Algebraic aspects of hyperbolic volume

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When considering volumes of hyperbolic orbifolds and manifolds $Q = \mathbb{H}^n/\Gamma$ as given by the volumes of fundamental polyhedra for the discontinuous action of the group $\Gamma \subset \mathrm{Isom}\mathbb{H}^n$, several algebraic aspects (beside the obvious analytical ones) appear. A rough picture is provided by the following chart.

polylogarithms

Bloch group | *ζ_F- and L-values*

Dehn invariants | **Hyperbolic volume** | *Prasad's formula*

Hilbert's 3rd Problem | *small volume space forms*

polyhedral volume
Schläfli's formula | *(non-)arithmetic discrete*
reflection groups

Euler characteristic

In even dimensions, by the (generalised) Theorem of Gauss-Bonnet, the volume of Q is proportional to the Euler characteristic $\chi(Q)$ which can be computed in different combinatorial and homological ways. When $n = 2$, there is a very satisfactory picture about the area spectrum of hyperbolic orbifolds. For $n \geq 4$ and if Q is an orientable and arithmetically defined n-orbifold, one has Prasad's important volume formula. For even $n \geq 4$, Belolipetsky in 2004, 2006 exploited this formula and deduced explicit values for the minimal volume orbifolds (for a survey with more details, see [4], for example). In particular, Belolipetsky showed that the quotient of $\mathbb{H}^4$ by the rotation subgroup of the arithmetic discrete reflection group with fundamental Coxeter simplex S given by the Coxeter diagram $[5,3,3,3]$ is the unique compact oriented arithmetic orbifold of minimal volume. Other optimality results in dimension 4 are known when looking at arbitrary orbifolds and manifolds with cusps (see [3] and [6]). Furthermore, the identification of minimal volume cusped orientable arithmetic hyperbolic n-orbifolds of even dimension up to $n = 18$ has recently been established in [5]. However, in the compact smooth case, the smallest known 4-manifold has Euler characteristic equal to 8, only, and was constructed by Conder and Maclachlan [1]. In fact, by means of the computer package MAGMA, they found a suitable torsionfree subgroup of the Coxeter simplex group given by $[5,3,3,3]$ above.

In odd dimensions, there is a structural difference in view of the volume spectra of hyperbolic orbifolds and manifolds for $n = 3$ and $n > 3$, respectively, and much more is known in the low dimensional case $n = 3$. By results of Jørgensen, Thurston and Gromov, the volume spectrum for hyperbolic 3-manifolds is non-discrete (well-ordered of order type ω^ω and with limits points given by cusped manifolds) while, by a result of Wang, the volume spectrum for hyperbolic n-manifolds is discrete if $n \neq 3$. Furthermore, minimality results are known for the small part of all the diverse restricted volume spectra in dimension 3 (see also [4]). For example, the Gieseking manifold built from the ideal regular tetrahedron is the unique cusped 3-manifold of minimal volume. However, an analogous optimality result in the non-compact case for $n = 5$ is not known.

In [2, Theorem 1.3], Goncharov proved a substantial generalisation to $n = 5$ of a result of Dupont, Sah, Neumann and Thurston for $n = 3$ which can be stated in terms of the Bloch group $B_2(F)$ of a field F as follows. For each finite volume hyperbolic 5-manifold M there are finitely many algebraic numbers z_i, $i \in I$, satisfying an algebraic identity of the form

$$\sum_{i\in I} \{z_i\}_2 \otimes z_i = 0 \quad \text{in} \quad B_2(\overline{\mathbb{Q}}) \otimes \overline{\mathbb{Q}}^* \ , \tag{1}$$

in such a way that

$$\mathrm{vol}_5(M) = \sum_{i\in I} \mathcal{L}_3(z_i) \ ,$$

where $\mathcal{L}_3(z)$ denotes a certain generalised trilogarithm function.
However, for Goncharov's result, in particular in the compact setting, we do not dispose of any non-trivial example in view of (1) (meaning $z_1 \neq 1$ and $|I| > 1$).

There is a construction due to Ratcliffe and Tschantz [7], [8] of a cusped hyperbolic 5-manifold M_1 (in-)directly related to the ideal right-angled polyhedron P^5 of Vinberg. They computed the volume of the manifold M_1 and obtained $7\zeta(3)/4$ (notice that $\mathcal{L}_3(1) = \zeta(3)$). In the ongoing work together with Conder, we constructed a cusped hyperbolic 5-manifold M_0 with a (conjectural) big degree of symmetry whose volume should be comparatively small.

References

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