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Commensurability of hyperbolic Coxeter groups: theory and computation

By

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Abstract

For hyperbolic Coxeter groups of finite covolume we review and present new theoretical and computational aspects of wide commensurability. We discuss separately the arithmetic and the non-arithmetic cases. Some worked examples are added as well as a panoramic view to hyperbolic Coxeter groups and their classification.

§ 1. Introduction

Consider two discrete groups $\Gamma_1, \Gamma_2 \subset \operatorname{Isom}\mathbb{H}^n$, $n \geq 3$, with fundamental polyhedra P_1, P_2 of finite volume, respectively. The groups Γ_1, Γ_2 are said to be *commensurable (in the wide sense)* if the intersection of Γ_1 with some conjugate Γ_2' of Γ_2 in $\operatorname{Isom}\mathbb{H}^n$ has finite index in both, Γ_1 and Γ_2'. In this case, the orbifolds $\mathbb{H}^n/\Gamma_1$ and $\mathbb{H}^n/\Gamma_2$ are covered, with finite sheets and up to isometry, by a common hyperbolic n-manifold. Or, in other words, there is a hyperbolic polyhedron $P \subset \mathbb{H}^n$ which is simultaneously glued by finitely many copies of P_1 and by finitely many copies of P_2. In particular, the quotient of the volumes of P_1 and P_2 is a rational number.

Consider a Coxeter group $\Gamma \subset \operatorname{Isom}\mathbb{H}^n$ of rank N, that is, Γ is a discrete group generated by finitely many reflections s_i , $1 \leq i \leq N$, in hyperplanes of $\mathbb{H}^n$. A fundamental polyhedron P for Γ is a *Coxeter polyhedron*, that is, a convex polyhedron with

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(non-zero) dihedral angles of the form π/m for integers $m \geq 2$. Such groups are characterised by a particularly nice presentation (see (2.2)) and provide – for small $N \geq n+1$ – an important class of hyperbolic n-orbifolds and n-manifolds of small volumes.

In [30], the hyperbolic Coxeter n-simplex groups (of rank $N = n+1$) of finite covolume were classified up to commensurability. They exist for $n \leq 9$, only. In [24], the authors resolved the commensurability problem for the considerably larger family of *hyperbolic Coxeter pyramid groups*, existing up to $n = 17$. They are of rank $N = n+2$ and have fundamental polyhedra which are combinatorially pyramids with apex neighborhood given by a product of two simplices of positive dimensions; they were discovered by Tumarkin [52], [54]. Among the 200 examples in this class are arithmetic and non-arithmetic groups. Furthermore, modulo finite index, all these groups have fundamental polyhedra which are (polarly and simply) truncated simplices, and at times they arise as amalgamated free products. This indicates why several different algebraic and geometric methods had to be developped in [24] in order to achieve the commensurability classification.

In this work we present various of these general methods allowing us to decide about commensurability of hyperbolic Coxeter groups. We illustrate the theory in detail by providing several typical and also new examples. In Section 2, we furnish the necessary background about hyperbolic Coxeter groups, including volume identities in three dimensions, and add a panoramic view to hyperbolic Coxeter polyhedra known so far. Since arithmeticity is a commensurability invariant, we discuss this aspect in a quite complete and self-contained way (see Section 4). For non-arithmetic Coxeter pyramid groups $\Gamma \subset \operatorname{Isom}\mathbb{H}^n$ with non-compact quotient space $\mathbb{H}^n/\Gamma$, the commensurability classification is based on geometric results exploiting the presence and the nature of Bieberbach groups and their full rank translational lattices in a thorough way. We summarise the corresponding results, only, and refer to [24, Section 4.1] for technical details and proofs. In the case of certain non-arithmetic groups in $\operatorname{Isom}\mathbb{H}^3$ which are seemingly incommensurable, we manage to provide a rigorous proof by means of their commensurator group (see Section 3.1) and an adequate covolume comparison (see Section 5.4).

At the end of the work are appended the list of the 23 non-compact Coxeter tetrahedra and their volumes, the list of Tumarkin's Coxeter pyramids as well as the classification tables of all arithmetic and non-arithmetic hyperbolic Coxeter pyramid groups, respectively.

Acknowledgements

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§ 2. Preliminaries

§ 2.1. Hyperbolic Coxeter groups and Coxeter polyhedra

Denote by $\mathbb{X}^n$ either the Euclidean space $\mathbb{E}^n$, the sphere $\mathbb{S}^n$, or the hyperbolic space $\mathbb{H}^n$, and let $\mathrm{Isom}\mathbb{X}^n$ be its isometry group. As for the sphere $\mathbb{S}^n \subset \mathbb{E}^{n+1}$, embed $\mathbb{H}^n$ in a quadratic space $\mathbb{Y}^{n+1}$. More concretely, we view $\mathbb{H}^n$ in the Lorentz-Minkowski space $\mathbb{E}^{n,1} = (\mathbb{R}^{n+1}, \langle x, y\rangle_{n,1} = \sum_{i=1}^{n} x_i y_i - x_{n+1}y_{n+1})$ of signature $(n, 1)$ so that

$$\mathbb{H}^n = \{\, x \in \mathbb{E}^{n,1} \mid \langle x, x\rangle_{n,1} = -1, x_{n+1} > 0 \,\}$$

(see [59], for example). The group $\mathrm{Isom}\mathbb{H}^n$ is then isomorphic to the group $PO(n, 1)$ of positive Lorentz-matrices. In the Euclidean case, we take the affine point of view and write $\mathbb{Y}^{n+1} = \mathbb{E}^n \times \{0\}$.

A *geometric Coxeter group* is a discrete subgroup $\Gamma \subset \mathrm{Isom}\mathbb{X}^n$ generated by finitely many reflections in hyperplanes of $\mathbb{X}^n$. The cardinality N of the set of generators is called the *rank* of the group Γ.

Let s_i be a generator of the geometric Coxeter group Γ acting on $\mathbb{X}^n$ as the reflection with respect to the hyperplane H_i $(1 \leq i \leq N)$. Associate to H_i a normal unit vector $e_i \in \mathbb{Y}^{n+1}$ such that

$$H_i = \{\, x \in \mathbb{X}^n \mid \langle x, e_i\rangle_{\mathbb{Y}^{n+1}} = 0 \,\},$$

and which bounds the closed half-space

$$H_i^- = \{\, x \in \mathbb{X}^n \mid \langle x, e_i\rangle_{\mathbb{Y}^{n+1}} \leq 0 \,\}.$$

A convex (closed) fundamental domain $P = P(\Gamma) \subset \mathbb{X}^n$ of Γ can be chosen to be given by the polyhedron

$$P = \bigcap_{i=1}^{N} H_i^- . \tag{2.1}$$

By Vinberg's work [59], the combinatorial, metrical and arithmetical properties of P and Γ, respectively, can be read off from the *Gram matrix* $G(P)$ of P formed by the products $\langle e_i, e_j\rangle_{\mathbb{Y}^{n+1}}$ $(1 \le i,j \le N)$. In the hyperbolic case, the product $\langle e_i, e_j\rangle_{n,1}$ characterises the mutual position of the hyperplanes H_i, H_j as follows.

$$-\langle e_i, e_j\rangle_{n,1} = \begin{cases} \cos\frac{\pi}{m_{ij}} & \text{if } H_i, H_j \text{ intersect at the angle } \frac{\pi}{m_{ij}} \text{ in } \mathbb{H}^n, \\ 1 & \text{if } H_i, H_j \text{ meet at } \partial\mathbb{H}^n, \\ \cosh l_{ij} & \text{if } H_i, H_j \text{ are at distance } l_{ij} \text{ in } \mathbb{H}^n. \end{cases}$$

A fundamental domain $P \subset \mathbb{X}^n$ as in (2.1) for a geometric Coxeter group is a *Coxeter polyhedron*, that is, a convex polyhedron in $\mathbb{X}^n$ all of whose dihedral angles are submultiples of π. Conversely, each Coxeter polyhedron in $\mathbb{X}^n$ gives rise to a geometric Coxeter group.

The geometric Coxeter group Γ has the presentation

$$\Gamma = \langle\, s_1, \dots, s_N \mid s_i^2\,,\, (s_i s_j)^{m_{ij}} \,\rangle \tag{2.2}$$

for integers $m_{ij} = m_{ji} \ge 2$ for $i \ne j$.

We restrict our attention to *cocompact* or *cofinite* geometric Coxeter groups, that is, we assume that the associated Coxeter polyhedra are compact or of finite volume in $\mathbb{X}^n$. In particular, hyperbolic Coxeter polyhedra are bounded by at least $n+1$ hyperplanes, appear as the convex hull of finitely many points in the extended hyperbolic space $\mathbb{H}^n \cup \partial\mathbb{H}^n$ and are *acute-angled* (less than or equal to $\frac{\pi}{2}$). An (*ordinary*) vertex $p \in \mathbb{H}^n$ of P is given by a positive definite principal submatrix of rank n of the Gram matrix $G(P)$ of P. Its vertex figure P_p is an $(n-1)$-dimensional spherical Coxeter polyhedron which is a product of $k \ge 1$ pairwise orthogonal lower-dimensional spherical Coxeter simplices. A vertex (*at infinity*) $q \in \partial\mathbb{H}^n$ of P is characterised by a positive semi-definite principal submatrix of rank $n-1$ of the Gram matrix $G(P)$. Its vertex figure P_q is a compact $(n-1)$-dimensional Euclidean Coxeter polyhedron which is a product of $l \ge 1$ pairwise orthogonal lower-dimensional Euclidean Coxeter simplices. The polyhedron P_q is a fundamental domain of the stabiliser Γ_q of q which is a crystallographic group containing a finite index translational lattice of rank $n-1$ by Bieberbach's Theorem.

Many of these properties can be read off from the *Coxeter graph* Σ of P and Γ. To each hyperplane H_i of P and to each generator $s_i \in \Gamma$ corresponds a node ν_i of Σ. Two nodes ν_i, ν_j are joined by an edge with label $m_{ij} \ge 3$ if $\measuredangle(H_i, H_j) = \pi/m_{ij}$ (the label 3 is usually omitted). If H_i, H_j are orthogonal, their nodes are not connected. If H_i, H_j meet at $\partial\mathbb{H}^n$, their nodes are joined by an edge with label ∞ (or by a bold edge); if they are at distance $l_{ij} > 0$ in $\mathbb{H}^n$, their nodes are joined by a dotted edge, often without the label l_{ij}. We will also use the *Coxeter symbol* for a Coxeter group. For example, $[p,q,r]$ is associated to a linear Coxeter graph with 3 edges of consecutive labels p,q,r, and the Coxeter symbol $[(p,q,r)]$ describes a cyclic graph with labels p,q,r. Often, we abbreviate further and write $[p^2]$ instead of $[p,p]$, and so on. The Coxeter symbol $[3^{i,j,k}]$

denotes a group with Y-shaped Coxeter graph with strings of i, j and k edges emanating from a common node. We assemble the different symbols into a single one in order to describe the different nature of portions of the Coxeter graph in question (see Figure 1 and also [29]).

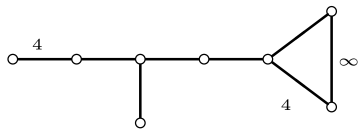


Figure 1: The Coxeter pyramid group $[4, 3^{1,1}, 3^2, (3, \infty, 4)]$ acting on $\mathbb{H}^6$

§ 2.2. A panoramic view to hyperbolic Coxeter polyhedra

The irreducible spherical and Euclidean Coxeter polyhedra are completely characterised [10]. They exist in any dimension $n \geq 2$, and correspond to the irreducible finite and affine Coxeter groups, respectively. Tables can be found in [61, pp. 202–203], for example.

Unlike their spherical and Euclidean counterparts, hyperbolic Coxeter groups are far from being classified. In this part, we provide an overview of known classification results. In the sequel, we shall make no distinction between a Coxeter group and the corresponding Coxeter polyhedron.

2.2.1. Dimensional bounds There are the following dimensional bounds for the finite volume case, due to Prokhorov-Kovanskij [45], and for the compact and arithmetic cases, due to Vinberg (see [61], for example).

Theorem 2.1. *There are no finite volume Coxeter polyhedra in $\mathbb{H}^n$ for $n \geq 996$.*

Theorem 2.2. *There are no compact Coxeter polyhedra in $\mathbb{H}^n$ for $n \geq 30$.*

Furthermore, if one restricts the context to particular families of hyperbolic Coxeter polyhedra, other dimensional bounds can be obtained. By an abuse of language, a Coxeter polyhedron is *arithmetic* if its associated Coxeter group is arithmetic (see Section 4).

Theorem 2.3. *There are no arithmetic Coxeter polyhedra in $\mathbb{H}^n$ for $n \geq 30$.*

2.2.2. Dimensions 2 and 3 The planar case is completely described by the following result due to Poincaré (see also [61, Chapter 3.2]).

Theorem 2.4. *Let $N \geq 3$ be an integer and $0 \leq \alpha_1, ..., \alpha_N < \pi$ be non-negative* *real numbers such that*

$$\alpha_1 + ... + \alpha_N < (N-2)\,\pi. \tag{2.3}$$

Then, there exists a hyperbolic N-gon $P \subset \mathbb{H}^2$ with angles $\alpha_1, ..., \alpha_N$. Conversely, if $0 \leq \alpha_1, ..., \alpha_N < \pi$ are the angles of an N-gon $P \subset \mathbb{H}^2$, then they satisfy (2.3).

In particular, the set of hyperbolic Coxeter polygons with N sides can be identified with the set of N-tuples given by $\left\{(p_1, ..., p_N) \mid 2 \leq p_i \leq \infty,\, \sum_{i=1}^{N} \frac{1}{p_i} < N-2\right\}$ over $\mathbb{Z}$.

The situation in the 3-dimensional case becomes already more subtle. For compact *acute-angled* polyhedra, that is, all dihedral angles are less than or equal to $\frac{\pi}{2}$, there is the following result due to Andreev [3], which has been fully proved by Roeder [47, 48].

Theorem 2.5. *Let $P \subset \mathbb{H}^3$ be a compact acute-angled polyhedron with $N \geq 5$ facets and $M \geq 5$ edges with corresponding dihedral angles $\alpha_1, ..., \alpha_M$. Then,*

1. *For all $i = 1, ..., M$, $\alpha_i > 0$.*
2. *If three edges e_i, e_j, e_k meet at a vertex, then $\alpha_i + \alpha_j + \alpha_k > \pi$.*
3. *For any prismatic 3-circuit with intersecting edges e_i, e_j, e_k, one has $\alpha_i + \alpha_j + \alpha_k < \pi$.*
4. *For any prismatic 4-circuit with intersecting edges e_i, e_j, e_k, e_l, one has $\alpha_i + \alpha_j + \alpha_k + \alpha_l < 2\pi$.*
5. *For any quadrilateral facet F bounded successively by edges e_i, e_j, e_k, e_l such that $e_{ij}, e_{jk}, e_{kl}, e_{li}$ are the remaining edges of P based at the vertices of F (e_{pq} is based at the intersection of e_p and e_q), then*
$$\alpha_i + \alpha_k + \alpha_{ij} + \alpha_{jk} + \alpha_{kl} + \alpha_{li} < 3\pi$$
and
$$\alpha_j + \alpha_l + \alpha_{ij} + \alpha_{jk} + \alpha_{kl} + \alpha_{li} < 3\pi.$$

Furthermore, the converse holds, i.e. any abstract 3-polyhedron satisfying the conditions above can be realised as a compact acute-angled hyperbolic 3-polyhedron, and is unique up to isometry.

Andreev also extended his result to the non-compact case [4]. Let us mention that the special case of *ideal* polyhedra, that is, polyhedra all of whose vertices lie on $\partial\mathbb{H}^n$, admits a simple formulation using the dual of a polyhedron: Recall that the *dual* of a polyhedron $P \subset \mathbb{H}^3$ is the polyhedron P^* such that the set of vertices of P is in bijection with the set of facets of P^*, and vice-versa. Furthermore, for any edge e of P associated to a dihedral angle α, the corresponding edge e^* of P^* supports a dihedral angle α^* given by $\alpha^* = \pi - \alpha$. The result, due to Rivin [46] (see also [21]), reads as follows.

Theorem 2.6. *Let $P \subset \mathbb{H}^3$ be an ideal polyhedron. Then, its dual $P^* \subset \mathbb{H}^3$ satisfies the following conditions.*

1. *For any dihedral angle α^* of P^*, one has $0 < \alpha^* < \pi$.*
2. *If the edges $e_1^*, ..., e_k^*$ with associated dihedral angles $\alpha_1^*, ..., \alpha_k^*$ form the boundary of a facet of P^*, then*
$$\sum_{i=1}^{k} \alpha_i^* = 2\pi.$$
3. *If the edges $e_1^*, ..., e_k^*$ with associated dihedral angles $\alpha_1^*, ..., \alpha_k^*$ form a closed circuit in P^* but do not bound a facet, then*
$$\sum_{i=1}^{k} \alpha_i^* > 2\pi.$$

Moreover, any polyhedron $P^* \subset \mathbb{H}^3$ satisfying the above conditions $(1) - (3)$ is the dual of some ideal polyhedron $P \subset \mathbb{H}^3$, which is unique up to isometry.

2.2.3. Classifications in terms of the number of facets

Let $P \subset \mathbb{H}^n$ be a finite volume Coxeter polyhedron with $N \geq n+1$ facets. Complete classifications have been obtained for $N = n+1$ and $N = n+2$ only. For $N = n+1$ and hyperbolic Coxeter simplices, the classification is due to Lannér in compact case and to Koszul in the finite volume case.

Theorem 2.7 $(N = n+1)$**.**

- *Compact hyperbolic Coxeter simplices exist in dimensions $n = 2, 3$ and 4 only. For $n = 3, 4$, there are finitely many of them.*
- *Finite-volume non-compact hyperbolic Coxeter simplices exist in dimensions $n = 2, ..., 9$ only. For $3 \leq n \leq 9$, there are finitely many of them.*

Tables can be found in [61, p. 203 and pp. 206–208], for example. For $n = 3$, the non-compact Coxeter tetrahedra of finite volume are listed in Appendix Appendix A.

If $N = n + 2$, then P is either a prism, or a product of two simplices of positive dimensions, or a pyramid over the product of two simplices of positive dimensions. The respective classification is due to Kaplinskaja [32], Esselmann [14] and Tumarkin [54].

Theorem 2.8 $(N = n+2)$**.**

- *Compact, respectively finite-volume, Coxeter prisms in $\mathbb{H}^n$ exist only for $n \leq 5$. For $4 \leq n \leq 5$, there are finitely many of them.*

- *Compact Coxeter polyhedra with $n+2$ facets in $\mathbb{H}^n$ which are not prisms exist only for $n \le 4$. For $n = 4$, there are exactly 7 of them.*
- *Finite-volume Coxeter polyhedra with $n+2$ facets in $\mathbb{H}^n$ which are not prisms exist only for $n \le 17$. For $3 \le n \le 17$, there are finitely many of them.*

Tables can be found in [32, pp. 89–90], [61, p. 61], [14, p. 230] respectively while the list [54] of Tumarkin's Coxeter pyramids which are pyramids over a product of two simplices of positive dimensions is attached in Appendix Appendix B. Notice that among the non-compact Coxeter polyhedra with $n+2$ facets in $\mathbb{H}^n$, $n \ge 3$, the polyhedra of Tumarkin are the only ones apart from the Coxeter polyhedron $\widetilde{P} \subset \mathbb{H}^4$ depicted in Figure 2 which is combinatorially a product of two triangles.

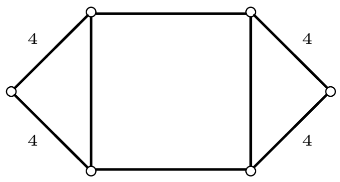


Figure 2: The Coxeter polyhedron $\widetilde{P} \subset \mathbb{H}^4$

In this work, the family of Tumarkin's pyramids will be of special interest. Such a pyramid is not compact since its apex, with a neighborhood being a cone over a product of two Euclidean simplices, has to be a point at infinity. In the associated Coxeter graph Σ, the node separating Σ into the two disjoint corresponding Euclidean Coxeter subgraphs is *encircled*. Let us just mention a few particular features of the family of Tumarkin's Coxeter pyramid groups. It has 200 members, with examples up to dimension $n = 17$, and comprises non-arithmetic groups up to dimension 10. There is a single but very distinguished group Γ_* in dimension 17.

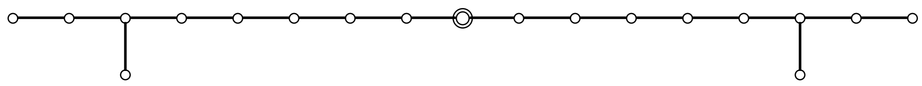


Figure 3: The graph of the Coxeter pyramid $P_* = [3^{2,1}, 3^{12}, 3^{1,2}]$ in $\mathbb{H}^{17}$

The group Γ_* is closely related to the even unimodular group $PSO(\mathrm{II}_{17,1})$ in the following way. Denote by P_* the Coxeter polyhedron associated to Γ_* whose Coxeter graph is given in Figure 3. By a result of Emery [12, Theorem 1], the orbifold $\mathbb{H}^{17}/PSO(\mathrm{II}_{17,1})$ is the (unique up to isometry) hyperbolic n-space form of minimal volume among all orientable arithmetic hyperbolic n-orbifolds for $n \ge 2$. Its volume can be identified with the one of P_* and computed according to (see [12, Section 3])

$$\mathrm{vol}_{17}(P_*) = \mathrm{vol}_{17}(\mathbb{H}^{17}/PSO(\mathrm{II}_{17,1})) = \frac{691 \cdot 3617}{2^{38} \cdot 3^{10} \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17}\, \zeta(9)\,.$$

The top-dimensional non-arithmetical Coxeter pyramid group $\Gamma_{3,4}$ is the subgroup of $\mathrm{Isom}\,\mathbb{H}^{10}$ given by the Coxeter symbol $[3^{2,1},3^6,(3,\infty,4)]$ and by the Coxeter graph in Figure 4 (for $k=3$, $l=4$). It turns out to be the free product with amalgamation

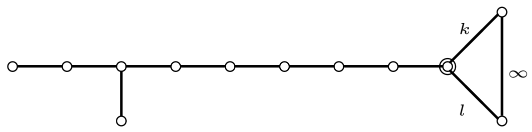


Figure 4: The Coxeter pyramid groups $\Gamma_{k,l} \subset \mathrm{Isom}\,\mathbb{H}^{10}$

$\Gamma_{3,4} = \widehat{\Gamma}_{2,3} \star_\Phi \widehat{\Gamma}_{2,4}$ of the two (incommensurable) arithmetic Coxeter groups $\widehat{\Gamma}_{2,l} = [3^{2,1},3^6,l]$, $l=3,4$, of infinite covolume (see Figure 4 and Figure 5 ; cf. [60] and Section 4).

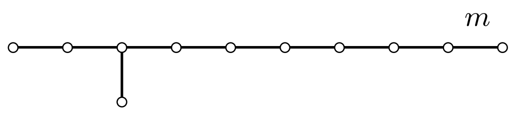


Figure 5: The (total) Coxeter simplex group $\widehat{\Gamma}_{2,m}$ of infinite covolume

The group $\Gamma_{3,4}$ is a mixture in the sense of Gromov and Piatetski-Shapiro [20] and has a Coxeter polyhedron $P_{3,4} \subset \mathbb{H}^{10}$ which is obtained by glueing the Coxeter pyramids $P_{2,3}, P_{2,4}$ with Coxeter graphs $\Gamma_{2,3} = [3^{2,1},3^6,3,\infty]$, $\Gamma_{2,4} = [3^{2,1},3^6,4,\infty]$ along their common Coxeter facet $F \subset \mathbb{H}^9$ with reflection group Φ and Coxeter graph given by Figure 6.

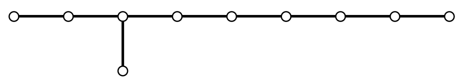


Figure 6: The common subgroup Φ of $\Gamma_{2,3}$ and $\Gamma_{2,4}$

For $N = n+3$, a complete classification has been obtained only in the compact case, due to results of Esselmann [13] and Tumarkin [55]. Such polyhedra exist in dimensions $n \leq 8$. Moreover, Tumarkin [53] proved that there are no finite-volume hyperbolic Coxeter polyhedra in dimensions $n \geq 17$, and that there is a unique such polyhedron in dimension $n = 16$.
For $N = n+4$, a dimensional bound is available for the compact case, only. In fact, such polyhedra do not exist in dimensions $n > 8$, and there is a unique compact hyperbolic Coxeter polyhedron in $\mathbb{H}^7$ with 11 facets. These results are due to Felikson and Tumarkin [15].
For $N \geq n+5$, no dimensional bound and no classification are available so far.

Hopes for complete classifications of Coxeter polyhedra with given numbers of facets have been dashed when the following two results, due to Allcock [2], were established.

Theorem 2.9. *There are infinitely many isometry classes of finite-volume Coxeter polyhedra in $\mathbb{H}^n$, for every $n \leq 19$. For $2 \leq n \leq 6$, they may be taken to be either compact or non-compact.*

Theorem 2.10. *For every $n \leq 19$, with the possible exceptions of $n = 16, 17$, the number of isometry classes of Coxeter polyhedra in $\mathbb{H}^n$ of volume $\leq V$ grows at least exponentially with respect to V. For $2 \leq n \leq 6$, they may be taken to be either compact or non-compact.*

Notice that the bound $n \leq 19$ is not a strict one. This is due to the fact that, except from Borcherds' particular example in dimension $n = 21$ [6], all *known* Coxeter polyhedra occur in dimensions $n \leq 19$.

One important tool in Allcock's approach is the so-called *doubling trick*, which allows one to construct new Coxeter polyhedra by gluing together congruent copies of the same Coxeter polyhedron along isometric facets satisfying certain angular conditions.

2.2.4. Classifications and dimensional bounds for polyhedra with specific combinatorics

Another way to attack the classification question is to consider polyhedra with prescribed combinatorial and metrical properties. There are a couple of results in this direction.

In [25], Im Hof studied and classified so-called *crystallographic Napier cycles of type d*, which are $(d-1)$-fold truncated Coxeter orthoschemes. He proved that they exist only for $d = 1, 2, 3$ and in dimensions $n \leq 9$, and he provided a complete classification [25, pp. 540–544].

Schlettwein [49] classified the so-called *edge-finite* hyperbolic Coxeter truncated simplices, that is, simplices whose edges are not entirely cut off during the truncation procedure. Such polyhedra exist in dimensions $2 \leq n \leq 5$ for both the compact and finite-volume cases. Tables can be found in [49, pp. 41–44].

The family of hyperbolic Coxeter pyramids has been investigated by McLeod [40], based on previous works of Tumarkin and Vinberg. In particular, he showed that hyperbolic Coxeter pyramids with $n + p$ facets exist only for $1 \leq p \leq 4$, and he completed the classification for the case $p = 4$ (for $1 \leq p \leq 3$, the pyramids were part of already available classifications, see Section 2.2.3).

It is remarkable that many of the known Coxeter polyhedra can be interpreted as (not necessarily edge-finite) truncated simplices [26].

A further class of polyhedra which has been investigated is the class of *n-cubes*, that is, polyhedra which are combinatorially equivalent to the standard cube $[0,1]^n \subset \mathbb{R}^n$. Jacquemet [28] proved that there are no finite-volume Coxeter n-cubes in $\mathbb{H}^n$ for $n \geq 10$, and no compact ones in $\mathbb{H}^n$ for $n \geq 9$. Moreover, he proved that the *ideal* hyperbolic Coxeter n-cubes exist for $n = 2$ and also $n = 3$ only, and provided a classification.

Beside polyhedra of fixed combinatorial type, some results are available for *simple* polyhedra $P \subset \mathbb{H}^n$ of finite volume, which means that for $1 \leq k \leq n$, each $(n-k)$-face is contained in exactly k bounding hyperplanes of P. All three results below are due to Felikson and Tumarkin (see [16], [17] and [18]).

Theorem 2.11. *There are no simple ideal Coxeter polyhedra in $\mathbb{H}^n$ for $n \geq 9$.*

Theorem 2.12. *Let $P \subset \mathbb{H}^n$ be a simple non-compact Coxeter polyhedron. If $n > 9$, then P has a pair of disjoint facets, and if $n \leq 9$, then either P has a pair of disjoint facets, or P is a simplex, or P is the product of two simplices.*

The corresponding result for compact (and therefore simple) Coxeter polyhedra is even stronger.

Theorem 2.13. *Let $P \subset \mathbb{H}^n$ be a compact Coxeter polyhedron. If $n > 4$, then P has a pair of disjoint facets, and if $n \leq 4$, then either P has a pair of disjoint facets, or P is a simplex, or P has $n+2$ facets and is one of the seven Esselmann polytopes.*

Based on these facts and in order to refine the classification problem for Coxeter polyhedra based on Allcock's results, Felikson and Tumarkin [18] introduced the notion of *essential* Coxeter polyhedra, playing the role of indecomposable 'building blocks' that can be used in order to build other Coxeter polyhedra. More precisely, let $n \geq 4$ and consider a polyhedron $P \subset \mathbb{H}^n$ with $f \geq n+1$ facets and p pairs of disjoint facets. If $p \leq f - n - 2$, then P is called an essential polyhedron. The following result gives a new hope for a classification of hyperbolic Coxeter polyhedra.

Theorem 2.14. *Let $\mathfrak{P}$ be the set of essential hyperbolic Coxeter polyhedra. Then* $\operatorname{card}(\mathfrak{P}) < \infty$.

§ 2.3. Some hyperbolic volume identities

Consider a Coxeter group $\Gamma \subset \operatorname{Isom}\mathbb{H}^n$ with Coxeter polyhedron $P = P(\Gamma)$. The *covolume* of Γ is defined to be the volume of P.
For n even, it is well known that the covolume of Γ is proportional to the absolute value of the Euler characteristic $\chi(\Gamma)$ and given by

$$\operatorname{covol}_n\Gamma = \operatorname{vol}_n(P) = \frac{(2\pi)^{\frac{n}{2}}}{1 \cdot 3 \cdot \ldots \cdot (2n-1)} \cdot |\chi(\Gamma)| \,. \tag{2.4}$$

The program *CoxIter* developped by Guglielmetti [22] provides the value of $\chi(\Gamma)$, and therefore the covolume of Γ in the even-dimensional case, for a group Γ with prescribed Coxeter graph. More concretely, the program gives the combinatorial structure in terms of the f-vector of $P = P(\Gamma)$, whose k-th component is the number of k-faces of P. It also allows one to decide whether Γ is cocompact or cofinite and whether it is arithmetic (see Section 4).

For n odd, volume computations are much more difficult to handle due to the lack of closed formulas and the implication of transcendental functions such as polylogarithms.

Let us consider some simple cases when $n = 3$. It is known that hyperbolic volume can be expressed in terms of the *Lobachevsky function* $Л(\omega)$ which is related to the dilogarithm function $\mathrm{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2}$ by means of

$$Л(\omega) = \frac{1}{2} \sum_{r=1}^{\infty} \frac{\sin(2r\omega)}{r^2} = -\int_0^{\omega} \log \mid 2\sin t \mid dt \ , \ \omega \in \mathbb{R} \ . \tag{2.5}$$

The Lobachevsky function satisfies the following three *essential* functional properties.

- $Л(x)$ is odd.
- $Л(x)$ is π-periodic.
- $Л(x)$ satisfies for each integer $m \neq 0$ the *distribution law*

$$Л(m\,x) = m \cdot \sum_{k=0}^{m-1} Л(x + \frac{k\pi}{m}) \ . \tag{2.6}$$

In this context, let us mention the following conjectures of Milnor [51, Chapter 7], [43].

Conjectures (Milnor)**.**

(A) Every rational linear relation between the real numbers $Л(x)$ with $x \in \mathbb{Q}\pi$ is a consequence of the three essential functional equations above.

(B) Fixing some denominator $N \geq 3$, the real numbers $Л(k\pi/N)$ with k relatively prime to N and $0 < k < N/2$ are linearly independent over $\mathbb{Q}$.

Next we consider some concrete cases. Let $[p, q, r]$ be a non-compact Coxeter orthoscheme in $\mathbb{H}^3$, with one vertex at infinity so that $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$, say. In Figure 7 we depict such an orthoscheme in the upper half space model with vertex ∞. Its volume can be expressed according to (see [34, p. 562], for example)

$$\mathrm{vol}_3([p, q, r]) = \frac{1}{4}\{Л(\frac{\pi}{p} + \frac{\pi}{r}) + Л(\frac{\pi}{p} - \frac{\pi}{r}) + 2\ Л(\frac{\pi}{q})\} \ , \ \frac{1}{q} + \frac{1}{r} \geq \frac{1}{2} \ . \tag{2.7}$$

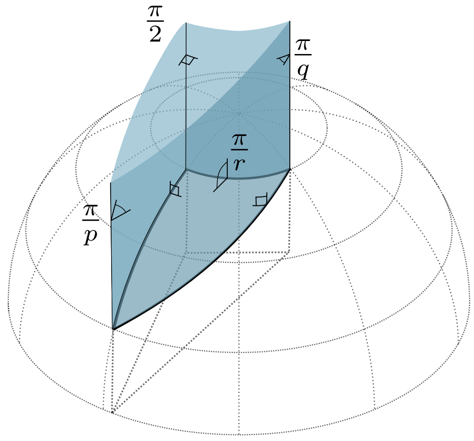


Figure 7: The Coxeter orthoscheme $[p,q,r] \subset \mathbb{H}^3$ with $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$

In particular, the volumes of the Coxeter orthoschemes $[3,3,6]$ and $[3,4,4]$ are given by

$$\begin{aligned} \mathrm{vol}_3([3,3,6]) &= \frac{1}{8}\ Л(\frac{\pi}{3}) \simeq 0.04229\ , \\ \mathrm{vol}_3([3,4,4]) &= \frac{1}{6}\ Л(\frac{\pi}{4}) \simeq 0.07633\ . \end{aligned} \tag{2.8}$$

Remark. In [1] and based on Meyerhoff's work [42], Adams proved that the 1-cusped 3-orbifold $\mathbb{H}^3/[3,3,6]$ is the (unique) non-compact hyperbolic orbifold of minimal volume.

Consider an ideal hyperbolic tetrahedron $S_\infty = S_\infty(\alpha,\beta,\gamma) \subset \mathbb{H}^3$ which is characterised by the dihedral angles $\alpha,\beta,\gamma \in]0,\frac{\pi}{2}]$, each one sitting at a pair of opposite edges, and satisfying $\alpha+\beta+\gamma=\pi$. It can be decomposed into 3 isometric pairs of orthoschemes, each having 2 vertices at infinity, in such a way that the identity (2.7) provides the following volume formula.

$$\mathrm{vol}_3(S_\infty) = Л(\alpha) + Л(\beta) + Л(\gamma)\ ,\ \ \alpha+\beta+\gamma=\pi\ . \tag{2.9}$$

As a consequence, a *regular* ideal tetrahedron S_∞, characterised by $\alpha=\beta=\gamma=\frac{\pi}{3}$, is of volume $3\,Л(\frac{\pi}{3})$. In [30], the covolumes of all hyperbolic Coxeter n-simplex groups, $3 \le n \le 9$, were determined; for the values in the case of $n=3$, see Appendix Appendix A.

Next, consider one of the 7 ideal Coxeter cubes in $\mathbb{H}^3$ (see [28]). It can be dissected into 4 ideal tetrahedra according to Figure 8. In the case of the Coxeter cube W with Coxeter graph given by Figure 9 and edge lengths l_1, l_2, l_3 of pairs of opposite facets satisfying

$$\cosh l_1 = \cosh l_2 = \frac{5}{2} \quad , \quad \cosh l_3 = \frac{2\sqrt{3}}{3}\ , \tag{2.10}$$

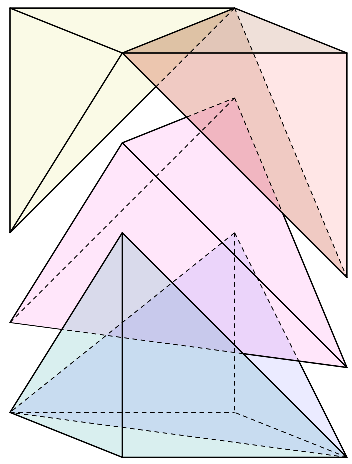


Figure 8: Dissection of a cube into tetrahedra

the formula (2.9) yields the following.

$$\mathrm{vol}_3(W) = 10\ Л(\frac{\pi}{3})\ . \tag{2.11}$$

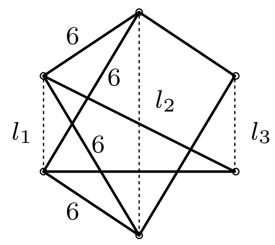
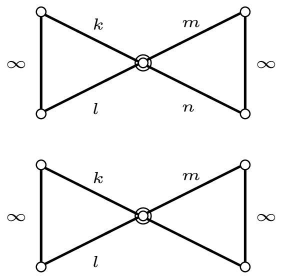


Figure 9: An ideal Coxeter cube $W \subset \mathbb{H}^3$

For pyramids $P \subset \mathbb{H}^3$, there is also a closed formula in terms of the dihedral angles due to Vinberg [61, pp. 129–130] (up to minor sign errors). In the special case of a pyramid $P = P(\alpha_1, \ldots, \alpha_4)$ whose apex q at infinity is the intersection of 4 edges with right (interior) dihedral angles, Vinberg's formula can be stated as follows in terms of the dihedral angles $\alpha_1, \ldots, \alpha_4$ opposite to q (circularly enumerated with indices modulo 4).

$$\begin{aligned} 2\,\mathrm{vol}_3(P) = \sum_{k=1}^{4} \Big\{\, & Л\,\big((\frac{\pi}{2} + \alpha_k + \alpha_{k+1})/2\big) + Л\,\big((\frac{\pi}{2} + \alpha_k - \alpha_{k+1})/2\big) \\ & + Л\,\big((\frac{\pi}{2} - \alpha_k + \alpha_{k+1})/2\big) + Л\,\big((\frac{\pi}{2} - \alpha_k - \alpha_{k+1})/2\big) \Big\}\ . \end{aligned} \tag{2.12}$$

As an application, the covolumes of all Coxeter pyramid groups in $\mathrm{Isom}\,\mathbb{H}^3$ given by the Coxeter graphs according to Figure 10 can be explicitly determined. In particular, the



$k = 2, 3, 4\,;$ $\quad m = 2, 3, 4\,;$
$l = 3, 4\,;$ $\quad n = 3, 4\,.$

$k = 5, 6\,;$ $\quad m = 2, 3\,.$
$l = 2, 3, 4, 5, 6\,;$

Figure 10: The Coxeter pyramids $[(k, \infty, l), (m, \infty, n)]$ and $[(k, \infty, l), (m, \infty, 3)]$ in $\mathbb{H}^3$

covolumes of the two (non-arithmetic) Coxeter pyramid groups Γ_1 and Γ_2 with Coxeter graphs given by Figure 11 is as follows.

Γ_1 $\qquad$ Γ_2

Figure 11: Two (non-arithmetic) Coxeter pyramid groups

$$
\begin{aligned}
\text{(2.13)} \qquad \mathrm{covol}_3(\Gamma_1) &= \frac{1}{3}\,Л(\frac{\pi}{4}) + \frac{1}{8}\,Л(\frac{\pi}{6}) + Л(\frac{5\pi}{24}) - Л(\frac{\pi}{24}) \;\simeq\; 0.40362\;, \\
\mathrm{covol}_3(\Gamma_2) &= \quad Л(\frac{\pi}{4}) + \frac{1}{8}\,Л(\frac{\pi}{6}) + Л(\frac{5\pi}{24}) - Л(\frac{\pi}{24}) \;\simeq\; 0.70894\;.
\end{aligned}
$$

For the (non-arithmetic) Coxeter pyramid group Γ_T with Coxeter graph given by Figure 12, the covolume is equal to

$$
\text{(2.14)} \qquad \mathrm{covol}_3(\Gamma_T) = \frac{5}{4}\,Л(\frac{\pi}{3}) + \frac{1}{3}\,Л(\frac{\pi}{4}) \;\simeq\; 0.57555\;.
$$

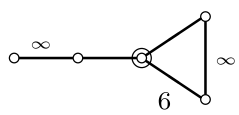


Figure 12: The (non-arithmetic) Coxeter pyramid group Γ_T

§ 3. Commensurable hyperbolic Coxeter groups

Two discrete subgroups $G_1, G_2 \subset \mathrm{Isom}\,\mathbb{H}^n$ are said to be *commensurable in the wide sense* or just *commensurable*, for abbreviation, if the intersection $G_1 \cap G_2'$ of G_1

with some conjugate G_2' of G_2 in $\mathrm{Isom}\,\mathbb{H}^n$ is of finite index both in G_1 and G_2'. As a consequence, the associated orbifolds $\mathbb{H}^n/G_1$ and $\mathbb{H}^n/G_2$ admit a common finite sheeted cover up to isometry.
Notice that the intersection $H := G_1 \cap G_2'$ contains a non-trivial normal subgroup N of finite index in G_1, the *normal core* of H, which is given by $\cap_{\gamma\in G_1}\gamma\, H\,\gamma^{-1}$.

It is not difficult to see that commensurability is an equivalence relation which is preserved when passing to a finite index subgroup. In particular, one can study commensurability for orientation preserving subgroups (of index two) or - by Selberg's Lemma - by passing to a finite index torsion-free subgroup. This is particularly convenient for dimensions $n = 2$ resp. $n = 3$ where the group $PSO(n, 1)$ of orientation preserving isometries is isomorphic to $PSL(2, \mathbb{R})$ resp. $PSL(2, \mathbb{C})$.

§ 3.1. Commensurability criteria

It follows from the definition that the commensurability relation on the set $\mathcal{G}$ of discrete groups $G \subset \mathrm{Isom}\,\mathbb{H}^n$ leaves invariant properties such as cocompactness, cofiniteness and arithmeticity (for the arithmetic aspects, see Section 4).

An important characterisation of (non-)arithmeticity is due to Margulis (see [38, Theorem 10.3.5], for example). Consider the *commensurator*

$$\mathrm{Comm}(G) = \{\, \gamma \in \mathrm{Isom}\,\mathbb{H}^n \mid G \cap \gamma G\gamma^{-1} \text{ has finite index in } G \text{ and } \gamma G\gamma^{-1} \,\} \tag{3.1}$$

of a cofinite discrete group $G \subset \mathrm{Isom}\,\mathbb{H}^n$, $n \geq 3$. Then, the result of Margulis can be stated in the following form.

Theorem 3.1. *The commensurator* $\mathrm{Comm}(G) \subset \mathrm{Isom}\mathbb{H}^n$ *is a discrete subgroup in* $\mathrm{Isom}\mathbb{H}^n$ *containing G with finite index if and only if G is non-arithmetic.*

In particular, for G non-arithmetic, the commensurator $\mathrm{Comm}(G)$ is the maximal element in the commensurability class of G so that all non-arithmetic hyperbolic orbifolds with fundamental groups commensurable to G cover a smallest common quotient.

Suppose that a finite volume hyperbolic orbifold $Q = \mathbb{H}^n/G$ has a *cusp*, that is, the group G has a fundamental polyhedron with an ideal vertex $q \in \partial\mathbb{H}^n$. Therefore, the stabiliser $G_q \subset G$ is a crystallographic group containing a translational lattice $\Lambda \cong \mathbb{E}^{n-1}$ of finite index. Let $U_q \subset Q$ be the maximal embedded cusp neighborhood. Then, the quotient $\mathrm{vol}_n(U_q)/\mathrm{vol}_n(Q)$ is called the *cusp density* of q in Q. It is known that for 1-cusped orbifolds $Q = \mathbb{H}^n/G$ with discrete commensurator $\mathrm{Comm}(G)$, the cusp density is a commensurability invariant (see [19, Section 2]). However, since (transcendental) volume expressions are involved, this property is of limited value (see (2.5) and (2.13), for example). In the case of 1-cusped quotients by non-arithmetic Coxeter pyramid groups (see Section 5.3), we present results whose proofs are based on a different reasoning without volume computation but using crystallography (see Theorem 5.4 and Lemma 5.6).

In fact, volume considerations rarely help to judge about commensurability in a rigorous way but relate such questions to analytical number theory. As a general principle, if the covolume quotient of two groups in $\mathcal{G}$ is an irrational number, then the groups cannot be commensurable. However, already for $n = 3$, such criteria are very difficult to handle and closely related to Milnor's Conjectures presented in Section 2.3. Furthermore, for $n = 2k$, the above observation is void in view of (2.4).

A natural conjugacy invariant is the (ordinary) trace field $\mathrm{Tr}(G)$ of G in $GL(n+1,\mathbb{R})$ which is defined to be the field generated by all the traces of matrices in $G \subset PO(n,1)$. In the particular case of *Kleinian* groups, that is, of cofinite discrete groups $G \subset PSL(2,\mathbb{C})$, one can sometimes avoid trace computations by considering the *invariant trace field* $kG = \mathbb{Q}(\mathrm{Tr}(G^{(2)}))$, generated by the traces of all squared elements γ^2, $\gamma \in G$, and the *invariant quaternion algebra* AG over kG. Both, the field kG and the algebra AG are commensurability invariants (see [44]). Furthermore, kG is a finite non-real extension of $\mathbb{Q}$ (see [38, Theorem 3.3.7]), and if the group G is not cocompact (containing parabolic elements), then the algebra AG is isomorphic to the matrix algebra $Mat(2,kG)$ (see [38, Theorem 3.3.8]).
In the case when the group G is an amalgamated free product of the form $G = G_1 \star_H G_2$, where H is a non-elementary Kleinian group, then kG is a composite of fields according to (see [38, Theorem 5.6.1])

$$kG = kG_1 \cdot kG_2 \, .$$

In this context, consider hyperbolic Coxeter groups $\Gamma \subset \mathrm{Isom}\,\mathbb{H}^n$ which are amalgamated free products $\widehat{\Gamma}_1 \star_\Phi \widehat{\Gamma}_2$ such that $\Phi \subset \mathrm{Isom}\,\mathbb{H}^{n-1}$ is itself a cofinite Coxeter group whose fundamental Coxeter polyhedron F is a common facet of the fundamental Coxeter polyhedra P_1 and P_2 of Γ_1 and Γ_2 (for an example, see the groups $\Gamma_{k,l} \subset \mathrm{Isom}\,\mathbb{H}^{10}$ as described by Figures 4, 5 and 6 and the explanation given in Section 2.2). Geometrically, a fundamental polyhedron for $\widehat{\Gamma}_1 \star_\Phi \widehat{\Gamma}_2$ is the Coxeter polyhedron arising by glueing together P_1 and P_2 along their common facet F. Then, the following general result of Karrass and Solitar [33, Theorem 10] is very useful to decide about commensurability, in particular in the case of Tumarkin's Coxeter pyramid groups (for details, see Section 5.3, Lemma 5.6).

Theorem 3.2. *Let $G = A \star_U B$ be a free product with amalgamated subgroup U, and let H be a finitely generated subgroup of G containing a normal subgroup N of G such that $N \not\leq U$. Then, H is of finite index in G if and only if the intersection of U with each conjugate of H is of finite index in U.*

§ 4. Arithmetical aspects

In [36], Maclachlan described an effective way to decide about the commensurability of two *arithmetic* discrete subgroups of $\mathrm{Isom}\,\mathbb{H}^n$. The key point is that the classification of arithmetic hyperbolic groups is equivalent to the classification of their underlying quadratic forms up to similarity. In turn, the classification of these quadratic forms can be achieved using a complete set of invariants based on quaternion algebras. In this section, we present the necessary algebraic background, discuss the result of Maclachlan and provide a few examples. In Appendix Appendix C we provide the commensurability classes of arithmetic hyperbolic Coxeter pyramid groups.

§ 4.1. Arithmetic groups

In order to motivate the precise definition of an arithmetic group (of the simplest type), we start with an example. Let $f_n(x) = x_1^2 + \ldots + x_n^2 - x_{n+1}^2$ be the standard Lorentzian quadratic form on $\mathbb{R}^{n+1}$ and consider the cone $C = \{x \in \mathbb{R}^{n+1} \mid f_n(x) < 0\}$. We denote by $O(f_n, \mathbb{Z}) \subset GL(n+1, \mathbb{R})$ the group of linear transformations of $\mathbb{R}^{n+1}$ with coefficients in $\mathbb{Z}$ which preserve the quadratic form f_n. The index two subgroup $O^+(f_n, \mathbb{Z})$ consisting of elements of $O(f_n, \mathbb{Z})$ which preserve the two connected components $C^{\pm} = \{x \in C \mid x_{n+1} \lesseqgtr 0\}$ of the cone C is the prototype of an arithmetic group. Moreover, it is well known that this discrete group has finite covolume in $O^+(n,1)$. Note that the condition that the transformation has coefficients in $\mathbb{Z}$ is equivalent to invariance requirement of the standard $\mathbb{Z}$-lattice in $\mathbb{R}^{n+1}$.

Now, let K be any totally real number field, $\mathcal{O}_K$ its ring of integers, and let V be an $(n+1)$-dimensional vector space over K endowed with an *admissible* quadratic form f of signature $(n,1)$, that is, all conjugates f^σ of f by the non-trivial Galois embedding $\sigma : K \longrightarrow \mathbb{R}$ are positive definite. The cone $C_f = \{x \in V \otimes \mathbb{R} : f(x) < 0\}$ has two connected components $C_f^{\pm}$ and gives rise to the Minkowski model $C_f^+/\mathbb{R}^*$ of the hyperbolic n-space $\mathbb{H}^n$. We now let

$$O(f) = \big\{T \in GL(V \otimes \mathbb{R}) \mid f \circ T(x) = f(x),\ \forall x \in V \otimes \mathbb{R}\big\}.$$

Finally, for a $\mathcal{O}_K$-lattice L of full rank in V and for $O(f)_K$ the K-points of $O(f)$, we consider the group

$$O(f, L) := \big\{T \in O(f)_K \mid T(C_f^+) = C_f^+,\ T(L) = L\big\}.$$

It is a discrete subgroup of $\mathrm{Isom}\,\mathbb{H}^n$ of finite covolume (see [7]).

Definition 4.1. A discrete group $\Gamma \subset \mathrm{Isom}\,\mathbb{H}^n$ is an *arithmetic group of the simplest type* if there exist K, f and L as above such that Γ is commensurable to $O(f, L)$. In this setting, we say that Γ is *defined over* K and that f is the *quadratic form associated to* Γ.

Remarks 4.2 ([61, Chapter 6]).

1. If such a group is non-cocompact, then it must be defined over $\mathbb{Q}$.
2. If Γ is defined over $\mathbb{Q}$, then it is non-cocompact if and only if the lattice is isotropic (see also [20, Section 2.3]). In particular, if Γ is non-compact then $n \geq 4$ (this follows from [9, Chapter 4, Lemma 2.7] and the Hasse-Minkowski theorem).
3. There is a slightly more general definition of discrete arithmetic subgroups of $\operatorname{Isom}\mathbb{H}^n$. However, when a discrete subgroup of $\operatorname{Isom}\mathbb{H}^n$ is arithmetic and contains reflections, then it is of the simplest type. Therefore, we will refer to *arithmetic of group of the simplest type* as *arithmetic group.*

Vinberg presented in [61] a criterion to decide whether a hyperbolic Coxeter group is arithmetic or not. In order to present the criterion, we need the following definition.

Definition 4.3. Let $A := (A_{i,j}) \in Mat(n, K)$ be a square matrix with coefficients in a field K. A *cycle* of length k, or *k-cycle*, in A is a product $A_{i_1,i_2} \cdot A_{i_2,i_3} \cdot \ldots \cdot A_{i_{k-1},i_k} \cdot A_{i_k,i_1}$. Such a cycle is denoted by $A_{(i_1,\ldots,i_k)}$. If the i_j are all distinct, the cycle is called *irreducible.*

Theorem 4.4. *Let Γ be a hyperbolic Coxeter group of finite covolume in* $\operatorname{Isom}\mathbb{H}^n$ *and let G be its Gram matrix. Let $\tilde{K}$ be the field generated by the entries of G and let K be the field generated by the (irreducible) cycles in $2G$. Then, Γ is arithmetic if and only if*

- *$\tilde{K}$ is a totally real number field;*
- *for each Galois embedding $\sigma : \tilde{K} \longrightarrow \mathbb{R}$ with $\sigma|_K \neq id$, the matrix G^σ is positive semi-definite;*
- *the cycles in $2G$ are algebraic integers in K.*

In this case, the field of definition of Γ is K.

If the group Γ is a non-cocompact hyperbolic Coxeter group, then the criterion can be simplified as follows.

Proposition 4.5 ([22, Proposition 1.13])**.** *Let $\Gamma = \langle s_1, \ldots, s_N \rangle$ be a non-cocompact hyperbolic Coxeter group of finite covolume, G its Gram matrix and Σ its Coxeter graph. Then, Γ is arithmetic if and only if the two following conditions are satisfied.*

1. *The graph Σ contains only edges with labels in $\{\infty, 2, 3, 4, 6\}$ and dotted edges.*

2. $4 \cdot G_{i,j}^2 \in \mathbb{Z}$.

3. For every simple cycle $(s_{i_1}, \dots, s_{i_k}, s_{i_1})$ *in the graph* Σ*, the product* $\big(2G\big)_{(i_1,\dots,i_k)}$ *is a rational integer. Moreover, it is sufficient to test this condition in the graph obtained by collapsing every non-closed path.*

Remark. The condition $4 \cdot G_{i,j}^2 \in \mathbb{Z}$ (which must hold for a non-cocompact arithmetic Coxeter group) is a strong condition for the weights of the dotted lines.

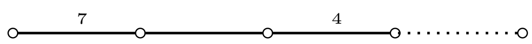


Figure 13: A 3-dimensional compact Coxeter prism

Example 4.6. We consider the cocompact group Γ given in Figure 13 whose fundamental polyhedron is a triangular Coxeter prism (or a simply truncated Coxeter orthoscheme). Its Gram matrix G is given by

$$G = \begin{pmatrix} 1 & -\cos\frac{\pi}{7} & 0 & 0 & 0 \\ -\cos\frac{\pi}{7} & 1 & -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & 1 & -\sqrt{\frac{\cos\frac{2\pi}{7}-1}{2\cos\frac{2\pi}{7}-1}} \\ 0 & 0 & 0 & -\sqrt{\frac{\cos\frac{2\pi}{7}-1}{2\cos\frac{2\pi}{7}-1}} & 1 \end{pmatrix}.$$

The field $\tilde{K}$ generated by the entries of G is $\mathbb{Q}\left(\cos\frac{\pi}{7}, \sqrt{2}, \sqrt{\alpha}\right)$, where

$$\alpha = (2\cos^2\left(\frac{\pi}{7}\right) - 1)/(4\cos^2\left(\frac{\pi}{7}\right) - 3) = (\cos\frac{2\pi}{7})/(2\cos\frac{2\pi}{7} - 1),$$

and the field K generated by the cycles is $\mathbb{Q}\left(\cos\frac{2\pi}{7}\right)$. Since the Galois conjugates of α are all positive, the first condition of Theorem 4.4 is satisfied. Since the minimal polynomial of $\cos\frac{\pi}{7}$ is $p(x) = 8x^3 - 4x^2 - 4x + 1$ (up to normalisation), the field extension $\tilde{K}/K$ has degree 12. Now, a Galois embedding $\sigma : \tilde{K} \longrightarrow \mathbb{R}$ which is not the identity on $\mathbb{Q}\left(\cos\frac{2\pi}{7}\right)$ has to send $\cos\frac{\pi}{7}$ to one of the two other roots of $p(x)$. Therefore, we have to consider the 8 possible embeddings. For each of them, numerical computations show that the matrix G^σ is positive semi-definite. Finally, since the cycles in $2G$ are algebraic integers, the group Γ is arithmetic.

§ 4.2. Algebraic background

In this section we present basic facts about the Brauer group and quaternion algebras which are needed to compute the invariants.

4.2.1. The Brauer group Let K be a field and let A be a finite dimensional central simple algebra over K (the center of A is K and A has no proper non-trivial two-sided ideal). By Wedderburn's theorem, there exists a unique (up to isomorphism) division algebra D over K and a unique integer n such that $A \cong Mat(n, D)$. This allows to define an equivalence relation on the set of isomorphism classes of central simple algebras over K: two algebras $A \cong Mat(n, D)$ and $A' \cong Mat(n', D')$ are said to be *Brauer equivalent* if and only if $D \cong D'$. The quotient set is endowed with the structure of an abelian group as follows:

$$[A] \cdot [B] = [A \otimes_K B].$$

We remark that the neutral element is the class of $Mat(m, K)$ and $[A]^{-1} = \left[A^{\text{op}}\right]$, where A^{op} denotes the opposite algebra of A. Note that we will often write $A \cdot B$ instead of $[A] \cdot [B]$.
The Brauer group is denoted by $\operatorname{Br} K$. All its elements are torsion and the 2-torsion is generated by quaternions algebras (see [41]).

4.2.2. Quaternion algebras Let K be a field of characteristic different of two. A *quaternion algebra* over K is a four-dimensional central simple algebra over K. Since the characteristic of K is different from two, there exist a K-basis $\{1, i, j, k\}$ of A and two non-zero elements a and b of K such that the multiplication in A is given by the following rules:

$$i^2 = a, \quad j^2 = b, \quad ij = -ji = k.$$

We then write $A = (a, b)_K$ or just (a, b) if there is no confusion about the base field. We sometimes call $(a, b)_K$ the *Hilbert symbol* of the quaternion algebra. This is kind of unfortunate because we also have the *Hilbert symbol* of a field K, which is the function $K^* \times K^* \longrightarrow \{-1, 1\}$ defined as follows.

$$(a, b) = \begin{cases} 1 & \text{if } ax^2 + by^2 - z^2 = 0 \text{ has a non-trivial solution in } K^3 \\ -1 & \text{otherwise.} \end{cases}$$

We will use this function later when speaking about the ramification of rational quaternion algebras.
For an element $q = x + yi + zj + tk$, with $x, y, z, t \in K$, the standard involution $\overline{q} = x - yi - zj - tk$ gives rise to the norm

$$N : A \longrightarrow K, \quad q \longmapsto N(q) = q \cdot \overline{q} = x^2 - ay^2 - bz^2 + abt^2.$$

Since an element $q \in A$ is invertible if and only if $N(q) \neq 0$, we have the following proposition.

Proposition 4.7 ([35, Chapter III, Theorem 2.7]). *Let $A = (a, b)_K$ be a quaternion algebra. Then, the following properties are equivalent:*

- *A is a division algebra;*
- *the norm $N : A \longrightarrow K$ has no non-trivial zero;*
- *the equation $aX^2 + bY^2 = 1$ has no solution in $K \times K$;*
- *the equation $aX^2 + bY^2 - Z^2 = 0$ has only the trivial solution.*

Moreover, if A is not a division algebra, then $A \cong Mat(2; K)$.

Therefore, deciding whether a given quaternion algebra is a division algebra or not reduces to a purely number theoretical question. We will come back to this question later.

Proposition 4.8. *For every $a, b, c \in K^*$, we have the following isomorphisms of quaternion algebras:*

$$\begin{aligned}
&(a, b) \cong (b, a), \quad (a, c^2 b) \cong (a, b), \quad (a, a) \cong (a, -1) \\
&(a, 1) \cong (a, -a) \cong (a, 1 - a) \cong (1, 1) \cong 1 \\
&(a, b) \cdot (a, c) \cong (a, bc) \cdot Mat(2; K), \quad (a, b)^2 \cong Mat(4; K).
\end{aligned}$$

We note that the last two relations can be rewritten in the Brauer group as follows

$$(a, bc) = (a, b) \cdot (a, c), \quad (a, b)^2 = 1.$$

Proposition 4.9 ([56, chapitre I, Théorème 2.9; chapitre III, Section 3]). *If K is a number field, and if B_1 and B_2 are quaternion algebras over K, there exists a quaternion algebra B such that $B_1 \cdot B_2 = B$ in $\operatorname{Br} K$.*

4.2.3. Isomorphism classes of quaternion algebras We will see below that the question of the commensurability of two arithmetic Coxeter subgroups of $\operatorname{Isom} \mathbb{H}^n$ reduces almost to deciding whether two quaternions algebras are isomorphic. Hence, in this section, we investigate the isomorphism classes of quaternion algebras.

First, it is worth to mention that the isomorphism classes of quaternion algebras are not determined by Hilbert symbols (for example, we have $(5, 3)_{\mathbb{Q}} \cong (-10, 33)_{\mathbb{Q}}$). However, we will see that there is an efficient way to produce a set which completely describe the quaternion algebra: the ramification set.

Let K be a number field. Recall that a *place* of K is an equivalence class of absolute values[1]: two non-trivial absolute values $|\cdot|_1, |\cdot|_2 : K \longrightarrow \mathbb{R}$ are *equivalent* if there exists

[1]Some authors use the word *(multiplicative) valuation* for what we call *absolute value*. This is why a place is often denoted by v.

some number $e \in \mathbb{R}$ such that $|x|_1 = |x|_2^e$ for all $x \in K$. We can easily create two kind of places:

Infinite places Any Galois embedding $\sigma : K \longrightarrow \mathbb{R}$ yields a place by composition with the usual absolute value. Similarly, any complex Galois embedding $\sigma : K \longrightarrow \mathbb{C}$ gives rise to a place by composition with the modulus. These places are called *infinite places.*
Note that in our setting, since the number field K is supposed to be totally real, we only get real embeddings.

Finite places Let $\mathcal{P}$ be a prime ideal of $\mathcal{O}_K$. This defines a valuation on $\mathcal{O}_K$ as follows:

$$\eta_{\mathcal{P}} \, : \, \mathcal{O}_K \longrightarrow \mathbb{Z} \cup \{\infty\}, \quad \eta_{\mathcal{P}}(x) = \sup\{r \in \mathbb{N} : x \in \mathcal{P}^r\}.$$

This valuation can be extended to K by setting $\eta_{\mathcal{P}}(x/y) = \eta_{\mathcal{P}}(x) - \eta_{\mathcal{P}}(y)$. Now, we pick any $0 < \lambda < 1$ and define the associated absolute value

$$|\cdot|_{\mathcal{P}} : K \longrightarrow \mathbb{R}, \quad |x|_{\mathcal{P}} = \lambda^{\eta_{\mathcal{P}}(x)}.$$

Note that the place associated to this absolute value is independent of the choice of λ. The places defined in this way are called *finite places.*

Using Ostrowski's theorem and theorems about extensions of absolute values, one gets the following standard result.

Theorem 4.10. *Let K be a number field. The two constructions explained above give all the places on K.*

We will denote by $\Omega(K)$ (respectively $\Omega_\infty(K)$ and $\Omega_f(K)$) the set of all places (respectively infinite places and finite places) of K. If $v \in \Omega(K)$ is a place, we denote by K_v the completion of K with respect to v. For a quaternion algebra B over K, we write B_v for $B \otimes_K K_v$, which is a quaternion algebra over K_v. When the place v comes from a prime ideal $\mathcal{P}$ of $\mathcal{O}_K$, we will sometimes write $K_{\mathcal{P}}$ instead of K_v and $B_{\mathcal{P}}$ instead of B_v.
The fact that B_v is either a division algebra or a matrix algebra (see Proposition 4.7) motivates the following definition.

Definition 4.11. Let B be a quaternion algebra defined over a number field K. The *ramification set* of B, denoted by $\operatorname{Ram} B$, is defined as follows:

$$\operatorname{Ram} B = \big\{v \in \Omega(K) \,|\, B_v := B \otimes_K K_v \text{ is a division algebra}\big\}.$$

We will also write

$$\operatorname{Ram}_f B := \operatorname{Ram} B \cap \Omega_f(K), \quad \operatorname{Ram}_\infty B = \operatorname{Ram} B \cap \Omega_\infty(K).$$

Theorem 4.12 ([56, Chapter III, Theorem 3.1]). *Let B be a quaternion algebra defined over a number field K. The ramification set* $\operatorname{Ram} B$ *of B is a finite set of even cardinality. Conversely, if $R \subset \Omega(K)$ is a finite set of even cardinality, there exists, up to isomorphism, a unique quaternion algebra B' over K such that* $\operatorname{Ram} B' = R$.

Remark. Using Proposition 4.7 it is easy to compute the infinite ramification $\operatorname{Ram}_\infty B$ of a quaternion algebra $B = (a,b)_K$. Indeed, if $\sigma : K \longrightarrow \mathbb{R}$ is a Galois embedding and if v is the corresponding absolute value, then $B_v \cong \big(\sigma(a), \sigma(b)\big)$. Thus, $v \in \operatorname{Ram}_\infty (a,b)_K$ if and only if $\sigma(a) > 0$ and $\sigma(b) > 0$.

Remark. When $K = \mathbb{Q}$, the previous theorem comes from classical results such as the Hasse-Minkowski principle (since two quaternion algebras are isomorphic if the quadratic spaces induced by their norms are isomorphic) and Hilbert's reciprocity law.

Finally, let us mention a result which helps for computations in the case $K = \mathbb{Q}$ (see also Proposition 4.15).

Proposition 4.13 ([56, p. 78]). *Let B_1 and B_2 be two quaternion algebras over a number field K and let B be such that $B_1 \cdot B_2 = B \in \operatorname{Br} K$ (see Proposition 4.9). Then, we have*

$$\operatorname{Ram} B = \big(\operatorname{Ram} B_1 \cup \operatorname{Ram} B_2\big) \setminus \big(\operatorname{Ram} B_1 \cap \operatorname{Ram} B_2\big).$$

When dealing with arithmetic groups of odd dimensions, we will need to compute the ramification of quaternion algebras over a quadratic extension of a number field.

Proposition 4.14. *Let K be a number field and let $L = K(\sqrt{\delta})$ be a quadratic extension of K. Let B be a quaternion algebra over K and define $A := B \otimes_K L$. Then, the ramification sets at finite places of A and B are related as follows:*

$$\operatorname{Ram}_f A = \{\mathfrak{P}_1, \mathfrak{P}_1', \ldots, \mathfrak{P}_r, \mathfrak{P}_r'\},$$

where each pair $\mathfrak{P}_i, \mathfrak{P}_i'$ is a pair of prime ideals of $\mathcal{O}_L$ which lie above a prime ideal $\mathcal{P}_i$ of $\mathcal{O}_K$ such that B is ramified at $\mathcal{P}_i$, and $\mathcal{P}_i$ splits completely.

Proof. Let $\mathfrak{P}$ be a prime ideal of $\mathcal{O}_L$ and let $\mathcal{P} := \mathfrak{P} \cap \mathcal{O}_K$. We also consider the completions $L_\mathfrak{P}$ (respectively $K_\mathcal{P}$) of L (respectively K) with respect to the valuation defined by $\mathfrak{P}$ (respectively $\mathcal{P}$). We first note that $A_\mathfrak{P} \cong B_\mathcal{P} \otimes_{K_\mathcal{P}} L_\mathfrak{P}$. Indeed, we have

$$A_\mathfrak{P} \cong A \otimes_L L_\mathfrak{P} = (B \otimes_K L) \otimes_L L_\mathfrak{P} \cong B \otimes_K L_\mathfrak{P} \cong B_\mathcal{P} \otimes_{K_\mathcal{P}} L_\mathfrak{P}.$$

By classical results, we then have three possibilities for the ideal $\mathcal{P}\mathcal{O}_L$:

1. **Inert case** The ideal $\mathcal{P}\mathcal{O}_L$ is prime, meaning that $\mathcal{P}\mathcal{O}_L = \mathfrak{P}$.
2. **Ramified case** We have $\mathcal{P}\mathcal{O}_L = \mathfrak{P}^2$.
3. **Split case** There exists another prime ideal $\mathfrak{P}'$ above $\mathcal{P}$ such that $\mathcal{P}\mathcal{O}_L = \mathfrak{P}\mathfrak{P}'$.

In the first two cases, $[L_{\mathfrak{P}} : K_{\mathcal{P}}] = 2$ (see, for example, [9, Chapter I, 5, Proposition 3]) which implies that $A_{\mathfrak{P}}$ is a matrix algebra (see [56, chapitre II, Théorème 1.3]). In the last case, we have $K_{\mathcal{P}} \cong L_{\mathfrak{P}}$ (again by [9]) and thus $A_{\mathfrak{P}} \cong B_{\mathcal{P}}$. Therefore, A is ramified at $\mathfrak{P}$ if and only if B is ramified at $\mathcal{P}$ if and only if A is ramified at $\mathfrak{P}'$, as required. □

Computing the ramification set when $K = \mathbb{Q}$. In this part we consider a quaternion algebra $B = (a, b)$ over $\mathbb{Q}$, and we explain how to compute its ramification. In this setting, the finite places are the p-adic valuations, and there is exactly one infinite place, denoted by $|\cdot|_\infty$, corresponding to the usual absolute value. By virtue of Proposition 4.7, it is clear that B is ramified at ∞ if and only if $a < 0$ and $b < 0$. Moreover, using Proposition 4.8 and Proposition 4.13, we see that we only have to compute the ramification set for quaternion algebras which are of one of the following forms:

$$(-1, q), \quad (p, q), \quad (-p, q), \quad \forall p, q \in \mathbb{P},$$

where $\mathbb{P}$ denotes the set of prime numbers.

Proposition 4.15. *We have* $\mathrm{Ram}(-1, 2) = \emptyset$ *and* $\mathrm{Ram}(-1, -2) = \{2, \infty\}$.

If q is a prime number different from two, then we have the following ramification sets:

	$q \equiv 1$ *mod* 8	$q \equiv 3$ *mod* 8	$q \equiv 5$ *mod* 8	$q \equiv 7$ *mod* 8
$(-1, q)$	$\emptyset$	$\{2, q\}$	$\emptyset$	$\{2, q\}$
$(-1, -q)$	$\{2, \infty\}$	$\{q, \infty\}$	$\{2, \infty\}$	$\{q, \infty\}$
$(2, -q)$	$\emptyset$	$\{2, q\}$	$\{2, q\}$	$\emptyset$
$(-2, q)$	$\emptyset$	$\emptyset$	$\{2, q\}$	$\{2, q\}$

Finally, let $q_1, q_2 \in \mathbb{P} \setminus \{2\}$ be two distinct prime numbers. The ramification set of the quaternion algebra $(-q_1, q_2)$ is as follows:

	$q_2 \equiv 1$ *(mod 4)*	$q_2 \equiv 3$ *(mod 4)*
$q_1 \equiv 1$ *(mod 4)*	$\{q_1, q_2\}$ *if* $\left(\frac{q_1}{q_2}\right) = -1$ $\emptyset$ *otherwise*	$\{2, q_1\}$ *if* $\left(\frac{q_1}{q_2}\right) = -1$ $\{2, q_2\}$ *otherwise*
$q_1 \equiv 3$ *(mod 4)*	$\{q_1, q_2\}$ *if* $\left(\frac{q_1}{q_2}\right) = -1$ $\emptyset$ *otherwise*	$\{q_1, q_2\}$ *if* $\left(\frac{q_1}{q_2}\right) = 1$ $\emptyset$ *otherwise*

where $\left(\frac{a}{b}\right)$ *denotes the Legendre symbol of a and b.*

Proof. For $x, y \in \mathbb{Q}$ we have the Hilbert symbol $(x,y)_p \in \{-1,1\}$ (see, for example, [9]) and for a diagonal quadratic form $f = \langle a_1, \dots, a_n\rangle$, we define

$$c(f)_p = \prod_{i<j}(a_i, a_j)_p, \quad c(B)_p = c\big(\langle 1, -a, -b, ab\rangle\big)_p.$$

By [9, Lemme 2.6, page 59]), we get

$$p \in \mathrm{Ram}(B) \Leftrightarrow c(B)_p = \begin{cases} 1 & p = 2, \infty \\ -1 & p \text{ odd.} \end{cases}$$

Therefore, to find the ramification set it is sufficient to compute the Hilbert symbols $c(B)_p$. To compute these symbols, we can use [50, Part I, Chapter III, Theorem 1]. We note that if $p \in \operatorname{Ram} B$, then we must have $p = 2$ or $p \mid a$ or $p \mid b$.
It follows that the Hilbert symbols satisfy

$$\begin{aligned} c\big((-1,2)\big)_2 &= (-1,-2)_2 = -1, & c\big((-1,-2)\big)_2 &= (2,-1)_2 = 1, \\ c\big((-1,q)\big)_2 &= (-q,-q)_2 = (-1)^{\varepsilon(-q)}, & c\big((-1,q)\big)_q &= (-q,-q)_q = (-1)^{\varepsilon(q)}, \\ c\big((-1,-q)\big)_2 &= (q,q)_2 = (-1)^{\varepsilon(q)}, & c\big((-1,-q)\big)_q &= (q,q)_q = (-1)^{\varepsilon(q)}, \\ c\big((2,-q)\big)_q &= (-1)^{\omega(q)}, & c\big((-2,q)\big)_q &= (-1)^{\varepsilon(q)+\omega(q)}, \end{aligned}$$

$$\begin{aligned} c\big((2,-q)\big)_2 &= (q,-1)_2 \cdot (-2,-q)_2 = (-1)^{1+\omega(-q)}, \\ c\big((-2,q)\big)_2 &= (2,-1)_2 \cdot (-q,-2)_2 = (-1)^{\varepsilon(-q)+\omega(-q)}, \end{aligned}$$

where

$$\varepsilon(n) = \begin{cases} 0 & \text{if } n \equiv 1 \pmod 4 \\ 1 & \text{if } n \equiv 3 \pmod 4 \end{cases}, \qquad \omega(n) = \begin{cases} 0 & \text{if } n \equiv \pm 1 \pmod 8 \\ 1 & \text{if } n \equiv \pm 3 \pmod 8. \end{cases}$$

Finally, if $q_1, q_2 \in \mathbb{P} \setminus \{2\}$ are two distinct prime numbers, we have

$$\begin{aligned} c\big((-q_1,q_2)\big)_2 &= (q_1,-1)_2 \cdot (-q_2,-q_1)_2 = (-1)^{\varepsilon(q_1)+\varepsilon(-q_1)\cdot\varepsilon(-q_2)} \\ c\big((-q_1,q_2)\big)_{q_1} &= (q_1,-1)_{q_1} \cdot (-q_2,-q_1)_{q_1} = \left(\frac{q_2}{q_1}\right) \\ c\big((-q_1,q_2)\big)_{q_2} &= (q_1,-1)_{q_2} \cdot (-q_2,-q_1)_{q_2} = (-1)^{\varepsilon(q_2)} \cdot \left(\frac{q_1}{q_2}\right), \end{aligned}$$

where $\left(\frac{a}{b}\right)$ denotes the Legendre symbol of a and b. □

4.2.4. Clifford algebras In this section, we present basic facts about Clifford algebras. These algebras are used to prove Maclachlan's theorem (cf. Theorem 4.20) but are not used for the computations of the invariants. The reader who only wants to compute the invariants can skip this part.

Let (V, f) be a quadratic space defined over some field K. A *Clifford algebra* associated to (V, f) is a unitary associative algebra over K denoted by $\mathrm{Cl}(V, f)$, together with a K-linear map $i : V \longrightarrow \mathrm{Cl}(V, f)$ which satisfy the following:

- For every $v \in V$, $i(v)^2 = f(v) \cdot 1_{\mathrm{Cl}(V,f)}$.
- The following *universal property* is satisfied: if A is another unitary associative K-algebra provided with a K-linear function $i_A : V \longrightarrow A$ such that $i_A(v)^2 = f(v) \cdot 1_A$, then there exists a unique morphism of K algebras $\psi : \mathrm{Cl}(V, f) \longrightarrow A$ such that the following diagram commutes.

$$\begin{tikzcd} V \arrow[r, "i_A"] \arrow[d, "i"'] & A \\ \mathrm{Cl}(V,f) \arrow[ur, dashed, "\psi"'] & \end{tikzcd}$$

Because of the universal property, if the Clifford algebra $\mathrm{Cl}(V, f)$ exists, then it is unique up to isomorphism. In fact, the algebra can be constructed explicitly as follows. We start with the tensor algebra $T(V) = \bigoplus_{i\in\mathbb{N}} V^{\otimes n}$, where $V^{\otimes n} = V \otimes \ldots \otimes V$, n times, and $V^{\otimes 0} = K$. Then we consider the ideal I in $T(V)$ generated by the elements $v^2 - f(v)$. Finally, we see that the quotient $T(V)/_I$ satisfies the universal property as required. Thus, we have $\mathrm{Cl}(V, f) = T(V)/_I$. When working in $\mathrm{Cl}(V, f)$ we will abbreviate and write the product $a \otimes b$ in the form $a \cdot b$. It is easy to see that if $v_1, \ldots, v_n$ is a K-basis of V, then

$$\left\{ v_{i_1} \cdot \ldots \cdot v_{i_k} \,|\, 1 \leq i_1 < i_2 < \ldots < i_k \leq n \right\}$$

is a K-basis of $\mathrm{Cl}(V, f)$. In particular, $\dim_k \mathrm{Cl}(V, f) = 2^{\dim_K V}$.

The natural $\mathbb{N}$-grading on the tensor algebra $T(V)$ induces a $\left(\mathbb{Z}/_{2\mathbb{Z}}\right)$-grading on $\mathrm{Cl}(V, f)$ which is compatible with the algebra structure. In other words, the vector space

$$\mathrm{Cl}_0(V, f) := \mathrm{span}_K \left\{ v_{i_1} \cdot \ldots \cdot v_{i_k} \,|\, 1 \leq i_1 < i_2 < \ldots < i_k \leq n, k \text{ even} \right\}$$

is a sub-algebra of $\mathrm{Cl}(V, f)$ called the *even* part of $\mathrm{Cl}(V, f)$. More canonically, $\mathrm{Cl}_0(V, f)$ is the image of $\bigoplus_{i\in\mathbb{N}} V^{\otimes 2i}$ by the quotient map.

Example 4.16. Let $V = \mathbb{R}^n$ be endowed with the quadratic form $f(v) = -\|v\|^2$, where $\|\cdot\|$ is the euclidean norm. For $n = 1$ we have $\mathrm{Cl}(V, f) = \mathbb{C}$ and for $n = 2$ we get $\mathrm{Cl}(V, f) = (-1, -1)_{\mathbb{R}}$, the quaternions of Hamilton. In the latter case, we have $\mathrm{Cl}_0(V, f) \cong \mathbb{C}$.

Remark. In some cases, it is useful to construct the algebra by requiring $v^2 = -f(v)$ instead of $v^2 = f(v)$. The choice depends on the authors.

§ 4.3. Classification

Let $\Gamma_1, \Gamma_2 \subset \operatorname{Isom} \mathbb{H}^n$ be two arithmetic Coxeter groups defined over the number fields K_1 and K_2, respectively, and let f_1 and f_2 be their underlying quadratic forms. A result of Gromov and Piatetski-Shapiro (see [20, Section 2.6]) implies the following result:

$$\Gamma_1 \sim \Gamma_2 \Leftrightarrow K_1 = K_2 \text{ and } f_1 \sim f_2,$$

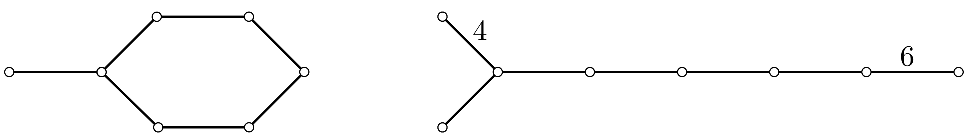
where $\Gamma_1 \sim \Gamma_2$ means that Γ_1 and Γ_2 are commensurable and where $f_1 \sim f_2$ means that the two quadratic forms are *similar* (meaning that there exists $c \in K_1 = K_2$ such that $c \cdot f_1 \cong f_2$). Therefore, deciding whether Γ_1 is commensurable to Γ_2 can be done in two steps. First, we have to find the defining fields and the underlying quadratic forms (see [57, Section II, Theorem 2]). Secondly, we have to detect the similarity of the forms. The main references for this section are [36] and [57].

Remark. An immediate consequence of the result of Gromov and Piatetski-Shapiro is the following: when n is odd, a necessary condition for Γ_1 and Γ_2 to be commensurable is that $K_1 = K_2$ and that the quotient $\det f_1 / \det f_2$ is a square in K_1.

We will present the two points separately and illustrate them with an example: the goal is to decide whether the two (non-cocompact, cofinite) subgroups $\Gamma_1^6, \Gamma_2^6 \subset \operatorname{Isom} \mathbb{H}^6$ presented in Figure 14 are commensurable or not. We first remark that these two groups are arithmetic by virtue of Proposition 4.5:

- The Gram matrix of the first one has entries in $\mathbb{Z}\left[\frac{1}{2}\right]$.
- The second graph contains only weights $3, 4$ and 6 and no cycles.

The group Γ_1^6 is the simplex group $\overline{P}_6$ and has covolume $\frac{13\pi^3}{1{,}360{,}800}$. The group Γ_2^6 is one of the pyramids given by Tumarkin in [54] and has covolume $\frac{13\pi^3}{604{,}800}$. Since both groups are non-cocompact, the fields of definition equal $\mathbb{Q}$.



Γ_1^6 Γ_2^6

Figure 14: Two arithmetic Coxeter subgroups of $\operatorname{Isom} \mathbb{H}^6$

4.3.1. Finding the defining field and the underlying quadratic form Let $\Gamma \subset \operatorname{Isom} \mathbb{H}^n$ be an arithmetic Coxeter group of rank N. We want to find the defining field K and the underlying quadratic form f of Γ. We consider the associated fundamental polyhedron $P(\Gamma)$ and the outward-pointing normal vectors $e_i \in \mathbb{R}^{n+1}$ of the bounding hyperplanes H_i of $P(\Gamma)$, which means that $P(\Gamma) = \bigcap_{i=1}^N H_{e_i}^-$ (see (2.1)). Finally, we denote by $G(\Gamma) = (a_{i,j}) \in Mat(\mathbb{R}, N)$ the Gram matrix of Γ, meaning that $a_{i,j} = \langle e_i, e_j \rangle$.
Using Vinberg's arithmeticity Criterion (see Theorem 4.4), we can determine the defining field K of Γ. Now, consider the K-subvector space V spanned by all the vectors

$$v_{i_1,\ldots,i_k} = a_{1,i_1} \cdot a_{i_1,i_2} \cdot \ldots \cdot a_{i_{k-1},i_k} e_{i_k}, \quad \{i_1,\ldots,i_k\} \subset \{1,\ldots,N\}. \tag{4.1}$$

The restriction of the Lorentzian form f_0 to V is an admissible quadratic space of dimension $n+1$ over K. Moreover, Γ is commensurable to $O(f, L)$, where L is the lattice spanned over K by a basis $v_1, \ldots, v_{n+1}$ of V and f is the quadratic form defined by the matrix whose entries are the Lorentzian products $\langle v_i, v_j \rangle$ of signature $(n, 1)$.

Remark. The task of finding the outward-pointing normal vectors of the polyhedron $P(\Gamma)$ is difficult: one has to find the vectors $e_1, \ldots, e_N$ of $\mathbb{R}^{n+1}$ such that $\langle e_i, e_j \rangle = a_{i,j}$. This is basically equivalent to solve a system of $\frac{r(r+1)}{2}$ quadratic equations with $r(n+1)$ unknowns. Assuming that $e_1 = (0, \ldots, 0, 1)$ and that the first vectors contain a lot of zeroes, we can guess the remaining vectors using software such as Mathematica or Maple. Another possibility is to compute sufficiently good approximations of the vectors e_i using numerical methods and then use the LLL algorithm to find the exact components. This approach is described in [5].

The normal vectors of the polyhedra $P(\Gamma_1^6)$ and $P(\Gamma_2^6)$ associated to the groups Γ_1^6 and Γ_2^6 (see Figure 14) can be easily determined by means of Remark 4.3.1 and are given in Figure 15. We know that the normal vectors of the simplex Γ_1^6 are linearly independent and they all arise as vectors according to (4.1), up to a rescaling by a rational number. Therefore, we can take $e_1, \ldots, e_7$ as a $\mathbb{Q}$-basis of the space V. Thus, in this case, the underlying quadratic form is the form induced by the Gram matrix of Γ_1^6. In the case of the group Γ_2^6, we find the following vectors:

$$e_1, \ldots, e_8, \sqrt{2} \cdot e_2, \ldots, \sqrt{2} \cdot e_7, \sqrt{6} \cdot e_8.$$

We see that the vectors $v_1 := e_1, v_2 := \sqrt{2} \cdot e_2, \ldots, v_7 := \sqrt{2} \cdot e_7$ are linearly independent over $\mathbb{Q}$. Hence, the matrix of the quadratic form is given by

$$(\langle v_i, v_j\rangle) = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

Finally, we determine the diagonal forms of these two quadratic forms (this can be done using a software such as Sage):

$$f_1^6 = \langle 1, 3, 6, 10, 15, 21, -21\rangle,$$
$$f_2^6 = \langle 1, 1, 2, 2, 2, 6, -1\rangle.$$

$$\begin{aligned}
e_1 &= (1,0,0,0,0,0,0)\\
e_2 &= \Big(-\frac{1}{2}, \frac{1}{6}(6-\sqrt{6}), \frac{1}{8}(-5+3\sqrt{6}),\\
&\quad \frac{1}{24}(-15\sqrt{2}+22\sqrt{3}), \frac{1}{24}(-9+11\sqrt{6}),\\
&\quad \frac{1}{2}, \frac{1}{12}(15-11\sqrt{6})\Big)\\
e_3 &= (0,-1,2,2\sqrt{2},2,0,-4)\\
e_4 &= \Big(0,0,-\frac{11}{8},-\frac{3}{4\sqrt{2}},-\frac{5}{8},0,\frac{5}{4}\Big)\\
e_5 &= (0,0,0,-\sqrt{2},0,0,1)\\
e_6 &= \Big(0,0,0,0,-1,\frac{1}{2},\frac{1}{2}\Big)\\
e_7 &= (0,0,0,0,0,-1,0)
\end{aligned}$$

(a) Normal vectors of $P(\Gamma_1^6)$

$$\begin{aligned}
e_1 &= (1,0,0,0,0,0,0)\\
e_2 &= \Big(0,-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0,0,0,0\Big)\\
e_3 &= \Big(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0,0,0,0,0\Big)\\
e_4 &= \Big(0,-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},-\frac{\sqrt{3}}{2},-\frac{1}{2},0,-1\Big)\\
e_5 &= \Big(0,0,0,\frac{\sqrt{3}}{2},-\frac{1}{2},0,0\Big)\\
e_6 &= \Big(0,0,0,0,1,\frac{1}{2},\frac{1}{2}\Big)\\
e_7 &= (0,0,0,0,0,-1,0)\\
e_8 &= \Big(0,0,0,-\frac{1}{2},-\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2},-\frac{\sqrt{3}}{2}\Big)
\end{aligned}$$

(b) Normal vectors of $P(\Gamma_2^6)$

Figure 15: Outward-pointing normal vectors for the examples $P(\Gamma_1^6)$ and $P(\Gamma_2^6)$ in dimension 6

4.3.2. Constructing the invariants Before we give Maclachlan's result (cf. Theorem 4.20), we briefly explain where the invariants come from. The reader only interested in computations can skip this section. Since the commensurability classes of arithmetic groups are determined by similarity classes of quadratic forms, we define, for a totally real number field K, the following map:

$$\begin{aligned}\Theta : \mathrm{SimQF}(K, n+1) &\longrightarrow \mathrm{IsomAlg}(K, 2^n)\\ \big[(V, f)\big] &\longmapsto \mathrm{Cl}_0(V, f),\end{aligned}$$

where $\mathrm{SimQF}(K, n+1)$ is the set of similarity classes of quadratic spaces over K of dimension $n+1$, and $\mathrm{IsomAlg}(K, 2^n)$ is the set of isomorphism classes of algebras over K of dimension 2^n. If we restrict the map Θ to the quadratic forms f of signature $(n, 1)$ whose conjugates f^σ under non-trivial Galois embeddings $\sigma : K \longrightarrow \mathbb{R}$ are positive definite, then we get an injective map (see [36, Theorems 6.1 and 6.2]). Hence, we have to be able to decide whether the even parts of the Clifford algebras (see Section 4.2.4) associated to the underlying quadratic forms are isomorphic as algebras. Luckily for us, in this setting, there is an efficient way to decide that using the Witt and the Hasse invariants.

Definition 4.17. Let (V, f) be a quadratic space over a field K. The *Witt invariant* of (V, f), denoted by $c(V, f)$, or only $c(f)$, is the element of $\mathrm{Br}\, K$ defined as follows:

$$c(V, f) := \begin{cases} \big[\,\mathrm{Cl}_0(V, f)\big] & \text{if } \dim V \text{ is odd,}\\ \big[\,\mathrm{Cl}(V, f)\big] & \text{if } \dim V \text{ is even.}\end{cases}$$

Definition 4.18. The *Hasse invariant* of a diagonal quadratic form $\langle a_1, \dots, a_n\rangle$ over K is the element of the Brauer group defined as follows:

$$\bigotimes_{i<j} (a_i, a_j)_K\,.$$

It can be shown (see [35, Chapter V, Proposition 3.18]) that two isomorphic quadratic forms have the same Hasse invariant. Therefore, we can define the Hasse invariant of quadratic space (V, f) to be the Hasse invariant of any diagonalisation of f. It will be denoted by $s(V, f)$ or just $s(f)$.

Proposition 4.19 ([35, Proposition 3.20])**.** *For a given quadratic space (V, f), the Hasse invariant and the Witt invariant are related as follows:*

$$c(f) = \begin{cases} s(f) & \dim V \equiv 1, 2 \ (mod\ 8)\\ s(f)\cdot(-1, -\det f) & \dim V \equiv 3, 4 \ (mod\ 8)\\ s(f)\cdot(-1, -1) & \dim V \equiv 5, 6 \ (mod\ 8)\\ s(f)\cdot(-1, \det f) & \dim V \equiv 7, 8 \ (mod\ 8)\end{cases}$$

Remark. If K is a number field, then Proposition 4.9 implies that we can choose a representative of $s(f)$ and of $c(f)$, respectively, which is a quaternion algebra. However, in some cases, finding such a quaternion algebra may be difficult.

The even case We suppose that n is even. Let Γ be an arithmetic group defined over a totally real number field K with underlying quadratic form f. Consider the Witt invariant $c(f)$ and a quaternion algebra B which represents $c(f)$. Then, we have

$$\Theta(f) = \mathrm{Cl}_0(f) \cong Mat\big(2^{(n-2)/2}; B\big),$$

where the last isomorphism comes from [36, Theorem 7.1]. Hence, $\Theta(f)$ is a central simple algebra over K whose class in the Brauer group is B (and thus it is completely determined by B). In particular, we get the following theorem due to C. Maclachlan.

Theorem 4.20 ([36, Theorem 7.2])**.** *When the dimension n is even, the commensurability class of an arithmetic group of the simplest type is completely determined by the isomorphism class of a quaternion algebra which represents the Witt invariant of its underlying quadratic form.*

The odd case When n is odd, the situation is more complicated. We consider, as before, Γ, f, $c(f)$ and B. We also consider the signed determinant $\delta = (-1)^{n(n+1)/2} \cdot \det f$ of f. Since f is admissible (i.e. all its conjugates by non-trivial Galois embeddings are positive definite), $\det f = \delta \cdot (-1)^{n(n+1)/2}$ is negative but all its non-trivial conjugates are positive. We distinguish now two cases:

- δ is a square in K^* :
  Then, δ cannot have any non-trivial conjugates which implies $K = \mathbb{Q}$. Moreover, we must have $n \equiv 1 \pmod 4$. We finally have $\mathrm{Cl}_0(f) \cong Mat\big(2^{(n-3)/2}; B\big) \times Mat\big(2^{(n-3)/2}; B\big)$.
- δ is not a square in K^* :
  In this case, we have $\mathrm{Cl}_0(f) \cong Mat\big(2^{(n-3)/2}; B \otimes_K K(\sqrt{\delta})\big)$.

About the ramification at infinite places As we saw above, commensurability classes are in fact determined by the ramification set of certain quaternion algebras (either the Witt invariant or the Witt invariant over a quadratic extension of the base field). However, the ramification at infinite places is completely independant of the Witt invariant. Indeed, using admissibility of the quadratic form, Proposition 4.19 and the fact that $\sigma(\det f) > 0$ for every $\sigma : K \longrightarrow \mathbb{R}$ with $\sigma \neq \mathrm{id}$, we find

$$\mathrm{Ram}_\infty\, c(f) = \begin{cases} \emptyset & \dim V \equiv 1, 2 \pmod 8 \\ \Omega_\infty(K) \setminus \{\mathrm{id}\} & \dim V \equiv 3, 4 \pmod 8 \\ \Omega_\infty(K) & \dim V \equiv 5, 6 \pmod 8 \\ \{\mathrm{id}\} & \dim V \equiv 7, 8 \pmod 8 \end{cases}$$

4.3.3. Computing the invariants In this section, we exlain how to compute the invariants which completely determine the commensurability class of an arithmetic hyperbolic Coxeter group.

Let $\Gamma \subset \operatorname{Isom} \mathbb{H}^n$ be an arithmetic Coxeter group defined over the totally real number field K. Let f be the underlying quadratic form (see Section 4.3.1) and $\delta = (-1)^{n(n+1)/2} \cdot \det f$ be the signed determinant of f. From a diagonalised form of f compute the Hasse invariant $s(f)$ of f (see Definition 4.18) and the Witt invariant $c(f)$ (see Proposition 4.19). Finally, choose a representative B of $c(f)$ which is a quaternion algebra (see Remarks 4.21 below). Then, a complete invariant for the commensurability class of Γ is given in Figure 16.

n	complete invariant
even	$\{K, \operatorname{Ram}_f B\}$
odd	δ not a square in K^*: $\{K, \delta, \operatorname{Ram}_f \left(B \otimes_K K(\sqrt{\delta})\right)\}$
	δ is a square in K^*: $\{\mathbb{Q}, \operatorname{Ram}_f \left(B\right)\}$

Figure 16: Complete invariant for an arithmetic group $\Gamma \subset \operatorname{Isom} \mathbb{H}^n$

Remarks 4.21.

- Sometimes, it can be tricky to find a representative of $c(f)$ which is a quaternion algebra. However, this step is not really important as we are only interested in the ramification of $c(f)$ (see Proposition 4.13).
- When n is odd and when δ is not a square in K^*, one can use Proposition 4.14 to compute the ramification of $B \otimes_K K(\sqrt{\delta})$ (as a quaternion algebra over $K(\sqrt{\delta})$).

4.3.4. Finishing the examples in dimension 6

We come back to our two examples in dimension 6 defined over the $\mathbb{Q}$ and with quadratic forms (see Figure 14)

$$f_1^6 = \langle 1, 3, 6, 10, 15, 21, -21 \rangle,$$
$$f_2^6 = \langle 1, 1, 2, 2, 2, 6, -1 \rangle.$$

We first compute the Hasse invariant of the two quadratic forms. Recall that for a quadratic form $f = \langle a_1, \ldots, a_m \rangle$, we have $s(f) = \bigotimes_{i<j} (a_i, a_j) \in \operatorname{Br} K$. Using the

properties of Proposition 4.8, we compute:

$$\begin{aligned} s(f_1^6) = (3,6)\cdot(3,10)\cdot(3,15)\cdot&\overbrace{(3,21)\cdot(3,-21)}^{(3,-1)} \\ &\cdot(6,10)\cdot(6,15)\cdot(6,-1) \\ &\cdot(10,15)\cdot(10,-1) \\ &\cdot(15,-1) \\ &=(5,-1). \end{aligned}$$

Similarly, we find $s(f_2^6) = 1$. By Proposition 4.19, we get $c(f_1^6) = (5,-1)\cdot(-1,-3) = (-1,-15)$ and $c(f_2^6) = 1\cdot(-1,-3)$. We find with Proposition 4.15 the followings ramifications sets:

$$\mathrm{Ram}(-1,-3) = \{3,\infty\}, \quad \mathrm{Ram}(-1,5) = \emptyset.$$

Finally, Proposition 4.13 yields $\mathrm{Ram}\, c(f_1^6) = \{3,\infty\} = \mathrm{Ram}\, c(f_2^6)$, which implies that the groups Γ_1^6 and Γ_2^6 are commensurable.

4.3.5. Worked example (compact case) We study the commensurability of three particular cocompact arithmetic subgroups Γ_i^4, $i = 1,2,3$, of $\mathrm{Isom}\,\mathbb{H}^4$. These groups correspond to maximal subgroups generated by reflections in the groups of units of three (reflexive) quadratic forms with base field $K = \mathbb{Q}[\sqrt{5}]$. They were found using Vinberg's algorithm (see [58]). For simplicity, we write $\Theta = \frac{1+\sqrt{5}}{2}$ for the generator of the ring of integers of $\mathcal{O}_K = \mathbb{Z}[\Theta]$. The first group Γ_1^4 is the simplex group $[5,3,3,4]$ and is associated to the quadratic form $f_1 = \langle -\Theta, 1, 1, 1, 1\rangle$ (see [8]). The two groups Γ_2^4, Γ_3^4 are new and associated to the quadratic forms $f_2 = \langle -\Theta, 1, 1, 1, 2+\Theta\rangle$ and $f_3 = \langle -\Theta, 1, 1, 2+\Theta, 2+\Theta\rangle$ respectively. The Coxeter graphs of the groups Γ_1^4 and Γ_2^4 are presented in Figure 17.

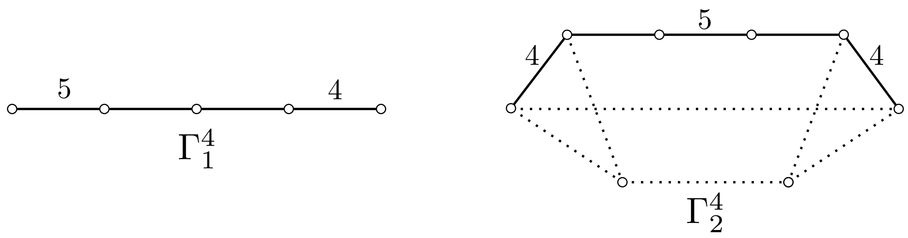


Figure 17: $\Gamma_1^4, \Gamma_2^4 \subset \mathrm{Isom}\,\mathbb{H}^4$

The Coxeter graph of the third group Γ_3^4 is more complicated: it has 16 vertices and 72 edges. In Figure 18 we depict the non-dotted edges of the graph while the dotted edges can be computed using the normal vectors given in Figure 19 (see Remark 4.3.1). The invariants of the three groups are presented in Figure 20.

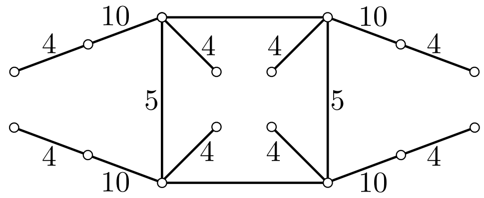


Figure 18: $\Gamma_3^4 \subset \operatorname{Isom}\mathbb{H}^4$

$$
\begin{aligned}
e_1 &= (0,-1,1,0,0), & e_4 &= (0,0,0,0,-1),\\
e_2 &= (0,0,-1,0,0), & e_5 &= (-1+1\cdot\Theta, 1\cdot\Theta, 0,0,0),\\
e_3 &= (0,0,0,-1,1), & e_6 &= (1,0,0,1,0),
\end{aligned}
$$

$$
\begin{aligned}
e_7 &= (1+3\cdot\Theta, 2+1\cdot\Theta, 0, 1+1\cdot\Theta, 1+1\cdot\Theta),\\
e_8 &= (2+3\cdot\Theta, 1+2\cdot\Theta, 2\cdot\Theta, 1+1\cdot\Theta, 1+1\cdot\Theta),\\
e_9 &= (4+7\cdot\Theta, 3+4\cdot\Theta, 3+4\cdot\Theta, 2+3\cdot\Theta, 2\cdot\Theta),\\
e_{10} &= (2+5\cdot\Theta, 1+2\cdot\Theta, 1+2\cdot\Theta, 1+2\cdot\Theta, 1+2\cdot\Theta),\\
e_{11} &= (9+12\cdot\Theta, 4+7\cdot\Theta, 4+7\cdot\Theta, 3+6\cdot\Theta, 2+4\cdot\Theta),\\
e_{12} &= (10+15\cdot\Theta, 4+7\cdot\Theta, 4+7\cdot\Theta, 4+7\cdot\Theta, 3+6\cdot\Theta),\\
e_{13} &= (11+18\cdot\Theta, 7+11\cdot\Theta, 7+11\cdot\Theta, 4+7\cdot\Theta, 4+5\cdot\Theta),\\
e_{14} &= (7+10\cdot\Theta, 4+6\cdot\Theta, 4+6\cdot\Theta, 3+4\cdot\Theta, 2+3\cdot\Theta),\\
e_{15} &= (7+11\cdot\Theta, 4+7\cdot\Theta, 4+5\cdot\Theta, 3+5\cdot\Theta, 2+3\cdot\Theta),\\
e_{16} &= (15+25\cdot\Theta, 10+15\cdot\Theta, 7+11\cdot\Theta, 7+11\cdot\Theta, 5+7\cdot\Theta).
\end{aligned}
$$

Figure 19: Normal vectors for $P(\Gamma_3^4)$

Since the dimension of the space is even, the commensurability class is completely determined by the ramification set of the Witt invariant $c(f_i)$, $i=1,2,3$, as indicated in Theorem 4.20. Direct computations (see Figure 20) show that the Witt invariants are given by

$$c(f_1) = (-1,-1), \quad c(f_2) = (-\Theta, 2+\Theta)\cdot(-1,-1), \quad c(f_3) = (-1,-2-\Theta).$$

Therefore, in order to decide about the commensurability of the groups, we have to compute the finite ramification of the quaternion algebras $(-1,-1)_K$, $(-\Theta,2+\Theta)_K$ and $(-1,2+\Theta)_K$. By Proposition 4.14, we get $\operatorname{Ram}_f(-1,-1)=\emptyset$. Since $N_{K/Q}(-\Theta)=-1$, the element $-\Theta$ is invertible, and $N_{K/Q}(2+\Theta)=5$ implies that $2+\Theta$ is a prime element of $\mathcal{O}_K$ lying over 5. For a quaternion algebra $(a,b)_K$, a necessary condition for a prime ideal $\mathcal{P}\subset\mathcal{O}_K$ to belong to the ramification set is that $\mathcal{P}\mid\langle 2ab\rangle=(2ab)\mathcal{O}_K$. In our case, since 2 remains prime in $\mathcal{O}_K$, the only candidates are 2 and $\sqrt{5}$. Moreover, since the ramification set has even cardinality and since $|\operatorname{Ram}_\infty B_i|=2$, then either $\operatorname{Ram}_f B_i=\emptyset$ or $\operatorname{Ram}_f B_i=\{2,\sqrt{5}\}$. To compute the ramification, we will use the following theorem.

Theorem 4.22 ([5, Theorem 16]). *Let K be a number field. Let $\mathcal{P}$ be a prime ideal of $\mathcal{O}_K$ and let $a, b \in \mathcal{O}_K$ such that the valuations satisfy $\eta_{\mathcal{P}}(a), \eta_{\mathcal{P}}(b) \in \{0, 1\}$. Define the integer m as follows:*

- *if $\mathcal{P} \mid \langle 2 \rangle$, $m = 2\eta_{\mathcal{P}}(2) + 3$;*
- *if $\mathcal{P} \nmid \langle 2 \rangle$, then $m = 1$ if $\eta_{\mathcal{P}}(a) = \eta_{\mathcal{P}}(b) = 0$ and $m = 3$ otherwise.*

Furthermore, let S be a finite set of representatives for the ring $\mathcal{O}_K/\mathcal{P}^m$.
Then, $\mathcal{P} \notin \mathrm{Ram}_f(a, b)_K$, i.e. (a, b) splits at $\mathcal{P}$, if and only if there exists a triple $(X, Y, Z) \in S^3$ such that the two following conditions are satisfied:

- *$aX^2 + bY^2 - Z^2 = 0$ or $m \leq \eta_{\mathcal{P}}(aX^2 + bY^2 - Z^2)$;*
- *$\eta_{\mathcal{P}}(X) = 0$ or $\eta_{\mathcal{P}}(Y) = 0$ or $\eta_{\mathcal{P}}(Z) = 0$.*

Remark. Since $(ac^2, b) \cong (a, b)$, the condition $\eta_{\mathcal{P}}(a), \eta_{\mathcal{P}}(b) \in \{0, 1\}$ is not a restriction.

We start with the group Γ_2^4. We take $\mathcal{P} = \langle \sqrt{5} \rangle$ and $a = -\Theta$, $b = 2 + \Theta$, which gives $\eta_{\mathcal{P}}(a) = 0$ and $\eta_{\mathcal{P}}(b) = 1$ and thus $m = 3$. The quotient $\mathcal{O}_K/\mathcal{P}^3$ has 125 elements which can be described has follows (see [11, Theorem 1]):

$$\mathcal{O}_K/\mathcal{P}^3 = \left\{ (x + y\sqrt{5})\mathcal{P}^3 \mid 0 \leq x < 25, 0 \leq y < 5 \right\}.$$

Using a computer program, we can check using Theorem 4.22 that $\sqrt{5}$ belongs to $\mathrm{Ram}_f(-\Theta, 2 + \Theta)$ and thus $\sqrt{5} \in \mathrm{Ram}_f B_2$. Therefore, we have $\mathrm{Ram}_f B_2 = \{2, \sqrt{5}\}$. On the other hand, since the equation $(2 + \Theta)X^2 - Y^2 - Z^2 = 0$ is satisfied with $X = Y = -1 + \Theta$ and $Z = 1$ (and thus $\eta_{\mathcal{P}}(X) = \eta_{\mathcal{P}}(Y) = \eta_{\mathcal{P}}(Z) = 0$), then $\sqrt{5} \notin \mathrm{Ram}_f(-1, 2 + \Theta)$ which implies that $\mathrm{Ram}_f B_3 = \emptyset$. Hence, the simplex group Γ_1^4 is commensurable to the group Γ_3^4 but *not* to Γ_2^4.

Example 4.23. In [24], the Coxeter pyramid groups of Tumarkin (including all, arithmetic and non-arithmetic groups) have been classified up to commensurability. In Appendix Appendix D, we provide the list with the commensurability classes in the arithmetic case.

§ 4.4. Case $n = 3$ (arithmetic and non-arithmetic)

When $n = 3$, we have two interesting features:

- the computation of the invariants is easier;
- some of the results are true even when the group is not arithmetic.

invariant	Γ_1^4	Γ_2^4	Γ_3^4
f_i	$\langle -\Theta, 1, 1, 1, 1\rangle$	$\langle -\Theta, 1, 1, 1, 2+\Theta\rangle$	$\langle -\Theta, 1, 1, 2+\Theta, 2+\Theta\rangle$
$f-$ vector	$(5, 10, 10, 5, 1)$	$(14, 28, 22, 8, 1)$	$(48, 96, 64, 16, 1)$
covolume	$\frac{17\pi^2}{21{,}600}$	$\frac{\pi^2}{60}$	$\frac{221\pi^2}{900}$
$s(f_i)$	1	$(-\Theta, 2+\Theta)$	$(-1, 2+\Theta)$
$c(f_i)$	$(-1, -1)$	$(-\Theta, 2+\Theta)\cdot(-1,-1)$	$(-1, 2+\Theta)\cdot(-1,-1)$
$\mathrm{Ram}_f\, B_i$	$\emptyset$	$\{2, \sqrt{5}\}$	$\emptyset$

Figure 20: Invariants of the three cocompact groups $\Gamma_i^4 \subset \mathrm{Isom}\,\mathbb{H}^4$

A standard reference for the three-dimensional case is [38]. Let $\Gamma \subset \mathrm{Isom}\,\mathbb{H}^3$ be a cofinite Coxeter group of rank N. As above, we denote by $G(\Gamma) = (a_{i,j})$ the Gram matrix of Γ and by e_i the normal vectors of the bounding hyperplanes of the fundamental polyhedron $P(\Gamma)$ of Γ. We let K be the field generated by the cycles in $G(\Gamma)$:

$$K = \mathbb{Q}\big(a_{i_1,i_2} \cdot \ldots \cdot a_{i_{k-1},i_k} \cdot a_{i_k,i_1} \,|\, i_j \in \{1, \ldots, N\}\big).$$

We also consider the K-vector space $M(\Gamma)$ spanned by the vectors

$$a_{1,i_1} \cdot a_{i_1,i_2} \cdot \ldots \cdot a_{i_{k-1},i_k} \cdot e_{i_k}.$$

The space $M(\Gamma)$, endowed with the restriction of the Lorentzian form, is a quadratic space of signature $(3, 1)$. Let δ be the signed determinant of this quadratic space. Now, the *invariant trace field* $k\Gamma^+ = \mathbb{Q}(\mathrm{Tr}({\Gamma^+}^{(2)}))$ of the rotation subgroup Γ^+ of Γ, which is a commensurability invariant by [38, Theorem 3.3.4], can be characterised as follows (cf. Section 3.1).

Theorem 4.24 ([38, Theorem 10.4.1])**.** $k\Gamma^+ = K(\sqrt{\delta})$.

Let $\langle -a_1, a_2, a_3, a_4\rangle$, $a_i > 0$, be a diagonal form of the quadratic form corresponding to $M(\Gamma)$.

Definition 4.25. The quaternion algebra $(-a_1 \cdot a_2, -a_1 \cdot a_3)_{k\Gamma^+}$ over the invariant trace field $k\Gamma^+$ is called the *invariant quaternion algebra* of Γ. It is denoted by $A\Gamma^+$.

Theorem 4.26 ([37, Theorems 2.1 and 3.1])**.** *The invariant quaternion algebra $A\Gamma$ of Γ is a commensurability invariant.*

Remark. Let $\Gamma^+ \subset PSL(2,\mathbb{C})$ be the rotational subgroup of a Coxeter group Γ acting on $\mathbb{H}^3$ such that the quotient $\mathbb{H}^3/\Gamma$ is not compact but of finite volume. Then the invariant quaternion algebra $A\Gamma^+$ equals $Mat(2, k\Gamma^+) = 1_{\mathrm{Br}(k\Gamma^+)}$ (see [38, Theorem 3.3.8]).

Remark. If Γ is arithmetic, then the invariant trace field and the invariant quaternion algebra correspond to what we get using the Witt invariant and thus form a *complete* set of invariants. Indeed, since we are only interested in the similarity class of the quadratic form, we can suppose that it can be written as $\langle 1, \tilde{b}, \tilde{c}, -\tilde{d}\rangle$, with $\tilde{b}, \tilde{c}, \tilde{d} > 0$. We then let $\delta = -bcd$ be the determinant of the form which gives

$$\langle 1, \tilde{b}, \tilde{c}, -\tilde{d}\rangle \cong \langle 1, -\delta\tilde{c}\tilde{d}, -\delta\tilde{b}\tilde{d}, \delta\tilde{b}\tilde{c}\rangle = \langle 1, -\delta c, -\delta d, \delta cd\rangle,$$

with $c := \tilde{c}\tilde{d}$, $d := \tilde{b}\tilde{d}$. Then, a direct computation shows that the Witt invariant of this last quadratic form is $(c, d) = (\tilde{c}\tilde{d}, \tilde{b}\tilde{d})$.

Example 4.27. Consider the ideal Coxeter cube $W \subset \mathbb{H}^3$ providing the Coxeter group Γ_W of rank $N = 6$ with graph Σ given by Figure 9. The distances l_1, l_2, l_3 of opposite facets are given by (2.10). Observe that the group Γ_W is *not* arithmetic in view of Vinberg's criterion reproduced in Proposition 4.5. The Gram matrix $G = G(W)$ is given by

$$G = \begin{pmatrix} 1 & -\sqrt{3}/2 & 0 & -1/2 & -\sqrt{3}/2 & -5/2 \\ -\sqrt{3}/2 & 1 & -1/2 & 0 & -5/2 & -\sqrt{3}/2 \\ 0 & -1/2 & 1 & -2\sqrt{3}/3 & -1/2 & 0 \\ -1/2 & 0 & -2\sqrt{3}/3 & 1 & 0 & -1/2 \\ -\sqrt{3}/2 & -5/2 & -1/2 & 0 & 1 & -\sqrt{3}/2 \\ -5/2 & -\sqrt{3}/2 & 0 & -1/2 & -\sqrt{3}/2 & 1 \end{pmatrix}.$$

In particular, the field K generated by the irreducible cycles of G is $\mathbb{Q}$. Moreover, one can see that the rows $e_1, ..., e_6$ of the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 & 0 \\ 0 & -1 & -\sqrt{3}/3 & \sqrt{3}/3 \\ -1/2 & -\sqrt{3}/2 & 7/4 & 7/4 \\ -\sqrt{3}/2 & -13/2 & 5\sqrt{3}/2 & 9\sqrt{3}/2 \\ -5/2 & -7\sqrt{3}/2 & 13/4 & 29/4 \end{pmatrix}$$

are normal vectors for the facets of W. It follows that the vectors described in (4.1) are equal to

$$v_1 = e_1, \quad v_2 = \sqrt{3}\, e_2, \quad v_3 = \sqrt{3}\, e_3, \quad v_4 = e_4, \quad v_5 = \sqrt{3}\, e_5, \quad v_6 = e_6.$$

One can check that $\{v_1, ..., v_4\}$ is a basis of $\mathbb{R}^4$, giving rise to the diagonal form $q = \langle 1, 3, 7, -7\rangle$ of signed determinant $\delta = -3$ (recall that δ is defined up to an integer square).
We conclude that the invariant trace field $k\Gamma_W^+$ of the rotation subgroup Γ_W^+ is given by $k\Gamma_W^+ = \mathbb{Q}(\sqrt{3}i)$.

Example 4.28. There are 32 non-isometric hyperbolic Coxeter tetrahedra giving rise to arithmetic and non-arithmetic, cocompact and non-cocompact hyperbolic reflection groups of finite covolume in $\mathrm{Isom}\mathbb{H}^3$. Their commensurability classification can be found in [38, Section 13] and [30].

Example 4.29. In [24], the Coxeter pyramid groups of Tumarkin have been classified up to commensurability.
Among the non-arithmetic examples in $\mathrm{Isom}\mathbb{H}^3$ is the pyramid group Γ_T with graph given by Figure 21 (see Section 2.3). As in Example 4.27 one determines the invariant

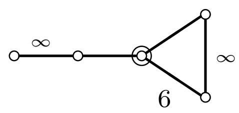


Figure 21: The non-arithmetic Coxeter pyramid group Γ_T

trace field of $k\Gamma_T^+$ and finds that $k\Gamma_T^+ = \mathbb{Q}(\sqrt{3}i)$.
In Appendix Appendix D, we provide the list with the commensurability classes in the non-arithmetic case.

We shall see below that the groups Γ_C and Γ_T having identical invariant trace field (and invariant quaternion algebra) are *not* commensurable.

§ 5. Some other aspects

Consider cofinite hyperbolic Coxeter groups in $PO(n, 1)$ for $n \geq 3$. There are methods to decide about commensurability in certain cases, and this independently of arithmetic considerations. In the sequel, we shall present some of these aspects.

§ 5.1. Comparison of trace fields related to Coxeter elements

A natural conjugacy invariant is the ordinary trace field $\mathrm{Tr}(G)$ of G in $\mathrm{Isom}\,\mathbb{H}^n$ which is defined to be the field generated by all the traces of matrices in $G \subset PO(n, 1)$. For a hyperbolic Coxeter group Γ with generators $s_1, \ldots, s_N$ and a *Coxeter element* $c = s_1 \cdot \ldots \cdot s_N$, one observes that each finite index subgroup of Γ must contain c^k for some positive integer k. Following ideas and methods in [30, p. 132], where all hyperbolic

Coxeter simplex groups (of rank $N = n + 1$) are classified up to commensurability, incommensurability for this family of Coxeter groups can be tested as follows. Let $\Delta_1, \Delta_2 \subset \mathrm{Isom}\mathbb{H}^n$ be Coxeter simplex groups with Coxeter elements c_1 and c_2, and let $T_i^k = \mathbb{Q}(\mathrm{tr}(c_i^k)) \subset \mathrm{Tr}(\Delta_i)$, $i = 1, 2$, be the fields generated by the traces of the k-th powers of c_1 and c_2, $k \in \mathbb{N}$. If

$$T_1^k \nsubseteq \mathrm{Tr}(\Delta_2) \text{ for all } k \in \mathbb{N} \quad \text{or} \quad T_2^l \nsubseteq \mathrm{Tr}(\Delta_1) \text{ for all } l \in \mathbb{N}, \tag{5.1}$$

then Δ_1 and Δ_2 are not commensurable. This property can be extended to Coxeter pyramid groups in the following way.

Let $\Gamma_1, \Gamma_2 \subset \mathrm{Isom}\mathbb{H}^n$ be two Coxeter pyramid groups (of rank $n + 2$). Up to finite index, each of them can be identified with a group generated by reflections in a *polarly truncated* Coxeter simplex (see [27, Section 4.1]). Let $P_1 = \bigcap_{i=1}^{n+2} H_i^-$ be a fundamental polyhedron for Γ_1, and let $S_1 = \{s_1, ..., s_{n+2}\}$ be the set of generators of Γ_1 such that each s_i is a reflection in the hyperplane $H_i \subset \mathbb{H}^n$ with unit normal vector u_i pointing outward from P_1, say. Associated to Γ_1 is a Coxeter total simplex group $\widehat{\Gamma_1}$ (for the definition of a *total simplex*, see [27, Definition 4.2]; an illustrative example is given by the group $\Gamma_{2,m}$ according to Figure 4 with total simplex $\widehat{\Gamma}_m$ according to Figure 5 for $m = 3, 4$). Without loss of generality, we can suppose that $\widehat{\Gamma_1}$ is generated by the reflections $s_1, ..., s_{n+1}$, so that the vectors $u_1, ..., u_{n+1}$ are linearly independent. Let $\widehat{G_1}$ and G_1 be the respective Gram matrices of $\widehat{\Gamma_1}$ and Γ_1. By construction, the matrix $\widehat{G_1}$ is the top-left principal submatrix of size $n + 1$ in G_1. Now, for $i = 1, ..., n + 1$, the matrix of s_i with respect to the canonical basis of $\mathbb{R}^{n+1}$ is $R_{1,i} := I - 2A_{1,i}$, where $A_{1,i}$ is obtained by replacing the i-th line of the zero matrix of size $n + 1$ by the i-th line of $\widehat{G_1}$. Moreover, the vector u_{n+2} normal to the truncating polar hyperplane H_{n+2} is given by

$$u_{n+2} = \frac{\sum_{j=1}^{n+1} \mathrm{cof}_{j,n+1}(\widehat{G_1})\, u_j}{\sqrt{\det(\widehat{G_1})\ \mathrm{cof}_{n+1,n+1}(\widehat{G_1})}} \tag{5.2}$$

(see [27, (4.7)]). Here, $\mathrm{cof}_{i,j}(M) = (-1)^{i+j}\det M_{ij}$ is the (i, j)-th cofactor arising from deleting the i-th line and the j-th column from the square matrix M. Notice that u_{n+2} can be interpreted as an *ultra-ideal vertex* with Lorentz norm 1, say v_{n+1}, of the total simplex associated to $\widehat{\Gamma_1}$ as described in [27, Section 4.1.2]. It follows that the matrix of s_{n+2} with respect to the canonical basis of $\mathbb{R}^{n+1}$ equals $R_{1,n+2} := I - 2B_1$, where B_1 is given by $[B_1]_{i,j} = 0$ if $j \neq n + 1$ and $[B_1]_{i,n+1} = \frac{\mathrm{cof}_{i,n+1}(\widehat{G_1})}{\mathrm{cof}_{n+1,n+1}(\widehat{G_1})}$, $1 \leq i \leq n + 1$.
Let $U := (u_1, ..., u_{n+1}) \in GL(n + 1, \mathbb{R})$ be the matrix whose i-th column is u_i, $i = 1, ..., n + 1$. Then, $UR_{1,i}U^{-1} \in O(n, 1)$ for $1 \leq i \leq n + 2$. The group generated by $R_{1,1}, ..., R_{1,n+2}$ is a matrix representation of Γ_1 in $GL(n + 1,\ \mathbb{Q}(\widehat{\Gamma_1}))$, with Coxeter element $C_1 := \Pi_{i=1}^{n+2} R_{1,i}$.

Similarly, one obtains a matrix representation of Γ_2 in $GL(n+1, \mathbb{Q}(\widehat{\Gamma_2}))$ with Coxeter element $C_2 = \Pi_{i=1}^{n+2} R_{2,i}$.
Now, as in the simplex case, if the corresponding condition (5.1) is satisfied, the groups Γ_1 and Γ_2 are incommensurable (as subgroups of $GL(n+1, \mathbb{R})$).

Example 5.1. Consider the hyperbolic Coxeter pyramid groups Γ_1, Γ_2 and Γ_3 given by Figure 22.

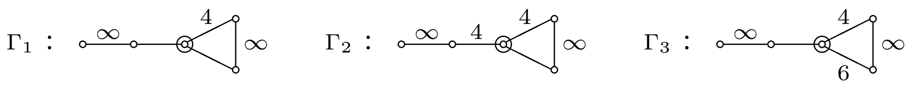


Figure 22: Three non-arithmetic Coxeter pyramid groups acting on $\mathbb{H}^3$

By removing the first node on the left of Γ_i, one obtains the Coxeter graph of the Coxeter total simplex group $\widehat{\Gamma_i}$ associated to Γ_i. Let G_i, $i = 1, 2, 3$, be the Gram matrix of Γ_i, and $\widehat{G_i}$ be the Gram matrix of $\widehat{\Gamma_i}$. For $i = 1, 2, 3$ and $j = 1, ..., 5$, we compute the matrix representations $R_{i,j}$ as above. In particular, the product $C_i := \Pi_{j=1}^{5} R_{i,j} \in GL(4, \mathbb{Q}(\widehat{G_i}))$ is a matrix representation for a Coxeter element of Γ_i. The characteristic polynomial $\chi_i = \chi(C_i)$ is of the form

$$\chi_i(t) = (t-1)(t+1)(t^2 - 2\alpha_i t + 1),$$

where the coefficients α_i are given by the following table.

i	1	2	3
α_i	$3+\sqrt{2}$	$7/2+\sqrt{2}$	$4+\sqrt{6}$

Then, the eigenvalues $\lambda_{i,k}$, $k = 1, ..., 4$, of C_i are given by

$$\lambda_{i,1} = 1, \quad \lambda_{i,2} = -1, \quad \lambda_{i,3} = \alpha_i + \sqrt{\alpha_i^2 - 1}, \quad \lambda_{i,4} = \alpha_i - \sqrt{\alpha_i^2 - 1}.$$

Hence, the trace $\mathrm{tr}(C_i^k)$, for $k \geq 0$, is given by

$$\begin{aligned}
\sum_{l=1}^{4} \lambda_{i,l}^k &= 1 + (-1)^k + \sum_{m=0}^{k} \binom{k}{m} (1 + (-1)^m)\, \alpha_i^{k-m} \left(\sqrt{\alpha_i^2 - 1}\right)^m \\
&= 1 + (-1)^k + 2 \sum_{\substack{m=0 \\ m \text{ even}}}^{k} \binom{k}{m} \alpha_i^{k-m} \left(\sqrt{\alpha_i^2 - 1}\right)^m \\
&= 1 + (-1)^k + 2 \sum_{\substack{m=0 \\ m \text{ even}}}^{k} \binom{k}{m} \alpha_i^{k-m} \left(\alpha_i^2 - 1\right)^{m/2} \ .
\end{aligned}$$

Since $\alpha_i^2 - 1 > 0$ for $i = 1,2,3$, each term of the sum consists of a product of (powers of) algebraic numbers of positive rational part and positive coefficients on $\sqrt{2}$ and $\sqrt{6}$, respectively, so that we obtain the following fields for the groups Γ_1, Γ_2 and Γ_3 :

$$T_1^k = \mathbb{Q}(\sqrt{2}), \quad T_2^k = \mathbb{Q}(\sqrt{2}), \quad T_3^k = \mathbb{Q}(\sqrt{6}), \quad \text{for all } k \in \mathbb{N}.$$

For $i = 1,2,3$, let $\mathbb{Q}(G_i)$ be the field generated by the coefficients of the Gram matrix G_i. Observe that $\mathrm{Tr}(\Gamma_1), \mathrm{Tr}(\Gamma_2) \subset \mathbb{Q}(\widehat{G_1}) = \mathbb{Q}(\widehat{G_2}) = \mathbb{Q}(\sqrt{2})$ and that $\mathrm{Tr}(\Gamma_3) \subset \mathbb{Q}(\widehat{G_3}) = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Then, by (5.1), the group Γ_3 is incommensurable to Γ_1 and to Γ_2. Moreover, commensurability between Γ_1 and Γ_2 cannot be decided by this approach. In Section 5.4, we shall see that the groups Γ_1 and Γ_2 are incommensurable as well.

Example 5.2. Consider the non-cocompact Coxeter pyramid groups Γ_4 and Γ_5 in $\mathrm{Isom}\,\mathbb{H}^4$ given by the graphs according to Figure 23.

Figure 23: Two Coxeter pyramid groups acting on $\mathbb{H}^4$

By removing the first nodes on the left of Γ_4 and Γ_5, we obtain graphs of rank 5 Coxeter groups $\widehat{\Gamma_4}$ and $\widehat{\Gamma_5}$, respectively. In particular, one has $\mathrm{Tr}(\Gamma_4) \subset \mathbb{Q}(\widehat{G_5}) = \mathbb{Q}(\sqrt{5})$ and $\mathrm{Tr}(\Gamma_5) \subset \mathbb{Q}(\widehat{G_5}) = \mathbb{Q}(\sqrt{2})$. Let C_4 and C_5 be the respective matrix representations of Coxeter elements of Γ_4 and Γ_5, obtained as described above. For example, one can take

$$C_4 = \begin{pmatrix} 3 & 1+\sqrt{5} & 0 & 0 & 1/2\left(3+5\sqrt{5}\right) \\ 2 & 1/2\left(1+\sqrt{5}\right) & 0 & 0 & 1/2\left(1+3\sqrt{5}\right) \\ 0 & 1/2\left(1+\sqrt{5}\right) & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

and

$$C_5 = \begin{pmatrix} 5+2\sqrt{2} & \sqrt{2} & 0 & 0 & 3+3\sqrt{2} \\ 2+\sqrt{2} & 0 & 0 & 0 & 1+\sqrt{2} \\ \sqrt{2} & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then, the characteristic polynomials $\chi(C_4)$ and $\chi(C_5)$ are given by

$$\begin{aligned} \chi(C_4)(t) &= \tfrac{1}{2}\,(t-1)\left(2t^4 - (5+\sqrt{5})t^3 - (8+2\sqrt{5})t^2 - (5+\sqrt{5})t + 2\right), \\ \chi(C_5)(t) &= \ (t-1)\left(t^4 - (4+2\sqrt{2})t^3 - (7+4\sqrt{2})t^2 - (4+2\sqrt{2})t + 1\right). \end{aligned}$$

Their eigenvalues $\lambda_{i,j}$, $i = 4, 5$, $j = 1, ..., 5$, are given by

$$\lambda_{i,1} = 1\,, \quad \lambda_{i,2} = \frac{1}{4}(\alpha_i + \beta_i + \sqrt{\gamma_i + \delta_i})\,, \quad \lambda_{i,3} = \frac{1}{4}(\alpha_i + \beta_i - \sqrt{\gamma_i + \delta_i})\,,$$
$$\lambda_{i,4} = \frac{1}{4}(\alpha_i - \beta_i + \sqrt{\gamma_i - \delta_i})\,, \quad \lambda_{i,5} = \frac{1}{4}(\alpha_i - \beta_i - \sqrt{\gamma_i - \delta_i})\,,$$

with

$$\alpha_4 = \frac{5+\sqrt{5}}{2}\,, \quad \beta_4 = \frac{1}{2}\sqrt{126 + 26\sqrt{5}}\,,$$
$$\gamma_4 = 23 + 9\sqrt{5}\,, \quad \delta_4 = \frac{2\sqrt{2}(95 + 32\sqrt{5})}{\sqrt{63 + 13\sqrt{5}}}\,,$$

and

$$\alpha_5 = 2\,(2 + \sqrt{2})\,, \quad \beta_5 = 2\,\sqrt{15 + 8\sqrt{2}}\,,$$
$$\gamma_5 = 2\,(17 + 12\sqrt{2})\,, \quad \delta_5 = 4\,\frac{46 + 31\sqrt{2}}{\sqrt{15 + 8\sqrt{2}}}\,.$$

Then, a procedure similar to the one already used above shows that for $i = 4, 5$ and $k \in \mathbb{N}$, one has

$$\operatorname{tr}(C_i^k) = 4^{2-k} \sum_{\substack{j=0 \\ j \text{ even}}}^{k} \sum_{l=0}^{k-j} \sum_{\substack{m=0 \\ m+l \text{ even}}}^{j/2} \binom{k}{j} \binom{k-j}{l} \binom{j/2}{m} \alpha_i^{k-j-l}\, \beta_i^l\, \gamma_i^{j/2-m}\, \delta_i^m.$$

Since for each term of the sum the integer $m + l$ has to be even, the integer $l - m$ is even as well, and one observes that

$$\beta_4^l\, \delta_5^m = 2^{\frac{m+l}{2}}(63 + 13\sqrt{5})^{\frac{l-m}{2}}(95 + 32\sqrt{5})^m$$

and that

$$\beta_5^l\, \delta_6^m = 2^{l+2m}(15 + 8\sqrt{2})^{\frac{l-m}{2}}(46 + 31\sqrt{2})^m\,.$$

Hence, for $i = 4, 5$, each term of the sum is an algebraic number with positive rational part and positive coefficient on $\sqrt{5}$ and $\sqrt{2}$, respectively, so that one obtains the following fields :

$$T_4^k = \mathbb{Q}(\sqrt{5}), \quad T_5^k = \mathbb{Q}(\sqrt{2}) \quad \text{for all } k \in \mathbb{N}\,.$$

As a consequence of condition (5.1), the groups Γ_4 and Γ_5 are incommensurable.

In a similar way, the trace fields associated to Coxeter elements can be determined for all Coxeter pyramid groups in $\mathrm{Isom}\mathbb{H}^n$ (see [27, Section 5.2]). In Table 1, the fields T_i of all *non-arithmetic* Coxeter pyramid groups $\Gamma_i \subset \mathrm{Isom}\mathbb{H}^4$, $i = 1, \ldots, 5$, are recapitulated.

i	Coxeter symbol	T_i
1	$[4, 4, 3, (3, \infty, 4)]$	$\mathbb{Q}(\sqrt{2})$
2	$[6, 3, 3, (3, \infty, 4)]$	$\mathbb{Q}(\sqrt{2})$
3	$[6, 3, 3, 5, \infty]$	$\mathbb{Q}(\sqrt{5})$
4	$[6, 3, 3, (3, \infty, 5)]$	$\mathbb{Q}(\sqrt{5})$
5	$[6, 3, 3, (4, \infty, 5)]$	$\mathbb{Q}(\sqrt{2}, \sqrt{5})$

Table 1: The non-arithmetic Coxeter pyramid groups $\Gamma_i \subset \operatorname{Isom} \mathbb{H}^4$ and their fields T_i

The methods described and exploited above for hyperbolic Coxeter pyramid groups, being simply and polarly truncated Coxeter simplex groups, can be generalised to *arbitrary Coxeter groups of rank* $N \geq n+2$ in $\operatorname{Isom} \mathbb{H}^n$ as follows.

Theorem 5.3. *Let $i = 1, 2$, and let $\Gamma_i \subset \operatorname{Isom} \mathbb{H}^n$ be a hyperbolic Coxeter group with natural generating reflection system $S_i = \{s_1, \ldots, s_{N_i}\}$ of rank $N_i \geq n+1$ and Coxeter element $c_i = s_1 \cdot \ldots \cdot s_{N_i}$. Let $T_i^k = \mathbb{Q}(\operatorname{tr}(c_i^k))$ be the field generated by the traces of the k-th powers of c_i, $k \in \mathbb{N}$. If*

$$T_1^k \not\subseteq \operatorname{Tr}(\Gamma_2) \text{ for all } k \in \mathbb{N} \quad \text{or} \quad T_2^l \not\subseteq \operatorname{Tr}(\Gamma_1) \text{ for all } l \in \mathbb{N},$$

then the groups Γ_1, Γ_2 are incommensurable (as subgroups of $GL(n+1; \mathbb{R})$).

Proof. Consider a fundamental Coxeter polyhedron $P = \bigcap_{i=1}^N H_i^- \subset \mathbb{H}^n$ of $\Gamma \in \{\Gamma_1, \Gamma_2\}$. It suffices to consider the case $N \geq n+2$. Suppose that the set of generators $= \{s_1, \ldots, s_N\}$ of Γ is such that s_i is the reflection with respect to the hyperplane H_i with normal unit vector u_i pointing outwards from P, and let G be the Gram matrix of P. Without loss of generality, we can assume that the vectors $u_1, ..., u_{n+1}$ are linearly independent providing a basis of $\mathbb{R}^{n+1}$. Let $\widehat{G}$ be their associated Gram matrix. Then, $\widehat{G}$ is the top-left principal submatrix of size $n+1$ of G.
Recall that for $i = 1, ..., N$, one has $s_i(x) = x - 2\langle x, u_i \rangle\, u_i$ for all $x \in \mathbb{H}^n$. Hence, for $i = 1, ..., n+1$, the matrix of s_i with respect to the canonical basis of $\mathbb{R}^{n+1}$ is $R_i := I - 2A_i$, where A_i is obtained by replacing the i-th line of the zero matrix of size $n+1$ by the i-th line of $\widehat{G}$.
Now, let $j \in \{n+2, ..., N\}$. Since $u_1, ..., u_{n+1}$ form a basis of $\mathbb{R}^{n+1}$, we can write $u_j = \sum_{k=1}^{n+1} \lambda_k^{(j)} u_k$, with uniquely determined coefficients $\lambda_k^{(j)} \in \mathbb{R}$. Hence, for $i = 1, ..., n+1$,

we get

$$s_j(u_i) = u_i - 2\langle u_i, \sum_{k=1}^{n+1} \lambda_k^{(j)} u_k \rangle \sum_{l=1}^{n+1} \lambda_l^{(j)} u_l = u_i - 2 \sum_{l=1}^{n+1} \sum_{k=1}^{n+1} \lambda_k^{(j)} \lambda_l^{(j)} \langle u_i, u_k \rangle u_l.$$

Observe that $\langle u_i, u_k \rangle = [\widehat{G}]_{i,j}$ for all $i, k \in \{1, ..., n+1\}$.
Let $B_j \in GL(n+1, \mathbb{R})$ be the matrix given by $[B_j]_{l,i} = \sum_{k=1}^{n+1} \lambda_k^{(j)} \lambda_l^{(j)} [\widehat{G}]_{i,k}$, $1 \leq l, i \leq n+1$. Then, the matrix of s_j with respect to the canonical basis of $\mathbb{R}^{n+1}$ equals $R_j := I - 2B_j$, $j = n+2, ..., N$.
As in the case $N = n+2$, let $U = (u_1, ..., u_{n+1}) \in GL(n+1, \mathbb{R})$ be the matrix whose i-th column is u_i, $i = 1, ..., n+1$. Then, $UR_iU^{-1} \in \mathrm{O}(n, 1)$ for $1 \leq i \leq N$. The group generated by $R_1, ..., R_N$ is a matrix representation of Γ in $GL(n+1, \mathbb{Q}(\widehat{\Gamma}))$, with Coxeter element $C = \Pi_{i=1}^N R_i$.

□

§ 5.2. Finite index subgroups

As mentionned in Section 3, commensurability is preserved by passing to finite index subgroups. When searching for finite index subgroups of $\Gamma \subset PO(n, 1)$, there are some general criteria, for example those due to Maxwell [39, Proposition 3.1, Proposition 3.2]. In particular, consider a Coxeter pyramid group Γ with Euclidean Coxeter subgroups of type $\widetilde{F}_4 = [3, 4, 3, 3]$ and Δ (see Figure 24 for the Coxeter graph Σ of Γ), Maxwell's result shows that the hyperbolic Coxeter pyramid group with Euclidean Coxeter subgroups $\widetilde{B}_4 = [4, 3, 3^{1,1}]$ (replacing $\widetilde{F}_4$) and Δ is a subgroup of index 3 in Γ.

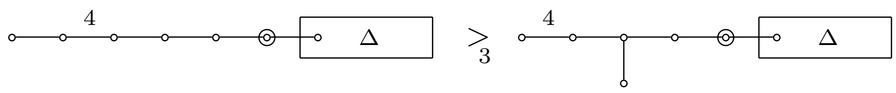


Figure 24: The group $\Gamma = [\widetilde{F}_4, 3, 3, \Delta]$ and its subgroup $[\widetilde{B}_4, 3, 3, \Delta]$ of index 3

There are also some ad hoc results based on looking at additional hyperplanes bisecting Coxeter polyhedra into smaller Coxeter polyhedra or at Coxeter groups related to higher Bianchi groups. In this way, in dimension 3, a natural bisection shows that the Coxeter pyramid groups $[\infty, 3, 3, \infty]$ resp. $[\infty, 4, 4, \infty]$ are subgroups of index 2 in the Coxeter simplex groups $[3, 4, 4]$ resp. $[4, 4, 4]$, and the latter group is related to the last but one by an index 3 subgroup relation arising by a tetrahedral trisection. In Table 2 and in Table 3, we provide the subgroup relations of all arithmetic hyperbolic Coxeter simplex groups in $\mathrm{Isom}\mathbb{H}^3$. These groups fall into 2 commensurability classes, one containing the group $[3, 4, 4]$ of covolume Л$(\pi/4)/6$ and one containing the group $[3, 3, 6]$ of covolume Л$(\pi/3)/8$ (see (2.8)).

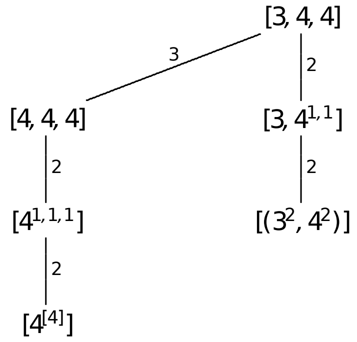


Table 2: Coxeter tetrahedral groups commensurable with $[3, 4, 4]$

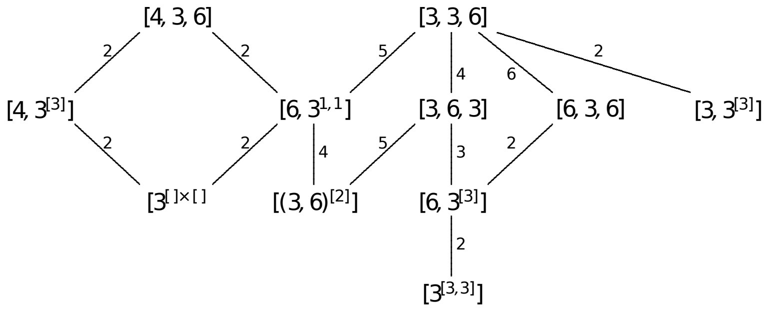


Table 3: Coxeter tetrahedral groups commensurable with $[3, 3, 6]$

As an example in dimension 4, by [31, p. 172], the Coxeter pyramid group $[\infty, 3, 3, 4, 4]$ given by Figure 25 is a subgroup of index 3 in the Coxeter simplex group $[3, 4, 3, 4]$; the latter group generates the symmetry group of the right-angled ideal regular 24-cell and is also related to the quaternionic modular group $PSL(2, \mathbb{H})$.

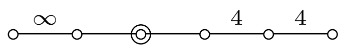


Figure 25: The Coxeter pyramid group $[\infty, 3, 3, 4, 4]$ acting on $\mathbb{H}^4$

§ 5.3. Commensurability of amalgamated free products

Consider hyperbolic Coxeter groups $\Gamma \subset \operatorname{Isom}\mathbb{H}^n$ which are amalgamated free products $\widehat{\Gamma}_1 \star_\Phi \widehat{\Gamma}_2$ such that $\Phi \subset \operatorname{Isom}\mathbb{H}^{n-1}$ is itself a cofinite Coxeter group whose fundamental Coxeter polyhedron F is a common facet of the fundamental Coxeter polyhedra P_1 and P_2 of Γ_1 and Γ_2 (see Section 2.2). Geometrically, a fundamental polyhedron for $\widehat{\Gamma}_1 \star_\Phi \widehat{\Gamma}_2$ is the Coxeter polyhedron arising by glueing together P_1 and P_2 along their common facet F. Among the Coxeter pyramid groups of Tumarkin, there are several examples of this form which can be characterised by Coxeter graphs as given in Figure 26 (see also Figures 11 and 12). For Coxeter groups given as amalgamated free products according to Figure 26, the following incommensurability result can be derived.

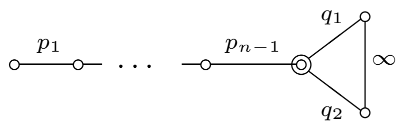


Figure 26: The free product $\Gamma = [p_1, \ldots, p_{n-1}, (q_1, \infty, q_2)] = \widehat{\Theta}_1 \star_\Phi \widehat{\Theta}_2$ of $\widehat{\Theta}_i = [p_1, \ldots, p_{n-1}, q_i]$ amalgamated by $\Phi = [p_1, \ldots, p_{n-1}]$

Theorem 5.4 ([24, Proposition 1]). *Let Γ be a hyperbolic Coxeter pyramid group with $n+2$ generators such that Γ is the free product of the Coxeter orthoscheme groups $\widehat{\Theta}_1 = [p_1, \ldots, p_{n-1}, q_1]$ and $\widehat{\Theta}_2 = [p_1, \ldots, p_{n-1}, q_2]$ amalgamated by their common Coxeter subgroup $\Phi = [p_1, \ldots, p_{n-1}]$, where $p_1 = \infty$ for $n = 3$. Suppose that $\mathbb{H}^n/\Gamma$ is 1-cusped. Then, the following holds.*
(1) If $q_1 = q_2 =: q$ and $\Theta := [p_1, \ldots, p_{n-1}, q, \infty]$, then Γ is a subgroup of index 2 in Θ.
(2) If $q_1 \neq q_2$, then Γ is incommensurable to $\Theta_k := [p_1, \ldots, p_{n-1}, q_k, \infty]$ for $k = 1$ and $k = 2$.

Example 5.5. Using Theorem 5.4, we see that the two (non-arithmetic) Coxeter pyramid groups $\Gamma = [6, 3, 3, (3, \infty, 5)]$ and $\Gamma' = [6, 3, 3, 5, \infty]$ in $\mathrm{Isom}(\mathbb{H}^4)$, both with 1-cusped orbit spaces, have identical trace field $\mathbb{Q}(\sqrt{5})$ but are not commensurable.

As for the groups $\Gamma_1 = [4, 4, 3,\ (3, \infty, 4)]$ and $\Gamma_2 = [6, 3, 3, (3, \infty, 4)]$ in $\mathrm{Isom}(\mathbb{H}^4)$, which are both non-arithmetic free products with amalgamation having trace field $\mathbb{Q}(\sqrt{2})$, Theorem 5.4 does not apply. In fact, the 4-orbifold $\mathbb{H}^4/\Gamma_1$ has 2 cusps. Inspired by the fact that the Euclidean lattices $[4, 4]$ and $[6, 3]$ are inequivalent, and in view of the result of Karrass and Solitar (see Section 3.1), we proved the following result by a geometric reasoning.

Lemma 5.6 ([24, Lemma 2]). *The two non-arithmetic Coxeter pyramid groups $\Gamma_1 = [4, 4, 3, (3, \infty, 4)]$ with 2-cusped quotient and $\Gamma_2 = [6, 3, 3, (3, \infty, 4)]$ with 1-cusped quotient are incommensurable in* $\mathrm{Isom}\mathbb{H}^4$.

§ 5.4. Commensurator and incommensurable non-arithmetic groups

Consider the two non-arithmetic Coxeter groups Γ_W and Γ_T arising as reflection groups associated to the ideal Coxeter cube W given by Figure 9 and the pyramid T

given by Figure 12 in $\mathbb{H}^3$. Their rotation subgroups have identical invariant trace field and invariant quaternion algebra, respectively. By having a look at the covolume ratio based on the values (2.11) and (2.14), an accurate numerical check indicates that the quotient

$$\begin{aligned}\omega := \frac{\mathrm{covol}_3(\Gamma_T)}{\mathrm{covol}_3(\Gamma_W)} &= \frac{\frac{5}{4}\ \text{Л}(\frac{\pi}{3}) + \frac{1}{3}\ \text{Л}(\frac{\pi}{4})}{10\ \text{Л}(\frac{\pi}{3})} = \frac{1}{8} + \frac{1}{30}\,\frac{\text{Л}(\frac{\pi}{4})}{\text{Л}(\frac{\pi}{3})} \\ &\sim 0.1701240538565287\end{aligned} \tag{5.3}$$

is an *irrational* number implying the incommensurability of the groups Γ_T and Γ_W.

Let us prove rigorously that ω defined by (5.3) is irrational. We assume the contrary and suppose that Γ_T and Γ_W are commensurable. Let C denote their commensurator (see Section 3.1). Since Γ_T and Γ_W are *non-arithmetic*, by Margulis' Theorem 3.1, the group C is a discrete subgroup of $\mathrm{Isom}\,\mathbb{H}^3$ containing both groups as subgroups of finite indices. Furthermore, C is a non-cocompact but cofinite (non-arithmetic) group so that its covolume is universally bounded from below by the minimal covolume $\text{Л}(\pi/3)/8$ in this class which is realised by the tetrahedral group $[3,3,6]$ (see [42] and [34, Table 2]). This allows us to rewrite (5.3) according to

$$\omega = \frac{\mathrm{covol}_3(\Gamma_T)}{\mathrm{covol}_3(\Gamma_W)} = \frac{\mathrm{covol}_3(\Gamma_T)/\mathrm{covol}_3(C)}{\mathrm{covol}_3(\Gamma_W)/\mathrm{covol}_3(C)} = \frac{[C:\Gamma_T]}{[C:\Gamma_W]}\ . \tag{5.4}$$

Now, since $\mathrm{covol}_3(\Gamma_W) = 10\ \text{Л}(\pi/3)$ and $\mathrm{covol}_3([3,3,6]) = \frac{1}{8}\ \text{Л}(\pi/3)$, we get that $[C:\Gamma_W] < 80$. Using this and (5.4), it is easy to check that there is no rational solution ω to (5.3) with an approximate value $\omega \sim 0.1701240538565287$. This provides the desired contradiction. A similar reasoning based on the covolume expressions (2.13) allows us to prove that the two non-arithmetic Coxeter pyramid groups $[\infty,3,(3,\infty,4)]$ with 1-cusped quotient and $[\infty,3,(4,\infty,4)]$ with 2-cusped quotient, both groups having identical invariant trace field and invariant quaternion algebra, are incommensurable in $\mathrm{Isom}\mathbb{H}^3$. In this way, the commensurability classification for the family of Tumarkin's Coxeter pyramid groups could be finalised. In Appendix Appendix D, we provide the commensurability classes of all *non-arithmetic* Coxeter pyramid groups.

Appendices

§ Appendix A. The non-compact hyperbolic Coxeter tetrahedra

In [29] and [30], the covolumes and commensurability classes of all hyperbolic Coxeter n-simplex groups (existing for $n \leq 9$) are listed. In this section, we provide the list of the 23 non-compact hyperbolic Coxeter tetrahedral groups together with their covolumes (see Table 4 and Table 5).

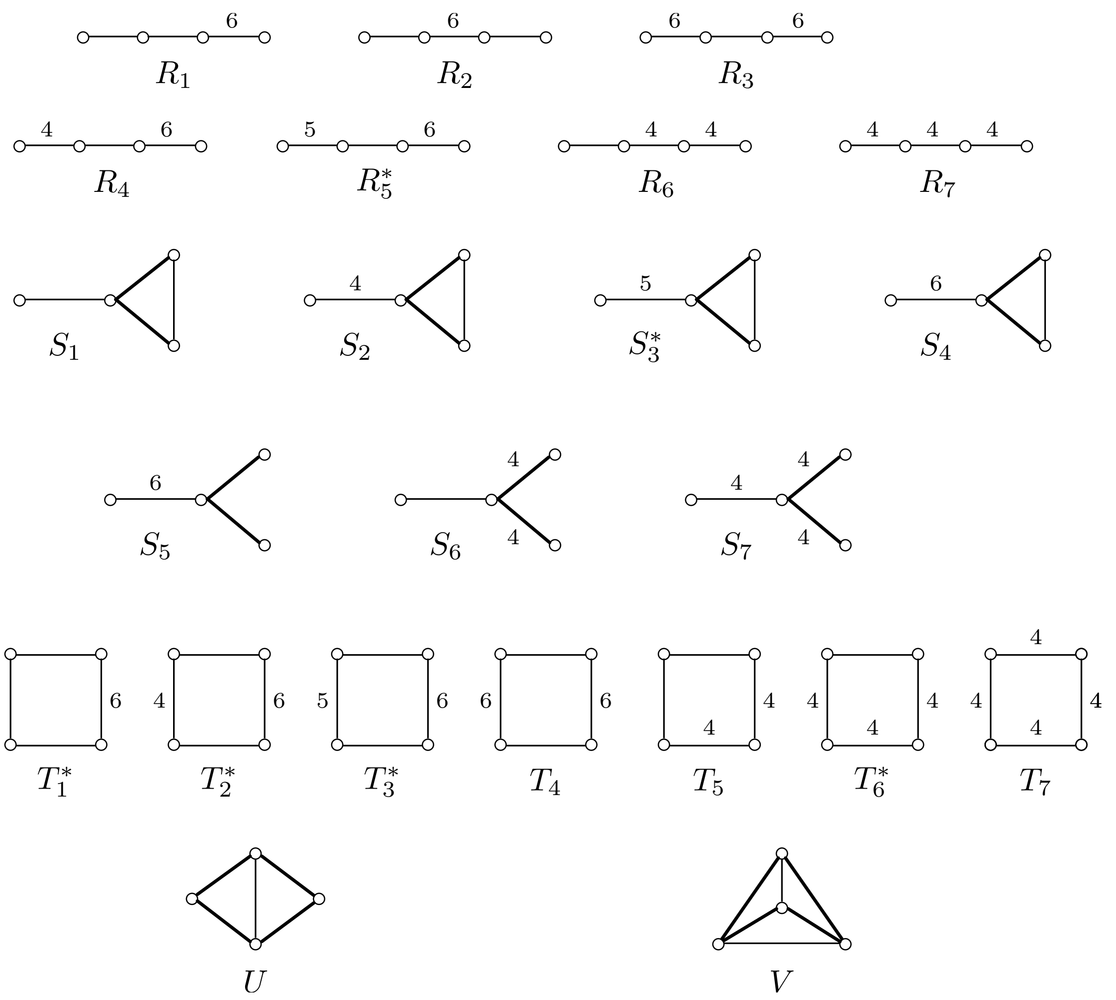


Table 4: The 23 non-compact hyperbolic Coxeter tetrahedral groups of finite covolume where * indicates non-arithmeticity

§ Appendix B. Tumarkin's Coxeter pyramid groups

The classification of the cofinite Coxeter groups in $\operatorname{Isom} \mathbb{H}^n$ of rank $n+2$ whose fundamental polyhedra are pyramids over a product of two simplices of positive dimensions is due to Tumarkin [52], [54]. The results are summarised in the following three tables (see [54, Section 4]).

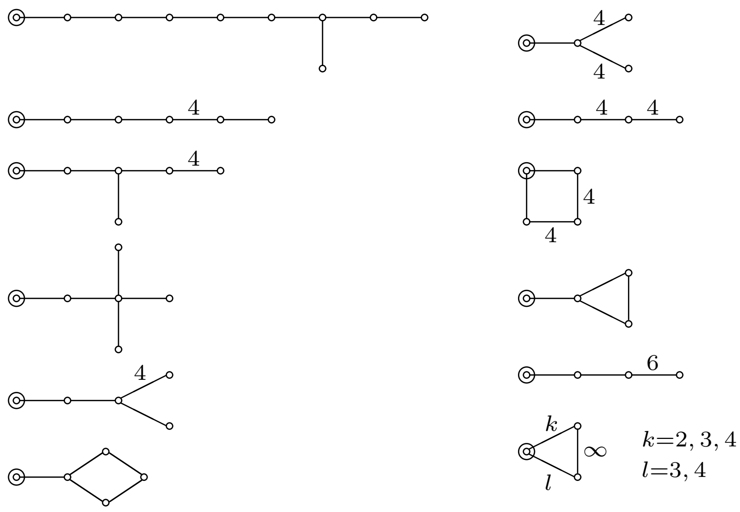


Figure 27: Glueing together any two graphs by the encircled node yields the graph of a hyperbolic Coxeter pyramid group

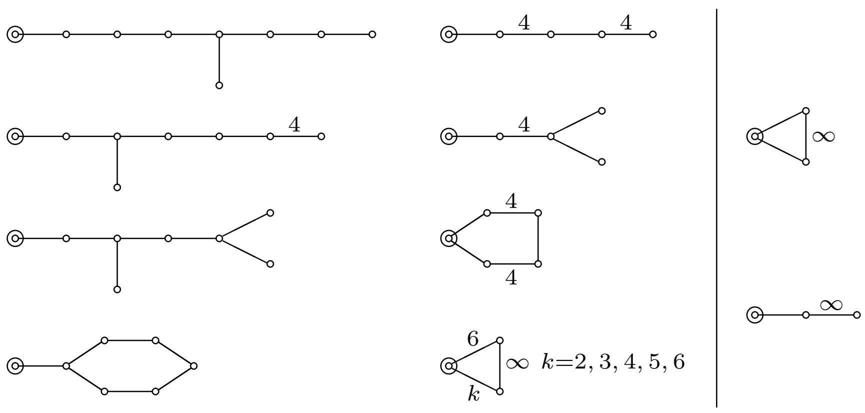


Figure 28: Glueing together any graph from the left side with any graph from the right side by identifying the encircled node yields the graph of a hyperbolic Coxeter pyramid group

Name	Coxeter symbol	Volume	Value $\simeq$
R_1	$[3,3,6]$	$\frac{1}{8}$ Л$(\frac{1}{3}\pi)$	0.0422892336
R_6	$[3,4,4]$	$\frac{1}{6}$ Л$(\frac{1}{4}\pi)$	0.0763304662
S_1	$[3,3^{[3]}]$	$\frac{1}{4}$ Л$(\frac{1}{3}\pi)$	0.0845784672
R_4	$[4,3,6]$	$\frac{5}{16}$ Л$(\frac{1}{3}\pi)$	0.1057230840
S_6	$[3,4^{1,1}]$	$\frac{1}{3}$ Л$(\frac{1}{4}\pi)$	0.1526609324
R_2	$[3,6,3]$	$\frac{1}{2}$ Л$(\frac{1}{3}\pi)$	0.1691569344
R_5^*	$[5,3,6]$	$\frac{1}{2}$ Л$(\frac{1}{3}\pi)+\frac{1}{4}$ Л$(\frac{11}{30}\pi)-\frac{1}{4}$ Л$(\frac{1}{30}\pi)$	0.1715016613
S_2	$[4,3^{[3]}]$	$\frac{5}{8}$ Л$(\frac{1}{3}\pi)$	0.2114461680
S_5	$[6,3^{1,1}]$	$\frac{5}{8}$ Л$(\frac{1}{3}\pi)$	0.2114461680
R_7	$[4,4,4]$	$\frac{1}{2}$ Л$(\frac{1}{4}\pi)$	0.2289913985
R_3	$[6,3,6]$	$\frac{3}{4}$ Л$(\frac{1}{3}\pi)$	0.2537354016
T_5	$[(3^2,4^2)]$	$\frac{2}{3}$ Л$(\frac{1}{4}\pi)$	0.3053218647
S_3^*	$[5,3^{[3]}]$	Л$(\frac{1}{3}\pi)+\frac{1}{2}$ Л$(\frac{11}{30}\pi)-\frac{1}{2}$ Л$(\frac{1}{30}\pi)$	0.3430033226
T_1^*	$[(3^3,6)]$	$\frac{5}{8}$ Л$(\frac{1}{3}\pi)+\frac{1}{3}$ Л$(\frac{1}{4}\pi)$	0.3641071004
U	$[3^{[\,]\times[\,]}]$	$\frac{5}{4}$ Л$(\frac{1}{3}\pi)$	0.4228923360
S_7	$[4^{1,1,1}]$	Л$(\frac{1}{4}\pi)$	0.4579827971
S_4	$[6,3^{[3]}]$	$\frac{3}{2}$ Л$(\frac{1}{3}\pi)$	0.5074708032
T_2^*	$[(3,4,3,6)]$	$\frac{13}{16}$ Л$(\frac{1}{3}\pi)-\frac{1}{6}$ Л$(\frac{1}{4}\pi)+\frac{2}{3}$ Л$(\frac{1}{8}\pi)$	0.5258402692
T_6^*	$[(3,4^3)]$	$\frac{1}{2}$ Л$(\frac{1}{4}\pi)+\frac{2}{3}$ Л$(\frac{1}{8}\pi)$	0.5562821156
T_3^*	$[(3,5,3,6)]$	Л$(\frac{1}{3}\pi)+$Л$(\frac{11}{60}\pi)-$Л$(\frac{1}{60}\pi)$	0.6729858045
T_4	$[(3,6)^{[2]}]$	$\frac{5}{2}$ Л$(\frac{1}{3}\pi)$	0.8457846720
T_7	$[4^{[4]}]$	2 Л$(\frac{1}{4}\pi)$	0.9159655942
V	$[3^{[3,3]}]$	3 Л$(\frac{1}{3}\pi)$	1.0149416064

Table 5: Covolumes of the 23 non-compact hyperbolic Coxeter tetrahedral groups – arranged by increasing order

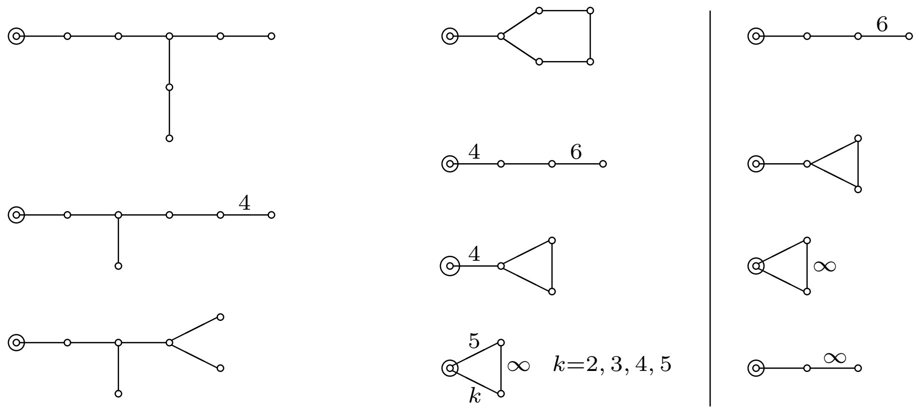


Figure 29: Glueing together any graph from the left side with any graph from the right side by identifying the encircled node yields the graph of a hyperbolic Coxeter pyramid group

§ Appendix C. Commensurability classes of arithmetic hyperbolic Coxeter pyramid groups

n	$\mathcal{A}_n^1 \div \alpha_n^1$	$\mathcal{A}_n^2 \div \alpha_n^2$	$\mathcal{A}_n^3 \div \alpha_n^3$	$\mathcal{A}_n^4 \div \alpha_n^4$
3	$[(3,\infty,3),(4,\infty,4)]$ 4	$[(3,\infty,3),(6,\infty,6)]$ 4	$[(3,\infty,3),(3,\infty,3)]$ 6	
4	$[6,3,3,3,\infty]$ 4	$[4,4,3,3,\infty]$ 20		
5	$[3^{[3]},3^2,3^{[3]}]$ 3	$[3^{[4]},3,(3,\infty,3)]$ 4	$[(3,4^2,3),3,3^{[3]}]$ 6	$[(3,4^2,3),(3,4^2,3)]$ 20
6	$[3^{[5]},3,(3,\infty,3)]$ 2	$[3^{[4]},3^2,3^{[3]}]$ 4	$[3^{[4]},3,(3,4^2,3)]$ 18	
7	$[3^{[5]},3^2,3^{[3]}]$ 2	$[3^{1,1},3^{1,2},(3,\infty,3)]$ 4	$[3^{[6]},3,(3,\infty,3)]$ 8	$[3^{[4]},3^2,3^{[4]}]$ 12
8	$[3^{2,2},3^3,(3,\infty,3)]$ 16			
9	$[3^{2,2},3^4,3^{[3]}]$ 10			
10	$[3^{2,1},3^6,(3,\infty,3)]$ 4			
11	$[3^{2,1},3^7,(3,\infty,3)]$ 2	$[3^{2,1},3^6,(3,4^2,3)]$ 3		
12	$[3^{2,1},3^6,3^{[4]}]$ 2			
13	$[3^{2,1},3^8,3^{1,1,1}]$ 3			
17	$[3^{2,1},3^{12},3^{1,2}]$ 1			

Table 6: Commensurability classes $\mathcal{A}_n^k$ with representatives and cardinalities α_n^k in the arithmetic case

§ Appendix D. Commensurability classes of non-arithmetic hyperbolic Coxeter pyramid groups

n	$\lvert\mathcal{N}_n\rvert = 1$	$\lvert\mathcal{N}_n\rvert = 2$	$\lvert\mathcal{N}_n\rvert = 3$	$\lvert\mathcal{N}_n\rvert = 4$
3	$[(3,\infty,4),(3,\infty,4)]$	$[\infty,3,(3,\infty,k)]$ for $k=4,5,6$ $[\infty,3,(l,\infty,m)]$ for $4 \leq l < m \leq 6$ $[\infty,4,(3,\infty,4)]$		$[\infty,3,5,\infty]$
4		$[6,3^2,(k,\infty,l)]$ for $3 \leq k < l \leq 5$	$[4^2,3,(3,\infty,4)]$	$[6,3^2,5,\infty]$
5		$[4,3^{2,1},(3,\infty,4)]$		
6			$[3,4,3^3,(3,\infty,4)]$	
10	$[3^{2,1},3^6,(3,\infty,4)]$			

Table 7: Commensurability classes $\mathcal{N}_n$ in the non-arithmetic case

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