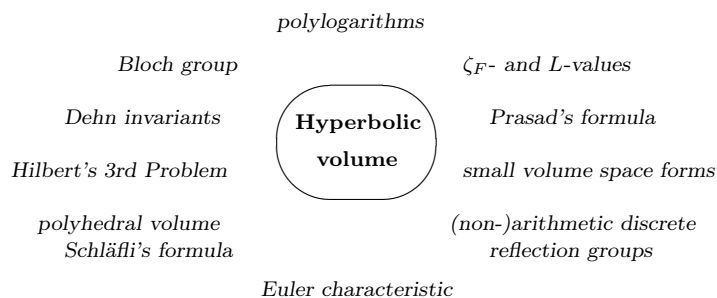


## Algebraic aspects of hyperbolic volume

RUTH KELLERHALS

When considering volumes of hyperbolic orbifolds and manifolds  $Q = \mathbb{H}^n/\Gamma$  as given by the volumes of fundamental polyhedra for the discontinuous action of the group  $\Gamma \subset \text{Isom}\mathbb{H}^n$ , several algebraic aspects (beside the obvious analytical ones) appear. A rough picture is provided by the following chart.



In even dimensions, by the (generalised) Theorem of Gauss-Bonnet, the volume of  $Q$  is proportional to the Euler characteristic  $\chi(Q)$  which can be computed in different combinatorial and homological ways. When  $n = 2$ , there is a very satisfactory picture about the area spectrum of hyperbolic orbifolds. For  $n \geq 4$  and if  $Q$  is an orientable and arithmetically defined  $n$ -orbifold, one has Prasad's important volume formula. For even  $n \geq 4$ , Belolipetsky in 2004, 2006 exploited this formula and deduced explicit values for the minimal volume orbifolds (for a survey with more details, see [4], for example). In particular, Belolipetsky showed that the quotient of  $\mathbb{H}^4$  by the rotation subgroup of the arithmetic discrete reflection group with fundamental Coxeter simplex  $S$  given by the Coxeter diagram  $[5, 3, 3, 3]$  is the unique compact oriented arithmetic orbifold of minimal volume. Other optimality results in dimension 4 are known when looking at arbitrary orbifolds and manifolds with cusps (see [3] and [6]). Furthermore, the identification of minimal volume cusped orientable arithmetic hyperbolic  $n$ -orbifolds of even dimension up to  $n = 18$  has recently been established in [5]. However, in the compact smooth case, the smallest known 4-manifold has Euler characteristic equal to 8, only, and was constructed by Conder and Maclachlan [1]. In fact, by means of the computer package MAGMA, they found a suitable torsionfree subgroup of the Coxeter simplex group given by  $[5, 3, 3, 3]$  above.

In odd dimensions, there is a structural difference in view of the volume spectra of hyperbolic orbifolds and manifolds for  $n = 3$  and  $n > 3$ , respectively, and much more is known in the low dimensional case  $n = 3$ . By results of Jørgensen, Thurston and Gromov, the volume spectrum for hyperbolic 3-manifolds is non-discrete (well-ordered of order type  $\omega^\omega$  and with limits points given by cusped manifolds) while, by a result of Wang, the volume spectrum for hyperbolic  $n$ -manifolds is discrete if  $n \neq 3$ . Furthermore, minimality results are known for the

small part of all the diverse restricted volume spectra in dimension 3 (see also [4]). For example, the Gieseking manifold built from the ideal regular tetrahedron is the unique cusped 3-manifold of minimal volume. However, an analogous optimality result in the non-compact case for  $n = 5$  is not known.

In [2, Theorem 1.3], Goncharov proved a substantial generalisation to  $n = 5$  of a result of Dupont, Sah, Neumann and Thurston for  $n = 3$  which can be stated in terms of the Bloch group  $B_2(F)$  of a field  $F$  as follows. For each finite volume hyperbolic 5-manifold  $M$  there are finitely many algebraic numbers  $z_i, i \in I$ , satisfying an algebraic identity of the form

$$(1) \quad \sum_{i \in I} \{z_i\}_2 \otimes z_i = 0 \quad \text{in} \quad B_2(\overline{\mathbb{Q}}) \otimes \overline{\mathbb{Q}}^*,$$

in such a way that

$$\text{vol}_5(M) = \sum_{i \in I} \mathcal{L}_3(z_i),$$

where  $\mathcal{L}_3(z)$  denotes a certain generalised trilogarithm function.

However, for Goncharov's result, in particular in the compact setting, we do not dispose of any non-trivial example in view of (1) (meaning  $z_1 \neq 1$  and  $|I| > 1$ ).

There is a construction due to Ratcliffe and Tschantz [7], [8] of a cusped hyperbolic 5-manifold  $M_1$  (in-)directly related to the ideal right-angled polyhedron  $P^5$  of Vinberg. They computed the volume of the manifold  $M_1$  and obtained  $7\zeta(3)/4$  (notice that  $\mathcal{L}_3(1) = \zeta(3)$ ). In the ongoing work together with Conder, we constructed a cusped hyperbolic 5-manifold  $M_0$  with a (conjectural) big degree of symmetry whose volume should be comparatively small.

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