

CoxIter – Computing invariants of hyperbolic Coxeter groups

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ABSTRACT

CoxIter is a computer program designed to compute invariants of hyperbolic Coxeter groups. Given such a group, the program determines whether it is cocompact or of finite covolume, whether it is arithmetic in the non-cocompact case, and whether it provides the Euler characteristic and the combinatorial structure of the associated fundamental polyhedron. The aim of this paper is to present the theoretical background for the program. The source code is available online as supplementary material with the published article and on the author's website (<http://coxiter.rgug.ch>).

[Supplementary materials are available with this article.](#)

Introduction

Let \mathcal{H}^n be the hyperbolic n -space, and let $\text{Isom } \mathcal{H}^n$ be the group of isometries of \mathcal{H}^n . For a given discrete hyperbolic Coxeter group $\Gamma < \text{Isom } \mathcal{H}^n$ and its associated fundamental polyhedron $P \subset \mathcal{H}^n$, we are interested in geometrical and combinatorial properties of P . We want to know whether P is compact, has finite volume and, if the answer is yes, what its volume is. We also want to find the combinatorial structure of P , namely, the number of vertices, edges, 2-faces, and so on. Finally, it is interesting to find out whether Γ is arithmetic, that is, if Γ is commensurable to the reflection group of the automorphism group of a quadratic form of signature $(n, 1)$. Most of these questions can be answered by studying finite and affine subgroups of Γ , but this involves a huge number of computations.

This article presents the algorithms used in CoxIter, a computer program written in C++ designed to compute these invariants. The program is published under a free license (the GNU General Public License v3) and can be used freely in various projects. The source code and the documentation are available as supplementary material with the online version of this article and on the author's website.

The input of CoxIter is the graph of a hyperbolic Coxeter group (encoded in a simple way in a text file, see [Appendix A](#)) and a typical output can be the following.

```
Reading file: ../graphs/14-vinb85.coxiter
  Number of vertices: 17
  Dimension: 14
  Vertices: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17
  Field generated by the entries of the Gram matrix: Q[sqrt(2)]
```

File read

Information

```
Cocompact: no
Finite covolume: yes
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Arithmetic: yes
f-vector: (94, 704, 2695, 6825, 12579, 17633, 19215, 16425, 11009,
5733, 2275, 665, 135, 17, 1)
Number of vertices at infinity: 5
Alternating sum of the components of the f-vector: 0
Euler characteristic: -87757/289236647411712000
Covolume: pi^7 * 87757/305359330843607040000
    
```

In the first section, we give the theoretical background that is needed to present the algorithms and their implementation in the program. These algorithms are explained in the second section. In the third part, we give some applications of CoxIter: we use the program to compute the invariants of a few Coxeter graphs in higher dimensions, including recently found groups of Vinberg in dimension eighteen. In the last section, we explain how CoxIter was tested.

1. Theoretical background

In this section we present the theoretical tools needed to implement the algorithms in CoxIter.

1.1. Hyperbolic space

We denote by $\mathbb{R}^{n,1}$ the space \mathbb{R}^{n+1} endowed with the *Lorentzian product* given by

$$(x = (x_0, x_1, \dots, x_n), y = (y_0, y_1, \dots, y_n)) \mapsto \langle x, y \rangle = -x_0y_0 + \sum_{i=1}^n x_iy_i.$$

For a vector $x \in \mathbb{R}^{n,1}$, we call the scalar $\|x\|^2 := \langle x, x \rangle \in \mathbb{R}$ the square of the *Lorentzian norm* of x . Then, we let $\mathcal{H}^n \subset \mathbb{R}^{n,1}$ be the set defined by

$$\mathcal{H}^n := \{x \in \mathbb{R}^{n,1} : \langle x, x \rangle = -1, x_0 > 0\} \subset \mathbb{R}^{n,1}.$$

If we endow \mathcal{H}^n with the map $d : \mathcal{H}^n \times \mathcal{H}^n \rightarrow \mathbb{R}$, which sends (x, y) to $\operatorname{arcosh}(-\langle x, y \rangle)$, then (\mathcal{H}^n, d) is a metric space, which is the *Minkowski model* of the hyperbolic n -space. The *boundary* of \mathcal{H}^n can be identified with the set

$$\partial\mathcal{H}^n = \left\{ x \in \mathbb{R}^{n,1} : \langle x, x \rangle = 0, \sum_{i=0}^n x_i^2 = 1, x_0 > 0 \right\},$$

and we let $\overline{\mathcal{H}^n} := \partial\mathcal{H}^n \cup \mathcal{H}^n$.

A *hyperplane* of \mathcal{H}^n is given by the intersection of the orthogonal complement (with respect to the Lorentzian product) of a vector v of Lorentzian norm one and \mathcal{H}^n . We denote such a hyperplane by H_v . Note that such a hyperplane splits \mathcal{H}^n into two half-spaces $H_v^+ := \{x \in \mathcal{H}^n : \langle v, x \rangle \geq 0\}$ and $H_v^- := \{x \in \mathcal{H}^n : \langle v, x \rangle \leq 0\}$, whose intersection is H_v . The relative behaviour of two distinct hyperplanes H_v and H_w can be described by means of the Lorentzian product of v and w (see [17, Theorems 3.2.6, 3.2.7 and 3.2.9] or [23, § 1.1]).

- The hyperplanes intersect if and only if $|\langle v, w \rangle| < 1$. In this case, the acute dihedral angle between them is given by $\arccos(|\langle v, w \rangle|)$.
- The hyperplanes are *parallel* if and only if $|\langle v, w \rangle| = 1$. In this case their dihedral angle is zero.
- The hyperplanes are *ultraparallel* if and only if $|\langle v, w \rangle| > 1$. In this setting, the two hyperplanes share a common perpendicular of length $\operatorname{arcosh}(|\langle v, w \rangle|)$.

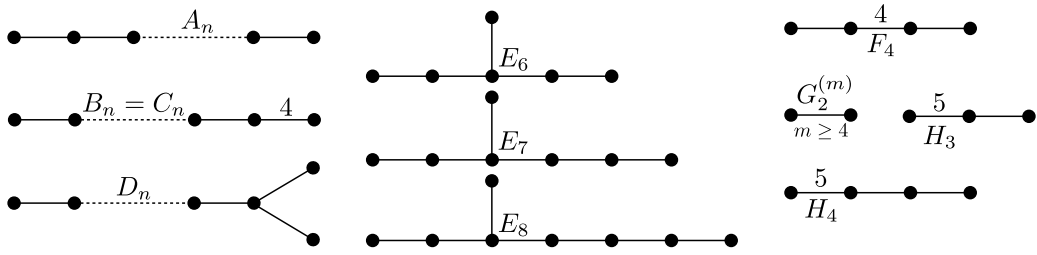


FIGURE 1. Irreducible finite Coxeter groups.

1.2. Abstract Coxeter groups

DEFINITION 1. A Coxeter group is a finitely presented group given by the presentation

$$\langle s_1, \dots, s_d : (s_i s_j)^{m_{ij}} = 1 \rangle,$$

where $m_{ij} = 1$ if and only if $i = j$, and $m_{ij} = m_{ji} \in \{2, 3, \dots\} \cup \{\infty\}$ if $i \neq j$. By $m_{ij} = \infty$, we mean that there is no relation between s_i and s_j . If Γ is such a group and $S := \{s_1, \dots, s_d\}$ is the set of generators, we will refer to the group as (Γ, S) , or just Γ if there is no ambiguity on S . When it is convenient, we use the notation $m(s_i, s_j) := m_{ij}$.

If S' is a subset of S , we denote the subgroup of Γ generated by the elements of S' by $\Gamma_{S'}$.

DEFINITION 2. An easy way to represent a Coxeter group $(\Gamma, S = \{s_1, \dots, s_d\})$ is by its Coxeter diagram, or Coxeter graph. The Coxeter diagram of Γ is the graph whose vertices correspond to the elements of S . Moreover, there is an edge between the vertices s_i and s_j if and only if $m(s_i, s_j) \geq 3$. We label such an edge with $m(s_i, s_j)$ if $m(s_i, s_j) \geq 4$. Sometimes, instead of labelling the edge, we use a double edge if $m(s_i, s_j) = 4$ and a bold (or heavy) edge if $m(s_i, s_j) = \infty$.

NOTATION 1. Let $(\Gamma, S = \{s_1, \dots, s_d\})$ be a Coxeter group. We can associate to (Γ, S) the real $d \times d$ matrix $G(\Gamma, S) = (g_{ij}) \in \text{Mat}(d, \mathbb{R})$ defined as follows: $g_{ii} = 1$, $g_{ij} = -1$ if $m(s_i, s_j) = \infty$ and $g_{ij} = -\cos(\pi/m(s_i, s_j))$ otherwise.

EXAMPLES 1.

- It is well known that the symmetric group S_n is isomorphic to the Coxeter group A_{n-1} (see Figure 1): the i th node of A_{n-1} corresponds to the transposition $(i, i + 1)$ of S_n .
- The dihedral group of $2n$ elements is isomorphic to $G_2^{(n)}$ (see Figure 1).
- If P is a polyhedron (see Definition 4) in \mathbb{R}^n whose dihedral angles are integer submultiples of π and if $\Gamma < \text{Isom } \mathbb{R}^n$ is the (discrete) group generated by the reflections in the facets of P , then Γ is a Coxeter group.

DEFINITION 3. A Coxeter group is said to be irreducible if its Coxeter diagram is connected. It is easy to see that a Coxeter group is the product of the groups corresponding to the connected components of its Coxeter diagram.

The finite irreducible Coxeter groups were classified by Coxeter [4] by means of their Coxeter diagrams. The classification is presented in Figure 1.

1.3. Reflection groups

In this section, we present the connection between abstract Coxeter groups and discrete reflection groups in the facets (sides of codimension one) of convex finite-sided polyhedra.

In what follows, X denotes the unit n -sphere S^n , the Euclidean n -space E^n or \mathcal{H}^n .

A vector $v \in E^n$ and a real number $t \in \mathbb{R}$ determine an (affine) *hyperplane* $H_{v,t} := \{x \in E^n : \langle x, v \rangle = t\}$, denoted by H_v if $t = 0$, and such a hyperplane splits E^n into two *half-spaces* $H_{v,t}^+ := \{x \in E^n : \langle v, x \rangle \geq t\}$ and $H_{v,t}^- := \{x \in E^n : \langle v, x \rangle \leq t\}$, whose intersection is $H_{v,t}$. A *hyperplane* in S^n is the intersection of a hyperplane H_v of E^{n+1} with S^n .

DEFINITION 4. A *polyhedron* P in X is a subset of X which is the intersection of a finite number of half-spaces, each one bounded by a hyperplane. Moreover, we require that P has non-empty interior and that no half-space contains the intersection of the others.

DEFINITION 5. Let H_v be a hyperplane of X . The reflection $r_v = r_{H_v}$ associated to the hyperplane H_v is defined as (recall that we choose v with norm one)

$$\begin{aligned} r_{H_v} : X &\longrightarrow X \\ x &\longmapsto x - 2\langle x, v \rangle v. \end{aligned}$$

DEFINITION 6. Let $P = \bigcap_{i=1}^d H_{v_i}^-$ be a polyhedron in X and $\Gamma \leq \text{Isom } X$. We say that Γ is the *reflection group* associated to P if Γ is generated by the reflections in the facets of P : that is, $\Gamma = \langle r_{H_{v_1}}, \dots, r_{H_{v_d}} \rangle$.

Let $P \subset X$ be a polyhedron and let $\Gamma \leq \text{Isom } X$ be its associated reflection group. If Γ is discrete, then all the dihedral angles of P are submultiples of π (that is, zero or of the form π/k where $k \in \mathbb{Z}$). For any facet S_i of P , we let H_{v_i} be the hyperplane containing S_i such that $P \subset H_{v_i}^-$. Now, if the dihedral angle between two adjacent facets S_i and S_j of P is $\pi/\theta(S_i, S_j)$, then the order of $r_{v_i} \circ r_{v_j}$ in Γ is $\theta(S_i, S_j)$ (see [17, Theorem 7.1.2]). Suppose, now, that P is of finite volume and has facets $\mathcal{S} = \{S_1, \dots, S_d\}$. Then, by [17, Theorem 7.1.4],

$$\langle r_1, \dots, r_d | r_i^2 = 1, (r_i \cdot r_j)^{\theta(S_i, S_j)} = 1 \rangle \cong \Gamma.$$

The matrix $G = G(\Gamma, \mathcal{S})$ of a Coxeter group (Γ, \mathcal{S}) (see Notation 1) induces a quadratic form called the *Tits form* on \mathbb{R}^d , where $d = |\mathcal{S}|$, via $x \mapsto x^t G x$. If the matrix G is indecomposable (meaning that we cannot transform G to a block diagonal matrix with permutations of the rows and columns of G) or, equivalently, if the graph of the group is connected, then we get information about the group by looking at the signature (n, p, q) of the quadratic form, where n (respectively, p, q) is the number of positive (respectively, negative, zero) eigenvalues of G . For a positive n , we are interested in the following cases.

$p = q = 0$ (the quadratic form is positive definite). In this case, the group is finite (see the classification of irreducible finite Coxeter groups in Figure 1). It can be shown that Γ can be realized as a discrete group of isometries of the n -sphere S^n . Thus, Γ is called *spherical*.

$p = 0, q > 0$ (the quadratic form is positive semidefinite). The group can be realized as a discrete subgroup of $\text{Isom } E^n$. Hence, it is said to be *affine* (or *euclidean* or *parabolic*). The classification of irreducible affine Coxeter groups is given in Figure 2 (see, for example, [2, Chapter 6, § 4.3, Theorem 4]).

$p = 1, q \geq 0$. The group can be realized as a discrete subgroup of $\text{Isom } \mathcal{H}^n$ (see details in § 1.4). Thus, it is called *hyperbolic*.

Now, if the matrix G is decomposable into a diagonal block matrix with blocks G_1, \dots, G_l , we say that the group is *spherical* (respectively, *affine*) if each block G_i is *spherical* (respectively,

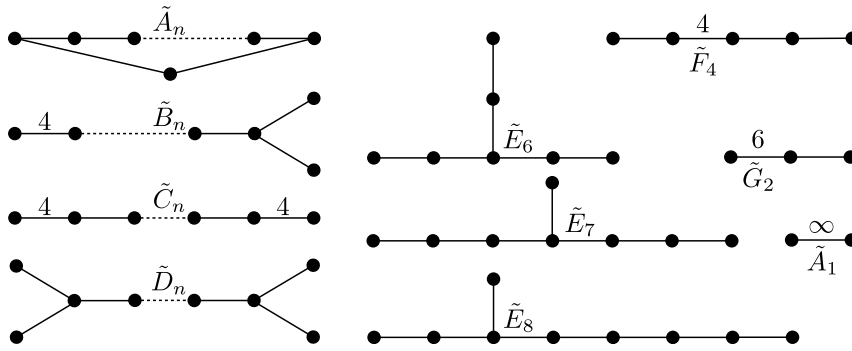


FIGURE 2. Irreducible affine Coxeter groups.

affine). If the matrix G has signature $(n, 1, q)$ for some n and q , we say that the group is hyperbolic.

If \mathcal{G} is the Coxeter diagram of Γ , then a subdiagram of \mathcal{G} is called spherical (respectively, affine, hyperbolic) if the corresponding subgroup of Γ is spherical (respectively, affine, hyperbolic).

1.4. Hyperbolic Coxeter groups and hyperbolic Coxeter polyhedra

In this section, we present some concepts related to hyperbolic Coxeter groups and polyhedra. Let $P \subset \mathcal{H}^n$ be a polyhedron. Hence, there exist vectors v_1, \dots, v_r of Lorentzian norm one and hyperplanes $H_i = \langle v_i \rangle^\perp$ such that $P = \bigcap_{i=1}^r H_i^-$, where H_i^- is the half-space delimited by H_i and given by $H_i^- := \{x \in \mathcal{H}^n : \langle x, v_i \rangle \leq 0\}$.

REMARK 1. Unlike in the Euclidean space, there exist unbounded hyperbolic Coxeter polyhedra of finite volume. Such a polyhedron is the convex hull of a finite set of points $x_1, \dots, x_k \in \overline{\mathcal{H}^n}$ with at least one $x_i \in \partial\mathcal{H}^n$. Such a vertex is called a vertex at infinity of the polyhedron (see [17, §6.4] for more details).

DEFINITION 7. The Gram matrix of the polyhedron $P \subset \mathcal{H}^n$ is the matrix $G = G(P) \in \text{Mat}(r, \mathbb{R})$ defined as $G = (\langle v_i, v_j \rangle)_{1 \leq i, j \leq r}$.

DEFINITION 8. Suppose now that P is a hyperbolic Coxeter polyhedron, that is, the dihedral angles are either zero or submultiples of π . The Coxeter diagram, or Coxeter graph, of P is the graph whose vertices s_i correspond to the hyperplanes H_i . For two hyperplanes H_i and H_j , there is between the vertices s_i and s_j :

- a dotted line if H_i and H_j are ultraparallel (labelled by the hyperbolic cosine of the length of the common perpendicular of the two hyperplanes);
- a line labelled with ∞ if H_i and H_j are parallel (we use sometimes a bold, or heavy, edge with no labelling); and
- a line if the dihedral angle is π/m_{ij} with $m_{ij} \geq 3$, which is simple if $m_{ij} = 3$ and is labelled by m_{ij} if $m_{ij} > 3$. Note that, sometimes, instead of labelling the edge, we use a double edge if $m_{ij} = 4$.

Note that we do not distinguish between the Gram matrix of a hyperbolic Coxeter polyhedron, the corresponding Coxeter diagram and the associated hyperbolic Coxeter group.

There is the following natural question: given a Coxeter graph, is it possible to find a hyperbolic Coxeter polyhedron whose graph is the given one? The answer is yes if the Gram

matrix of the graph has the correct signature, as explained in the next theorem.

THEOREM 1.1 [23, Theorem 2.1]. *Let $G = (G_{ij})$ be an indecomposable symmetric matrix of signature $(n, 1, k)$ such that:*

- $G_{ii} = 1$ for every i ; and
- $G_{ij} \leq 0$ for every $i \neq j$.

Then, there exists a convex polyhedron P in \mathcal{H}^n whose Gram matrix is equal to G . Moreover, P is unique up to isometry. We will refer to P as the polyhedron associated to G .

Therefore, given a hyperbolic Coxeter group Γ , we can speak of geometrical properties of its associated polyhedron $P \subset \mathcal{H}^n$. If the polyhedron P is compact, then the group Γ is called *cocompact*. Note that there is a criterion to decide if Γ is cocompact or not (see § 1.6). In a similar way, if P is of finite volume (with respect to the hyperbolic metric), we say that Γ is *cofinite* or of *finite covolume*. In this case, the *covolume* of Γ is the volume of P . As before, there is a nice criterion to decide whether the group is of finite covolume or not (see § 1.6).

It has been shown that hyperbolic Coxeter groups of finite covolume do not exist in dimensions above 995 (see [16, Theorem C]), but examples of such groups are known only up to dimension 21 (see [1, Example 5]). In the cocompact case, it is known that such groups do not exist if the dimension is greater than 29 (see [22, Theorem 1]). However, examples of such groups are known only up to dimension 8 (see [3] for the arithmetic cocompact hyperbolic Coxeter group in dimension 8 and [14] for a list of cocompact groups in dimensions 5–8).

1.5. Euler characteristic, f -vector and volume

In what follows, unless stated otherwise, $\Gamma < \text{Isom } \mathcal{H}^n$ denotes a hyperbolic Coxeter group with finite set of generator S . Let P be its associated fundamental convex polyhedron.

NOTATION 2. Let G be a finitely generated group with generating set S . We denote by l_S the *length function* of G with respect to S : for an element $g \in G$,

$$l_S(g) = \min\{k \in \mathbb{N} : \exists g_1, \dots, g_k \in S \cup S^{-1} \text{ such that } g = g_1 \cdot \dots \cdot g_k\}.$$

DEFINITION 9. Let G be a finitely generated group with generating set S . The *growth series* of G is the formal power series $f_S(x) = \sum_{g \in G} x^{l_S(g)}$. This series is also called the *Poincaré series* of G .

The next proposition gives the key tool which is used by CoxIter to compute the orbifold Euler characteristic of \mathcal{H}^n/Γ .

PROPOSITION 1.2 [12, (1.2) and (1.3)]. *Let (Γ, S) be an abstract Coxeter group and let $\mathcal{F} = \{T \subset S : \Gamma_T \text{ is finite}\}$. Then*

$$\chi(\Gamma) = \sum_{T \in \mathcal{F}} \frac{(-1)^{|T|}}{f_T(1)},$$

where f_T is the growth series of the group $\Gamma_T \leq \Gamma$ generated by T (see Definition 1).

In order to compute $\chi(\Gamma)$, we see, by using the classification of finite Coxeter groups (see Figure 1), that it is sufficient to know the value $f_T(1)$ for all irreducible finite Coxeter groups. With the growth series (see [12, Table 1], for example), we find the values given in Table 1, page 760.

The next result relates the Euler characteristic of a hyperbolic Coxeter group $\Gamma < \text{Isom } \mathcal{H}^n$ to its covolume when n is even.

PROPOSITION 1.3 [12, (1.4)]. *When n is even,*

$$\text{covolume}(\Gamma) = (-1)^{n/2} \cdot \frac{\pi^{n/2} \cdot 2^n \cdot (n/2)!}{n!} \cdot \chi(\Gamma).$$

REMARK 2. When n is odd, $\chi(\Gamma) = 0$.

We are also interested in the combinatorial properties of the Coxeter polyhedron P associated to Γ . Thus, we consider the vector $(f_0, \dots, f_{n-1}, 1) \in \mathbb{Z}^{n+1}$, where f_i is the number of faces of dimension i of P . This vector is called the f -vector of P (or f -vector of Γ). To compute it, we will use the following results.

THEOREM 1.4 [23, Theorem 3.1]. *Let \mathcal{G} be the Coxeter diagram of Γ . There is a bijective correspondence between spherical subdiagrams of rank k of \mathcal{G} and faces of codimension k of P .*

THEOREM 1.5 [23, Theorem 3.2]. *Let \mathcal{G} be the Coxeter diagram of Γ . There is a bijective correspondence between parabolic subdiagrams of rank $n - 1$ of \mathcal{G} and vertices at infinity of P .*

As a test for the output of the program we also use the following classical result of Euler–Schläfli.

PROPOSITION 1.6 [15]. *For a polyhedron (not necessarily in \mathcal{H}^n), we have the equality*

$$\sum_{i=0}^n f_i = 1 - (-1)^n.$$

1.6. Compactness and finite volume criterion

Since a polyhedron P is compact if and only if it is the convex hull of a finite number of vertices in \mathcal{H}^n (also called *ordinary vertices*), we have the following result.

PROPOSITION 1.7 [23, Proposition 4.2]. *The polyhedron P is compact if and only if the following conditions are satisfied.*

- P contains at least one vertex (that is, face of dimension zero) in \mathcal{H}^n .
- For every vertex of P and every edge of P emanating from it there is another vertex of P on that edge.

Since a polyhedron P is of finite volume if and only if it is the convex hull of a finite number of vertices in $\overline{\mathcal{H}^n}$, we have the following result.

PROPOSITION 1.8 [23, Proposition 4.2]. *The polyhedron P has finite volume if and only if the following conditions are satisfied.*

TABLE 1. Orders of finite Coxeter groups.

Group	$f_S(1)$	Group	$f_S(1)$
A_m	$(m + 1)!$	E_6	51 840
B_m	$2^m \cdot m!$	E_7	2 903 040
D_m	$2^{m-1} \cdot m!$	E_8	696 729 600
G_m	$2m$	H_3	120
F_4	1152	H_4	14 400

- P contains at least one vertex (ordinary or at infinity).
- For every vertex (ordinary or at infinity) of P and every edge of P emanating from it there is another vertex of P (ordinary or at infinity) on that edge.

Using Theorems 1.4 and 1.5 we deduce the following two criteria.

PROPOSITION 1.9 (Cocompactness criterion in CoxIter). *Let $\Gamma < \text{Isom } \mathcal{H}^n$ be a Coxeter group and \mathcal{G} be its Coxeter diagram. The group Γ is cocompact if and only if the following conditions hold.*

- \mathcal{G} contains at least one spherical subdiagram of rank n .
- Each spherical subdiagram of rank $n - 1$ of \mathcal{G} can be extended in exactly two ways to a spherical subdiagram of rank n of \mathcal{G} .

PROPOSITION 1.10 (Finite covolume criterion in CoxIter). *Let $\Gamma < \text{Isom } \mathcal{H}^n$ be a Coxeter group and \mathcal{G} be its Coxeter diagram. The group Γ has finite covolume if and only if the following conditions hold.*

- \mathcal{G} contains at least one spherical subdiagram of rank n or one parabolic subdiagram of rank $n - 1$.
- Each spherical subdiagram of rank $n - 1$ of \mathcal{G} can be extended in exactly two ways to one of the following type of subdiagrams of \mathcal{G} :
 - * a spherical diagram of rank n ; or
 - * a parabolic diagram of rank $n - 1$.

1.7. Arithmeticity

DEFINITION 10. Let $G := (G_{i,j}) \in \text{Mat}(n, K)$ be a square matrix with coefficients in a field K . A cycle of length k , or k -cycle, in G is a product $G_{i_1, i_2} \cdot G_{i_2, i_3} \cdot \dots \cdot G_{i_{k-1}, i_k} \cdot G_{i_k, i_1}$. Such a cycle is denoted by $G_{(i_1, \dots, i_k)}$. If the i_j are all distinct, the cycle is called *irreducible*.

Observe that each cycle is a product of irreducible cycles.

DEFINITION 11. Let \mathcal{G} be an undirected graph. A cycle, or *closed walk*, in \mathcal{G} is a sequence $(v_{i_1}, \dots, v_{i_m}, v_{i_1})$ of adjacent vertices of \mathcal{G} . We say that the cycle is *simple* if all the v_{i_j} are different.

Using Theorem 3.1 and the remarks of [24], we get the following result.

THEOREM 1.11 [24]. *Let Γ be a non-cocompact hyperbolic Coxeter group of finite covolume and let G be its Gram matrix. Then Γ is arithmetic if and only if all the cycles of the matrix $2 \cdot G$ are rational integers.*

COROLLARY 1.12. *Let $\Gamma = (W, S)$ be a non-cocompact hyperbolic Coxeter group of finite covolume and let $G' = 2 \cdot G$, where G is its Gram matrix. If the Coxeter graph of Γ contains no dotted edges, then we have the following result.*

Γ is arithmetic if and only if the following two conditions are satisfied:

- (i) for every $s, t \in S$, we have $m(s, t) \in \{\infty, 2, 3, 4, 6\}$; and
- (ii) every irreducible cycle $G'_{(i_1, \dots, i_k)}$ of length at least 3 in G' lies in \mathbb{Z} .

In particular, a necessary condition for Γ to be arithmetic is $G \in \text{Mat}(|S|, \mathbb{Q}[\sqrt{2}, \sqrt{3}])$.

From now on, we suppose that $\Gamma = (W, S)$ is a non-cocompact hyperbolic Coxeter group of finite covolume which satisfies condition (i) of Corollary 1.12 and we let $G' = 2 \cdot G$. If $s \in S$

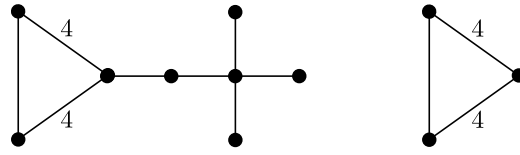


FIGURE 3. Two equivalent graphs for the arithmeticity criterion.

is a leaf of the Coxeter graph (meaning that $m(s, t) = 2$ for every $t \in S \setminus \{s\}$ except for one vertex), then it is clear that s cannot be a member of a non-zero irreducible cycle. Therefore, we can forget this vertex for the test. Applying this idea successively, we see that we can collapse every non-closed path of the graph. For example, the two Coxeter graphs of Figure 3 are equivalent for the arithmeticity criterion.

Now, if $G'_{(i_1, \dots, i_k)}$ is an irreducible k -cycle, then it corresponds to a simple cycle in the Coxeter graph (see Definition 11) if and only if it is non-zero. Thus, it is sufficient to consider simple cycles in the Coxeter graph. This can be summarized in the next proposition.

PROPOSITION 1.13 (Arithmetic criterion in CoxIter). *Let $\Gamma = (W, S = \{s_1, \dots, s_d\})$ be a non-cocompact hyperbolic Coxeter group of finite covolume and let \mathcal{G} be its Coxeter graph. We suppose that \mathcal{G} contains no dotted line. Then, Γ is arithmetic if and only if the two following conditions are satisfied.*

- (i) For every $s, t \in S$, we have $m(s, t) \in \{\infty, 2, 3, 4, 6\}$.
- (ii) For every simple cycle (s_1, \dots, s_k, s_1) in \mathcal{G} , the product $2^k \cdot \prod_{i=1}^{k-1} \cos(\pi/m(s_i, s_{i+1})) \cdot \cos(\pi/m(s_k, s_1))$ is an integer. Moreover, it is sufficient to test this condition in the graph obtained by collapsing every non-closed path.

REMARK 3. If the graph contains dotted lines, we cannot decide the arithmeticity of the group only by looking at the weights $m(s, t)$ in the graph. In order to extend Proposition 1.13, we need to know all the values in the Gram matrix. If there is a dotted line between vertices i and j , a necessary condition for arithmeticity is that $4 \cdot G_{ij}^2 \in \mathbb{Z}$.

2. Algorithms

In what follows, we use the notations of the program. In particular, the r vertices of the Coxeter graph are labelled $0, \dots, r - 1$.

2.1. Euler characteristic and the f -vector

Here, we present the main steps for the computation of the Euler characteristic and the f -vector. The main steps are the following.

- (1) We find all paths (simple walks with all edges labelled with a 3) starting from every vertex. Note that a single vertex is such a path.
- (2) We extend these paths to finite and affine irreducible Coxeter groups.
- (3) We compute all the possible products of groups.
- (4) We count these products with their orders and multiplicities. The Euler characteristic is then computed with Proposition 1.2 (by taking into account the orders of finite irreducible spherical Coxeter groups in Table 1).

Note that, using (1)–(3), and Theorems 1.4 and 1.5, we can compute the f -vector of Γ .

First, we use a *depth-first search* algorithm to explore the graph from every vertex and to find any subgraph A_m (function `CoxIter::DFS`). For each such A_m , we try to extend it to other

connected spherical and Euclidean graphs (function `CoxIter::addGraphsFromPath`).

Once the connected graphs are found, we compute all the possible products. Counting them with their multiplicities gives the f -vector and the Euler characteristic.

2.2. Arithmeticity

To check the arithmeticity of a given Coxeter group, we use Proposition 1.13. The code for the test is in the `Arithmeticity` class.

First, we check whether the graph contains no dotted edge, if it is non-cocompact and if all the edge labels $m(s, t)$ lie in the set $\{\infty, 2, 3, 4, 6\}$. These verifications are done in the beginning of the `Arithmeticity::test` function. At this point, we know that the coefficients of the matrix $2 \cdot G$ can take the values $\{0, -1, -2, 2, -\sqrt{2}, -\sqrt{3}\}$. We use multiple calls to the function `Arithmeticity::collapseQueues` to collapse all non-closed paths of the graph (regardless of the labels which appear in these paths) to reduce the computation time of the determination of the cycles.

For each vertex v , we look for simple cycles passing through v (note that we only look at cycles which contain vertices j with $v \leq j$ to avoid multiple counting of the graphs). This is done by using `Arithmeticity::findCycles(v,v)`. For each cycle, we see if the number of edges labelled with 4 and 6, respectively, along the cycle is even (labels 3 and ∞ are not an obstacle to arithmeticity at this point).

As explained in Remark 3, if the graph contains dotted lines, we need to determine all the relevant entries of the Gram matrix in order to decide about the arithmeticity of the group. In this case, `CoxIter` will indicate what are the conditions for the group to be arithmetic. The program would give conditions in the following way.

```
11m3: weight of the dotted line between hyperplanes 2 and 4
10m5: weight of the dotted line between hyperplanes 1 and 6
```

The group is arithmetic if and only if all the following values lie in \mathbb{Z} :

```
4 * 11m3^2
4 * 10m5^2
2^3 * 11m3 * 10m5
```

3. Using `CoxIter`: some examples

In this section, we present some worked examples in dimensions 13, 16, 17, 18 and 19. In dimension 18 we compute the invariants for three groups given by Vinberg in a recent article [25]. The complete documentation of the program can be found online at <http://coxiter.rgug.ch>.

3.1. Two arithmetic groups of Vinberg

We start with the arithmetic groups $\Sigma_{16} < \text{Isom } \mathcal{H}^{16}$ and $\Sigma_{17} < \text{Isom } \mathcal{H}^{17}$ (see Figure 4 and [23]).

The program is called with parameters ‘-fv -compactness -arithmeticity’ (we want to test the cocompactness, the arithmeticity and finite covolume) and the output is given in Table 2. We see that the values match the theoretical results. For the Euler characteristic, see [18, Theorem 22], and for a correction factor 2^k , where k is the number of symmetry axis of the graph, see [19, § 6]. In our case, $k = 2$.

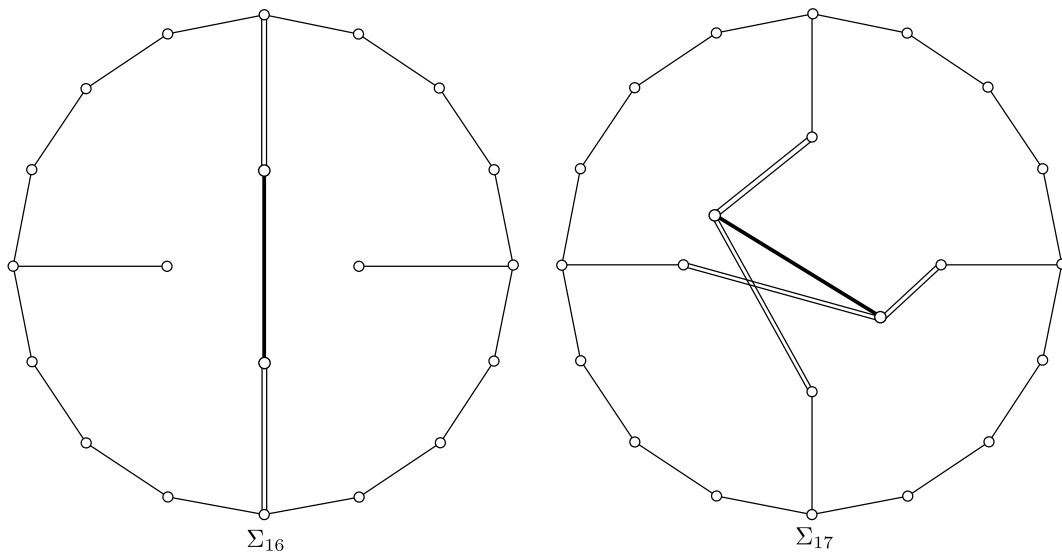


FIGURE 4. Σ_{16} and Σ_{17} .

3.2. *An arithmetic group of McLeod*

In his paper [13], McLeod constructed the maximal reflection groups in the automorphism groups of the quadratic forms $-3x_0 + x_1^2 + \dots + x_n^2$ for $2 \leq n \leq 13$, using Vinberg’s algorithm. We give here the invariants for $n = 13$ (see Figure 5). The program is called with parameters ‘-fv -compactness’ (we want to test the cocompactness and finite covolume), and the output is given in Table 3.

3.3. *A free product with amalgamation in dimension 18*

In a recent paper [25], Vinberg explains how to construct a non-arithmetic group Γ in dimension 18 as a free product with amalgamation $\Gamma = \Gamma_1 \star_{\Gamma_0} \Gamma_2$, where Γ_1 and Γ_2 are two non-commensurable arithmetic hyperbolic Coxeter groups (Γ is a mixture in the sense of Gromov–Piatetski–Shapiro). Geometrically, we suppose that the two associated polyhedra P_1 and P_2 of the groups $\Gamma_1, \Gamma_2 < \text{Isom } \mathcal{H}^{18}$ have an isometric facet P_0 . In this setting, the associated polyhedron P of Γ is the gluing of P_1 and P_2 along their common facet P_0 . We use

TABLE 2. *Output of CoxIter for Σ_{16} and Σ_{17} .*

Invariant	Σ_{16}	Σ_{17}
Cocompact	No	No
Finite covolume	Yes	Yes
Arithmetic	Yes	Yes
f-vector	(325, 2804, 11 914, 33 164, 67 410, 105 462, 130 646, 130 062, 104 670, 68 042, 35 490, 14 658, 4690, 1122, 189, 20, 1)	(807, 7586, 33 960, 98 184, 206 120, 332 982, 427 584, 444 428, 377 232, 262 050, 148 500, 68 076, 24 884, 7089, 1518, 230, 22, 1)
Alt. sum of the comp. of the f-vector	0	2
Euler characteristic	$\frac{642\,332\,179}{2\,360\,171\,042\,879\,569\,920\,000}$	0
Running time	0.6 s	2.7 s

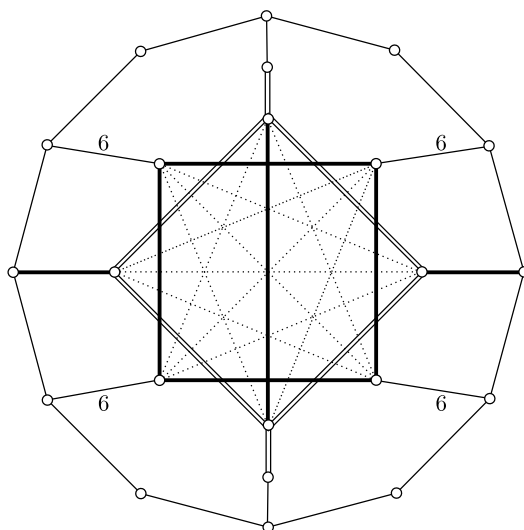


FIGURE 5. Group of reflections related to automorphism group of the quadratic form $-3x_0 + x_1^2 + \dots + x_{13}^2$.

CoxIter to compute the invariants of the groups Γ_1, Γ_2 and Γ and we check that the covolume of Γ is indeed the sum of the covolumes of the two components.

3.3.1. *First component of the product.* Information about the construction of the first component Γ_1 of the product are given in [10] and [25]. The generators $\{s_1, \dots, s_{37}\}$ of Γ_1 are given by the roots of a certain integral lattice L_1 (in fact, L_1 is the unique odd unimodular quadratic lattice of signature $(18, 1)$).

We use the terminology and present the results of [10]. There are 22 long roots (that is roots of norm two) and fifteen short roots (that is roots of norm one). The action of the symmetry group of the graph (which is isometric to S_4 ; see below) splits the short roots into two orbits of size three and twelve. The three roots are called roots of the *first kind* while the twelve other elements are called roots of the *second kind*. The diagram Σ^* of the long roots is presented in Figure 6(a). If we consider the six ‘long edges’ (or ‘tetrahedral edges’) of Figure 6(b), then each pair of opposite edges gives rise to a root of the first kind. Hence, we get the three vertices of the first kind 18, 25, 26 connected, respectively, to vertices (17, 22), (1, 9) and (5, 13) by a double edge. The Figure 6(c) depicts the connection of one vertex of the second kind with the graph Σ^* . We then let the symmetry group of Σ^* (see below) act on the triple of

TABLE 3. Output of CoxIter for the reflection group corresponding to the automorphism group of the quadratic form $-3x_0 + x_1^2 + \dots + x_{13}^2$ (see Figure 5).

Invariant	Value
Cocompact	No
Finite covolume	Yes
f-vector	(413, 2964, 10 238, 22 761, 36 024, 42 265, 37 380, 25 005, 12 556, 4641, 1218, 213, 22, 1)
Alt. sum of the comp. of the f-vector	2
Euler characteristic	0
Running time	0.18 s

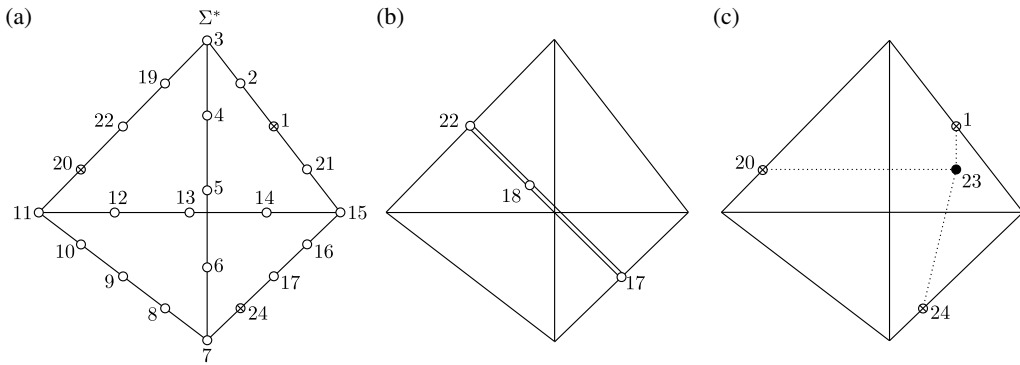


FIGURE 6. Construction of $\Gamma_1 < \text{Isom } \mathcal{H}^{18}$: first component of the product.

vertices $(1, 20, 24)$ and find 11 other triples, corresponding to the 11 other vertices of the second kind. The 12 triples of vertices are the following: $(1, 20, 24)$, $(1, 6, 12)$, $(5, 16, 20)$, $(5, 10, 21)$, $(9, 16, 19)$, $(9, 4, 14)$, $(13, 2, 8)$, $(13, 19, 24)$, $(17, 2, 10)$, $(17, 4, 12)$, $(22, 8, 21)$, $(22, 6, 14)$. Now, vertices of the first kind are connected among themselves by lines labelled with an ∞ , while vertices of the second kind are connected among themselves by broken edges. Each vertex of the first kind is connected to each vertex of the second kind either by an ∞ or by a broken edge according to the rule explained in [10, § 2]. Finally, we found a Coxeter graph $\Sigma(\Gamma_1)$ with 37 vertices and 171 edges. When given to CoxIter, it produces the output given in Table 4.

LEMMA 3.1 [10]. *The symmetry group of the graph $\Sigma(\Gamma_1)$ is isometric to S_4 .*

Proof. To determine the symmetry group of the graph of the group Γ_1 , we first compute the group of symmetries $\text{Sym } \Sigma^*$ of the graph Σ^* , presented in Figure 6. It is easy to see that $\text{Sym } \Sigma^*$ has order 24. We consider the automorphisms

$$\begin{aligned} \sigma_1 &= (22\ 9)(1\ 17) \\ \sigma_2 &= (1\ 5)(9\ 13) \\ \sigma_3 &= (22\ 1)(9\ 17) \end{aligned}$$

and we see that $\sigma_1 \circ \sigma_2$ has order three, $\sigma_2 \circ \sigma_3$ has order three and $\sigma_1 \circ \sigma_3$ has order two. Therefore, $\text{Sym } \Sigma^* = S_4$. Finally, as explained in [10], we have $\text{Sym } \Sigma^* \cong \text{Sym } \Sigma$. \square

TABLE 4. Output of CoxIter for Γ_1 .

Invariant	Value
Cocompact	No
Finite covolume	Yes
f-vector	$(3839, 37\ 842, 177\ 812, 540\ 624, 1\ 197\ 240, 2\ 050\ 008, 2\ 807\ 602, 3\ 135\ 528, 2\ 883\ 540, 2\ 189\ 924, 1\ 369\ 854, 700\ 352, 288\ 801, 94\ 113, 23\ 497, 4282, 525, 37, 1)$
Alt. sum of the comp. of the f-vector	0
Euler characteristic	$-\frac{109\ 638\ 854\ 849}{22\ 028\ 263\ 066\ 875\ 985\ 920\ 000}$
Volume	$\pi^9 \cdot \frac{109\ 638\ 854\ 849}{1\ 482\ 580\ 623\ 111\ 880\ 900\ 608\ 000\ 000}$
Running time	55 s

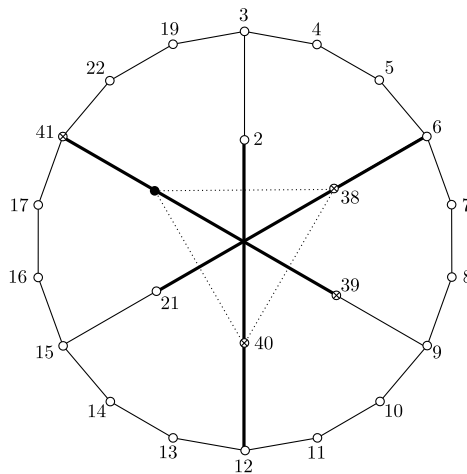


FIGURE 7. $\Gamma_2 < \text{Isom } \mathcal{H}^{18}$: second component of the product.

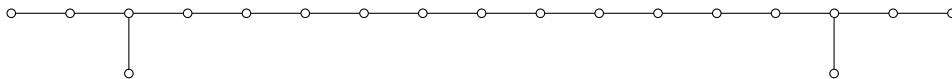


FIGURE 8. $\Gamma_0 < \text{Isom } \mathcal{H}^{17}$.

3.3.2. *Second component of the product.* The lattice for the second group is $L_2 = L_0 \oplus \mathbb{Z}e \subset \mathbb{R}^{18,1}$, where:

- L_0 is the unique even unimodular quadratic lattice of signature $(17, 1)$ (its maximal subgroup generated by reflections is Γ_0 , the hyperbolic Coxeter group whose associated fundamental polyhedron is P_0 , the common facet of P_1 and P_2); and
- e is a long root of norm two.

The graph of the group $\Gamma_2 < \text{Isom } \mathcal{H}^{18}$ is presented in Figure 7 and its computed invariants in Table 5.

3.3.3. *Amalgamated product.* As written previously, we want to construct the amalgamated product of the groups Γ_1 and Γ_2 . We see that they have a common hyperbolic subgroup Γ_0 of signature $(17, 1, 1)$ (see Figure 8). The hyperplane in \mathcal{H}^{18} , which contains the associated polytope P_0 , is the one corresponding to the black dot in the graphs of Γ_1 and Γ_2 .

TABLE 5. Output of CoxIter for Γ_2 .

Invariant	Value
Cocompact	No
Finite covolume	Yes
f-vector	(535, 5160, 24 876, 79 590, 188 352, 348 012, 517 247, 628 599, 629 544, 520 631, 354 651, 197 676, 89 148, 31 977, 8892, 1843, 267, 24, 1)
Alt. sum of the comp. of the f-vector	0
Euler characteristic	$-\frac{109\ 638\ 854\ 849}{22\ 600\ 997\ 906\ 614\ 761\ 553\ 920\ 000}$
Volume	$\pi^9 \cdot \frac{109\ 638\ 854\ 849}{1\ 521\ 127\ 719\ 312\ 789\ 804\ 023\ 808\ 000\ 000}$
Running time	3.2 s

A presentation for the product $\Gamma = \Gamma_1 \star_{\Gamma_0} \Gamma_2$ is obtained as follows.

- Start with the presentation of $\Gamma_1 = \langle s_1, \dots, s_{37} \mid (s_i \cdot s_j)^{m_{ij}} \rangle$.
- Remove the vertex 23, corresponding to the hyperplanes containing P_0 .
- Add four generators $s_{38}, s_{39}, s_{40}, s_{41}$ corresponding to the crossed vertices of Γ_2 .
- The relations between these four new generators and the generators $s_2, \dots, s_{17}, s_{19}, s_{21}, s_{22}$ are according to Figure 7.
- There is no relation between any of the generators $s_{38}, s_{39}, s_{40}, s_{41}$ and any of $s_1, s_{18}, s_{20}, s_{24}, \dots, s_{37}$, meaning that the corresponding m_{ij} are infinity. Note that at this point we do not know if the corresponding hyperplanes are parallel or ultraparallel (or, equivalently, if in the graph we have dotted or bold edges). This means that we cannot fully determine the f -vector (that is the number f_0 of vertices is unknown). However, this does not influence the computation for the Euler characteristic.

The output of CoxIter for the product Γ is given in Table 6. As expected, we find the equality $\chi(\Gamma) = \chi(\Gamma_1) + \chi(\Gamma_2)$. Moreover, using Proposition 1.6, we find that the number f_0 of vertices of the polyhedron is 4212.

3.4. An arithmetic group in dimension 19

Kaplinskaja and Vinberg described in [10] the construction of an arithmetic hyperbolic Coxeter group Γ_{19} in $\text{Isom } \mathcal{H}^{19}$ related to the standard quadratic form $-x_0^2 + x_1^2 + \dots + x_{19}^2$. This construction is similar to the first component of the free amalgamated product in § 3.3.1.

In this section, we use the terminology of [10]. There are 25 long roots (depicted in the diagram Σ^* of Figure 9), five roots of the first kind (such as the vertex 19 in Figure 9(b)) and twenty roots of the second kind (such as the vertex 24 in Figure 9(c)). In order to explain how to connect the vertices of the first and second kind to the graph Σ^* , we first compute the automorphism group of the graph Σ^* .

LEMMA 3.2 [10]. *The automorphism group of Σ^* is isomorphic to S_5 .*

Proof. Using the same method as in Lemma 3.1, we find the four generators of $\text{Sym } \Sigma^*$ in order to conclude that

$$\begin{aligned} \sigma_1 &= (27\ 4)(14\ 20)(16\ 18)(2\ 12)(21\ 22)(26\ 10) \\ \sigma_2 &= (4\ 6)(8\ 20)(2\ 25)(10\ 21)(16\ 22)(18\ 26) \\ \sigma_3 &= (6\ 25)(4\ 16)(14\ 20)(2\ 22)(12\ 21)(18\ 27) \\ \sigma_4 &= (2\ 20)(18\ 26)(8\ 25)(21\ 22)(12\ 14)(10\ 16). \end{aligned} \quad \square$$

3.4.1. *Vertices of the first and second kind.* In order to determine the vertices of the first kind and their connections to the graph Σ^* , we consider the triple of vertices (27, 26, 18)

TABLE 6. Output of CoxIter for Γ .

Invariant	Value
f -vector	(4212, 41 464, 195 047, 594 510, 1321 044, 2 271 012, 3 123 717, 3 503 919, 3 235 974, 2 467 015, 1 548 147, 793 376, 327 561, 106 710, 26 575, 4814, 583, 40, 1)
Euler characteristic	$-\frac{8\ 661\ 469\ 533\ 071}{1\ 738\ 538\ 300\ 508\ 827\ 811\ 840\ 000}$
Volume	$\pi^9 \cdot \frac{8\ 661\ 469\ 533\ 071}{117\ 009\ 824\ 562\ 522\ 292\ 617\ 216\ 000\ 000}$
Running time	74 s

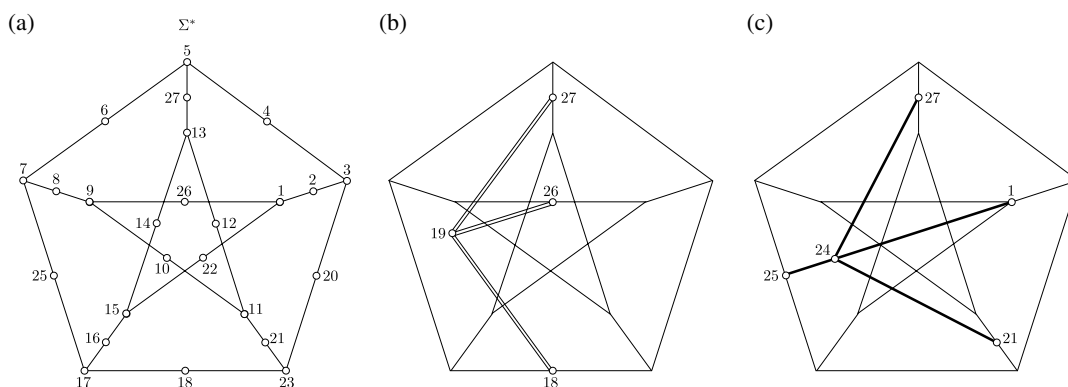


FIGURE 9. Construction of $\Sigma_{19} < \text{Isom } \mathcal{H}^{19}$.

(see Figure 9(b)) and look at its images under action of $\text{Sym } \Sigma^* \cong S_5$. We find five triples which correspond to five vertices of the first kind. These are

$$(27, 26, 18), \quad (2, 12, 25), \quad (21, 22, 6), \quad (16, 10, 4), \quad (8, 14, 20).$$

The vertices of the first kind are connected among themselves by bold edges.

To determine the vertices of the second kind and their connections to the graph Σ^* , we consider the quadruple of vertices $(27, 1, 21, 25)$ (see Figure 9(c)) and look at its images under the action of $\text{Sym } \Sigma^*$. We find twenty quadruples which correspond to the twenty vertices of the second kind. These are

$$\begin{aligned} &(1, 21, 25, 27), \quad (8, 22, 23, 27), \quad (4, 8, 15, 21), \quad (4, 11, 22, 25), \quad (2, 10, 17, 27) \\ &(9, 16, 20, 27), \quad (4, 9, 14, 18), \quad (4, 12, 17, 26), \quad (2, 8, 13, 18), \quad (13, 20, 25, 26) \\ &(3, 10, 14, 25), \quad (3, 8, 12, 16), \quad (1, 6, 12, 18), \quad (6, 14, 23, 26), \quad (2, 6, 11, 16) \\ &(6, 10, 15, 20), \quad (2, 7, 14, 21), \quad (7, 12, 20, 22), \quad (5, 16, 21, 26), \quad (5, 10, 18, 22). \end{aligned}$$

The vertices of the second kind are connected among themselves by broken edges.

A vertex of the first kind and a vertex of the second kind are connected by a double edge or a broken edge according to the rule described in [10, §2, p. 196].

3.4.2. *Summary and computations.* Finally, the connections between the 50 vertices of the group Γ_{19} are as follows:

- 30 simple edges in Σ^* ;
- a complete graph with 5 vertices (vertices of the first kind, bold edges);
- 15 double edges (vertices of the first kind and Σ^*);
- a complete graph with 20 vertices (vertices of the second kind, broken edges);
- 80 bold edges (vertices of the second kind and Σ^*); and
- 100 edges between vertices of the first and second kind.

The output of CoxIter for the group Γ_{19} is presented in Table 7.

TABLE 7. Output of CoxIter for Γ_{19} .

Invariant	Value
Cocompact	No
Finite covolume	Yes
f-vector	(27 841, 292 340, 1 429 615, 4 465 955, 10 081 519, 17 518 035, 24 310 230, 27 542 850, 25 791 030, 20 062 168, 12 956 240, 6 908 365, 3 009 960, 1 054 645, 290 315, 60 660, 9125, 905, 50, 1)
Alt. sum of the comp. of the f-vector	2

Appendix A. Example of the introduction

The output presented in the introduction is the group Γ generated by reflections in the group of units of the Lorentzian quadratic form $-x_0^2 + x_1^2 + \dots + x_{14}^2$ of signature (14, 1) described in [21]. We describe the group Γ by means of the text file ‘14-vinb85.coxiter’, as shown in Figure A.1.

Remarks.

- The first line indicates that the Coxeter group Γ has 17 generators and that it is a subgroup of $\text{Isom } \mathcal{H}^{14}$.
- Each of the remaining 17 lines describes one edge of the Coxeter diagram: the first two numbers are the labels of the generators and the third one is the label of the edge (a ‘0’ indicates a bold edge).

Appendix B. Program testing and some values

To test the accuracy of the program, we ran it on a collection of (around 800) groups for which some of the invariants were known. These graphs can be found in the graphs/folder of the source code. Except for a few graphs, the name of each file goes

dimension – reference_page number – name or number of the graph.

The reference is as given in Table C.1.

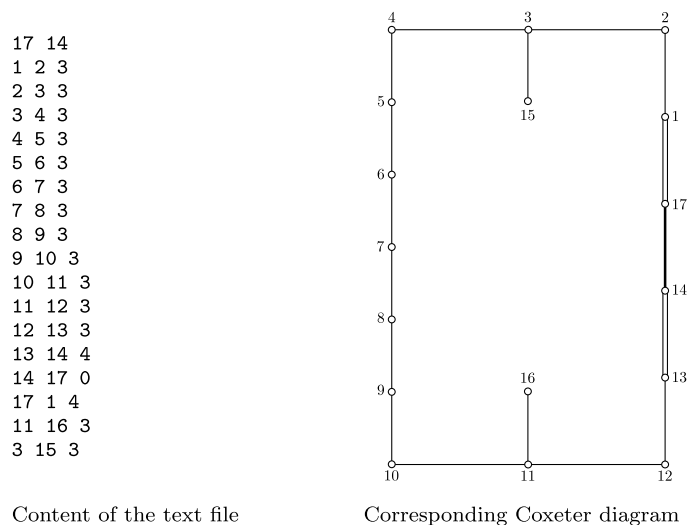


FIGURE A.1. 14-vinb85.coxiter

B.1. *Euler characteristic*

The groups of Table C.2 were tested.

B.2. *f-vector*

The *f*-vectors of the groups of Table C.3 were tested. The alternating sum of the components of the *f*-vectors of the groups [8, 20], [14, Appendix C] and [23] were tested (see Proposition 1.6).

B.3. *Cocompactness and finite covolume criterion*

The groups of Table C.4 were tested for cocompactness and the ones of Table C.5 were tested for the finite covolume.

B.4. *Arithmeticity*

The groups of Table C.6 were tested.

B.5. *Some more complicated Coxeter graphs*

Using cocompact groups in dimensions five and six we constructed twelve sequences $\mathcal{S}_i = \{\Gamma_0 < \Gamma_1 < \dots < \Gamma_9\}$ of ten hyperbolic Coxeter groups such that Γ_i is of index two in Γ_{i+1} . These groups allowed us to test CoxIter on more complicated graphs (the number of nodes ranges from 7 to 775).

Appendix C. *Tables*

TABLE C.1. *Correspondence between bibliography and file names.*

Reference in the paper	Name of the file
[5]	ess96
[6]	ftz07
[8]	jkrt
[13]	mcl11
[14]	per09
[20]	tum04

TABLE C.2. *Euler characteristic of some groups.*

Reference	Group	Theoretical value
[8, 20, 23]		
[14, Appendix C]	All groups of odd dimension	Remark 2
[6]	13 compact simplicial prisms in \mathcal{H}^4 , page 117	[6]
[8]	14 groups in \mathcal{H}^4	[8]
[8]	3 groups in \mathcal{H}^6	[8]
[14, Appendix C]	Bugaenko $P_6 \subset \mathcal{H}^6$, Bugaenko $P_8 \subset \mathcal{H}^8$	[11]
[8]	4 groups in \mathcal{H}^8	[8]
[23]	Σ_{10} and a subgroup	[17, Theorem 22]
[7]	P_1^{10}	[7]
[23]	Σ_{12} and a subgroup	[17, Theorem 22]
[23]	Σ_{14}	[17, Theorem 22]
[23]	Σ_{16}	[17, Theorem 22]

TABLE C.3. *f*-vector of some groups.

Group	Theoretical value
All groups of [8]	Simplexes
All pyramids in \mathcal{H}^3 of [20]	
Bugaenko P_6	[11]
Bugaenko P_8	[26]
Birectified 5-simplex	http://en.wikipedia.org/wiki/Rectified_5-simplexes

TABLE C.4. *Cocompactness*.

Reference	Cocompact	Groups
[5]	Yes	Groups in $\text{Isom } \mathcal{H}^4$
[14, Appendix C]	Yes	All groups
[6]	Yes	13 compact simplicial prisms in \mathcal{H}^4 , page 117
[9]	Yes	$\overline{BH}_3, \overline{J}_3, \overline{DH}_3, \overline{AB}_3, \overline{K}_3, \overline{AH}_3, \overline{BB}_3$ $\overline{BH}_3, \overline{HH}_3, \overline{H}_4, \overline{BH}_4, \overline{DH}_4, \overline{K}_4, \overline{AF}_4$
[20]	No	All groups
[20]	No	All groups obtained by removing a polar
[23]	No	$\{\Sigma_n : 10 \leq n \leq 17\}$
[9]	No	$\overline{V}_3, \overline{R}_3, \overline{P}_3, \overline{BV}_3, \overline{O}_3, \overline{Y}_3, \overline{HV}_3, \overline{BP}_3, \overline{DV}_3, \overline{N}_3, \overline{Z}_3$ $\overline{BR}_3, \overline{HP}_3, \overline{AV}_3, \overline{DP}_3, \overline{M}_3, \overline{VP}_3, \overline{BV}_3, \overline{CR}_3, \overline{HV}_3$ $\overline{VV}_3, \overline{RR}_3, \overline{PP}_3, \overline{S}_4, \overline{R}_4, \overline{P}_4, \overline{O}_4, \overline{N}_4, \overline{M}_4, \overline{BP}_4, \overline{FR}_4$ $\overline{DP}_4, \overline{U}_5, \overline{S}_5, \overline{X}_5, \overline{Q}_5, \overline{R}_5, \overline{P}_5, \overline{O}_5, \overline{N}_5, \overline{AU}_5, \overline{M}_5$ $\overline{L}_5, \overline{UR}_5, \overline{S}_6, \overline{Q}_6, \overline{P}_6, \overline{T}_7, \overline{S}_7, \overline{Q}_7, \overline{P}_7, \overline{T}_8, \overline{S}_8, \overline{Q}_8, \overline{P}_8$

TABLE C.5. *Finite covolume*.

Reference	Finite covolume	Groups
[6]	Yes	13 compact simplicial prisms in \mathcal{H}^4 , page 117
[8]	Yes	All groups
[14, Appendix C]	Yes	All groups
[20]	Yes	All groups
[23]	Yes	$\{\Sigma_n : 10 \leq n \leq 17\}$
[20]	No	Every polar was removed to create 387 groups which do not have finite covolume

TABLE C.6. *Arithmeticity*.

Reference	Arithmetic	Groups
[8]	Yes (see [9])	$\overline{S}_6, \overline{Q}_6, \overline{P}_6, \overline{T}_7, \overline{S}_7, \overline{Q}_7, \overline{P}_7, \overline{T}_8, \overline{S}_8, \overline{Q}_8, \overline{P}_8$
[23]	Yes	$\{\Sigma_n : 10 \leq n \leq 17\}$
[20]		All groups

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