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# On hyperbolic Coxeter $n$-cubes

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For my grandfather Charles

**ABSTRACT**

Besides simplices, $n$-cubes form an important class of simple poly-hedra. Unlike hyperbolic Coxeter simplices, hyperbolic Coxeter $n$-cubes are not classified. In this work, we first show that there are no Coxeter $n$-cubes in $\mathbb{H}^n$ for $n \geq 10$. Then, we show that the *ideal* ones exist only for $n = 2$ and 3, and provide a classification. The methods used are of combinatorial and algebraic nature, using properties of a Coxeter graph, its Schläfli matrix, and the Gram matrix of a polyhedron.

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## 1. Introduction

Let $\mathbb{H}^n$ be the $n$-dimensional real hyperbolic space. A *hyperbolic Coxeter polyhedron* $\mathcal{P} \subset \mathbb{H}^n$ is a finite-volume convex polyhedron whose angles are of the form $\frac{\pi}{k}$ for some $k \in \{2, \dots, \infty\}$. Identifying the facets of a hyperbolic Coxeter polyhedron by using the reflections in their supporting hyperplanes is a simple way to construct hyperbolic $n$-manifolds and $n$-orbifolds. In the known cases, such polyhedra are responsible for minimal volume hyperbolic manifolds and orbifolds [13].

In contrast to the spherical and Euclidean cases, hyperbolic Coxeter polyhedra cannot exist any more if $n \geq 996$ and they are far from being classified. In fact, comprehensive lists are available only if the number of facets of $\mathcal{P}$ equals $n + 1$ or $n + 2$ [16]. For example, hyperbolic Coxeter simplices exist only for $2 \leq n \leq 9$ (see [17, pp. 205–208]).

Hyperbolic $n$-cubes are simple polyhedra bounded by $2n$ facets in $\mathbb{H}^n$. Unlike simplices, they have no simplex facet. Hyperbolic manifolds and orbifolds can be constructed by identifying isometric facets of hyperbolic $n$-cubes in a suitable way. This has been performed by Aitchison–Rubinstein [1] and Everitt [4] to produce complete hyperbolic 3-manifolds by using regular ideal 3-cubes, and by Choi–Hodgson–Lee [3] to construct hyperbolic 3-orbifolds by using compact hyperbolic Coxeter

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3-cubes with angles $\pi/2$ or $\pi/3$. Notably, hyperbolic manifolds which can be decomposed into regular ideal cubes (so-called *cubical manifolds*) are not necessarily decomposable into regular ideal tetrahedra (see [6, Remark 3.7], for example).

In this paper, we shall study and partially classify hyperbolic Coxeter $n$-cubes. The methods which we develop are essentially of combinatorial and algebraic nature, using *Coxeter graphs* and their *Schläfli matrix*. The fact that a cube can be interpreted as the intersection of two simplicial cones makes this approach particularly tractable.

We first show that there is no hyperbolic Coxeter $n$-cube in $\mathbb{H}^n$ for $n \geq 10$. Next, we focus on *ideal* hyperbolic Coxeter $n$-cubes. We show that they exist only for $n = 2$ and $n = 3$, and classify them completely. In particular, there is a one-parameter family of ideal hyperbolic Coxeter squares, and there are precisely seven ideal hyperbolic Coxeter 3-cubes. We compute the volumes of the ideal hyperbolic Coxeter 3-cubes, which turn out to be rational multiples either of $Л(\frac{\pi}{3})$ or of $Л(\frac{\pi}{4})$, where $Л$ is the Lobachevsky function. Finally, we discuss briefly the problem of classifying non-ideal hyperbolic Coxeter $n$-cubes, and exhibit three infinite families of (compact and non-compact) hyperbolic Coxeter 3-cubes.

## 2. Hyperbolic Coxeter *n*-cubes

We denote by $\mathbb{H}^n$ the hyperbolic space of dimension $n$, and by $\partial\mathbb{H}^n$ its boundary. We set $\overline{\mathbb{H}^n} = \mathbb{H}^n \cup \partial\mathbb{H}^n$.

### 2.1. Hyperbolic Coxeter polyhedra

Let $\mathbb{X}^n \in \{\mathbb{S}^n, \mathbb{E}^n, \mathbb{H}^n\}$ be one of the three standard geometric spaces of constant curvature. A *Coxeter (n-)polyhedron* is a convex, finite-volume $n$-polyhedron $\mathcal{P} \subset \mathbb{X}^n$ whose dihedral angles are of the form $\frac{\pi}{k_{ij}}$, for $k_{ij} \in \{2, \ldots, \infty\}$. Standard references about Coxeter polyhedra and their properties are [15,17].

In the sequel, we assume that $\mathcal{P} \subset \mathbb{H}^n$ is a hyperbolic Coxeter polyhedron (of finite volume). Then, it is bounded by finitely many hyperplanes, say $H_1, \ldots, H_N$. For $i = 1, \ldots, N$, the *facet (or $(n-1)$-face) $F_i$* of $\mathcal{P}$ is the intersection $F_i = \mathcal{P} \cap H_i$. For $0 \leq k \leq n-2$, a *k-face* of $\mathcal{P}$ is a facet of a $(k+1)$-face of $\mathcal{P}$. A *vertex* is a 0-face, and an *edge* is a 1-face of $\mathcal{P}$.

If a vertex $v$ of $\mathcal{P}$ lies on $\partial\mathbb{H}^n$ we call $v$ an *ideal vertex* of $\mathcal{P}$. Then, $\mathcal{P}$ is a *non-compact* polyhedron. If all vertices of $\mathcal{P}$ lie on $\partial\mathbb{H}^n$, then $\mathcal{P}$ is said to be *ideal*.

Let $v \in \mathbb{H}^n$ be an ordinary vertex of $\mathcal{P}$. The *vertex figure, or link, $L(v)$* of $v$ is the intersection

$$L(v) = \mathcal{P} \cap \mathcal{S}_\rho(v),$$

where $\mathcal{S}_\rho(v)$ is a sphere with centre $v$ and radius $\rho > 0$ not containing any other vertex of $\mathcal{P}$ and not intersecting any facet of $\mathcal{P}$ not incident to $v$. In particular, $L(v)$ is a spherical Coxeter $(n-1)$-polyhedron.

Similarly, the link $L(v)$ of an ideal vertex $v \in \partial\mathbb{H}^n$ is defined as the intersection of $\mathcal{P}$ with a sufficiently small horosphere centred at $v$. In particular, $L(v)$ is a Euclidean Coxeter $(n-1)$-polyhedron [17, I, Chapter 6.2].

The Gram matrix of $\mathcal{P}$ is the matrix $G = G(\mathcal{P}) = (g_{ij})_{1 \leq i,j \leq N}$ given by

$$g_{ij} = \begin{cases} 1, & \text{if } j = i, \\ -\cos\dfrac{\pi}{k_{ij}}, & \text{if } H_i \text{ and } H_j \text{ intersect in } \mathbb{H}^n \text{ with angle } \dfrac{\pi}{k_{ij}}, \\ -1, & \text{if } H_i \text{ and } H_j \text{ are parallel}, \\ -\cosh l_{ij}, & \text{if } d(H_i, H_j) = l_{ij} > 0. \end{cases}$$

The matrix $G$ is real, symmetric, and of signature $(n, 1)$ [17, Chapter 6.2].

A Coxeter polyhedron $\mathcal{P} \subset \mathbb{H}^n$ is often described by its *Coxeter graph* $\Gamma = \Gamma(\mathcal{P})$ as follows. A node $i$ in $\Gamma$ represents the bounding hyperplane $H_i$ of $\mathcal{P}$. Two nodes $i$ and $j$ are joined by an edge

with weight $2 \leq k_{ij} \leq \infty$ if $H_i$ and $H_j$ intersect in $\overline{\mathbb{H}^n}$ with angle $\frac{\pi}{k_{ij}}$. If the hyperplanes $H_i$ and $H_j$ have a common perpendicular of length $l_{ij} > 0$ in $\mathbb{H}^n$, the nodes $i$ and $j$ are joined by a dotted edge, sometimes labelled cosh $l_{ij}$. In practice and in the following discussion, an edge of weight 2 is omitted, and an edge of weight 3 is written without its weight. The *rank* of $\Gamma$ denotes the number of its nodes.

The Coxeter graphs of indecomposable spherical (resp. Euclidean) Coxeter polyhedra are well known. These polyhedra exist in any dimension and are completely classified. The corresponding graphs are called *elliptic* (resp. *parabolic*). They can be found in [17, pp. 202–203], for example.

## 2.2. Hyperbolic Coxeter n-cubes

A hyperbolic *n-cube*, $n \geq 2$, is a polyhedron $\mathcal{C} \subset \mathbb{H}^n$ whose closure $\overline{\mathcal{C}} \subset \overline{\mathbb{H}^n}$ is combinatorially equivalent to the standard cube $[0, 1]^n \subset \mathbb{R}^n$. In particular, an *n*-cube has $2^n$ vertices, and is bounded by $n$ pairs of mutually disjoint hyperplanes. It is *cubical*, i.e. its $k$-faces are $k$-cubes, $2 \leq k \leq n - 1$. Moreover, the number $f_k(\mathcal{C})$ of $k$-faces of $\mathcal{C}$ is given by (see [7, Chapter 4.4], for example)

$$f_k(\mathcal{C}) = 2^{n-k} \binom{n}{k}, \quad 0 \leq k \leq n.$$

For $n \geq 2$, let $\mathcal{C} \subset \mathbb{H}^n$ be an *n*-cube bounded by hyperplanes $H_1, \ldots, H_{2n}$ such that the hyperplane $H_i$ intersects all hyperplanes except $H_{2n-i+1}$ for $i = 1, \ldots, 2n$. The set $\mathfrak{H} = \{H_1, \ldots, H_{2n}\}$ can be partitioned into two families of $n$ concurrent hyperplanes in $2^{n-1}$ different ways. Let $\mathfrak{H} = \mathfrak{H}_1 \sqcup \mathfrak{H}_2$ be such a partition. Then, for $i = 1, 2$, the hyperplanes in $\mathfrak{H}_i$ form a simplicial cone in $\mathbb{H}^n$ based at a vertex of $\mathcal{C}$, say $\mathfrak{v}_i$. The vertices $\mathfrak{v}_1$ and $\mathfrak{v}_2$ lie on a (spatial) diagonal of $\mathcal{C}$. We say that they are *opposite* in $\mathcal{C}$. In this way, we can label the vertices $p_1, \ldots, p_{2^n}$ of $\mathcal{C}$ such that $p_i$ and $p_{2^n-i+1}$ are opposite in $\mathcal{C}$, $i = 1, \ldots, 2^n$. For example, one can write

$$p_1 = \bigcap_{i=1}^{n} H_i \quad \text{and} \quad p_{2^n} = \bigcap_{i=n+1}^{2n} H_i.$$

Moreover, for any vertex $p_i \in \mathcal{C}$, the graph of the vertex figure $F(p_i)$ is the subgraph of $\Gamma(\mathcal{C})$ of rank $n$ spanned by the nodes representing the hyperplanes in $\mathfrak{H}$ which contain $p_i$.

**Theorem 1.** *There are no Coxeter n-cubes in $\mathbb{H}^n$ for $n \geq 10$.*

**Proof.** Let $\mathcal{C} \subset \mathbb{H}^n$ be a Coxeter *n*-cube with graph $\Gamma = \Gamma(\mathcal{C})$. Let $V = V(\Gamma)$ be the set of vertices of $\Gamma$ and $E = E(\Gamma)$ its set of edges (that is, its dotted edges and its solid edges, of weight greater than or equal to 3). Then, $|V| = 2n$ and $|E| \leq \binom{2n}{2} = n(2n - 1)$. Let $e$ be the number of edges of $\Gamma$ which are *not* dotted edges. Then $e \leq 2n(n - 1)$, since $\Gamma$ contains exactly $n$ dotted edges.

Because the associated Gram matrix $G = G(\mathcal{C}) = G(\Gamma)$ has signature $(n, 1)$, it follows that if $\Gamma' \subset \Gamma$ is a rank-4 subgraph having two dotted edges, then $\Gamma'$ is connected (see also [17, I, Chapter 6.2]). Hence, one must have

$$\binom{n}{2} = \frac{n(n-1)}{2} \leq e. \tag{1}$$

Moreover, the graph of the figure of any of the $2^n$ vertices of $\mathcal{C}$ is a subgraph of $\Gamma$ of rank $n$ which is either elliptic or parabolic. Conversely, any rank-$n$ subgraph $\Gamma'$ of $\Gamma$ not containing a dotted edge is the graph of the figure of a vertex of $\mathcal{C}$ (and is therefore either elliptic or parabolic). Observe that such $\Gamma'$ is spanned by the vertices obtained by selecting for each dotted edge of $\Gamma$ one of the two vertices it connects. Hence, any non-dotted edge of $\Gamma$ belongs to the graph of precisely $2^{n-2}$ vertex figures. Since any elliptic or parabolic Coxeter graph of rank $n$ has at most $n$ edges, one deduces that

$$e \leq \frac{2^n \, n}{2^{n-2}} = 4n. \tag{2}$$

From (1) and (2), one deduces that one must have

$$\frac{n(n-1)}{2} \leq 4n.$$

This inequality holds only for $n \leq 9$. □

**Corollary 1.** *There are no compact Coxeter n-cubes in $\mathbb{H}^n$ for $n \geq 9$.*

**Proof.** The vertex figure of an ordinary vertex $p \in \mathbb{H}^n$ is a spherical Coxeter $(n-1)$-simplex. Since the graph of such a polyhedron has at most $n-1$ vertices, Eq. (2) in the proof of Theorem 1 can be replaced by $e \leq 4(n-1)$. □

**Remark 1.** If $\mathcal{C}$ is an ideal $n$-cube in $\mathbb{H}^n$, then all its vertex figures are Euclidean simplices. The graph of any such polyhedron is a connected parabolic Coxeter graph of rank $n$, with $n$ edges (if it is isomorphic to $\tilde{A}_{n-1}$) or with $n-1$ edges (in all other cases). Hence, the number $e$ of edges of $\Gamma$ which are not dotted edges satisfies $4(n-1) \leq e \leq 4n$.

## 3. Ideal hyperbolic Coxeter *n*-cubes

In this section, we focus on the class of *ideal* Coxeter $n$-cubes in $\mathbb{H}^n$.

### 3.1. The graph of an ideal hyperbolic Coxeter n-cube

Let $\Gamma = (V, E)$ be a graph with set of vertices $V = \{v_1, \ldots, v_{2n}\}$ and set of edges $E = \{(v_i, v_j) \mid 1 \leq i < j \leq 2n\}$ such that the edges of the form $(v_i, v_{2n-i+1})$, $i = 1, \ldots, n$ are dotted edges of $\Gamma$. To each dotted edge $(v_i, v_{2n-i+1})$ we assign a weight $\cosh l_i := \cosh l_{i,2n-i+1} \in \mathbb{R}_{>0}$, and each non-dotted edge $(v_i, v_j)$ is labelled with an integer weight $m_{ij} \geq 3$. The *Schläfli matrix* $S = S(\Gamma)$ of $\Gamma$ is the symmetric matrix $S = (s_{ij})_{1 \leq i,j \leq 2n} \in \mathrm{Mat}(2n \times 2n, \mathbb{R})$ given by

$$s_{ij} = \begin{cases} 1, & \text{if } j = i, \\ -\cosh l_i, & \text{if } j = 2n-i+1, \\ -\cos \dfrac{\pi}{m_{ij}}, & \text{if } (v_i, v_j) \in E, \ j \neq i, 2n-i+1, \\ 0, & \text{otherwise.} \end{cases}$$

Our goal is to interpret $S$ as the Gram matrix of an ideal Coxeter $n$-cube $\mathcal{C} \subset \mathbb{H}^n$. More precisely, in such a case, any vertex $v_i \in V$ corresponds to a facet $F_i$ of $\mathcal{C}$, the facets $F_i$ and $F_{2n-i+1}$, have a common perpendicular of length $l_i$, $i = 1, \ldots, n$, and the angle between the facets $F_i$ and $F_j$, $j \neq 2n-i+1$, is equal to $\frac{\pi}{m_{ij}}$ if $(v_i, v_j) \in E$ and to $\frac{\pi}{2}$ otherwise.

Let $\Gamma$ be a graph as above, with Schläfli matrix $S = S(\Gamma)$. In order to have $S = G(\mathcal{C})$ for an ideal Coxeter $n$-cube $\mathcal{C} \subset \mathbb{H}^n$, $\Gamma$ must satisfy the following conditions (which are in fact necessary conditions for the existence of any hyperbolic polyhedron of finite volume, see [17, I, Chapter 6.2], for example).

(1) The signature of $S$ equals $(n, 1)$.
(2) Any subgraph of $\Gamma$ corresponding to the figure of a vertex of $\mathcal{C}$ is a connected parabolic Coxeter graph.

Conversely, Conditions (1) and (2) are sufficient in order for $\Gamma$ to be the graph of a hyperbolic Coxeter $n$-cube $\mathcal{C} = \mathcal{C}(\Gamma)$. In particular, Condition (2) ensures that $\mathcal{C}$ has finite volume, as it is the convex hull of $2^n$ vertices in $\mathbb{H}^n$.

Moreover, the following necessary condition will be particularly useful in order to discard certain cases.

(3) Let $\Gamma_1$ and $\Gamma_2$ be two *indefinite* subgraphs of $\Gamma$ (i.e. $\Gamma_i$ contains at least one connected component which is neither elliptic nor parabolic, $i = 1, 2$). Then, $\Gamma_1$ and $\Gamma_2$ are connected in $\Gamma$.

Notice that for *n*-cubes, Condition (3) implies

(3′) Any pair of dotted edges is connected by an edge in $\Gamma$,

a condition we have already used in the proof of Theorem 1.

In the sequel, we call Condition (2) *parabolicity* and Condition (3) (resp. (3′)) *signature obstruction*.

Our approach is the following. We first focus on Condition (2). Start with a graph $\Gamma^{(0)}$ with $2n$ vertices, say $v_1, \ldots, v_{2n}$, such that the vertices $v_i$ and $v_{2n-i+1}$, $i = 1, \ldots, n$, are connected by a dotted edge, and such that $\Gamma^{(0)}$ has no other edge (which is equivalent to supposing that the remaining edges of $\Gamma^{(0)}$ have weight 2). Let $\sigma^{(0)} := \langle v_1, \ldots, v_n \rangle \subset \Gamma$ be the subgraph of $\Gamma^{(0)}$ spanned by the vertices $v_1, \ldots, v_n$. Add $n-1$ or $n$ edges to $\Gamma^{(0)}$ so that $\sigma^{(0)}$ turns into a connected parabolic Coxeter graph, say $\Sigma^{(0)}$. Denote by $\Gamma^{(1)}$ the graph obtained from $\Gamma^{(0)}$ by replacing $\sigma^{(0)}$ by $\Sigma^{(0)}$. Next, consider a rank $n$ subgraph $\sigma^{(1)} \subset \Gamma^{(1)}$, $\sigma^{(1)} \neq \Sigma^{(0)}$, containing no dotted edge, and add edges to $\Gamma^{(1)}$ so that $\sigma^{(1)}$ turns into a connected parabolic Coxeter graph, say $\Sigma^{(2)}$. This leads a graph $\Gamma^{(2)}$. After at most $2^n$ steps, this procedure either yields a graph $\Gamma$ satisfying Condition (2), or allows us to claim that such a graph does not exist. At this stage, Condition (3) may be useful in order to reduce the list of graphs.

Let $\Gamma$ be a graph obtained by the procedure described in the previous paragraph, and satisfying Conditions (2) and (3). The weights of all edges of $\Gamma$ are fixed, except those of its dotted edges. Finally, we look at Condition (1).

Let $\chi_S$ be the characteristic polynomial of $S$. Then, one has

$$\chi_S(t) = \sum_{i=0}^{2n} a_i t^i \in \mathbb{R}[t], \tag{3}$$

where the coefficients $a_0, \ldots, a_{2n}$ depend on $l_1, \ldots, l_n$. Furthermore, the condition $\mathrm{sign}(S) = (n, 1)$ implies that

$$a_0 = \cdots = a_{n-2} = 0, \tag{4}$$

since 0 is an eigenvalue of multiplicity $n-1$ of $S$. Eqs. (3) and (4) provide a system of $n-1$ equations with respect to the unknowns $l_1, \ldots, l_n$, which has to be solved in order to decide the realizability of $\Gamma$ as the graph of an ideal Coxeter *n*-cube in $\mathbb{H}^n$. This will be worked out in the next sections.

### 3.2. Ideal Coxeter squares and 3-cubes

As a warm-up, we classify all ideal Coxeter squares and 3-cubes. Let us point out that such polyhedra can be described by using results due to Poincaré (case $n = 2$), respectively Andreev (case $n = 3$), see [17, I, Chapter 6.2], for examples. In particular, Andreev's theorem gives a list of angular and combinatorial conditions, which does not provide any information of the respective lengths between the pairs of non-intersecting facets, and which does not provide a complete classification.

In this section, the *signature of a graph* $\Gamma$ will denote the signature of the associated Schläfli matrix $S(\Gamma)$.

**Ideal Coxeter squares**

There is only one parabolic Coxeter graph of rank 2: $\widetilde{A}_1$. If $\mathcal{C} \subset \mathbb{H}^2$ is an ideal Coxeter square, then its graph $\Gamma$ can only be of the following type:
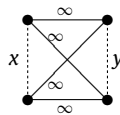


where the weights $x$ and $y$ correspond to the respective lengths between the two pairs of ultra-parallel sides of $\mathcal{C}$.

A straightforward application of the procedure described in Section 3.1 shows that ideal hyperbolic Coxeter squares form a one-parameter family $\mathcal{C}(x)$, $x > 0$, of polygons in $\mathbb{H}^2$ whose lengths $l_1, l_2$ between the two pairs of non-intersecting sides are given by

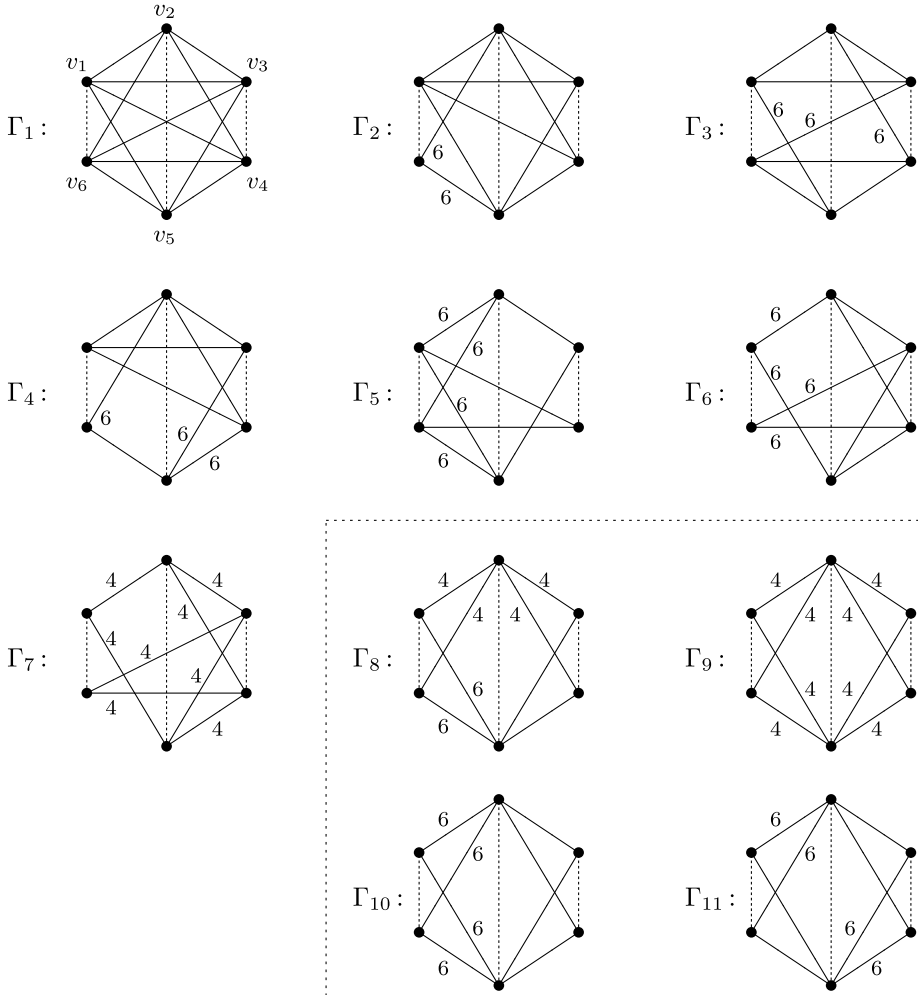$$l_1 = x \quad \text{and} \quad \cosh l_2 = 1 + \frac{4}{-1 + \cosh x}.$$

**Fig. 1.** Potential graphs of ideal hyperbolic Coxeter 3-cubes.

**Remark 2.** Consider the upper half-plane model $\mathcal{U}^2 \subset \mathbb{C} \cup \{\infty\}$ of the hyperbolic plane $\mathbb{H}^2$. Then, the vertices $(v_1, v_2, v_3, v_4)$ of any ideal quadrilateral can be mapped (by a unique Möbius transformation) onto $(-1, 0, \infty, z)$, $z \in \mathbb{R} \setminus \{-1, 0\}$. The parameter $z$ is given by $z = \frac{1}{1-[v_1,v_2,v_3,v_4]} - 1$, where $[v_1, v_2, v_3, v_4]$ is the *cross-ratio* of $(v_1, v_2, v_3, v_4)$. This leads to an alternative interpretation of the one-parameter family $\mathcal{C}(x)$, $x > 0$, of ideal hyperbolic Coxeter squares.

**Ideal Coxeter 3-cubes**

Let $\Gamma$ be the graph of an ideal Coxeter 3-cube $\mathcal{C} \subset \mathbb{H}^3$. Then, $\Gamma$ has 6 vertices, say $v_1, \ldots, v_6$, corresponding to the hyperplanes bounding $\mathcal{C}$, as well as 3 dotted edges (between the vertices $v_1$ and $v_6$, $v_2$ and $v_5$, and $v_3$ and $v_4$) corresponding to the 3 pairs of ultra-parallel faces of $\mathcal{C}$. The vertex figures of $\mathcal{C}$ correspond to those subgraphs of $\Gamma$ of rank 3 which do not contain any dotted edge. There are 3 different parabolic Coxeter graphs of rank 3: $\widetilde{A}_2$, $\widetilde{C}_2$ and $\widetilde{G}_2$.

By applying the procedure described in Section 3.1, one finds the 11 potential graphs enlisted on Fig. 1.

The graphs $\Gamma_8$, $\Gamma_9$, $\Gamma_{10}$ and $\Gamma_{11}$ contain each a subgraph which consist in two disjoint dotted edges. Hence, they have to be removed from the list due to the signature obstruction.

**Table 1**
Weights of the dotted edges in the graphs $\Gamma_1, \ldots, \Gamma_7$.

| Graph | $\cosh l_1$ | $\cosh l_2$ | $\cosh l_3$ |
|---|---|---|---|
| $\Gamma_1$ | 2 | 2 | 2 |
| $\Gamma_2$ | $\sqrt{3}$ | $\frac{7}{2}$ | $\frac{5}{4}$ |
| $\Gamma_3$ | $\sqrt{3}$ | $\sqrt{3}$ | $\sqrt{3}$ |
| $\Gamma_4$ | $\frac{3\sqrt{3}}{4}$ | $2\sqrt{3}$ | $\frac{3}{2}$ |
| $\Gamma_5$ | $\frac{5}{2}$ | $\frac{5}{2}$ | $\frac{2\sqrt{3}}{3}$ |
| $\Gamma_6$ | 2 | $\frac{3}{2}$ | $\frac{3}{2}$ |
| $\Gamma_7$ | $\sqrt{2}$ | 2 | 2 |

At this point, we can deduce from Andreev's Theorem [17, I, Chapter 6.2, Theorem 2.8] that the graphs $\Gamma_1, \ldots, \Gamma_7$ are the graphs of hyperbolic Coxeter 3-cubes $\mathcal{C}_1, \ldots, \mathcal{C}_7$. This, however, is not sufficient in order to determine the corresponding Gram matrices.

Let us consider the graph $\Gamma_1$. Its Schläfli matrix $S_1 = S(\Gamma_1)$ is given by

$$S_1 = \begin{pmatrix} 1 & -1/2 & -1/2 & -1/2 & -1/2 & a \\ -1/2 & 1 & -1/2 & -1/2 & b & -1/2 \\ -1/2 & -1/2 & 1 & c & -1/2 & -1/2 \\ -1/2 & -1/2 & c & 1 & -1/2 & -1/2 \\ -1/2 & b & -1/2 & -1/2 & 1 & -1/2 \\ a & -1/2 & -1/2 & -1/2 & -1/2 & 1 \end{pmatrix},$$

where $a = -\cosh l_1$, $b = -\cosh l_2$ and $c = -\cosh l_3$ depend on the weights $l_1$, $l_2$ and $l_3$ of the dotted edges of $\Gamma_1$.

In order to be the Gram matrix of a hyperbolic polyhedron in $\mathbb{H}^3$, $S_1$ has to have signature $(3, 1)$. In particular, it has to have the eigenvalue $\lambda_1 = 0$ with multiplicity 2. The characteristic polynomial $\chi_1 = \chi_{S_1}$ is given by

$$\chi_1(t) = -(t + a - 1)(t + b - 1)(t + c - 1)(-4 + ab + ac + bc + abc$$
$$- t\,(2a + 2b + 2c + ab + ac + bc) + t^2(3 + a + b + c) - t^3),$$

for $t \in \mathbb{R}$. Since $a, b, c < -1$, the eigenvalue $\lambda_1 = 0$ must be a root of multiplicity at least 2 of the factor

$$-4 + ab + ac + bc + abc - t\,(2a + 2b + 2c + ab + ac + bc) + t^2\,(3 + a + b + c) - t^3,$$

which yields the system

$$\begin{cases} -4 + ab + ac + bc + abc &= 0 \\ 2a + 2b + 2c + ab + ac + bc &= 0. \end{cases}$$

Since $a, b, c < -1$, this system admits the unique solution $a = b = c = -2$ (this can for example be obtained by using a symmetry argument). One can check that the matrix obtained by replacing the coefficients $a$, $b$, $c$ by $-2$ in $S_1$ has signature $(3, 1)$. As an outcome, one deduces that the graph $\Gamma_1$ is the graph of an ideal hyperbolic Coxeter cube $\mathcal{C}_1$ with $l_1 = l_2 = l_3 = \text{arcosh } 2$ (in fact, $\mathcal{C}_1$ is the *regular* ideal Coxeter 3-cube).

Similar computations with the remaining graphs show that the graphs $\Gamma_1$ to $\Gamma_7$ are the graphs of the ideal Coxeter 3-cubes in $\mathbb{H}^3$. The corresponding values of $\cosh l_1$, $\cosh l_2$ and $\cosh l_3$ are provided in Table 1.

Let $\text{Л} : \mathbb{R} \to \mathbb{R}$ be the *Lobachevsky function* given by

$$\text{Л}(x) := -\int_0^x \log |2 \sin t|\, dt.$$

By a result of Milnor [14, Lemma 2], the volume of an ideal tetrahedron $\mathcal{T}(\alpha, \beta, \gamma) \subset \mathbb{H}^3$ with angles $\alpha, \beta, \gamma > 0$ is given by

$$\mathcal{T}(\alpha, \beta, \gamma) = \text{Л}(\alpha) + \text{Л}(\beta) + \text{Л}(\gamma).$$

**Table 2**
Volumes of the ideal hyperbolic Coxeter 3-cubes $\mathcal{C}_1, \ldots, \mathcal{C}_7$.

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| vol $\mathcal{C}_i$ | $15 \text{Л}(\frac{\pi}{3})$ | $\frac{27}{2} \text{Л}(\frac{\pi}{3})$ | $\frac{27}{2} \text{Л}(\frac{\pi}{3})$ | $13 \text{Л}(\frac{\pi}{3})$ | $12 \text{Л}(\frac{\pi}{3})$ | $13 \text{Л}(\frac{\pi}{3})$ | $10 \text{Л}(\frac{\pi}{4})$ |

The volume of an ideal hyperbolic 3-cube can be computed by using a suitable decomposition into ideal tetrahedra, and analytic properties of the Lobachevsky function [17, I, Chapter 7.3]. The explicit formula reads as follows.

**Proposition 1.** *Let $\mathcal{C} \subset \mathbb{H}^3$ be an ideal hyperbolic 3-cube with faces $F_i$, $i = 1, \ldots, 6$, such that $F_i$ is opposite to $F_{7-i}$ in $\mathcal{C}$, $i = 1, \ldots, 3$. Let $\alpha_{ij}$, $1 \leq i < j \leq 6$, $i + j \neq 6$, denote the dihedral angles of $\mathcal{C}$. Then, one has*

$$\text{vol}(\mathcal{C}) = \sum_{\substack{1 \leq i < j \leq 6 \\ i+j \neq 7}} \text{Л}(\alpha_{ij}) - \sum_{1 \leq i < j \leq 3} \text{Л}\left(\alpha_{ij} + \alpha_{7-i,7-j}\right). \tag{5}$$

Let $\mathcal{C}_1, \ldots, \mathcal{C}_7$ be the ideal Coxeter 3-cubes with respective graphs $\Gamma_1, \ldots, \Gamma_7$. The volumes vol $\mathcal{C}_i$, $i = 1, \ldots, 7$ are collected in Table 2.

**Remark 3.** Formula (5) and Table 2 correct the corresponding formula [9, Formula (3.5), p.36] and table [9, Table 3.2, p.37], which contained a mistake.

For $i = 1, \ldots, 7$, let $W_i < \text{Isom}(\mathbb{H}^3)$ be the Coxeter group with fundamental polyhedron $\mathcal{C}_i$.

**Remark 4.** By using Fig. 1 and Table 1, it follows from Vinberg's arithmeticity criterion for Coxeter groups [17, II, Chapter 6.3] that the Coxeter groups $W_1, W_3, W_6$ and $W_7$ are arithmetic. The Coxeter groups $W_2, W_4$ and $W_5$ are non-arithmetic, but they are quasi-arithmetic.

**Remark 5.** The Coxeter groups $W_1, W_3$ and $W_6$ are commensurable to the Coxeter simplex group [3, 3, 6]. Moreover, the group $W_7$ is commensurable to the Coxeter simplex group [3, 4, 4] (see [9, Section 5.4]).

The cubes $\mathcal{C}_2$ and $\mathcal{C}_3$ (respectively, the cubes $\mathcal{C}_4$ and $\mathcal{C}_6$) can be dissected into 108 (resp. 104) isometric copies of the Coxeter simplex $\mathcal{T}$ associated to the Coxeter group [3, 3, 6]. However, it follows from Remark 4 that the corresponding groups $W_2$ and $W_3$ (respectively, $W_4$ and $W_6$) are not commensurable.

### 3.3. Absence of ideal Coxeter n-cubes in $\mathbb{H}^n$, $n \geq 4$

This section is devoted to the proof of the following result.

**Theorem 2.** *There are no ideal hyperbolic Coxeter n-cubes for $n \geq 4$.*

**Proof.** By a result due to Felikson and Tumarkin [5], there is no simple ideal hyperbolic Coxeter polyhedron in $\mathbb{H}^n$ for $n \geq 9$. Hence, it suffices to prove the assertion for $4 \leq n \leq 8$ only. We will proceed dimension by dimension, by using the notation and the procedure described in Section 3.1. Details of the case exhaustion can be found in [9, Chapter 3.3].

**Dimension 4**

Let $\Gamma$ be the graph of an ideal 4-cube $\mathcal{C} \subset \mathbb{H}^4$, with vertices $v_1, \ldots, v_8$ and dotted edges $(v_1, v_8)$, $(v_2, v_7)$, $(v_3, v_6)$ and $(v_4, v_5)$. Then, $\Gamma$ must satisfy the conditions $(1) - (3)$ described in Section 3.1. As for Condition (2), notice that there are 3 connected parabolic Coxeter graphs of rank 4 which may appear as subgraphs $\langle v_i, v_j, v_k, v_l \rangle \subset \Gamma$: $\widetilde{A}_3, \widetilde{B}_3$ and $\widetilde{C}_3$.

First, suppose that $\Gamma$ has a subgraph isomorphic to $\widetilde{B}_3$, say $\langle v_1, v_2, v_3, v_4 \rangle$. Without loss of generality, we can suppose that $m_{12} = m_{23} = 3$, $m_{24} = 4$, and $m_{13} = m_{14} = m_{34} = 2$. Then,

by considering the subgraph $\langle v_1, v_2, v_3, v_5 \rangle$, one deduces that one must have either $m_{25} = 4$ and $m_{15} = m_{35} = 2$ (so that $\langle v_1, v_2, v_3, v_5 \rangle$ is isomorphic to $\widetilde{B}_3$), or $m_{15} = m_{35} = 3$ and $m_{25} = 2$ (so that $\langle v_1, v_2, v_3, v_5 \rangle$ is isomorphic to $\widetilde{A}_3$).

(1) *Suppose that $m_{25} = 4$ and $m_{15} = m_{35} = 2$.* Then, by parabolicity, one deduces by considering the subgraph $\langle v_2, v_3, v_4, v_8 \rangle$ that one must have $m_{48} = 2$. In the same way, the subgraph $\langle v_2, v_3, v_5, v_8 \rangle$ cannot be parabolic unless $m_{58} = 2$.

Since $m_{14} = m_{15} = m_{48} = m_{58} = 2$, the dotted edges $(v_1, v_8)$ and $(v_4, v_5)$ will be disconnected. Hence, by the signature obstruction, the graph $\Gamma$ cannot describe an ideal hyperbolic 4-cube.

(2) *Suppose that $m_{15} = m_{35} = 3$ and $m_{25} = 2$.* Then, by parabolicity, we have the following dichotomy for the subgraph $\langle v_1, v_2, v_5, v_6 \rangle$:

    (2.1) If $m_{16} = 4$ and $m_{26} = m_{25} = 2$, then we have two possibilities coming from the subgraph $\langle v_1, v_3, v_4, v_7 \rangle$:

        (2.1.1) If $m_{17} = 3 = m_{37}$ and $m_{47} = 4$, then by considering the subgraphs $\langle v_1, v_4, v_6, v_7 \rangle$ and $\langle v_1, v_5, v_6, v_7 \rangle$, we deduce by parabolicity that one must have $m_{7,6} = 2 = m_{7,5}$. Moreover, by parabolicity again, the subgraph $\langle v_2, v_3, v_5, v_8 \rangle$ leads to $m_{28} = m_{58} = 3$ and $m_{38} = 2$, and the subgraph $\langle v_3, v_4, v_7, v_8 \rangle$ to $m_{78} = 3$ and $m_{38} = m_{48} = 2$. Finally, for the subgraph $\langle v_5, v_6, v_7, v_8 \rangle$, parabolicity forces $m_{68} = 4$, so that we obtain the following graph $\Gamma_1$:



        (2.2.2) If $m_{17} = 4$ or $m_{37} = 4$, then the subgraph $\langle v_1, v_3, v_5, v_7 \rangle$ is not parabolic, which contradicts Condition (2).

    (2.2) If $m_{16} = 2$ and $m_{26} = 3 = m_{56}$, then we have two possibilities in order to have a parabolic subgraph $\langle v_2, v_3, v_4, v_8 \rangle$:

        (2.2.1) If $m_{28} = 3$ and $m_{38} = m_{48} = 2$, then, the parabolicity of the subgraph $\langle v_2, v_4, v_6, v_8 \rangle$ forces $m_{68} = 2$. Then, the dotted edges $(v_1, v_8)$ and $(v_3, v_6)$ are disconnected, which contradicts the signature obstruction.

        (2.2.2) If $m_{28} = m_{48} = 2$ and $m_{38} = 4$, then one can easily determine the remaining edge weights and get the following graph $\Gamma_2$:



Next, suppose that $\Gamma$ has a subgraph which is isomorphic to $\widetilde{A}_3$, say $\langle v_1, v_2, v_3, v_4 \rangle$, but no subgraph isomorphic to $\widetilde{B}_3$. We can suppose that $m_{12} = m_{23} = m_{34} = m_{14} = 3$ and $m_{13} = m_{24} = 2$. Then, the parabolicity of the subgraph $\langle v_1, v_2, v_3, v_5 \rangle$ implies that $m_{25} = 2$ and $m_{15} = m_{35} = 3$, and the parabolicity of the subgraph $\langle v_1, v_3, v_4, v_7 \rangle$ forces $m_{17} = m_{37} = 3$ and $m_{47} = 2$. The subgraph

$\langle v_1, v_3, v_5, v_7 \rangle$ also has to be parabolic, so that $m_{57} = 2$, which implies that the dotted edges $(v_2, v_7)$ and $(v_4, v_5)$ are disconnected. By the signature obstruction, this implies that $\Gamma$ has no subgraph isomorphic to $\tilde{A}_3$.

Finally, suppose that all parabolic rank 4 subgraphs of $\Gamma$ are isomorphic to $\tilde{C}_3$. We start by supposing that $m_{23} = 3$, $m_{12} = m_{34} = 4$ and $m_{13} = m_{14} = m_{24} = 2$. Then, by parabolicity, the subgraphs $\langle v_1, v_2, v_3, v_5 \rangle$ and $\langle v_2, v_3, v_4, v_8 \rangle$ lead to $m_{35} = m_{28} = 4$ and $m_{15} = m_{25} = 2 = m_{38} = m_{48}$, so that by considering the subgraph $\langle v_2, v_3, v_5, v_8 \rangle$, we deduce $m_{58} = 2$. Hence, the dotted edges $(v_1, v_8)$ and $(v_4, v_5)$ are disconnected, which violates the signature obstruction.

It remains to consider more closely the graphs $\Gamma_1$ and $\Gamma_2$ obtained above and satisfying Conditions (2) and (3) from Section 3.1. In view of Condition (1), we have to determine the weights of the various dotted edges in these graphs. To this end, one first computes the respective characteristic polynomials and then the coefficients of their constant, linear and quadratic terms (see (3) and (4)). In contrast with the case of dimension 3 (see Section 3.2), the resulting systems of equations with respect to the weights of the dotted edges turn out to have no solution. Hence, there is no ideal 4-cube in $\mathbb{H}^4$.

## Dimension 5

Consider the graph $\Gamma$ of an ideal Coxeter 5-cube, with vertices $v_1, \ldots, v_{10}$ and with dotted edges $(v_1, v_{10})$, $(v_2, v_9)$, $(v_3, v_8)$, $(v_4, v_7)$ and $(v_5, v_6)$. Any rank 5 subgraph of $\Gamma$ *not* containing any dotted edge has to be parabolic, i.e. it has to be isomorphic to $\tilde{A}_4$, $\tilde{B}_4$, $\tilde{C}_4$, $\tilde{D}_4$ or $\tilde{F}_4$. The strategy here is similar to the one we have used for dimension 4, but quite longer, since we have to deal with 5 possible parabolic graphs.

## Dimensions 6, 7 and 8

Let $\mathcal{C} \subset \mathbb{H}^n$ be an ideal Coxeter $n$-cube, $n \geq 2$. Then, for $2 \leq k \leq n$, any $k$-face of $\mathcal{C}$ is an ideal $k$-cube (not necessarily of Coxeter type). The following property is a consequence of an observation due to Borcherds [2, Example 5.6] and will be useful in order to determine when a $k$-face of $\mathcal{C}$ is a Coxeter polyhedron: if the graph $\Gamma$ of $\mathcal{C}$ has an elliptic subgraph $\Gamma'$ of rank $n - k$ with no component of type $A_l$, $l \geq 1$, or $D_5$, then the $k$-face $F \subset \mathcal{C}$ corresponding to $\Gamma'$ is a Coxeter polyhedron itself.

The parabolic graphs $\tilde{B}_{n-1}$ and $\tilde{C}_{n-1}$, $n = 6, 7, 8$, contain an elliptic subgraph of type $I_2(4)$, $B_3$ and $B_4$, respectively. Since, as we have seen, there is no ideal hyperbolic Coxeter 4-cube, the above observation allows us to deduce that the graphs $\tilde{B}_{n-1}$ and $\tilde{C}_{n-1}$, $n = 6, 7, 8$ cannot occur as parabolic subgraphs of the graph of an ideal Coxeter $n$-cube in $\mathbb{H}^n$, $n = 6, 7, 8$. Hence, for $n = 6$, the only possible rank 6 parabolic subgraphs are $\tilde{A}_5$ and $\tilde{D}_5$, for $n = 7$, the only possible rank 7 parabolic subgraphs are $\tilde{A}_6$, $\tilde{D}_6$ and $\tilde{E}_6$, and for $n = 8$, the only possible rank 8 parabolic subgraphs are $\tilde{A}_7$, $\tilde{D}_7$ and $\tilde{E}_7$.

The different subgraph chasings in these cases are much shorter than for dimensions 4 and 5. Because of the high proportion of edges of weight 2 in parabolic graphs of higher rank, the parabolicity condition (2) of Section 3.1 already suffices in order to proceed. □

## 4. Beyond the ideal case

Consider a hyperbolic Coxeter $n$-cube $\mathcal{C} \subset \mathbb{H}^n$. In contrast to an ideal vertex, an ordinary vertex of $\mathcal{C}$ has a vertex figure whose graph is not necessarily connected. This implies that the number of possible figures for any ordinary vertex is *a priori* infinite, because of the rank 2 elliptic graph $I_2(p)$, $p \geq 3$. Notice that even if one excludes subgraphs of this type, the number of possibilities remains high in comparison with the ideal case.

Nevertheless, the procedure described in Section 3 can be used in order to produce new examples. As an illustration, we exhibit new families of (both compact and non-compact) hyperbolic Coxeter 3-cubes.

From works of Im Hof [8], we have one family of hyperbolic Coxeter 3-cubes $\mathcal{C}_{p,q,r}$ ($3 \leq p, q, r \leq \infty$), so-called *Lambert cubes*, characterized by a cyclic Coxeter graph as depicted on Fig. 2.

Their volumes have been determined by Kellerhals [12].

For $3 \leq p \leq \infty$, consider the graphs $\Sigma_1(p)$, $\Sigma_2(p)$ and $\Sigma_3(p)$ given in Fig. 3.

By using a similar procedure as in Section 3.2, one can show that the graph $\Sigma_i(p)$ turns out to be the graph of a compact (if $i = 1$), resp. non-compact (if $i = 2, 3$), hyperbolic Coxeter 3-cube $\mathcal{K}_i(p)$. The weight of the respective dotted edges are given in Table 3.
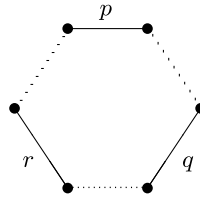
**Fig. 2.** Graph of a Lambert cube $\mathcal{C}_{p,q,r}$.
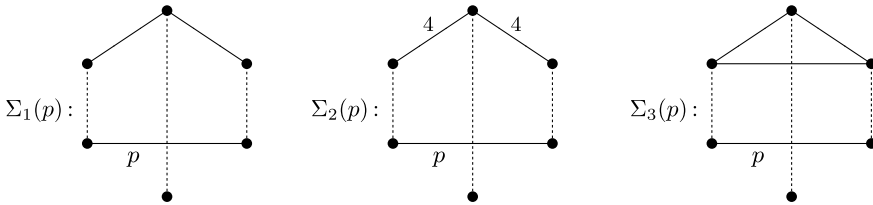


**Fig. 3.** Graphs of hyperbolic Coxeter 3-cubes ($3 \leq p \leq \infty$).

**Table 3**
Weights of the dotted edges in the graphs $\Sigma_i(p)$.

| Graph | $\cosh l_1$ | $\cosh l_2$ | $\cosh l_3$ |
|---|---|---|---|
| $\Sigma_1(p)$ | $\sqrt{2}\cos\frac{\pi}{2p}$ | $\frac{1}{3}\sqrt{3+\frac{1}{\cos\frac{\pi}{p}}}$ | $\sqrt{2}\cos\frac{\pi}{2p}$ |
| $\Sigma_2(p)$ | $\sqrt{2}\cos\frac{\pi}{2p}$ | $\frac{\sqrt{2}}{2}\sqrt{1+\frac{1}{\cos\frac{\pi}{p}}}$ | $\sqrt{2}\cos\frac{\pi}{2p}$ |
| $\Sigma_3(p)$ | $\sqrt{3}\cos\frac{\pi}{2p}$ | $\frac{\sqrt{3}}{2}\sqrt{1+\frac{1}{1+2\cos\frac{\pi}{p}}}$ | $\sqrt{3}\cos\frac{\pi}{2p}$ |

**Remark 6.** In fact, each of the 3-cubes $\mathcal{K}_i(p)$, $i = 1, 2, 3$, $3 \leq p \leq \infty$, can be decomposed (by using a suitable diagonal hyperplane) into two isometric copies of Coxeter simplicial prisms [11].

**Remark 7.** In a recent joint work with Steve Tschantz [10], the procedure developed in this paper could be programmed in Mathematica® in order to provide a complete classification of the hyperbolic Coxeter $n$-cubes of finite volume (compact and non-compact).

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