OLD AND NEW ABOUT HILBERT'S THIRD PROBLEM

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This is a summary of a survey talk about Hilbert's third problem on scissors congruence and analogous questions in hyperbolic geometry. The interested reader finds some selected publications for further information in the bibliography.

1. Introduction and some history. In the list of 23 problems proposed by David Hilbert during the Second International Congress of Mathematicians held in Paris in 1900, the third problem plays a special role, and does so in several respects.

In contrast to the other problems, the third one deals with elementary geometrical questions about the foundations of geometry. Actually, in 1899, Hilbert had just finished writing the book *Grundlagen der Geometrie* and was interested in how to teach geometry. In this context, Hilbert mentioned that—contrary to the planar case—volume computations in three-dimensional Euclidean geometry are always based on some limiting process and on methods of exhaustion. He asked for a rigorous proof that one cannot construct a theory of polyhedral volume without the continuity axiom. His precise formulation goes as follows.

The equality of the volumes of two tetrahedra of equal bases and equal altitudes. *In two letters to Gerling, Gauss expresses his regret that certain theorems of solid geometry depend upon the method of exhaustion, that is, in modern phraseology, upon the axiom of continuity (or upon the axioms of Archimedes). Gauss mentions in particular the theorem of Euclid, that triangular pyramids of equal altitudes are to each other as their bases. Now, the analogous problem in the plane has been solved. Gerling also succeeded in proving the equality of volume of symmetrical polyhedra by dividing them into congruent parts. Nevertheless, it seems to me probable that a general proof of this kind for the theorem of Euclid just mentioned is impossible, and it should be our task to give a rigorous proof of its impossibility. This would be obtained, as soon as we succeeded in "specifying two tetrahedra of equal bases and equal altitudes which can in no way be split up into congruent tetrahedra, and which cannot be combined with congruent tetrahedra to form two polyhedra which themselves could be split up into congruent tetrahedra."*

In the very same year, Max Dehn confirmed Hilbert's conjecture by constructing two polyhedra of equal volume which are *not* equidecomposable. I will come back to Dehn's solution below.

For a long while, the problem was forgotten until some Swiss mathematicians started to work on related questions. Among those were

– Jean-Pierre Sydler, a student of Heinz Hopf at the ETH Zürich, who extended the work of Dehn in a completely satisfactory way. For this, he obtained the Gold Medal of the Danish Academy of Sciences in 1966;

– Hugo Hadwiger and his group at the University of Bern who contributed by extending Hilbert's problem to Euclidean spaces of arbitrary dimensions.

In 1974, during the conference on *mathematical developments arising from Hilbert problems* at DeKalb, there was a talk held about the third problem. Unfortunately, the speaker did not submit his manuscript for publication in the proceedings (see [12]).

In recent years, the mathematicians B. Jessen, A. Thorup, J. Dupont of the Danish school, C.-H. Sah, P. Cartier, J. Milnor, J.-L. Cathelineau, A. Goncharov, and others showed active interest in this circle of questions.

2. Reformulation and results. Although Hilbert's third problem deals with solid geometry and polyhedral volume, it is basically an *algebraic* problem. Let $X^n = S^n, E^n$ or H^n be the standard space of constant curvature $1, 0$, or -1. The *scissors congruence* or *polytope group* $\mathcal{P}(X^n)$ of X^n is the abelian group generated by the symbols $[P]$ for each polytope $P \subset X^n$ subject to the relations

$$[P \sqcup Q] = [P] + [Q], \quad \text{where } P \sqcup Q \text{ is the disjoint union of } P, Q;$$
$$[g(P)] = [P], \quad \forall g \in \mathrm{Iso}\,(X^n).$$

The problem can now be stated as follows: *Find a complete system of invariants for the scissors congruence or groups.* There is the following criterion.

Proposition 2.1 (Zylev). *$[P] = [Q] \Leftrightarrow P \sim Q$ equidecomposable, that is, $\exists P = P_1 \sqcup \cdots \sqcup P_n, Q = Q_1 \sqcup \cdots \sqcup Q_n$ such that $g_k(P_k) = Q_k$ for some $g_1, \ldots, g_n \in \mathrm{Iso}(X^n)$.*

For $n = 2$ and in all geometries, a classical result due to Farkas Bolyai and P. Gerwien says that polygonal area separates points in $\mathcal{P}(X^2)$.

Lemma 2.2. *Let P, $Q \subset X^2$ be two polygons. Then, $[P] = [Q]$ if and only if $vol_2(P) = vol_2(Q)$.*

3. Dehn's solution for E^3 and the theorem of Dehn-Sydler. We now present a short outline of Dehn's proof and note that he profited from a hint of Bricard. Dehn discovered—beside polyhedral volume—another scissors congruence invariant, the so-called *Dehn invariant.*

Let $P \subset E^3$ be a Euclidean polyhedron with edges $e_1, \ldots, e_r$ of lengths $l_1, \ldots, l_r$ and dihedral angles $\alpha_1, \ldots, \alpha_r$ attached at $e_1, \ldots, e_r$. The Dehn invariant is then defined by

$$D(P) = \sum_{i=1}^{r} l_i \otimes_{\mathbb{Z}} \alpha_i \in \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}/\pi\mathbb{Z}.$$

It is obvious that $D(\text{prism}) = 0$. Now, a necessary condition for two polyhedra $P, Q \subset E^3$ to be equidecomposable is that

$$\mathrm{vol}_3(P) = \mathrm{vol}_3(Q), \qquad D(P) = D(Q). \tag{$*$}$$

Dehn's solution consists of the construction of the following counter-example. Let $P := S_{\text{reg}}(2\alpha)$ be a regular tetrahedron of edge length 1, that is, $\cos(2\alpha) = 1/3$ and α is irrational. On the other hand, choose a regular cube Q with $\mathrm{vol}_3(Q) = \mathrm{vol}_3(P)$. Then, P and Q cannot be scissors congruent since $D(Q) = 0$ while $D(P) = 6 \otimes 2\alpha \neq 0$ (see ($*$)).

In 1965, after 20 years of hard work, Sydler proved that the conditions ($*$) are also sufficient.

Theorem 3.1 (Dehn-Sydler). *Let $P, Q \subset E^3$ be two polyhedra. Then, $[P] = [Q]$ if and only if $\mathrm{vol}_3(P) = \mathrm{vol}_3(Q)$ and $D(P) = D(Q)$.*

In 1968, Jessen [9] found a much simpler proof of Sydler's result. Moreover, only a few years later, he discovered that the analogous problem for E^4 can be reduced to the case of E^3 as follows.

Let $P \subset E^4$ be a Euclidean polytope with polygonal faces $p_1, \ldots, p_r$ of areas $f_1, \ldots, f_r$ and with dihedral angles $\alpha_1, \ldots, \alpha_r$ attached at $p_1, \ldots, p_r$. Consider the *Dehn invariant* defined, similarly as above, by

$$D(P) = \sum_{i=1}^{r} f_i \otimes_{\mathbb{Z}} \alpha_i \in \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}/\pi\mathbb{Z}.$$

Theorem 3.2 (Jessen). *Let P, $Q \subset E^4$ be two polytopes. Then, $[P] = [Q]$ if and only if $\mathrm{vol}_4(P) = \mathrm{vol}_4(Q)$ and $D(P) = D(Q)$.*

The proof is based essentially on the reducibility to the 3-dimensional result by using the properties that

(a) in E^4: $P \sim [0,1] \times Q$ for some polyhedron $Q \subset E^3$;

(b) $D(P) = D(I \times Q) = D(Q)$.

However, for arbitrary spaces $X^n = S^n, E^n$ or H^n, $n \geq 3$, and $X^n \neq E^3, E^4$, the *generalized third problem of Hilbert* asking for a complete system of invariants for $\mathcal{P}(X^n)$ is unresolved.

4. Some developments concerning $\mathcal{P}(H^n)$. In the last few years, Hilbert's third problem experienced some revival. This is mainly due to the interplay with the cohomology of Lie groups made discrete, number theory and polylogarithms, algebraic K—theory and Borel groups, and the motivic interpretation of the non-Euclidean Dehn complex. For example, it was shown that the theorem of Dehn-Sydler is equivalent to the fact that

$$H_2(SO(3), \mathbb{R}^3) = 0.$$

In this article, it is impossible to discuss these relations (see [3, 7]). However, in the following, I would like to indicate how some *geometrical* ideas of Sydler and Jessen can be adapted to describe $\mathcal{P}(H^3)$.

Consider hyperbolic space H^n with boundary ∂H^n of *points at infinity*. For this space, there are different polytope groups. While $\mathcal{P}(H^n)$ denotes the usual polytope group, $\mathcal{P}(\overline{H^n})$ is built upon polytopes with vertices possibly at infinity, and $\mathcal{P}(\overline{H^n})_\infty$ is generated by hyperbolic simplices all of whose vertices are at infinity (such simplices are termed *ideal*).

Moreover, in any n-space of constant curvature, one can decompose convex polytopes and simplices into *orthoschemes*; these are certain orthogonal simplices which generalize the notion of right-angled triangles in some way. They possess exactly two among the $n+1$ vertices which might be at infinity (in the extremal case, they are termed *doubly asymptotic*). Orthoschemes are very basic and fundamental in the following sense.

Proposition 4.1 (Debrunner-Sah). (1) $\mathcal{P}(H^n)$ *is generated by orthoschemes.*
(2) $\mathcal{P}(\overline{H^{2m+1}})$ *is generated by doubly asymptotic orthoschemes.*
(3) *For* $n \geq 3 : \mathcal{P}(H^n) \simeq \mathcal{P}(\overline{H^n}) \simeq \mathcal{P}(\overline{H^n})_\infty$.

Now, the notion of the Dehn invariant can be extended, to include the case of asymptotic, hyperbolic polyhedra. For example, consider an ideal tetrahedron

$$S_\infty(z) \subset \overline{H^3} = \left(P_1(\mathbb{C}) \times \mathbb{R}_+, ds^2 = \frac{|dz|^2}{(\operatorname{Im} z)^2}\right)$$

with vertices $0, 1, \infty, z$ in the upper half space model. $S_\infty(z)$ has three pairs of dihedral angles attached at opposite edges, namely

$$\alpha_1 := \arg z, \qquad \alpha_2 := \arg\left(1 - \frac{1}{z}\right), \qquad \alpha_3 := \arg \frac{1}{1-z} = \pi - (\alpha_1 + \alpha_2).$$

It can be seen that the formula

$$D(S_\infty(z)) = 2 \sum_{i=1}^{3} \log(2 \sin \alpha_i) \otimes_{\mathbb{Z}} \alpha_i$$

extends Dehn's invariant of a hyperbolic tetrahedron if all vertices tend to boundary points. Hence, the generalized third Hilbert problem for hyperbolic space goes as follows.

Conjecture 4.2. $P \sim Q$ *in* $\overline{H^3} \Leftrightarrow \operatorname{vol}_3(P) = \operatorname{vol}_3(Q)$, $D(P) = D(Q)$.

Sydler's original papers (cf. [14]) are difficult to read. The simplification of Jessen still reflects the principal geometrical idea as expressed by the role of orthogonal simplices and the fundamental lemma. That is, consider an orthoscheme $R(a,b;\lambda) \subset E^3$ defined by the parameters

$$a = \sin^2 \alpha_1, \quad b = \sin^2 \alpha_3 \quad \text{whence } \cos^2 \alpha_2 = a \cdot b,$$

as well as

$$\lambda := l_1 \cdot \tan \alpha_1 = l_2 \cdot \cot \alpha_2 = l_3 \cdot \tan \alpha_3.$$

Here, α_1, α_2, and α_3 denote the non-right dihedral angles of R attached at edges of lengths l_1, l_2, l_3 (see Figure 4.1).

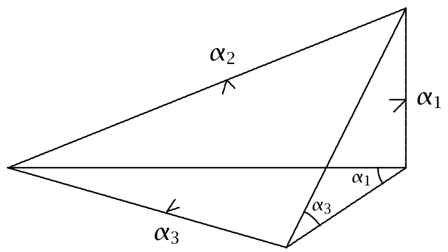

FIGURE 4.1.

It follows that

$$\mathrm{vol}_3\,(R(a,b;\lambda)) = \frac{\lambda^3}{6}(v(ab) - v(a) - v(b)), \quad \text{where } v(x) := \frac{1-x}{x}.$$

For $0 < a,b,c < 1$, put

$$X := R(a,b;\lambda) + R(ab,c;\lambda), \qquad Y := R(a,c;\lambda) + R(ac,b;\lambda).$$

An easy calculation shows that $\mathrm{vol}_3(X) = \mathrm{vol}_3(Y)$ and $D(X) = D(Y)$.

THEOREM 4.3 (The fundamental lemma)**.** *$X \sim Y$, that is, $R(a,b;\lambda) + R(ab,c;\lambda) \sim R(a,c;\lambda) + R(ac,b;\lambda)$.*

Given this lemma the Dehn-Sydler theorem, according to Jessen, is roughly proven as follows:

(1) Let $\mathcal{X} < \mathcal{P}(E^3)$ be generated by the prisms in E^3, that is, $\mathcal{X} \subset \ker(D)$, and suppose $\mathcal{P}(E^3)/\mathcal{X}$ admits the structure of a real vector space. Moreover, suppose

$$\mathrm{vol}_3 : \mathcal{X} \longrightarrow \mathbb{R} \quad \text{is bijective.}$$

Then, one shows that $\mathcal{X} = \ker(D)$ which finally yields that

$$\mathrm{vol}_3 \times D : \mathcal{P}(E^3) \longrightarrow \mathbb{R} \times (\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}/\pi\mathbb{Z}) \quad \text{is injective.}$$

(2) Next, one observes that the class of orthoschemes are generators of $\mathcal{P}(E^3)/\mathfrak{T}$ and that the fundamental lemma provides several important algebraic relations. A crucial implication of these are the following properties, stated and proved in a very elegant algebraic setting by Jessen and Thorup.

Theorem 4.4 (Jessen-Thorup)**.** *Let $F:(0,1)\times(0,1)\longrightarrow V$ be a mapping to a real vector space V satisfying*

$$F(a,b)=F(b,a),\qquad F(a,b)+F(ab,c)=F(b,c)+F(a,bc).$$

Then, there is a mapping $f:(0,1)\longrightarrow V$ such that $F(a,b)=f(ab)-f(a)-f(b)$.

Theorem 4.5 (Jessen-Thorup)**.** *Let $G:(0,\infty)\times(0,\infty)\longrightarrow V$ be such that*

$$G(a,b)=G(b,a),\qquad G(a,b)+G(a+b,c)=G(b,c)+G(a,b+c),$$
$$G(ac,bc)=cG(a,b).$$

Then, there is a mapping $g:(0,\infty)\longrightarrow V$ such that $G(a,b)=g(a+b)-g(a)-g(b)$.

(3) To finish the proof, take a polyhedral basis $\{Q_r\}$ of $\mathcal{P}(E^3)/\mathfrak{T}$ so that, for each polyhedron P, we have

$$P\sim_{\mathfrak{T}}\sum_r m_rQ_r\quad\text{for some } m_r\in\mathbb{R}.$$

In particular, we obtain $R\sim_{\mathfrak{T}}\sum_r F_rQ_r$ with $F_r=F_r(a,b)$ satisfying the condition of Jessen-Thorup in Theorem 4.4. Therefore, there is a function $f_r(x)$ which is additive, annulates π and is such that

$$f_r(P):=\sum_{l\text{ edge of }P} l\otimes f_r(\alpha)$$

represents Dehn's invariant. Since R is a generator, one deduces that $P\sim_{\mathfrak{T}}\sum_r f_r(P)Q_r$. Finally, one gets $A\sim_{\mathfrak{T}}\sum_r f_r(A)Q_r=\sum_r f_r(B)Q_r\sim_{\mathfrak{T}}B$.

Now turn to the hyperbolic analogue. Let $R(a,b;\mu)\subset H^3$ denote a hyperbolic orthoscheme with parameters a,b as above and consider the additional parameter

$$\mu=\frac{\cos^2\alpha_2-\sin^2\alpha_1\sin^2\alpha_3}{\cos^2\alpha_1\cos^2\alpha_3}=:\tan^2\theta,\qquad\theta\in\left[0,\frac{\pi}{2}\right].$$

One checks that

$$\mu=\tanh l_1\cdot\tan\alpha_1=\tanh l_2\cdot\cot\alpha_2=\tanh l_3\cdot\tan\alpha_3,$$

and that

$$\cos^2\alpha_2=a\bigcirc_\mu b:=ab+\mu^2(1-a)(1-b).$$

For $0 < a, b, c < 1$, put

$$U := R(a,b;\mu) + R(a \bigcirc_\mu b, c;\mu), \qquad V := R(a,c;\mu) + R(a \bigcirc_\mu c, b;\mu).$$

Again, it follows that $\mathrm{vol}_3(U) = \mathrm{vol}_3(V)$ and $D(U) = D(V)$.

Question 4.6 (Analogue of the fundamental lemma for H^3). $U \sim V$, that is, $R(a,b;\mu) + R(a \bigcirc_\mu b, c;\mu) \sim R(a,c;\mu) + R(a \bigcirc_\mu c, b;\mu)$?

In order to simplify the question, the following observation is useful. Computing the volume of R (an expression in *dilogarithm functions*), one finds that

$$\mathrm{vol}_3(R(a,b;\mu)) = \mathrm{vol}_3(R_\infty(a)) + \mathrm{vol}_3(R_\infty(b)) - \mathrm{vol}_3\left(R_\infty(a \bigcirc_\mu b)\right),$$

where $R_\infty(a)$ denotes a simply asymptotic orthoscheme with dihedral angles α_1, θ, $(\pi/2) - \theta$. Therefore, a way to study Hilbert's third problem for hyperbolic 3-space, is to investigate the above question for $\mathcal{P}(\overline{H^3})$ and to profit from the isomorphisms in Proposition 4.1.

5. Algebraic K-theoretical aspects—a brief account. The group $\mathcal{P}(H^3)$ admits further equivalent and very elegant interpretations. The following is a very short tour around these ideas (due to Sah, Dupont, Thurston, and others).

Let $\mathcal{P}(\mathbb{C})$ be the abelian group generated by $\{z\}$, $z \in \mathbb{C}$, such that

(1) $\{0\} = \{1\} = 0$;

(2) $\forall a \neq b \in \mathbb{C} \setminus \{0,1\} : \{a\} - \{b\} + \{a/b\} - \{1 - a/1 - b\} + \{b(1-a)/a(1-b)\} = 0$.

This group can be isomorphically identified with the group $\mathcal{T}(\mathbb{C})$ generated by ideal tetrahedra, that is, $\mathcal{T}(\mathbb{C})$ is the group of 4-tuples (p_0, p_1, p_2, p_3), $p_i \in P_1(\mathbb{C})$, with

(1′) $(g(p_0), g(p_1), g(p_2), g(p_3)) = (p_0, p_1, p_2, p_3)$, $\forall g \in \mathrm{PSL}(2,\mathbb{C})$,

(2′) $\forall p_0, \dots, p_4 \in \mathbb{C}$ disjoint:

$$\sum_{i=1}^{4} (-1)^i (p_0, \dots, \widehat{p_i}, \dots, p_4) = 0.$$

The identification is then performed by using the map

$$(p_0, p_1, p_2, p_3) \longmapsto \left\{ z := r(p_0, p_1, p_2, p_3) \right\} \quad (r \text{ denotes cross-ratio}).$$

Furthermore, one has

$$\mathrm{vol}_3(p_0, p_1, p_2, p_3) = \mathrm{vol}_3\left(\infty, 0, 1, r(p_0, p_1, p_2, p_3)\right) = D(z),$$

where

$$D(z) := \log|z| \arg(1-z) - \mathrm{Im}\, Li_2(z)$$

denotes the Bloch-Wigner dilogarithm (a modification of the classical dilogarithm $Li_2(z)$) which satisfies the 5-term relation of Spence-Abel

$$\sum_{i=1}^{4}(-1)^i D\Big(r(p_0,\dots,\widehat{p_i},\dots,p_4)\Big) = 0.$$

Now, the following isomorphisms can be established.

$$\mathcal{P}(H^3) \simeq \mathcal{P}(\overline{H^3})_\infty \simeq \mathcal{P}(\mathbb{C})/_{\langle\{z\}+\{\bar z\}\rangle} \simeq \mathcal{P}(\mathbb{C})^-,$$

where the exponent $^-$ describes the eigenspace to the eigenvalue -1 with respect to complex conjugation. Next, consider the following mappings:

$$\rho : \mathcal{P}(\mathbb{C}) \longrightarrow \Lambda^2_{\mathbb{Z}}\mathbb{C}^\times = \mathbb{C}^\times \otimes_{\mathbb{Z}} \mathbb{C}^\times/\langle a\otimes b + b\otimes a\rangle, \qquad \rho(\{z\}) := z\wedge(1-z),$$
$$\widetilde{D} : \mathcal{P}(\mathbb{C}) \longrightarrow \mathbb{R}, \qquad \widetilde{D}(\{z\}) := D(z).$$

They are related to volume and Dehn's invariant according to the following picture:

(a) The composition $\mathcal{P}(H^3) \longrightarrow \mathcal{P}(\mathbb{C})^- \xrightarrow{\widetilde{D}} \mathbb{R}$ is the volume.

(b) The composition $\mathcal{P}(H^3) \longrightarrow \mathcal{P}(\mathbb{C})^- \xrightarrow{\rho^-} (\Lambda^2_{\mathbb{Z}}\mathbb{C}^\times)^- \simeq \mathbb{R}\otimes_{\mathbb{Z}}\mathbb{R}/\pi\mathbb{Z}$ is Dehn's invariant.

Finally, consider the *Bloch group* defined by $B(\mathbb{C}) := \ker(\mathcal{P}(\mathbb{C}) \xrightarrow{\rho} \Lambda^2_{\mathbb{Z}}\mathbb{C}^\times)$. Again, let $B(\mathbb{C})^-$ denote the negative part of $B(\mathbb{C})$ with respect to the involution induced by the conjugation (actually, the Bloch group can be identified with the group of polyhedra with vanishing Dehn invariant). In this setting, Hilbert's third problem for H^3 can be reformulated very efficiently as follows:

Is the mapping $\widetilde{D} : B(\mathbb{C})^- \longrightarrow \mathbb{R}$ injective?

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