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Calculations in Teichmüller TQFT

Rinat Kashaev
(joint work with Joergen Andersen)

During the last 3 years I and J.E. Andersen formulated a TQFT model based on the quantum Teichmüller theory. The model is based on a special function called Faddeev's quantum dilogarithm defined by the formula

$$\Phi_b(x) = \exp\left(\frac{1}{4}\int_C \frac{e^{2ixz}}{\sinh(zb)\sinh(zb^{-1})}\frac{dz}{z}\right)$$

Where the contour C is given by $C = \mathbb{R} + i\epsilon$. The function is extended to the complex plane by analytic continuation.

The geometric input is given by a triangulated 3-manifold where each tetrahedron is provided by the structure of an ideal hyperbolic tetrahedron so that we have a 3-manifold with conical singularities along some of the edges of the triangulation along which the total dihedral angle is different from 2π. By calculations in particular simple examples we conjecture that the partition function of our model decays exponentially, the decay rate being given by the hyperbolic volume of the corresponding (underlying) 3-manifold with conical singularities. Namely the formula reads as follows:

$$|Z_b(x)| \sim e^{-\mathrm{vol}(X)/2\pi b^2} \quad b \to 0$$

Commensurability of hyperbolic Coxeter groups

Ruth Kellerhals
(joint work with Rafael Guglielmetti, Matthieu Jacquemet)

This work deals with the determination of the wide commensurability classes of a certain large family $\mathcal{P}$ of discrete groups of isometries of n-dimensional hyperbolic space $\mathbb{H}^n$. For $n > 2$ this family consists of all hyperbolic Coxeter n-pyramid groups of finite covolume. It is a finite set as shown by Tumarkin [11], [12] who listed them in 2004. For the basic notions of geometric Coxeter group theory, including Coxeter graphs, combinatorics, criteria for finite covolume (cofiniteness) and arithmeticity, we refer to Vinberg's seminal work as summarised in [14] and [15].

In the sequel we abbreviate the terminology and shall use the term *commensurable* for two groups $\Gamma_1, \Gamma_2 \subset \mathrm{Isom}(\mathbb{H}^n)$ if there exists an element $\gamma \in \mathrm{Isom}(\mathbb{H}^n)$ such that the intersection $\Gamma_1 \cap \gamma\Gamma_2\gamma^{-1}$ is of finite index in both Γ_1 and $\gamma\Gamma_2\gamma^{-1}$. In particular, any subgroup of finite index is commensurable to its supergroup. Furthermore, commensurability is an equivalence relation preserving properties such as cocompactness, cofiniteness and arithmeticity.

For $n = 2$ and $n = 3$, any discrete subgroup $\Gamma \subset \mathrm{PSL}_2(k)$ of orientation preserving hyperbolic isometries, where $k = \mathbb{R}$ or $k = \mathbb{C}$, gives rise to a subalgebra of the matrix group $\mathrm{M}_2(k)$, and is in fact a quaternion algebra. For *arithmetic* subgroups $\Gamma \subset \mathrm{PSL}_2(k)$, these quaternion algebras are defined over number fields, and their classification up to commensurability corresponds to the classification up to isomorphism of the quaternion algebras (see also [13]). In this way, Takeuchi [10] classified the arithmetic triangle groups while Maclachlan and Reid [7] determined the commensurability classes of all cocompact arithmetic Coxeter tetrahedral groups.

In [4] and [5], together with Johnson, Ratcliffe and Tschantz, we determined all subgroup relations and covolumes of hyperbolic Coxeter n-simplex groups and classified them up to commensurability. These are groups generated by the reflections in the hyperplanes bounding hyperbolic n-simplices whose dihedral angles are all of the form π/m for an integer $m \geq 2$ and which are of finite volume. When assuming $n > 2$, this family comprises finitely many examples, including some non-arithmetic ones. Notice that they exist in $\mathrm{Isom}(\mathbb{H}^n)$ for $n \leq 9$, only.

Let us return to the class $\mathcal{P}$ of Tumarkin's hyperbolic Coxeter n-pyramid groups of finite covolume. They are generated by $n + 2$ reflections in the hyperplanes bounding an n-dimensional Coxeter pyramid with an apex on the boundary $\partial\mathbb{H}^n$ at infinity whose horospherical neighborhood is a product of two simplices, each of dimension ≥ 2. A nice combinatorial-metrical feature of such a pyramid is that it relates to a hyperbolic truncated Coxeter simplex (for more details, see [3]). Observe that there is no classification of hyperbolic Coxeter groups with more than $n+2$ generators which are not cocompact but of finite volume. By Vinberg's arithmeticity criterion, one checks easily that there are non-arithmetic elements in $\mathcal{P}$. For example, the group $\Gamma_4 \subset \mathrm{Isom}(\mathbb{H}^{10})$ described by the Coxeter graph in Figure 1 is the top-dimensional non-arithmetic group in $\mathcal{P}$. Observe that the group $\Gamma_3 \subset \mathrm{Isom}(\mathbb{H}^{10})$ given by the same graph after replacement of the edge with weight 4 by an edge without weight (or equivalently by an edge with weight 3) is an arithmetic element in $\mathcal{P}$.

The classification results of Tumarkin show that $\mathcal{P}$ contains groups acting on $\mathbb{H}^n$ for $n \leq 17$, only.

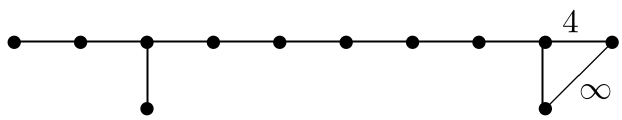

FIGURE 1. The non-arithmetic Coxeter pyramid group acting on $\mathbb{H}^{10}$

In fact, the group Γ_* given by the Coxeter graph in Figure 2 is the (single) top-dimensional group in $\mathcal{P}$. The orientation preserving subgroup Γ'_* of Γ_* is distinguished by the amazing fact that the quotient space $\mathbb{H}^n/\Gamma'_*$ built upon the 17-dimensional pyramid P_* has minimal volume among ALL orientable arithmetic hyperbolic n-orbifolds, and that it is as such unique. This result is due to Emery [1] who computed the minimal volume according to

$$\mathrm{vol}_{17}(\mathbb{H}^n/\Gamma'_*) = \mathrm{vol}_{17}(P_*) = \frac{691 \cdot 2617}{2^{38} \cdot 3^{10} \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17}\,\zeta(9)$$

by a clever exploitation of Prasad's volume formula and other sophisticated tools.

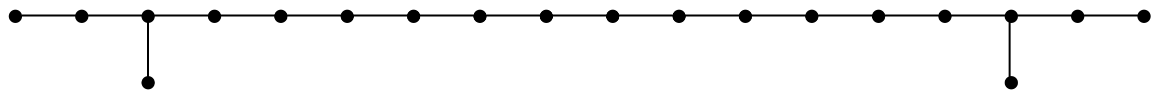

Figure 2. The single Coxeter pyramid group Γ_* acting on $\mathbb{H}^{17}$

Let us return to and discuss the commensurability classification of elements in $\mathcal{P}$. For the large subset of arithmetic groups in $\mathcal{P}$, we exploit the results of Maclachlan [7] about commensurability of discrete arithmetic hyperbolic groups in the special case of Coxeter groups in $\mathcal{P}$. The results in [7] show that the commensurability classes in $\mathcal{P}$ are in one-to-one correspondence with the isomorphism classes of quaternion algebras over certain number fields. We determined explicitly the classes and identified representatives in terms of certain hyperbolic Coxeter simplex groups whenever possible. Some representatives are related to Mcleod's Coxeter groups which appear as a subgroup of the automorphism group of the Lorentzian lattice $-3x_0^2 + x_1^2 + \cdots + x_n^2$ for certain n. All their covolumes were determined by Ratcliffe and Tschantz [9]. Therefore, by exploiting the volume results in [4] and [9], we can proceed in order to list the volumes of all arithmetic Coxeter pyramids and to find all subgroup relations.

In the case of the non-arithmetic elements in $\mathcal{P}$ which show up mainly in low dimensions, different ad hoc methods such as scissors congruences, glueings in the spirit of Gromov-Piatetski Shapiro [2] for groups such as Γ_l for $l = 2, 3, 4$, volume computations based on Schläfli's volume differential and ratio tests, are involved. This part of the work is also much inspired by [5].

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From isolated codes to isolated 3-manifolds

Matthias Kreck

After briefly explaining linear codes and the relation between self-dual codes and unimodular lattices I report about realization of codes by manifolds with a $\mathbb{Z}/p$-action. Then I address the question of explicit construction of manifolds with involution realizing special codes, in particular the Golay code whose unimodular lattice is the Leech lattice. Then I report a result by Manin about the distribution of codes where he defines isolated codes. This leads to the definition of isolated manifolds whic means it has a $\mathbb{Z}/p$-action and that the corresponding code is isolated. Only a few explicit examples of isolated manifolds are known. We finished by posing some obvious questions.

On Kauffman Bracket Skein Modules at Root of Unity

Thang Lê

1. Definitions

1.1. **Skein modules.** Suppose M is an oriented compact 3-manifold. The Kauffman bracket skein module (J. Przytycki, V. Turaev) of M is defined by

$$\mathcal{S}(M) = \mathbb{C}[t^{\pm 1}] - \text{span of framed unoriented links in } M/\text{relations (1) \& (2)}.$$

Here

$$L = tL_+ + t^{-1}L_- \tag{1}$$

$$L \sqcup U = -(t^2+t^{-2})L, \tag{2}$$