{10}\left(P_1^{10}\right) = \frac{W\left(P^{10}\right)}{(-1)^5\,\frac{2}{\Omega^{10}}} = \frac{\pi^{\frac{11}{2}}}{183\,936\,614\,400 \cdot \Gamma\left(\frac{11}{2}\right)} = \frac{\pi^5}{5431\,878\,144\,000},$$

where the Gamma function satisfies

$$\Gamma\left(\frac{11}{2}\right) = \frac{945\sqrt{\pi}}{32}.$$

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