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THE FCC LATTICE AND THE CUSPED HYPERBOLIC 4-ORBIFOLD OF MINIMAL VOLUME

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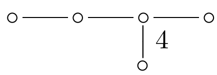
In memoriam H. S. M. Coxeter

Abstract

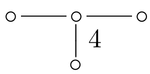
The 1-cusped hyperbolic coset space of H^4 by the Coxeter group $[4, 3^{2,1}]$ of volume $\pi^2/1440$ is the unique minimal volume orbifold among all non-compact complete hyperbolic 4-orbifolds. Our proof is geometric and based on horoball geometry combined with Gauss's characterization of the face centered cubic lattice packing as the densest one in euclidean 3-space.

Introduction

Let Q be a non-compact complete hyperbolic 4-orbifold of finite volume. Then, Q is the quotient of hyperbolic space H^4 by a discrete group $\Gamma < \mathrm{Iso}(H^4)$ with parabolic elements. In this article, we show that the quotient space Q_* of H^4 by the hyperbolic Coxeter group $\Gamma_* = [4, 3^{2,1}]$ with diagram

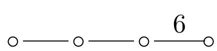
Σ_* :


is the unique non-compact hyperbolic 4-orbifold of minimal volume. The orbifold Q_* is isometric to a hyperbolic Coxeter 4-simplex of volume equal to $\pi^2/1440$ with precisely one vertex at infinity. Its vertex neighborhood is a cone over the euclidean tetrahedron Δ_{fcc}, which is a fundamental domain for the action of the symmetry group of the famous fcc lattice given by the parabolic Coxeter group $\Gamma_{\mathrm{fcc}} < \Gamma_*$ with diagram

Σ_{fcc} :


By a well-known result of Gauss, the fcc packing is the unique lattice packing of E^3 with maximal density $\pi/\sqrt{18}$. Indeed, our methods are based on results about crystallographic groups and lattice packings in E^3 as well as horoball geometry in hyperbolic space. In particular, a conjugacy class of a subgroup of parabolic type in Γ gives rise to a canonical cusp in Q which we control very well.

Our theorem generalizes the area minimality property of the triangle group $(2, 3, \infty)$. It is known that the quotient of H^2 by the group $(2, 3, \infty)$ with fundamental triangle of non-zero angles $\pi/2, \pi/3$ is the unique 2-orbifold of minimal area equal to $\pi/6$ (cf. [**2**, § 10]). Meyerhoff [**12**] showed that among the non-compact oriented hyperbolic 3-orbifolds, the oriented double cover of the quotient of H^3 by the Coxeter group $[3, 3, 6]$ with diagram



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is of minimal volume. The methods which we develop allow to conclude that the space $H^3/[3,3,6]$ is the hyperbolic 3-orbifold of minimal volume and as such is unique. It is of volume equal to $\frac{1}{8}\,Л(\frac{1}{3}\pi) \simeq 0.04229$, where Л denotes the Lobachevsky function (cf. § 2.2).

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1. *Preliminaries*

1.1. *Hyperbolic space and isometries*

Let H^n denote the standard hyperbolic space realized in the upper half-space model $E^n_+ = \{\, x = (x_1, \ldots, x_n) \mid x_n > 0 \,\}$ equipped with the hyperbolic metric $ds^2 = 1/x_n^2\,(dx_1^2 + \cdots + dx_n^2)$. Points at infinity are elements of the boundary $\partial H^n = \widehat{E}^{n-1} = E^{n-1} \cup \{\infty\}$. In this model, a hyperbolic r-sphere $S_p(r)$ centered at $p = (p_1, \ldots, p_n)$ is a euclidean $(p_n \sinh r)$-sphere centered at $(p_1, \ldots, p_{n-1}, p_n \cosh r)$. In the limiting case when p tends to a point at infinity, we obtain a horosphere based at a point at infinity of ∂H^n which is either a euclidean sphere $S_q(\infty)$ internally tangent at $q \in E^{n-1}$ or a hyperplane $S_\infty(\rho) = \{\, x \in E^n_+ \mid x_n = \rho \,\}$, for some $\rho > 0$, based at ∞. Consider a horosphere of the type $S_\infty(\rho)$, for example. It bounds a horoball $B_\infty(\rho)$ in H^n of infinite volume. Moreover, it carries a euclidean metric in a natural way given by

$$ds_0^2 = \frac{1}{\rho^2}\,(dx_1^2 + \cdots + dx_{n-1}^2), \tag{1}$$

where the subscript 0 refers to the flatness of the metric. Therefore, $S_\infty(\rho)$ can be identified with a copy of E^{n-1}. We obtain a unified picture by passing to the ball model realized in the unit ball $B \subset E^n$ equipped with the corresponding metric

$$ds_B^2 = 4\frac{dx_1^2 + \cdots + dx_n^2}{(1 - |x|^2)^2}\,. \tag{2}$$

Here, all horospheres are euclidean spheres internally tangent to boundary S^{n-1} of points at infinity.

The group $\mathrm{Iso}(H^n)$ of hyperbolic isometries consists of Möbius transformations leaving invariant E^n_+. By Poincaré extension [**2**, § 3.3], it is isomorphic to the group of Möbius transformations of $\widehat{E}^{n-1}$. According to the fixed point behavior, elements $\gamma \in \mathrm{Iso}(H^n)$ fall into the three conjugacy classes of elliptic, parabolic and loxodromic elements. More precisely, γ is elliptic if it possesses at least one fixed point in H^n, and γ is parabolic if it has precisely one fixed point (called parabolic fixed point) which lies in ∂H^n. Suppose that the parabolic element γ fixes ∞. Then, γ acts as a euclidean isometry on each horosphere $S_\infty(\rho)$ for $\rho > 0$.

1.2. *Some horoball geometry*

In H^n, consider the horoball $B_\infty(\rho)$ based at ∞ with euclidean distance ρ from the ground space E^{n-1} and a horoball $B_x = B_x(\rho)$ based at $x \in E^{n-1}$ with euclidean diameter ρ. Let l be a geodesic with endpoints x and $y \in E^{n-1} \setminus \{x\}$. Put $a = (x, \rho)$ and $b = (y, \rho)$, and denote by $d = \mathrm{dist}_0(a, b)$ the euclidean distance on $S_\infty(\rho) = \partial B_\infty(\rho)$ from a to b. By (1), $d = \mathrm{dist}_0(x, y)/\rho$. Suppose without loss of generality that $d \geqslant 1$. Let $p \in \partial B_x$ be the intersection point of l with the horosphere $S_x = \partial B_x$. By generalizing a result of Adams [**1**, Lemma 4.3], we can identify the euclidean distance $\mathrm{dist}_0(a, p)$ on S_x as follows.

Lemma 1.

$$\mathrm{dist}_0(a, p) = \frac{1}{d}\,. \tag{3}$$

Proof. Let g be the semicircle centered at x and ending at y. Let $\gamma \in \mathrm{Iso}(H^n)$ be the half-turn around g. Then, $\gamma(l)$ is the geodesic line from y to ∞, and γB_x is a horoball $B_\infty(\tau)$ for some $\tau > 0$. Since γ is an isometry, the hyperbolic distance from the horosphere boundary $S_\infty(\tau)$ to g is given by

$$\mathrm{dist}(S_\infty(\tau), g) = \log \frac{\tau}{\rho d} = \log \frac{\rho d}{\rho} = \mathrm{dist}(S_x, g),$$

that is, $\tau = \rho\, d^2$. Moreover, the points a and p are sent to $A = (x, \tau)$ and $P = (y, \tau)$, respectively. By (1) and the properties of γ, we deduce that

$$\mathrm{dist}_0(a, p) = \mathrm{dist}_0(A, P) = \frac{1}{\tau}\,\mathrm{dist}_0(x, y) = \frac{\rho}{\rho\, d^2}\,\mathrm{dist}_0(a, b) = \frac{1}{d}\,. \qquad \square$$

LEMMA 2. *Let $B_x(\rho)$ be a horoball of diameter $\rho > 0$ in H^n. Then, the interior of its upper hemisphere is an open ball of radius* 1 *with respect to the euclidean metric on the boundary $S_x(\rho)$.*

Proof. Write $B = B_x(\rho)$ and $a = (x, \rho)$. Let l be a geodesic starting at x and passing through a point p of the equator set of B. Denote the endpoint of l by y. Then, $\mathrm{dist}_0(x, y) = \rho$. Let g be the semicircle centered at x and starting from y. Then, the half-turn $\gamma \in \mathrm{Iso}(H^n)$ around g sends l to the geodesic from y to ∞ and the horoball B to the horoball $B_\infty(\rho)$. Moreover, $b := (y, \rho) = \gamma(p)$. We have to show that $\mathrm{dist}_0(a, p) = 1$ on $S_x(\rho)$. By the properties of the isometry γ and by (1), $\mathrm{dist}_0(a, p) = \mathrm{dist}_0(a, b) = \mathrm{dist}_0(x, y)/\rho = \rho/\rho = 1$ which finishes the proof. $\square$

1.3. *Groups of parabolic type and associated horoballs*

Consider a subgroup $\Gamma < \mathrm{Iso}(H^n)$. The group Γ is said to be elementary if it has a finite orbit in $\overline{H^n}$. More specifically, an elementary discrete subgroup $\Gamma < \mathrm{Iso}(H^n)$ is of elliptic or parabolic type if it has a finite orbit in H^n or fixes precisely one point in ∂H^n and has no other finite orbit in $\overline{H^n}$. These notions are conjugacy invariant characterizations. Let $\Gamma < \mathrm{Iso}(H^n)$ be a finite covolume discrete group containing a parabolic element fixing ∞. Then, the stabilizer Γ_∞ is of parabolic type acting cocompactly as discrete subgroup of $\mathrm{Iso}(E^{n-1})$ on each horosphere $S_\rho(\infty)$ for $\rho > 0$. Hence, Γ_∞ is a crystallographic group, that is, the subgroup of translations in Γ_∞ has finite index and corresponds to a lattice Λ of E^{n-1}.

In Λ, consider a vector of shortest length $\mu > 0$, and define the horoball

$$B_\infty(\mu) := \{\, x \in H^n \mid x_n > \mu \,\}\,. \tag{4}$$

It turns out that $B_\infty(\mu)$ is precisely invariant with respect to Γ projecting to an embedded cusp neighborhood in the quotient space H^n/Γ. The set $B_\infty(\mu)$ is called a canonical horoball associated to ∞, and $C = B_\infty(\mu)/\Gamma_\infty$ is called a canonical cusp in H^n/Γ. Canonical horoballs and cusps associated to inequivalent parabolic fixed points are disjoint [7, Proposition 3.3].

Expand the canonical cusp C by diminishing μ in such a way that it touches itself. Such a cusp is called the maximal cusp associated to Γ_∞. It is of the form $B_\infty(\rho)/\Gamma_\infty$ for some $\rho \leqslant \mu$.

Finally, consider a horoball $\gamma B_\infty(\rho) = B_q(\rho)$, $\gamma \in \Gamma \setminus \Gamma_\infty$, based at a point $q \in E^{n-1}$ which is of diameter ρ. It touches $B_\infty(\rho)$ and is called a fullsized horoball associated to ∞.

1.4. *Crystallographic groups and the fcc lattice*

Consider crystallographic groups in $\mathrm{Iso}(E^3)$ with their associated lattices Λ and point groups. Fedorov and Schoenflies classified and described all 219 crystallographic groups acting on E^3 (cf. [4]). Associated to these are 14 different (Bravais) lattices grouped together into seven crystal classes by taking into account their axial symmetries. The lattices of highest symmetry

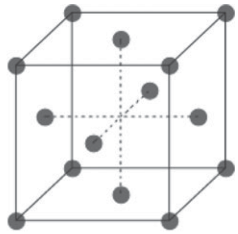

FIGURE 1. *The* fcc *lattice*

are the three cubic lattices, the simple cubic or sc lattice, the body centered cubic or bcc lattice and the face centered cubic or fcc lattice. More concretely, the fcc lattice (cf. Figure 1) is the translation group Λ_{fcc} generated by the basis of vectors

$$\frac{1}{\sqrt{2}}\,(1,1,0),\ \frac{1}{\sqrt{2}}\,(1,0,1),\ \frac{1}{\sqrt{2}}\,(0,1,1) \tag{5}$$

in E^3. Notice that the four points consisting of 0 and the three in (5) define a regular tetrahedron in E^3. The symmetry group Γ_{fcc} of the fcc lattice has a point group of maximal order which, in fact, is the group of symmetries of a regular octahedron. A fundamental domain of Γ_{fcc} is given by the simplex Δ_{fcc} with vertices

$$a=(0,0,0),\quad b=\frac{1}{\sqrt{2}}(1,0,0),\quad c=\frac{1}{2\sqrt{2}}(1,1,0),\quad d=\frac{1}{2\sqrt{2}}\,(1,1,1). \tag{6}$$

Moreover, the fcc lattice underlies the single quasi-regular honeycomb in E^3 with Schläfli symbol $\left\{3, {3 \atop 4}\right\}$ consisting of regular tetrahedra and octahedra (cf. [**6**, §4.7]). All the faces are triangles, each one belonging to one regular tetrahedron $\{3,3\}$ and one regular octahedron $\{3,4\}$.

1.5. *Geometric Coxeter groups*

Let $X^n = S^n, E^n$ or H^n, and consider a discrete subgroup $\Gamma < \mathrm{Iso}(X^n)$. Simplest examples are the groups generated by reflections R in hyperplanes H of X^n called (geometric) Coxeter groups. More specifically, an elliptic, parabolic or hyperbolic Coxeter group Γ_{C} is a discrete group generated by finitely many reflections R in hyperplanes H of S^n, E^n or H^n subject to relations

$$R^2 = 1; \quad (RR')^p = 1 \quad \text{if } \angle(H,H') = \pi/p \ \text{ for } p \in \mathbb{N},\, p > 1.$$

A convex fundamental polytope of Γ_{C} is called a Coxeter polytope and will be denoted by P_{C}. Its dihedral angles are of the form π/p for integers $p \geqslant 2$. The Coxeter diagram Σ of Γ_{C} or of P_{C} is a labeled graph, nodes i,k of which correspond to generators R_i, R_k of Γ_{C} or hyperplanes H_i, H_k of P_{C}. Suppose that $(R_iR_k)^p = 1$. If $p = 2$, the nodes i,k are not connected. For $p = 3$ and for $p \geqslant 4$, they are connected by a single edge and by an edge marked p, respectively. Notice that hyperbolic Coxeter diagrams of finite covolume are connected graphs. The most basic examples are Coxeter groups with linear diagrams of rank n and $n+1$. Such Coxeter groups generate the symmetry group of regular n-polytopes and honeycombs in X^n. Of particular interest will be the elliptic Coxeter group $A_4 = [3,3,3]$ given by diagram $\circ\!-\!\!-\!\circ\!-\!\!-\!\circ\!-\!\!-\!\circ$ of rank 4 acting on S^3, the parabolic Coxeter group $\Gamma_{\mathrm{fcc}} = [4, 3^{1,1}]$ of rank 4 given by diagram

$$\Sigma_{\mathrm{fcc}} : \begin{array}{ccccc} \circ & \text{---} & \circ & \text{---} & \circ \\ & & |\,4 & & \\ & & \circ & & \end{array} \tag{7}$$

acting on E^3 and the hyperbolic Coxeter group $\Gamma_* = [4, 3^{2,1}]$ of rank 5 acting on H^4 given by the diagram

$$\Sigma_* : \circ\!\!-\!\!\!-\!\!\!-\!\!\circ\!\!-\!\!\!-\!\!\!-\!\!\underset{\underset{\circ}{|\,4}}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\circ . \tag{8}$$

The group A_4 is associated to the honeycomb $\{3,3,3\}$ of S^3 by regular tetrahedra $\{3,3\}$ of dihedral angle $2\pi/3$ (cf. [**14**, pp. 214–215]). A fundamental domain of Γ_{fcc} is the Coxeter simplex Δ_{fcc} with vertices (6) and with diagram (7). Observe that Γ_* contains A_4 and Γ_{fcc} as elliptic and parabolic subgroups, respectively. Moreover, Γ_* has a simplex fundamental domain with one vertex at infinity. Its covolume equals $\pi^2/1440 \simeq 0.00685$ and is the (only) smallest covolume of hyperbolic Coxeter groups with simplex fundamental domain (cf. [**8**]).

1.6. *Sphere packings and the simplicial density bound*

Let $\mathcal{B}$ be a packing of X^n with r-balls B (for balls in S^n, we assume that $r < \pi/4$). Associate to each ball B of $\mathcal{B}$ its Dirichlet–Voronoĭ cell or DV-cell D consisting of all points closer to B than to any other ball of $\mathcal{B}$. Consider the local density of B in D given by $\mathrm{ld}(B, D) = \mathrm{vol}_n(B)/\mathrm{vol}_n(D) < 1$. It can be estimated from above by the simplicial density function $d_n(r)$. For its definition, consider $n+1$ r-balls B mutually touching one another. Their centers give rise to a regular n-simplex S_{reg} of edge length $2r$ and of dihedral angle 2α, say. The simplicial density function is given by

$$d_n(r) = (n+1)\frac{\mathrm{vol}_n(B \cap S_{\mathrm{reg}})}{\mathrm{vol}_n(S_{\mathrm{reg}})} .$$

In the euclidean case, the simplicial density function $d_n(r)$ does not depend on r, and we write $d_n = d_n(r)$. By results of Coxeter, Rogers and Böröczky (cf. [**3**], for example), the local density can be estimated as follows:

$$\mathrm{ld}_n(B, D) \leqslant d_n(r) \quad \forall\ B \in \mathcal{B} .$$

In particular, $d_3 \simeq 0.77964$. In the special case of lattice packings of E^3, a well-known result of Gauss says that the maximal (global) density δ_3 for lattice packings equals

$$\delta_3 = \pi/3\sqrt{2} \simeq 0.74048, \tag{9}$$

and it is uniquely attained by the fcc lattice packing (cf. [**5**]). Consider basis (5) generating the fcc lattice and the packing $\mathcal{B}_{\mathrm{fcc}}$ of E^3 by balls B of radius $\frac{1}{2}$ centered at the different lattice points. The DV-cell of B is a rhombic dodecahedron with characteristic simplex Δ_{fcc}, the center stabilizer of which is the Coxeter group $\Pi_\diamondsuit$ of order 48 with diagram

$$\Sigma_\diamondsuit : \circ\!\!-\!\!\!-\!\!\!-\!\!\circ\overset{4}{\!\!-\!\!\!-\!\!\!-\!\!}\circ .$$

The lattice packing density $\mathrm{vol}_3(B)/\mathrm{vol}_3(D)$ equals δ_3. As a consequence, let $\mathcal{B}_\Lambda$ be an arbitrary lattice packing of E^3 by balls B of radius $\frac{1}{2}$. Denote by d_Λ the packing density of $\mathcal{B}_\Lambda$. Then, for the DV-cell D of B, we obtain

$$\mathrm{vol}_3(D) = \frac{\mathrm{vol}_3(B)}{d_\Lambda} \geqslant \frac{\mathrm{vol}_3(B)}{\delta_3} = \frac{\pi/6}{\pi/3\sqrt{2}} = \frac{1}{\sqrt{2}} . \tag{10}$$

Another important optimality property of the fcc lattice packing $\mathcal{B}_{\mathrm{fcc}}$ is the fact that for each ball there are three further balls in $\mathcal{B}_{\mathrm{fcc}}$ such that the four balls mutually touch one another. The four centers form a regular tetrahedron of edge length 1 (cf. [**3**, § 4]).

In the hyperbolic case, we also allow arrangements $\mathcal{B}_\infty$ by horoballs B_∞ of infinite radius. Consider a horoball B_∞ and a point $p \in H^n$. Then, $\mathrm{dist}(p, B_\infty)$ is defined to be the length of the unique perpendicular from p to the horosphere S_∞ bounding B_∞, where $\mathrm{dist}(p, B_\infty)$ is taken negative for $p \in B_\infty$. The DV-cell D_∞ of B_∞ is defined to be the convex body

$$D_\infty = \{\, p \in H^n \mid \operatorname{dist}(p, B_\infty) \leqslant \operatorname{dist}(p, B'_\infty)\ \forall B'_\infty \in \mathcal{B}_\infty \,\}\,.$$

Since both B_∞ and D_∞ are of infinite volume, the notion of local density has to be modified (cf. [**3**, § 6] or [**10**, § 2.2]). As for the simplicial density function, consider $n+1$ horoballs B_∞ that are mutually tangent. The convex hull of their base points at infinity forms an ideal regular simplex $S^\infty_{\text{reg}} \subset \overline{H^n}$ with dihedral angle given by $2\alpha^n_\infty = \arccos(1/n-1)$. Define

$$d_n(\infty) = (n+1)\frac{\operatorname{vol}_n(B_\infty \cap S^\infty_{\text{reg}})}{\operatorname{vol}_n(S^\infty_{\text{reg}})}\,.$$

The following relation exists between $d_n(\infty)$ and the volume μ_n of an ideal regular hyperbolic n-simplex (cf. [**10**, Theorem 3.2]):

$$d_n(\infty) = \frac{n+1}{n-1}\cdot\frac{\sqrt{n}}{2^{n-1}}\cdot\prod_{k=2}^{n-1}\left(\frac{k-1}{k+1}\right)^{n-k/2}\cdot\frac{1}{\mu_n}\,.$$

By combining this result with [**9**, (14.57)], we derive the value

$$d_4(\infty) \simeq 0.73046. \tag{11}$$

By a result of Böröczky [**3**, § 6], the local horoball density $\operatorname{ld}_n(B_\infty, D_\infty)$ of any element $B_\infty \in \mathcal{B}_\infty$ with respect to its DV-cell D_∞ is bounded from above by $d_n(\infty)$.

2. *Hyperbolic orbifolds*

2.1. *Basic notions*

Denote by Q a complete hyperbolic n-orbifold, that is, $Q = H^n/\Gamma$ with $\Gamma < \operatorname{Iso}(H^n)$ a discrete group of hyperbolic isometries (cf. [**13**, § 13]). In the sequel, we tacitly suppose a hyperbolic orbifold to be complete and of finite volume.

Denote by $\pi : H^n \to Q = H^n/\Gamma$ the natural projection. Let $q \in Q$ and choose a point $p \in \pi^{-1}(q)$. The stabilizer Γ_p of p in Γ is a finite group. If $p' \in \pi^{-1}(q)$ is another point lying above q, then the stabilizer $\Gamma_{p'}$ is Γ-conjugate and therefore isomorphic to Γ_p. In the sequel, we often do not distinguish between these groups and introduce the notion of isotropy group or stabilizer Γ_q of q.

The singular locus $S \subset Q$ consists of all points $q \in Q$ with $\Gamma_q \neq 1$. The set S is closed with empty interior. Let Q be non-compact. Then, the group Γ contains parabolic elements giving rise to subgroups of parabolic type and canonical cusps in Q (cf. § 1.3). Let $\mathcal{C}$ denote the finite set of (disjoint) canonical cusps C in Q. Then, $\operatorname{vol}_n(Q) \geqslant \operatorname{vol}_n(\mathcal{C}) = \sum_{C\in\mathcal{C}} \operatorname{vol}_n(C)$. As in the case of oriented hyperbolic n-orbifolds with empty singular set (cf. [**11**, § 3]), one verifies the following improvement.

Lemma 3. *Let Q be a hyperbolic n-orbifold. Then,*

$$\operatorname{vol}_n(Q) \geqslant \frac{\operatorname{vol}_n(\mathcal{C})}{d_n(\infty)}\,. \tag{12}$$

Next, consider a single element $C \in \mathcal{C}$. For simplicity, suppose that Γ has non-trivial stabilizer Γ_∞. It acts as crystallographic group on E^{n-1} with translational lattice Λ of rank $n-1$ and of finite index $i_\Lambda = [\Gamma_\infty : \Lambda]$. Denote by $\mu > 0$ the shortest translational length in Λ and consider a Dirichlet fundamental domain $P \subset E^{n-1}$ for Λ. Then, P contains a ball $B_0 = B(\mu/2)$ which is part of the lattice packing $\mathcal{B}_\Lambda = \{\, \gamma B_0 \mid \gamma \in \Lambda \,\}$ of E^{n-1}. In fact, P is the DV-cell of B_0. Denote by d_Λ the corresponding lattice density.

Lemma 4. *Let $C = B_\infty(\mu)/\Gamma_\infty \subset Q$ be a canonical cusp. Then*

$$\mathrm{vol}_n(C) = \frac{\mathrm{vol}_{n-1}(P)}{(n-1)\cdot\mu^{n-1}\cdot i_\Lambda} = \frac{\mathrm{vol}_{n-2}(S^{n-2})}{2^{n-1}\cdot(n-1)^2\cdot i_\Lambda\cdot d_\Lambda}\,. \tag{13}$$

Proof. By Section 1.6, we have

$$\mathrm{vol}_{n-1}(P) = \frac{\mathrm{vol}_{n-1}(B_0)}{d_\Lambda} = \frac{\mathrm{vol}_{n-2}(S^{n-2})\cdot\mu^{n-1}}{2^{n-1}\cdot(n-1)\cdot d_\Lambda}\,.$$

For the action of the Poincaré extension of Λ on $B_\infty(\mu)$, a fundamental domain F is of the form

$$F = \{\, x = (x_1,\dots,x_n)\in H^n \mid (x_1,\dots,x_{n-1})\in P\ ;\ x_n > \mu \,\}$$

and of volume

$$\mathrm{vol}_n(F) = \int_F \frac{dx_1\cdots dx_n}{x_n^n} = \mathrm{vol}_{n-1}(P)\cdot\int_\mu^\infty \frac{dx_n}{x_n^n} = \frac{\mathrm{vol}_{n-1}(P)}{(n-1)\cdot\mu^{n-1}}\,.$$

Therefore,

$$\mathrm{vol}_n(F) = \frac{\mathrm{vol}_{n-2}(S^{n-2})}{2^{n-1}\cdot(n-1)^2\cdot d_\Lambda}\,.$$

For the canonical cusp $C = B_\infty(\mu)/\Gamma_\infty$, we deduce

$$\mathrm{vol}_n(C) = \frac{\mathrm{vol}_n(F)}{i_\Lambda} = \frac{\mathrm{vol}_{n-1}(P)}{(n-1)\cdot\mu^{n-1}\cdot i_\Lambda} = \frac{\mathrm{vol}_{n-2}(S^{n-2})}{2^{n-1}\cdot(n-1)^2\cdot i_\Lambda\cdot d_\Lambda}\,. \qquad \square$$

2.2. *Examples*

An important class of hyperbolic orbifolds arises as quotients of H^n by hyperbolic Coxeter groups. Let us consider the non-compact case only. It is not difficult to see (cf. [2, § 10], for example) that the planar Coxeter group $\Gamma_*^2 = (2,3,\infty)$ given by diagram

$$\Sigma_*^2 : \circ\!\!-\!\!-\!\!\circ\overset{\infty}{-\!\!-}\!\!\circ$$

is of minimal covolume equal to $\pi/6$. More precisely, the coset space H^2/Γ_*^2 is the unique minimal volume orbifold among all non-compact hyperbolic 2-orbifolds.

The Coxeter group Γ_*^3 given by diagram

$$\Sigma_*^3 : \circ\!\!-\!\!-\!\!\circ\!\!-\!\!-\!\!\circ\overset{6}{-\!\!-}\!\!\circ$$

is not cocompact but of finite covolume equal to $\frac{1}{8}\,\text{Л}(\frac{1}{3}\pi) \simeq 0.04229$. Here

$$\text{Л}(\alpha) = -\int_0^\alpha \log|2\sin t|\,dt = \frac{1}{2}\sum_{r=1}^\infty \frac{\sin(2r\alpha)}{r^2}$$

denotes the Lobachevsky function (cf. [9, for example]). Meyerhoff [12] showed that the oriented double cover of the coset space H^3/Γ_*^3 is a minimal volume orbifold among all non-compact oriented hyperbolic 3-orbifolds.

Finally, consider the hyperbolic Coxeter group $\Gamma_* := \Gamma_*^4$ given by diagram (8). It is of covolume $v_* = \pi^2/1440 \simeq 0.00685$ (cf. Section 1.5). We shall prove that the coset space $Q_* := H^4/\Gamma_*$ is the unique minimal volume orbifold among all non-compact hyperbolic 4-orbifolds.

2.3. *Small-volume hyperbolic* 4*-orbifolds*

Let $Q = H^4/\Gamma$, $\Gamma < \mathrm{Iso}(H^4)$, be a non-compact hyperbolic 4-orbifold of volume

$$\mathrm{vol}_4(Q) \leqslant 0.012\,. \tag{14}$$

For short we say that Q is of small volume.

Lemma 5. *Let Q be a non-compact hyperbolic 4-orbifold of small volume. Then, Q has only one cusp.*

Proof. It suffices to show that Q has only one canonical cusp. Write $Q = H^4/\Gamma$ with $\Gamma < \mathrm{Iso}(H^4)$ and denote by $\mathcal{C}$ its set of $k \geqslant 1$ canonical cusps. By Lemma 3, (11) and (14)

$$0.012 \geqslant \mathrm{vol}_4(Q) \geqslant \frac{\mathrm{vol}_4(\mathcal{C})}{d_4(\infty)} > 1.36 \cdot \mathrm{vol}_4(\mathcal{C}). \tag{15}$$

Consider an arbitrary element $C \in \mathcal{C}$ arising as quotient of a canonical horoball in H^4 by some subgroup $\Gamma_q < \Gamma$ of parabolic type with translational lattice Λ (cf. Section 1.3). By (13), and since $i_\Lambda \leqslant 48$ (cf. [**4**, p. 72]) and $d_\Lambda \leqslant \delta_3 = \pi/\sqrt{18}$, we deduce that

$$\mathrm{vol}_4(C) = \frac{\mathrm{vol}_2(S^2)}{8 \cdot 9 \cdot i_\Lambda \cdot d_\Lambda} > 0.00491. \tag{16}$$

By combining (15) and (16), $0.012 \geqslant \mathrm{vol}_4(Q) > 1.36 \cdot k \cdot 0.00491 > k \cdot 0.00667$ which implies that $k = 1$. □

Let $Q = H^4/\Gamma$ be a non-compact orbifold of small volume. By Lemma 5, Q has only one cusp. We normalize this situation and assume that $\Gamma_\infty \neq 1$. Denote by Λ its translational subgroup and suppose that its minimal translational length equals 1. Let $P \subset E^3$ be a Dirichlet fundamental domain for the action of Λ. Finally, let $B = B_\infty(\rho)$, $\rho \leqslant 1$, be the maximal horoball associated to ∞. Hence, $C = B_\infty(\rho)/\Gamma_\infty \subset Q$. We investigate image horoballs of B under the action of Γ. A fullsized horoball is a ball of diameter ρ based at a point of E^3.

Lemma 6. *Let Q be a non-compact hyperbolic 4-orbifold of small volume. Then, every non-fullsized horoball is tangent to a larger horoball.*

Proof (compare also [**1**, Lemma 4.8]). Let B_x be a non-fullsized image of B based at $x \in E^3$ and not tangent to any larger horoball. Hence, the upper hemisphere of B_x contains no tangency point with any other image horosphere of ∂B. By Lemma 2, this upper hemisphere is an open ball of radius 1 with respect to the euclidean metric on ∂B_x. Sending B_x to B by an isometry of Γ, we obtain an open ball of radius 1 on ∂B, containing no tangency point of B with any of its fullsized images. Consider a fundamental domain $P \subset E^3$ for the action of Λ. By the maximality of B, at least one point of P is the base point of a fullsized horoball. We conclude that — modulo the action of Λ — P contains at least two disjoint euclidean balls of radius $\rho/2$. In other words, we find two disjoint open balls of radius $\rho/2$ in E^3 denoted by K_1 and K_2 which project to two disjoint open balls of radius $\rho/2$ in the quotient E^3/Λ. Now, consider the lattice packing

$$\mathcal{B}_\Lambda = \{\, \gamma K_1,\, \gamma K_2 \mid \gamma \in \Lambda \,\}$$

of E^3 satisfying the density bound

$$\frac{2\,\mathrm{vol}_3(K_1)}{\mathrm{vol}_3(P)} = \frac{\pi\,\rho^3}{3\,\mathrm{vol}_3(P)} \leqslant d_3\,.$$

By Section 1.6, we deduce that $\mathrm{vol}_3(P) \geqslant \sqrt{2}\,\rho^3$. By (11)–(13), and since $i_\Lambda \leqslant 48$, we obtain the following lower volume bound for Q:

$$\mathrm{vol}_4(Q) \geqslant \frac{\mathrm{vol}_4(C)}{d_4(\infty)} = \frac{\mathrm{vol}_3(P)}{3 \cdot \rho^3 \cdot i_\Lambda \cdot d_4(\infty)} > 0.013,$$

which, by (14), contradicts the small volume assumption. □

LEMMA 7. *Let Q be a non-compact hyperbolic 4-orbifold of small volume. Then, the translational lattice Λ permutes all fullsized horoballs.*

Proof (compare also [1, Lemma 4.7]). By the proof of Lemma 6, the quotient E^3/Λ cannot contain two disjoint open balls of radius $\rho/2$ and so there is only one Λ-orbit of fullsized horoballs. □

LEMMA 8. *Let Q be a non-compact hyperbolic 4-orbifold of small volume. Then, there are largest non-fullsized horoballs and they are tangent to fullsized horoballs.*

Proof. By Lemma 6, it suffices to exclude the possibility of an infinite chain of tangent horoballs γB, $\gamma \in \Gamma \setminus \Gamma_\infty$, getting larger and larger but not fullsized. The existence of such a chain would imply an infinite number of tangent horoballs of euclidean diameter in $[\delta - \varepsilon, \delta]$ for some $\delta < \rho$ and $\varepsilon > 0$ and not being tangent to another larger horoball. Consider a fundamental domain $P \subset E^3$ for the action of Λ. The prism $P \times [0, \delta]$ has finite euclidean volume and cannot contain an infinite number of tangent horoballs all having a euclidean diameter in $[\delta - \varepsilon, \delta]$. By Lemma 6 and the fact that P yields a lattice tiling of E^3, we deduce that all these horoballs must be of the same euclidean diameter. This implies the absence of tangency points on their upper hemispheres. A similar argument as in the proof of Lemma 6 yields now a contradiction to the small volume assumption. □

3. *The cusped hyperbolic 4-orbifold of minimal volume*

3.1. Let $Q_0 = H^4/\Gamma_0$, $\Gamma_0 < \mathrm{Iso}(H^4)$, be a non-compact hyperbolic 4-orbifold of minimal volume (cf. [14, Part II, Chapter 7, § 3], for example). Then, $\mathrm{vol}_4(Q_0) \leqslant v_* = \pi^2/1440 \simeq 0.00685$.

PROPOSITION 1. *Let $Q_0 = H^4/\Gamma_0$ be a hyperbolic 4-orbifold of minimal volume. Then, a maximal subgroup of parabolic type in Γ_0 is isomorphic to the crystallographic group Γ_{fcc}, the symmetry group of the* fcc *lattice.*

Proof. Modulo conjugation, assume that the stabilizer $\Gamma_\infty < \Gamma_0$ is non-trivial and that its lattice Λ of index $i_\Lambda = [\Gamma_\infty : \Lambda]$ has minimal translational length 1. Let P be a fundamental parallelepiped of Λ. Then, by Lemmata 3, 4 and by (11),

$$0.00685 \geqslant \mathrm{vol}_4(Q_0) \geqslant \frac{\mathrm{vol}_3(P)}{3 \cdot i_\Lambda \cdot d_4(\infty)} > 0.45 \cdot \frac{\mathrm{vol}_3(P)}{i_\Lambda} . \tag{17}$$

The crystallographic groups and the associated Bravais lattices in E^3 are well known (cf. [4]). In particular, for non-cubical lattices Λ, $i_\Lambda \leqslant 24$ so that (17) together with (10) imply the inequality

$$\mathrm{vol}_4(Q_0) \geqslant 0.45 \cdot \frac{1/\sqrt{2}}{24} > 0.01 > 0.00685.$$

As a consequence, the lattice Λ associated to Γ_∞ is cubical with $i_\Lambda = 48$. For the sc lattice Λ_{sc} with minimal translational length 1, $\mathrm{vol}_3(P) = 1$ which, by (17), has to be excluded. For the bcc lattice Λ_{bcc} with minimal translational length 1, one has $\mathrm{vol}_3(P) = 4/\sqrt{27}$ which yields a contradiction to (17) as well. Therefore, Λ is equal to the fcc lattice Λ_{fcc}. By Lemma 7, its 0-orbit is identical to the Γ_∞-orbit of 0. Hence, Γ_∞ is isomorphic to Γ_{fcc} (cf. § 1.4; [4, p. 161]). □

LEMMA 9. *Let Q_0 be a non-compact hyperbolic 4-orbifold of minimal volume. Then, the canonical horoball is maximal.*

Proof. By Lemma 5, $Q_0 = H^4/\Gamma_0$ has only one cusp which we assume to be maximal. Denote it by M. Modulo conjugation, suppose that the stabilizer $\Gamma_\infty < \Gamma_0$ is non-trivial. By Proposition 1, Γ_∞ is isomorphic to the crystallographic group Γ_{fcc} the translational lattice Λ of which is of index 48. Suppose that the minimal translational length of Λ equals 1. The point group of Γ_∞ can be identified with $\Pi_\diamond$. A fundamental domain of Λ_∞ is a rhombic dodecahedron. It is the DV-cell of a ball of radius $\frac{1}{2}$ giving rise—by the action of the center stabilizer $\Pi_\diamond$—to the 12 symmetry axes passing through the ball center and the centers (or kissing points) of the rhombic facets.

Now, let $B = B_\infty(\rho)$ denote the maximal horoball based at ∞ with $\rho \leqslant 1$ covering M. By (4), we have to show that $\rho = 1$. Suppose that $\rho \neq 1$. Then, fullsized horoballs are pairwise disjoint. We shall derive the inequality $\rho \leqslant 1/\sqrt{2}$ which yields the volume bound $\mathrm{vol}_4(Q_0) \geqslant \mathrm{vol}_4(M)/d_4(\infty) > 0.019 > v_*$ (cf. proof of Lemma 4) in contradiction to the minimal volume assumption.

Without loss of generality, we may suppose that one fullsized horoball, say B_0, is based at 0. Let B_1 be one of the 12 fullsized horoballs at minimal euclidean distance from B_0. Its base point $x_1 \in E^3$ satisfies $d_0(0, x_1) = 1$.

Next, there is an element $\gamma \in \Gamma_0 \setminus \Gamma_\infty$ such that $\gamma B_0 = B$. Consider the fullsized horoball $\gamma B =: B'$. The image $\gamma B_1 =: B_1'$ is one of the 12 largest non-fullsized horoballs touching B'. It belongs to the packing of H^4 by disjoint horoballs induced by the action of Γ_0 on B. In fact, B_1' is a ball of euclidean diameter ρ^3, the base point $\gamma(x_1)$ of which is at euclidean distance ρ^2 from the base point $\gamma(\infty)$ of B'. To see this, denote by τ the euclidean diameter of B_1' and by p the touching point of B_1' with B'. By (1) and Lemma 1,

$$\mathrm{dist}_0((\gamma(\infty), \rho), p) = \frac{\rho}{\mathrm{dist}_0(\gamma(\infty), \gamma(x_1))} = \frac{\rho}{\sqrt{\rho\,\tau}} = \frac{\sqrt{\rho}}{\sqrt{\tau}}\,. \tag{18}$$

Since $\mathrm{dist}_0((\gamma(\infty), \rho), p)$ equals the minimal translation length $1/\rho$ of $\gamma\Lambda\gamma^{-1}$, (18) implies that $\tau = \rho^3$ and $\mathrm{dist}_0(\gamma(\infty), \gamma(x_1)) = \rho^2$.

By Lemma 7, the horoball B' is based at a lattice point $\lambda(0)$ for some $\lambda \in \Lambda$. Consider the inverse image $\lambda^{-1}(B_1')$ of B_1'. We show that its base point y is collinear with 0 and the base point x of a fullsized horoball at minimal translational distance 1 from $B_0 = \lambda^{-1}(B')$ implying that $2\rho^2 \leqslant 1$ as desired. To this end, it is sufficient to show that y lies on one of the 12 symmetry axes of the DV-cell of B_0 as described above. Consider the stabilizer $\Pi_\diamond$ of 0 given by diagram

$$\Sigma_\diamond : \circ\!\!-\!\!\!-\!\!\!-\!\!\circ\overset{4}{-\!\!\!-\!\!\!-}\!\!\circ$$

which is of order 48 and the fundamental domain of which is a cone based at 0 bounded by the mirrors of the three generating reflections with dihedral angles $\pi/2, \pi/3$ and $\pi/4$. We look at the possible positions of y in the cone. The only way to get length 12 for the $\Pi_\diamond$-orbit of point y is for y to lie on the intersection line l of the mirror planes forming the angle $\pi/2$. In this case, the product of the corresponding reflections is a rotation of order 2 fixing l pointwise. Finally, observe that l is identical to one of the 12 symmetry axes passing though 0 and the centers of the rhombic facets. □

REMARK 1. Denote by σ the reflection with respect to the hemisphere H_0 bisecting B and B_0. The above proof shows that

$$\sigma \in \Gamma_0.$$

Indeed, consider the Möbius transformation $\gamma \in \Gamma_0 \setminus \Gamma_\infty$ with $\gamma B_0 = B$. Then, we can write $\gamma = \psi \circ \sigma$, where ψ denotes a euclidean isometry fixing ∞ (cf. [**2**, § 3.5]). Consider the composition $\phi := \lambda^{-1} \circ \gamma = \lambda^{-1} \circ \psi \circ \sigma$, where $\lambda \in \Lambda$ denotes the translation above. Then, ϕ fixes the touching point $(0, 1)$ of B_0 with B and permutes the 12 fullsized horoballs tangent to B_0. As H_0 touches ∂H^4 at their 12 base points, σ fixes these points. Therefore, $\lambda^{-1} \circ \psi$

permutes the 12 base points and fixes their centroid 0 as well as ∞. As a consequence, $\lambda^{-1} \circ \psi$ permutes the 12 face centers and fixes the center of the rhombic dodecahedron so that it belongs to its symmetry group $\Pi_{\diamondsuit}$. Hence, $\sigma \in \Gamma_0$ as asserted.

3.2. Let $Q_0 = H^4/\Gamma_0$, $\Gamma_0 < \mathrm{Iso}(H^4)$, be a non-compact hyperbolic 4-orbifold of minimal volume. By Lemma 5, Q_0 has precisely one canonical cusp C which we may assume to be of the form $C = B_\infty(1)/\Gamma_{\mathrm{fcc}}$. By Lemma 9, the canonical horoball $B = B_\infty(1)$ is maximal. Therefore, some of the images, $B_q = \gamma B$, $\gamma \in \Gamma_0 \setminus \Gamma_{\mathrm{fcc}}$, are fullsized of diameter 1. Assume without loss of generality that one of these fullsized horoballs is based at $0 \in \partial H^4$ and call it B_0. Then, the base points $q \in \partial H^4$ belong to the Λ_{fcc}-orbit of 0. Orthogonal projection of the fullsized horoballs onto the horosphere ∂B gives the fcc lattice packing of balls of radius $\frac{1}{2}$ of E^3. A fundamental domain of the action of Γ_{fcc} on E^3 is given by the Coxeter simplex Δ_{fcc} with vertices a, b, c, d according to (6). Here, a is the center of a ball in the fcc lattice packing and tangency point with the fullsized horoball B_0. Obviously, the diameter ρ of Δ_{fcc} equals $\mathrm{dist}_0(a, b) = 1/\sqrt{2}$. By Poincaré extension, a Dirichlet fundamental domain for the action of Γ_{fcc} must be contained in the cylinder $Z = \Delta_{\mathrm{fcc}} \times]0, \infty[$ of width ρ.

Proposition 2. *Let $Q_0 = H^4/\Gamma_0$ be a non-compact hyperbolic 4-orbifold of minimal volume. Then, the Ford domain of Γ_0 is a hyperbolic 4-simplex with precisely one vertex at infinity.*

Proof. We will show that the effect of elements in $\Gamma_0 \setminus \Gamma_{\mathrm{fcc}}$ induces a cut of Z by one additional geodesic hemisphere H so that a fundamental domain for Γ_0 is a simplex. The simplex will have ∞ as single vertex at infinity and its local structure is a cone over the tetrahedron Δ_{fcc}. To this end, consider the horoball packing $\mathcal{B}_\infty = \{ \gamma B \mid \gamma \in \Gamma_0 \}$ of H^4 and study the DV-cell D (of infinite volume) of element B. In fact, a Ford fundamental domain for Γ_0 is the intersection $D \cap Z$. By construction of Δ_{fcc} (cf. also Section 1.4), B_0 is the only fullsized horoball in $\mathcal{B}_\infty$ with non-empty intersection with Z. Consider the bisecting hyperplane H_0 of B and B_0 passing therefore through $(a, 1)$. We will show that $H = H_0$. More concretely, we will show that the bisecting hyperplanes associated to the largest horoballs in $\mathcal{B}_\infty$ of diameter less than 1 do not affect the codimension 1 face complex of $D \cap Z$. By Lemma 8, there is a largest horoball $B' \in \mathcal{B}_\infty$ of diameter $\delta < 1$ which touches B_0. Denote by x its base point. An easy calculation shows that $\mathrm{dist}_0(0, x) = |x| = \sqrt{\delta}$. Let H' be the bisecting hyperplane of B' and B. Consider the intersection $L = H_0 \cap H'$ in the 2-plane E determined by $0, x, \infty$. By symmetry, it is sufficient to show that point $s = (u, v) := E \cap L$ satisfies $|u| \geqslant \rho = 1/\sqrt{2}$. Suppose without loss of generality that $u > 0$. First, we determine the radius r of the half-circle $E \cap H'$. In fact, since the hyperbolic distances from H' to B' and B, respectively, are identical, we deduce that

$$\log \frac{1}{r} = \log \frac{r}{\delta}, \quad \text{that is,} \ r = \sqrt{\delta}. \tag{19}$$

Next, since $s = (u, v) \in H_0 \cap H'$, we obtain $u^2 + v^2 = 1$, and by (19), $(u - \sqrt{\delta})^2 + v^2 = \delta$. That is,

$$u = \frac{1}{2\sqrt{\delta}}\,. \tag{20}$$

Consider the group $\widetilde{\Gamma}$ conjugate to Γ_{fcc} acting on the horosphere ∂B_0 with lattice $\tilde{\Lambda}$ of shortest translational length 1 isomorphic to Λ_{fcc}. By Lemma 2, the euclidean distance from the north pole $(a, 1) \in \partial B_0$ to the tangent point $p = \partial B_0 \cap \partial B'$ equals the second minimal translational length $\sqrt{2}$ of $\tilde{\Lambda}$ (cf. Section 1.4). By Lemma 1, we conclude that $\sqrt{\delta} = 1/\sqrt{2}$. By (20), $u = 1/\sqrt{2}$. The situation is illustrated in Figure 2. Therefore, we have showed that the hyperplane H' has empty intersection with the interior of $Z \cap D$. $\square$

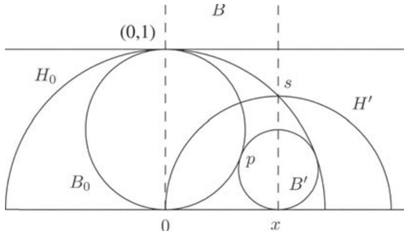

FIGURE 2.

We know that a minimal volume non-compact hyperbolic 4-orbifold is the quotient of H^4 by a discrete group $\Gamma_0 < \mathrm{Iso}(H^4)$ with simplex fundamental domain $S \subset H^4$ having precisely one vertex at infinity. The structure at infinity is a cone over the Coxeter tetrahedron Δ_{fcc} which itself is a fundamental domain of the parabolic Coxeter group $\Gamma_{\mathrm{fcc}} = [4, 3^{1,1}]$ with the following diagram (cf. (7)).

$$\Sigma_{\mathrm{fcc}} : \begin{array}{c} \circ\!\!-\!\!\!-\!\!\!-\!\!\circ\!\!-\!\!\!-\!\!\!-\!\!\circ \\ \;\;|\,4 \\ \;\;\circ \end{array}$$

The next result distinguishes one of the remaining ordinary vertices of S as being a center of high regularity (cf. Section 1.5).

PROPOSITION 3. *Let $Q_0 = H^4/\Gamma_0$ be a non-compact hyperbolic 4-orbifold of minimal volume. Then, Γ_0 contains the elliptic Coxeter group $A_4 = [3, 3, 3]$.*

Proof. Consider the fundamental simplex $S \subset H^4$ as above. Suppose that the vertex at infinity is ∞ and denote by B_∞ the associated canonical horoball. The vertex neighborhood of ∞ in S is a chimney in B_∞ over the Coxeter tetrahedron $\Delta_{\mathrm{fcc}} \subset \partial B_\infty$, the vertices of which may be chosen to be a, b, c, d according to (6) (cf. also Figure 1). The tetrahedron Δ_{fcc} is a fundamental domain of the symmetry group $\Gamma_{\mathrm{fcc}} < \Gamma_0$ associated to the fcc lattice packing $\mathcal{B}_{\mathrm{fcc}}$ of E^3 with balls of radius $1/2$. Observe that vertex d is equidistant from four mutually tangent balls of $\mathcal{B}_{\mathrm{fcc}}$ which, together with their centers, are permuted by means of the subgroup $A_3 = [3, 3]$ of Γ_{fcc} (cf. (7)). After Poincaré extension, these balls become four mutually tangent fullsized horoballs $B_0, \ldots, B_3$ in H^4. By Remark 1, Section 3.1, the reflection σ with respect to the unit hemisphere H_0 (cf. Figure 2) belongs to Γ_0. The element σ transposes B_0 and $B_\infty =: B_4$ and fixes the horoballs B_1, B_2, B_3 since H_0 touches ∂H^4 at their base points. The passage to the ball model (B, ds_B^2) (cf. (2)) turns the horoballs $B_0, \ldots, B_4$ into five mutually tangent horoballs. Modulo an isometry, we may assume that their radical point z (intersection point of the five bisector hyperplanes) equals the origin $0 \in B$ of E^4 and the five horoballs are congruent in E^4. Their euclidean centers form a regular simplex S_{reg} with center $z = 0$ in E^4, the symmetry group of which is generated by the reflections of A_3 and σ. Therefore, the Coxeter group $A_4 = [3, 3, 3]$ is a subgroup of Γ_0. □

3.3. We are now ready to prove our main result.

THEOREM. *The non-compact hyperbolic 4-orbifold of minimal volume is given by $Q_* = H^4/\Gamma_*$, where Γ_* is the Coxeter group*

$$\Sigma_* : \begin{array}{c} \circ\!\!-\!\!\!-\!\!\!-\!\!\circ\!\!-\!\!\!-\!\!\!-\!\!\circ\!\!-\!\!\!-\!\!\!-\!\!\circ . \\ \qquad\qquad\;\;|\,4 \\ \qquad\qquad\;\;\circ \end{array}$$

Q_* is unique, 1-cusped and of volume $v_* = \pi^2/1440$.

Proof. Let $Q_0 = H^4/\Gamma_0$, $\Gamma_0 < \mathrm{Iso}(H^4)$, denote a non-compact hyperbolic 4-orbifold of minimal volume. By Propositions 1, 2 and 3, a fundamental domain of Γ_0 is a simplex bounded by five hyperplanes with precisely one vertex at infinity. Moreover, Γ_0 contains Coxeter subgroups Γ_{fcc} and A_4 given by diagrams

$$\Sigma_{\mathrm{fcc}} : \begin{tikzcd} \circ \arrow[r, no head] & \circ \arrow[r, no head] \arrow[d, no head, "4"] & \circ \\ & \circ & \end{tikzcd} \quad \text{and} \quad \Sigma_4 : \begin{tikzcd} \circ \arrow[r, no head] & \circ \arrow[r, no head] & \circ \arrow[r, no head] & \circ \end{tikzcd} \tag{21}$$

respectively, each being generated by reflections in four of the five bounding hyperplanes and fixing two of the five vertex figures of S. However, in a 4-simplex, the passage to one vertex figure corresponds to the omission of one node together with its connecting edges in the Vinberg graph of order 5 and valence less than or equal to 4 (cf. [**14**, Part I, Chapter 6; Part II, Chapter 5, § 1.3]). Hence, the only hyperbolic 4-simplex bounded by hyperplanes giving rise to reflections grouped together to satisfy (21) is the Coxeter simplex associated to the Coxeter group Γ_* with diagram Σ_*. □

REMARK 2. It is well known that the Coxeter group Γ_* is arithmetic. Consult [**14**, p. 226 ff] concerning an arithmeticity criterion for hyperbolic Coxeter groups.

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