The Gauss-Bonnet formula for hyperbolic manifolds of finite volume

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Abstract. Let M denote an even-dimensional non-compact hyperbolic manifold of finite volume. We show that such manifolds are candidates for minimal volume. Generalizing H. Hopf's ideas around the Curvatura integra for compact Clifford-Klein space forms, we present an elementary combinatorial-metrical proof of the Gauss-Bonnet formula for M. In contrast to former results of G. Harder and M. Gromov, our approach doesn't make use of the arithmetical and differential geometrical machinery.

0. Introduction

Among all complete Riemannian manifolds, the hyperbolic ones, equipped with a metric g_{-1} of sectional curvatures -1, are distinguished by their diversity and many nice extremal properties. For example, consider a compact hyperbolic manifold (M, g_{-1}) of dimension $n \geq 3$ which admits a Riemannian metric g of sectional curvatures $|K| \leq 1$. Then, $\mathrm{vol}_n(M, g) \geq \mathrm{vol}_n(M, g_{-1})$, and equality holds only if g is isometric to g_{-1} (cf. [BCG]).

In the following, let $n \geq 2$ be even. Then, the volume of an $n-$dimensional compact hyperbolic manifold M is related to the Euler-characteristic $\chi(M)$ by the Gauss-Bonnet formula

$$\mathrm{vol}_n(M) = (-1)^{\frac{n}{2}} \cdot \frac{\Omega_n}{2} \cdot \chi(M) \quad , \tag{0.1}$$

where Ω_n denotes the volume of the standard unit $n-$sphere. A first proof for (0.1) was given by H. Hopf [Ho] (cf. also [AW], [C]). If M is assumed to be oriented, we show that $\chi(M)$ is an even number (cf. Theorem 1.2) so that, by (0.1), $\mathrm{vol}_n(M)$ is a natural multiple of Ω_n. In the case of compact hyperbolic surfaces M_p of genus $p > 1$, this is reflected by the well-known relation $\chi(M_p) = 2\,(1-p)$.

On the other hand, there are examples of hyperbolic manifolds of volume given by the least possible value $\Omega_n/2$ according to (0.1). For $n = 2$, such surfaces are classified; they fall into precisely 4 homeomorphism classes. For $n = 4$, there are constructions due to J. Ratcliffe and S. Tschantz [RT] based on the ideal 24−cell. All these spaces are examples of non-compact hyperbolic manifolds with finitely many unbounded ends of finite volume (cf. §1).

In this note, we present an elementary combinatorial-metrical proof of the Gauss-Bonnet formula (0.1) for a hyperbolic $n-$manifold of finite volume. Our methods are based on Hopf's ideas around the Curvatura integra for compact Clifford-Klein space forms. The essential ingredients of his proof are the existence of a finite geodesic triangulation and that each top-dimensional simplex T in the triangulation has a generalized angle sum $W(T)$ satisfying

$$W(T) = (-1)^{\frac{n}{2}} \cdot \frac{2}{\Omega_n} \cdot \mathrm{vol}_n(T) \quad , \tag{0.2}$$

in analogy to former results of H. Poincaré [Po] and L. Schläfli in the spherical context (cf. [K], [Pe]).

We first consider the structure of a non-compact hyperbolic manifold M^n of finite volume at infinity. Each end is diffeomorphic to a cusp $\bar{N}^n = N^{n-1} \times [0,\infty)$, where N^{n-1} is a closed manifold. For such cusps, we introduce and compute the Euler-characteristic $\chi(\bar{N}^n)$ combinatorially. We associate to $\bar{N}^n$ a certain finite cell decomposition induced by a finite triangulation of the boundary N^{n-1}. Specializing to hyperbolic cusps, it allows to express the Euler-characteristic $\chi(M^n)$ of M^n by means of some generalized finite triangulation consisting of possibly asymptotic hyperbolic simplices. For each $n-$dimensional element of the triangulation, the notion of generalized angle sum of Hopf-Poincaré can be extended in a natural way such that (0.2) remains valid (cf. Corollary 3.2). After these preparations, the proof of (0.1) follows easily (cf. §3.2).

Our approach to (0.1) is purely combinatorial-metrical and differs from earlier and more general results in the case $n = 2$ due to A. Huber and S. Cohn-Vossen (for references and a short proof, cf. [Ro]), and for n arbitrary, due to G. Harder [Ha] and M. Gromov [Gro]. They proved generalized Gauss-Bonnet theorems of the form

$$\chi(M) = \int_M E(g) \tag{0.3}$$

for Riemannian manifolds (M, g) of finite volume with Euler integrand $E(g)$ using the machinery of arithmetic and differential geometry. Harder considered quotients of Riemannian symmetric spaces X of non-compact type and rank ≥ 2 by arithmetically defined discrete, torsionfree groups of isometries in X which are of finite cohomological dimension. Gromov studied Riemannian manifolds M of finite volume with bounded non-positive sectional curvatures. These manifolds are of bounded cohomology such that $\chi(M) = \sum_{r=0}^{n} (-1)^r \, b_r(M)$ is well-defined.

Acknowledgement. We would like to thank Ernst Ruh and Domingo Toledo for some helpful discussions*.

1. Hyperbolic manifolds of finite volume

1.0. *Hyperbolic manifolds.* Let M^n, $n \geq 2$, denote a hyperbolic manifold, that is, a complete Riemannian manifold of constant sectional curvature -1, of finite volume. M^n is therefore a Clifford-Klein space form of type $M^n = H^n/\Gamma$, where H^n is the standard hyperbolic $n-$space, and $\Gamma < Iso(H^n)$ is a discrete, torsionfree group of hyperbolic isometries. If not explicitly stated, we always assume M^n and all other manifolds considered below to be without boundary. As general reference for the following, we refer to [Ra, §6].

Denote by $P \subset H^n$ a fundamental (convex closed) polyhedron for the action of Γ on H^n so that we can write $\mathrm{vol}_n(M^n) = \mathrm{vol}_n(P) < \infty$. For example, take the closure of a Dirichlet domain for Γ. It is known that the inclusion map $i : P \longrightarrow H^n$ induces a homeomorphism κ from the quotient space $P/\Gamma = \{\, \Gamma x \cap P \mid x \in P \,\}$ to $H^n/\Gamma = M^n$ making the diagram

$$\begin{array}{ccc} P & \xrightarrow{i} & H^n \\ \Big\downarrow \Theta & & \Big\downarrow \pi \\ P/\Gamma & \xrightarrow{\kappa} & H^n/\Gamma \end{array}$$

commutative. If M^n is compact, the polyhedron P is compact and convex hull of finitely many points in H^n. Moreover, each vertex figure of P (arising as intersection of P with a sufficiently small sphere centered at the vertex) is a spherical polyhedron. If M^n is non-compact and of finite volume, P is a *generalized* polyhedron. That is, P is the convex hull of finitely many points in extended hyperbolic space $\bar{H}^n = H^n \cup \partial H^n$, and the vertex figure associated to each vertex belonging to ∂H^n (arising as intersection of P with a sufficiently small horosphere based at the vertex) is a compact Euclidean polyhedron. Actually, by [EP], one can construct an *ideal* fundamental polyhedron $P_\infty \subset H^n$ for Γ, that is, *all* vertices of P_∞ belong to ∂H^n.

Now, the group Γ is generated by the set

$$\Phi := \{\, \gamma \in \Gamma \mid P \cap \gamma(P) \text{ is a side of } P \,\} \quad ,$$

* First author partially supported by Schweizerischer Nationalfonds No. 2000-050579.97.
1991 *Mathematics Subject Classification.* 53C22; 53C25; 57S30; 51N25; 15A45.

which is a side-pairing for P. To each side $S \in P$ corresponds precisely one element $\gamma = \gamma_S \in \Phi$ with $S = P \cap \gamma(P)$ and $S' := \gamma^{-1}S \in P$.

Two points $x, x' \in P$ are said to be $\Phi-$related, which we abbreviate by $x \simeq x'$, if there is a side $S \in P$ such that $x \in S$, $x' \in S'$ and $\gamma_S(x') = x$. For $x, y \in P$, define the relation $x \sim_\Phi y$ if and only if there exist finitely many $x_1, \ldots, x_r \in P$ with $r \geq 1$ such that $x = x_1 \simeq \cdots \simeq x_r = y$. This is an equivalence relation, and the set of equivalence classes $[x]$ is denoted by P/Φ. There is the following connection to the quotient space P/Γ above (cf. [Ra, Theorem 6.7.5]).

Proposition 1.1.

Let $P \subset H^n$ be a fundamental polyhedron for the discrete group $\Gamma < Iso(H^n)$. Then, P/Γ is homeomorphic to P/Φ. More precisely, for each $x \in P$, $[x] = \Gamma x \cap P$.

1.1. *The compact case.* Let $M^n = H^n/\Gamma$, $n \geq 2$, denote a compact oriented hyperbolic manifold. For $n = 2$, a hyperbolic surface M is characterized topologically by having genus $p > 1$. Another topological invariant is the Euler-characteristic $\chi(M) = a_0 + a_1 - a_2$ given as alternating sum of the numbers of vertices, edges and faces appearing in a finite triangulation of M.

Let $P \subset H^2$ be a fundamental polygon for the group action of Γ. Then, we can find a $4p-$sided fundamental polygon P_0 normalizing P (cf. [N, §8]). From this, one deduces the well-known relation $\chi(M) = 2(1-p)$ between the Euler-characteristic $\chi(M)$ and the genus p of M. This implies that $\chi(M)$ is an even number. By the classical Gauss-Bonnet formula, the area of a compact hyperbolic surface M is given by

$$\mathrm{vol}_2(M) = -2\pi\chi(M) = 4\pi(p-1) \quad .$$

Hence, $\mathrm{vol}_2(M)$ is a natural multiple of $\Omega_2 = 4\pi$, where

$$\Omega_k = 2\pi^{\frac{k+1}{2}}/\Gamma(\frac{k+1}{2})$$

is the volume of the standard unit $k-$sphere.

A similar situation holds in higher even dimensions. Let M^n denote a compact hyperbolic manifold of even dimension $n \geq 2$. Then, M^n can be finitely triangulated by a_r simplices of dimension r, $0 \leq r \leq n$ (cf. [M, Theorem 10.6]). Consider its Euler-characteristic

$$\chi(M^n) = \sum_{r=0}^{n} (-1)^k a_k \quad . \tag{1.1}$$

By the Euler-Poincaré formula (cf. [AH, V.2, No. 5]), $\chi(M^n)$ is expressible in terms of singular homology according to

$$\chi(M^n) = \sum_{r=0}^{n} (-1)^r b_r \quad ,$$

where $b_r = \dim H_r(M^n;\mathbb{R})$ is the $r-$th Betti number of M^n. First, we prove the following result which seems to have been unnoticed up to now*.

Theorem 1.2.

Let M^n denote an oriented compact hyperbolic manifold of even dimension $n \geq 2$. Then, the Euler-characteristic $\chi(M^n)$ is an even number.

Proof: Let X^n denote an oriented compact manifold of even dimension n. By Poincaré duality, we have

$$\chi(X^n) \equiv b_{\frac{n}{2}} \quad \text{modulo } 2 \quad . \tag{1.2}$$

We distinguish between the two cases $n \equiv 0$ and $n \equiv 2$ modulo 4. Let $n = 4k+2$ for an integer $k \geq 0$. The cup-product furnishes the middle cohomology group $H^{2k+1}(X^{4k+2};\mathbb{R})$ with a non-degenerate skew-symmetric bilinear and therefore symplectic form. This implies that b_{2k+1} is even. By (1.2), we deduce that (cf. also [Gre, (26.11)])

$$\chi(X^{4k+2}) \equiv b_{2k+1} \equiv 0 \quad \text{modulo } 2 \quad .$$

Let $n = 4k$ for some $k \in \mathbb{N}$. An oriented compact smooth manifold X of dimension $4k$ has a signature $\operatorname{sign}(X) \in \mathbb{Z}$ which is defined by

$$\operatorname{sign}(X) = b_+ - b_- \quad ,$$

where b_+ and b_- are the numbers of positive and negative eigenvalues of the symmetric matrix associated to the cup-pairing $\cup$ on $H^{2k}(X;\mathbb{R})$. Therefore, $\operatorname{sign}(X)$ is the signature of the quadratic form defined by $\cup$. By Hirzebruch's signature theorem (cf. [Hi, Theorem 8.2.2]),

$$\operatorname{sign}(X) = L(X) \quad ,$$

where $L(X)$ is the $L-$genus given by a certain linear combination of the Pontrjagin classes $p_j \in H^{4j}(X,\mathbb{R})$, $j = 1,\dots,k$, evaluated on the fundamental cycle $[X]$ of X. More precisely,

$$L(X) = L_k(p_1,\dots,p_k)\,[X] \quad ,$$

where $\{L_j\}$ denotes the multiplicative sequence corresponding to the power series of $\sqrt{z}/\tanh\sqrt{z}$ (cf. [Hi, §1.5]). For example,

$$\begin{aligned} L_1 &= \frac{1}{3}\,p_1 \quad , \\ L_2 &= \frac{1}{45}\,(\,7p_2 - p_1^2\,) \quad . \end{aligned}$$

* After writing the paper, the authors learned that John G. Ratcliffe has found independently a proof of Theorem 1.2 which will appear in the *Handbook of Geometric topology*.

Now, specialize to an oriented compact hyperbolic $4k-$manifold M. Then, all Pontrjagin classes $p_j(TM) \in H^{4j}(M,\mathbb{R})$, $j = 1,\ldots,k$, vanish. This follows from the Chern-Weil theory treating characteristic classes for Riemannian manifolds in terms of their curvature form (cf. [S, vol. V, §13]) and by using the fact that M has constant curvature (cf. [S, vol. V, Corollary 44]). Therefore, $\mathrm{sign}(M) = 0$.
On the other hand, we have $b_{2k} = b_+ + b_-$. By (1.2), this yields

$$\chi(M) \equiv b_{2k} \equiv b_+ - b_- = \mathrm{sign}(M) \quad \text{modulo } 2 \quad .$$

Since $\mathrm{sign}(M)$ vanishes, the Euler-characteristic $\chi(M)$ must therefore be even.

q.e.d.

Remarks. (i) The proof shows that Theorem 1.2 remains valid for arbitrary even-dimensional oriented compact Riemannian manifolds of constant sectional curvature.
(ii) The condition of constant curvature cannot be dropped. For example, $P_{2m}(\mathbb{C})$ is even-dimensional with $\chi(P_{2m}(\mathbb{C})) = 2m+1$ (cf. [Hi, §4.10]). Indeed, $P_{2m}(\mathbb{C})$ equipped with the Fubini-Study metric has variable $(\frac{1}{4},1)-$pinched sectional curvatures.

For even-dimensional compact Clifford-Klein space forms, H. Hopf [Ho] proved a generalized Gauss-Bonnet formula. We state the result for hyperbolic manifolds.

Theorem 1.3.

Let M^n denote a compact hyperbolic manifold of even dimension $n \geq 2$. Then,

$$\mathrm{vol}_n(M^n) = (-1)^{\frac{n}{2}} \cdot \frac{\Omega_n}{2} \cdot \chi(M^n) \quad . \tag{1.3}$$

For a proof of Theorem 1.3, we refer to §3 below where we treat the more general finite volume case (cf. Theorem 3.3).

Corollary 1.4.

Let M^n denote an oriented compact hyperbolic manifold of even dimension $n \geq 2$. Then, $vol_n(M^n)$ is a natural multiple of Ω_n.

Example. M. Davis [D] (cf. also [Ra, pp. 505–506]) constructed geometrically an oriented compact hyperbolic $4-$manifold M_D with Euler-characteristic equal to 26 and $\mathrm{vol}_4(M_D) = 13\,\Omega_4$. M_D arises by identifying suitably the sides of the $120-$cell $\{5,3,3\}$ with dihedral angles $\frac{2\pi}{5}$ in hyperbolic $4-$space, a regular compact polyhedron with 120 dodecahedra as $3-$dimensional faces.

Corollary 1.5.

Let M^n *be an oriented hyperbolic manifold of even dimension* $n \geq 2$ *with* $vol_n(M^n) = \Omega_n/2$ *. Then,* M^n *is non-compact.*

If M^n is a compact hyperbolic manifold of odd dimension n, one can show by purely topological means that $\chi(M^n) = 0$ (cf. [Pe, Satz 16, p. 343], for example).

1.2. *The non-compact case.* Let M^n be a non-compact hyperbolic manifold of finite volume. Then, M^n is a quotient of H^n by a discrete, torsionfree group $\Gamma < Iso(H^n)$ containing parabolic elements. The lemma of Margulis (cf. [Ra, Theorem 12.5.1]) characterizing Γ yields a decomposition

$$M^n = M_0 \cup C_1 \cup \cdots \cup C_m \tag{1.4}$$

of M^n into a compact part M_0 and $m \geq 1$ disjoint unbounded ends $C_1, \ldots, C_m$ of finite volume. m is often called the number of punctures of M^n. Each component C_i , $1 \leq i \leq m$, is diffeomorphic to a manifold of type $N_i^{n-1} \times (0, \infty)$ where N_i^{n-1} is a compact Euclidean manifold; more concretely, to C_i corresponds a precisely invariant horoball B_i (with Euclidean boundary sphere!) in the universal covering space H^n which is based at the fixed point $q_i \in \partial H^n$ of some parabolic element in Γ so that one can write $C_i = B_i/\Gamma_{q_i}$.

Example 1. The non-compact hyperbolic surfaces of volume $\Omega_2/2 = 2\pi$ are classified. Up to homeomorphisms, there are precisely 4 different surfaces. The thrice punctured sphere and the torus with one puncture form the orientable examples, while the Klein bottle and the projective plane each with one puncture yield the non-orientable representatives.

Example 2. The simply punctured non-orientable Gieseking manifold M_G has minimal volume among all non-compact hyperbolic $3-$manifolds (cf. [A]). M_G has an ideal regular tetrahedron with dihedral angle $\pi/3$ as fundamental domain.

Example 3. There are many non-compact hyperbolic $4-$manifolds of volume $\Omega_4/2$. Ratcliffe and Tschantz [RT] constructed such manifolds with at most 6 punctures by identifying suitably the faces of an ideal $24-$cell. This is a regular polyhedron in $\bar{H}^4$ of dihedral angle $\pi/2$ with 24 octahedral faces.

2. The Euler-characteristic for cusps

2.0. *Cusps.* Let N^{n-1} denote an $(n-1)-$dimensional closed smooth manifold. A *cusp* $\bar{N}^n$ is an $n-$dimensional bounded smooth manifold diffeomorphic to the product $N^{n-1} \times [0, \infty)$. The boundary manifold N^{n-1} of $\bar{N}^n$ admits a finite smooth triangulation which can be extended to yield a smooth triangulation of $\bar{N}^n$ (cf. [M, Theorem 10.6]).

For simplicity, we work in the smooth case although all considerations are valid in the C^r-category for $r \geq 1$.

2.1. *The canonical cell decomposition of a cusp.* Let $\bar{N}^n$ be a cusp diffeomorphic to a product manifold $N^{n-1} \times [0,\infty)$ as in §2.0. Choose a finite triangulation $f : K \longrightarrow N^{n-1}$. It gives rise to a cell decomposition of $\bar{N}^n$ as follows. Denote by

$$\{\, \sigma_i^r \,\}_{\substack{0 \leq r \leq n-1 \\ 1 \leq i \leq a_r(K)}}$$

the set of simplices of K and put $e_i^{r+1} := f(int(\sigma_i^r)) \times (0,\infty)$. Then, the union

$$\{\, f(int(\sigma_i^r)) \,\}_{\substack{0 \leq r \leq n-1 \\ 1 \leq i \leq a_r(K)}} \bigcup \{\, e_i^{r+1} \,\}_{\substack{0 \leq r \leq n-1 \\ 1 \leq i \leq a_r(K)}} \tag{2.1}$$

is a cell decomposition Z of $\bar{N}^n$ (i.e., a family of disjoint non-empty subsets, each homeomorphic to $\mathbb{R}^n$ whose union equals $\bar{N}^n$). Z is called the *canonical cell decomposition* of the cusp $\bar{N}^n$. Observe that Z consists of finitely many bounded and unbounded cells. For the Euler-characteristic $\chi(\bar{N}^n)$, we put

$$\chi(\bar{N}^n) := \chi(Z) := \#\{\, e \in Z \mid dim(e) \text{ even} \,\} - \#\{\, e \in Z \mid dim(e) \text{ odd} \,\} \quad . \tag{2.2}$$

Proposition 2.1.

Let $\bar{N}^n = N^{n-1} \times [0,\infty)$ be a cusp together with a triangulation $f : K \longrightarrow N^{n-1}$. Let Z denote the the canonical cell decomposition associated to K. Then,

$$\chi(\bar{N}^n) = 0 \quad .$$

Proof: Consider the set (2.1) of elements in the canonical cell decomposition Z of $\bar{N}^n$. To each $r-$cell $f(int(\sigma_i^r)) \subset f(K)$ corresponds precisely one $(r+1)-$cell $e_i^{r+1} = f(int(\sigma_i^r)) \times (0,\infty)$ for $r = 0,\ldots,n-1$. By (2.2), this proves the remaining equality above.

q.e.d.

Let $\bar{M}^n$ denote an $n-$dimensional compact smooth manifold with boundary whose boundary components are $N_1^{n-1},\ldots,N_m^{n-1}$. Let $\bar{N}_1^n,\ldots,\bar{N}_m^n$ denote the cusps over them. Then, by glueing together the corresponding boundary components, we obtain a smooth manifold

$$M^n := \bar{M}^n \cup_\phi \left(\cup_{r=1}^m \bar{N}_r^n \right) \quad .$$

We make use of the Mayer-Vietoris property of χ and see that

$$\chi(M^n) = \chi(\bar{M}^n) - \sum_{r=1}^m \chi(N_r^{n-1}) \quad . \tag{2.3}$$

2.2. *Hyperbolic manifolds with cusps.* Let $M^n = H^n/\Gamma$ denote a non-compact hyperbolic manifold of finite volume. By §1.2, M^n is of the form

$$M^n = \bar{M}^n \cup_\phi (\bar{C}_1 \cup \cdots \cup \bar{C}_m) \quad , \tag{2.4}$$

where $\bar{M}^n$ is a compact manifold with boundary and $\bar{C}_1, \ldots, \bar{C}_m$ are cusps given by

$$\bar{C}_i = \bar{N}_i^n = N_i^{n-1} \times [0, \infty) \quad , \quad i = 1, \ldots, m \quad ,$$

where N_i^{n-1} is a compact Euclidean manifold with vanishing Euler-characteristic(cf. (1.4)). By (2.3), we have $\chi(M^n) = \chi(\bar{M}^n)$.

In the following, we interpret $\chi(M^n)$ as the Euler-characteristic $\chi_{K,\tau}(M^n)$ of a certain finite triangulation (K, τ) of M^n. Such a triangulation will be constructed from a fundamental domain $P \subset H^n$ for Γ.

First, let K denote a *generalized simplicial complex* in H^n, that is, K consists of finitely many generalized simplices in H^n (cf. §1.0) such that, with each generalized simplex, all its faces are contained in K, and the intersection of two generalized simplices of K is a common generalized simplex. A *generalized triangulation* of M^n is a pair (K, τ) consisting of a generalized simplicial complex K with $n-$simplices $\sigma_1, \ldots, \sigma_l \subset H^n$ and of a family $\tau = \{\tau_1, \ldots, \tau_l\}$ of mappings $\tau_i : \sigma_i \to M^n$, $i = 1, \ldots, l$, satisfying the following conditions:

1. For $i = 1, \ldots, l$, τ_i is a homeomorphism from σ_i onto $\tau_i(\sigma_i)$. The images of σ_i and of its faces under τ_i are called *generalized simplex* and *faces in* M^n.
2. $M^n = \cup_{i=1}^l \tau_i(\sigma_i)$.
3. For $i \neq j$, the intersection $\tau_i(\sigma_i) \cap \tau_j(\sigma_j)$ is either empty or a common face of the generalized simplices $\tau_i(\sigma_i)$ and $\tau_j(\sigma_j)$ in M^n.

For $1 \leq r \leq n$, denote by $a^r_{K,\tau}(M^n)$ the number of $r-$dimensional faces in M^n of a generalized triangulation (K, τ) of M^n. Hence, $a^n_{K,\tau}(M^n) = l$. Let $a^0_{K,\tau}(M^n)$ be the number of (ordinary) vertices of $\sigma_1, \ldots, \sigma_l$ in H^n, and define the Euler-characteristic

$$\chi_{K,\tau}(M^n) := \sum_{r=0}^{n} (-1)^r \, a^r_{K,\tau}(M^n) \tag{2.5}$$

of M^n with respect to (K, τ). Obviously, for compact hyperbolic manifolds M^n and for each generalized triangulation (K, τ) of M^n, $\chi_{K,\tau}(M^n)$ coincides with the standard Euler-characteristic $\chi(M^n)$ of M^n.

Let $M^n = H^n/\Gamma$ be a non-compact hyperbolic manifold of finite volume as in (2.4). Denote by $P \subset H^n$ a fundamental polyhedron for Γ. P is a generalized polyhedron whose vertices in ∂H^n are denoted by $p_1, \ldots, p_k$. Observe that $k \geq m$. Starting from P, we associate to M^n a generalized triangulation as follows. Consider the projection (cf. §1.0)

$$\Theta \quad : \quad P \longrightarrow P/\Gamma = M^n \quad ,$$

and let $P_0 \subset P$ denote the compact polyhedron which is mapped by Θ onto $\bar{M}^n \subset M^n$ (cf. (2.4)). The set

$$R := \overline{(P - P_0)} \cap P_0 \subset \partial P_0$$

is the union of finitely many $(n-1)-$dimensional compact polyhedra $R_1, \ldots, R_k \subset H^n$. Here, R_i, $1 \le i \le k$, is numerated in such a way that the vertex cone of P at p_i has base R_i.

Now, decompose P_0 barycentrically several times so that the projection $\Theta|_{P_0}$ induces a triangulation of $\bar{M}^n$. Notice that $M^n = H^n/\Gamma$ is homeomorphic to P/Γ which, by Proposition 1.1, is homeomorphic to P/Φ with Φ a side-pairing for P. Hence, by this process, $R \subset P_0$ is also dissected into simplices; and this simplicial decomposition yields a triangulation of the boundary $N_1^{n-1} \cup \cdots \cup N_m^{n-1}$ of $\bar{M}^n$.

The simplicial dissection of P_0 induces one of the fundamental polyhedron P by adding to the simplices of P_0 the vertex cones at p_i and base in all simplices of $R_i \subset P_0$ for $i = 1, \ldots, k$. This decomposition of P is a generalized simplicial complex denoted by P'. Moreover, the pair (P', Θ) is a generalized triangulation of M^n.

Finally, we see that $\chi(M^n) = \chi_{P',\Theta}(M^n)$ since $\chi(\bar{M}^n) = \chi_{P',\Theta}(\bar{M}^n)$, and the decomposition of the cusps $\bar{C}_1, \ldots, \bar{C}_m$ of M^n corresponds to the canonical cell decomposition constructed in §2.1 .

3. The Gauss-Bonnet formula for hyperbolic manifolds of finite volume

3.0. Let M^n, $n \ge 2$, denote a hyperbolic manifold of finite volume. By §2.2, M^n has a generalized triangulation consisting of finitely many elements. Each element of dimension n is a generalized simplex in M^n and therefore homeomorphic image of a generalized simplex in H^n (cf. §1.0).

In the following, we associate to each element of such a triangulation a higher dimensional analogue of the angle sum in triangles. The alternating sum of all those can be related to both, the volume and the Euler-characteristic of M^n.

3.1. *Generalized angle sums.* Let $T^n \subset H^n$ denote a hyperbolic $n-$simplex. T^n is the convex hull of $n+1$ points in H^n. For $k = 0, \ldots, n-1$, denote by $\{T_i^k\}_{i=1}^{\nu_k}$ the set of all $k-$dimensional faces of T^n, with $\nu_k = \binom{n+1}{k+1}$.

For all indices i, k, choose an interior point $x_i^k \in T_i^k$ and center an $(n-1)-$sphere $S(x_i^k, \epsilon)$ of radius ϵ at x_i^k such that $S(x_i^k, \epsilon)$ intersects only those faces of T^n containing T_i^k as subset. Then, $T^n \cap S(x_i^k, \epsilon)$ is an $(n-1)-$dimensional subset of the sphere $S(x_i^k, \epsilon)$. Now, the value

$$\gamma_i^k = \gamma_i^k(T_i^k) := \frac{\mathrm{vol}_{n-1}(T^n \cap S(x_i^k, \epsilon))}{\mathrm{vol}_{n-1}(S(x_i^k, \epsilon))} \quad , \quad 1 \le i \le \nu_k \quad ; \quad 0 \le k \le n-1 \quad , \qquad (3.1)$$

is independent of the choice of x_i^k and the radius ϵ; γ_i^k is called the $(n-k)-$*fold interior angle of* T^n *at the apex* T_i^k. γ_i^k can also be interpreted as follows. The $(n-k)-$dimensional plane through x_i^k orthogonal to T_i^k intersects $S(x_i^k,\epsilon)$ in an $(n-k-1)-$dimensional sphere $s(x_i^k,\epsilon)$. The intersection $T^n \cap s(x_i^k,\epsilon)$ is an $(n-k-1)-$dimensional simplex on the sphere $s(x_i^k,\epsilon)$ whose dihedral angles, by construction, coincide with certain ones of T^n (cf. also [Pe, p. 330]). Moreover, we see that

$$\gamma_i^k = \frac{\mathrm{vol}_{n-k-1}(T^n \cap s(x_i^k,\epsilon))}{\mathrm{vol}_{n-k-1}(s(x_i^k,\epsilon))} \quad , \quad 1 \le i \le \nu_k \quad ; \quad 0 \le k \le n-1 \quad . \tag{3.1$'$}$$

The definition (3.1) (or (3.1)') can be extended for a generalized simplex in $\bar{H}^n = H^n \cup \partial H^n$. This has to be verified only for $k=0$. Observe that, for $i=1,\dots,\nu_0$, $T^n \cap S(x_i^k,\epsilon)$ is the vertex figure of T^n associated to T_i^0 (cf. §1.0). As such, it is a spherical $(n-1)-$simplex whose dihedral angles coincide with some of T^n, and γ_i^0 equals its volume normalized by $\mathrm{vol}_{n-1}(S(x_i^k,\epsilon))$. Now, if we let a vertex of T^n tend to a point in ∂H^n, its vertex figure deforms to a horospherical sector and yields a compact Euclidean $(n-1)-$simplex. By analytical continuation (cf. [G, Chapter 7, §2.1]), its $n-$fold interior angle tends to 0. Therefore, γ_i^k is defined for a generalized simplex in $\bar{H}^n$ with $\gamma_i^0 = 0$ if its apex belongs to ∂H^n.

Let $T^n \subset \bar{H}^n$ be a generalized $n-$simplex. Collect all $(n-k)-$fold interior angles of T^n and put

$$w_k := \begin{cases} \sum\limits_{i=1}^{\nu_k} \gamma_i^k & \text{for} \quad k=0,\dots,n-1 \quad ; \\ 1 & \text{for} \quad k=n \quad . \end{cases}$$

Since $\gamma_i^{n-1} = 1/2$ we have $w_{n-1} = (n+1)/2$, and w_0 has contributions from ordinary vertices in H^n, only.

The *generalized angle sum* $W(T^n)$ of T^n is defined by

$$W(T^n) := \sum_{k=0}^{n} (-1)^k w_k \quad .$$

Consider an ordinary $n-$simplex $T^n \subset H^n$. The set $w_0,\dots,w_n$ satisfies the complete system of linear independent relations (cf. [Pe, Satz 4], [Po])

$$w_{n-2l-1} + \sum_{r=0}^{l} (-1)^{r+1} a_{2r+1} \binom{n-2l+2r+1}{2r+1} w_{n-2l+2r} = 0 \quad \text{for} \quad 0 \le 2l < n \quad , \tag{3.2}$$

where

$$a_{2r+1} = \frac{(-1)^r}{r+1} (2^{2r+2}-1)\, B_{2r+2}$$

are closely related to the coefficients in the power series of $\tan x$ (cf. [Pe, §5]), with $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42, \ldots$ the Bernoulli numbers. By these relations, one deduces (cf. [Pe, Satz 4])

$$W(T^n) = \begin{cases} 2 \sum\limits_{r=0}^{[\frac{n}{2}]} (-1)^r \, a_{2r+1} \, w_{2r} & \text{for } n \text{ even} \quad , \\ 0 & \text{for } n \text{ odd} \quad . \end{cases} \tag{3.3}$$

Example. Let $T \subset H^2$ be a triangle with angles $\alpha, \beta, \gamma \in (0, \pi)$. Then, $2\pi \, W(T) = \alpha + \beta + \gamma - \pi$. On the other hand, it is well-known that $\mathrm{vol}_2(T) = \pi - (\alpha + \beta + \gamma)$. Hence,

$$W(T) = -\frac{1}{2\pi} \, \mathrm{vol}_2(T) \quad . \tag{3.4}$$

Considering spherical simplices only, H. Poincaré [Po] observed that (3.4) can be generalized to higher dimensions. One should mention that this observation goes back to L. Schläfli who formulated it as a reduction principle (cf. [K], [Pe], for example). H. Hopf [Ho, Satz III] extended these results to all geometries of constant curvature $K \in \{-1, 0, +1\}$ by taking into account the analytical dependence of the ingredients with respect to K.

Theorem 3.1. [Ho, Satz III], [Po]

Let $T^n \subset H^n$ denote an $n-$simplex with generalized angle sum $W(T^n)$. Then,

$$W(T^n) = \begin{cases} (-1)^{\frac{n}{2}} \cdot \frac{2}{\Omega_n} \cdot \mathrm{vol}_n(T^n) & \text{for } n \text{ even} \quad ; \\ 0 & \text{for } n \text{ odd} \quad . \end{cases} \tag{3.5}$$

Now, (3.5) still holds in the extended case when $T^n \subset \bar{H}^n$ is a generalized simplex. This follows since both, $\mathrm{vol}_n(T^n)$ and $W(T^n)$ defined by means of interior angles through (3.1)', are analytical functions in the dihedral angles of T^n (cf. [G, Chapter 7, §2.1]). Therefore, we obtain (cf. also [K])

Corollary 3.2.

Let $T^n \subset \bar{H}^n$ denote a generalized $n-$simplex. Then, $W(T^n)$ satisfies (3.5).

3.2. *The generalized Gauss-Bonnet formula.* We are now ready to extend the Gauss-Bonnet formula (1.3) for hyperbolic manifolds of finite volume following the elementary approach of H. Hopf [Ho].

Theorem 3.3.

Let $n \in \mathbb{N}$ be even, and denote by Ω_n the volume of the unit $n-$sphere. Let M^n be a hyperbolic manifold of finite volume with Euler-characteristic $\chi(M^n)$. Then,

$$\mathrm{vol}_n(M^n) = (-1)^{\frac{n}{2}} \cdot \frac{\Omega_n}{2} \cdot \chi(M^n) \quad . \tag{3.6}$$

Proof: Write $M^n = H^n/\Gamma$ with Γ a discrete, torsionfree group of hyperbolic isometries. Let $P \subset H^n$ be a fundamental polyhedron for Γ, and denote by $\Theta : P \longrightarrow M^n$ the projection. P is a generalized polyhedron which can be dissected to obtain a generalized simplicial complex P' and a generalized triangulation (P', Θ) for M^n (cf. §2.2).

Denote by $\alpha_r := a^r_{P',\Theta}(M^n)$, $0 \le r \le n$, the number of $r-$dimensional elements of the triangulation (P', Θ) of M^n. By §2.2, we know that $\chi_{P',\Theta}(M^n) = \sum_{r=0}^{n} (-1)^r \alpha_r = \chi(M^n)$.

Let $T^n_1, \ldots, T^n_{\alpha_n}$ be the $n-$dimensional elements of (P', Θ) with generalized angle sums $W_1, \ldots, W_{\alpha_n}$. By definition,

$$\sum_{l=1}^{\alpha_n} W_l = \sum_{l=1}^{\alpha_n} \sum_{r=0}^{n} (-1)^r w^l_r = \sum_{r=0}^{n} (-1)^r \sum_{l=1}^{\alpha_n} w^l_r \quad .$$

Now, for each r with $r \le n-1$, $\sum_{l=1}^{\alpha_n} w^l_r$ is the sum of all $(n-r)-$ fold interior angles of the triangulating elements. Those angles among them which - after the side identification of P - share a common apex simplex in the triangulation of M^n add up to 1. Therefore, we have

$$\sum_{l=1}^{\alpha_n} w^l_r = \alpha_r \quad \text{for} \quad r = 0, \ldots, n-1 \quad .$$

Since $w_n = 1$, we get

$$\sum_{l=1}^{\alpha_n} W_l = \sum_{r=0}^{n} (-1)^r \alpha_r = \chi(M^n) \quad .$$

By Corollary 3.2,

$$W_l = W(T^n_l) = (-1)^{\frac{n}{2}} \cdot \frac{2}{\Omega_n} \cdot \mathrm{vol}_n(T^n_l) \quad .$$

Therefore, we deduce that

$$\begin{aligned} \mathrm{vol}_n(M^n) = \mathrm{vol}_n(P) &= \sum_{l=1}^{\alpha_n} \mathrm{vol}_n(T^n_l) \\ &= (-1)^{\frac{n}{2}} \cdot \frac{\Omega_n}{2} \cdot \sum_{l=1}^{\alpha_n} W_l = (-1)^{\frac{n}{2}} \cdot \frac{\Omega_n}{2} \cdot \chi(M^n) \quad . \end{aligned}$$

q.e.d.

Following the same strategy as in the proof of Theorem 3.3, one sees that a hyperbolic manifold M^n of finite volume satisfies

$$\chi(M^n) = 0 \quad \text{if } n \text{ is odd} \quad . \tag{3.7}$$

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