# Polytopal Complexes in Spaces of Constant Curvature 

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## Introduction

The basic objects of study in this thesis are even-dimensional polytopes and polytopal complexes in spaces of constant curvature. The connections between combinatorics, angles and volume are in the center of the investigations.
More precisely, we will develop volume formulas for polytopes in the spherical space $\mathbb{S}^{2 m}$ and hyperbolic space $\mathbb{H}^{2 m}$ of even dimension which only depend on the combinatorics and the odddimensional angles of the polytope. These types of formulas are usually called reduction formulas because they reduce the volume problem in spherical and hyperbolic space of even dimension to the determination of odd-dimensional spherical volumes. The general reduction formula for a $2 m$-dimensional polytope $P$ in $\mathbb{X}^{2 m}=\mathbb{S}^{2 m}$ with constant curvature $K=1, \mathbb{E}^{2 m}$ with $K=0$ or $\mathbb{H}^{2 m}$ with $K=-1$ can be written as

$$
2 c_{2 m}^{-1} K^{m} \operatorname{vol}_{\mathbb{X} 2 m}(P)=\sum_{\substack{P^{2 j} \in \Omega^{2 j}(P) \\ j=0, \ldots, m}} \sigma^{2 j}\left(P^{2 j}\right) \alpha_{2 m-2 j-1}\left(P^{2 j}\right)
$$

where $c_{2 m}$ denotes the volume of the $2 m$-dimensional unit sphere, $\Omega^{2 j}(P)$ is the set of all $2 j$ dimensional ordinary faces of $P, \alpha_{2 m-2 j-1}\left(P^{2 j}\right)$ is the ( $2 m-2 j-1$ )-dimensional (normalized) angle in the apex $P^{2 j}$ and $\sigma^{2 j}\left(P^{2 j}\right)$ are rational numbers which only depend on the combinatorics of the face $P^{2 j}$. If we fix a combinatorial type, the main problem is the explicit determination of these rational numbers which reflect in a yet mysterious way the combinatorics of the polytope $P$. These coefficients $\sigma^{2 j}\left(P^{2 j}\right)$ can also be viewed as combinatorial invariants which map the set of all $2 j$-dimensional polytopes $\mathbf{P}^{2 \mathrm{j}}$ into the rational numbers

$$
\begin{aligned}
\sigma^{2 j}: \mathbf{P}^{\mathbf{2 j}} & \longrightarrow \mathbb{Q} \\
P^{2 j} & \longmapsto \sigma^{2 j}\left(P^{2 j}\right) .
\end{aligned}
$$

In honour to Ludwig Schläfli these invariants are called Schläfli invariants.
We point out that little is known about volume functions for odd-dimensional polytopes in the spherical and hyperbolic space. About 1935 Coxeter [C] introduced the function

$$
S(\alpha, \beta, \gamma):=\sum_{r=1}^{\infty} \frac{(-X)^{r}}{r^{2}}(\cos 2 r \alpha-\cos 2 r \beta+\cos 2 r \gamma-1)-\alpha^{2}+\beta^{2}-\gamma^{2}
$$

where

$$
X=\frac{\sin \alpha \sin \gamma-D}{\sin \alpha \sin \gamma+D} \quad \text { with } \quad D=\sqrt{\cos ^{2} \alpha \cos ^{2} \gamma-\cos ^{2} \beta}
$$

Coxeter combined results of Schläfli [Sch] for spherical and Lobatschewsky [L] for hyperbolic orthoschemes and proved that the volume of an orthoscheme $R=R(\alpha, \beta, \gamma)$ in $\mathbb{S}^{3}$ or $\mathbb{H}^{3}$ can be written as

$$
\operatorname{vol}_{\mathbb{X}^{3}}(R)= \begin{cases}\frac{1}{4} S\left(\frac{\pi}{2}-\alpha, \beta, \frac{\pi}{2}-\gamma\right) & , \quad \mathbb{X}^{3}=\mathbb{S}^{3} \\ \frac{i}{4} S\left(\frac{\pi}{2}-\alpha, \beta, \frac{\pi}{2}-\gamma\right) & , \quad \mathbb{X}^{3}=\mathbb{H}^{3}\end{cases}
$$

The reduction formula $(\star)$ arises in two different ways. The first way was followed by Schläfli [Sch]. He used the so-called Schläfli Differential Formula to prove ( $\star$ ) by induction. Furthermore,
he identified the numbers $\sigma^{2 j}\left(T^{2 j}\right)$ for a simplex $T^{2 j}$ as the modified tangent numbers. Therefore the reduction formula for simplices is often called Schläfli's Reduction Formula. Moreover, Schläfli noticed that the numbers $\sigma^{2 j}\left(P^{2 j}\right)$ can always be determined by recursion formulas.
The second way was followed by E. Peschl [Pe]. He used the so-called Poincaré Formula $[\mathrm{Po}]$ to prove ( $\star$ ) for simplices by combinatorial methods. Poincaré's Formula can be written as

$$
W(P)= \begin{cases}2 K^{m} c_{2 m}^{-1} \text { vol }_{\mathbb{X} 2 m}(P) & , \quad n=2 m \\ 0 & , \quad n=2 m+1 \text { even }\end{cases}
$$

where $W(P)$ denotes the generalized (alternating) angle sum of the polytope $P$. This formula can be viewed as an angle analog of Eulers Polyhedron Theorem and it gives a volume formula for even-dimensional polytopes in spherical and hyperbolic space.
A first aim of this thesis is the determination of the numbers $\sigma^{2 j}\left(P^{2 j}\right)$ for the class of simple and simplicial polytopes. These polytopes have an easy combinatorial structure and the duality allows us to transfer results from one class to the other. The method of the determination of the numbers $\sigma^{2 j}\left(P^{2 j}\right)$ can be described as follows. We decompose $P^{2 j}$ suitably into simplices. Then we use Schläfli's Reduction Formula and add all the volumes and the decomposition angles.
In dimension two this is quite easy. We see that the sum of the decomposition angles are angles of the decomposed polygon.


Of course, we also see that we can simplify our lives by choosing a suitable decomposition. For dimensions bigger or equal to three the connections between the decomposition angles and the angles of the polytope are no longer obvious.


Hence the general problem here is to understand how angles in the decomposition add to angles in the polytope.

We will solve this problem by developing a general combinatorial calculus. This calculus allows us to transfer linear relations between angles and combinatorial values of polytopes to polytopal complexes which are decomposable into such "bricks". In more details, let $\Pi$ be an $n$-dimensional polytopal complex in $\mathbb{X}^{n}$ and $\mathcal{D}=\mathcal{D}(\Pi)$ a polytopal decomposition of $\Pi$. Furthermore, let

$$
0=F_{D}=\sum_{\substack{D^{i} \in \Omega^{i}(\mathcal{D}) \\ i=0, \ldots, n}} \phi\left(D^{i}\right) \alpha_{n-i-1}\left(D^{i}\right)+\kappa(D) \operatorname{vol}_{\mathbb{X} n}(D)
$$

be a linear relation for each element $D \in \mathcal{D}$. Here $\phi\left(D^{i}\right)$ and $\kappa(D)$ denote arbitrary real numbers which depend on the face $D^{i}$ or $D$. Then this calculus allows us to combine all these relations to a relation for the complex $\Pi$ of the form

$$
0=F_{\Pi}=\sum_{\substack{P_{i} \in \Omega^{i}(\Pi) \\ i=0, \ldots, n}} \Phi\left(P^{i}\right) \alpha_{n-i-1}^{\Pi}\left(P^{i}\right)+\kappa \operatorname{vol}_{\mathbb{X}^{n}}(\Pi),
$$

where $\alpha_{n-i-1}^{\Pi}\left(P^{i}\right)$ denotes the complex angle of $\Pi$ in the apex $P^{i}$. The main problem is to see that a sum of decomposition angles in $\mathcal{D}$ with the same apex gives a complex angle of a certain dimension. This dimension depends on the dimension of the face of $\Pi$, which contains the (open) apex in the relative interior. This gives us a reduction formula and the determination of the coefficients $\sigma^{2 j}\left(P^{2 j}\right)$ is then a purely combinatorial problem. We find out that

$$
\sigma^{2 j}\left(P^{2 j}\right)=\left\{\begin{array}{cl}
2 \sum_{k=0}^{j}(-1)^{k} a_{2 k+1} a^{2 k-1}\left(P^{2 j}\right) & , \quad P^{2 j} \text { simplicial } \\
2 \sum_{k=0}^{j}(-1)^{k} a_{2 k+1} a^{2 j-2 k}\left(P^{2 j}\right) & , \quad P^{2 j} \text { simple }
\end{array}\right.
$$

where $a_{2 k+1}(k \geq 0)$ is a modified tangent number and $a^{l}\left(P^{2 j}\right)$ the number of $l$-dimensional faces of $P^{2 j}$. By using combinatorial relations the facevectors of particular (well-known) polytopes wee see that

$$
\sigma^{2 j}\left(P^{2 j}\right)=\left\{\begin{array}{cl}
2(-1)^{j} a_{2 j+1} & , \quad P^{2 j} \text { simplex } \\
(-1)^{j} E_{2 j} & , \quad P^{2 j} \text { cube or cross polytope }
\end{array}\right.
$$

where $E_{2 j}$ is a Euler number. This method is only successful here because the combinatorial structure of $P^{2 j}$ is not too complicated and applied to arbitrary polytopes it supplies a volume formula which depends also on even-dimensional angles.
So for the description of the Schläfli invariants for arbitrary polytopes we must go another way. We take up the idea of Schläfli that all invariants $\sigma^{2 j}\left(P^{2 j}\right)$ can be determined by recursion formulas. A second aim of this thesis is to work out this idea. These recursion formulas can be constructed by mapping a spherical polytope to a hyperplane. We thus get a tesselation of this hyperplane which has the same combinatorics as the boundary of the polytope. Furthermore, this tesselation can be viewed as the boundary of a degenerated spherical polytope and all angles of this polytope are of measure one half. Hence all angles disappear in the recursion formula and we get a relations between the Schläfli invariants of different dimensions

$$
\sigma^{2 m}(P)=1-\frac{1}{2} \sum_{j=0}^{m-1} \sum_{P^{2 j} \in \Omega^{2 j}(P)} \sigma^{2 j}\left(P^{2 j}\right)
$$

Polytopes in spaces of constant curvature have connections with discrete groups $\Gamma$ of isometries and with geometric orbifolds. In more details, we can construct (generalized) polytopes for each discrete group. These polytopes are closures of fundamental domains and the simplest construction method is the Dirichlet construction. If these (generalized) polytopes have only finitely many faces, they are called fundamental polytopes and the important invariant of the group $\Gamma$, the covolume $\operatorname{covol}(\Gamma)$, can be computed as the volume of these fundamental polytopes.

Furthermore, the faces (of a fixed dimension) of a fundamental polytope can be arranged in pairwise disjoint sets of $\Gamma$-equivalent faces

$$
\Omega^{d}(P)=C_{1} \cup C_{2} \cup \ldots \cup C_{k(d)}
$$

for $0 \leq d \leq n-1$. The angles with apexes at the faces of one of these subsets $C_{i}$ add up to a rational number

$$
\sum_{P^{d} \in C_{i}} \alpha_{n-d-1}\left(P^{d}\right)=\frac{1}{g_{i}^{d}}
$$

where $g_{i}^{d}$ is the number of elements in the (setwise) stabilizer subgroup of the face $P^{d}$ for $i=1, \ldots, k(d)$. This condition is usually called the cycle condition. The stabilizer subgroups can be viewed as discrete subgroups of $O(d)$, so they are finite. Of course, all elements in $C_{i}$ are $\Gamma$-equivalent and all stabilizer subgroups of elements in $C_{i}$ are conjugated to each other.
A third aim of this thesis is to construct volume formulas for fundamental polytopes which depend on orders of stabilizer subgroups instead of angles by using the cycle condition. A two-dimensional formula of this type was constructed by Carl Ludwig Siegel [S]

$$
2 K c_{2}^{-1} \operatorname{vol}_{\mathbb{X}^{2}}(P)=\sum_{i=1}^{\mu^{0}}\left(\frac{1}{g_{i}^{0}}-\frac{1}{2} l_{i}^{0}\right)+1-\frac{1}{2} a_{i n f}^{0}(P)
$$

where $\mu^{0}$ denotes the number of equivalence classes of ordinary vertices, $l_{i}^{0}$ the number of vertices in an equivalence class $C_{i}$ and $a_{\text {inf }}^{0}(P)$ the number of vertices at infinity of $P$. Siegel used this formula to determine a lower bound for the covolumes of discrete subgroups $\Gamma$ of hyperbolic isometries. Furthermore, he showed that this minimum is realized by the Coxeter group $[2,3,7]$

$$
\operatorname{covol}(\Gamma) \geq \frac{\pi}{42}=\operatorname{covol}\left(\circ-\frac{7}{\square}\right)
$$

In the last part of this thesis we will try to show the utility of the constructed formulas by exhibiting explicit examples. So we compute the covolume of Coxeter groups, classified by I. M. Kaplinskaja [Kapl], F. Esselmann [Es] and P. Tumarkin [T], and of the group $\operatorname{PSL}\left(2, \mathbb{Z}\left[i_{1}, i_{2}\right]\right)$ (in a quite natural way) by computing the volume of a fundamental polytope. Here the group $P S L\left(2, \mathbb{Z}\left[i_{1}, i_{2}\right]\right)$ can be viewed as the 4-dimensional generalization of the well-known discrete subgroup $\operatorname{PSL}(2, \mathbb{Z})<\operatorname{Iso}\left(\mathbb{H}^{2}\right)$ and its covolume is equal to $\frac{\pi^{2}}{72}$.


Also by computing the volume of a fundamental polytope we establish the volumes of the ideal 24 -cell manifold, the Davis manifold [D] and the Ivanšić manifolds [I].

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## 1 Three Metric Spaces

In this section we will define the Euclidean space $\mathbb{E}^{n}$, the spherical space $\mathbb{S}^{n}$ and the hyperbolic space $\mathbb{H}^{n}$ as metric spaces. Especially in the case of the hyperbolic space it is very helpful to use several models. To avoid confusion we will use the following notations. We shall denote by $\mathbb{H}^{n}$ "the" hyperbolic space without interpretation in one of the models, by $\mathbb{H}^{n}$ the hyperbolic space in the vector-space model, by $\mathbb{B}^{n}$ the hyperbolic space in the ball model, by $\mathbb{D}^{n}$ the hyperbolic space in the projective model and by $I^{n}$ the hyperbolic space in the upper halfspace model. Furthermore, we have bijective (and by construction isometric) maps between the different models.

$$
\mathbb{H}^{n}: \mathbb{D}^{n} \xrightarrow{\mu} \mathbb{H}^{n} \stackrel{\eta}{\longleftarrow} \mathbb{B}^{n} \stackrel{\zeta}{\longleftarrow} \mathbb{U}^{n}
$$

For the details of all notions in this section see for instance [R], sections 1-5.
From the viewpoint of differential geometry the three spaces $\mathbb{E}^{n}, \mathbb{S}^{n}$ and $\mathbb{H}^{n}$ are the only complete, connected and simply-connected Riemannian manifolds with constant sectional curvature $K=$ 0,1 or -1 (up to an isometry of Riemannian manifolds).
For more details you may use [V2].

### 1.1 The Euclidean Space $\mathbb{E}^{n}$

Denoting the coordinates in the space $\mathbb{R}^{n}$ by $x_{1}, \ldots, x_{n}$, we define the scalar product, the (induced) norm and the (induced) metric by the formulae

$$
\begin{aligned}
\langle x, y\rangle_{n} & :=x_{1} y_{1}+\ldots+x_{n} y_{n} \\
\|x\|_{n} & :=\sqrt{\langle x, x\rangle_{n}} \\
d(x, y) & :=\|x-y\|_{n}
\end{aligned}
$$

for all $x, y \in \mathbb{R}^{n}$. The metric space $\left(\mathbb{R}^{n}, d\right)$, or simply $\mathbb{E}^{n}$, is called the Euclidean space. The notions plane, hyperplane, etc. in the space $\mathbb{E}^{n}$ are defined in the well-known way.
From the viewpoint of differential geometry the space $\mathbb{E}^{n}$ can be viewed as a complete, connected and simply-connected Riemannian manifolds with constant sectional curvature $K=0$.

### 1.2 The Spherical Space $\mathbb{S}^{n}$

We define

$$
\begin{aligned}
X_{1}^{n} & :=S^{n}(0,1) \\
& =\left\{x \in \mathbb{R}^{n+1}:\langle x, x\rangle_{n+1}=1\right\}
\end{aligned}
$$

as the set of all points in $\mathbb{R}^{n+1}$ with distance 1 from the origin. Now we can define an intrinsic metric on $X_{1}^{n}$ by

$$
\begin{aligned}
d_{S}: X_{1}^{n} \times X_{1}^{n} & \longrightarrow \mathbb{R} \\
(x, y) & \longmapsto \theta(x, y),
\end{aligned}
$$

where $\theta(x, y)$ is the Euclidean angle between $x$ and $y$, defined in the well-known way by

$$
\cos d_{S}(x, y)=\langle x, y\rangle_{n+1}
$$

The metric space $\left(X_{1}^{n}, d_{S}\right)$, or simply $\mathbb{S}^{n}$, is called the spherical space.
From the viewpoint of differential geometry the space $\mathbb{S}^{n}$ can be viewed as a complete, connected and simply-connected Riemannian manifolds with constant sectional curvature $K=1$.

### 1.3 The Vector Space Model of $\mathbb{H}^{n}$

The Lorentz-Minkowski Space Denote the coordinates in the space $\mathbb{R}^{n+1}$ by $x_{0}, x_{1}, \ldots, x_{n}$. We introduce another bilinear form and a pseudonorm on $\mathbb{R}^{n+1}$ by the formulae

$$
\begin{aligned}
\langle x, y\rangle_{1, n} & :=-x_{0} y_{0}+x_{1} y_{1}+\ldots+x_{n} y_{n} \\
\|x\|_{1, n} & :=\sqrt{\langle x, x\rangle_{1, n}} \in \mathbb{C}
\end{aligned}
$$

for all $x, y \in \mathbb{R}^{n+1}$. The $(n+1)$-dimensional vector space $\mathbb{R}^{n+1}$ with the bilinear form $\langle,\rangle_{1, n}$ is called the $(n+1)$-dimensional Lorentz-Minkowski space, denoted by $\mathbb{R}^{1, n}$. With reference to this pseudonorm, the space $\mathbb{R}^{1, n}$ decomposes into three parts:

1. The light cone is the set

$$
\begin{aligned}
C^{n} & :=\left\{x \in \mathbb{R}^{1, n}:\|x\|_{1, n}=0\right\} \\
& =\left\{x \in \mathbb{R}^{n+1}: x_{0}^{2}=x_{1}^{2}+\ldots+x_{n}^{2}\right\}
\end{aligned}
$$

and all vectors $x \in C^{n}$ are called light-like or parabolic.
2. The exterior of the light cone is the set

$$
\begin{aligned}
E C^{n} & :=\left\{x \in \mathbb{R}^{1, n}:\|x\|_{1, n} \in \mathbb{R} \text { and }\|x\|_{1, n}>0\right\} \\
& =\left\{x \in \mathbb{R}^{n+1}: x_{0}^{2}<x_{1}^{2}+\ldots+x_{n}^{2}\right\}
\end{aligned}
$$

and all vectors $x \in A C^{n}$ are called space-like or elliptic.
3. The interior of the light cone is the set

$$
\begin{aligned}
I C^{n} & :=\left\{x \in \mathbb{R}^{1, n}:\|x\|_{1, n} \text { positive imaginary }\right\} \\
& =\left\{x \in \mathbb{R}^{n+1}: x_{0}^{2}>x_{1}^{2}+\ldots+x_{n}^{2}\right\}
\end{aligned}
$$

and all vectors $x \in I C^{n}$ are called time-like or hyperbolic. A time-like vector $x \in I C^{n}$ is called positive (resp. negative), if $x_{0}>0$ (resp. $x_{0}<0$ ).
The connected component of $I C^{n}$ consisting of all positive (resp. negative) vectors is denoted by $I C_{+}^{n}$ (resp. $I C_{-}^{n}$ ).

A vector subspace $V$ of $\mathbb{R}^{1, n}$ is called time-like if $V$ has a time-like vector, space-like if every nonzero vector in $V$ is space-like and light-like otherwise.
If $x$ and $y$ are two positive time-like vektors in $\mathbb{R}^{1, n}$ then we have

$$
\langle x, y\rangle_{1, n} \leq \underbrace{\|x\|_{1, n}}_{\text {neg. real }} \cdot \underbrace{\|y\|_{1, n}}_{\text {pos. im. }}
$$

with equality if and only if $x$ and $y$ are linearly dependent. Hence, there is an uniqu real number $\eta(x, y)$, the space-like Lorentzian angle between $x$ and $y$, such that

$$
\langle x, y\rangle_{1, n}=\|x\|_{1, n} \cdot\|y\|_{1, n} \cdot \cosh \eta(x, y) .
$$

The Hyperbolic Space in the Vector Space Model $\mathbb{H}^{n}$ We define

$$
\begin{aligned}
X_{-1}^{n} & :=\left\{x \in \mathbb{R}^{n+1}:\langle x \cdot x\rangle_{1, n}=-1\right\} \\
H^{n} & :=\left\{x \in X_{-1}^{n}: x_{0}>0\right\}
\end{aligned}
$$

Now we can understand the hyperbolic space in the vector-space model as the (metric) space $\left(H^{n}, d_{H}\right)$, or simply $\mathbb{H}^{n}$, with

$$
\begin{aligned}
d_{H}: H^{n} \times H^{n} & \longrightarrow \mathbb{R} \\
(x, y) & \longmapsto \eta(x, y) .
\end{aligned}
$$

Indeed, $d_{H}$ is a metric on the set $H^{n}$ and for all $x, y \in H^{n}$ we have the identity

$$
\cosh d_{H}(x, y)=-\langle x, y\rangle_{1, n}
$$

From the viewpoint of differential geometry the space $\mathbb{H}^{n}$ can be viewed as a complete, connected and simply-connected Riemannian manifolds with constant sectional curvature $K=-1$.

### 1.4 Planes in $\mathbb{S}^{n}$ and $\mathbb{H}^{n}$

To simplify the notations we write

$$
\langle,\rangle_{\mathbb{X}^{n}}=\left\{\begin{array}{lll}
\langle,\rangle_{n+1} & , & \mathbb{X}^{n}=\mathbb{S}^{n} \\
\langle,\rangle_{n} & , & \mathbb{X}^{n}=\mathbb{E}^{n} \\
\langle,\rangle_{1, n} & , & \mathbb{X}^{n}=\mathbb{H}^{n}
\end{array}\right.
$$

Let $\mathbb{X}^{n}=\mathbb{S}^{n}$ or $\mathbb{H}^{n}$. A $k$-dimensional plane $F$ in $\mathbb{X}^{n}$ is a non-empty intersection of $\mathbb{X}^{n}$ with a $(k+1)$-dimensional vector subspace $U_{F}$ of $\mathbb{R}^{n+1}$, called the defining subspace of the plane $F$. An $(n-1)$-dimensional plane in $\mathbb{X}^{n}$ is called a hyperplane. Every hyperplane $H$ in $\mathbb{X}^{n}$ is determined by an $n$-dimensional vector subspace $U_{H}$ in $\mathbb{R}^{n+1}$ and divides the whole space $\mathbb{X}^{n}$ into two closed half-spaces, denoted by $H^{+}$and $H^{-}$. In particular, the hyperplane $H$ and the closed half-spaces $H^{+}$and $H^{-}$are given by

$$
\begin{aligned}
H & =\left\{x \in \mathbb{R}^{n+1}:\langle x, e\rangle_{\mathbb{X}^{n}}=0\right\} \cap \mathbb{X}^{n} \\
H^{+} & =\left\{x \in \mathbb{R}^{n+1}:\langle x, e\rangle_{\mathbb{X}^{n}} \geq 0\right\} \cap \mathbb{X}^{n} \\
H^{-} & =\left\{x \in \mathbb{R}^{n+1}:\langle x, e\rangle_{\mathbb{X}^{n}} \leq 0\right\} \cap \mathbb{X}^{n},
\end{aligned}
$$

where $e$ is a unit normal vector of the defining subspace $U_{H}$ of $H$. For any set $K \subset \mathbb{X}^{n}$ we denote by $\langle K\rangle$ the intersection of all planes in $\mathbb{X}^{n}$ which contain the set $K$. Of course, $\left.<K\right\rangle$ itself is a plane.

### 1.5 Other Models of $\mathbb{H}^{n}$

The Ball Model We denote by $\zeta$ the stereographic projection from the point $-e_{0}$, which maps the $n$-dimensional unit ball $B^{n}:=\left\{x \in \mathbb{R}^{n+1}: \|\left. x\right|_{n+1}<1\right.$ and $\left.x_{0}=0\right\}$ (embedded in the space $\mathbb{R}^{n+1}$ ), bijectively on the set $H^{n}$ by:

$$
\begin{aligned}
\eta: B^{n} & \longrightarrow H^{n} \\
\left(0, x_{1}, \ldots, x_{n}\right) & \longmapsto \frac{1}{1-\|x\|_{n+1}^{2}}\left(1+\|x\|_{n+1}^{2}, 2 x_{1}, \ldots, 2 x_{n}\right) .
\end{aligned}
$$

The inverse of $\eta$ is the map:

$$
\begin{aligned}
\eta^{-1}: H^{n} & \longrightarrow B^{n} \\
\left(y_{0}, y_{1}, \ldots, y_{n}\right) & \longmapsto \frac{1}{1+y_{0}}\left(y_{1}, \ldots, y_{n}\right) .
\end{aligned}
$$

We can define the so-called Poincaré metric on $B^{n}$ by

$$
d_{B}(x, y):=d_{H}(\eta(x), \eta(y))
$$

for all $x, y \in B^{n}$. The metric space ( $\underline{B}^{n}, d_{B}$ ), or simply $\mathbb{B}^{n}$, is called the (conformal) ball model of the hyperbolic space. The closure $\overline{\mathbb{B}^{n}}$ of $\mathbb{B}^{n}$ is the natural compactification of $\mathbb{B}^{n}$ and points in $\partial \mathbb{B}^{n}:=\mathbb{B}^{n}-\overline{\mathbb{B}^{n}}$ are called points at infinity. Of course, $\partial \mathbb{B}^{n}$ is homeomorphic to $S^{n-1}(0,1)$. Furthermore, for the metric $d_{B}$ we have the following result (see $[\mathrm{R}]$ Theorem 4.5.1.).

Theorem 1.5.1 The metric $d_{B}$ is given by

$$
\cosh d_{B}(x, y)=1+\frac{2\|x-y\|_{n}^{2}}{\left(1-\|x\|_{n}^{2}\right)\left(1-\|y\|_{n}^{2}\right)}
$$

The Projective Disk Model In this case we denote the $n$-dimensional unit ball by $D^{n}:=$ $\left\{x \in \mathbb{R}^{n+1}:\|x\|_{n+1}<1\right.$ and $\left.x_{0}=0\right\}$ (embedded in the space $\mathbb{R}^{n+1}$ ). The gnonomic projection $\mu$ maps the set $D^{n}$ bijectively on the set $H^{n}$ :

$$
\begin{aligned}
\mu: D^{n} & \longrightarrow H^{n} \\
x & \longmapsto \frac{x+e_{0}}{\left|\left|\left|x+e_{0} \|_{1, n}\right|\right.\right.}
\end{aligned}
$$

The inverse of $\mu$ is the map:

$$
\begin{aligned}
\mu^{-1}: H^{n} & \longrightarrow D^{n} \\
\left(x_{0}, x_{1}, \ldots, x_{n}\right) & \longmapsto \frac{1}{x_{0}}\left(x_{1}, \ldots, x_{n}\right) .
\end{aligned}
$$

The map $\mu$ is a decomposition of a vertical translation of $D^{n}$ by $e_{0}$ and a radial projection with center 0 . We can define a metric on $D^{n}$ by

$$
d_{D}(x, y):=d_{H}(\mu(x), \mu(y))
$$

for all $x, y \in D^{n}$. The metric space $\left(D^{n}, d_{D}\right)$, or simply $\mathbb{I}^{n}$, is called the projective disk model of the hyperbolic space. The closure of $\mathbb{D}^{n}$ is denoted by $\overline{\mathbb{D}^{n}}$ and $\partial \mathbb{D}^{n}:=\mathbb{D}^{n}-\overline{\mathbb{D}^{n}}$ is homeomorphic to $S^{n-1}(0,1)$.

The Upper Half-Space Model We denote by $U^{n}$ the $n$-dimensional upper half-space $U^{n}:=$ $\left\{x \in \mathbb{R}^{n}: x_{n}>0\right\}$. Let $\zeta$ be the standard transformation from the upper half-space $U^{n}$ to the unit ball $B^{n}$. This means that $\zeta=\sigma \rho$ where $\rho$ is the reflection of $\hat{\mathbb{R}}^{n}$ in the hyperplane $\mathbb{R}^{n-1}$ and $\sigma$ is the reflection of $\mathbb{R}^{n}$ in the sphere $S\left(e_{n}, \sqrt{2}\right)$ :

$$
\begin{aligned}
\rho: U^{n} & \longrightarrow-U^{n} \\
\left(x_{1}, x_{2}, \ldots, x_{n}\right) & \longmapsto\left(-x_{1}, x_{2}, \ldots, x_{n}\right) \\
\sigma:-U^{n} & \longrightarrow B^{n} \\
x & \longmapsto e_{1}+\frac{2}{\left\|x-e_{1}\right\|_{n}^{2}}\left(x-e_{1}\right) .
\end{aligned}
$$

As usual, we can define a metric on $U^{n}$ by

$$
d_{U}(x, y):=d_{B}(\zeta(x), \zeta(y))
$$

for all $x, y \in U^{n}$. The metric space $\left(U^{n}, d_{U}\right)$, or simply $I U^{n}$, is called the upper half-space model of the hyperbolic space. The closure of $I U^{n}$ is denoted by $\overline{U^{n}}$ and $\partial U^{n}:=I U^{n}-\overline{U^{n}}$ is homeomorphic to $\mathbb{R}^{n-1} \cup\{\infty\}$. Furthermore, for the metric $d_{U}$ we have the following result (see [R] Theorem 4.6.1.).

Theorem 1.5.2 The metric $d_{U}$ is given by

$$
\cosh d_{U}(x, y)=1+\frac{\|x-y\|_{n}^{2}}{2 x_{0} y_{0}}
$$

## 2 The Groups of Isometries

In this section we will study the groups of isometries of the three spaces $\mathbb{E}^{n}, \mathbb{S}^{n}$ and $\mathbb{H}^{n}$. Furthermore, we will describe several representations of the group of isometries of the hyperbolic space in the different models by so-called Möbius transformations.

### 2.1 The Group of Isometries of $\mathbb{E}^{n}$

Let $O(n)$ be the group of orthogonal transformations and $T_{n}$ the group of translations in $\mathbb{R}^{n}$. Then the group of isometries $\operatorname{Iso}\left(\mathbb{E}^{n}\right)$ of the space $\mathbb{E}^{n}$ is the semidirect product

$$
\operatorname{Iso}\left(\mathbb{E}^{n}\right)=T_{n} \lambda O(n) .
$$

### 2.2 The Group of Isometries of $\mathbb{S}^{n}$

The group $O(n+1)$ of all orthogonal transformations of $\mathbb{R}^{n+1}$ maps the sphere $X_{1}^{n}$ bijectively on itself. Then the group of isometries Iso $\left(\mathbb{S}^{n}\right)$ of the space $\mathbb{S}^{n}$ is the restriction of the action of $O(n+1)$ on the sphere

$$
\operatorname{Iso}\left(\mathbb{S}^{n}\right)=\left.O(n+1)\right|_{\mathbb{S}^{n}}
$$

### 2.3 The Group of Isometries of $I H^{n}$

Let $O(1, n)$ be the group of all linear transformations in $\mathbb{R}^{n+1}$ which preserve the bilinear form $\langle,\rangle_{1, n}$. Of course, this group maps the set $X_{-1}^{n}$ bijectively on itself but the elements may exchange the two connected components. Let $O(1, n)^{\prime}$ be the subgroup of $O(1, n)$ (of index 2) which maps $H^{n}\left(\subset X_{-1}^{n}\right)$ bijectively on itself. Then the group of isometries Iso $\left(H^{n}\right)$ of the space $\mathbb{H}^{n}$ is

$$
\operatorname{Iso}\left(\mathbb{H}^{n}\right)=\left.O(1, n)^{\prime}\right|_{\mathbb{H}^{n}}
$$

### 2.4 The Group of Isometries of $I U^{n}$

Clearly, we can define the groups of isometries in the several models of the hyperbolic space by conjugation with the isometric maps $\mu, \eta$ or $\zeta$, respectively. Another direct way to define the groups of isometries for the upper half-space model and the ball model is the use of the so-called Möbius transformations. Since we only make use of the upper half-space model in this thesis we will explain it in this case. For more details see [R].

Möbius Transformations of $\hat{\mathbb{R}}^{n}$ Let $\mathbb{R}^{n}$ be the Euclidean space with the canonical bilinear form (and induced metric) and $\hat{\mathbb{R}}^{n}=\mathbb{R}^{n} \cup\{\infty\}$ the one-point compactification of $\mathbb{R}^{n}$.
A sphere $\sum$ in $\hat{\mathbb{R}}^{n}$ is defined as either a Euclidean sphere

$$
S(a, r)=\left\{x \in \mathbb{R}^{n}:\|x-a\|_{n}=r\right\}
$$

or an extended hyperplane

$$
\begin{aligned}
\hat{P}(a, t) & =P(a, t) \cup\{\infty\} \\
& =\left\{x \in \mathbb{R}^{n}:\langle x, a\rangle_{n}=t\right\} \cup\{\infty\} .
\end{aligned}
$$

A reflection $\rho=\rho_{a, t}$ in an extended hyperplane $\hat{P}(a, t)$ is defined as

$$
\rho(x)=\left\{\begin{array}{lll}
\infty & , \quad x=\infty \\
x+2\left(t-\langle x, a\rangle_{n}\right) a & , \quad \text { otherwise }
\end{array} .\right.
$$

A reflection (or inversion) $\sigma=\sigma_{a, r}$ in a sphere $S(a, r)$ is defined as

$$
\sigma(x)=\left\{\begin{array}{lll}
\infty & , \quad x=a \\
a & , \quad x=\infty \\
a+\frac{r^{2}}{\|x-a\|_{n}^{2}}(x-a) & , \quad \text { otherwise }
\end{array}\right.
$$

Definition 2.4.1 The composition of finitely many reflections in extended hyperplanes and spheres is called a Möbius transformation.

Reflections in hyperplanes as well as reflections in spheres are conformal maps, which means that they preserve angles. So every Möbius transformation is a conformal map.
Möbius transformations form a topological group, which is called the general Möbius group $G M(n)$, and we denote the subgroup of orientation preserving elements in $G M(n)$ by $G M^{+}(n)$. Of course, $\operatorname{Iso}\left(\mathbb{E}^{n}\right)$ and $\operatorname{Sim}\left(\mathbb{E}^{n}\right)$ are subgroups of $G M(n)$ (each element is a decomposition of finitely many reflections in hyperplanes).
Let $u, v, x, y$ be points in $\hat{\mathbb{R}}^{n}$ with $u \neq v$ and $x \neq y$. The cross ratio $[u, v, x, y]$ is defined as

$$
[u, v, x, y]=\frac{d_{c}(u, x) d_{c}(v, y)}{d_{c}(u, v) d_{c}(x, y)}
$$

where $d_{c}$ denotes the chordal metric on $\hat{\mathbb{R}}^{n}$. A map $\phi: \hat{\mathbb{R}}^{n} \rightarrow \hat{\mathbb{R}}^{n}$ is a Möbius transformation if and only if it preserves cross ratios.
If $\phi$ is in $G M(n)$ with $\phi(\infty)=\infty$, then it can be written as

$$
\phi(x)=b+k A x
$$

with $b \in \mathbb{R}^{n}, k>0$ and $A \in O(n)$, which means that $\phi$ is a Euclidean similarity.
Let $\phi \in G M(n)$ with $\phi(\infty) \neq \infty$ and $a=\phi^{-1}(\infty)$. Then the composition $\phi \sigma_{a, r}$ of $\phi$ with the reflection $\sigma_{a, r}$ in the sphere $S(a, r)$ satisfies the equality $\phi \sigma_{a, r}(\infty)=\infty$ and so we get

$$
\phi(x)=b+k A \sigma_{a, r}(x)
$$

with $b \in \mathbb{R}^{n}, k>0$ and $A \in O(n)$. Furthermore, $\phi$ acts on $S(a, r \sqrt{k})$ as a Euclidean isometry:

$$
\begin{aligned}
\|\phi(x)-\phi(y)\|_{n} & =\frac{k r^{2}\|x-y\|_{n}}{\|x-a\|_{n}\|y-a\|_{n}} \\
& =\|x-y\|_{n}
\end{aligned}
$$

for all $x, y \in S(a, r \sqrt{k})$. This sphere is uniquely determined by this property and it is called the isometric sphere of $\phi$. We get

$$
\phi(x)=b+A \sigma_{a, r \sqrt{k}}(x)
$$

Let $\phi \in G M(n)$ be a Möbius transformation. The map $\phi$ is a composition of finitely many reflections $\rho(a, t)$ in extended hyperplanes $\hat{P}(a, t)$ and reflections $\sigma_{b, r}$ in spheres $S(b, r)$. It is
possible to extend $\phi$ to an element $\tilde{\phi} \in G M(n+1)$ in a canonical way and it is enough to define the extension of the following two types of transformations: We embed $\hat{\mathbb{R}}^{n}$ into $\hat{\mathbb{R}}^{n+1}$ by $x \mapsto \tilde{x}=(x, 0)$ and define $\tilde{\rho}=\tilde{\rho}_{\tilde{a}, t}$ as the reflection of $\hat{\mathbb{R}}^{n+1}$ in the extended hyperplane $\hat{P}(\tilde{a}, t)$ and $\tilde{\sigma}=\tilde{\sigma}_{\tilde{b}, r}$ as the reflection of $\hat{\mathbb{R}}^{n+1}$ in the sphere $S(\tilde{b}, r)$.

Definition 2.4.2 Let $\phi \in G M(n)$ be a Möbius transformation. Then the so-called Poincaré extension $\tilde{\phi} \in G M(n+1)$ is defined as the composition of the canonical extensions of the reflections in $\phi$.

## Möbius Transformations of $I U^{n}$

Definition 2.4.3 A Möbius transformation of the upper half-space $I U^{n}$ is a Möbius transformation of $\hat{\mathbb{R}}^{n}$, that leaves $\mathbb{U}^{n}$ invariant.

Let $G M\left(U^{n}\right)$ be the set of all Möbius transformations of the upper half-space $I U^{n}$. Of course, $G M\left(U^{n}\right)$ is a subgroup of $G M(n)$. Furthermore, $G M\left(U^{n}\right)$ is isomorphic to $G M(n-1)$ (by Poincaré extension) and every Möbius transformation of $I U^{n}$ is a composition of reflections of $\hat{\mathbb{R}}^{n}$ in spheres orthogonal to $\mathbb{R}^{n-1}$.

Let $\bar{I}^{n}$ be the closed upper half-space. We see that each element in $G M\left(U^{n}\right)$ leaves the closed upper half-space $\bar{I}^{n}$ invariant and so has a fixed point in $\bar{I}^{n}$ by Brouwer's Fixed Point Theorem. Now we will classify the elements of $G M\left(I U^{n}\right)$ by their fixed points.

Definition 2.4.4 Let $\phi$ be an element in $G M\left(I U^{n}\right)$. Then $\phi$ is said to be

1. elliptic if $\phi$ fixes a point of $I U^{n}$,
2. parabolic if $\phi$ fixes no point of $\mathbb{U}^{n}$ and fixes a unique point in $\hat{\mathbb{R}}^{n-1}$ or
3. hyperbolic if $\phi$ fixes no point of $I U^{n}$ and fixes two points of $\hat{\mathbb{R}}^{n-1}$.

Of course, these properties depend only on the conjugacy class of $\phi$ in $G M\left(U^{n}\right)$. Now we have the following important results (see $[\mathrm{R}]$ Theorems 4.7.1, 4.7.2 and 4.7.4).

Theorem 2.4.1 Let $\phi$ be an element in $G M\left(I U^{n}\right)$.

1. $\phi$ is elliptic if and only if $\phi$ is conjugated in $G M\left(U^{n}\right)$ to an orthogonal transformation of $\mathbb{E}^{n}$.
2. $\phi$ is parabolic if and only if $\phi$ is conjugated in $G M\left(I U^{n}\right)$ to the Poincaré extension of a fixed point free isometry $\psi$ of $\mathbb{E}^{n-1}$ of the form $\psi(x)=b+A x$ with $b \neq 0$ and $A \in O(n-1)$.
3. $\phi$ is hyperbolic if and only if $\phi$ is conjugated in $G M\left(I U^{n}\right)$ to the Poincaré extension of a similarity $\psi$ of $\mathbb{E}^{n-1}$ of the form $\psi(x)=k A x$ with $k>1$ and $A \in O(n-1)$.

Elementary Groups Let $G$ be a subgroup of $G M\left(I U^{n}\right)$. Then $G$ is called elementary if $G$ has a finite orbit in the closed upper half-space $\bar{I}^{n}$. We divide the elementary subgroups of $G M\left(I U^{n}\right)$ into three different types.

Definition 2.4.5 Let $G$ be an elementary subgroup of $G M\left(I U^{n}\right)$. Then $G$ is said to be of

1. elliptic type if $G$ has a finite orbit in $I U^{n}$,
2. parabolic type if $G$ fixes a point in $\hat{\mathbb{R}}^{n-1}$ and has no other finite orbit in $\overline{\mathbb{U}}^{n}$ or
3. hyperbolic type if $G$ is neither of elliptic nor of parabolic type.

Now we have the following important results (see [R], Theorems 5.5.2, 5.5.5 and 5.5.7).

Theorem 2.4.2 Let $G$ be an elementary discrete subgroup of $G M\left(I U^{n}\right)$. Then $G$ is of

1. elliptic type if and only if $G$ is conjugated in $G M\left(I U^{n}\right)$ to a finite subgroup of $O(n)$.
2. parabolic type if and only if $G$ is conjugated in $G M\left(I U^{n}\right)$ to an infinite discrete subgroup of $\operatorname{Iso}\left(\mathbb{E}^{n}\right)$.
3. hyperbolic type if and only if $G$ is conjugated in $G M\left(U^{n}\right)$ to an infinite discrete subgroup of $\operatorname{Sim}\left(\mathbb{E}^{n}\right)$, that leaves the set $\{0, \infty\}$ invariant.

We have the following result (see [R], Paragraph 12.1, Lemma 2).

Lemma 2.4.1 If $G$ is a discrete subgroup of $G M\left(I U^{n}\right)$ all of whose elements are elliptic, then $G$ is elementary of elliptic type and so conjugated to a finite subgroup of $O(n)$.

The Group of Isometries of $I U^{n} \quad$ Now we can describe the group of isometries of the (metric) upper half-space $I U^{n}$ in terms of Möbius transformations of $I U^{n}$. We have the following result (see [R], Theorem 4.6.2.).

Theorem 2.4.3 Every element in $G M\left(I U^{n}\right)$ restricts to an isometry of the upper half-space $I U^{n}$ and every isometry of $\mathbb{U}^{n}$ extends to a unique Möbius transformation of $I U^{n}$.

So the groups $\operatorname{Iso}\left(\mathbb{H}^{n}\right)$ and $G M\left(I^{n}\right)$ are isomorphic and the two spaces $\left(\mathbb{H}^{n}, \operatorname{Iso}\left(\mathbb{H}^{n}\right)\right)$ and $\left(I U^{n}, G M\left(I U^{n}\right)\right)$ can be viewed as isomorphic homogeneous spaces.

### 2.5 Clifford Matrices and Möbius Transformations

There is another way to describe hyperbolic isometries in the upper half-space model. We do this by using Clifford matrices, which are strongly related to the Möbius transformations. In the following we record results of L. Ahlfors [A] and P.L. Waterman [W].

The Clifford Algebra Let $C_{n}$ be the Clifford algebra which is the real associative algebra, generated by $n$ elements $i_{1}, i_{2}, \ldots, i_{n}$ subjected to the relations

$$
\begin{aligned}
i_{k} i_{l} & =-i_{l} i_{k} \\
i_{k}^{2} & =-1,
\end{aligned}
$$

for all $k \neq l$. Furthermore, let $i_{0}=1$. Every element $a$ in $C_{n}$ can be expressed uniquely in the form

$$
a=\sum a_{I} I
$$

where $I=i_{v_{1}} i_{v_{2}} \cdots i_{v_{k}}$ with $1 \leq v_{1}<v_{2}<\ldots<v_{k} \leq n$ and $a_{I} \in \mathbb{R}$. We define the Euclidean norm $|a|$ of $a \in C_{n}$ by $|a|=\sqrt{\sum a_{I}^{2}}$. As with complex numbers we distinguish the real and the purely imaginary parts $a=a_{\mathbb{R}}+a_{C}$ with $a_{\mathbb{R}}=a_{0}$. An element $a$ in $C_{n}$ is called pure if $a_{\mathbb{R}}=0$. We define

$$
\mathbb{S}^{2^{n}-1}=\left\{a \in C_{n}:|a|=1\right\}
$$

the unit sphere of $C_{n}$. Let $C_{n}^{*}$ be the group of units in $C_{n}$ (this is the set of invertible elements). There are three involutions of $C_{n}$ :

1. *: replaces each $I=i_{v_{1}} i_{v_{2}} \ldots i_{v_{k}}$ with $i_{v_{k}} i_{v_{k-1}} \ldots i_{v_{1}}$. This map determines an anti-automorphism of $C_{n}$ :

$$
\begin{aligned}
(a+b)^{*} & =a^{*}+b^{*} \\
(a b)^{*} & =b^{*} a^{*} .
\end{aligned}
$$

2. ' : replaces each $i_{k}$ with $-i_{k}$ for $k>0$. This map determines an automorphism of $C_{n}$ :

$$
\begin{aligned}
(a+b)^{\prime} & =a^{\prime}+b^{\prime} \\
(a b)^{\prime} & =a^{\prime} b^{\prime} .
\end{aligned}
$$

3. ${ }^{-}: \bar{a}=\left(a^{\prime}\right)^{*}=\left(a^{*}\right)^{\prime}$. This map determines an anti-automorphism of $C_{n}$ :

$$
\begin{aligned}
(\overline{a+b}) & =\bar{a}+\bar{b} \\
(\overline{a b}) & =\bar{b} \bar{a} .
\end{aligned}
$$

Clifford numbers of the form $x=x_{0}+x_{1} i_{1}+\ldots+x_{n} i_{n}$ are called vectors. Clearly, they form an $(n+1)$-dimensional subspace of the $2^{n}$-dimensional real vector space $C_{n}$, which we identify with $\mathbb{R}^{n+1}$. For vectors we have $x^{*}=x, \bar{x}=x^{\prime}$ and $x \bar{x}=\bar{x} x=|x|^{2}$. So nonzero vectors $x$ are invertible with $x^{-1}=\bar{x} /|x|^{2}$.

Thus products of nonzero vectors are also invertible and form a multiplicative group, which we call the Clifford group $\Gamma_{n}$.

Clifford Multiplication and Orthogonal Maps Now we have the following connection between orthogonal maps and multiplication with invertible Clifford numbers (see [W], Theorems 2 and 3).

Theorem 2.5.1 If $a \in C_{n}$ is invertible and axa ${ }^{\prime-1} \in \mathbb{R}^{n+1}$ for all $x \in \mathbb{R}^{n+1}$, then

$$
\begin{aligned}
\rho_{a}: \mathbb{R}^{n+1} & \longrightarrow \mathbb{R}^{n+1} \\
x & \longmapsto a^{\prime} a^{\prime-1}
\end{aligned}
$$

is orthogonal. Furthermore, if $a$ is in $\mathbb{R}^{n+1}-\{0\}$, then $\rho_{a}$ is the decomposition of the reflection $R_{1}$ in the perpendicular through 0 to 1 and the reflection $R_{a}$ in the perpendicular through 0 to a.

Theorem 2.5.2 The map

$$
\begin{aligned}
\phi: \Gamma_{n} & \longrightarrow O(n+1) \\
a & \longmapsto \rho_{a}
\end{aligned}
$$

is onto $S O(n+1)$ with kernel $\mathbb{R}-\{0\}$.

## Clifford Matrices

Definition 2.5.1 We define by

$$
\begin{aligned}
G L\left(2, C_{n}\right)= & \left\{T=\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right): a, b, c, d \in C_{n} \text { and } T\right. \text { induces a bijection from } \\
& \hat{R}^{n+2} \longrightarrow \hat{\mathbb{R}}^{n+2} \\
& \left.x \longmapsto(a x+b)(c x+d)^{-1}\right\}
\end{aligned}
$$

the general linear group of $2 \times 2$ matrices with entries in the Clifford algebra $C_{n}$. Furthermore, let $T$ be an element in $G L\left(2, C_{n}\right)$. Then the pseudo-determinant $\Delta$ of $T$ is defined by

$$
\Delta(T)=a d^{*}-b c^{*} .
$$

We can show (compare [W], Lemma 10) that any $T \in G L\left(2, C_{n}\right)$ is a decomposition of a:

$$
\left.\begin{array}{lll}
\text { translation } & \left(\begin{array}{cc}
1 & \mu \\
0 & 1
\end{array}\right) & x \mapsto x+\mu \\
\text { inversion } & \left(\begin{array}{cc}
0 & 1 \\
-1 & 0
\end{array}\right) & x \mapsto-x^{-1}=-\frac{\bar{x}}{|x|^{2}}
\end{array}\right]
$$

Since the Möbius group $G M(n+1)$ of the space $\hat{R}^{n+1}$ is generated by transformations of the above types we have the following result (see [W], Theorem 4).

Theorem 2.5.3 The group $G L\left(2, C_{n}\right)$ acts on $\hat{\mathbb{R}}^{n+1}$ by

$$
\begin{aligned}
T: \hat{\mathbb{R}}^{n+1} & \longrightarrow \hat{\mathbb{R}}^{n+1} \\
x & \longmapsto(a x+b)(c x+d)^{-1}
\end{aligned}
$$

as the group of Möbius transformations with kernel $\{\lambda I: \lambda \in \mathbb{R}-\{0\}\}$.

Thus with

$$
\begin{aligned}
P G L\left(2, C_{n}\right) & :=G L\left(2, C_{n}\right) /\{\lambda I: \lambda \in \mathbb{R}-\{0\}\} \\
S L\left(2, C_{n}\right) & :=\left\{T \in G L\left(2, C_{n}\right): \Delta(T)=1\right\} \\
P S L\left(2, C_{n}\right) & :=S L\left(2, C_{n}\right) /\{ \pm I\}
\end{aligned}
$$

we have the following result (see [W], Theorem 5).

Theorem 2.5.4 The group $P G L\left(2, C_{n}\right)$ is isomorphic to the full group of Möbius transformations and $P S L\left(2, C_{n}\right)$ to the group of orientation preserving Möbius transformations of $\hat{\mathbb{R}}^{n+1}$ :

$$
\begin{aligned}
P G L\left(2, C_{n}\right) & \cong G M(n+1) \\
P S L\left(2, C_{n}\right) & \cong G M^{+}(n+1)
\end{aligned}
$$

Furthermore, the group $S L\left(2, C_{n}\right)$ preserves hyperbolic $(n+2)$-space $\mathbb{U}^{n+2}=\left\{x \in \mathbb{R}^{n+2}\right.$ : $\left.x_{n+1}>0\right\}$ in the upper half-space model and the metric in $U^{n+2}$ (compare L. Ahlfors [A], section 2.5).

Theorem 2.5.5 The group $P S L\left(2, C_{n}\right)$ acts on the hyperbolic space $U^{n+2}$ as the group of orientation preserving isometries.

## 3 Polytopal Complexes

In this section we will define polytopes and polytopal complexes. For the combinatorial point of view the classical book $[\mathrm{AH}]$ and the modern book $[\mathrm{Z}]$ are good references. For the geometrical point of view see $[\mathrm{R}]$ or [V2] and for more information about algebraic topology (CW-complexes, Euler-Poincaré Characteristic) see [Ma].

### 3.1 Combinatorics

Polytopes and Polytopal Complexes Throughout this chapter let $\mathbb{X}^{n}$ be $\mathbb{S}^{n}, \mathbb{E}^{n}$ or $\mathbb{H}^{n}$, if not specified otherwise.
Let $U$ be a $\mathbb{X}^{n}$-convex set in $\mathbb{X}^{n}$ and $\langle U\rangle$ the intersection of all hyperplanes in $\mathbb{X}^{n}$, containing the set $U$. Then the relative interior of $U$, denoted by $r i(U)$, is the interior of $U$ in the plane $\langle U\rangle$. The closure of the set $U$ is denoted by $\operatorname{cl}(U)$ and the set $r b(U):=c l(U)-r i(U)$ is called relative boundary of $U$.

Definition 3.1.1 An n-dimensional (convex and generalized) polytope $P$ in $\mathbb{X}^{n}$ is

- the $\mathbb{S}^{n}$-convex hull of finitely many points for $\mathbb{X}^{n}=\mathbb{S}^{n}$ which are contained in an open hemisphere,
- the $\mathbb{E}^{n}$-convex hull of finitely many points for $\mathbb{X}^{n}=\mathbb{E}^{n}$ and
- the $\mathbb{H}^{n}$-convex hull of finitely many ordinary points and points at infinity for $\mathbb{X}^{n}=\mathbb{H}^{n}$
which containes an open set of $\mathbb{X}^{n}$.

Then $P$ may not be compact but it is always of finite volume. It turns out that $P$ is the intersection of finitely many half-spaces (see [Z], Theorem 1.1)

$$
P=\bigcap_{i \in I} H_{i}^{-}
$$

An $n$-dimensional simplex $T$ in $\mathbb{X}^{n}$ is an $n$-dimensional polytope $\mathbb{X}^{n}$, which is the convex hull of $(n+1)$ points. A supporting hyperplane $H$ of $P$ is a hyperplane in $\mathbb{X}^{n}$ such that $P$ is contained in $H^{-}$or $H^{+}$and $P \cap H \neq \emptyset$. The intersection of a supporting hyperplane $H$ of $P$ with $P$ is called a face of $P$. The dimension of a face is the minimal dimension of a plane in $\mathbb{X}^{n}$, which contains this face.

Definition 3.1.2 A generalized polytopal complex $\Pi$ in $\mathbb{X}^{n}$ is a set of polytopes in $\mathbb{X}^{n}$ such that

1. if a polytope belongs to $\Pi$ then so do all its faces,
2. the intersection of two polytopes in $\Pi$ is a face of both polytopes and
3. the complex $\Pi$ is locally finite.

A polytopal complex $\Pi$ in $\mathbb{X}^{n}$ is a generalized polytopal complex $\Pi$ in $\mathbb{X}^{n}$, consisting of finitely many polytopes.

A simplicial complex in $\mathbb{X}^{n}$ is a polytopal complex in $\mathbb{X}^{n}$ which consists only of simplices. Further $|\Pi|$ denotes the underlying topological space of $\Pi$. The dimension of $\Pi$ is the maximal dimension of an element in $\Pi$. The complex $\Pi$ is called pure if all elements of $\Pi$ are included in an element of the dimension of $\Pi$.

The boundary $\partial \Pi$ of a pure $n$-complex is the pure ( $n-1$ )-complex consisting of all $(n-1)$ dimensional faces of $\Pi$ (and their faces), which are not contained in exactly two $n$-dimensional faces of $\Pi$.

Let $\Pi$ be a pure polytopal complex of dimension $n$ in $\mathbb{X}^{n}$ with more than one maximally dimensional element and $0 \leq k<n$. Then $\Pi$ is called $k$-connected if for each pair $P, Q$ in $\Pi$ of maximal dimension $n$ there is a finite sequence $P_{1}, \ldots, P_{m}$ in $\Pi$ with

1. $\operatorname{dim}\left(P_{i}\right)=n$ for each $i=1, \ldots, m$;
2. $P_{1}=P, P_{m}=Q$ and
3. $P_{i-1}$ and $P_{i}$ share a common side of dimension $=k$ for each $i>0$.

If $\Pi$ is $(n-1)$-connected we call $\Pi$ connected.
Let $\Pi$ be a polytopal complex in $\mathbb{X}^{n}$. For all $d$ with $0 \leq d \leq n$ we denote by $\mathcal{G}^{d}(\Pi)$ the $d$-skeleton of $\Pi$, which is the polytopal complex consisting of all elements in $\Pi$ of dimension $\leq d$ :

$$
\mathcal{G}^{d}(\Pi):=\{P \in \Pi: \operatorname{dim}(P) \leq d\} .
$$

Let $\Pi$ be a polytopal complex in $\mathbb{X}^{n}$. A polytopal decomposition $\mathcal{D}=\mathcal{D}(\Pi)$ is a polytopal complex, such that $|\mathcal{D}|=\Pi$ and each element of $\mathcal{D}$ is contained in an element of $\Pi$. If $\Pi$ is pure (for instance if $\Pi$ is a polytope) then $\mathcal{D}$ is pure and it includes only finitely many decomposition polytopes of each dimension.
One of the simplest and most symmetrical decompositions is the barycentric decomposition $\mathcal{B}(\Pi)$ of a polytopal complex $\Pi$.

Example 3.1.1 Let $P$ be the 2-dimensional polytope in Figure 1. The polytope $P$ is a pure polytopal complex with four 0-dimensional elements $P_{1}^{0}, P_{2}^{0}, P_{3}^{0}, P_{4}^{0}$; four 1-dimensional elements $P_{1}^{1}=\operatorname{conv}\left(P_{1}^{0}, P_{2}^{0}\right), P_{2}^{1}=\operatorname{conv}\left(P_{2}^{0}, P_{3}^{0}\right), P_{3}^{1}=\operatorname{conv}\left(P_{3}^{0}, P_{4}^{0}\right), P_{4}^{1}=\operatorname{conv}\left(P_{4}^{0}, P_{1}^{0}\right) ;$ and one maximally dimensional element $P_{1}^{2}=\operatorname{conv}\left(P_{1}^{0}, P_{2}^{0}, P_{3}^{0}, P_{4}^{0}\right)$.


Figure 1: A 2-dimensional Polytope

Example 3.1.2 Let $\Pi$ be the 2-dimensional polytopal complex in Figure 2. The complex consists of eight 0-dimensional elements $P_{1}^{0}, P_{2}^{0}, P_{3}^{0}, P_{4}^{0}, P_{5}^{0}, P_{6}^{0}, P_{7}^{0}, P_{8}^{0}$; ten 1-dimensional elements $P_{1}^{1}=\operatorname{conv}\left(P_{1}^{0}, P_{2}^{0}\right), P_{2}^{1}=\operatorname{conv}\left(P_{2}^{0}, P_{3}^{0}\right), P_{3}^{1}=\operatorname{conv}\left(P_{3}^{0}, P_{1}^{0}\right), P_{4}^{1}=\operatorname{conv}\left(P_{1}^{0}, P_{4}^{0}\right), P_{5}^{1}=$ $\operatorname{conv}\left(P_{4}^{0}, P_{5}^{0}\right), P_{6}^{1}=\operatorname{conv}\left(P_{5}^{0}, P_{1}^{0}\right), P_{7}^{1}=\operatorname{conv}\left(P_{1}^{0}, P_{6}^{0}\right), P_{8}^{1}=\operatorname{conv}\left(P_{6}^{0}, P_{7}^{0}\right), P_{9}^{1}=\operatorname{conv}\left(P_{7}^{0}, P_{8}^{0}\right)$, $P_{10}^{1}=\operatorname{conv}\left(P_{8}^{0}, P_{1}^{0}\right)$ and three maximally dimensional elements $P_{1}^{2}=\operatorname{conv}\left(P_{1}^{0}, P_{2}^{0}, P_{3}^{0}\right), P_{2}^{2}=$ $\operatorname{conv}\left(P_{1}^{0}, P_{4}^{0}, P_{5}^{0}\right)$ and $P_{3}^{2}=\operatorname{conv}\left(P_{1}^{0}, P_{6}^{0}, P_{7}^{0}, P_{8}^{0}\right)$.
Furthermore, each element in $\Pi$ is included in an element of the dimension of $\Pi$; so $\Pi$ is pure and 0-connected.


Figure 2: Polytopal Complex

For all $d$ with $1 \leq d \leq n$ let $a^{d}(\Pi)$ be the number of $d$-dimensional faces, $a_{\text {ord }}^{0}(\Pi)$ the number of ordinary 0 -dimensional faces and $a_{\text {inf }}^{0}(\Pi)$ the number of points at infinity, which are contained in $\Pi$. Clearly for $\Pi \subseteq \mathbb{S}^{n}$ we have $a_{\text {inf }}^{0}(\Pi)=0$. Futhermore let $a^{0}(\Pi):=a_{\text {ord }}^{0}(\Pi)+a_{\text {inf }}^{0}(\Pi)$.
Every polytope can be viewed as a polytopal complex. For example, we have the following well-known result (compare [G], chapter 4).

Lemma 3.1.1 Let $T$ be an n-dimensional simplex, $W$ an $n$-dimensional cube and $W^{*}$ an $n$ dimensional cross polytope (dual cube). Then we have

$$
a^{k}(T)=\binom{n+1}{k+1}, \quad a^{k}(W)=2^{n-k}\binom{n}{k}, \quad a^{k}\left(W^{*}\right)=2^{k+1}\binom{n}{k+1} ;
$$

for all $k$ with $0 \leq k \leq n$.
For all $d$ with $1 \leq d \leq n$ let

$$
\Omega^{d}(\Pi):=\left\{P_{1}^{d}, P_{2}^{d}, \ldots, P_{a^{d}(\Pi)}^{d}\right\}
$$

be the set of $d$-dimensional ordinary faces

$$
\Omega^{0}(\Pi):=\left\{P_{1}^{0}, P_{2}^{0}, \ldots, P_{a_{o r d}^{0}(\Pi)}^{0}\right\}
$$

the set of ordinary vertices and

$$
\Upsilon^{0}(\Pi):=\left\{p_{1}^{0}, p_{2}^{0}, \ldots, p_{a_{i n f}^{0}(\Pi)}^{0}\right\}
$$

the set of points at infinity in $\Pi$. Furthermore, we define by $\Omega(\Pi)$ the set of all ordinary faces of the complex $\Pi$.

If we consider geometrical properties of the polytopal complexes there are great differences between ordinary vertices and vertices at infinity. So the above notation indicates clearly the distinction of the two types of points. However, if we consider combinatorial properties there is no difference between these two types of vertices.

## The Face Poset

Definition 3.1.3 We denote by $\boldsymbol{\Pi}^{\mathbf{n}}$ (resp. $\mathbf{P}^{\mathbf{n}}$ ) the set of all pure $n$-dimensional polytopal complexes (resp. n-dimensional polytopes) in the spaces $\mathbb{S}^{n}, \mathbb{E}^{n}$ and $\mathbb{H}^{n}$.

Let $\Pi$ be an element of $\Pi^{n}$. The face poset $F(\Pi)$ of $\Pi$ is the set of all faces of $\Pi$, partially ordered by inclusion. Let $\Pi_{1}$ and $\Pi_{2}$ be elements in $\Pi^{\mathbf{n}}$. Then $\Pi_{1}$ and $\Pi_{2}$ are called combinatorially isomorphic, if the two face posets $\mathrm{F}\left(\Pi_{1}\right)$ and $\mathrm{F}\left(\Pi_{2}\right)$ are isomorphic (as partially ordered sets). Two elements $P_{1}$ and $P_{2}$ in $\mathbf{P}^{\mathbf{n}}$ are said to be combinatorially isomorphic, denoted by $P_{1} \sim P_{2}$, if they are combinatorially isomorphic as polytopal complexes. These are equivalence relations on the sets $\Pi^{\mathbf{n}}$ and $\mathbf{P}^{\mathbf{n}}$, respectively.

For all $P \in \mathbf{P}^{\mathbf{n}}$ we denote by $C l(P)$ the combinatorial equivalence class, to which the polytope $P$ belongs. The set of all equivalence classes is denoted by $\mathbf{P}_{\sim}^{\mathbf{n}}$ and $\pi: \mathbf{P}^{\mathbf{n}} \longrightarrow \mathbf{P}_{\sim}^{\mathbf{n}}$ is the quotient map.
Let $P=\bigcap_{i \in I} H_{i}^{-} \subseteq \mathbb{X}^{n}$ be an $n$-dimensional polytope. For all $d$-faces $P^{d}$ of $\Omega^{d}(P)$ with $0 \leq d \leq n$ we put

$$
I\left(P^{d}\right):=\left\{i \in I: P^{d} \subset H_{i}\right\}
$$

The family of all subsets of $I$ of this form can be partially ordered by inclusion. We denote this partially ordered set by $\mathcal{F}(P)$ and call it the complex of the polytope $P$. If $P$ is in $\mathbb{H}^{n}$, then the vertices at infinity of $P$ are of special interest. For all $p^{0} \in \Upsilon^{0}(P)$ let

$$
I\left(p^{0}\right):=\left\{i \in I: p^{0} \subset \bar{H}_{i}\right\}
$$

We denote by $\overline{\mathcal{F}}(P)$ the collection of all subsets of $I$ obtained by adding all subsets of the form $I\left(p^{0}\right)$, for $p^{0} \in \Upsilon^{0}(P)$, to $\mathcal{F}(P)$. Also $\overline{\mathcal{F}}(P)$ can be partially ordered by inclusion and it is called the extended complex of the polytope $P$. It is easy to see that $\overline{\mathcal{F}}(P)$ is anti-isomorphic to the face poset $\mathrm{F}(P)$ of $P$.

## Combinatorial Invariants for Polytopes

Definition 3.1.4 Let $J$ be a set. A map $j: \mathbf{P}^{\mathbf{n}} \longrightarrow J$ is called a combinatorial $n$-invariant if there exists a map $j_{*}: \mathbf{P}_{\sim}^{n} \longrightarrow J$ such that $j_{*} \circ \pi=j$. A combinatorial invariant $j$ is called complete if $j_{*}$ is injective.

The numbers $a^{i}(P)$ for $0 \leq i \leq n-1$ are of course incomplete combinatorial invariants (for $n>2$ ). The isomorphism class of the face poset $F(P)$ is, by definition, a complete combinatorial invariant.
Another complete combinatorial invariant arises by considering the incidence matrix $M(P)$ of a polytope $P \in \mathbf{P}^{\mathbf{n}}$, defined as follows. Let $P$ be an element in $\mathbf{P}^{\mathbf{n}}$,

$$
\begin{aligned}
\Omega^{0}(P) \cup \Upsilon^{0}(P) & =\left\{P_{1}^{0}, \ldots, P_{a^{0}(P)}^{0}\right\} \\
\Omega^{n-1}(P) & =\left\{P_{1}^{n-1}, \ldots, P_{a^{n-1}(P)}^{n-1}\right\}
\end{aligned}
$$

(we do not distinguish between ordinary vertices and vertices at infinity). The matrix

$$
M(P)=\left(m_{i j}\right)
$$

is defined by

$$
m_{i j}=\left\{\begin{array}{lll}
1 & , & P_{i}^{0} \text { is a face of } P_{j}^{n-1} \\
0 & , & \text { otherwise }
\end{array}\right.
$$

Then the map $j: \mathbf{P}^{\mathbf{n}} \longrightarrow$ Mat, $P \longmapsto M(P)$ from $\mathbf{P}^{\mathbf{n}}$ in the set of all matrices Mat, is a complete combinatorial invariant and so the structure of $F(P)$ can be deduced from the matrix $M(P)$.

### 3.2 The Euler-Poincaré Characteristic

Definition 3.2.1 Let $\Pi$ be a pure $n$-dimensional polytopal complex in $\mathbb{X}^{n}$. We define the combinatorial Euler-Poincaré characteristic $\chi_{c}(\Pi)$ and the geometrical Euler-Poincaré characteristic $\chi_{g}(\Pi)$ of $\Pi$ by

$$
\begin{aligned}
\chi_{c}(\Pi) & :=\sum_{v=0}^{n}(-1)^{v} a^{v}(\Pi) \\
\chi_{g}(\Pi) & :=\sum_{v=1}^{n}(-1)^{v} a^{v}(\Pi)+a_{o r d}^{0}(\Pi) .
\end{aligned}
$$

In the cases where there is no difference between $\chi_{c}$ and $\chi_{g}$ we will write simply $\chi$. Of course, for polytopal complexes without vertices at infinity we have $\chi(\Pi)=\chi_{c}(\Pi)=\chi_{g}(\Pi)$ and in general we have the relation $\chi_{c}(\Pi)=\chi_{g}(\Pi)+a_{i n f}^{0}(\Pi)$.
If $\Pi$ is a compact polytopal complex in $\mathbb{X}^{n}$ then the pair $(|\Pi|, \Pi)$ can be viewed as a finite CW-complex. For details see [Ma], section IV.2. Especially, the Euler-Poincaré characteristic $\chi(\Pi)$ is a topological invariant of the space $|\Pi|$ and so is independent of the decomposition of the topological space $|\Pi|$. Hence we have the following result (see [Ma], Theorem IV.3.6).

Theorem 3.2.1 Let $\Pi$ and $\Pi^{\prime}$ be $n$-dimensional pure compact polytopal complexes in $\mathbb{X}^{n}$ such that $|\Pi|$ is homeomorphic to $\left|\Pi^{\prime}\right|$. Then

$$
\chi(\Pi)=\chi\left(\Pi^{\prime}\right) .
$$

Furthermore, we can deduce the following lemma.

Lemma 3.2.1 Let $\Pi$ be an n-dimensional pure polytopal complex in $\mathbb{X}^{n}$ and $\Pi^{\prime}$ a polytopal decomposition of $\Pi$. Then

$$
\begin{aligned}
& \chi_{c}(\Pi)=\chi_{c}\left(\Pi^{\prime}\right) \\
& \chi_{g}(\Pi)=\chi_{g}\left(\Pi^{\prime}\right)
\end{aligned}
$$

Proof: If $\Pi$ is compact the result follows immediately from Theorem 3.2.1. Hence we only have to consider the case where $\Pi$ is a noncompact polytopal complex in $\mathbb{H}^{n}$. We work in the model $\mathbb{I D}^{n}$ of the hyperbolic space and we denote by $\bar{\Pi}$ the compactification of $\Pi$ in the ambient space $\mathbb{R}^{n} \supset \mathbb{D}^{n}$. Then $\bar{\Pi}$ and $\overline{\Pi^{\prime}}$ can be viewed as a Euclidean polytopal complex where $\overline{\Pi^{\prime}}$ is a polytopal decomposition of $\bar{\Pi}$ with $\chi_{c}(\bar{\Pi})=\chi_{c}(\Pi)$ and $\chi_{c}\left(\overline{\Pi^{\prime}}\right)=\chi_{c}\left(\Pi^{\prime}\right)$. With Theorem 3.2.1 we have $\chi_{c}(\bar{\Pi})=\chi_{c}\left(\overline{\Pi^{\prime}}\right)$ and the first equation follows.
Furthermore, the number $a_{i n f}^{0}(\Pi)$ does not change under an arbitrary decomposition and so we find with the first part of the proof

$$
\begin{aligned}
\chi_{g}(\Pi) & =\chi_{c}(\Pi)-a_{i n f}^{0}(\Pi) \\
& =\chi_{c}\left(\Pi^{\prime}\right)-a_{i n f}^{0}\left(\Pi^{\prime}\right) \\
& =\chi_{g}\left(\Pi^{\prime}\right)
\end{aligned}
$$

In the special case where the polytopal complex is a polytope with all of its faces we can deduce the following lemma (compare [Ma], section IV.3).

Lemma 3.2.2 Let $P$ be an n-dimensional polytope in $\mathbb{X}^{n}, \mathcal{D}=\mathcal{D}(P)$ a polytopal decomposition of $P$ and $\partial \mathcal{D}=\mathcal{D}(P) \cap \partial P$ the decomposition complex of the boundary of $P$. Then

$$
\begin{aligned}
\chi_{c}(\mathcal{D}) & =\chi_{g}(\mathcal{D})+a_{\text {inf }}^{0}(P)
\end{aligned}=1 .
$$

Proof: If $P$ is a compact polytope in $\mathbb{X}^{n}$ then $|\mathcal{D}|$ is homeomorphic to the closed $n$-dimensional unit ball and $\chi_{c}(\mathcal{D})=1$. Furthermore, $|\partial \mathcal{D}|$ is homeomorphic to the $(n-1)$-dimensional sphere and $\chi_{c}(\partial \mathcal{D})=1+(-1)^{n-1}$.
Let $P$ be a noncompact hyperbolic polytope in the model $\mathbb{D}^{n}$ of the hyperbolic space. We denote by $\overline{\mathcal{D}}$ (resp. $\overline{\partial \mathcal{D}}$ ) the compactification of $\mathcal{D}$ (resp. $\partial \mathcal{D}$ ). These two complexes can be viewed as Euclidean polytopal complexes. We get $\chi_{c}(\mathcal{D})=\chi_{c}(\overline{\mathcal{D}}), \chi_{c}(\partial \mathcal{D})=\chi_{c}(\overline{\partial \mathcal{D}})$ and with the first part of the proof the lemma follows.

The Euler Polyhedron Theorem In the proof of the previous lemma we have used the equation $\chi_{c}(\mathcal{D})=1$ for a polytopal decomposition $\mathcal{D}=\mathcal{D}(P)$ of a polytope $P$ in $\mathbb{X}^{n}$. This is a generalization of the well-known Euler Polyhedron Theorem.
This Theorem was discovered by L. Euler for 3-dimensional polytopes (polyhedrons) in 1752. It is interesting that this result was known to R. Descartes about hundred years earlier. In the middle of the nineteenth century L. Schläfli generalized Euler's Polyhedron Theorem to polytopes of all dimensions (compare [Sch], page 190).
There are many different proofs for this nice theorem. For instance, we can use the fact that the boundary of all polytopes is a so-called shellable polytopal complex (see [Z], Corollary 8.17). For another proof which works with intersections of the polytope with hyperplanes see [Br], Theorem 16.1.

Theorem 3.2.2 (Euler Polyhedron Theorem) Let $P$ be an $n$-dimensional polytope in $\mathbb{X}^{n}$. Then

$$
\chi_{c}(P)=a_{i n f}^{0}(P)+a_{\text {ord }}^{0}(P)-a^{1}(P)+\cdots+(-1)^{n-1} a^{n-1}(P)+(-1)^{n}=1 .
$$

### 3.3 Angles

Normalized Angles Let $P$ be an $n$-dimensional polytope in $\mathbb{X}^{n}$. In the following we will define the notion of an $(n-k-1)$-dimensional angle of $P$ at a face $P^{k}$. So let $P^{k}$ be an element in $\Omega^{k}(P)$ for $0 \leq k \leq n-1$; this means it has no vertex at infinity of $P$. Further let $x$ be an interior point of $P^{k}$ and $K$ the $(n-k)$-dimensional plane passing through $x$ and orthogonal to the plane $<P^{k}>$. It is possible to find an $\epsilon>0$, such that the sphere $S^{n-1}(x, \epsilon) \subset \mathbb{X}^{n}$ with center $x$ and radius $\epsilon>0$ only intersects faces of $P$ that are incident with $P^{k}$. Then the ( $n-k-1$ )-dimensional sphere $S^{n-k-1}(x, \epsilon) \subset K$ intersects also only faces of $P$ that are incident with $P^{k}$.

Definition 3.3.1 The ( $n-k-1$ )-dimensional (normalized) angle of $P$ at a face $P^{k}$ is defined as

$$
\begin{aligned}
\alpha_{-1}(P \mid P)=\alpha_{-1}(P) & :=1 \\
\alpha_{n-k-1}\left(P^{k} \mid P\right)=\alpha_{n-k-1}\left(P^{k}\right) & :=c_{n-k-1}(\epsilon)^{-1} \operatorname{vol}\left(S^{n-k-1}(x, \epsilon) \cap P\right) \\
\alpha_{n}(\emptyset \mid P)=\quad \alpha_{n}(P) & :=c_{n}^{-1} \operatorname{vol}_{\mathbb{X}^{n}( }(P)
\end{aligned}
$$

for $0 \leq k \leq n-1$, where the constant $c_{m}(\epsilon)$ denotes the volume of the $m$-dimensional sphere of radius $\epsilon$ and $c_{m}:=c_{m}(1)$. A 1-dimensional angle of $P$ is also called a dihedral angle.

Remark 3.3.1 All angles of an $n$-dimensional polytope $P$ can also be measured ( $n-1$ )-dimensionally. Let $P^{k} \in \Omega^{k}(P)$ for $0 \leq k \leq n$. We use the same notations as above. The real number $\epsilon>0$ is chosen in such a way that the sphere $S^{n-1}(x, \epsilon)$ only intersects faces of $P$ that are incident with $P^{k}$. Then we have

$$
\alpha_{n-k-1}\left(P^{k}\right)=c_{n-1}(\epsilon)^{-1} \operatorname{vol}\left(S^{n-1}(x, \epsilon) \cap P\right)
$$

for all $0 \leq k \leq n-1$ (compare $[P e]$ ).

The face $P^{k}$ of $P$ is called the apex of the angle $\alpha_{n-k-1}\left(P^{k} \mid P\right)$. This angle does not depend on the choice of $x$ and $\epsilon>0$, and it is normed in such a way, that the whole sphere will be measured as 1 . Furthermore, we define by

$$
\beta_{n-k-1}\left(P^{k} \mid P\right):=\frac{1}{2}-\alpha_{n-k-1}\left(P^{k} \mid P\right)
$$

the exterior angle of $P$ with apex $P^{k}$ for $0 \leq k \leq n-1$.

Face Figures The intersection $S^{n-k-1}(x, \epsilon) \cap P$ is an $(n-k-1)$-dimensional spherical polytope in $S^{n-k-1}(x, \epsilon)$ (see $[\mathrm{R}]$ ). By simple radial projection $p r$ we can map it on the unit sphere $S^{n-k-1}(x, 1) \equiv \mathbb{S}^{n-k-1}$ and get an $(n-k-1)$-dimensional spherical polytope in the sense of our definition. We define the face figure or the $\operatorname{link} L\left(P^{k}\right)$ as

$$
L\left(P^{k}\right):=\operatorname{pr}\left(S^{n-k-1}(x, \epsilon) \cap P\right)
$$

for all $P^{k}$ in $\Omega^{k}(P)$ with $0 \leq k \leq n-1$.
Let $p^{0}$ be an element in $\Upsilon^{0}(P)$ and $\sum$ a horosphere with basepoint $p^{0}$ such that $\sum$ meets only the sides of $P$ incident with $p^{0}$. The intersection

$$
L\left(p^{0}\right):=\sum \cap P
$$

is also called face figure or link of $P$ in $p^{0}$. The Euclidean geometry of $L\left(p^{0}\right)$ is uniquely determined by $p^{0}$ up to a similarity (induced by a radial projection from $p^{0}$ ). So $L\left(p^{0}\right)$ can be viewed as an $(n-1)$-dimensional Euclidean polytope.

Angle Sums Let $P$ be a polytope in $\mathbb{X}^{n}$. Then the ( $n-k-1$ )-dimensional angle sum of $P$ is defined as the sum of all $(n-k-1)$-dimensional angles of $P$ for all $0 \leq k \leq n$. This means:

$$
\begin{aligned}
\omega_{n-k-1}(P) & :=\sum_{P^{k} \in \Omega^{k}(P)} \alpha_{n-k-1}\left(P^{k} \mid P\right) \text { for } k=0, \ldots, n ; \\
\omega_{n}(P) & :=c_{n}^{-1} \operatorname{vol}_{\mathbb{X}^{n}}(P) .
\end{aligned}
$$

So we get for instance $\omega_{0}(P)=\frac{a^{n-1}(P)}{2}$ and $\omega_{-1}(P)=1$. Furthermore, we define the generalized angle sum $W(P)$ of $P$ as

$$
\begin{aligned}
W(P) & :=\sum_{i=0}^{n}(-1)^{i} \omega_{n-i-1}(P) \\
& =\omega_{n-1}(P)-\omega_{n-2}(P)+\ldots+(-1)^{n} \omega_{-1}(P)
\end{aligned}
$$

Complex Angles The notion of an angle of $P$ at a face $P^{k}$ can be generalized to pure polytopal complexes $\Pi$ of dimension $n$ in $\mathbb{X}^{n}$ as follows. Let $P^{k}$ be an element in $\Omega^{k}(\Pi)$ for $0 \leq k \leq n$. Then $P^{k}$ is included in a finite number of elements $P_{1}^{n}, \ldots, P_{h}^{n}$ in $\Omega^{n}(\Pi)$ of maximal dimension such that $P^{k}=P_{1}^{n} \cap \ldots \cap P_{h}^{n}$. Furthermore, the number $h$ is maximal which means that there are no other elements in $\Omega^{n}(\Pi)$ containing $P^{k}$.

Definition 3.3.2 The $(n-k-1)$-dimensional complex angle of $\Pi$ at a face $P^{k}$ is defined as

$$
\alpha_{n-k-1}^{\Pi}\left(P^{k}\right):=\sum_{i=1}^{h} \alpha_{n-k-1}\left(P^{k} \mid P_{i}^{n}\right)
$$

for $0 \leq k \leq n$.
Defining $\alpha_{n-k-1}\left(P^{k} \mid P^{n}\right)=0$ if $P^{k}$ is not a face of the polytope $P^{n}$ we can write the above equation as

$$
\alpha_{n-k-1}^{\Pi}\left(P^{k}\right)=\sum_{P^{n} \in \Omega^{n}(\Pi)} \alpha_{n-k-1}\left(P^{k} \mid P^{n}\right) .
$$

We can also speak of the exterior complex angle and define

$$
\beta_{n-k-1}^{\Pi}\left(P^{k}\right):=\frac{1}{2}-\alpha_{n-k-1}^{\Pi}\left(P^{k} \mid P\right)
$$

for $0 \leq k \leq n-1$. A simple computation shows that for all pure polytopal complexes $\Pi$ and for all $k$ with $0 \leq k \leq n$ we have

$$
\sum_{P^{n} \in \Omega^{n}(\Pi)} \sum_{P^{k} \in \Omega^{k}\left(P^{n}\right)} \alpha_{n-k-1}\left(P^{k} \mid P^{n}\right)=\sum_{P^{k} \in \Omega^{k}(\Pi)} \alpha_{n-k-1}^{\Pi}\left(P^{k}\right) .
$$

Example 3.3.1 Let $\Pi$ be the 2-dimensional polytopal complex in Example 3.1.2 (see Figures 2 and 3). We use the same notations $P_{1}^{2}=\operatorname{conv}\left(P_{1}^{0}, P_{2}^{0}, P_{3}^{0}\right), P_{2}^{2}=\operatorname{conv}\left(P_{1}^{0}, P_{4}^{0}, P_{5}^{0}\right)$ and $P_{1}^{2}=\operatorname{conv}\left(P_{1}^{0}, P_{6}^{0}, P_{7}^{0}, P_{8}^{0}\right)$. For example we have

$$
\begin{aligned}
\sum_{P^{2} \in \Omega^{2}(\Pi)} \sum_{P^{0} \in \Omega^{0}\left(P^{2}\right)} \alpha_{1}\left(P^{0} \mid P^{2}\right)= & \sum_{P^{0} \in \Omega^{0}\left(P_{1}^{2}\right)} \alpha_{1}\left(P^{0} \mid P_{1}^{2}\right)+\sum_{P^{0} \in \Omega^{0}\left(P_{2}^{2}\right)} \alpha_{1}\left(P^{0} \mid P_{2}^{2}\right) \\
& +\sum_{P^{0} \in \Omega^{0}\left(P_{3}^{2}\right)} \alpha_{1}\left(P^{0} \mid P_{3}^{2}\right) \\
= & \alpha_{1}\left(P_{1}^{0} \mid P_{1}^{2}\right)+\alpha_{1}\left(P_{2}^{0} \mid P_{1}^{2}\right)+\alpha_{1}\left(P_{3}^{0} \mid P_{1}^{2}\right) \\
& +\alpha_{1}\left(P_{1}^{0} \mid P_{2}^{2}\right)+\alpha_{1}\left(P_{4}^{0} \mid P_{2}^{2}\right)+\alpha_{1}\left(P_{5}^{0} \mid P_{2}^{2}\right) \\
& +\alpha_{1}\left(P_{1}^{0} \mid P_{3}^{2}\right)+\alpha_{1}\left(P_{6}^{0} \mid P_{3}^{2}\right)+\alpha_{1}\left(P_{7}^{0} \mid P_{3}^{2}\right)+\alpha_{1}\left(P_{8}^{0} \mid P_{3}^{2}\right) \\
= & \sum_{i=1}^{8} \alpha_{1}^{\Pi}\left(P_{i}^{0}\right) \\
= & \sum_{P^{0} \in \Omega^{0}(\Pi)} \alpha_{1}^{\Pi}\left(P^{0}\right),
\end{aligned}
$$

where we use that $\alpha_{1}^{\Pi}\left(P_{1}^{0}\right)=\alpha_{1}\left(P_{1}^{0} \mid P_{1}^{2}\right)+\alpha_{1}\left(P_{1}^{0} \mid P_{2}^{2}\right)+\alpha_{1}\left(P_{1}^{0} \mid P_{3}^{2}\right)$.


Figure 3: The Complex Angle in $P_{1}^{0}$

## 4 Combinatorial Numbers

The combinatorial numbers described in this section are coefficients in the Taylor series of analytic functions. The tangent numbers $T_{2 n+1}$ will appear in Schläfli's Reduction Formula for simplices if we normalize the angles in such a way, that the generalized octand is measured as 1 (compare L. Schläfli [Sch]; here denoted as $a_{n}$ ) and the modified tangent numbers $a_{2 n+1}$ if we normalize the angles in such a way, that the whole sphere is measured as 1 (compare E. Peschl $[\mathrm{Pe}])$. The Euler numbers will appear in the Reduction Formula for cubes.
For the study of number series the book $[\mathrm{S}]$ and on line version [OL] are very helpfull.
The Bernoulli numbers, the tangent numbers and the Euler numbers play also an important role in algebraic geometry. For instance, they are used to describe topological invariants of oriented (even-dimensional) manifolds (compare [Hir], chapter 8.9).

Thus we will recall some definitions. A complex or real function $f$ is called analytic in a point $z_{0}$ in $\mathbb{C}$ or $\mathbb{R}$ if $f$ can be expanded as a power series in $z$, which converges to the function in a neighborhood of $z_{0}$. If $f$ is analytic in a point $z_{0}$ then the power series is the Taylor series and so $f$ can be written as

$$
f(z)=\sum_{n=0}^{\infty} \frac{f^{(n)}\left(z_{0}\right)}{n!}\left(z-z_{0}\right)^{n}
$$

The complex number $\frac{f^{(n)}\left(z_{0}\right)}{n!}$ is called the $n$-th coefficient of the Taylor series for $f$ in $z_{0}$ and the number $f^{(n)}\left(z_{0}\right)$ is called the $n$-th reduced coefficient of the Taylor series for $f$ in $z_{0}$.

### 4.1 The Stirling Numbers

Let $X$ be a finite set with $n$ elements. A family of subsets $A_{1}, \ldots, A_{m}$ of $X$ is called a partition of $X$ if

- $A_{i} \neq \emptyset$ for all $i$;
- $A_{i} \cap A_{j}=\emptyset$ for all $i, j$ with $i \neq j$;
- $\cup_{i=1}^{m} A_{i}=X$.

The sets $A_{i}$ are called the classes of the partition.
For all $m \leq n$ the number $S_{n}^{m}$ of partitions of a set of $n$ objects into $m$ classes is called Stirling number (of the second kind). We have (see [Bc]):

1. $S_{n}^{1}=S_{n}^{n}=1$;
2. $S_{n+1}^{m}=S_{n}^{m-1}+m S_{n}^{m}$ for all $1<m<n$;
3. the number of surjections of $X$ into a set $A$ with $|A|=m$ is equal to $m!S_{n}^{m}$ and
4. 

$$
\begin{aligned}
S_{n}^{m} & =\frac{1}{m!} \sum_{v=0}^{m}(-1)^{m-v}\binom{m}{v} v^{n} \\
& =\frac{1}{m!} \sum_{v=0}^{m}(-1)^{v}\binom{m}{v}(m-v)^{n} .
\end{aligned}
$$

### 4.2 The Square-Root Numbers

The real function $f(x)=\sqrt{1+x}$ is analytic in the point $x=0$. The square-root numbers $q_{n}$ $(n \geq 0)$ are defined as the coefficients of the Taylor series of $f$ at $x=0$

$$
\sqrt{1+x}=\sum_{n=0}^{\infty} q_{n} x^{n}
$$

with

$$
\begin{aligned}
& q_{0}=1 \\
& q_{1}=\frac{1}{2} \\
& q_{n}=(-1)^{n} \frac{1 \cdot 3 \cdot 5 \cdots 2 n-3}{2 \cdot 4 \cdot 6 \cdots 2 n} .
\end{aligned}
$$

By a direct calculation we get the following result.
Lemma 4.2.1 We have

$$
\begin{aligned}
& q_{n+1}=-\frac{2 n-1}{2 n+2} q_{n} \\
& q_{n+1}=\frac{(-1)^{n}}{n+1}\binom{2 n}{n} \frac{1}{2^{2 n+1}}
\end{aligned}
$$

for all $n$ with $1 \leq n<\infty$.
For instance, we have the following values:

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $q_{n}$ | 1 | $\frac{1}{2}$ | $-\frac{1}{8}$ | $\frac{1}{16}$ | $-\frac{5}{128}$ | $\frac{7}{256}$ | $-\frac{21}{1024}$ | $\frac{33}{2048}$ | $-\frac{429}{32768}$ | $\frac{715}{65536}$ | $-\frac{2431}{262144}$ | $\frac{4199}{524288}$ |

### 4.3 The Bernoulli Numbers

The complex function $f(z)=\frac{z}{\exp z-1}$ has a removable singularity in the point $z=0$. The Bernoulli numbers $B_{n}(n \geq 0)$ are defined as the reduced coefficients of the Taylor series of $f(z)$

$$
\frac{z}{\exp z-1}=\sum_{n=0}^{\infty} \frac{B_{n}}{n!} z^{n}
$$

Comparing this with the series of the exponential function we get the recursion:

$$
\begin{aligned}
B_{0} & =1 \\
0 & =\sum_{v=0}^{n-1}\binom{n}{v} B_{v} .
\end{aligned}
$$

For instance, we have the following values:

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $B_{n}$ | 1 | $-\frac{1}{2}$ | $\frac{1}{6}$ | 0 | $-\frac{1}{30}$ | 0 | $\frac{1}{42}$ | 0 | $-\frac{1}{30}$ | 0 | $\frac{5}{66}$ | 0 | $-\frac{691}{2730}$ | 0 | $\frac{7}{6}$ | 0 | $-\frac{3617}{510}$ |

### 4.4 The Tangent Numbers

The complex functions $f(z)=\tan (z)$ and $g(z)=\tanh (z)$ are analytic in the point $z=0$. The tangent numbers $T_{2 n+1}(n \geq 0)$ are defined as the reduced coefficients of the Taylor series of $\tan (z)($ or $\tanh (z))$

$$
\begin{aligned}
\tan (z) & =\sum_{n=0}^{\infty} \frac{T_{2 n+1}}{(2 n+1)!} z^{2 n+1} \\
\tanh (z) & =\sum_{n=0}^{\infty}(-1)^{n} \frac{T_{2 n+1}}{(2 n+1)!} z^{2 n+1}
\end{aligned}
$$

Comparing these series with those of $\sin (z)$ and $\cos (z)$ we get the recursion

$$
\begin{aligned}
T_{1} & =1 \\
(-1)^{n} T_{2 n+1} & =1-\sum_{v=0}^{n-1}(-1)^{v}\binom{2 n+1}{2 v+1} T_{2 v+1}
\end{aligned}
$$

It follows by induction that all tangent numbers are integers. For instance, we have the following values:

| $m$ | 1 | 3 | 5 | 7 | 9 | 11 | 13 | 15 | 17 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | 1 |  | 16 | 272 | 7936 | 353792 | 22368256 | 1903757312 | 209865342976 |
| $T_{m}$ | 1 | 2 | 16 | 272 |  |  |  |  |  |

Furthermore, we have the following relations between tangent and Bernoulli numbers (see [Sch], page 243 ).

Theorem 4.4.1 For all integers $n \geq 0$ we have

$$
T_{2 n+1}=(-1)^{n} \frac{2^{2 n+1}\left(2^{2 n+2}-1\right)}{n+1} B_{2 n+2}
$$

### 4.5 The Euler Numbers

The complex functions $f(z)=\cos ^{-1}(z)$ and $g(z)=\cosh ^{-1}(z)$ are analytic in the point $z=0$. The Euler numbers $E_{2 n}(n \geq 0)$ are defined as the reduced coefficients of the Taylor series of $\cos ^{-1}(z)\left(\right.$ or $\left.\cosh ^{-1}(z)\right)$ in 0

$$
\begin{aligned}
& \frac{1}{\cos z}=\sum_{n=0}^{\infty} \frac{E_{2 n}}{(2 n)!} z^{2 n} \\
& \frac{1}{\cosh z}=\sum_{n=0}^{\infty}(-1)^{n} \frac{E_{2 n}}{(2 n)!} z^{2 n}
\end{aligned}
$$

We get the recursion

$$
\begin{aligned}
E_{0} & =1 \\
(-1)^{n} E_{2 n} & =\sum_{v=0}^{n-1}(-1)^{v+1}\binom{2 n}{2 v} E_{2 v}
\end{aligned}
$$

For instance, we have the following values:

| $m$ | 0 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $E_{m}$ | 1 | 1 | 5 | 61 | 1385 | 50521 | 2702765 | 199360981 | 19391512145 | 2404879675441 |

Furthermore, we have the following relation between the Euler and the tangent numbers (see [N], page 52):

Lemma 4.5.1 We have

$$
(-1)^{n} E_{2 n}=\sum_{v=0}^{n}(-1)^{v}\binom{2 n}{2 v} T_{2 v+1}
$$

for all $n \geq 0$.

### 4.6 Related Numbers

For all intergers $n \geq 0$ we define the rational numbers $a_{2 n+1}(n \geq 0)$ by

$$
\begin{aligned}
a_{2 n+1} & =\frac{1}{2^{2 n+1}} T_{2 n+1} \\
G_{2 n+1} & =\frac{n+1}{2^{2 n}} T_{2 n+1}
\end{aligned}
$$

The numbers $G_{2 n+1}$ are called the Genocchi numbers. The numbers $a_{2 n+1}$ are the coefficients of the Taylor series of the function $\tan (z / 2)$. We have

$$
\begin{aligned}
\tan \left(\frac{z}{2}\right) & =\sum_{n=0}^{\infty} \frac{a_{2 n+1}}{(2 n+1)!} z^{2 n+1} \\
& =\sum_{n=0}^{\infty} \frac{G_{2 n+1}}{(2 n+2)!} z^{2 n+1}
\end{aligned}
$$

Thus we can easily modify the recursion formula for the tangent numbers to obtain

$$
\begin{aligned}
a_{1} & =\frac{1}{2} \\
(-1)^{n} a_{2 n+1} & =\frac{1}{2}-\frac{1}{2} \sum_{v=0}^{n-1}(-1)^{v}\binom{2 n+1}{2 v+1} a_{2 v+1}
\end{aligned}
$$

Furthermore, by a direct calculation we get

Lemma 4.6.1 We have

$$
\begin{aligned}
& (-1)^{n} a_{2 n+1}=\sum_{v=0}^{n}(-1)^{v}\binom{2 n+1}{2 v} a_{2 v+1} \\
& (-1)^{n} a_{2 n+1}=-\frac{1}{2 n} \sum_{v=0}^{n-1}(-1)^{v}\binom{2 n+1}{2 v} a_{2 v+1}
\end{aligned}
$$

for all $n \geq 1$.

For instance, we have the following values:

| $m$ | 1 | 3 | 5 | 7 | 9 | 11 | 13 | 15 | 17 | 19 | 21 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $a_{m}$ | $\frac{1}{2}$ | $\frac{1}{4}$ | $\frac{1}{2}$ | $\frac{17}{8}$ | $\frac{31}{2}$ | $\frac{691}{4}$ | $\frac{5461}{2}$ | $\frac{929569}{16}$ | $\frac{3202291}{2}$ | $\frac{221930581}{4}$ | $\frac{4722116521}{2}$ |
| $G_{m}$ | 1 | 1 | 3 | 17 | 155 | 2073 | 38227 | 929569 | 28820619 | 11096552905 | 51943281731 |

### 4.7 Zick-Zack Permutations

The Euler numbers $E_{2 n}$ and tangent numbers $T_{2 n+1}$ play an essential role for counting the number of zick-zack permutations. Thus they have a pure combinatorial description. A permutation

$$
\left(\begin{array}{ccc}
1 & \cdots & m \\
k_{1} & \cdots & k_{m}
\end{array}\right)
$$

of $m$ elements is called a zick-zack permutation if

$$
\left(k_{v}-k_{v-1}\right)\left(k_{v+1}-k_{v}\right)<0
$$

for all $2 \leq v \leq m-1$. We denote the number of zick-zack permutations of $m$ elements by $Z(m)$. We have the following result (compare [En]).

Lemma 4.7.1 We have for all $m>1$

$$
Z(m)=\frac{1}{4} \sum_{v=0}^{m-1}\binom{m-1}{v} Z(v) Z(m-v-1)
$$

and furthermore

$$
Z(m)=\left\{\begin{array}{lll}
2 E_{m} & , \quad m \text { even } \\
2 T_{m} & , & m \text { odd }
\end{array} .\right.
$$

## 5 The Gram Matrix and Acute-Angled Polytopes

In this section we record well-known facts about Gram matrices and the connection of the Gram matrix of acute-angled polytopes with their combinatorial structure. For more details see [V1] and [V2].

### 5.1 General Facts

Let $P=\cap_{i \in I} H_{i}^{-}$be an $n$-dimensional polytope in $\mathbb{X}^{n}$, where we use the vector-space model $\mathbb{H}^{n}$ in the hyperbolic case. Let $e_{i}$ be the unit normal vector of the defining subspace $U_{H_{i}}$, directed inwards with respect to $P$ for all $i \in I$. The Gram matrix of $P$ is the matrix

$$
G(P)=\left(\left\langle e_{i}, e_{j}\right\rangle_{\mathbb{X}^{n}}\right)_{i, j \in I}
$$

Clearly, $G(P)$ is symmetric with 1's along the diagonal.
For $\mathbb{X}^{n}=\mathbb{S}^{n}$ the matrix $G(P)$ is positive semidefinite of rank $\leq n+1$, and $G(P)$ defines the polytope $P$ up to an isometry. For all $i, j \in I$ with $i \neq j$ we have $\left\langle e_{i}, e_{j}\right)_{\mathbb{S}^{n}}=-\cos \left(\alpha_{i j}\right)$ with $\alpha_{i j}=\angle\left(H_{i}, H_{j}\right)$ equal to the intersection angle formed by $H_{i}$ and $H_{j}$.
For $\mathbb{X}^{n}=\mathbb{E}^{n}$ the matrix $G(P)$ is positive semidefinite of rank $\leq n$. For all $i, j \in I$ with $i \neq j$ we have

$$
\left\langle e_{i}, e_{j}\right\rangle_{\mathbb{E}^{n}}=\left\{\begin{array}{rll}
-\cos \left(\alpha_{i j}\right) & , & H_{i}, H_{j} \text { intersect with angle } \alpha_{i j} \\
-1 & , & H_{i}, H_{j} \text { parallel }
\end{array} .\right.
$$

For $\mathbb{X}^{n}=\mathbb{H}^{n}$ the matrix $G(P)$ is indefinite of rank $n+1$ and signature $(1, n)$; and $G(P)$ defines the polytope $P$ up to an isometry. The entries $\left\langle e_{i}, e_{j}\right\rangle_{\mathbb{H}^{n}}$ for $i \neq j$ have the following geometrical meaning:

$$
\left\langle e_{i}, e_{j}\right\rangle_{\mathbb{H}^{n}}=\left\{\begin{array}{rll}
-\cos \left(\alpha_{i j}\right) & , & H_{i}, H_{j} \text { intersect with angle } \alpha_{i j} \\
-1 & , & H_{i}, H_{j} \text { parallel } \\
-\cosh \left(l_{i j}\right) & , & H_{i}, H_{j} \text { ultraparallel with common perpendicular of length } l_{i j}
\end{array} .\right.
$$

### 5.2 The Gram Matrix of an Acute-Angled Polytope

An $n$-dimensional polytope $P$ in $\mathbb{X}^{n}$ is called acute-angled if all dihedral angles of $P$ are of measure less than or equal to $1 / 4$. In this case $P$ is simple, which means that each ordinary ( $n-k$ )-dimensional face of $P$ is contained in exactly $k$ distinct ( $n-1$ )-faces.
A polytope $P$ is called a Coxeter polytope if all its dihedral angles have measure $\frac{1}{2 p}$ for $p \in \mathbb{N}$ with $p \geq 2$. Any acute-angled polytope in $\mathbb{S}^{n}$ is a simplex. The Gram matrix $G(P)$ of $P$ is positive definite of rank $=n+1$, and it has only non-positive entries off the diagonal.
Any acute-angled polytope in $\mathbb{E}^{n}$ is a direct product of simplices. The Gram matrix $G(P)$ of $P$ is positive semidefinite of rank $n$, and it has only non-positive entries off the diagonal. In particular $G(P)$ is parabolic, which means that by permutating of the rows and columns $G(P)$ can be brought into the form

$$
\left(\begin{array}{cccc}
A_{1} & & & 0 \\
& A_{2} & & \\
& & \ddots & \\
0 & & & A_{k}
\end{array}\right)
$$

where $A_{1}, A_{2}, \ldots, A_{k}, 1 \leq k \leq n$ are degenerate indecomposable positive-semidefinite matrices.

For an acute-angled polytope $P$ in $\mathbb{H}^{n}$ there is in general no simple combinatorial description. The Gram matrix $G(P)$ of $P$ is indecomposable of signature $(1, n)$ and it has only non-positive entries off the diagonal. The order of $G(P)$ can be arbitrarily large.
Let $P$ be a polytope in $\mathbb{X}^{n}$. Then $P$ is called a Coxeter polytope if all its dihedral angles (1-dimensional angles) are of the form $\frac{1}{2 p}$ for $p \in I N$ with $p \geq 2$.

### 5.3 Gram Matrix and Combinatorial Structure

Let $P=\bigcap_{i \in I} H_{i}^{-} \subset \mathbb{X}^{n}$ be an $n$-dimensional acute-angled polytope, $G=G(P)$ its Gram matrix, $\mathcal{F}(P)$ the complex of $P$ and $\overline{\mathcal{F}}(P)$ the extended complex of $P$. For each subset $J \subset I$ we use the following notation: $G_{J}$ is the principal submatrix of $G$ formed by the rows and columns whose indices belong to $J$. The following results are due to E. Vinberg ([V1], Theorems 3.1. and 3.2.).

Theorem 5.3.1 Let $J$ be a subset of $I$. Then $J \in \mathcal{F}(P)$ if and only if the matrix $G_{J}$ is positive definite.

Theorem 5.3.2 Let $J$ be a subset of $I$. Then $J \in \overline{\mathcal{F}}(P)-\mathcal{F}(P)$ if and only if the matrix $G_{J}$ is parabolic of rank $n-1$.

With these two results we are able to reconstruct the combinatorial structure of the polytope $P$ in terms of the incidence matrix (which is a complete combinatorial invariant) from its Gram matrix. Let $P=\cap_{i \in I} H_{i}^{-}$be an $n$-dimensional acute-angled polytope in $\mathbb{X}^{n}$ and $G=G(P)$ its Gram matrix. We use the following algorithm, called the Gram matrix-Incidence matrix Algorithm (GIA), defined as follows:

1. Determination of the set of all positive definite and parabolic submatrices of rank $n-1$ of G

$$
\left\{G_{J_{j}}: j=1, \ldots, a^{0}(P)\right\}
$$

where $J_{j} \subset I$ are the corresponding subsets of $I$ for $j=1, \ldots, a^{0}(P)$. This set is bijective to the set of all vertices (ordinary and at infinity) of the polytope $P$.
2. Of course, the set

$$
\left\{H_{i}: i=1, \ldots, a^{n-1}(P)\right\} .
$$

is bijective to the set of all $(n-1)$-dimensional faces of $P$.
3. Now we can derive the incidence matrix $M(P)=\left(m_{i j}\right)$ of $P$ by

$$
m_{j i}= \begin{cases}1 & , \quad i \in J_{j} \\ 0 & , \\ \text { otherwise }\end{cases}
$$

for all $1 \leq i \leq a^{n-1}(P)$ and $1 \leq j \leq a^{0}(P)$.

## 6 Schemes

Let $S$ be a graph with vertices $\left\{v_{i}\right\}_{i \in I}$. Then $S$ is called a scheme if each edge $v_{i} v_{j}$ has a positive weight $c_{i j}$. If there is no edge between two vertices we speak of an edge with zero weight. A subscheme of $S$ is a subgraph of $S$ in which every edge carries the same weight as $S$.
The number of vertices of $S$ is the order of $S$. If $\left\{v_{i}\right\}_{i \in I}$ is the set of vertives of $S$, we denote by $S_{J}$ the subscheme of $S$ with vertex set $\left\{v_{i}\right\}_{i \in J}$ for all subsets $J \subseteq I$. Of course, we have $S_{I}=S$.
To each scheme $S$ one can associate a symmetric matrix $A(S)=\left(a_{i j}\right)_{i, j \in I}$ with $a_{i i}=1$ and $a_{i j}=$ $-c_{i j}$ for all $i, j \in I$ with $i \neq j$. A scheme $S$ is connected if and only if $A(S)$ is indecomposable.
The rank, determinant and signature of $S$ can be transferred from $A(S)$ to $S$ and vice versa. Then $S$ is called elliptic, if $A(S)$ is positive definite, parabolic if $A(S)$ is parabolic or hyperbolic if $A(S)$ has index of inertia -1 .

Definition 6.0.1 The scheme $S$ of an acute-angled polytope $P$ in $\mathbb{X}^{n}$ is the scheme corresponding to its Gram-matrix $G(P)$.

Let $S$ be the scheme of an $n$-dimensional acute-angled polytope in $\mathbb{X}^{n}$. The Theorems 5.3.1 and 5.3.2 can be translated in the language of schemes.

Theorem 6.0.3 Let $J$ be a subset of $I$. Then $J$ is in $\mathcal{F}(P)$ if and only if the subscheme $S_{J}$ is elliptic. Furthermore, $J$ is in $\overline{\mathcal{F}}(P)-\mathcal{F}(P)$ if and only if $S_{J}$ is parabolic of rank $n-1$.

Of course, the Gram matrix-Incidence-Algorithm (GIA) can be modified easily to a SchemeIncidence matrix-Algorithm (SIA). The advantage of using schemes instead of the Gram matrix is reflected in the case of Coxeter polytopes, because we can use the classification results of elliptic and parabolic schemes.
If $P^{k}$ (or $p^{0}$ ) is an arbitrary face of $P$ with $0 \leq k \leq n-1$ we denote by $S\left(P^{k}\right)$ (or $S\left(p^{0}\right)$ ) the scheme of the face figure $L\left(P^{k}\right)\left(\right.$ or $L\left(p^{0}\right)$ ) of $P$. This is an elliptic scheme in the first case ( $P^{k}$ is an ordinary face of $P$ ) and a parabolic scheme in the second case ( $p^{0}$ is a vertex at infinity of $P)$.

## 7 Discrete Subgroups of $\operatorname{Iso}\left(\mathbb{X}^{n}\right)$

The groups which are generated by the reflections in the facets of a Coxeter polytope are simple cases of discrete subgroups of $\operatorname{Iso}\left(\mathbb{X}^{n}\right)$. In this section we give an introduction to discrete subgroups of $\operatorname{Iso}\left(\mathbb{X}^{n}\right)$ for $\mathbb{X}^{n}=\mathbb{S}^{n}, \mathbb{E}^{n}$ or $\mathbb{H}^{n}$. Furthermore, we define the notion of a fundamental polytope of a discrete group and the connections with tesselations.

### 7.1 General Facts

A family of subsets of $\mathbb{X}^{n}$ is called locally finite, if for each point there is a neighbourhood intersecting only finitely many subsets of this family. A subgroup $\Gamma<\operatorname{Iso}\left(\mathbb{X}^{n}\right)$ is called discrete if the family $\{\gamma K: \gamma \in \Gamma\}$ is locally finite for each compact set $K \subset \mathbb{X}^{n}$.
In the following let $\Gamma<\operatorname{Iso}\left(\mathbb{X}^{n}\right)$ always be a discrete group. Of course, $\Gamma$ acts on the space $\mathbb{X}^{n}$ via

$$
\begin{aligned}
\Gamma \times \mathbb{X}^{n} & \longrightarrow \mathbb{X}^{n} \\
(\gamma, x) & \mapsto \gamma x:=\gamma(x) .
\end{aligned}
$$

Now $\Gamma$ is a discrete group if and only if $\Gamma$ acts discontinously on $\mathbb{X}^{n}$ : for all compact sets $K \subset \mathbb{X}^{n}$, the set $K \cap \gamma K$ is nonempty for at most finitely many $\gamma \in \Gamma$.
Let us extend the notion of compact sets. A generalized compact set of $\mathbb{X}^{n}$ is either an ordinary compact set or for $\mathbb{X}^{n}=\mathbb{H}^{n}$, interpreted in the model $\mathbb{D}^{n}$, a set $K$ in $\mathbb{D}^{n}$ such that

- $\partial D^{n} \cap \bar{K}$ is a finite set of points
- $K \cup\left\{\partial \mathbb{D}^{n} \cap \bar{K}\right\}$ is an ordinary compact set in $\mathbb{E}^{n}$.

Let $K \subset \mathbb{X}^{n}$ be an generalized compact set. We define by

$$
\begin{aligned}
\Gamma_{K} & =\operatorname{Stab}(K, \Gamma) \\
& :=\{\gamma \in \Gamma: \gamma K=K\}
\end{aligned}
$$

the stabilizer of $K$ in $\Gamma$.
We prefer the second notation in the cases where the the symbol of the set $K$,,carries,, many indices.
Then $\Gamma_{K}$ is a subgroup of $\Gamma$. Furthermore, we define the subgroup $\Gamma_{K}^{\prime}<\Gamma_{K}$ of all elements in $\Gamma_{K}$, which fix the set $K$ pointwise:

$$
\begin{aligned}
\Gamma_{K}^{\prime} & =\operatorname{Stab}_{p}(K, \Gamma) \\
& :=\{\gamma \in \Gamma: \gamma k=k \text { for all } k \in K\} .
\end{aligned}
$$

Of course, $\Gamma_{K}$ and $\Gamma_{K}^{\prime}$ are discrete groups. The family of subsets

$$
\Gamma K:=\{\gamma K: \gamma \in \Gamma\}
$$

of $\mathbb{X}^{n}$ is called the $\Gamma$-orbit through $K$ or the cycle through $K$. The elements in $\Gamma K$ are called $\Gamma$-equivalent.

The map

$$
\begin{array}{lll}
\Gamma & \longrightarrow & \Gamma K \\
\gamma & \mapsto & \gamma K
\end{array}
$$

is surjective with kernel $\Gamma_{K}$ and thus induces a bijection

$$
\Gamma / \Gamma_{K} \quad \longrightarrow \quad \Gamma K
$$

It follows that

$$
\begin{aligned}
\operatorname{ord}(\Gamma K) & =\left[\Gamma: \Gamma_{K}\right] \\
& =\frac{\operatorname{ord}(\Gamma)}{\operatorname{ord}\left(\Gamma_{K}\right)}
\end{aligned}
$$

Lemma 7.1.1 For all discrete subgroups of $\Gamma<\operatorname{Iso}\left(\mathbb{X}^{n}\right)$ and all ordinary compact sets $K$ in $\mathbb{X}^{n}$ each of the groups $\Gamma_{K}^{\prime}$ and $\Gamma_{K}$ is conjugated to a finite subgroup of $O(n)$.

## Proof:

- $\Gamma_{K}^{\prime}$ is a subgroup of $O(n)$. This is clear for $\mathbb{X}^{n}=\mathbb{S}^{n}$ or $\mathbb{E}^{n}$. For $\mathbb{X}^{n}=\mathbb{H}^{n}$ the group $\Gamma_{K}^{\prime}$ is an elementary group of elliptic type ( $\Gamma_{K}^{\prime}$ has at least one fixed point). Hence $\Gamma_{K}^{\prime}$ is isomorphic (conjugated) to a subgroup of $O(n)$. Clearly $\Gamma_{K}^{\prime}$ is finite, because it is discrete.
- Let $\gamma$ be an element in $\Gamma_{K}$. Then $\gamma$ maps the compact set $K$ onto itself and has a fixed point in $K$ (by Browers Fixed Point Theorem). Hence $\gamma$ is conjugated to an orthogonal map and we see that each element in $\Gamma_{K}$ is elliptic and of finite order ( $\Gamma_{K}$ is discrete). Then $\Gamma_{K}$ is conjugated to a finite subgroup of $O(n)$ (compare Lemma 2.4.1).

If $K$ is a compact set in $\mathbb{X}^{n}$ then also $\gamma K$ is a compact set in $\mathbb{X}^{n}$ for all elements $\gamma \in \Gamma$ and the stabilizer $\Gamma_{\gamma K}$ is conjugated to $\Gamma_{K}$ :

$$
\Gamma_{\gamma K}=\gamma \Gamma_{K} \gamma^{-1}
$$

It is often enough to consider compact sets $K=\{x\}$ consisting of one point only. In this case we have $\Gamma_{K}=\Gamma_{K}^{\prime}=: \Gamma_{x}$ and the following important result ( $[\mathrm{R}]$, Theorem 5.3.4).

Theorem 7.1.1 Let $\mathbb{X}^{n}=\mathbb{S}^{n}$ or $\mathbb{H}^{n}$ and $\Gamma<\operatorname{Iso}\left(\mathbb{X}^{n}\right)$ an arbitrary subgroup. Then $\Gamma$ is discrete if and only if

- each stabilizer subgroup of $\Gamma$ is finite, and
- $\Gamma x$ is a closed and discrete subset of $\mathbb{X}^{n}$ for all $x \in \mathbb{X}^{n}$.

Let $\Gamma<\operatorname{Iso}\left(\mathbb{X}^{n}\right)$ be discrete. Then there exists at least one point $x \in \mathbb{X}^{n}$ such that $\Gamma_{x}$ is trivial. If the group $\Gamma_{x}$ is trivial for all $x \in \mathbb{X}^{n}$, the action of $\Gamma$ on $\mathbb{X}^{n}$ is called fixed point free.

### 7.2 Fundamental Regions of Discrete Groups

Let $\Gamma<\operatorname{Iso}\left(\mathbb{X}^{n}\right)$ be a discrete subgroup of isometries in $\mathbb{X}^{n}$. A subset $R \subset \mathbb{X}^{n}$ is called a (locally finite) fundamental region for $\Gamma$ if

1. $R$ is closed in $\mathbb{X}^{n}$;
2. the elements in $\{\gamma \operatorname{ri}(R): \gamma \in \Gamma\}$ are mutually disjoint;
3. $\mathbb{X}^{n}=\cup_{\gamma \in \Gamma} \gamma R$;
4. the family $\{\gamma R: \gamma \in \Gamma\}$ is a locally finite family of subsets in $\mathbb{X}^{n}$.

If $R$ is connected, then $R$ is called a fundamental domain for $\Gamma$. A fundamental domain $R$ for $\Gamma$ is called a fundamental polytope for $\Gamma$ if $R$ is an $n$-dimensional polytope in $\mathbb{X}^{n}$. A fundamental polytope $P$ for $\Gamma$ is called normal, if there exists $\gamma \in \Gamma$ with $P^{n-1}=P \cap \gamma P$ for each ( $n-1$ )-face $P^{n-1}$ of $P$.
A fundamental region $R$ for $\Gamma$ is called proper, if $\partial R$ is a set of measure zero. All proper fundamental regions $R$ for $\Gamma$ have the same volume. So we can define the covolume of $\Gamma$ as

$$
\operatorname{covol}(\Gamma):=\operatorname{vol}_{\mathbb{X}^{n}}(R)
$$

for any proper fundamental region $R$ for $\Gamma$ (if there exists one). Of course fundamental polytopes for $\Gamma$ are proper fundamental domains for $\Gamma$. Discrete groups $\Gamma<\operatorname{Iso}\left(\mathbb{X}^{n}\right)$ with $\operatorname{covol}(\Gamma)<\infty$ are called of finite covolume or crystallographic. If one fundamental region for $\Gamma$ is compact, then so are all other fundamental regions for $\Gamma$. In this case $\Gamma$ is called cocompact or uniform.

Example 7.2.1 The modular group $\operatorname{PSL}(2, \mathbb{Z})$ can be viewed as a discrete subgroup of $\operatorname{Iso}\left(\mathbb{H}^{2}\right)$, generated (for instance) by the elements

$$
\begin{array}{ll}
\left(\begin{array}{cc}
1 & 1 \\
0 & 1
\end{array}\right) & : \quad z \mapsto z+1 \\
\left(\begin{array}{cc}
0 & 1 \\
-1 & 0
\end{array}\right) \quad: & z \mapsto-\frac{1}{z}
\end{array}
$$

In Picture 4 one can see a (canonical) proper fundamental polytope $P_{m}=\operatorname{conv}(A, B, \infty)$ for $\operatorname{PSL}(2, \mathbb{Z})$, where $A=-1 / 2+i \sqrt{3} / 2$ and $B=1 / 2+i \sqrt{3} / 2$. The polytope $P_{m}$ is a triangle with one vertex at infinity and the three 1-dimensional angles $\alpha_{1}(\infty)=0, \alpha_{1}(A)=1 / 6$ and $\alpha_{1}(B)=1 / 6$; so $P_{m}$ is a Coxeter polytope. Furthermore, $\operatorname{PSL}(2, \mathbb{Z})$ can be viewed as the subgroup of orientation preserving elements of the Coxeter group $\Gamma$, generated by the reflections in the faces of the polytope $P_{m}^{+}=\operatorname{conv}(A, C, \infty)$ (compare Picture 5). We have $[G: \operatorname{PSL}(2, \mathbb{Z})]=2$ and $\operatorname{vol}_{\mathbb{H}^{2}}\left(P_{m}\right)=2 \operatorname{vol}_{\mathbb{H}^{2}}\left(P_{m}^{+}\right)$.

### 7.3 Tesselations and Discrete Groups

A tesselation of the space $\mathbb{X}^{n}$ is a collection $\Phi$ of $n$-dimensional polytopes in $\mathbb{X}^{n}$ such that

1. the interiors of the polytopes in $\Phi$ are mutually disjoint;
2. $\mathbb{X}^{n}=|\Phi|$;


Figure 4: The Fundamental Polytope $P_{m}$ of $\operatorname{PSL}(2, \mathbb{Z})$


Figure 5: The Polytope $P_{m}^{+}$
3. $\Phi$ is a locally finite family of subsets in $\mathbb{X}^{n}$.

A tesselation $\Phi$ of $\mathbb{X}^{n}$ is called normal if each $(n-1)$-dimensional face of a polytope in $\Phi$ is contained in exactly two polytopes of $\Phi$.

Now we have the following important connection between tesselations and discrete groups (compare [R], Theorem 6.7.1.).

Theorem 7.3.1 Let $P$ be an n-dimensional polytope in $\mathbb{X}^{n}$ and let $\Gamma$ be a group of isometries of $\mathbb{X}^{n}$. Then $\Gamma$ is discrete and $P$ is a (normal) fundamental polytope for $\Gamma$ if and only if

$$
\Phi=\{\gamma P: \gamma \in \Gamma\}
$$

is a (normal) tesselation.
Of course, each normal tesselation of $\mathbb{X}^{n}$ is connected. Furthermore, we have the following result (see [R], Theorem 6.7.3. and 6.7.4.).

Theorem 7.3.2 Let $P$ be a normal fundamental polytope for a discrete group $\Gamma$ of isometries of $\mathbb{X}^{n}$. Then $\Gamma$ is finitely generated by the set

$$
\Psi=\{\gamma \in \Gamma: P \cap \gamma P \text { is a face of } P\}
$$

Example 7.3.1 Let $P_{m}$ be the ("canonical") fundamental polytope for the discrete group $P S L(2, \mathbb{Z})$ $<$ Iso $\left(I^{2}\right)$ described in Example 7.2.1 (see Figure 4). Then the set

$$
\Phi:=\{\gamma P: \gamma \in P S L(2, \mathbb{Z})\}
$$

is a normal tesselation of the upper half-space $\mathbb{U}^{2}$ (see Figure 6).


Figure 6: A PSL $(2, \mathbb{Z})$-Tesselation

Furthermore, the group $P S L(2, \mathbb{Z})$ is finitely generated by the set

$$
\begin{aligned}
\Psi & =\left\{\gamma \in \operatorname{PSL}(2, \mathbb{Z}): P_{m} \cap \gamma P_{m} \text { is a face of } P\right\} \\
& =\{z \mapsto z+1, z \mapsto z-1, z \mapsto-1 / z\}
\end{aligned}
$$

## 8 Geometric Orbifolds

In this section we will develop the theory of geometric orbifolds and explain the connections with discrete groups. For the details see [R], section 13 or [Kapo], section 6 . Especially, we show that geometric orbifolds can be viewed as geometrical interpretations of discrete subgroups of $\operatorname{Iso}\left(\mathbb{X}^{n}\right)$.

### 8.1 Definitions

In the following let $\mathbb{X}^{n}=\mathbb{S}^{n}, \mathbb{E}^{n}$ or $\mathbb{H}^{n}, G<\operatorname{Iso}\left(\mathbb{X}^{n}\right)$ an arbitrary subgroup of the group of isometries of $\mathbb{X}^{n}$ and $M$ a Hausdorff space.
An $\left(\mathbb{X}^{n}, G\right)$-orbifold atlas for $M$ is a collection

$$
\Phi=\left\{\left(U_{i}, \phi_{i}\right): i \in I\right\}
$$

of sets $U_{i} \subset M$ and maps $\phi_{i}: U_{i} \longrightarrow \mathbb{X}^{n} / \Gamma_{i}$ which are called charts, such that for all $i \in I$ :

1. The set $U_{i}$, which is called coordinate neighbourhood, is an open connected subset of $M$ and $\Gamma_{i}$ is a discrete subgroup of $\operatorname{Iso}\left(\mathbb{X}^{n}\right)$.
2. The chart $\phi_{i}$ maps the coordinate neighbourhood $U_{i}$ homeomorphically onto an open subset of $\mathbb{X}^{n} / \Gamma_{i}$.
3. $M=\bigcup_{i \in I} U_{i}$.
4. If $U_{i}$ and $U_{j}$ overlap, the map

$$
\phi_{j} \circ \phi_{i}^{-1} \quad: \quad \phi_{i}\left(U_{i} \cap U_{j}\right) \longrightarrow \phi_{j}\left(U_{i} \cap U_{j}\right),
$$

called coordinate change, has the following property: If $x, y$ are in $\mathbb{X}^{n}$ with

$$
\phi_{j} \circ \phi_{i}^{-1}\left(\Gamma_{i} x\right)=\Gamma_{j} y,
$$

then there exists an element $g \in G$ with $g x=y$ and which lifts $\phi_{j} \circ \phi_{i}^{-1}$ in a neighbourhood of $x$. More precisely, we have

$$
\phi_{j} \circ \phi_{i}^{-1}\left(\Gamma_{i} w\right)=\Gamma_{j} g w
$$

for all $w$ in a neighbourhood of $x$ (see Figure 7).
An $\left(\mathbb{X}^{n}, G\right)$-orbifold structure for a Hausdorff space $M$ is a maximal $\left(\mathbb{X}^{n}, G\right)$-orbifold atlas for $M$ (for each ( $\mathbb{X}^{n}, G$ )-orbifold atlas for $M$ there exists a unique maximal ( $\mathbb{X}^{n}, G$ )-orbifold atlas for $M$, which contains this atlas). An $\left(\mathbb{X}^{n}, G\right)$-orbifold $M$ is a Hausdorff space $M$ with a ( $\left.\mathbb{X}^{n}, G\right)$-orbifold structure for $M$. An $\left(\mathbb{X}^{n}, \operatorname{Iso}\left(\mathbb{X}^{n}\right)\right.$ )-orbifold is called

$$
\begin{aligned}
\text { spherical orbifold } & : \Longleftrightarrow \mathbb{X}^{n}=\mathbb{S}^{n} ; \\
\text { Euclidean orbifold } & : \Longleftrightarrow \mathbb{X}^{n}=\mathbb{E}^{n} ; \\
\text { hyperbolic orbifold } & : \Longleftrightarrow \mathbb{X}^{n}=\mathbb{H}^{n}
\end{aligned}
$$

Let $M$ be an $\left(\mathbb{X}^{n}, G\right)$-orbifold and $u \in M$. A chart for $(M, u)$ is a chart $\phi: U \rightarrow \mathbb{X}^{n} / \Gamma$ for $M$ with $u \in U$. If

$$
\begin{array}{rll}
\phi_{i}: U_{i} \rightarrow \mathbb{X}^{n} / \Gamma_{i} & \text { with } & \phi_{i}(u)=\pi_{i}(x) \text { and } \\
\phi_{j}: U_{j} \rightarrow \mathbb{X}^{n} / \Gamma_{j} & \text { with } & \phi_{j}(u)=\pi_{j}(y)
\end{array}
$$



Figure 7: The Coordinate Change
are charts for $(M, u)$, then the stabilizer groups $\Gamma_{x}<\Gamma_{i}$ and $\Gamma_{y}<\Gamma_{j}$ are conjugated. This means that there exists $g \in G$ with $g \Gamma_{x} g^{-1}=\Gamma_{y}$ and $g x=y$, lifting $\phi_{j} \phi_{i}^{-1}$ to a neighbourhood of $x$. In particular, we have $\operatorname{ord}\left(\Gamma_{x}\right)=\operatorname{ord}\left(\Gamma_{y}\right)$. So we can define the order of a point $u \in M$ as the order of the stabilizer group $\Gamma_{x}$ for any chart $\phi: U \rightarrow \mathbb{X}^{n} / \Gamma$ for $(M, u)$ with $\phi(u)=\pi(x)$. We call

$$
\begin{aligned}
& M_{\text {ord }}:=\{u \in M: \operatorname{ord}(u)=1\} \\
& M_{\text {sin }}:=\{u \in M: \operatorname{ord}(u)>1\}
\end{aligned}
$$

the ordinary and the singular set of $M$. Of course, $M$ is the disjoint union of $M_{\text {ord }}$ and $M_{\text {sin }} ; M_{\text {ord }}$ is open and dense and $M_{\text {sin }}$ is closed and nowhere dense in $M$. In the special case $M_{\text {sin }}=\{\emptyset\}$, $M$ is called a geometric manifold .
Let $c:[a, b] \rightarrow M$ be a curve in an $\left(\mathbb{X}^{n}, G\right)$-orbifold $M$. We can define the $\mathbb{X}^{n}$-length $\|c\|$ of $c$ via charts in a canonical way. A curve $c$ in $M$ is called $\mathbb{X}^{n}$-rectifiable if $\|c\|<\infty$. Then the function

$$
\begin{aligned}
d: M \times M & \longrightarrow \mathbb{R} \\
(u, v) & \longmapsto \inf _{c}\|c\|
\end{aligned}
$$

where $c$ varies over all $\mathbb{X}^{n}$-rectifiable curves from $u$ to $v$, is a metric on $M$. Hence $M$ is a metric space with an inner length metric. An $\left(\mathbb{X}^{n}, G\right)$-orbifold $M$ is called complete if $M$ is a complete metric space.

## $8.2\left(\mathbb{X}^{n}, G\right)$-Equivalences

Let $M$ and $N$ be two $\left(\mathbb{X}^{n}, G\right)$-orbifolds. A map

$$
\eta: M \quad \longrightarrow N
$$

is called an $\left(\mathbb{X}^{n}, G\right)$-map if the following holds:

1. $\eta$ is continous and
2. for all charts $\phi: U \rightarrow \mathbb{X}^{n} / \Gamma$ for $M$ and $\psi: V \rightarrow \mathbb{X}^{n} / \Lambda$ for $N$ with $U \cap \eta^{-1}(V) \neq \emptyset$, the map

$$
\psi \circ \eta \circ \phi^{-1}: \phi\left(U \cap \eta^{-1}(V)\right) \quad \longrightarrow \quad \psi(\eta(U) \cap V)
$$

has the following properties: If $x, y$ are in $\mathbb{X}^{n}$ with $\psi \circ \eta \circ \phi^{-1}(\Gamma x)=\Lambda y$, then there exists an element $g \in G$ such that

- $g x=y$ and
- $g$ lifts $\psi \circ \eta \circ \phi^{-1}$ to a neighbourhood of $x$ (Figure 8).

An $\left(\mathbb{X}^{n}, G\right)$-map is called an $\left(\mathbb{X}^{n}, G\right)$-equivalence, if it is a homeomorphism.


Figure 8: A $\left(\mathbb{X}^{n}, G\right)$-map
Now we have the following result (compare [R], Theorem 13.3.10.).
Theorem 8.2.1 Let $\mathbb{X}^{n}=\mathbb{S}^{n}, \mathbb{E}^{n}$ or $\mathbb{H}^{n}$ and $G<\operatorname{Iso}\left(\mathbb{X}^{n}\right)$. Furthermore let $M$ be a complete connected $\left(\mathbb{X}^{n}, G\right)$-orbifold. Then there exists a discrete group $\Gamma<\operatorname{Iso}\left(\mathbb{X}^{n}\right)$ such that $M$ and $\mathbb{X}^{n} / \Gamma$ are $\left(\mathbb{X}^{n}, G\right)$-equivalent.

Hence we can view a complete connected $\left(\mathbb{X}^{n}, \operatorname{Iso}\left(\mathbb{X}^{n}\right)\right)$-orbifold as another interpretation of a discrete subgroup of $\operatorname{Iso}\left(\mathbb{X}^{n}\right)$.
Let $\Gamma<\operatorname{Iso}\left(\mathbb{X}^{n}\right)$ be discrete. Then the projection map $\pi: \mathbb{X}^{n} \longrightarrow \mathbb{X}^{n} / \Gamma$ induces an isometry from $B(x, r) / \Gamma_{x}$ onto $B(\pi(x), r)$ for all $r$ such that $0<r \leq \frac{1}{4} \operatorname{dist}(x, \Gamma x-\{x\})$ (compare [R], Theorem 13.1.1.). Furthermore, the group $\Gamma_{x}$ is a discrete subgroup of Iso $\left(\mathbb{X}^{n}\right)$ with a fixed point in $\mathbb{X}^{n}$. It is easy to see that $\Gamma_{x}$ is a finite subgroup of $O(n)$ for each space $\mathbb{X}^{n}=\mathbb{S}^{n}, \mathbb{E}^{n}$ or $\mathbb{H}^{n}$. In order to understand the local structure of an orbifold near a point $z=\pi(x)$ it is enough to investigate the quotient space $B(x, r) / \Gamma_{x}$ for a suitable $r>0$.

### 8.3 Local Structure of Hyperbolic 2-Orbifolds

We describe the local structure of a hyperbolic 2-orbifold $M$ near a singular point $z$ of $M$ (compare [Kapo]). A finite group of $O(2)$ is conjugated to the group $\mathbb{Z}_{2}$ generated by a reflection, a cyclic group $C_{q}$ generated by a rotation of order $q \geq 2$ or a dihedral group $D_{q}$ generated by two reflections whose product has order $q \geq 2$. Let $U$ be a neighbourhood of $z$. Then we have the following types of local structures near $z$ (Table 1).

| Local stucture near $z$ | $U \cong \mathbb{H}^{2} / \Gamma$ |
| :---: | :---: |
| Boundary (Reflector) | $\mathbb{H}^{2} / \mathbb{Z}_{2}: \mathbb{Z}_{2}=\mathbb{Z} / 2 \mathbb{Z}$ |
| Cone Point | $\mathbb{H}^{2} / C_{q}: C_{q}$ cyclic group |
| Cone Reflector | $\mathbb{H}^{2} / D_{p}: D_{p}$ dihedral group $(p>2)$ |

Table 1: The Local Structures of a 2-Orbifold


Figure 9: Hyperbolic 2-Orbifold

## 9 Combinatorics and Angles

### 9.1 The Combinatorics of Polytopal Complexes

In this section we will develop some connections between the combinatorics of polytopal complexes and their decompositions. Particularly we decompose the set of all decomposition polytopes of a polytopal complex $\Pi$ into subsets. Each subset consists of all decomposition polytopes (of several dimensions), which are included in a skeleton of fixed dimension but not in a lowerdimensional skeleton of $\Pi$.

Definition 9.1.1 Let $\Pi$ be an $n$-dimensional polytopal complex in $\mathbb{X}^{n}$, $l$ an integer with $0 \leq l \leq$ $n, \mathcal{D}=\mathcal{D}(P)$ a decomposition of $\Pi$ and $D \in \Omega(\mathcal{D})$ an element of $\mathcal{D}$. Then $D$ is called element in $\mathcal{G}^{l}(\Pi)$, if $|D| \subset\left|\mathcal{G}^{l}(\Pi)\right|$ and $|D| \not \subset\left|\mathcal{G}^{l-1}(\Pi)\right|$. We will denote this inclusion by $D<\mathcal{G}^{l}(\Pi)$.

In the special case $\Pi=\mathcal{D}$ the fact $D<\mathcal{G}^{l}(\Pi)$ means that $D$ is an $l$-dimensional polytope in $\Pi$.

Lemma 9.1.1 Let $\Pi$ be an n-dimensional polytopal complex in $\mathbb{X}^{n}, \mathcal{D}=\mathcal{D}(P)$ a decomposition of $\Pi$ and $D \in \Omega^{k}(\mathcal{D})$ for $0 \leq k \leq n$. Then $D \in \mathcal{G}^{l}(\Pi)$ for an $l$ with $k \leq l \leq n$ if and only if there is a uniquely determined $P^{l} \in \Omega^{l}(\Pi)$ such that $\operatorname{ri}(D) \subset r i\left(P^{l}\right)$.

## Proof:

- Let $D \in \Omega^{k}(\mathcal{D})$ with $D<\mathcal{G}^{l}(\Pi)$. Then $|D| \subset\left|\mathcal{G}^{l}(\Pi)\right|$ and of course, there is an element $P^{l} \in \Omega^{l}(\Pi)$ with $|D| \subset\left|P^{l}\right|$. We have to show that $r i(D) \subset r i\left(P^{l}\right)$. But the condition $|D| \subset\left|P^{l}\right|$ means either that $r i(D) \subset r i\left(P^{l}\right)$, or that $|D| \subset\left|\partial P^{l}\right| \subset\left|\mathcal{G}^{l-1}(\Pi)\right|$ which is impossible.
Furthermore, the element $P^{l}$ is uniquely determined. Let $Q^{l} \in \Omega^{l}(\Pi)$ be a second element different from $P^{l}$ with $r i(D) \subset \operatorname{ri}\left(Q^{l}\right)$. Then we have $|D| \subset\left|P^{l} \cap Q^{l}\right| \subset\left|\mathcal{G}^{l-1}(\Pi)\right|$ and this is impossible.
- Let $D \in \Omega^{k}(\mathcal{D})$ such that there exists a uniquely determined $P^{l} \in \Omega^{l}(\Pi)$ with $r i(D) \subset$ ri $\left(P^{l}\right)$. Then we have $|D| \subset\left|P^{l}\right|$ and so $|D| \subset\left|\mathcal{G}^{l}(\Pi)\right|$. Now we assume that $|D| \subset$ $\left|\mathcal{G}^{l-1}(\Pi)\right|$. Then we have $|D| \subset\left|P^{l-1}\right|$ for an element $P^{l-1} \in \Omega^{l-1}(\Pi)$ which implies ri $(D) \not \subset r i\left(P^{l}\right)$ for all $P^{l} \in \Omega^{l}(\Pi)$ in contradiction to the given conditions.

Of course, the decomposition $\mathcal{D}$ of a polytopal complex $\Pi$ induces (canonically) a decomposition of each of the polytopes in this complex. Let $P^{l}$ be an element in $\Omega^{l}(\Pi)$. Then we denote by $\mathcal{D} \cap P^{l}$ the decomposition of the polytope $P^{l}$ induced by $\mathcal{D}$.

Lemma 9.1.2 Let $\Pi$ be an n-dimensional polytopal complex in $\mathbb{X}^{n}, \mathcal{D}=\mathcal{D}(\Pi)$ a decomposition of $\Pi, D \in \Omega^{k}(\mathcal{D})$ with $0 \leq k \leq n$ and $P^{l} \in \Omega^{l}(\Pi)$ with $k \leq l \leq n$. Then we have:

$$
\left\{D \in \Omega^{k}(\mathcal{D}): \operatorname{ri}(D) \subset \operatorname{ri}\left(P^{l}\right)\right\}=\left\{D \in \Omega^{k}\left(\mathcal{D} \cap P^{l}\right): D<\mathcal{G}^{l}\left(P^{l}\right)\right\}
$$

Proof:

- Let $D \in \Omega^{k}(\mathcal{D})$ with $r i(D) \subset r i\left(P^{l}\right)$. Then $D$ is also a decomposition polytope in the decomposition $\mathcal{D} \cap P^{l}$ of $P^{l}$ and $|D| \subset\left|\mathcal{G}^{l}\left(P^{l}\right)\right|=\left|P^{l}\right|$. Now we assume that $|D| \subset$ $\left|\mathcal{G}^{l-1}\left(P^{l}\right)\right|=\left|\partial P^{l}\right|$. Then this implies that ri $(D) \not \subset r i\left(P^{l}\right)$.
- Let $D \in \Omega^{k}\left(\mathcal{D} \cap P^{l}\right)$, and $D<\mathcal{G}^{l}\left(P^{l}\right)$. Then of course $D \in \Omega^{k}(\mathcal{D})$. Furthermore, $D<G^{l}\left(P^{l}\right)$ means $|D| \not \subset\left|\mathcal{G}^{l-1}\left(P^{l}\right)\right|$, and so $D$ is not a decomposition polytope in the boundary of $P^{l}$. Thus ri $(D) \subset r i\left(P^{l}\right)$.

Let $\Pi$ be an $n$-dimensional polytopal complex in $\mathbb{X}^{n}$ and $\mathcal{D}=\mathcal{D}(\Pi)$ a decomposition of $\Pi$. The set $\Omega(\mathcal{D})$ of all faces of the polytopal complex $\mathcal{D}$ splits into disjoint subsets

$$
\Omega(\mathcal{D})=\Omega_{0}(\Pi, \mathcal{D}) \cup \Omega_{1}(\Pi, \mathcal{D}) \cup \ldots \cup \Omega_{n}(\Pi, \mathcal{D})
$$

where the set $\Omega_{l}(\Pi, \mathcal{D})$ is defined as:

$$
\Omega_{l}(\Pi, \mathcal{D}):=\left\{D \in \mathcal{D}: D<\mathcal{G}^{l}(\Pi)\right\}
$$

for all $l=0,1, \ldots, n$. In the same way the set $\Omega^{k}(\mathcal{D})$ splits into $(n-k+1)$ pairwise disjoint subsets

$$
\Omega^{k}(\mathcal{D})=\bigcup_{l=k}^{n} \Omega_{l}^{k}(\Pi, \mathcal{D})
$$

for all $0 \leq k \leq n$. The set $\Omega_{l}^{k}(\Pi, \mathcal{D})$ is defined by

$$
\Omega_{l}^{k}(\Pi, \mathcal{D})=\left\{D \in \Omega^{k}(\mathcal{D}): D<\mathcal{G}^{l}(\Pi)\right\}
$$

for all $k \leq l \leq n$.
Lemma 9.1.3 The sets $\Omega_{l}^{k}(\Pi, \mathcal{D})$ are disjoint unions according to

$$
\Omega_{l}^{k}(\Pi, \mathcal{D})=\bigcup_{P^{l} \in \Omega^{l}(\Pi)}\left\{D \in \Omega^{k}(\mathcal{D}): r i(D) \subset r i\left(P^{l}\right)\right\}
$$

for all $k$ with $0 \leq k \leq n$ and all $l$ with $k \leq l \leq n$.

## Proof:

It is clear that the union on the right-hand side is disjoint.

- Let $D \in \Omega_{l}^{k}(\Pi, \mathcal{D})$. Then there exists a unique face $P^{l} \in \Omega^{l}(\Pi)$ such that ri $(D) \subset r i\left(P^{l}\right)$ (with Lemma 9.1.1) and we have the first inclusion.
- Let $D \in \Omega^{k}(\mathcal{D})$ and $P^{l} \in \Omega^{l}(\Pi)$ with $r i(D) \subset \operatorname{ri}\left(P^{l}\right)$. It is clear that $|D| \subset\left|\mathcal{G}^{l}(\Pi)\right|$ and $|D| \not \subset\left|\mathcal{G}^{l-1}(\Pi)\right|$ with Lemma 9.1.1.

Example 9.1.1 Let $\Pi=\operatorname{conv}\left(P_{1}^{0}, P_{2}^{0}, P_{3}^{0}, P_{4}^{0}\right)$ be the 2-dimensional polytope in Example 3.1.1 and $\mathcal{D}=\mathcal{D}(P)$ the polytopal decomposition in Figure 10. We use the notation $\overline{A B}:=\operatorname{conv}(A, B)$ for a pair of points $A$ and $B$. Then

$$
\begin{aligned}
& \Omega_{0}^{0}(\Pi, \mathcal{D})=\left\{P_{1}^{0}, P_{2}^{0}, P_{3}^{0}, P_{4}^{0}\right\} \\
& \Omega_{1}^{0}(\Pi, \mathcal{D})=\{Q, R\} \\
& \Omega_{2}^{0}(\Pi, \mathcal{D})=\emptyset \\
& \Omega_{1}^{1}(\Pi, \mathcal{D})=\left\{\overline{P_{1}^{0} P_{2}^{0}}, \overline{P_{2}^{0} Q}, \overline{Q P_{3}^{0}}, \overline{P_{3}^{0} P_{4}^{0}}, \overline{P_{4}^{0} R}, \overline{R P_{1}^{0}}\right\} \\
& \Omega_{2}^{1}(\Pi, \mathcal{D})=\{\overline{Q R}\} \\
& \Omega_{2}^{2}(\Pi, \mathcal{D})=\left\{\operatorname{conv}\left(Q, R, P_{1}^{0}, P_{2}^{0}\right), \operatorname{conv}\left(Q, R, P_{4}^{0}, P_{3}^{0}\right)\right\} .
\end{aligned}
$$



Figure 10: Decomposition of a Polytope

## Example 9.1.2 The Cone-Decomposition and the Decomposition of a Cone

An n-dimensional cone $C$ in $\mathbb{X}^{n}$ is a special kind of a polytope. It can be written as the $\mathbb{X}^{n}$-convex hull of an ( $n-1$-dimensional polytope $\tilde{C}$ and a point $m$ with $m \notin \tilde{C}$; so $C=\operatorname{conv}(m, \tilde{C})$. For all $d$ with $0 \leq d \leq n-1$ we can divide the set $\Omega^{d}(C)$ into two subsets as follows: $\Omega^{d}(C)^{\prime}$ is the set of d-dimensional faces of $C$ which are contained in $\tilde{C}$ and $\Omega^{d}(C)^{\prime \prime}:=\Omega^{d}(C)-\Omega^{d}(C)^{\prime}$. Clearly, we have $\Omega^{0}(C)^{\prime \prime}=\{m\}$ and $\Omega^{n-1}(C)^{\prime}=\{\tilde{C}\}$.
Now we will describe two special kinds of polytopal decompositions. Let $P$ be a polytope in $\mathbb{X}^{n}$. Then we denote by $\mathcal{K}=\mathcal{K}(P)$ the pure polytopal complex we get from $P$ by cone decomposition. All maximally dimensional elements in $\mathcal{K}$ are of the form $\operatorname{conv}\left(b(P), P^{n-1}\right)$, where $b(P)$ is the barycenter and $P^{n-1}$ is an $(n-1)$-dimensional face of $P$. For the sets of faces in $\mathcal{K}$ we have:

$$
\begin{aligned}
\Omega^{0}(\mathcal{K}) & =\Omega^{0}(P) \cup\{b(P)\} \\
\cdot & \cdot \\
\Omega^{i}(\mathcal{K}) & =\Omega^{i}(P) \cup\left\{\operatorname{conv}\left(b(P), P^{i-1}\right): P^{i-1} \in \Omega^{i-1}(P)\right\} \\
\cdot & \cdot \cdot \\
\Omega^{n}(\mathcal{K}) & =\left\{\operatorname{conv}\left(b(P), P^{n-1}\right): P^{n-1} \in \Omega^{n-1}(P)\right\}
\end{aligned}
$$

If $C=\operatorname{conv}(m, \tilde{C})$ is a cone in $\mathbb{X}^{n}$, we denote by $\mathcal{S}=\mathcal{S}(C)$ the pure simplicial complex that can be constructed from $C$ in the following way. Firstly we decompose $\tilde{C}$ barycentrically without decomposition of the 1-dimensional faces. Secondly we construct all cones with basis equal to
one of the decomposition simplices in $\tilde{C}$ and center $m$. If $C$ is a cone in $\mathbb{X}^{4}$, then for the sets of faces in $\mathcal{S}$ we obviously have:

$$
\begin{aligned}
& \Omega_{0}^{0}(C, \mathcal{S})=\Omega^{0}(C) \\
& \Omega_{1}^{0}(C, \mathcal{S})=\emptyset \\
& \Omega_{2}^{0}(C, \mathcal{S})=\left\{b\left(C^{2}\right): C^{2} \in \Omega^{2}(C)^{\prime}\right\} \\
& \Omega_{3}^{0}(C, \mathcal{S})=\{b(\tilde{C})\} \\
& \Omega_{4}^{0}(C, \mathcal{S})=\emptyset \\
& \Omega_{1}^{1}(C, \mathcal{S})=\Omega^{1}(C) \\
& \Omega_{2}^{1}(C, \mathcal{S})=\left\{\operatorname{conv}\left(b\left(C^{2}\right), C^{0}\right): C^{0} \in \Omega^{0}(C)^{\prime}, C^{2} \in \Omega^{2}(C)^{\prime}\right\} \\
& \Omega_{3}^{1}(C, \mathcal{S})=\left\{\operatorname{conv}\left(b(\tilde{C}), C^{0}\right): C^{0} \in \Omega^{0}(C)^{\prime}\right\} \cup \\
& \left\{\operatorname{conv}\left(b(\tilde{C}), b\left(C^{2}\right)\right): C^{2} \in \Omega^{2}(C)^{\prime}\right\} \cup \\
& \left\{\operatorname{conv}\left(b\left(C^{2}\right), m\right): C^{2} \in \Omega^{2}(C)^{\prime}\right\} \\
& \Omega_{4}^{1}(C, \mathcal{S})=\{\operatorname{conv}(b(\tilde{C}), m)\} \\
& \Omega_{2}^{2}(C, \mathcal{S})=\Omega^{2}(C)^{\prime \prime} \cup\left\{\operatorname{conv}\left(b\left(C^{2}\right), C^{1}\right): C^{1} \in \Omega^{1}(C)^{\prime}\right\} \\
& \Omega_{3}^{2}(C, \mathcal{S})=\left\{\operatorname{conv}\left(b(\tilde{C}), C^{1}\right): C^{1} \in \Omega^{1}(C)^{\prime}\right\} \cup \\
& \left\{\operatorname{conv}\left(b(\tilde{C}), b\left(C^{2}\right), C^{0}\right): C^{0} \in \Omega^{0}(C)^{\prime}, C^{2} \in \Omega^{2}(C)^{\prime},\left|C^{0}\right| \subset\left|C^{2}\right|\right\} \cup \\
& \left\{\operatorname{conv}\left(b\left(C^{2}\right), C^{0}, m\right): C^{0} \in \Omega^{0}(C)^{\prime}, C^{2} \in \Omega^{2}(C)^{\prime},\left|C^{0}\right| \subset\left|C^{2}\right|\right\} \\
& \Omega_{4}^{2}(C, \mathcal{S})=\left\{\operatorname{conv}\left(b(\tilde{C}), C^{0}, m\right): C^{0} \in \Omega^{0}(C)^{\prime}\right\} \cup \\
& \left\{\operatorname{conv}\left(b(\tilde{C}), b\left(C^{2}\right), m\right): C^{2} \in \Omega^{2}(C)^{\prime}\right\} \\
& \Omega_{3}^{3}(C, \mathcal{S})=\left\{\operatorname{conv}\left(b\left(C^{2}\right), C^{1}, m\right): C^{1} \in \Omega^{1}(C)^{\prime}, C^{2} \in \Omega^{2}(C)^{\prime},\left|C^{1}\right| \subset\left|C^{2}\right|\right\} \cup \\
& \left\{\operatorname{conv}\left(b(\tilde{C}), b\left(C^{2}\right), C^{1}\right): C^{1} \in \Omega^{1}(C)^{\prime}, C^{2} \in \Omega^{2}(C)^{\prime},\left|C^{1}\right| \subset\left|C^{2}\right|\right\} \\
& \Omega_{4}^{3}(C, \mathcal{S})=\left\{\operatorname{conv}\left(b(\tilde{C}), C^{1}, m\right): C^{1} \in \Omega^{1}(C)^{\prime}\right\} \cup \\
& \left\{\operatorname{conv}\left(b(\tilde{C}), b\left(C^{2}\right), C^{0}, m\right): C^{0} \in \Omega^{0}(C)^{\prime}, C^{2} \in \Omega^{2}(C)^{\prime},\left|C^{0}\right| \subset\left|C^{2}\right|\right\} \\
& \Omega_{4}^{4}(C, \mathcal{S})=\left\{\operatorname{conv}\left(b(\tilde{C}), b\left(C^{2}\right), C^{1}, m\right): C^{1} \in \Omega^{1}(C)^{\prime}, C^{2} \in \Omega^{2}(C)^{\prime},\left|C^{1}\right| \subset\left|C^{2}\right|\right\}
\end{aligned}
$$

Definition 9.1.2 Let $\Pi$ be an $n$-dimensional polytopal complex in $\mathbb{X}^{n}, \mathcal{D}=\mathcal{D}(\Pi)$ a polytopal decomposition and $k$ and $l$ integers with $0 \leq k \leq l \leq n$. Then we define the non-negative integer $Z_{l}^{k}(\Pi, \mathcal{D})$ as:

$$
\begin{aligned}
Z_{l}^{k}(\Pi, \mathcal{D}) & :=\sharp\left\{D \in \Omega^{k}(\mathcal{D}): D<\mathcal{G}^{l}(\Pi)\right\} \\
& =\sharp \Omega_{l}^{k}(\Pi, \mathcal{D}) .
\end{aligned}
$$

In particular we have $Z_{l}^{k}(\Pi, \mathcal{D})=0$ for all $k>l$.
This means that $Z_{l}^{k}(\Pi, \mathcal{D})$ is the number of $k$-dimensional decomposition polytopes in $\mathcal{D}$ which are elements in the $l$-skeleton $\mathcal{G}^{l}(\Pi)$ of $\Pi$.

Theorem 9.1. 1 Let $\Pi$ be an n-dimensional polytopal complex in $\mathbb{X}^{n}$ and $\mathcal{D}=\mathcal{D}(\Pi)$ a polytopal decomposition of $\Pi$. Then we have

$$
\begin{aligned}
a_{\text {ord }}^{0}(\mathcal{D}) & =\sum_{l=0}^{n} Z_{l}^{0}(\Pi, \mathcal{D}), \\
a^{k}(\mathcal{D}) & =\sum_{l=k}^{n} Z_{l}^{k}(\Pi, \mathcal{D}) \text { and } \\
Z_{l}^{k}(\Pi, \mathcal{D}) & =\sum_{P^{l} \in \Omega^{l}(\Pi)} Z_{l}^{k}\left(P^{l}, \mathcal{D} \cap P^{l}\right)
\end{aligned}
$$

for all $k$ with $1 \leq k \leq n$ and all $l$ with $k \leq l \leq n$. The notation $\mathcal{D} \cap P^{l}$ means the decomposition of the polytope $P^{l}$ induced by $\mathcal{D}$.

Proof: The second (and also the first) equation follows from the observation that

$$
\begin{aligned}
a^{k}(\mathcal{D}) & =\sharp \Omega^{k}(\mathcal{D}) \\
& =\sharp\left\{\Omega_{k}^{k}(\Pi, \mathcal{D}) \cup \ldots \cup \Omega_{n}^{k}(\Pi, \mathcal{D})\right\}
\end{aligned}
$$

and that the union is disjoint. For the third equation we note that

$$
\begin{aligned}
Z_{l}^{k}(\Pi, \mathcal{D}) & =\sharp \Omega_{l}^{k}(\Pi, \mathcal{D}) \\
& =\sum_{P^{l} \in \Omega^{l}(\Pi)} \sharp\left\{D \in \Omega^{k}(\mathcal{D}): r i(D) \subset r i\left(P^{l}\right)\right\} \\
& =\sum_{P^{l} \in \Omega^{l}(\Pi)} \sharp\left\{D \in \Omega^{k}\left(\mathcal{D} \cap P^{l}\right): D<\mathcal{G}^{l}\left(P^{l}\right)\right\} \\
& =\sum_{P^{l} \in \Omega^{l}(\Pi)} Z_{l}^{k}\left(P^{l}, \mathcal{D} \cap P^{l}\right),
\end{aligned}
$$

where we have used Lemma 9.1.3 in the second step and Lemma 9.1.2 in the third step.
For a shorter description we write in the polytopal case (with the same notations):

$$
z\left(k, P^{l}, \mathcal{D}\right):=Z_{l}^{k}\left(P^{l}, \mathcal{D} \cap P^{l}\right)
$$

this is the number of $k$-dimensional decomposition polytopes in $\mathcal{D}$, whose relative interior is contained in the relative interior of $P^{l}$.

Example 9.1.3 Consider the decomposition of the polytope $P$ in Example 9.1.1. For instance we have

$$
\begin{aligned}
z(0, P, \mathcal{D}) & =0 \\
z(1, P, \mathcal{D}) & =1 \\
z(2, P, \mathcal{D}) & =2 \\
z\left(0, \overline{P_{1}^{0} P_{4}^{0}}, \mathcal{D}\right) & =1 \\
z\left(1, \overline{P_{1}^{0} P_{4}^{0}}, \mathcal{D}\right) & =2 .
\end{aligned}
$$

Lemma 9.1.4 Let $P \subset \mathbb{X}^{n}$ be a polytope and $\mathcal{D}=\mathcal{D}(P)$ a polytopal decomposition of $P$. Then for all $k$ with $0 \leq k \leq n$ and for all l-dimensional faces $P^{l} \in \Omega^{l}(P)$ of $P$ with $0 \leq l \leq n$ we have

$$
z\left(k, P^{l}, \mathcal{D}\right)=\sharp\left\{D \in \Omega_{l}^{k}(\Pi, \mathcal{D}): \operatorname{ri}(D) \subset \operatorname{ri}\left(P^{l}\right)\right\} .
$$

Proof: We have

$$
\begin{aligned}
z\left(k, P^{l}, \mathcal{D}\right) & =Z_{l}^{k}\left(P^{l}, \mathcal{D} \cap P^{l}\right) \\
& =\sharp\left\{D \in \Omega^{k}\left(\mathcal{D} \cap P^{l}\right): D<\mathcal{G}^{l}\left(P^{l}\right)\right\} \\
& =\sharp\left\{D \in \Omega^{k}(\mathcal{D}): \operatorname{ri}(D) \subset r i\left(P^{l}\right)\right\} \\
& =\sharp\left\{D \in \Omega_{l}^{k}(\Pi, \mathcal{D}): \operatorname{ri}(D) \subset \operatorname{ri}\left(P^{l}\right)\right\} ;
\end{aligned}
$$

where we have used Lemma 9.1.2 in the third step.
Now we will consider the first barycentric decomposition $\mathcal{B}(\Pi)$ of a polytopal complex $\Pi$ Of course, $\mathcal{B}(\Pi)$ has the nice property that $\mathcal{B}(\Pi) \cap P^{l}=\mathcal{B}\left(P^{l}\right)$ for all elements $P^{l} \in \Omega^{l}(\Pi)$ !

Definition 9.1.3 Let $\Pi$ be an $n$-dimensional polytopal complex in $\mathbb{X}^{n}$ and $k$ and $l$ integers with $0 \leq k \leq l \leq n$. Then we define the positive integer $B_{l}^{k}(\Pi)$ as:

$$
B_{l}^{k}(\Pi) \quad:=\quad Z_{l}^{k}(\Pi, \mathcal{B}(\Pi))
$$

This means that $B_{l}^{k}(\Pi)$ is the number of $k$-dimensional decomposition polytopes in $\mathcal{B}(\Pi)$ which lie in the $l$-skeleton $\mathcal{G}(\Pi)$ of $\Pi$.

Proposition 9.1.1 Let $\Pi$ be an $n$-dimensional polytopal complex in $\mathbb{X}^{n}$. Then

$$
\begin{aligned}
a_{\text {ord }}^{0}(\mathcal{B}(\Pi)) & =\sum_{l=0}^{n} B_{l}^{0}(\Pi) \\
a^{k}(\mathcal{B}(\Pi)) & =\sum_{l=k}^{n} B_{l}^{k}(\Pi) \text { and } \\
B_{l}^{k}(\Pi) & =\sum_{P^{l} \in \Omega^{l}(\Pi)} B_{l}^{k}\left(P^{l}\right)
\end{aligned}
$$

for all $k$ with $1 \leq k \leq n$ and all $l$ with $k \leq l \leq n$.
Proof: The first and the second equation is a simple conclusion of Theorem 9.1.1. For the third equation we have:

$$
\begin{aligned}
B_{l}^{k}(\Pi) & =Z_{l}^{k}(\Pi, \mathcal{B}(\Pi)) \\
& =\sum_{P^{l} \in \Omega^{l}(\Pi)} Z_{l}^{k}\left(P^{l}, \mathcal{B}(\Pi) \cap P^{l}\right) \\
& =\sum_{P^{l} \in \Omega^{l}(\Pi)} Z_{l}^{k}\left(P^{l}, \mathcal{B}\left(P^{l}\right)\right) \\
& =\sum_{P^{l} \in \Omega^{l}(\Pi)} B_{l}^{k}\left(P^{l}\right)
\end{aligned}
$$

Definition 9.1.4 Let $P^{n}$ be an $n$-dimensional polytope in $\mathbb{X}^{n}$ and $k$ an integer with $0 \leq k \leq n$. Then we define

$$
\begin{aligned}
b\left(k, P^{n}\right) & :=B_{n}^{k}\left(P^{n}\right) \\
& =z\left(k, P^{n}, \mathcal{B}\left(P^{n}\right)\right) .
\end{aligned}
$$

Now we develop an important property of the decompositions.
Theorem 9.1.2 Let $\Pi$ be a pure n-dimensional polytopal complex in $\mathbb{X}^{n}, \mathcal{D}=\mathcal{D}(\Pi)$ a decomposition of $\Pi$ and $P^{l}$ an arbitrary element in $\Omega^{l}(\Pi)$ for $0 \leq l \leq n$. Then

$$
\sum_{k=0}^{l}(-1)^{k} z\left(k, P^{l}, \mathcal{D}\right)=(-1)^{l}
$$

Proof: Let $P^{l}$ be an arbitrary element in $\Omega^{l}(\Pi)$ with $0 \leq l \leq n$. The number $z\left(k, P^{l}, \mathcal{D}\right)$ of $k$-dimensional polytopes in $\mathcal{D}$ whose relative interiors are contained in the relative interior of $P^{l}$ is equal to the number $a^{k}\left(\mathcal{D} \cap P^{l}\right)-a^{k}\left(\partial \mathcal{D} \cap P^{l}\right)$ for all $k=0, \ldots, l$. We get

$$
\begin{aligned}
\sum_{k=0}^{l}(-1)^{k} z\left(k, P^{l}, \mathcal{D}\right) & =\sum_{k=0}^{l}(-1)^{k} a^{k}\left(\mathcal{D} \cap P^{l}\right)-\sum_{k=0}^{l-1}(-1)^{k} a^{k}\left(\partial \mathcal{D} \cap P^{l}\right) \\
& =\chi_{g}\left(\mathcal{D} \cap P^{l}\right)-\chi_{g}\left(\partial \mathcal{D} \cap P^{l}\right) \\
& =1-a_{i n f}^{0}(P)-\left(1+(-1)^{l-1}-a_{i n f}^{0}(P)\right) \\
& =(-1)^{l}
\end{aligned}
$$

where we have used Lemma 3.2.2 in the third step.
Theorem 9.1.3 Let $P=P^{n}$ be an $n$-dimensional polytope in $\mathbb{X}^{n}$. Then we have

$$
\begin{aligned}
b(0, P) & =1 \text { and } \\
b(k, P) & =\sum_{v=0}^{n-k} \sum_{\substack{P u \in R^{u}(P) \\
u=k-1+v}} b\left(k-1, P_{i}^{u}\right)
\end{aligned}
$$

for all $k$ with $1 \leq k \leq n$.
Proof: Each of the $k$-dimensional decomposition simplices of $\Omega^{k}\left(\mathcal{B}\left(P^{n}\right)\right)$ in the interior of $P$ is a pyramide with base equal to a uniquely determined decomposition simplex of $\Omega^{k-1}\left(\mathcal{B}\left(P^{n}\right)\right)$. In contrast, each $(k-1)$-dimensional decomposition simplex in the interior of a proper face of $P$ determines a uniquely determined $k$-dimensional decomposition simplex in the interior of $P$.

For an $n$-dimensional polytope $P=P^{n}$ in $\mathbb{X}^{n}$ we define a new combinatorial number $f(k, P)$ for all $k$ with $1 \leq k \leq n$ :

$$
f(k, P):=\sum_{i=k}^{n} \sum_{P^{i} \in \Omega^{i}(P)} b\left(k, P^{i}\right) .
$$

It is easy to see that $f(k, P)$ is the number of all $k$-dimensional decomposition simplices in $\mathcal{B}^{1}\left(P^{n}\right)$. For all $k>n$ we put $f(k, P)=b(k, P)=0$.

## Special Cases:

1. Let $P=T$ be an $n$-dimensional simplex and for all $k$ with $0 \leq k \leq n$ we define $b(k, n):=$ $b(k, T)$ and $f(k, n):=f(k, T)$. All faces of $T$ are simplices of lower dimensions and the number of $k$-dimensional faces of $T$ is equal to $a^{k}(T)=\binom{n+1}{k+1}$ (compare Lemma 3.1.1). So we get with Theorem 9.1.3

$$
\begin{aligned}
b(k, n) & =\sum_{l=0}^{n-k}\binom{n+1}{k+l} b(k-1, k-1+l) \\
& =\sum_{l=0}^{n-k}\binom{n+1}{n-l} b(k-1, n-1-l) .
\end{aligned}
$$

2. Let $P=W$ be an $n$-dimensional cube. All faces of $W$ are cubes of lower dimensions and the number of $k$-dimensional faces of $W$ is equal to $a^{k}(W)=2^{n-k}\binom{n}{k}$ (compare Lemma 3.1.1). So we get with Theorem 9.1.3

$$
b(k, W)=\sum_{l=0}^{n-k} 2^{n-k+1-l}\binom{n}{k-1+l} b\left(k-1, W^{k-1+l}\right) .
$$

### 9.2 The Numbers $f(k, n)$ and $b(k, n)$

In this section we will determine the numbers $f(k, n)$ and $b(k, n)$ by translation of combinatorialgeometrical questions into combinatorial-stochastical questions, which are easier to solve.
Let $T=T^{n}$ be an $n$-dimensional simplex in $\mathbb{X}^{n}$ with vertices $\Omega^{0}(T)=\left\{T_{0}^{0}, \ldots, T_{n}^{0}\right\}, \mathcal{P}:=$ $\mathcal{P}\left(\Omega^{0}(T)\right)$ the power set of $\Omega^{0}(T)$ and $\mathcal{B}:=\mathcal{B}(T)$ the barycentric subdivision of $T$. Then the map

$$
\begin{aligned}
\rho: \Omega(T) & \longrightarrow \mathcal{P} \\
\operatorname{conv}\left(T_{i_{0}}^{0}, \ldots, T_{i_{k}}^{0}\right) & \longmapsto\left\{T_{i_{0}}^{0}, \ldots, T_{i_{k}}^{0}\right\}
\end{aligned}
$$

is bijective and each element in $\mathcal{B}$ can be written as the convex hull of barycenters of faces of $T$. This means that

$$
\operatorname{conv}\left(b\left(T_{i_{0}}^{n\left(i_{0}\right)}\right), \ldots, b\left(T_{i_{k}}^{n\left(i_{k}\right)}\right)\right) \in \Omega^{k}(\mathcal{B})
$$

if and only if there exists a permutation $\sigma \in S_{k+1}$ with

$$
\rho\left(T_{\sigma\left(i_{0}\right)}^{n\left(\sigma\left(i_{0}\right)\right)}\right) \subset \rho\left(T_{\sigma\left(i_{1}\right)}^{\left.n\left(i_{1}\right)\right)}\right) \subset \ldots \subset \rho\left(T_{\sigma\left(i_{k}\right)}^{n\left(\sigma\left(i_{k}\right)\right)}\right) .
$$

Now we can extend the results of D. G. Hoel (compare [Hoe]).
Theorem 9.2.1 Let $T$ be an $n$-dimensional simplex in $\mathbb{X}^{n}$ and $k$ in $\mathbb{N}$ with $0 \leq k \leq n$. Then

$$
\begin{aligned}
f(k, n) & =\sum_{i=0}^{k+1}(-1)^{i}\binom{k+1}{i}(k+2-i)^{n+1} \\
b(k, n) & =\sum_{i=0}^{k+1}(-1)^{i}\binom{k+1}{i}(k+1-i)^{n+1} \\
& =(k+1)!S_{n+1}^{k+1}
\end{aligned}
$$

where $S_{n+1}^{k+1}$ is a Stirling number of the second order.

Proof: We will use a simpler notation. We find out that $f(k, n)$ is equal to the number of different collections $\left\{f_{0}, \ldots, f_{k}\right\}$ with

- $f_{i} \in \mathcal{P}$ for all $i$;
- $f_{i} \neq f_{j}$ for all $i \neq j$ and
- $f_{0} \subset \ldots \subset f_{k}$.

The set of these collections will be denoted by $L_{f}$.
Furthermore, $b(k, n)$ is equal to the number of different collections $\left\{f_{0}, \ldots, f_{k}\right\}$ with

- $f_{i} \in \mathcal{P}$ for all $i$;
- $f_{i} \neq f_{j}$ for all $i \neq j$;
- $f_{0} \subset \ldots \subset f_{k}$ and
- $f_{k}=\left\{T_{0}^{0}, \ldots, T_{n}^{0}\right\}$.

The set of these collections will be denoted by $L_{b}$. Of course, $L_{b} \subset L_{f}$.
Let $\left\{f_{0}, \ldots, f_{k}\right\}$ be a collection. Then we define inductively the following sets:

$$
\begin{aligned}
E_{0} & :=f_{0} \\
E_{i} & :=f_{i}-f_{i-1} \text { for } i=1, \ldots, k \\
E_{k+1} & :=V-f_{k} .
\end{aligned}
$$

The sets $E_{0}, E_{1}, \ldots, E_{k+1}$ are pairwise disjoint.
Let $\left\{f_{0}, \ldots, f_{k}\right\}$ be a subset of $L_{b}$. The assigned sets $E_{0}, E_{1}, \ldots, E_{k}$ are not empty and neither the set $E_{k+1}\left(f_{k}=\left(v_{0}, \ldots, v_{n}\right)=V\right)$. This means, that $b(k, n)=\sharp L_{b}$ is equal to the number of possibilities of arranging $(n+1)$ balls (the elements of $V)$ in $(k+1)$ cells $\left(E_{0}, E_{1}, \ldots, E_{k}\right)$ in such a way that no cell is empty.
Let $\left\{f_{0}, \ldots, f_{k}\right\}$ be a subset of $L_{f}$. The assigned sets $E_{0}, E_{1}, \ldots, E_{k}$ are not empty but the set $E_{k+1}$ is empty if and only if $\left\{f_{0}, \ldots, f_{k}\right\} \subset L_{b}$. This means that $f(k, n)=\sharp L_{f}$ is equal to the number of possibilities of arranging $(n+1)$ balls (the elements of $V$ ) in $(k+2)$ cells $\left(E_{0}, E_{1}, \ldots, E_{k+1}\right)$ in such a way that the first $(k+1)$ cells $\left(E_{0}, E_{1}, \ldots, E_{k}\right)$ are not-empty.
We denote the number of possibilities of arranging $n$ balls in $k$ cells in such a way that no cell is empty by $N(k, n)$. Then with ([F]) we get:

$$
N(k, n)=\sum_{i=0}^{k}(-1)^{i}\binom{k}{i}(k-i)^{n} .
$$

We deduce that

$$
\begin{aligned}
f(k, n) & =N(k+1, n+1)+N(k+2, n+1), \\
b(k, n) & =N(k+1, n+1)
\end{aligned}
$$

and the theorem follows.
Let $T$ be a regular $n$-dimensional simplex in $\mathbb{X}^{n}, b(T)$ the barycenter of $T$ and $\left\{H_{1}, \ldots, H_{k}\right\}$ the hyperplanes of symmetry of $T$. The (from these planes) induced decomposition of $T$ is the barycentric decomposition and this decomposition can be extended to a decomposition of the whole space $\mathbb{X}^{n}$ into $k$-dimensional open cones with apex $b(T)$. Let $h(k, n)$ be the number of $k$-dimensional cones in this decomposition of $\mathbb{X}^{n}$. Then we have

$$
\begin{aligned}
h(k, n) & =b(k, n) \\
& =(k+1)!S_{n+1}^{k+1}
\end{aligned}
$$

for all $k$ with $0 \leq k \leq n$.
Lemma 9.2.1 For all numbers $n \in \mathbb{N}$ and all $k$ with $0<k \leq n$ we have

$$
b(k, n)=(k+1)(b(k-1, n-1)+b(k, n-1))
$$

Proof: We have

$$
\begin{aligned}
(k+1)(b(k-1, n-1)+b(k, n-1)) & =(k+1)\left(k!S_{n}^{k}+(k+1)!S_{n}^{k-1}\right) \\
& =(k+1)!\left(S_{n}^{k}+(k+1) S_{n}^{k-1}\right) \\
& =(k+1)!S_{n+1}^{k+1} \\
& =b(k, n)
\end{aligned}
$$

where we have used a slight modification of equation (2) in Section 4.1.

### 9.3 Angles and Combinatorics

In this section we will develop some connections between the combinatorics and the angles of a polytopal complex.

Theorem 9.3.1 Let $\Pi$ be a pure $n$-dimensional complex, $k$ an integer with $0 \leq k \leq n$ and $\kappa^{k}$ a combinatorial $k$-invariant. Then we have

$$
\sum_{P^{n} \in \Omega^{n}(\Pi)} \sum_{P^{k} \in \Omega^{k}\left(P^{n}\right)} \kappa^{k}\left(P^{k}\right) \alpha_{n-k-1}\left(P^{k} \mid P^{n}\right)=\sum_{P^{k} \in \Omega^{k}(\Pi)} \kappa^{k}\left(P^{k}\right) \alpha_{n-k-1}^{\Pi}\left(P^{k}\right)
$$

Proof: Let $\Pi$ be a pure $n$-dimensional complex and $\kappa^{k}$ a combinatorial $k$-invariant for all $k$ with $0 \leq k \leq n$. Then by definition we have:

$$
\kappa^{k}\left(P^{k}\right) \alpha_{n-k-1}^{\Pi}\left(P^{k}\right)=\kappa^{k}\left(P^{k}\right) \sum_{P^{n} \in \Omega^{n}(\Pi)} \alpha_{n-k-1}\left(P^{k} \mid P^{n}\right)
$$

Summing over all $k$-dimensional elements in $\Omega^{k}(\Pi)$ we get

$$
\begin{aligned}
\sum_{P^{k} \in \Omega^{k}(\Pi)} \kappa^{k}\left(P^{k}\right) \alpha_{n-k-1}^{\Pi}\left(P^{k}\right)= & \sum_{P^{k} \in \Omega^{k}(\Pi)} \sum_{P^{n} \in \Omega^{n}(\Pi)} \kappa^{k}\left(P^{k}\right) \alpha_{n-k-1}\left(P^{k} \mid P^{n}\right) \\
= & \sum_{P^{n} \in \Omega^{n}(\Pi)} \sum_{\substack{P^{k} \in \Omega^{k}(\Pi) \\
P^{k} \in \Omega^{k}\left(P^{n}\right)}} \kappa^{k}\left(P^{k}\right) \alpha_{n-k-1}\left(P^{k} \mid P^{n}\right) \\
& +\sum_{P^{n} \in \Omega^{n}(\Pi)} \sum_{\substack{P^{k} \in \Omega^{k}(\Pi) \\
P^{k} \notin \Omega^{k}\left(P^{n}\right)}} \kappa^{k}\left(P^{k}\right) \alpha_{n-k-1}\left(P^{k} \mid P^{n}\right) \\
= & \sum_{P^{n} \in \Omega^{n}(\Pi)} \sum_{\substack{P^{k} \in \Omega^{k}(\Pi) \\
P^{k} \in \Omega^{k}\left(P^{n}\right)}} \kappa^{k}\left(P^{k}\right) \alpha_{n-k-1}\left(P^{k} \mid P^{n}\right)
\end{aligned}
$$

because $\alpha_{n-k-1}\left(P^{k} \mid P^{n}\right)=0$ for $P^{k} \notin \Omega^{k}\left(P^{n}\right)$. This completes the proof.
The following two theorems will show how the angles of the decomposition polytopes are related to the angles of the basic $n$-dimensional polytope. In particular, we cannot expect that the sum of all angles (of a fixed dimension) with the same apex in the decomposition will give an angle of the same dimension in the polytope. In the following we use the fact that all angles of arbitrary dimension can be measured as the volume of a part of an $(n-1)$-dimensional sphere $S^{n-1}(x, \epsilon)$ with center in an inner point $x$ of the apex of this angle.

Theorem 9.3.2 Let $P$ be an n-dimensional polytope in $\mathbb{X}^{n}, \mathcal{D}=\mathcal{D}(P)$ a polytopal decomposition of $P, D \in \Omega_{l}^{k}(P, \mathcal{D})$ with $0 \leq k \leq l \leq n$ and $P^{l}=P^{l}(D)$ the face of $P$ with $\operatorname{ri}(D) \subset$ ri $\left(P^{l}\right)$. Then we have

$$
\alpha_{n-k-1}^{\mathcal{D}}(D)=\alpha_{n-l-1}\left(P^{l}\right)
$$

Proof: For $D \in \Omega_{l}^{k}(P, \mathcal{D})$ where $0 \leq k \leq l \leq n$, let $P^{l}=P^{l}(D)$ be the uniquely determined face of $P$ with $r i(D) \subset \operatorname{ri}\left(P^{l}\right)$ (compare Lemma 9.1.1). We can find an inner point $x$ in $D$ and $\epsilon>0$ such that the sphere $S^{n-1}(x, \epsilon)$ only intersects elements in $\Omega(\mathcal{D})$ which are incident with $D$. Hence there are finitely many maximally dimensional elements $D_{1}^{n}, \ldots, D_{h}^{n} \in \Omega_{n}^{n}(P, \mathcal{D})$ such that

$$
\alpha_{n-k-1}^{\mathcal{D}}(D)=\sum_{i=1}^{h} \alpha_{n-k-1}\left(D \mid D_{i}^{n}\right)
$$

Here we have used that every angle of arbitrary dimension can be measured ( $n-1$ )-dimensionally.


Figure 11: A Decomposition of a Cube

Of course, $x \in \operatorname{ri}(D) \subset r i\left(P^{l}\right)$ is an inner point of $P^{l}$ and the sphere $S^{n-1}(x, \epsilon)$ intersects only elements in $\Omega(P)$ which are incident with $P^{l}$. Furthermore, the set $\left\{D_{i}^{n} \cap S^{n-1}(x, \epsilon): i=\right.$ $1, \ldots, l\}$ can be viewed as a polytopal decomposition of $P \cap S^{n-1}(x, \epsilon)$ and

$$
\alpha_{n-k-1}\left(D \mid D_{i}^{n}\right)=c_{n-1}(\epsilon)^{-1} \operatorname{vol}\left(D_{i}^{n} \cap S^{n-1}(x, \epsilon)\right)
$$

We denote by $c_{n-1}(\epsilon):=\operatorname{vol}\left(S^{n-1}(x, \epsilon)\right)$ the volume of the $(n-1)$-dimensional sphere $S^{n-1}(x, \epsilon)$ of radius $\epsilon$. It follows that

$$
\alpha_{n-k-1}^{\mathcal{D}}(D)=c_{n-1}(\epsilon)^{-1} \sum_{i=1}^{h} \operatorname{vol}\left(D_{i}^{n} \cap S^{n-1}(x, \epsilon)\right)
$$

$$
\begin{aligned}
& =c_{n-1}(\epsilon)^{-1} \operatorname{vol}\left(P^{n} \cap S^{n-1}(x, \epsilon)\right) \\
& =\alpha_{n-l-1}\left(P^{l}\right)
\end{aligned}
$$

Theorem 9.3.3 Let $\Pi$ be a pure $n$-dimensional polytopal complex in $\mathbb{X}^{n}, \mathcal{D}=\mathcal{D}(\Pi)$ a polytopal decomposition of $\Pi, D \in \Omega_{l}^{k}(\Pi, \mathcal{D})$ with $0 \leq k \leq l \leq n$ and $P^{l}=P^{l}(D)$ the face of $\Pi$ with $r i(D) \subset r i\left(P^{l}\right)$. Then we have

$$
\alpha_{n-k-1}^{\mathcal{D}}(D)=\alpha_{n-l-1}^{\Pi}\left(P^{l}\right)
$$

Proof: Let $D \in \Omega_{l}^{k}(\Pi, \mathcal{D})$ with $0 \leq k \leq l \leq n$ and $P^{l}=P^{l}(D)$ be the uniquely determined face of $\Pi$ with $r i(D) \subset r i\left(P^{l}\right)$ (compare Lemma 9.1.1). Then there are finitely many maximally dimensional elements $D_{1}^{n}, \ldots, D_{h}^{n} \in \Omega_{n}^{n}(\Pi, \mathcal{D})$ such that

$$
\alpha_{n-k-1}^{\mathcal{D}}(D)=\sum_{i=1}^{h} \alpha_{n-k-1}\left(D \mid D_{i}^{n}\right)
$$

Furthermore, there exist finitely many elements $P_{1}^{n}, \ldots, P_{q}^{n} \in \Omega^{n}(\Pi)$ and an arrangement

$$
\begin{array}{rll}
D_{1,1}^{n}, D_{1,2}^{n} & , \ldots, & D_{1, g(1)}^{n} \\
D_{2,1}^{n}, D_{2,2}^{n} & , \ldots, & D_{2, g(2)}^{n} \\
\ldots & & \\
D_{q, 1}^{n}, D_{q, 2}^{n} & , \ldots, & D_{q, g(q)}^{n}
\end{array}
$$

of the elements $D_{1}^{n}, \ldots, D_{h}^{n}$ such that $P_{i}^{n}$ is the union of $D_{i, 1}^{n}, \ldots, D_{i, g(i)}^{n}$ which means

$$
P_{i}^{n}=\bigcup_{j=1}^{g(i)} D_{i, j}^{n}
$$

for all $i=1, \ldots, q$. Of course, $P^{l}$ is a face of each $P_{i}^{n}$. We have

$$
\begin{aligned}
\sum_{s=1}^{g(i)} \alpha_{n-k-1}\left(D \mid D_{i, s}^{n}\right) & =\alpha_{n-k-1}\left(D \mid P_{i}^{n}\right) \\
& =\alpha_{n-k-1}^{\mathcal{D} \cap P_{i}^{n}}(D)
\end{aligned}
$$

for all $i=1, \ldots, q$, where $\mathcal{D} \cap P_{i}^{n}$ denotes the decomposition of $P_{i}^{n}$ induced by $\mathcal{D}$. We follow that

$$
\begin{aligned}
\alpha_{n-k-1}^{\mathcal{D}}(D) & =\sum_{i=1}^{h} \alpha_{n-k-1}\left(D \mid D_{i}^{n}\right) \\
& =\sum_{i=1}^{q} \sum_{s=1}^{g(i)} \alpha_{n-k-1}\left(D \mid D_{i, s}^{n}\right)
\end{aligned}
$$

$$
\begin{aligned}
& =\sum_{i=1}^{q} \alpha_{n-k-1}^{\mathcal{D} \cap P_{i}^{n}}(D) \\
& =\sum_{i=1}^{q} \alpha_{n-l-1}\left(P^{l} \mid P_{i}^{n}\right) \\
& =\alpha_{n-l-1}^{\Pi}\left(P^{l}\right),
\end{aligned}
$$

where we have used Theorem 9.3.1 in the forth step.
Furthermore we have the following generalization of the latest theorem.
Theorem 9.3.4 Let $\Pi$ be a pure n-dimensional complex in $\mathbb{X}^{n}$ and $\mathcal{D}=\mathcal{D}(\Pi)$ a polytopal decomposition of $\Pi$. Then we have for all $k$ with $0 \leq k \leq n$ and all $l$ with $k \leq l \leq n$ :

$$
\sum_{D \in \Omega_{l}^{k}(\Pi, \mathcal{D})} \phi(D) \alpha_{n-k-1}^{\mathcal{D}}(D)=\sum_{P^{l} \in \Omega^{l}(\Pi)}\left(\sum_{\substack{D \in \Omega_{l}^{k}(\Pi, \mathcal{D}) \\ r i(D) \subset r i(P l)}} \phi(D)\right) \alpha_{n-l-1}^{\Pi}\left(P^{l}\right),
$$

where all $\phi(D)$ are arbitrary real numbers, which only depend on the polytope $D$ (for instance, they are combinatorial $k$-invariants).

Proof: Let $\Pi \subset \mathbb{X}^{n}$ be a pure $n$-dimensional complex and $\mathcal{D}=\mathcal{D}(\Pi)$ be an arbitrary polytopal decomposition of $\Pi$. Then we have $|\mathcal{D}(\Pi)|=|\Pi|$. Furthermore let

$$
\begin{aligned}
\phi: \mathcal{D}(\Pi) & \longrightarrow \mathbb{R} \\
D & \longmapsto \phi(D)
\end{aligned}
$$

be an arbitrary map. For example, $\phi$ may be a combinatorial $k$-invariant (but in general $\phi$ is allowed to map combinatorially equal decomposition polytopes to different natural numbers) or the $k$-dimensional volume of the decomposition polytope $D$. Now let $k$ and $l$ be natural numbers with $0 \leq k \leq n$ and $k \leq l \leq n$. It is clear that for each $D \in \Omega_{l}^{k}(\Pi, \mathcal{D})$ there exists a uniquely determined $l$-dimensional face $P^{l}=P^{l}(D) \in \Omega^{l}(\Pi)$ such that $|D| \subset\left|P^{l}\right|$ and there exists no lower-dimensional face of $\Pi$ which also contains $D$ (compare Lemma 9.1.1). Now we sum over all elements $D$ in $\Omega_{l}^{k}(\Pi, \mathcal{D})$ and with with Theorem 9.3.3 in the first step and Lemma 9.1.3 in the second step we get

$$
\begin{aligned}
& \sum_{D \in \Omega_{l}^{k}(\Pi, \mathcal{D})} \phi(D) \alpha_{n-k-1}^{\mathcal{D}}(D) \\
= & \sum_{D \in \Omega_{l}^{k}(\Pi, \mathcal{D})} \phi(D) \alpha_{n-l-1}^{\Pi}\left(P^{l}(D)\right) \\
= & \left(\sum_{\substack{D \in \Omega_{l}^{k}(\Pi, \mathcal{D}) \\
r i(D) \subset r i(P l}} \phi(D)\right) \alpha_{n-l-1}^{\Pi}\left(P_{1}^{l}\right)+\ldots+\left(\sum_{\substack{D \in \Omega_{l}^{k}(\Pi, \mathcal{D}) \\
r i(D) \subset r i\left(P a_{a}^{l}(\Pi)\right.}} \phi(D)\right) \alpha_{n-l-1}^{\Pi}\left(P_{a^{l}(\Pi)}^{l}\right) \\
= & \sum_{P^{l} \in \Omega^{l}(\Pi)}\left(\sum_{\substack{D \in \Omega_{l}^{k}(\Pi, \mathcal{D}) \\
r i(D) \subset r i(P l)}} \phi(D)\right) \alpha_{n-l-1}^{\Pi}\left(P^{l}\right)
\end{aligned}
$$

where $\Omega^{l}(\Pi)=\left\{P_{1}^{l}, \ldots, P_{a^{l}(\Pi)}^{l}\right\}$ is the set of all $l$-dimensional faces of the complex $\Pi$ as usual. In the special case where all the numbers $\phi(D)$ are 1 we have the following result.

Corollary 9.3.1 With the same notations as in Theorem 9.3.4 we have for all pure dimensional complexes $\Pi$ and decompositions $\mathcal{D}$ :

$$
\sum_{D \in \Omega_{l}^{k}(\Pi, \mathcal{D})} \alpha_{n-k-1}^{\mathcal{D}}(D)=\sum_{P^{l} \in \Omega^{l}(P)} z\left(k, P^{l}, \mathcal{D}\right) \alpha_{n-l-1}^{\Pi}\left(P^{l}\right) .
$$

Proof: We have

$$
\begin{aligned}
\sum_{\substack{D \in \Omega_{l}^{k}(\Pi, \mathcal{D}) \\
\text { ri(D)Cri(Pl)}}} 1 & =\sharp\left\{D \in \Omega_{l}^{k}(\Pi, \mathcal{D}): r i(D) \subset r i\left(P^{l}\right)\right\} \\
& =z\left(k, P^{l}, \mathcal{D}\right)
\end{aligned}
$$

with Lemma 9.1.4, and the Corollary follows immediately.
In the special case where the pure complex $\Pi$ is a polytope $P$, the complex angle $\alpha_{n-l-1}^{\Pi}\left(P^{l}\right)$ is equal to the ordinary angle $\alpha_{n-l-1}\left(P^{l}\right)$ of $P$ in the face $P^{l}$. So we get the following result.

Corollary 9.3.2 Let $P$ be an n-polytope in $\mathbb{X}^{n}$ and $\mathcal{D}=\mathcal{D}(P)$ a polytopal decomposition of $P$. Then we have for all $k$ with $0 \leq k \leq n$ and all $l$ with $k \leq l \leq n$ :

$$
\sum_{D \in \Omega_{l}^{k}(P, \mathcal{D})} \phi(D) \alpha_{n-k-1}^{\mathcal{D}}(D)=\sum_{P^{l} \in \Omega^{l}(P)}\left(\sum_{\substack{\left.D \in \Omega_{l}^{k}(P, \mathcal{D}) \\ \text { ri( } D\right)\left(\mathrm{ri}\left(P^{l}\right)\right.}} \phi(D)\right) \alpha_{n-l-1}\left(P^{l}\right),
$$

where all $\phi(D)$ are arbitrary natural numbers, which only depend on the polytope $D$ (for instance, they are combinatorial $k$-invariants).

Example 9.3.1 Let $P=\operatorname{conv}\left(P_{1}^{0}, P_{2}^{0}, P_{3}^{0}, P_{4}^{0}\right)$ be a 2-dimensional polytope and $\mathcal{D}=\mathcal{D}(P)$ the polytopal decomposition in Figur 12. We denote by $D_{1}^{2}=\operatorname{conv}\left(Q, R, P_{1}^{0}, P_{2}^{0}\right)$ and $D_{2}^{2}=$ $\operatorname{conv}\left(Q, R, P_{3}^{0}, P_{4}^{0}\right)$ the two decomposition polytopes of maximal dimension in $\mathcal{D}$. Then we have

$$
\begin{aligned}
\sum_{D^{0} \in \Omega_{0}^{0}(P, \mathcal{D})} \alpha_{1}^{\mathcal{D}}\left(D^{0}\right) & =\alpha_{1}^{\mathcal{D}}\left(P_{1}^{0}\right)+\alpha_{1}^{\mathcal{D}}\left(P_{2}^{0}\right)+\alpha_{1}^{\mathcal{D}}\left(P_{3}^{0}\right)+\alpha_{1}^{\mathcal{D}}\left(P_{4}^{0}\right) \\
& =\sum_{P^{0} \in \Omega^{0}(P)} z\left(0, P^{0}, \mathcal{D}\right) \alpha_{1}\left(P^{0}\right) ; \\
\sum_{D^{0} \in \Omega_{1}^{0}(P, \mathcal{D})} \alpha_{1}^{\mathcal{D}}\left(D^{0}\right) & =\alpha_{1}^{\mathcal{D}}(R)+\alpha_{1}^{\mathcal{D}}(Q) \\
& =\alpha_{1}\left(R \mid D_{1}^{2}\right)+\alpha_{1}\left(R \mid D_{2}^{2}\right)+\alpha_{1}\left(Q \mid D_{1}^{2}\right)+\alpha_{1}\left(Q \mid D_{2}^{2}\right) ; \\
& =\frac{1}{2}+\frac{1}{2} \\
& =\alpha_{0}\left(\overline{P_{2}^{0} P_{3}^{0}} \mid P\right)+\alpha_{0}\left(\overline{P_{1}^{0} P_{4}^{0}} \mid P\right)
\end{aligned}
$$

$$
\begin{aligned}
& =\sum_{P^{1} \in \Omega^{1}(P)} z\left(0, P^{1}, \mathcal{D}\right) \alpha_{0}\left(P^{1}\right) ; \\
& \sum_{D^{0} \in \Omega_{2}^{0}(P, \mathcal{D})} \alpha_{1}^{\mathcal{D}}\left(D^{0}\right)=0 ; \\
& \sum_{D^{1} \in \Omega_{1}^{1}(P, \mathcal{D})} \alpha_{0}^{\mathcal{D}}\left(D^{1}\right)=\alpha_{0}^{\mathcal{D}}\left(\overline{P_{1}^{0} P_{2}^{0}}\right)+\alpha_{0}^{\mathcal{D}}\left(\overline{P_{2}^{0} Q}\right)+\alpha_{0}^{\mathcal{D}}\left(\overline{Q P_{3}^{0}}\right) \\
& +\alpha_{0}^{\mathcal{D}}\left(\overline{P_{3}^{0} P_{4}^{0}}\right)+\alpha_{0}^{\mathcal{D}}\left(\overline{P_{4}^{0} R}\right)+\alpha_{0}^{\mathcal{D}}\left(\overline{R P_{1}^{0}}\right) \\
& =6 \frac{1}{2} \\
& =\alpha_{0}\left(\overline{P_{1}^{0} P_{2}^{0}} \mid P\right)+2 \alpha_{0}^{\mathcal{D}}\left(\overline{P_{2}^{0} P_{3}^{0}} \mid P\right)+\alpha_{0}\left(\overline{P_{3}^{0} P_{4}^{0}} \mid P\right)+2 \alpha_{0}^{\mathcal{D}}\left(\overline{P_{1}^{0} P_{4}^{0}} \mid P\right) \\
& =\sum_{P^{1} \in \Omega^{1}(P)} z\left(1, P^{1}, \mathcal{D}\right) \alpha_{0}\left(P^{1}\right) ; \\
& \sum_{D^{1} \in \Omega_{2}^{1}(P, \mathcal{D})} \alpha_{0}^{\mathcal{D}}\left(D^{1}\right)=\alpha_{0}^{\mathcal{D}}(\overline{Q R}) \\
& =1 \\
& =\sum_{P^{2} \in \Omega^{2}(P)} z\left(1, P^{2}, \mathcal{D}\right) \alpha_{0}\left(P^{2}\right) \\
& =z(1, P, \mathcal{D}) \alpha_{-1}(P) ; \\
& \sum_{D^{2} \in \Omega_{2}^{2}(P, \mathcal{D})} \alpha_{-1}^{\mathcal{D}}\left(D^{2}\right)=\alpha_{-1}^{\mathcal{D}}\left(D_{1}^{2}\right)+\alpha_{-1}^{\mathcal{D}}\left(D_{2}^{2}\right) \\
& =2 \\
& =\sum_{P^{2} \in \Omega^{2}(P)} z\left(2, P^{2}, \mathcal{D}\right) \alpha_{-1}\left(P^{2}\right) \\
& =z(2, P, \mathcal{D}) \alpha_{-1}(P) .
\end{aligned}
$$



Figure 12: A Polytopal Decomposition

## 10 Simplices and Volume

In this section we will recall some well-known facts about simplices in the spaces $\mathbb{X}^{n}=\mathbb{S}^{n}$ or $\mathbb{H}^{n}$.

### 10.1 Schläfli's Differential Formula for Simplices

This formula was established by L. Schläfli for spherical simplices [Sch]. Much later, H. Kneser [Kn] gave a second proof for both spherical and hyperbolic simplices. This proof was worked out by J. Böhm (compare [BH]). Another functional analytic proof due to J. Milnor (compare [Mi]).

Theorem 10.1.1 (Schläfli's Differential Formula) Let $T$ be an $n$-dimensional simplex in the space $\mathbb{X}^{n}=\mathbb{S}^{n}$ or $\mathbb{H}^{n}(n \geq 2)$. If the simplex $T$ is deformed differentially in such a way that its combinatorial stucture does not change, then the volume of $T$ changes differentially and we have

$$
K d \operatorname{vol}_{\mathbb{X}^{n}}(T)=\frac{1}{n-1} \sum_{T^{n-2} \in \Omega^{n-2}(T)} \operatorname{vol}_{\mathbb{X}^{n}}\left(T^{n-2}\right) d \alpha_{1}\left(T^{n-2}\right) \quad \operatorname{vol}_{\mathbb{X}^{0}}\left(T^{0}\right)=1
$$

### 10.2 The Volume Function of a Simplex in $\mathbb{X}^{n}$

Let $T=T^{n}$ be an $n$-dimensional simplex in the space $\mathbb{X}^{n}=\mathbb{S}^{n}$ or $\mathbb{H}^{n}$. The Gram matrix $G=G(T)$ of $T$, which is defined by the dihedral angles of the simplex, defines $T$ up to an isometry. Therefore, the volume $\operatorname{vol}_{\mathbb{X}^{n}}(T)$ is a function of the dihedral angles.

Let $N=n(n+1) / 2$ and agree to number the coordinates of vectors in $\mathbb{C}^{N}$ by unordered pairs of integers $i, j$, where $i, j=0,1, \ldots, n ; i \neq j$. For each vector $\alpha \in(0, \pi)^{N} \subset \mathbb{C}^{N}$ we denote by $G(\alpha)$ the symmetric matrix of order $(n+1)$ with 1 's on the main diagonal and $-\cos \left(\alpha_{i j}\right)$ off it. Now we denote by $M_{+}$(respectively $M_{0}$ and $M_{-}$) the family of all sets of dihedral angles for simplices in $\mathbb{S}^{n}\left(\mathbb{E}^{n}, \mathbb{H}^{n}\right)$. Furthermore, let $M=M_{+} \cup M_{0} \cup M_{-}$and for each $\alpha \in M$ denote by $T(\alpha)$ the simplex whose dihedral angles are the entries of $\alpha$. Now we have the following result (compare [V1], Theorem 2.1, page 117).

Theorem 10.2.1 For each even integer $n \geq 0$ there exists an analytic function $v$ which is defined on an open subset of the space $\mathbb{C}^{N}$ containing the set $M$ and assumes the following values on $M$ :

$$
v(\alpha)=\left\{\begin{array}{lll}
\operatorname{vol}_{\mathbb{S}^{n}} T(\alpha) & , \quad \alpha \in M_{+} \\
0 & , \quad \alpha \in M_{0} . \\
(-1)^{\frac{n}{2}} \operatorname{vol}_{\mathbb{H}^{n}} T(\alpha) & , \quad \alpha \in M_{-} .
\end{array}\right.
$$

For each odd integer $n \geq 0$ there exists a two-valued function $v$ which is defined on an open subset of the space $\mathbb{C}^{N}$ containing the set $M$, ramifies on the set of $\alpha$ with $\operatorname{det} G(\alpha)=0$ and assumes the following values on $M$ :

$$
v(\alpha)=\left\{\begin{array}{lll} 
\pm \operatorname{vol}_{\mathbb{S}^{n}} T(\alpha) & , & \alpha \in M_{+} \\
0 & , & \alpha \in M_{0} \\
\pm i \operatorname{vol}_{\mathbb{H}^{n}} T(\alpha) & , & \alpha \in M_{-}
\end{array} .\right.
$$

### 10.3 Angle Sums and Poincaré's Formula

Let $T$ be an $n$-dimensional simplex in $\mathbb{S}^{n}$ and for all $-1 \leq k \leq n$ let $\omega_{n-k-1}$ be the sum of all ( $n-k-1$ )-dimensional angles and let $W(T)$ be the generalized angle sum of $P$. H. Poincaré [Po] proved in 1905 that

$$
W(T)=\left\{\begin{array}{lll}
2 c_{2 m}^{-1} \operatorname{vol}_{X 2 m}(T) & , & n=2 m \text { even } \\
0 & , & n=2 m+1 \text { odd }
\end{array} .\right.
$$

For the proof we can use the fact that the volume of a spherical simplex $T$ is equal to the volume of its antipodal simplex $T^{a}$. By expressing these volumes by the integrals of the characteristic functions we get the result (compare [V2], page 120). This result was generalized by H. Hopf to simplices in all spaces $\mathbb{X}^{n}$ of constant sectional curvature $K$ (compare [Hop], Theorem III) in the following way.

Theorem 10.3.1 (Poincaré's Formula for Simplices) Let $T$ be an n-dimensional simplex in the space $\mathbb{X}^{n}=\mathbb{S}^{n}, \mathbb{E}^{n}$ or $\mathbb{H}^{n}$. Then we have

$$
W(T)=\left\{\begin{array}{lll}
2 K^{m} & c_{2 m}^{-1} \operatorname{vol}_{\mathbb{X} 2 m}(T) & , \\
0 & , & n=2 m \text { even } \\
0 &
\end{array} .\right.
$$

Remark: If we use Theorem 10.2 .1 we can follow the first equation (for even $n=2 m$ ) in Theorem 10.3.1 from Poincaré's result in the spherical case. The function $v$ can be written as

$$
v(T(\alpha))=v(\alpha)=\frac{1}{2}(-1)^{m} c_{2 m} W(T(\alpha)) .
$$

### 10.4 Schläfli's Reduction Formula and Peschl's Relations for Simplices

Let $T$ be an $n$-dimensional simplex in $\mathbb{X}^{n}$. Each angle of $T$ is equal to the (normalized) volume of a spherical simplex. We can use Poincaré's Formula to eliminate the volumes of even-dimensional angles. So we get a volume formula for $T$ which depends only on the odd-dimensional angles and the combinatorics of $T$. This formula is called Schläfli's Reduction Formula, first proved by L. Schläfli [Sch] in 1901 by using his differential formula. E. Peschl proved this formula and other linear relations between the angle sums, called Peschl's Relations, in a purely combinatorial way (compare $[\mathrm{Pe}]$ ) in 1955 by using Poincaré's Formula.

Theorem 10.4.1 (Schläfi-Peschl Relations) Let $T$ be an $n$-dimensional simplex in $\mathbb{X}^{n}$. Then we have the following relations between the angle sums of $T$ :

$$
\omega_{2 l}(T)=\sum_{k=0}^{l}(-1)^{k} a_{2 k+1}\binom{n-2 l+2 k+1}{2 k+1} \omega_{2 l-2 k-1}(T)
$$

for all $l$ with $0 \leq l<[n / 2]$ and

$$
W(T)=\left\{\begin{array}{ll}
2 \sum_{k=0}^{m}(-1)^{k} a_{2 k+1} \omega_{2 m-2 k-1}(T) & , \quad n=2 m \quad \text { even } \\
0 & , \quad n=2 m+1 \text { odd }
\end{array} .\right.
$$

Furthermore, these relations are a complete system of linearly independent linear relations between the $\omega_{i}$ 's.

Remark: In the special case of a $2 m$-dimensional Euclidian simplex $T$ in $\mathbb{E}^{n}$ (this means $K=0$ ) we get

$$
\omega_{2 m-1}(T)=\sum_{k=1}^{m}(-1)^{k+1} a_{2 k+1} \omega_{2 m-2 k-1}(T)
$$

### 10.5 The Schläfli-Kellerhals Reduction Formula for Orthoschemes

Let $R$ be an orthoscheme (of degree 0 ) in the space $\mathbb{X}^{2 m}=\mathbb{S}^{2 m}$ or $\mathbb{H}^{2 m}$. We define $\Omega^{d}(R)^{*}$ as the set of all elements $R^{d} \in \Omega^{d}(R)$ such that the scheme $S\left(R^{d}\right)$ is elliptic and all of its components are of even order.
The special geometric properties of orthoschemes can be used to get a simpler reduction formula. This formula was proved by L. Schläffi [Sch] in 1901 for spherical orthoschemes and generalized by R. Kellerhals [Ke] in 1991 to hyperbolic orthoschemes of all degrees. Both proofs use Schläfli's Differential Formula. It is remarkable that this formula has the same structure for orthoschemes of all degrees though they are of different combinatorial types.

Theorem 10.5.1 (Schläfli-Kellerhals Formula) Let $R$ be a $2 m$-dimensional orthoscheme of degree 0 in the space $\mathbb{X}^{2 m}=\mathbb{S}^{2 m}$ or $\mathbb{H}^{2 m}$. Then

$$
2 K^{m} c_{2 m}^{-1} \operatorname{vol}_{\mathbb{X} 2 m}(R)=\sum_{k=0}^{m} q_{k+1} \omega_{2 m-2 k-1}^{*}(R)
$$

with

$$
\omega_{2 m-2 k-1}^{*}(R)=\sum_{R^{2 k} \in \Omega^{d}(R)^{*}} \alpha_{2 m-2 k-1}\left(R^{2 k}\right)
$$

The numbers $q_{k+1}$ are the coefficients of the Taylor expansion of $\sqrt{1+x}$ (compare Lemma 4.2.1).

## 11 Volumes and Angle Sums of Polytopal Complexes

It is clear that all polytopes, and so all polytopal complexes, can be decomposed into simplices. With the results of the previous section we are able to transfer facts from section 8 to polytopal complexes. Furthermore, we will derive some well-known theorems by this machinery.

### 11.1 Schläfli's Differential Formula for Polytopes

In this section we will generalize Schläfli's Differential Formula (Theorem 10.1.1) from simplices to polytopes. Schläfli remarked that this is possible by decomposition of a polytope into simplices (compare [Sch], page 273). Now we will work out this idea.

Theorem 11.1.1 (Schläfli's Differential Formula for Polytopes) Let P be an n-dimensional polytope in the space $\mathbb{X}^{n}=\mathbb{H}^{n}$ or $\mathbb{S}^{n}$. For all $P^{n-2}$ in $\Omega^{n-2}(P)$ let $\alpha_{1}\left(P^{n-2}\right)$ be the 1dimensional angle of $P$ with apex $P^{n-2}$. If the polytope $P$ is deformed differentially in such a way that its combinatorial stucture does not change, then the volume of $P$ changes differentially and we have

$$
K d v o l_{\mathbb{X}^{n}}(P)=\frac{1}{n-1} \sum_{P^{n-2} \in \Omega^{n-2}(P)} \operatorname{vol}_{\mathbb{X}^{n}}\left(P^{n-2}\right) d \alpha_{1}\left(P^{n-2}\right) \quad \operatorname{vol}_{\mathbb{X} 0}\left(P^{0}\right)=1
$$

Proof: We have proved this result in the special case where $P$ is a simplex (see Theorem 10.1.1). Now, let $P=P^{n}$ be an $n$-dimensional polytope in $\mathbb{X}^{n}$ and $\mathcal{D}=\mathcal{D}(P)$ an arbitrary simplicial decomposition of $P$. Then we have

$$
\begin{aligned}
\operatorname{Kdvol}_{\mathbb{X}^{n}}(P) & =\sum_{T^{n} \in \Omega^{n}(\mathcal{D})} K d \operatorname{vol}_{\mathbb{X}^{n}}\left(T^{n}\right) \\
& =\frac{1}{n-1} \sum_{T^{n} \in \Omega^{n}(\mathcal{D})} \sum_{T^{n-2} \in \Omega^{n-2}\left(T^{n}\right)} \operatorname{vol}_{\mathbb{X}^{n}}\left(T^{n-2}\right) d \alpha_{1}\left(T^{n-2}\right) \\
& =\frac{1}{n-1} \sum_{T^{n-2} \in \Omega^{n-2}(\mathcal{D})} \operatorname{vol}_{\mathbb{X}^{n} n}\left(T^{n-2}\right) d \alpha_{1}^{\mathcal{D}}\left(T^{n-2}\right) \\
& =\frac{1}{n-1} \sum_{i=n-2}^{n} \sum_{T^{n-2} \in \Omega_{i}^{n-2}(P, \mathcal{D})} \operatorname{vol}_{\mathbb{X}^{n}\left(T^{n-2}\right)} d \alpha_{1}^{\mathcal{D}}\left(T^{n-2}\right) \\
& =\frac{1}{n-1} \sum_{i=n-2}^{n} \sum_{P^{i} \in \Omega^{i}(P)}\left(\sum_{\begin{array}{c}
T^{n-2} \in \Omega_{i}^{n-2}(P, \mathcal{D}) \\
r i\left(T^{n-2}\right) \subset r i(P i)
\end{array}} \operatorname{vol}_{\mathbb{X}^{n}}\left(T^{n-2}\right)\right) d \alpha_{n-i-1}\left(P^{i}\right) .
\end{aligned}
$$

Here we have used Theorem 9.3.1 in the third step and Theorem 9.3.4 in the forth step. Of course, for $i=n-1$ and $n$ the differential $d \alpha_{n-i-1}\left(P^{i}\right)$ is equal to zero, because the angle $\alpha_{n-i-1}\left(P^{i}\right)$ is constant during the deformation. So we get

$$
K d v o l_{\mathbb{X}^{n}}(P)=\frac{1}{n-1} \sum_{P^{n-2} \in \Omega^{n-2}(P)} \underbrace{\left(\sum_{\sum_{T^{n-2} \in \Omega_{n-2}^{n-2}(P, \mathcal{D})}^{T_{1 i\left(T^{n-2}\right) C r i(P n-2}}} \operatorname{vol}_{\mathbb{X}^{n}}\left(T^{n-2}\right)\right.}_{\operatorname{vol}_{\mathbb{X}} n\left(P^{n-2}\right)} d \alpha_{1}\left(P^{n-2}\right)
$$

$$
=\frac{1}{n-1} \sum_{P^{n-2} \in \Omega^{n-2}(P)} \operatorname{vol}_{\mathbb{X}^{n}}\left(P^{n-2}\right) d \alpha_{1}\left(P^{n-2}\right) .
$$

### 11.2 Generalized Poincaré Relations

Theorem 11.2.1 Let $\Pi$ be a pure $n$-dimensional polytopal complex in $\mathbb{X}^{n}$ and $\operatorname{vol}_{\mathbb{X}}{ }^{2 m}(\Pi)$ be the volume of the geometric realization of $\Pi$ in $\mathbb{X}^{n}$. Then

$$
\sum_{\substack{P j \in \Omega^{j}(\Pi) \\
j=0, \ldots, n}}(-1)^{j} \alpha_{n-j-1}^{\Pi}\left(P^{j}\right)=\left\{\begin{array}{ll}
2 K^{m} c_{2 m}^{-1} \operatorname{vol}_{\mathbb{X}^{2 m}(\Pi)} & , \quad n=2 m \text { even } \\
0 & , \quad n=2 m+1 \text { odd }
\end{array} .\right.
$$

Proof: Let $\Pi$ be a pure $n$-dimensional polytopal complex and $\mathcal{D}=\mathcal{D}(\Pi)$ a simplicial decomposition of $\Pi$. We know that the generalized angle sum of each decomposition simplex is equal to zero in odd dimensions and proportional to the volume in even dimensions (compare Theorem 10.3.1). So we get:

$$
\begin{aligned}
& \left\{\begin{array}{lll}
2 K^{m} c_{2 m}^{-1} \operatorname{vol}_{\mathbb{X} 2 m}(\Pi) & , \quad n=2 m \text { even } \\
0 & , \quad n=2 m+1 \text { odd }
\end{array}\right. \\
& =\sum_{T^{n} \in \Omega^{n}(\mathcal{D})} W\left(T^{n}\right) \\
& =\sum_{T^{n} \in \Omega^{n}(\mathcal{D})} \sum_{i=0}^{n}(-1)^{i} \omega_{n-i-1}\left(T^{n}\right) \\
& =\sum_{T^{n} \in \Omega^{n}(\mathcal{D})} \sum_{i=0}^{n}(-1)^{i} \sum_{T^{i} \in \Omega^{i}\left(T^{n}\right)} \alpha_{n-i-1}\left(T^{i} \mid T^{n}\right) \\
& =\sum_{i=0}^{n}(-1)^{i} \sum_{T^{n} \in \Omega^{n}(\mathcal{D})} \sum_{T^{i} \in \Omega^{i}\left(T^{n}\right)} \alpha_{n-i-1}\left(T^{i} \mid T^{n}\right) \\
& =\sum_{i=0}^{n}(-1)^{i} \sum_{T^{i} \in \Omega^{i}(\mathcal{D})} \alpha_{n-i-1}^{\mathcal{D}}\left(T^{i}\right) \\
& =\sum_{i=0}^{n}(-1)^{i} \sum_{j=i}^{n} \sum_{T^{i} \in \Omega_{j}^{i}(\Pi, \mathcal{D})} \alpha_{n-i-1}^{\mathcal{D}}\left(T^{i}\right) \\
& =\sum_{i=0}^{n}(-1)^{i} \sum_{j=i}^{n} \sum_{P j \in \Omega^{j}(\Pi)} z\left(i, P^{j}, \mathcal{D}\right) \alpha_{n-j-1}^{\Pi}\left(P^{i}\right) \\
& =\sum_{i=0}^{n}(-1)^{i} \sum_{j=0}^{n} \sum_{P^{j} \in \Omega^{j}(\Pi)} z\left(i, P^{j}, \mathcal{D}\right) \alpha_{n-j-1}^{\Pi}\left(P^{i}\right) \\
& =\sum_{j=0}^{n} \sum_{P^{j} \in \Omega^{j}(\Pi)}\left(\sum_{i=0}^{n}(-1)^{i} z\left(i, P^{j}, \mathcal{D}\right)\right) \alpha_{n-j-1}^{\Pi}\left(P^{j}\right) \\
& =\sum_{\substack{P j \in \Omega^{j}(\Pi) \\
j=0, \ldots, n}}\left(\sum_{i=0}^{n}(-1)^{i} z\left(i, P^{j}, \mathcal{D}\right)\right) \alpha_{n-j-1}^{\Pi}\left(P^{j}\right) \text {. }
\end{aligned}
$$

We have used Theorem 9.3.1 in the fifth step and Theorem 9.3.4, resp. Corollary 9.3.2, in the seventh step. Furthermore, we have $z\left(i, P^{j}, \mathcal{D}\right)=0$ for all $i>j$ and with Theorem 9.1.2 the claim follows immediately.
If $\Pi$ is an $n$-dimensional polytope (with all of its faces) we get Poincaré's formula for polytopes.

Corollary 11.2.1 (Poincaré's Formula for Polytopes) Let $P$ be an $n$-dimensional polytope in $\mathbb{X}^{n}$. Then

$$
W(P)=\sum_{\substack{P j \in \Omega^{j}(P) \\
j=0, \ldots, n}}(-1)^{j} \alpha_{n-j-1}\left(P^{j}\right)=\left\{\begin{array}{ll}
2 K^{m} c_{2 m}^{-1} \operatorname{vol}_{\mathbb{X}^{2 m}}(P) & , \quad n=2 m \quad \text { even } \\
0 & , \quad n=2 m+1 \text { odd }
\end{array} .\right.
$$

With Theorem 11.2.1 we can easily compute the Euler-Characteristic of the sphere $\mathbb{S}^{n}$. But this is not an independent proof because we have used this result to deduce Theorem 11.2.1.

Corollary 11.2.2 Let $\Pi$ be a tesselation of the sphere $\mathbb{S}^{n}$. Then

$$
\chi(\Pi)=\left\{\begin{array}{lll}
2 & , & n=2 m \quad \text { even } \\
0 & , & n=2 m+1 \text { odd }
\end{array} .\right.
$$

Proof: We have $\operatorname{vol}_{\mathbb{S} n}(\Pi)=c_{n}$ and each complex angle of $\Pi$ is of measure 1 ( $\mathbb{S}^{n}$ has no boundary). Furthermore, the volume of $\mathbb{S}^{n}$ is equal to $c_{n}$. Hence we get with Theorem 11.2.1

$$
\begin{aligned}
& \begin{cases}2 & n \text { even } \\
0 & , \\
n \text { odd }\end{cases} \\
= & \sum_{\substack{P_{j} \in \Omega^{j}(\Pi) \\
j=0, \ldots, n}}(-1)^{j} \alpha_{n-j-1}^{\Pi}\left(P^{j}\right) \\
= & \sum_{\substack{P_{j} \in \Omega^{j}(P) \\
j=0, \ldots, n}}(-1)^{j} \\
= & a^{0}(\Pi)-a^{1}(\Pi)+\ldots+(-1)^{n} a^{n}(\Pi) \\
= & \chi(\Pi)
\end{aligned}
$$

Furthermore, from Corollary 11.2.1 we can derive the Gram-Sommerville Formula for Euclidean polytopes. In 1874 Gram proved it for polytopes in $\mathbb{E}^{3}$. A similar formula for $n$-dimensional polytopes was proved by Sommerville in 1927 (his proof contained a gap, which was removed by Grünbaum 1967). For a direct proof see for instance at [PT], page 143, Theorem 4 or [G], chapter 14.1.

Proposition 11.2.1 (Gram-Sommerville Formula) Let $P$ be an n-dimensional polytope in $\mathbb{E}^{n}$. Then

$$
W(P)=\omega_{n-1}(P)-\omega_{n-2}(P)+\cdots+(-1)^{n-1} \omega_{0}(P)+(-1)^{n} \omega_{-1}(P)=0
$$

Proof: We know that $K=0$ because $P$ is a Euclidean polytope. Furthermore, we have

$$
\sum_{P^{j} \in \Omega^{j}(P)}(-1)^{j} \alpha_{n-j-1}\left(P^{j}\right)=(-1)^{j} \omega_{n-j-1}(P)
$$

for all $j=0,1, \ldots, n$ and the proposition follows immediately.
Moreover, we can prove the Theorem of Gauß and Bonnet for polytopal $\mathbb{X}^{n}$-manifolds with or without boundary. The following proposition can be viewed as a generalization of results of H . Hopf (compare [Hop], equations (10) and (12)), but we focus our attention to the very special case of pure $n$-dimensional polytopal complexes in $\mathbb{X}^{n}$.

Proposition 11.2.2 (Gauß-Bonnet for Polytopal Complexes) Let $\Pi$ be a pure $n$-dimensional polytopal complex in $\mathbb{X}^{n}$ with boundary complex $\partial \Pi$. Then we have

$$
\chi_{g}(\Pi)-\frac{1}{2} \chi_{g}(\partial \Pi)-r= \begin{cases}2 K^{m} c_{2 m}^{-1} \operatorname{vol}_{\mathbb{X} 2 m}(\Pi) & , \quad n=2 m \quad \text { even } \\ 0 & , \quad n=2 m+1 \text { odd }\end{cases}
$$

where the number $r$ is defined as

$$
r:=\sum_{\substack{P j \in \Omega^{j}(\partial \Pi) \\ j=0, \ldots, n-2}}(-1)^{j} \beta_{n-j-1}^{\Pi}\left(P^{j}\right)
$$

Proof: Let $\Pi$ be a pure $n$-dimensional polytopal complex in $\mathbb{X}^{n}$ with boundary complex $\partial \Pi$. If $P^{j} \in \Omega^{j}(\Pi)-\Omega^{j}(\partial \Pi)$ for some $j=0, \ldots, n$ then the complex angle $\alpha_{n-j-1}^{\Pi}\left(P^{j}\right)$ is equal to one and we get

$$
\begin{aligned}
\sum_{\substack{P^{j} \in \Omega^{j}(\Pi)-\Omega^{j}(\partial \Pi) \\
j=0, \ldots, n}}(-1)^{j} & =\sum_{j=0}^{n}(-1)^{j} \sharp\left(\Omega^{j}(\Pi)-\Omega^{j}(\partial \Pi)\right) \\
& =\sum_{j=0}^{n}(-1)^{j}\left(a^{j}(\Pi)-a^{j}(\partial \Pi)\right) \\
& =\sum_{j=0}^{n}(-1)^{j} a^{j}(\Pi)-\sum_{j=0}^{n-1}(-1)^{j} a^{j}(\partial \Pi) \\
& =\chi_{g}(\Pi)-\chi_{g}(\partial \Pi)
\end{aligned}
$$

Then we have

$$
\begin{aligned}
\sum_{\substack{P^{j} \in \Omega^{j}(\Pi) \\
j=0, \ldots, n}}(-1)^{j} \alpha_{n-j-1}^{\Pi}\left(P^{j}\right)= & \sum_{\substack{P^{j} \in \Omega^{j}(\Pi)-\Omega^{j}(\partial \Pi) \\
j=0, \ldots, n}}(-1)^{j} \alpha_{n-j-1}^{\Pi}\left(P^{j}\right)+\sum_{\substack{P j \in \Omega^{j}(\partial \Pi) \\
j=0, \ldots, n-1}}(-1)^{j} \alpha_{n-j-1}^{\Pi}\left(P^{j}\right) \\
= & \chi_{g}(\Pi)-\chi_{g}(\partial \Pi)+\sum_{\substack{P^{j} \in \Omega^{j}(\partial \Pi) \\
j=0, \ldots, n-1}}(-1)^{j} \alpha_{n-j-1}^{\Pi}\left(P^{j}\right) \\
= & \chi_{g}(\Pi)-\chi_{g}(\partial \Pi)+(-1)^{n-1} \sum_{P^{n-1} \in \Omega^{n-1}(\partial \Pi)} \alpha_{0}^{\Pi}\left(P^{n-1}\right) \\
& +\sum_{\substack{P^{j} \in \Omega^{j}(\partial \Pi) \\
j=0, \ldots, n-2}}(-1)^{j} \alpha_{n-j-1}^{\Pi}\left(P^{j}\right)
\end{aligned}
$$

$$
\begin{aligned}
= & \chi_{g}(\Pi)-\chi_{g}(\partial \Pi)+\frac{1}{2}(-1)^{n-1} a^{n-1}(\partial \Pi) \\
& +\sum_{\substack{P j \in \Omega j \\
j=0, \ldots, n-2}}(-1)^{j} \alpha_{n-j-1}^{\Pi}\left(P^{j}\right) .
\end{aligned}
$$

We have exert with Lemma 3.2.1 in the second step and with $\alpha_{0}^{\Pi}\left(P^{n-1}\right)=\frac{1}{2}$ for all $P^{n-1} \in$ $\Omega^{n-1}(\partial \Pi)$ in the third step. Now we use the exterior angles $\beta_{n-j-1}^{\Pi}\left(P^{j}\right)=\frac{1}{2}-\alpha_{n-j-1}^{\Pi}\left(P^{j}\right)$ and develop the last sum. We find that

$$
\begin{aligned}
\sum_{\substack{P j \in \Omega j(\partial \Pi) \\
j=0, \ldots, n-2}}(-1)^{j} \alpha_{n-j-1}^{\Pi}\left(P^{j}\right) & =\sum_{\substack{P j \in \Omega j(\partial \Pi) \\
j=0, \ldots, n-2}}(-1)^{j}\left(\frac{1}{2}-\beta_{n-j-1}^{\Pi}\left(P^{j}\right)\right) \\
& =\frac{1}{2}\left(\chi_{g}(\partial \Pi)-(-1)^{n-1} a^{n-1}(\partial \Pi)\right)-\sum_{\substack{P j \in \Omega j(\partial \Pi) \\
j=0, \ldots, n-2}}(-1)^{j} \beta_{n-j-1}^{\Pi}\left(P^{j}\right) .
\end{aligned}
$$

So we get

$$
\sum_{\substack{P j \in \Omega j(\Pi) \\ j=0, \ldots, n}}(-1)^{j} \alpha_{n-j-1}^{\Pi}\left(P^{j}\right)=\chi_{g}(\Pi)-\frac{1}{2} \chi_{g}(\partial \Pi)-r ;
$$

and with Theorem 11.2.1 the desired result follows immediately. Furthermore, we remark that

$$
\chi_{g}(\Pi)-\frac{1}{2} \chi_{g}(\partial \Pi)=\chi_{c}(\Pi)-\frac{1}{2} \chi_{c}(\partial \Pi)-\frac{1}{2} a_{i n f}^{0}(\Pi) .
$$

Example 11.2.1 Let $\Pi$ be the 2-dimensional polytopal complex in Example 3.1.2 (see Figure 2). We can easily see that

$$
\begin{aligned}
\chi(\Pi) & =8-10+3=1 \\
\chi(\partial \Pi) & =8-10=-2 \\
r & =\sum_{i=1}^{8} \beta_{1}^{\Pi}\left(P_{i}^{0}\right)=4-\sum_{i=1}^{8} \alpha_{1}^{\Pi}\left(P_{i}^{0}\right),
\end{aligned}
$$

and so we get for the volume of $\Pi$ :

$$
2 K c_{2}^{-1} \operatorname{vol}_{\mathbb{X}_{2}(\Pi)}=\sum_{i=1}^{8} \alpha_{1}^{\Pi}\left(P_{i}^{0}\right)-2
$$

Example 11.2.2 Let $\Pi$ be the 2-dimensional polytopal complex in Example 9.1.1 (see Figure 10). We can easily see that

$$
\begin{aligned}
\chi(\Pi) & =6-7+2=1 \\
\chi(\partial \Pi) & =6-6=0 \\
r & =\sum_{i=1}^{8} \beta_{1}^{\Pi}\left(P_{i}^{j}\right)=2-\sum_{i=1}^{4} \alpha_{1}^{\Pi}\left(P_{i}^{0}\right),
\end{aligned}
$$

and so we get for the volume of $\Pi$ :

$$
2 K c_{2}^{-1} \operatorname{vol}_{\mathbb{X} 2}(\Pi)=\sum_{i=1}^{4} \alpha_{1}^{\Pi}\left(P_{i}^{0}\right)-1 .
$$

Proposition 11.2.3 (Gauß-Bonnet for Polytopes) Let $P$ be an n-dimensional polytope in $\mathbb{X}^{n}$. Then

$$
\frac{1}{2}\left(1+(-1)^{n}-a_{i n f}^{0}(P)\right)-r=\left\{\begin{array}{lll}
2 K^{m} c_{2 m}^{-1} \operatorname{vol}_{\mathbb{X} 2 m}(P) & , & n=2 m \\
0 & , & n=2 m+1 \text { oven }
\end{array},\right.
$$

where the number $r$ is defined as

$$
r:=\sum_{\substack{P j \in \Omega j(\partial P) \\ j=0, \ldots, n-2}}(-1)^{j} \beta_{n-j-1}\left(P^{j}\right) .
$$

Proof: The polytope $P$ in $\mathbb{X}^{n}$ is an $n$-dimensional polytopal complex with boundary complex $\partial P$. Thus we have with Lemma 3.2.2

$$
\begin{aligned}
\chi_{g}(P) & =\chi_{c}(P)-a_{i n f}^{0}(P) \\
& =1-a_{i n f}^{0}(P) \\
\chi_{g}(\partial P) & =\chi_{c}(\partial P)-a_{i n f}^{0}(P) \\
& =1+(-1)^{n}-a_{i n f}^{0}(P) .
\end{aligned}
$$

Hence, by a simple computation we get

$$
\chi_{g}(P)+\frac{1}{2} \chi_{g}(\partial P)=\frac{1}{2}\left(1+(-1)^{n}-a_{i n f}^{0}(P)\right),
$$

and the claim follows with Proposition 11.2.2.

Remark 11.2.1 Let $P$ be an $n$-dimensional polytope in $\mathbb{X}^{n}$. For even dimensions $n=2 m$ Proposition 11.2.3 can be written as

$$
1-r-\frac{1}{2} a_{i n f}^{0}(P)=2 K^{m} c_{2 m}^{-1} \operatorname{vol}_{\mathbb{X} 2 m}(P),
$$

and for odd dimensions $n=2 m+1$ we get

$$
r+\frac{1}{2} a_{i n f}^{0}(P)=0 .
$$

If $P$ is an n-dimensional polytope in the Euclidean space $\mathbb{E}^{n}(K=0)$ Proposition 11.2.3 can be written as

$$
r=\left\{\begin{array}{ll}
1 & , \\
0 \text { even } \\
0 & , \\
n \text { odd }
\end{array} .\right.
$$

This is a generalization of the well-known fact that the sum of all exterior angles of a Euclidean polygone ( $n=2$ ) is equal to 1 (or $2 \pi$ without normalization).

### 11.3 The Generalized Schläfli-Peschl Relations

Next we generalize Theorem 10.4.1 to polytopal complexes by decomposing the complex into simplices.

Theorem 11.3.1 (Generalized Schläfli-Peschl Relations I) Let $\Pi$ be a $2 m$-dimensional pure polytopal complex in $\mathbb{X}^{2 m}$ and $\mathcal{D}=\mathcal{D}(\Pi)$ a simplicial decomposition of $\Pi$. Then we have:
with

$$
\begin{gathered}
2 K^{m} c_{2 m}^{-1} \operatorname{vol}_{\mathbb{X} 2 m}(\Pi)=\sum_{\substack{P j \in \Omega^{j}(\Pi) \\
j=0, \ldots, 2 m}} E\left(P^{j}, \mathcal{D}\right) \alpha_{2 m-j-1}^{\Pi}\left(P^{j}\right) \\
E\left(P^{j}, \mathcal{D}\right)=2 \sum_{k=0}^{m}(-1)^{k} a_{2 k+1} z\left(2 k, P^{j}, \mathcal{D}\right) \cdot
\end{gathered}
$$

Proof: Let $\Pi$ be a $2 m$-dimensional pure polytopal complex in $\mathbb{X}^{2 m}$ and $\mathcal{D}=\mathcal{D}(\Pi)$ a simplicial decomposition of $\Pi$. Then with Theorem 10.3.1 and 10.4.1 we get

$$
\begin{aligned}
K^{m} c_{2 m}^{-1} \operatorname{vol}_{\mathbb{X}^{n}}(\Pi) & =K^{m} c_{2 m}^{-1} \sum_{T^{2 m} \in \Omega^{2 m}(\mathcal{D})} v o l_{\mathbb{X}^{n}}\left(T^{2 m}\right) \\
& =\sum_{T^{2 m} \in \Omega^{2 m}(\mathcal{D})} \sum_{k=0}^{m}(-1)^{k} a_{2 k+1} \sum_{T^{2 k} \in \Omega^{2 k}\left(T^{2 m}\right)} \alpha_{2 m-2 k-1}\left(T^{2 k} \mid T^{2 m}\right) \\
& =\sum_{k=0}^{m}(-1)^{k} a_{2 k+1} \sum_{T^{2 m} \in \Omega^{2 m}(\mathcal{D})} \sum_{T^{2 k} \in \Omega^{2 k}\left(T^{2 m}\right)} \alpha_{2 m-2 k-1}\left(T^{2 k} \mid T^{2 m}\right) \\
& =\sum_{k=0}^{m}(-1)^{k} a_{2 k+1} \sum_{T^{2 k} \in \Omega^{2 k}(\mathcal{D})} \alpha_{2 m-2 k-1}^{\mathcal{D}}\left(T^{2 k}\right) \\
& =\sum_{k=0}^{m}(-1)^{k} a_{2 k+1} \sum_{j=2 k}^{2 m} \sum_{T^{2 k} \in \Omega_{j}^{2 k}(\Pi, \mathcal{D})} \alpha_{2 m-2 k-1}^{\mathcal{D}}\left(T^{2 k}\right) \\
& =\sum_{k=0}^{m}(-1)^{k} a_{2 k+1} \sum_{j=2 k}^{2 m} \sum_{P^{j} \in \Omega^{j}(\Pi)} z\left(2 k, P^{j}, \mathcal{D}\right) \alpha_{2 m-j-1}^{\Pi}\left(P^{j}\right) \\
& =\sum_{k=0}^{m}(-1)^{k} a_{2 k+1} \sum_{j=0}^{2 m} \sum_{P_{j} \in \Omega^{j}(\Pi)} z\left(2 k, P^{j}, \mathcal{D}\right) \alpha_{2 m-j-1}^{\Pi}\left(P^{j}\right) \\
& =\sum_{\substack{P j \in \Omega^{j}(\Pi) \\
j=0, \ldots, 2 m}}\left[\sum_{k=0}^{m}(-1)^{k} a_{2 k+1} z\left(2 k, P^{j}, \mathcal{D}\right)\right] \alpha_{2 m-j-1}^{\Pi}\left(P^{j}\right) \\
& =\sum_{\substack{P^{2 j} \in \Omega^{2 j}(\Pi) \\
j=0, \ldots, m}}\left[\sum_{k=0}^{m}(-1)^{k} a_{2 k+1} z\left(2 k, P^{2 j}, \mathcal{D}\right)\right] \alpha_{2 m-2 j-1}^{\Pi}\left(P^{2 j}\right) \\
& +\sum_{\substack{P^{2 j+1 \in \Omega^{2 j+1}(\Pi)} \\
j=0, \ldots, m-1}}\left[\sum_{k=0}^{m}(-1)^{k} a_{2 k+1} z\left(2 k, P^{2 j+1}, \mathcal{D}\right)\right] \alpha_{2 m-2 j}^{\Pi}\left(P^{2 j+1}\right) .
\end{aligned}
$$

The fourth step follows from Theorem 9.3.1 and the sixth step from Theorem 9.3.4 and Corollary 9.3.2. In the seventh step we have used that $z\left(2 k, P^{j}, \mathcal{D}\right)=0$ for all $j<2 k$.

Corollary 11.3.1 (Reduction Formula for a 4-dimensional Cone) Let $C=\operatorname{conv}(m, \tilde{C})$ be a 4-dimensional cone in $\mathbb{X}^{4}=\mathbb{S}^{4}$ or $\mathbb{H}^{4}$. Furthermore, for all integers $d$ with $0 \leq d \leq 4$ let $\Omega^{d}(C)^{\prime}$ be the set of d-dimensional faces of $C$ which are contained in $\tilde{C}$ and $\Omega^{d}(C)^{\prime \prime}:=$ $\Omega^{d}(C)-\Omega^{d}(C)^{\prime}$. Then

$$
\begin{aligned}
2 c_{4}^{-1} \operatorname{vol}_{\mathbb{X}^{4}}(C)=\sum_{C^{0} \in \Omega^{0}(C)} \alpha_{3}\left(C^{0}\right) & +\sum_{C^{2} \in \Omega^{2}(C)^{\prime}}\left(1-\frac{1}{2} a^{0}\left(C^{2}\right)\right) \alpha_{1}\left(C^{2}\right)-\frac{1}{2} \sum_{C^{2} \in \Omega^{2}(C)^{\prime \prime}} \alpha_{1}\left(C^{2}\right) \\
& +\frac{1}{2}-\frac{1}{2}\left(a^{0}(\tilde{C})+a^{2}(\tilde{C})\right)+\frac{1}{4} \sum_{C^{2} \in \Omega^{2}(\tilde{C})} a^{0}\left(C^{2}\right) .
\end{aligned}
$$

Proof: Let $C=\operatorname{conv}(m, \tilde{C})$ be a 4 -dimensional cone in $\mathbb{X}^{4}=\mathbb{S}^{4}$ or $\mathbb{H}^{4}$ and $\mathcal{S}=\mathcal{S}(C)$ the decomposition of $C$, described in Example 9.1.2. We remark that

$$
\begin{aligned}
\sum_{C^{2} \in \Omega^{2}(\tilde{C})} a^{1}\left(C^{2}\right) & =\sum_{C^{2} \in \Omega^{2}(\tilde{C})} a^{0}\left(C^{2}\right) \\
& =2 a^{1}(\tilde{C})
\end{aligned}
$$

because we count all edges of $\tilde{C}$ twice. Then we have

$$
\begin{aligned}
& z\left(2 k, C^{1}, \mathcal{S}\right)=0 \text { for } C^{1} \in \Omega^{1}(C) \text { and } k=0,1,2 \\
& z\left(0, C^{3}, \mathcal{S}\right)=1 \text { for } C^{3}=\tilde{C} \\
& z\left(0, C^{3}, \mathcal{S}\right)=0 \text { for } C^{3} \in \Omega^{3}(C)^{\prime \prime} \\
& z\left(2, C^{3}, \mathcal{S}\right)=a^{0}\left(C^{3}\right)-1 \text { for } C^{3} \in \Omega^{3}(C)^{\prime \prime} \text { with } C^{3}=\operatorname{conv}\left(m, C^{2}\right) \\
& z\left(2, C^{3}, \mathcal{S}\right)=a^{1}\left(C^{3}\right)+\sum_{C^{2} \in \Omega^{2}\left(C^{3}\right)} a^{0}\left(C^{2}\right)=\frac{3}{2} \sum_{C^{2} \in \Omega^{2}(\tilde{C})} a^{0}\left(C^{2}\right) \text { for } C^{3}=\tilde{C} \\
& z\left(4, C^{3}, \mathcal{S}\right)=0 \text { for all } C^{3} \in \Omega^{3}(C) \\
& z\left(0, C^{0}, \mathcal{S}\right)=1 \text { for all } C^{0} \in \Omega^{0}(C) \\
& z\left(0, C^{2}, \mathcal{S}\right)=1 \text { for all } C^{2} \in \Omega^{2}(C)^{\prime} \\
& z\left(0, C^{2}, \mathcal{S}\right)=0 \text { for all } C^{2} \in \Omega^{2}(C)^{\prime \prime} \\
& z\left(2, C^{2}, \mathcal{S}\right)=1 \text { for all } C^{2} \in \Omega^{2}(C)^{\prime \prime} \\
& z\left(2, C^{2}, \mathcal{S}\right)=a^{1}\left(C^{2}\right)=a^{0}\left(C^{2}\right) \text { for all } C^{2} \in \Omega^{2}(C)^{\prime} \\
& z(0, C, \mathcal{S})=0 \\
& z(2, C, \mathcal{S})=a^{0}(\tilde{C})+a^{2}(\tilde{C}) \\
& z(4, C, \mathcal{S})= \\
& \sum^{2} \in \Omega^{2}(\tilde{C}) \\
& a^{1}\left(C^{2}\right)=\sum_{C^{2} \in \Omega^{2}(\tilde{C})} a^{0}\left(C^{2}\right)
\end{aligned}
$$

Since $a_{1}=1 / 2, a_{3}=1 / 4$ and $a_{5}=1 / 2$ we get

$$
\begin{aligned}
E\left(C^{0}, \mathcal{S}\right) & =\frac{1}{2} \\
E\left(C^{1}, \mathcal{S}\right) & =0 \\
E\left(C^{2}, \mathcal{B}\right) & =\frac{1}{2}-\frac{1}{4} a^{0}\left(C^{2}\right) \text { for all } C^{2} \in \Omega^{2}(C)^{\prime} \\
E\left(C^{2}, \mathcal{S}\right) & =-\frac{1}{4} \text { for all } C^{2} \in \Omega^{2}(C)^{\prime \prime} \\
E(\tilde{C}, \mathcal{S}) & =\frac{1}{2}-\frac{3}{8} \sum_{C^{2} \in \Omega^{2}(\tilde{C})} a^{0}\left(C^{2}\right) \\
E\left(C^{3}, \mathcal{S}\right) & =-\frac{1}{4} a^{0}\left(C^{2}\right) \text { for } C^{3}=\operatorname{conv}\left(m, C^{2}\right) \\
& =-\frac{1}{4}\left(a^{0}\left(C^{3}\right)-1\right) \\
E(C, \mathcal{S}) & =-\frac{1}{4}\left(a^{0}(\tilde{C})+a^{2}(\tilde{C})\right)+\frac{1}{2} \sum_{P^{2} \in \Omega^{2}(\tilde{C})} a^{0}\left(C^{2}\right)
\end{aligned}
$$

Thus we find for the volume of $C$ (compare Theorem 11.3.1) that

$$
\begin{aligned}
c_{4}^{-1} \operatorname{vol}_{\mathbb{X}^{4}}(C)= & \frac{1}{2} \sum_{C^{0} \in \Omega^{0}(C)} \alpha_{3}\left(C^{0}\right)+\sum_{C^{2} \in \Omega^{2}(C)^{\prime}}\left(\frac{1}{2}-\frac{1}{4} a^{0}\left(C^{2}\right)\right) \alpha_{1}\left(C^{2}\right)-\frac{1}{4} \sum_{C^{2} \in \Omega^{2}(C)^{\prime \prime}} \alpha_{1}\left(C^{2}\right) \\
& -\frac{1}{8} \sum_{C^{3} \in \Omega^{3}(C)^{\prime \prime}}\left(a^{0}\left(C^{3}\right)-1\right)+\frac{1}{2}\left(\frac{1}{2}-\frac{3}{8} \sum_{C^{2} \in \Omega^{2}(\tilde{C})} a^{0}\left(C^{2}\right)\right) \\
& -\frac{1}{4}\left(a^{0}(\tilde{C})+a^{2}(\tilde{C})\right)+\frac{1}{2} \sum_{C^{2} \in \Omega^{2}(\tilde{C})} a^{0}\left(C^{2}\right) .
\end{aligned}
$$

Finally, we use the identity

$$
\sum_{C^{3} \in \Omega^{3}(C)^{\prime \prime}}\left(a^{0}\left(C^{3}\right)-1\right)=\sum_{C^{2} \in \Omega^{2}(\tilde{C})} a^{0}\left(C^{2}\right)
$$

and the theorem follows immediately.
The following volume formula for 4-dimensional spherical polytopes $P$ is due to L. Schläfli (see [Sch], page 276) who proved it by a decomposition of $P$ into cones.

Corollary 11.3.2 (Reduction Formula for a 4-dimensional Polytope) Let $P$ be a 4-dimensional polytope in $\mathbb{X}^{4}=\mathbb{S}^{4}$ or $\mathbb{H}^{4}$. Then

$$
\begin{aligned}
2 c_{4}^{-1} \operatorname{vol}_{\mathbb{X}^{4}}(P)= & \sum_{P^{0} \in \Omega^{0}(P)} \alpha_{3}\left(P^{0}\right)+\sum_{P^{2} \in \Omega^{2}(P)}\left(1-\frac{1}{2} a^{0}\left(P^{2}\right)\right) \alpha_{1}\left(P^{2}\right) \\
& +1-\frac{1}{2}\left(a^{0}(P)+a^{2}(P)\right)+\frac{1}{4} \sum_{P^{2} \in \Omega^{2}(P)} a^{0}\left(P^{2}\right)
\end{aligned}
$$

## Proof:

Let $P$ be a 4 -dimensional polytope in $\mathbb{X}^{4}=\mathbb{S}^{4}$ or $\mathbb{H}^{4}$ and $\mathcal{B}=\mathcal{B}(P)$ the barycentric decomposition of $P$ where 1-dimensional faces of $P$ are not dissected. We remark that

$$
\begin{aligned}
\sum_{P^{2} \in \Omega^{2}\left(P^{3}\right)} a^{0}\left(P^{2}\right) & =\sum_{P^{2} \in \Omega^{2}\left(P^{3}\right)} a^{1}\left(P^{2}\right) \\
& =2 a^{1}\left(P^{3}\right)
\end{aligned}
$$

for all $P^{3} \in \Omega^{3}(P)$, because we count all edges of $P^{3}$ twice. Furthermore, each 2-dimensional face $P^{2}$ of $P$ is contained in exactly two 3 -dimensional faces of $P$ and so it intersects $2 a^{1}\left(P^{2}\right)=$ $2 a^{0}\left(P^{2}\right)$ different decomposition simplices. Hence

$$
\begin{aligned}
z(4, P, \mathcal{B}) & =2 \sum_{P^{2} \in \Omega^{2}(P)} a^{0}\left(P^{2}\right) \\
& =2 \sum_{P^{2} \in \Omega^{2}(P)} a^{1}\left(P^{2}\right)
\end{aligned}
$$

Then we have

$$
\begin{aligned}
& z\left(2 k, P^{1}, \mathcal{B}\right)=0 \text { for } P^{1} \in \Omega^{1}(P) \text { and } k=0,1,2 \\
& z\left(0, P^{3}, \mathcal{B}\right)=1 \text { for } P^{3} \in \Omega^{3}(P) \\
& z\left(2, P^{3}, \mathcal{B}\right)=a^{1}\left(P^{3}\right)+\sum_{P^{2} \in \Omega^{2}\left(P^{3}\right)} a^{0}\left(P^{2}\right)=\frac{3}{2} \sum_{P^{2} \in \Omega^{2}\left(P^{3}\right)} a^{0}\left(P^{2}\right) \text { for } P^{3} \in \Omega^{3}(P) \\
& z\left(4, P^{3}, \mathcal{B}\right)=0 \text { for } P^{3} \in \Omega^{3}(P) \\
& z\left(0, P^{0}, \mathcal{B}\right)=1 \text { for } P^{0} \in \Omega^{0}(P) \\
& z\left(0, P^{2}, \mathcal{B}\right)=1 \text { for } P^{2} \in \Omega^{2}(P) \\
& z\left(2, P^{2}, \mathcal{B}\right)=a^{0}\left(P^{2}\right) \text { for } P^{2} \in \Omega^{2}(P) \\
& z(0, P, \mathcal{B})=1 \\
& z(2, P, \mathcal{B})=\underbrace{a^{1}(P)}_{\mathbf{A}}+\underbrace{2 a^{2}(P)}_{\mathbf{B}}+\underbrace{}_{P_{P^{2} \in \Omega^{2}(P)}^{\sum_{\mathbf{C}}} a^{0}\left(P^{2}\right)}=\underbrace{\mathbf{D}}_{\sum_{P^{3} \in \Omega^{3}(P)} a^{0}\left(P^{3}\right)} \\
&=a^{1}(P)+2 a^{2}(P)+a_{P^{2} \in \Omega^{2}(P)}^{\sum_{0}\left(P^{2}\right)}+\sum_{P^{3} \in \Omega^{3}(P)}^{\sum_{0}}\left(2+a^{1}\left(P^{3}\right)-a^{2}\left(P^{3}\right)\right) \\
&=a^{1}(P)+2 a^{2}(P)+\sum_{P^{2} \in \Omega^{2}(P)} a^{0}\left(P^{2}\right)+a^{2} a^{3}(P)
\end{aligned}
$$

$$
\begin{aligned}
&+\underbrace{\sum_{P^{3} \in \Omega^{3}(P)} a^{1}\left(P^{3}\right)}_{\mathbf{E}}-\underbrace{\sum_{P^{3} \in \Omega^{3}(P)} a^{2}\left(P^{3}\right)}_{\mathbf{F}} \\
&=a^{1}(P)+2 a^{2}(P)+2 \sum_{P^{2} \in \Omega^{2}(P)} a^{0}\left(P^{2}\right) \\
& z(4, P, \mathcal{B})=2 \sum_{P^{2} \in \Omega^{2}(P)} a^{0}\left(P^{2}\right) .
\end{aligned}
$$

Each 2-dimensional decomposition polytope in the interior of $P$ is a cone with basis in a 1dimensional decomposition simplex in the boundary of $P$. The term $\mathbf{A}$ is the number of elements with basis equal to an edge of $P, \mathbf{B}$ is the number of elements with basis in a simplex $\operatorname{conv}($ (midpoint of a 3 -face) , (midpoint of a 2 -face) ), $\mathbf{C}$ is the number of elements with basis in a simplex conv ( (midpoint of a 2-face), (vertex of a 2 -face) ) and $\mathbf{D}$ is the number of elements with basis in a simplex conv( (midpoint of a 3-face), (vertex of a 3-face) ). Furthermore we use that

$$
\begin{aligned}
& \mathbf{E}=\sum_{P^{3} \in \Omega^{3}(P)} a^{1}\left(P^{3}\right)=\sum_{P^{2} \in \Omega^{2}(P)} a^{1}\left(P^{2}\right)=\sum_{P^{2} \in \Omega^{2}(P)} a^{0}\left(P^{2}\right) \\
& \mathbf{F}=\sum_{P^{3} \in \Omega^{3}(P)} a^{2}\left(P^{3}\right)=2 a^{2}(P)=\mathbf{B}
\end{aligned}
$$

So we get

$$
\begin{aligned}
& E\left(P^{0}, \mathcal{B}\right)=\frac{1}{2} \\
& E\left(P^{1}, \mathcal{B}\right)=0 \\
& E\left(P^{2}, \mathcal{B}\right)=\frac{1}{2}-\frac{1}{4} a^{0}\left(P^{2}\right) \\
& E\left(P^{3}, \mathcal{B}\right)=\frac{1}{2}-\frac{3}{8} \sum_{P^{2} \in \Omega^{2}\left(P^{3}\right)} a^{0}\left(P^{2}\right) \\
& E(P, \mathcal{B})=\frac{1}{2}-\frac{1}{4} a^{1}(P)-\frac{1}{2} a^{3}(P)+\frac{1}{2} \sum_{P^{2} \in \Omega^{2}(P)} a^{0}\left(P^{2}\right)
\end{aligned}
$$

and it follows for the volume of $P$ (compare Theorem 11.3.1) that

$$
\begin{aligned}
c_{4}^{-1} \operatorname{vol}_{\mathbb{X}^{4}}(P)= & \frac{1}{2} \sum_{P^{0} \in \Omega^{0}(P)} \alpha_{3}\left(P^{0}\right)+\sum_{P^{2} \in \Omega^{2}(P)}\left(\frac{1}{2}-\frac{1}{4} a^{0}\left(P^{2}\right)\right) \alpha_{1}\left(P^{2}\right) \\
& +\frac{1}{2} \sum_{P^{3} \in \Omega^{3}(P)}\left(\frac{1}{2}-\frac{3}{8} \sum_{P^{2} \in \Omega^{2}\left(P^{3}\right)} a^{0}\left(P^{2}\right)\right) \\
& +\frac{1}{2}-\frac{1}{4} a^{1}(P)-\frac{1}{2} a^{3}(P)+\frac{1}{2} \sum_{P^{2} \in \Omega^{2}(P)} a^{0}\left(P^{2}\right)
\end{aligned}
$$

Finally, we use the two identities

$$
\begin{aligned}
\sum_{P^{3} \in \Omega^{3}(P)} \sum_{P^{2} \in \Omega^{2}\left(P^{3}\right)} a^{0}\left(P^{2}\right) & =2 \sum_{P^{2} \in \Omega^{2}(P)} a^{0}\left(P^{2}\right) \\
a^{1}(P)+a^{3}(P) & =a^{0}(P)+a^{2}(P)
\end{aligned}
$$

and the theorem follows immediately.
Now we will generalize the remaining Schläfli-Peschl Relations (compare Theorem 10.4.1) from simplices to $n$-dimensional pure polytopal complexes.

Theorem 11.3.2 (Generalized Schläfli-Peschl Relations II) Let $\Pi$ be an n-dimensional pure polytopal complex in $\mathbb{X}^{n}$. Then

$$
\begin{aligned}
& \sum_{j=0}^{l} \sum_{\substack{P v \in \Omega^{v}(\Pi) \\
v=q+2 j}}\left[z\left(q-1, P^{v}, \mathcal{D}\right)\right. \\
& \left.+\sum_{p=0}^{j}(-1)^{p+1} a_{2 p+1}\binom{q+2 p+1}{2 p+1} z\left(q+2 p, P^{v}, \mathcal{D}\right)\right] \alpha_{2 l-2 j-1}^{\Pi}\left(P^{v}\right) \\
& +\sum_{j=0}^{l} \sum_{\substack{P u \in \Omega_{u}^{u}(\Pi) \\
u=q+2 j-1}}\left[z\left(q-1, P^{u}, \mathcal{D}\right)\right. \\
& \left.+\sum_{p=0}^{j-1}(-1)^{p+1} a_{2 p+1}\binom{q+2 p+1}{2 p+1} z\left(q+2 p, P^{u}, \mathcal{D}\right)\right] \alpha_{2 l-2 j}^{\Pi}\left(P^{u}\right) \\
& =0
\end{aligned}
$$

for all $l$ with $0 \leq 2 l<n$ and $q:=n-2 l$.

## Proof:

Let $\Pi$ be a $n$-dimensional pure complex in $\mathbb{X}^{n}$ and $\mathcal{D}=\mathcal{D}(\Pi)$ a simplicial decomposition of $\Pi$. Then with Theorem 10.4.1 we get for all $l$ with $0 \leq 2 l<n$ and for $q=q(m, l):=n-2 l$

$$
\begin{aligned}
0= & \sum_{T^{n} \in \Omega^{n}(\mathcal{D})} \omega_{2 l}\left(T^{n}\right) \\
& -\sum_{T^{n} \in \Omega^{n}(\mathcal{D})} \sum_{k=0}^{l}(-1)^{k} a_{2 k+1}\binom{q+2 k+1}{2 k+1} \sum_{\substack{T v \in \Omega^{v}\left(T^{n}\right) \\
v=q+2 k}} \alpha_{2 l-2 k-1}\left(T^{v}\right) \\
= & \sum_{T^{n} \in \Omega^{n}(\mathcal{D})} \sum_{\substack{T^{v} \in \Omega^{v}\left(T^{n}\right) \\
v=q-1}} \alpha_{2 l}\left(T^{v}\right) \\
& -\sum_{T^{n} \in \Omega^{n}(\mathcal{D})} \sum_{k=0}^{l}(-1)^{k} a_{2 k+1}\binom{q+2 k+1}{2 k+1} \sum_{\substack{T v \Omega^{v}\left(T^{n}\right) \\
v=q+2 k}} \alpha_{2 l-2 k-1}\left(T^{v}\right) \\
= & \sum_{\substack{T^{v} \in \Omega^{v}(\mathcal{D}) \\
v=q-1}} \alpha_{2 l}^{\mathcal{D}}\left(T^{v}\right)
\end{aligned}
$$

$$
\begin{aligned}
& -\sum_{k=0}^{m}(-1)^{k} a_{2 k+1}\binom{q+2 k+1}{2 k+1} \sum_{\substack{T_{v} \in \Omega^{v}(\mathcal{D}) \\
v=q+2 k}} \alpha_{2 l-2 k-1}^{\mathcal{D}}\left(T^{v}\right) \\
= & \sum_{j=q-1}^{n} \sum_{\substack{T^{v} \in \Omega_{j}^{v}(\mathcal{D}) \\
v=q-1}} \alpha_{2 l}^{\mathcal{D}}\left(T^{v}\right) \\
& -\sum_{k=0}^{l}(-1)^{k} a_{2 k+1}\binom{q+2 k+1}{2 k+1} \sum_{j=q+2 k}^{n} \sum_{\substack{T^{v} \in \Omega_{j}^{v}(\Pi, \mathcal{D}) \\
v=q+2 k}} \alpha_{2 l-2 k-1}^{\mathcal{D}}\left(T^{v}\right) \\
= & \sum_{j=q-1}^{n} \sum_{P^{j} \in \Omega^{j}(\Pi)} z\left(q-1, P^{j}, \mathcal{D}\right) \alpha_{n-j-1}^{\Pi}\left(P^{j}\right) \\
& -\sum_{k=0}^{l}(-1)^{k} a_{2 k+1}\binom{q+2 k+1}{2 k+1} \sum_{j=q+2 k}^{n} \sum_{P^{j} \in \Omega^{j}(\Pi)}^{n} z\left(q+2 k, P^{j}, \mathcal{D}\right) \alpha_{n-j-1}^{\Pi}\left(P^{j}\right) .
\end{aligned}
$$

If we rearrange the parts of the sums skillfully, the theorem follows.
Theorem 11.3.3 (Reduction Formula for Simplicial Polytopes) For a simplicial $2 m$-dimensional polytope $P$ in $\mathbb{X}^{2 m}$ we have

$$
2 K^{m} c_{2 m}^{-1} \operatorname{vol}_{\mathbb{X} 2 m}(P)=\sum_{\substack{P^{2 j} \in \Omega^{2 j}(P) \\ j=0, \ldots, m}} \sigma_{j}\left(P^{2 j}\right) \alpha_{2 m-2 j-1}\left(P^{2 j}\right),
$$

with

$$
\sigma_{j}\left(P^{2 j}\right)=2 \sum_{k=0}^{j}(-1)^{k} a_{2 k+1} a^{2 k-1}\left(P^{2 j}\right)= \begin{cases}(-1)^{j} 2 a_{2 j+1} & , \quad 0 \leq j<m \\ 2 \sum_{k=0}^{m}(-1)^{k} a_{2 k+1} a^{2 k-1}(P) & , j=m\end{cases}
$$

for all $0 \leq j \leq m$.
Proof: Let $P$ be a simplicial $2 m$-dimensional polytope in $\mathbb{X}^{2 m}$ and $\mathcal{K}=\mathcal{K}(P)$ the cone decomposition of $P$, described in Example 9.1.2. Of course, $\mathcal{K}$ is a simplicial complex, because each face of $P$ is a simplex. So we have for each $P^{2 j+1}$ in $\Omega^{2 j+1}(P)$ with $j=0, \ldots, m-1$

$$
z\left(2 k, P^{2 j+1}, \mathcal{K}\right)=0(k=0, \ldots, m)
$$

and for each $P^{2 j}$ in $\Omega^{2 j}(P)$ with $j=0, \ldots, m-1$

$$
z\left(2 k, P^{2 j}, \mathcal{K}\right)=\left\{\begin{array}{lll}
0 & , & k<j \text { or } k>j \\
1 & , & j=j
\end{array} .\right.
$$

Furthermore, we have for all $k=0, \ldots, m$

$$
\begin{aligned}
z\left(2 k, P^{2 m}, \mathcal{K}\right) & =z(2 k, P, \mathcal{K}) \\
& =a^{2 k-1}(P)
\end{aligned}
$$

with $a^{-1}(P):=1$. Each $2 k$-dimensional decomposition simplex is a cone with base in a $(2 k-1)$ dimensional face of $P$. It follows with Theorem 11.3.1 that

$$
E\left(P^{2 j+1}, \mathcal{K}\right)=2 \sum_{k=0}^{m}(-1)^{k} a_{2 k+1} z\left(2 k, P^{2 j+1}, \mathcal{K}\right)
$$

$$
\begin{aligned}
& =0 \\
\sigma_{j}\left(P^{2 j}\right) & :=E\left(P^{2 j}, \mathcal{K}\right) \\
& =2 \sum_{k=0}^{j}(-1)^{k} a_{2 k+1} z\left(2 k, P^{2 j}, \mathcal{K}\right) \\
& =(-1)^{j} 2 a_{2 j+1} \\
\sigma_{m}(P) & :=E(P, \mathcal{K}) \\
& =2 \sum_{k=0}^{m}(-1)^{k} a_{2 k+1} z(2 k, P, \mathcal{K}) \\
& =2 \sum_{k=0}^{m}(-1)^{k} a_{2 k+1} a^{2 k-1}(P)
\end{aligned}
$$

for all $j=0, \ldots, m-1$. We remark that for a $2 j$-dimensional (decomposition) simplex $T=T^{2 j}$ we have

$$
\begin{aligned}
(-1)^{j} 2 a_{2 j+1} & =2 \sum_{k=0}^{j}(-1)^{k} a_{2 k+1}\binom{2 j+1}{2 k} \\
& =2 \sum_{k=0}^{j}(-1)^{k} a_{2 k+1} a^{2 k-1}(T),
\end{aligned}
$$

and so we get a consistent description of the coefficients $\sigma_{j}\left(P^{2 j}\right)$ as

$$
\sigma_{j}\left(P^{2 j}\right)=2 \sum_{k=0}^{j}(-1)^{k} a_{2 k+1} a^{2 k-1}\left(P^{2 j}\right)
$$

for all $j$ with $0 \leq j \leq m$ (compare Lemma 4.6.1). This completes the proof.
Remark 11.3.1 (Determination of the tangent numbers) Let $T$ be a $2 m$-dimensional simplex in $\mathbb{X}^{2 m}$. The method used in the proof of Theorem 11.3 .3 allows to determine explicitly the combinatorial invariants $\sigma_{m}(T)$ appearing in Schläfli's Reduction Formula. Indeed, let $\mathcal{K}=\mathcal{K}(P)$ be the cone decomposition of $T$ and let $A_{2 j+1}(j \geq 0)$ be rational numbers such that

$$
2 K^{m} c_{2 m}^{-1} \operatorname{vol}_{\mathbb{X} 2 m}(T)=\sum_{\substack{T^{2 j} \in \Omega^{2 j}(T) \\ j=0, \ldots, m}} A_{2 j+1} \alpha_{2 m-2 j-1}\left(T^{2 j}\right) .
$$

Then we get in the same way as in the theorem

$$
\begin{aligned}
2 K^{m} c_{2 m}^{-1} \operatorname{vol}_{\mathbb{X} 2 m}(T) & =\sum_{\substack{T^{2 j} \in \Omega^{2 j}(T) \\
j=0, \ldots, m}} A_{2 j+1} \alpha_{2 m-2 j-1}\left(T^{2 j}\right) \\
& =\sum_{\substack{T^{2 j} \in \Omega^{2 j}(T) \\
j=0, \ldots, m}}\left(\sum_{k=0}^{m} A_{2 k+1} z\left(2 k, T^{2 j}, \mathcal{K}\right)\right) \alpha_{2 m-2 j-1}\left(T^{2 j}\right)
\end{aligned}
$$

where we have used that $z\left(2 k, T^{2 j+1}, \mathcal{K}\right)=0$ for all $j=0, \ldots, m-1$. Furthermore, the remaining decomposition numbers are given by

$$
z\left(2 k, T^{2 j}, \mathcal{K}\right)= \begin{cases}0 & , \quad k \leq j \\ 1 & , \quad k=j<m \\ a^{2 k-1}(T) & , \quad j=m\end{cases}
$$

where $a^{2 k-1}(T)=\binom{2 m+1}{2 k}$ is the number of $(2 k-1)$-dimensional faces of $T$. So we get the identity

$$
\begin{aligned}
& \sum_{\substack{T^{2 j} \in \Omega^{2 j}(T) \\
j=0, \ldots, m-1}} A_{2 j+1} \alpha_{2 m-2 j-1}\left(T^{2 j}\right)+A_{2 m+1} \\
= & \sum_{\substack{T^{2 j} \in \Omega^{2 j}(T) \\
j=0, \ldots, m-1}} A_{2 j+1} \alpha_{2 m-2 j-1}\left(T^{2 j}\right)+\sum_{k=0}^{m} A_{2 k+1}\binom{2 m+1}{2 k},
\end{aligned}
$$

and so

$$
A_{2 m+1}=-\frac{1}{2 m} \sum_{k=0}^{m-1} A_{2 k+1}\binom{2 m+1}{2 k}
$$

If we use $A_{1}=1$ we get $A_{2 m+1}=(-1)^{m} 2 a_{2 m+1}$ for all $m>1$ by induction (compare Lemma 4.6.1).

By using the well-known duality of simplicial and simple polytopes ( $a^{2 k-1} \leftrightarrow a^{2 j-2 k}$ ) we get the following result (compare the statement in [V2], page 122).

Corollary 11.3.3 (Reduction Formula for Simple Polytopes) For a simple $2 m$-dimensional polytope $P$ in $\mathbb{X}^{2 m}$ we have

$$
2 K^{m} c_{2 m}^{-1} \operatorname{vol}_{\mathbb{X} 2 m}(P)=\sum_{\substack{P^{2 j} \in \Omega^{2 j}(P) \\ j=0, \ldots, m}} \sigma_{j}\left(P^{2 j}\right) \alpha_{2 m-2 j-1}\left(P^{2 j}\right),
$$

with

$$
\sigma_{j}\left(P^{2 j}\right)=2 \sum_{k=0}^{j}(-1)^{k} a_{2 k+1} a^{2 j-2 k}\left(P^{2 j}\right) .
$$

Furthermore, we can determine the coefficients for cubes (and also for their dual polytopes).
Theorem 11.3.4 Let $W=W^{2 m}$ be a $2 m$-dimensional cube in $\mathbb{X}^{2 m}$. Then

$$
2 K^{m} c_{2 m}^{-1} \operatorname{vol}_{\mathbb{X}^{n}(W)}=\sum_{v=0}^{m}(-1)^{v} E_{2 v} \omega_{2 m-2 v-1}(W)
$$

with the Euler numbers $E_{2 v}$.
Proof: We know that a $2 m$-dimensional cube $W$ is simple and each $2 j$-dimensional face $W^{2 j}$ of $W$ is also a cube. Thus with Lemma 3.1.1 we get

$$
a^{2 j-2 k}\left(W^{2 j}\right)=2^{2 k}\binom{2 j}{2 k}
$$

With Corollary 11.3.3, Lemma 4.5.1 and Section 4.6 we find

$$
\begin{aligned}
\sigma_{j}\left(W^{2 j}\right) & =2 \sum_{k=0}^{j}(-1)^{k} a_{2 k+1} 2^{2 k}\binom{2 j}{2 k} \\
& =\sum_{k=0}^{j}(-1)^{k} T_{2 k+1}\binom{2 j}{2 k} \\
& =(-1)^{j} E_{2 j}
\end{aligned}
$$

and the theorem follows.

## 12 General Schläfli Reduction Formula

In the previous section we have constructed reduction formulas for 4 -dimensional polytopes (compare Corollary 11.3.2) and simplicial and simple polytopes of even dimension (compare Theorem 11.3.3 and Corollary 11.3.3) by decomposing into simplices. In the following section we will develop a general reduction formula for arbitrary polytopes $P$ in even-dimensional spaces $\mathbb{X}^{2 m}=\mathbb{S}^{2 m}$ or $\mathbb{H}^{2 m}$ without using decompositions of $P$. In the first part we show that such a formula must exist. We do this by describing a combinatorial algorithm which can be used to eliminate all even-dimensional angles in Poincaré's Formula. This algorithm produces a reduction formula and in the second part we will show that the combinatorial invariants in it are uniquely determined. Then in the third part we give a general description of these so-called Schläfli invariants.

### 12.1 Existence

Let $P$ be a $2 m$-dimensional polytope in $\mathbb{X}^{2 m}=\mathbb{S}^{2 m}$ or $\mathbb{H}^{2 m}$. For each element $P^{k} \in \Omega^{k}(P)$ $(0 \leq k \leq 2 m)$ let $x=x\left(P^{k}\right)$ be an interior point of the face $P^{k}$ and $<P^{k}>^{\perp}$ be the $(2 m-k)$ dimensional plane passing through $x$ orthogonal to the plane $\left\langle P^{k}\right\rangle$. Furthermore, let $\epsilon=$ $\epsilon\left(P^{k}\right)>0$ such that the sphere $S^{2 m-1}(x, \epsilon)$ only intersects faces of $P$ that are incident with $P^{k}$. Then also the sphere $S^{2 m-k-1}(x, \epsilon) \subset<P^{k}>^{\perp}$ only intersects faces of $P$ that are incident with $P^{k}$. We note that

$$
\alpha_{2 m-k-1}\left(P^{k}\right)=\frac{\operatorname{vol}\left(S^{2 m-k-1}(x, \epsilon) \cap P\right)}{\operatorname{vol}\left(S^{2 m-k-1}(x, \epsilon)\right)}=c_{2 m-k-1}^{-1} \operatorname{vol}_{\mathbb{S} 2 m-k-1}\left(L\left(P^{k}\right)\right)
$$

where $L\left(P^{k}\right)$ denotes the $(2 m-k-1)$-dimensional link in the face $P^{k}$ and $c_{2 m-k-1}$ the volume of the $(2 m-k-1)$-dimensional unit sphere. The link is a $(2 m-k-1)$-dimensional spherical polytope and with Poincaré's Formula 11.2 .1 we have

$$
\begin{aligned}
2 K^{m} c_{2 m}^{-1} \operatorname{vol}_{\mathbb{X} 2 m}(P) & =\sum_{\substack{P k \in \Omega^{k}(P) \\
k=0, \ldots, 2 m}}(-1)^{k} \alpha_{2 m-k-1}\left(P^{k}\right) \\
& =\sum_{\substack{P k \in \Omega^{k}(P) \\
k=0, \ldots, 2 m}}(-1)^{k} c_{2 m-k-1}^{-1} \operatorname{vol}_{\mathbb{S} 2 m-k-1}\left(L\left(P^{k}\right)\right) .
\end{aligned}
$$

Let $k$ be an odd number then $L\left(P^{k}\right)$ is an even dimensional spherical polytope. So we can eliminate the volume of this link by Poincaré's Formula. In detail, we have

$$
c_{2 m-k-1}^{-1} \operatorname{vol}_{\mathbb{S}^{2} 2 m-k-1}\left(L\left(P^{k}\right)\right)=\frac{1}{2} \sum_{\substack{Q^{j} \in \Omega j(L(P k)) \\ j=0, \ldots, 2 m-k-1}}(-1)^{j} \alpha_{2 m-k-j-2}\left(Q^{j} \mid L\left(P^{k}\right)\right)
$$

Furthermore, each of these angles $\alpha_{2 m-k-j-2}\left(Q^{j} \mid L\left(P^{k}\right)\right)$ of the link $L\left(P^{k}\right)$ is also a $(2 m-k-j-$ 2)-dimensional angle of the polytope $P$ with apex $P^{k+j+1}$ such that $Q^{j}=P^{k+j+1} \cap S^{2 m-k-1}(x, \epsilon)$.

Now we can reduce all $(2 m-k-1)$-dimensional angles of $P(k$ is odd) and we see that each angle of $P$ with the apex $P^{k+j+1}(j=0, \ldots, 2 m-k-1)$ will change according to

$$
\alpha_{2 m-k-j-2}\left(P^{k+j+1}\right) \leadsto \alpha_{2 m-k-j-2}\left(P^{k+j+1}\right)\left(1+\frac{1}{2}(-1)^{j} \kappa\left(k, P^{k+j+1}\right)\right),
$$

where $\kappa\left(k, P^{k+j+1}\right)=a^{k}\left(P^{k+j+1}\right)$ is the number of $k$-dimensional faces of $P$ which are also faces of the polytope $P^{k+j+1}$.

If we start with the top-dimensional angles of $P$ and reduce, as described all even-dimensional angles by Poincaré's Formula in top-down fashion, we get a general reduction formula

$$
2 K^{m} c_{2 m}^{-1} \operatorname{vol}_{\mathbb{X} 2 m}(P)=\sum_{\substack{P^{2 k} \in \Omega^{2 k}(P) \\ k=0, \ldots, m}} \sigma^{2 k}\left(P^{2 k}\right) \alpha_{2 m-2 k-1}\left(P^{2 k}\right)
$$

where $\sigma^{2 k}\left(P^{2 k}\right)$ is a rational combinatorial invariant, depending on the combinatorial structure of the face $P^{2 k}$ for all $0 \leq k \leq m$.

Furthermore, we call this way of angle reducing the Top to Down Algorithm (TDA). Of course, this is a purely combinatorial method and independent of the ambient space $\mathbb{X}^{2 m}$.

## Description for Simplices and Cubes

Now we will describe this algorithm for the simplest $2 m$-dimensional polytopes. Let $P$ be the $2 m$-dimensional simplex $T$ or the $2 m$-dimensional cube $C$. Both polytopes have the nice property that each face is also a polytope of the same type, but of lower dimension. So we can consider the angle sums instead of the angles and if we reduce all $(2 m-k-1)$-dimensional angles of $P$, we see that the $(2 m-k-j-2)$-dimensional angle sum $\omega_{2 m-k-j-2}(P)$ will change by

$$
\omega_{2 m-k-j-2}(P) \leadsto \omega_{2 m-k-j-2}(P)\left(1+\frac{1}{2}(-1)^{j} \kappa(k, j)\right)
$$

where

$$
\kappa(k, j)=a^{k}\left(P^{k+j+1}\right)=\left\{\begin{aligned}
\binom{k+j+2}{k+1} & , \quad P=T \\
2^{j+1}\binom{k+j+1}{k} & , \quad P=C
\end{aligned}\right.
$$

(compare Lemma 3.1.1). Now we start with the top even dimensional angles of $P$ and reduce them step by step. See the Appendix 16.2 and 16.3 for a detailed scheme for simplices and cubes.

By induction we see that in the simplex case

$$
\begin{aligned}
\sigma^{0}\left(T^{0}\right) & =1 \\
\sigma^{2 k}\left(T^{2 k}\right) & =1+\frac{1}{2} \sum_{j=1}^{k}\binom{2 k+1}{2 j} \mu_{2 j}
\end{aligned}
$$

for all $k \geq 1$, where the numbers $\mu_{2 j}$ are defined by recursion

$$
\mu_{2 j}=-1-\frac{1}{2} \sum_{v=1}^{j-1}\binom{2 j}{2 v} \mu_{2 v}
$$

for all $j \geq 2$ and $\mu_{2}=2$. So we can determine firstly all the numbers $\mu_{2 j}$ and secondly all the numbers $\sigma^{2 k}\left(T^{2 k}\right)$. In this way we find again that $\sigma^{2 k}\left(T^{2 k}\right)=2(-1)^{k} a_{2 k-1}$.

### 12.2 Uniqueness

In this section we will prove that the combinatorial invariants $\sigma^{2 j}$, constructed in the last section, are uniquely determined.

Theorem 12.2.1 The rational combinatorial invariants

$$
\sigma^{2 j}: \mathbf{P}^{2 \mathbf{j}} \longrightarrow \mathbb{Q}
$$

are uniquely determined for all $0 \leq j<\infty$ in the following sense: If there are combinatorial invariants $\kappa^{2 j}: \mathbf{P}^{\mathbf{2 j}} \longrightarrow \mathbb{Q}$ for all $0 \leq j \leq m$ such that for all $m$ with $0 \leq m<\infty$ and all $P$ in $\mathbf{P}^{\mathbf{2 m}}$ we have

$$
\sum_{j=0}^{m} \sum_{P^{2 j} \in \Omega^{2 j}(P)} \sigma^{2 j}\left(P^{2 j}\right) \alpha_{2 m-2 j-1}\left(P^{2 j}\right)=\sum_{j=0}^{m} \sum_{P^{2 j} \in \Omega^{2 j}(P)} \kappa^{2 j}\left(P^{2 j}\right) \alpha_{2 m-2 j-1}\left(P^{2 j}\right)
$$

then $\sigma^{2 j}=\kappa^{2 j}$ for all $0 \leq j \leq m$.

Proof: For all $0 \leq m<\infty$ and all polytopes $P$ in $\mathbf{P}^{\mathbf{2 m}}$ let

$$
\sum_{j=0}^{m} \sum_{P^{2 j} \in \Omega^{2 j}(P)} \sigma^{2 j}\left(P^{2 j}\right) \alpha_{2 m-2 j-1}\left(P^{2 j}\right)=\sum_{j=0}^{m} \sum_{P^{2 j} \in \Omega^{2 j}(P)} \kappa^{2 j}\left(P^{2 j}\right) \alpha_{2 m-2 j-1}\left(P^{2 j}\right) .
$$

Let $k$ be the smallest number with $0 \leq k \leq m$ such that $\sigma^{2 k} \neq \kappa^{2 k}$, that means there is at least one polytope $R$ in $\mathbf{P}^{\mathbf{2 k}}$ with $\sigma^{2 k}(R) \neq \kappa^{2 k}(R)$. By assumption we have for all $2 k$-dimensional polytopes $Q$ the identity

$$
\begin{aligned}
& \sum_{j=0}^{k-1} \sum_{P^{2 j} \in \Omega^{2 j}(Q)} \sigma^{2 j}\left(Q^{2 j}\right) \alpha_{2 m-2 j-1}\left(Q^{2 j}\right)+\sigma^{2 k}(Q) \\
= & \sum_{j=0}^{k-1} \sum_{P^{2 j} \in \Omega^{2 j}(Q)} \sigma^{2 j}\left(Q^{2 j}\right) \alpha_{2 m-2 j-1}\left(Q^{2 j}\right)+\kappa^{2 k}(Q)
\end{aligned}
$$

and of course we can conclude immediately that $\sigma^{2 k}(Q)=\kappa^{2 k}(Q)$ for all polytopes $Q$ in $\mathbf{P}^{2 \mathbf{k}}$. But this contradicts the fact that we have $\sigma^{2 k}(R) \neq \kappa^{2 k}(R)$ for the polytope $R$. So the invariants are uniquely determined.

Definition 12.2.1 Let $P=P^{2 m}$ be a $2 m$-dimensional polytope in $\mathbb{X}^{2 m}$. Then we call the number $\sigma^{2 m}(P)$ the $2 m$-dimensional Schläfli invariant of the polytope $P$.

### 12.3 Another Description of Schläfli's Invariants

Let $P$ be an $n$-dimensional polytope in $\mathbb{X}^{n}=\mathbb{S}^{n}, \mathbb{E}^{n}$ or $\mathbb{H}^{n}$. We will derive another general description of Schläfli's invariants, where this time we won't use decomposition methods. For this we introduce combinatorial numbers which are generalizations of the numbers $a^{k}(P)$.

Definition 12.3.1 Let $P$ be an $n$-dimensional polytope in $\mathbb{X}^{n}$ and $\left.\left.\left(l_{\mu}\right), l_{\mu}-1\right), \ldots, l_{1}, k\right)$ a $(\mu+1)$ tupel of integers with $n>l_{\mu}>l_{\mu-1}>\ldots>l_{1}>k \geq 0$ and $\mu \geq 1$. Then the positive integer $A\left(l_{\mu}, l_{\mu-1}, \ldots, l_{1} ; k\right)(P)$ is defined as

$$
\begin{aligned}
A(;)(P) & :=1 \\
A(; k)(P) & :=a^{k}(P) \\
A\left(l_{\mu}, l_{\mu-1}, \ldots, l_{1} ; k\right)(P) & :=\sum_{P^{l_{\mu} \in \Omega^{l_{\mu}}(P)}} \sum_{P^{l_{\mu-1}} \in \Omega^{l_{\mu-1}}\left(P^{l_{\mu}}\right)} \ldots . . \sum_{P^{l_{2} \in \Omega^{l_{2}}\left(P^{l_{3}}\right)}} \sum_{P_{1} \in \Omega^{l_{1}}\left(P^{l_{2}}\right)} a^{k}\left(P^{l_{1}}\right) .
\end{aligned}
$$

For all other $(\mu+1)$-tupel of positive integers let the combinatorial invariant be equal to zero.

For instance, we have

$$
\begin{aligned}
A(j ; k)(P) & =\sum_{P^{j} \in \Omega^{j}(P)} a^{k}\left(P^{j}\right) \\
& =\sum_{i=1}^{a^{j}(P)} a^{k}\left(P_{i}^{j}\right)
\end{aligned}
$$

for all $j>k \geq 0$, where $\Omega^{j}(P)=\left\{P_{1}^{j}, \ldots, P_{a^{j}(P)}^{j}\right\}$ is the set of $k$-dimensional faces of $P$. With a simple computation we get the following result.

Lemma 12.3.1 Then we have

$$
\begin{aligned}
A(; k)(P) & =\sum_{P^{k} \in \Omega^{k}(P)} A(;)\left(P^{k}\right) \\
A\left(l_{\mu}, l_{\mu-1}, \ldots, l_{1} ; k\right)(P) & =\sum_{P^{l_{\mu} \in \Omega^{l_{\mu}}(P)}} A\left(l_{\mu-1}, \ldots, l_{1} ; k\right)\left(P^{l_{\mu}}\right) .
\end{aligned}
$$

Definition 12.3.2 Let $P=P^{2 m}$ be an $2 m$-dimensional polytope in $\mathbb{X}^{2 m}=\mathbb{S}^{2 m}$, $\mathbb{E}^{2 m}$ or $\mathbb{H}^{2 m}$. The combinatorial invariant $A_{i}(P)$ is defined as

$$
\begin{aligned}
A_{0}(P) & = \\
A_{1}(P) & =\sum_{0 \leq k<m} A(; 2 k)(P) \\
A_{i}(P) & =\sum_{0 \leq k<f_{1}<\ldots<f_{i-1}<m} A\left(2 f_{i-1}, \ldots, 2 f_{1} ; 2 k\right)(P)
\end{aligned}
$$

for all $2 \leq i \leq m$. For all other $i$ let $A_{i}(P)$ be equal to zero.
Example 12.3.1 For an 2m-dimensional polytope $P=P^{2 m}$ and the values $m=0,1,2,3$ we have

$$
\begin{aligned}
& A_{0}\left(P^{0}\right)=1 \\
& A_{0}\left(P^{2}\right)=1 \\
& A_{1}\left(P^{2}\right)=\sum_{0 \leq k<1} A(; 2 k)(P) \\
& =a^{0}\left(P^{2}\right) \\
& A_{0}\left(P^{4}\right)=1 \\
& A_{1}\left(P^{4}\right)=\sum_{0 \leq k<2} A(; 2 k)(P) \\
& =a^{0}\left(P^{4}\right)+a^{2}\left(P^{4}\right) \\
& A_{2}\left(P^{4}\right)=\sum_{0 \leq k<f_{1}<2} A\left(2 f_{1} ; 2 k\right)(P)=A(2 ; 0)(P) \\
& =\sum_{P^{2} \in \Omega^{2}\left(P^{4}\right)} a^{0}\left(P^{2}\right) \\
& A_{0}\left(P^{6}\right)=1 \\
& A_{1}\left(P^{6}\right)=\sum_{0 \leq k<3} A(; 2 k)(P) \\
& =a^{0}\left(P^{6}\right)+a^{2}\left(P^{6}\right)+a^{4}\left(P^{6}\right) \\
& A_{2}\left(P^{6}\right)=\sum_{0 \leq k<f_{1}<3} A\left(2 f_{1} ; 2 k\right)=A(2 ; 0)(P)+A(4 ; 0)(P)+A(4 ; 2)(P) \\
& =\sum_{P^{2} \in \Omega^{2}\left(P^{6}\right)} a^{0}\left(P^{2}\right)+\sum_{P^{4} \in \Omega^{4}\left(P^{6}\right)} a^{0}\left(P^{4}\right)+\sum_{P^{4} \in \Omega^{4}\left(P^{6}\right)} a^{2}\left(P^{4}\right) \\
& A_{3}\left(P^{6}\right)=\sum_{0 \leq k<f_{1}<f_{2}<3} A\left(2 f_{2}, 2 f_{1} ; 2 k\right)=A(4,2 ; 0)(P) \\
& =\sum_{P^{4} \in \Omega^{4}\left(P^{6}\right)} \sum_{P^{2} \in \Omega^{2}\left(P^{4}\right)} a^{0}\left(P^{2}\right) \text {. }
\end{aligned}
$$

Lemma 12.3.2 Let $P=P^{2 m}$ be a $2 m$-dimensional polytope in $\mathbb{X}^{2 m}$. Then

$$
\sum_{j=0}^{m-1} \sum_{P^{2 j} \in \Omega^{2 j}(P)} A_{i}\left(P^{2 j}\right)=A_{i+1}(P)
$$

for all $0 \leq i \leq m-1$.

Proof: Let $i=0$. Then we have by definition

$$
\begin{aligned}
A_{1}(P) & =\sum_{0 \leq k<m} A(; 2 k)(P) \\
& =\sum_{j=0}^{m-1} \sum_{P^{2 j} \in \Omega^{2 j}(P)} A_{0}\left(P^{2 j}\right),
\end{aligned}
$$

where we have used Lemma 12.3 .1 in the second step. Now let $i$ be a natural number with $1 \leq i \leq m-1$. Then it follows by Lemma 12.3.1

$$
\begin{aligned}
A_{i+1}(P) & =\sum_{0 \leq k<f_{1}<\ldots<f_{i}<m} A\left(2 f_{i}, \ldots, 2 f_{1} ; 2 k\right)(P) \\
& =\sum_{0 \leq k<f_{1}<\ldots<f_{i}<m} \sum_{P^{2 f_{i} \in \Omega^{2} f_{i}(P)}} A\left(2 f_{i-1}, \ldots, 2 f_{1} ; 2 k\right)\left(P^{2 f_{i}}\right) \\
& =\sum_{\substack{0 \leq k<f_{1}<\ldots<f_{i-1}<f_{i} \\
f_{i}-1<f_{i}<m}} \sum_{P^{2 f_{i} \in \Omega^{2} f_{i}(P)}} A\left(2 f_{i-1}, \ldots, 2 f_{1} ; 2 k\right)\left(P^{2 f_{i}}\right) \\
& =\sum_{j=f_{i-1}+1}^{m-1} \sum_{P^{2 j} \in \Omega^{2 j}(P)} \sum_{\substack{0 \leq k<f_{1}<\ldots<f_{i-1} \\
f_{i-1}<j}} A\left(2 f_{i-1}, \ldots, 2 f_{1} ; 2 k\right)\left(P^{2 j}\right) \\
& =\sum_{j=0}^{m-1} \sum_{P^{2 j} \in \Omega^{2 j}(P)} \sum_{\substack{0 \leq k<f_{1}<\ldots<f_{i-1} \\
f_{i-1}<j}} A\left(2 f_{i-1}, \ldots, 2 f_{1} ; 2 k\right)\left(P^{2 j}\right),
\end{aligned}
$$

because $A\left(2 f_{i-1}, \ldots, 2 f_{1} ; 2 k\right)\left(P^{2 j}\right)$ is equal to zero for all $j$ with $2 j \leq 2 f_{i-1}$ by definition. So we get

$$
A_{i+1}(P)=\sum_{j=0}^{m-1} \sum_{P^{2 j} \in \Omega^{2 j}(P)} A_{i}\left(P^{2 j}\right)
$$

and this completes the proof.
The following theorem follows an idea of Schläfli (compare [Sch], page 280).
Theorem 12.3.1 Let $P=P^{2 m}$ be a $2 m$-dimensional polytope in $\mathbb{X}^{2 m}=\mathbb{S}^{2 m}$ or $\mathbb{H}^{2 m}$. Then

$$
2 c_{2 m}^{-1} K^{m} \operatorname{vol}_{\mathbb{X} 2 m}(P)=\sum_{j=0}^{m} \sum_{P^{2 j} \in \Omega^{2 j}(P)} \sigma^{2 j}\left(P^{2 j}\right) \alpha_{2 m-2 j-1}\left(P^{2 j}\right)
$$

where the Schläfli invariant $\sigma^{2 j}$ can be written as

$$
\sigma^{2 j}\left(P^{2 j}\right)=\sum_{p=0}^{j}(-1)^{p} \frac{1}{2^{p}} A_{p}(P)
$$

## Proof:

Firstly, we know that a reduction formula of the type above must exist. We observe that the combinatorial numbers $\sigma^{2 j}$ depend only on the combinatorial structure of the face $P^{2 j}$ of $P$. This means that $\sigma^{2 j}$ is independent of the geometry of the space $\mathbb{X}^{2 m}$ and therefore it suffices to prove the theorem only for the case $\mathbb{X}^{2 m}=\mathbb{S}^{2 m}$.
Let $P=P^{2 m}$ be a $2 m$-dimensional polytope in $\mathbb{S}^{2 m}$. Now we want to assign to $P$ a nice representative $P_{\mathbb{S}}$ in the equivalence class of combinatorially isomorphic polytopes $C l(P)$. We do this by mapping the boundary of $P$ onto the sphere $\mathbb{S}^{2 m-1} \subset \mathbb{S}^{2 m}$ such that the constructed tesselation of $\mathbb{S}^{2 m-1}$ is geodesic and has the same combinatorial structure like the boundary of $P$. This tesselation of $\mathbb{S}^{2 m-1}$ can be viewed as the boundary of a $2 m$-dimensional spherical polytope $P_{\mathbb{S}}$ which is an element in the set $C l(P)$.

Thus all angles of $P_{\mathbb{S}}$ are of measure $1 / 2$ and $P_{\mathbb{S}}$ is the nice representatative in the equivalence class $C l(P)$.

Now we can use the polytope $P_{\mathbb{S}}$ to develop a recursion formula for the Schläfli invariants. Let $P$ be an element in $C l\left(P_{\mathbb{S}}\right)$. Then we have

$$
\begin{aligned}
2 c_{2 m}^{-1} K^{m} \operatorname{vol}_{\mathbb{X} 2 m}\left(P_{\mathbb{S}}\right) & =1 \\
& =\frac{1}{2} \sum_{j=0}^{m-1} \sum_{P^{2 j} \in \Omega^{2 j}\left(P_{\mathbb{S}}\right)} \sigma^{2 j}\left(P^{2 j}\right)+\sigma^{2 m}\left(P_{\mathbb{S}}\right),
\end{aligned}
$$

where we have used that $\alpha_{2 m-2 j-1}\left(P^{2 j}\right)=1 / 2$ for $j=0,1, \cdots, m-1$ and $\alpha_{-1}\left(P_{\mathbb{S}}\right)=1$. Now $P$ has the same combinatorial structure like $P_{\mathbb{S}}$. So we get the recursion

$$
\sigma^{2 m}(P)=1-\frac{1}{2} \sum_{j=0}^{m-1} \sum_{P^{2 j} \in \Omega^{2 j}(P)} \sigma^{2 j}\left(P^{2 j}\right), \quad \sigma^{0}(P)=1
$$

We see that the Schläfli invariantes takes the claimed form (compare Corollary 11.3.2):

$$
\begin{aligned}
& \sigma^{0}\left(P^{0}\right)=1 \\
& \sigma^{2}\left(P^{2}\right)=1-\frac{1}{2} a^{0}\left(P^{2}\right) \\
& \sigma^{4}\left(P^{4}\right)=1-\frac{1}{2}\left(a^{0}(P)+a^{2}(P)\right)+\frac{1}{4} \sum_{P^{2} \in \Omega^{2}(P)} a^{0}\left(P^{2}\right) .
\end{aligned}
$$

Next we assume that for all $j$ with $0 \leq j \leq m-1$ we have

$$
\sigma^{2 j}\left(P^{2 j}\right)=\sum_{p=0}^{j}(-1)^{p} \frac{1}{2^{p}} A_{p}(P) .
$$

We use the above recursion formula and get for a $2 m$-dimensional polytope $P$

$$
\sigma^{2 m}(P)=1-\frac{1}{2} \sum_{j=0}^{m-1} \sum_{P^{2 j} \in \Omega^{2 j}(P)} \sigma^{2 j}\left(P^{2 j}\right)
$$

$$
\begin{aligned}
& =1-\frac{1}{2} \sum_{j=0}^{m-1} \sum_{P^{2 j} \in \Omega^{2 j}(P)} \sum_{p=0}^{j}(-1)^{p} \frac{1}{2^{p}} A_{p}\left(P^{2 j}\right) \\
& =1+\sum_{p=0}^{m-1}(-1)^{p+1} \frac{1}{2^{p+1}} \sum_{j=0}^{m-1} \sum_{P^{2 j} \in \Omega^{2 j}(P)} A_{p}\left(P^{2 j}\right) \\
& =1+\sum_{p=0}^{m-1}(-1)^{p+1} \frac{1}{2^{p+1}} A_{p+1}(P) \\
& =\sum_{p=0}^{m}(-1)^{p} \frac{1}{2^{p}} A_{p}(P),
\end{aligned}
$$

where we have used that $A_{p}\left(P^{2 j}\right)$ is equal to zero if $p$ is greater as $j$ in the third step and Lemma 12.3.2 in the forth step.

Example 12.3.2 For a $2 m$-dimensional polytope $P=P^{2 m}$ and the values $m=0,1,2$ we have

$$
\begin{aligned}
\sigma^{0}\left(P^{0}\right) & =1 \\
\sigma^{2}\left(P^{2}\right) & =\sum_{p=0}^{1}(-1)^{p} \frac{1}{2^{p}} A_{p}\left(P^{2}\right) \\
& =A_{0}\left(P^{2}\right)-\frac{1}{2} A_{1}\left(P^{2}\right) \\
& =1-\frac{1}{2} a^{0}\left(P^{2}\right) \\
\sigma^{4}\left(P^{4}\right) & =\sum_{p=0}^{2}(-1)^{p} \frac{1}{2^{p}} A_{p}\left(P^{4}\right) \\
& =A_{0}\left(P^{4}\right)-\frac{1}{2} A_{1}\left(P^{4}\right)+\frac{1}{4} A_{2}\left(P^{4}\right) \\
& =1-\frac{1}{2}\left(a^{0}\left(P^{4}\right)+a^{2}\left(P^{4}\right)\right)+\frac{1}{4} \sum_{P^{2} \in \Omega^{2}\left(P^{4}\right)} a^{0}\left(P^{2}\right)
\end{aligned}
$$

## 13 The Volume of a Fundamental Polytope

The set of faces of a fundamental polytope for a discrete group $\Gamma<$ Iso $\left(\mathbb{X}^{n}\right)$ splits into subsets of $\Gamma$-equivalent faces. The cycle condition, proved in the first part of this section, connects the angles in these subsets. In the second part of this section we combine our knowledge about the volume of polytopes with the cycle condition. Hence we can develop some special volume formulas for (normal) fundamental polytopes.

### 13.1 The Cycle Condition for Fundamental Polytopes

$\Gamma$-equivalence Let $\mathbb{X}^{n}=\mathbb{S}^{n}, \mathbb{E}^{n}$ or $\mathbb{H}^{n}$ and $\Gamma<$ Iso $\left(\mathbb{X}^{n}\right)$ be a discrete subgroup, such that there exists an $n$-dimensional normal fundamental polytope $P=P(\Gamma)$ for $\Gamma$ (compare section 7). This means that $\Gamma$ is of finite covolume, $\operatorname{covol}(\Gamma)=\operatorname{vol}_{\mathbb{X}^{n}}(P)<\infty$, and $P$ has finitely many faces in each dimension.

Now we want to establish an equivalence relation on the set of faces of a fixed dimension. In the following let $d$ always be an integer with $0 \leq d \leq n-1$. For all $P_{i}^{d}, P_{j}^{d} \in \Omega^{d}(P)$ let

$$
P_{i}^{d} \sim_{\Gamma} P_{j}^{d} \quad: \Leftrightarrow \quad \exists \gamma \in \Gamma: \gamma P_{i}^{d}=P_{j}^{d} ;
$$

and for all $p_{i}^{0}, p_{j}^{0} \in \Upsilon^{0}(P)$ let

$$
p_{i}^{0} \sim_{\Gamma} p_{j}^{0} \quad: \Leftrightarrow \quad \exists \gamma \in \Gamma: \gamma p_{i}^{0}=p_{j}^{0} .
$$

In fact this is an equivalence relation on the set of faces of $P$ of a fixed dimension and each set $\Omega^{d}(P)$ for $0 \leq d \leq n-1$ and $\Upsilon^{0}(P)$ decompose in equivalence classes, which we denote by $\Omega_{(i)}^{d}$ and $\Upsilon_{(i)}^{0}$.
Furthermore let $\mu^{d}$ be the number of equivalence classes in $\Omega^{d}(P)$ and $m^{0}$ be the number of equivalence classes in $\Upsilon^{0}(P)$.
Each equivalence class $\Omega_{(i)}^{d}$ or $\Upsilon_{(i)}^{0}$ contains only finitely many elements from $\Omega^{d}(P)$ or $\Upsilon^{0}(P)$ and we denote these numbers by $l_{i}^{d}$ or $f_{i}^{0}$ respectively.

We will use the following notation for the elements in a fixed equivalence class:

$$
\begin{aligned}
\Omega^{d} & =\{\underbrace{P_{(1) 1}^{d}, \ldots, P_{(1) l_{1}^{d}}^{d}}_{\Omega_{(1)}^{d}}, \ldots, \underbrace{P_{(i) 1}^{d}, \ldots, P_{(i) l_{i}^{d}}^{d}}_{\Omega_{(i)}^{d}}, \ldots, \underbrace{P_{\left(\mu^{d}\right) 1}^{d}, \ldots, P_{\left(\mu^{d}\right) l_{\mu^{d}}^{d}}^{d}}_{\Omega_{\left(\mu^{d}\right)}^{d}}\} \\
& =\Omega_{(1)}^{d} \cup \Omega_{(2)}^{d} \cup \ldots \cup \Omega_{\left(\mu^{d}\right)}^{d} \\
\Upsilon^{0} & =\{\underbrace{p_{(1) 1}^{0}, \ldots, p_{(1) f_{1}^{0}}^{0}}_{\Upsilon_{(1)}^{0}}, \ldots, \underbrace{p_{(i) 1}^{0}, \ldots, p_{(i) f_{i}^{0}}^{0}}_{\Upsilon_{(i)}^{0}}, \ldots, \underbrace{p_{\left(m^{0}\right) 1}^{0}, \ldots, p_{\left(m^{0}\right) f_{m^{0}}^{0}}^{0}}_{\Upsilon_{\left(m^{0}\right)}^{0}}\} \\
& =\Upsilon_{(1)}^{0} \cup \Upsilon_{(2)}^{0} \cup \ldots \cup \Upsilon_{\left(m^{0}\right)}^{0}
\end{aligned}
$$

and all unions are disjoint. Of course we have

$$
\begin{align*}
f_{1}^{0}+\cdots+f_{i}^{0}+\cdots+f_{m^{0}}^{0} & =a_{i n f}^{0}(P) \\
l_{1}^{0}+\cdots+l_{i}^{0}+\cdots+l_{\mu^{0}}^{0} & =a_{\text {ord }}^{0}(P) \text { and }  \tag{1}\\
l_{1}^{d}+\cdots+l_{i}^{d}+\cdots+l_{\mu^{d}}^{d} & =a^{d}(P)
\end{align*}
$$

for all $d$ with $1 \leq d \leq n-1$.
Furthermore, for all $i=1, \ldots, \mu^{d}$ the $d$-dimensional polytopes $P_{(i)}^{d} \in \Omega_{(i)}^{d}$ of $P$ are equivalent under the action of $\Gamma$ and they have equal combinatorial structure (in fact, they are isometric). So we can define:

$$
a^{k}\left(\Omega_{(i)}^{d}\right):=a^{k}\left(P_{(i)}^{d}\right)
$$

for all $k<d$ and for an arbitrary $P_{(i)}^{d} \in \Omega_{(i)}^{d}$, or more generally: If $\kappa^{d}$ is a combinatorial $d$-invariant in the family of $d$-dimensional polytopes, we can define

$$
\kappa^{d}\left(\Omega_{(i)}^{d}\right):=\kappa^{d}\left(P_{(i)}^{d}\right)
$$

for an arbitrary $P_{(i)}^{d} \in \Omega_{(i)}^{d}$.

Lemma 13.1.1 Let $P$ be a normal fundamental polytope for a discrete subgroup $\Gamma$ of isometries in $\mathbb{X}^{n}$ and let $F$ be an element in $\Omega^{d}(P)$ for some $d$ with $0 \leq d \leq n-1$. Then $\Gamma_{F}=\operatorname{Stab}(F, \Gamma)$ and $\Gamma_{F}^{\prime}=\operatorname{Stab}_{p}(F, \Gamma)$ are conjugated to finite subgroups of $O(n)$.

Proof: Let $\mathbb{X}^{n}=\mathbb{E}^{n}$ or $\mathbb{S}^{n}$. Then $F$ is an ordinary compact set in $\mathbb{X}^{n}$ and the result follows with Lemma 7.1.1.

Let $\mathbb{X}^{n}=\mathbb{H}^{n}$ and let $F \in \Omega^{d}(P)$ be an arbitrary face of $P$ (not a vertex at infinity). The set

$$
\Phi=\{\gamma P: \gamma \in \Gamma\}
$$

is a locally finite family of subsets in $\mathbb{H}^{n}$ and for all elements $\gamma \in \Gamma$ we have $\operatorname{ri}(P) \cap \operatorname{ri}(\gamma P)=\emptyset$ (compare section 7.3). We suppose that $\Gamma_{F}$ is an infinite group. Then the set

$$
\left\{\gamma P: \gamma \in \Gamma_{F}\right\} \subset \Phi
$$

is also an infinite set and all elements in it share the face $F$ in contradiction to the local finiteness of the collection $\Phi$. Hence $\Gamma_{F}$ (and also $\Gamma_{F}^{\prime}$ as a subgroup) is a finite and discrete group and so conjugated to a finite subgroup of $O(n)$.

The group $\Gamma<\operatorname{Iso}\left(\mathbb{X}^{n}\right)$ may have fixed points and there may exist elements in $\Gamma$, which fix faces (not necessarily pointwise!) of $P$. It is easy to see that the stabilizer subgroups of equivalent $d$-dimensional ordinary faces are conjugated, also isomorphic:

$$
\operatorname{Stab}\left(P_{(i) j}^{d}, \Gamma\right)=\gamma \operatorname{Stab}\left(P_{(i) k}^{d}, \Gamma\right) \gamma^{-1}
$$

for all $i=1, \ldots, \mu^{d}, j, k=1, \ldots, l_{i}^{d}$ and for some $\gamma$ in $\Gamma$. So we see that $d$-dimensional faces in the same equivalence class (which means $\Gamma$-equivalent) are fixed by the same number of elements in $\Gamma$. So we define for all $i=1, \ldots, \mu^{d}$

$$
\begin{aligned}
g_{i}^{d} & :=g_{i}^{d}\left(\Omega_{(i)}^{d}\right)=\sharp \operatorname{Stab}\left(P_{(i) j}^{d}, \Gamma\right) \\
g_{1}^{n} & :=1
\end{aligned}
$$

for an arbitrary $P_{(i) j}^{d}$ in $\Omega_{(i)}^{d}$ and for all $d=0, \ldots, n-1$.

The Cycle Conditions The following theorem can be viewed as a generalization of Theorem 9.3.4. in $[\mathrm{Be}]$ or of the second part of Theorem 6.7.7 in $[\mathrm{R}]$.

Theorem 13.1.1 (Cycle Condition) Let $\mathbb{X}^{n}=\mathbb{S}^{n}, \mathbb{E}^{n}$ or $\mathbb{H}^{n}$ and $P=P(\Gamma)$ be a normal ( $n$-dimensional) fundamental polytope for a discrete subgroup $\Gamma<I s o\left(\mathbb{X}^{n}\right)$. For all $d$ with $0 \leq d \leq n-1$ let $\kappa^{d}$ be a combinatorial d-invariant on the set $\mathbf{P}^{\mathbf{d}}$ of $d$-dimensional polytopes. Then

$$
\sum_{P_{(i)}^{d} \in \Omega_{(i)}^{d}} \kappa^{d}\left(P_{(i)}^{d}\right) \alpha_{n-d-1}\left(P_{(i)}^{d}\right)=\kappa^{d}\left(\Omega_{(i)}^{d}\right) \frac{1}{g_{i}^{d}}
$$

for all $i=1, \ldots, \mu^{d}$ and $j=1, \ldots, l_{i}^{d}$.
Proof: To simplify the notation: For a fixed $d$ with $0 \leq d \leq n-1$ and a fixed $i \in\left\{1, \ldots, \mu^{d}\right\}$ let

$$
\Omega_{(i)}^{d}=\left\{P_{1}, \ldots, P_{v}\right\}
$$

be a cycle of $\Gamma$-equivalent $d$-dimensional faces of $P$. This means, we have a set of elements

$$
\left\{\gamma_{1}=i d, \gamma_{2}, \ldots, \gamma_{v}\right\} \subset \Gamma
$$

such that $\gamma_{j} P_{j}=P_{1}$ for all $j=1, \ldots, v$. Furthermore, let $\Gamma_{1}:=\operatorname{Stab}\left(P_{1}, \Gamma\right)$ denote the stabilizer of the face $P_{1}$ (all elements in $\Gamma_{1}$ leave the set $P_{1}$ invariant). The group $\Gamma_{1}$ is finite by Lemma 13.1.1 and we have $g_{i}^{d}=\sharp \Gamma_{1}<\infty$.

Since we have $\kappa^{d}\left(\Omega_{(i)}^{d}\right)=\kappa^{d}\left(P_{j}\right)$ for all $P_{j}$, it is enough to show that

$$
\begin{aligned}
\sum_{P_{(i)}^{d} \in \Omega_{(i)}^{d}} \alpha_{n-d-1}\left(P_{(i)}^{d}\right) & =\sum_{j=1}^{v} \alpha_{n-d-1}\left(P_{j} \mid P\right) \\
& =\frac{1}{g_{i}^{d}}
\end{aligned}
$$

Of course, the polytope $\gamma_{j} P$ has $P_{1}=\gamma_{j} P_{j}$ as a $d$-face and for the $(n-d-1)$-dimensional angles of $P$ and $\gamma_{j} P$ in the apex $P_{1}$ we get

$$
\alpha_{n-d-1}\left(P_{1} \mid \gamma_{j} P\right)=\alpha_{n-d-1}\left(\gamma_{j}^{-1} P_{1} \mid P\right)
$$

for all $j=1, \ldots, v$.
Furthermore, let $h$ be an arbitrary element in $\Gamma$ and consider the element $P_{1}=\gamma_{j} P_{j} \in \Omega^{d}(P)$. Then we have the following properties:

- We have $P_{1} \in \Omega^{d}(h P)$ if and only if there exists some $j$ such that $h^{-1} P_{1}=P_{j}$.
$\Rightarrow$ Let $P_{1} \in \Omega^{d}(h P)$ for some $h \in \Gamma$. Then $h^{-1} P_{1}\left(\in \Omega^{d}(P)\right)$ is a $d$-dimensional face of $P$ which is $\Gamma$-equivalent to $P_{1}$. So there exists $P_{j} \in \Omega_{(i)}^{d}$ with $h^{-1} P_{1}=P_{j}$.
$\Leftarrow$ Let $h^{-1} P_{1}=P_{j}$. Then $P_{1}=h P_{j} \in \Omega^{d}(h P)$.
- We have $h^{-1} P_{1}=P_{j}$ if and only if the element $h \gamma_{j}^{-1}$ fixes the face $P_{1}$, which means that $h \gamma_{j}^{-1} \in \Gamma_{1}$.

$$
\Rightarrow \text { Let } h^{-1} P_{1}=P_{j}=\gamma_{j}^{-1} P_{1} \text {. It follows that } h \gamma_{j}^{-1} \in \Gamma_{1} .
$$

$\underset{P_{j}}{\Leftarrow}$ Let $h \gamma_{j}^{-1} \in \Gamma_{1}$ with $\gamma_{j}^{-1} P_{1}=P_{j}$. This means $h \gamma_{j}^{-1} P_{1}=P_{1}$ and $\gamma_{j}^{-1} P_{1}=h^{-1} P_{1}=$ $P_{j}$.

- The element $h \gamma_{j}^{-1}$ fixes the face $P_{1}$ (not necesarily pointwise) if and only if $h \in \Gamma_{1} \gamma_{j}$.

Thus we have

$$
P_{1} \in \Omega^{d}(h P) \quad \Longleftrightarrow \quad h \in \Gamma_{1} \gamma_{j}
$$

for some $j=1, \ldots, v$.
Now let $H:=\left\{h_{1}, \ldots, h_{t}\right\} \subset \Gamma$ be the finite set of all elements in $\Gamma$ with $P_{1} \in \Omega^{d}\left(h_{i} P\right)$ for $i=1, \ldots, t$. With the above observations we derive that

$$
H=\Gamma_{1} \gamma_{1} \cup \ldots \cup \Gamma_{1} \gamma_{v}
$$

and this union is pairwise disjoint.
We know that the set

$$
\Phi=\{\gamma P: \gamma \in \Gamma\}
$$

is a normal tesselation by Theorem 7.3.1. Of course, $\Phi$ can be viewed as a (generalized) pure $n$-dimensional polytopal complex in $\mathbb{X}^{n}$ and the complex angle $\alpha_{n-d-1}^{\Phi}\left(P_{1}\right)$ of this complex in the face $P_{1}$ is of measure 1. Furthermore, the elements in $\Gamma_{1}$ are conformal maps and so preserve the measure of an angle. So we get

$$
\begin{aligned}
1 & =\alpha_{n-d-1}^{\Phi}\left(P_{1}\right) \\
& =\sum_{\gamma \in H} \alpha_{n-d-1}\left(P_{1} \mid \gamma P\right) \\
& =\sum_{\gamma \in \Gamma_{1} \gamma_{1}} \alpha_{n-d-1}\left(P_{1} \mid \gamma P\right)+\ldots+\sum_{\gamma \in \Gamma_{1} \gamma_{v}} \alpha_{n-d-1}\left(P_{1} \mid \gamma P\right) \\
& =\sharp \Gamma_{1} \alpha_{n-d-1}\left(P_{1} \mid \gamma_{1} P\right)+\ldots+\sharp \Gamma_{1} \alpha_{n-d-1}\left(P_{1} \mid \gamma_{v} P\right) \\
& =\sharp \Gamma_{1}\left(\alpha_{n-d-1}\left(\gamma_{1}^{-1} P_{1} \mid P\right)+\ldots+\alpha_{n-d-1}\left(\gamma_{v}^{-1} P_{1} \mid P\right)\right) \\
& =g_{i}^{d} \sum_{j=1}^{v} \alpha_{n-d-1}\left(P_{j} \mid P\right) .
\end{aligned}
$$

Example 13.1.1 Let $P_{m}$ be the (canonical) normal fundamental polytope for the modular group $\operatorname{PSL}(2, \mathbb{Z})<\operatorname{Iso}\left(U^{2}\right)$ (see Figures 4 or 13). We use the following notations: $p_{1}^{0}=\infty, P_{1}^{0}=A$, $P_{2}^{0}=B, P_{1}^{1}=\operatorname{conv}(A, \infty), P_{2}^{1}=\operatorname{conv}(B, \infty)$ and $P_{3}^{1}=\operatorname{conv}(A, B)$.
Furthermore, we have $P_{1}^{0} \sim_{P S L(2, \mathbb{Z})} P_{2}^{0}$ and $P_{1}^{1} \sim_{P S L(2, \mathbb{Z})} P_{2}^{1}$ and we can write $p_{(1) 1}^{0}=\infty$ and $\Upsilon_{(1)}^{0}=\{\infty\} ; P_{(1) 1}^{0}=P_{1}^{0}, P_{(1) 2}^{0}=P_{2}^{0}$ and $\Omega_{(1)}^{0}=\left\{P_{1}^{0}, P_{2}^{0}\right\} ; P_{(1) 1}^{1}=P_{1}^{1}, P_{(1) 2}^{1}=P_{2}^{1}$, $\Omega_{(1)}^{1}=\left\{P_{1}^{1}, P_{2}^{1}\right\}$ and $\Omega_{(2)}^{1}=\left\{P_{3}^{1}\right\}$.

For the orders of the stabilizers we get

$$
\begin{aligned}
g_{1}^{0} & =g_{1}^{0}\left(\Omega_{(1)}^{0}\right)=3 \\
g_{1}^{1} & =g_{1}^{1}\left(\Omega_{(1)}^{1}\right)=1 \\
g_{2}^{1} & =g_{2}^{1}\left(\Omega_{(2)}^{1}\right)=2
\end{aligned}
$$

(you may count it in the tesselation). Furthermore, we have $l_{1}^{0}=2, l_{1}^{1}=2, l_{2}^{1}=1, \mu^{0}=1$ and $\mu^{1}=2$. So we get with the trivial combinatorial invariants:

$$
\begin{array}{ll}
\sum_{P_{(1)}^{0} \in \Omega_{(1)}^{0}} \alpha_{1}\left(P_{(1)}^{0}\right)=\alpha_{1}\left(P_{1}^{0}\right)+\alpha_{1}\left(P_{2}^{0}\right) & =\frac{1}{3} \\
\sum_{P_{(1)}^{1} \in \Omega_{(1)}^{1}} \alpha_{0}\left(P_{(1)}^{1}\right)=\alpha_{0}\left(P_{1}^{1}\right)+\alpha_{0}\left(P_{2}^{1}\right) & =1 \\
\sum_{P_{(2)}^{1} \in \Omega_{(2)}^{1}} \alpha_{0}\left(P_{(2)}^{1}\right)=\alpha_{0}\left(P_{3}^{1}\right) & =\frac{1}{2} .
\end{array}
$$



Figure 13: A Fundamental Polytope for $\operatorname{PSL}(2, \mathbb{Z})$

### 13.2 General Results

In this section we will combine the volume formulas for polytopes with the cycle conditions to get volume formulas for fundamental polytopes of discrete groups.

Theorem 13.2.1 Let $P$ be a n-dimensional normal fundamental polytope for a discrete group $\Gamma<\operatorname{Iso}\left(\mathbb{X}^{n}\right)$. Then we have

$$
\sum_{d=0}^{n}(-1)^{d}\left(\sum_{i=1}^{\mu^{d}} \frac{1}{g_{i}^{d}}\right)=\left\{\begin{array}{ll}
2 K^{m} c_{2 m}^{-1} \operatorname{vol}_{\mathbb{X}^{n}}(P) & , \quad n=2 m \quad \text { even } \\
0 & , \quad n=2 m+1 \text { odd }
\end{array} .\right.
$$

Proof: We use Poincaré's Formula for polytopes (compare Corollary 11.2.1) and the cycle conditions (compare Theorem 13.1.1) and get

$$
\begin{aligned}
& \left\{\begin{array}{lll}
2 K^{m} c_{2 m}^{-1} \operatorname{vol}_{\mathbb{X}^{n}}(P) & , & n=2 m \quad \text { even } \\
0 & , & n=2 m+1 \text { odd }
\end{array}\right. \\
& =\sum_{d=0}^{n} \sum_{P^{d} \in \Omega^{d}(P)}(-1)^{d} \alpha_{n-d-1}\left(P^{d}\right) \\
& =\sum_{d=0}^{n} \sum_{i=1}^{\mu^{d}} \sum_{P_{(i)}^{d} \in \Omega_{(i)}^{d}}(-1)^{d} \alpha_{n-d-1}\left(P_{(i)}^{d}\right) \\
& =\sum_{d=0}^{n} \sum_{i=1}^{\mu^{d}}(-1)^{d} \frac{1}{g_{i}^{d}} \\
& =\sum_{d=0}^{n}(-1)^{d}\left(\sum_{i=1}^{\mu^{d}} \frac{1}{g_{i}^{d}}\right) \text {, }
\end{aligned}
$$

and the theorem follows immediately.
In the special case where $\Gamma$ is torsionfree we get the following result, which is the well-known Theorem of Gauß and Bonnet.

Corollary 13.2.1 Let $P$ be a n-dimensional normal fundamental polytope for a discrete and torsionfree group $\Gamma<\operatorname{Iso}\left(\mathbb{X}^{n}\right)$ and let $\mu^{d}$ be the number of $\Gamma$-equivalence classes in $\Omega^{d}(P)$ for $d=0 . \ldots, n$. Then

$$
\sum_{d=0}^{n}(-1)^{d} \mu^{d}=\chi\left(\mathbb{X}^{n} / \Gamma\right)= \begin{cases}2 K^{m} c_{2 m}^{-1} \operatorname{vol}_{\mathbb{X} n}(P) & , \quad n=2 m \quad \text { even } \\ 0 & , \quad n=2 m+1 \text { odd }\end{cases}
$$

Now we will combine the General Schläfli Reduction Formula (compare 12.3.1) with the cycle conditions.

Theorem 13.2.2 Let $P=P^{2 m}$ be a $2 m$-dimensional normal fundamental polytope for a discrete group $\Gamma<\operatorname{Iso}\left(\mathbb{X}^{2 m}\right)$. Then we have

$$
2 K^{m} c_{2 m}^{-1} \operatorname{vol}_{\mathbb{X} 2 m}(P)=\sum_{d=0}^{m} \sum_{i=1}^{\mu^{2 d}} \sigma^{2 d}\left(\Omega_{(i)}^{2 d}\right) \frac{1}{g_{i}^{2 d}},
$$

where $\sigma^{2 d}$ denotes the Schläfli invariant.
Proof: We use the General Schläfli Reduction Formula 12.3.1 and the cycle conditions 13.1.1. By a simple computation we get

$$
\begin{aligned}
\sum_{d=0}^{m} \sum_{P^{2 d} \in \Omega^{2 d}} \sigma^{2 d}\left(P^{2 d}\right) \alpha_{2 m-2 d-1}\left(P^{2 d}\right) & =\sum_{d=0}^{m} \sum_{i=1}^{\mu^{2 d}} \sum_{P_{(i)}^{2 d} \in \Omega_{(i)}^{2 d}} \sigma^{2 d}\left(P_{(i)}^{2 d}\right) \alpha_{2 m-2 d-1}\left(P_{(i)}^{2 d}\right) \\
& =\sum_{d=0}^{m} \sum_{i=1}^{\mu^{2 d}} \sigma^{2 d}\left(\Omega_{(i)}^{2 d}\right) \frac{1}{g_{i}^{2 d}}
\end{aligned}
$$

The following volume formula was constructed by C. L. Siegel (compare [S]) for fundamental polygones in the hyperbolic plane. He used this formula to determine the discrete subgroup of Iso $\left(\mathbb{H}^{2}\right)$ of minimal covolume.

Corollary 13.2.2 Let $P=P^{2}$ be a 2-dimensional normal fundamental polytope for a discrete group $\Gamma$. Then

$$
2 K c_{2}^{-1} \operatorname{vol}_{\mathbb{X}^{2}}(P)=\sum_{i=1}^{\mu^{0}}\left(\frac{1}{g_{i}^{0}}-\frac{1}{2} l_{i}^{0}\right)+1-\frac{1}{2} a_{i n f}^{0}(P) .
$$

## Proof:

The Schläfli invariants are given by

$$
\begin{aligned}
\sigma^{0}\left(\Omega_{(i)}^{0}\right) & =1 \\
\sigma^{2}\left(\Omega_{(i)}^{2}\right) & =1-\frac{1}{2} a^{0}\left(\Omega_{(i)}^{2}\right) \\
& =1-\frac{1}{2} a_{o r d}^{0}(P)-\frac{1}{2} a_{i n f}^{0}(P)
\end{aligned}
$$

(compare Example 12.3.2) and for the volume of $P$ follows

$$
\begin{aligned}
2 K c_{2}^{-1} \operatorname{vol}_{\mathbb{X}^{2}}(P) & =\sum_{i=1}^{\mu^{0}} \sigma^{0}\left(\Omega_{(i)}^{0}\right) \frac{1}{g_{i}^{0}}+1-\frac{1}{2} a^{0}(P) \\
& =\sum_{i=1}^{\mu^{0}} \frac{1}{g_{i}^{0}}+1-\frac{1}{2} \sum_{i=1}^{\mu^{0}} l_{i}^{0}-\frac{1}{2} a_{i n f}^{0}(P) .
\end{aligned}
$$

So the result follows immediately.
If the group $\Gamma$ is also torsionfree we get

$$
2 K c_{2}^{-1} \operatorname{vol}_{\mathbb{X}^{2}}(P)=1+\mu^{0}-\frac{1}{2} a^{0}(P) .
$$

Example 13.2.1 For the normal fundamental polytope $P_{m}$ for the modular group $\operatorname{PSL}(2, \mathbb{Z})$ (see Example 13.1.1) we get (with $\mu^{0}=1, g_{1}^{0}=3, l_{1}^{0}=2$ and $a_{i n f}^{0}\left(P_{m}\right)=1$ )

$$
\operatorname{vol}_{\mathbb{H}^{2}}\left(P_{m}\right)=-2 \pi\left(\sum_{i=1}^{\mu^{0}}\left(\frac{1}{g_{i}^{0}}-\frac{1}{2} l_{i}^{0}\right)+1-\frac{1}{2} a_{i n f}^{0}\left(P_{m}\right)\right)=\frac{1}{3} \pi .
$$

Corollary 13.2.3 Let $P=P^{4}$ be a 4-dimensional normal fundamental polytope for a discrete group $\Gamma$. Then

$$
2 c_{4}^{-1} \operatorname{vol}_{\mathbb{X}^{4}}(P)=\sum_{i=1}^{\mu^{0}}\left(\frac{1}{g_{i}^{0}}-\frac{1}{2} l_{i}^{0}\right)+\sum_{i=1}^{\mu^{2}}\left(1-\frac{1}{2} a^{0}\left(\Omega_{(i)}^{2}\right)\right)\left(\frac{1}{g_{i}^{2}}-\frac{1}{2} l_{i}^{2}\right)+1-\frac{1}{2} a_{i n f}^{0}(P)
$$

Proof: The Schläfli invariants are given by

$$
\begin{aligned}
\sigma^{0}\left(\Omega_{(i)}^{0}\right) & =1 \\
\sigma^{2}\left(\Omega_{(i)}^{2}\right) & =1-\frac{1}{2} a^{0}\left(\Omega_{(i)}^{2}\right) \\
\sigma^{4}\left(\Omega_{(i)}^{4}\right) & =\sigma^{4}(P) \\
& =1-\frac{1}{2}\left(a^{0}(P)+a^{2}(P)\right)+\frac{1}{4} \sum_{P^{2} \in \Omega^{2}(P)} a^{0}\left(P^{2}\right)
\end{aligned}
$$

(compare Example 12.3.2) and for the volume of $P$ follows:

$$
\begin{aligned}
2 c_{4}^{-1} \operatorname{vol}_{\mathbb{X}^{4}}(P)= & \sum_{d=0}^{2} \sum_{i=1}^{\mu^{2 d}} \sigma^{2 d}\left(\Omega_{(i)}^{2 d}\right) \frac{1}{g_{i}^{2 d}} \\
= & \sum_{i=1}^{\mu^{0}} \frac{1}{g_{i}^{0}}+\sum_{i=1}^{\mu^{2}}\left(1-\frac{1}{2} a^{0}\left(\Omega_{(i)}^{2}\right)\right) \frac{1}{g_{i}^{2}} \\
& +1-\frac{1}{2}\left(a^{0}(P)+a^{2}(P)\right)+\frac{1}{4} \sum_{P^{2} \in \Omega^{2}(P)} a^{0}\left(P^{2}\right) .
\end{aligned}
$$

Now we use the identities

$$
\begin{aligned}
a^{0}(P) & =\sum_{i=1}^{\mu^{0}} l_{i}^{0}+a_{i n f}^{0}(P) \\
a^{2}(P) & =\sum_{i=1}^{\mu^{2}} l_{i}^{2} \\
\sum_{P^{2} \in \Omega^{2}(P)} a^{0}\left(P^{2}\right) & =\sum_{i=1}^{\mu^{2}} \sum_{P_{(i)}^{2} \in \Omega_{(i)}^{2}(P)} a^{0}\left(P_{(i)}^{2}\right) \\
& =\sum_{i=1}^{\mu^{2}} l_{i}^{2} a^{0}\left(\Omega_{(i)}^{2}\right)
\end{aligned}
$$

and the result follows immediately.
If the group $\Gamma$ is also torsionfree we get $g_{i}^{0}=1$ and $g_{i}^{2}=1$ for all $i \geq 1$. We have

$$
2 c_{4}^{-1} \operatorname{vol}_{\mathbb{X}^{4}}(P)=1-\frac{1}{2}\left(a^{0}(P)+a^{2}(P)\right)+\mu^{0}+\mu^{2}-\frac{1}{2} \sum_{i=1}^{\mu^{2}} a^{0}\left(\Omega_{(i)}^{2}\right)\left(1-\frac{1}{2} l_{i}^{2}\right) .
$$

### 13.3 Simple Fundamental Polytopes

In this section we will consider discrete groups with simple polytopes.

Theorem 13.3.1 Let $P$ be a simple $2 m$-dimensional normal fundamental polytope for a discrete group $\Gamma<\operatorname{Iso}\left(\mathbb{X}^{2 m}\right)$. Then

$$
\begin{aligned}
2 K^{m} c_{2 m}^{-1} \operatorname{vol}_{\mathbb{X} 2 m}(P)= & \sum_{d=0}^{m-1} \sum_{i=1}^{\mu^{2 d}}\left((-1)^{m-d} 2 a_{2 m-2 d+1} l_{i}^{2 d}+\sigma_{d}\left(\Omega_{(i)}^{2 d}\right) \frac{1}{g_{i}^{2 d}}\right) \\
& +1+(-1)^{m} 2 a_{2 m+1} a_{i n f}^{0}(P)
\end{aligned}
$$

with

$$
\sigma_{d}\left(\Omega_{(i)}^{2 d}\right)=2 \sum_{k=0}^{d}(-1)^{k} a_{2 k+1} a^{2 d-2 k}\left(\Omega_{(i)}^{2 d}\right) .
$$

Proof: We use the reduction formula for simple polytopes 11.3.3 and the cycle conditions 13.1.1 and get by a simple computation

$$
\begin{aligned}
2 K^{m} c_{2 m}^{-1} \operatorname{vol}_{\mathbb{X} 2 m}(P) & =\sum_{d=0}^{m} \sum_{P^{2 d} \in \Omega^{2 d}(P)} \sigma_{d}\left(P^{2 d}\right) \alpha_{2 m-2 d-1}\left(P^{2 d}\right) \\
& =\sum_{d=0}^{m} \sum_{i=1}^{\mu^{2 d}} \sum_{P_{(i)}^{2 d} \in \Omega_{(i)}^{2 d}} \sigma_{d}\left(P_{(i)}^{2 d}\right) \alpha_{2 m-2 d-1}\left(P_{(i)}^{2 d}\right) \\
& =\sum_{d=0}^{m} \sum_{i=1}^{\mu^{2 d}} \sigma_{d}\left(\Omega_{(i)}^{2 d}\right) \frac{1}{g_{i}^{2 d}} \\
& =\sum_{d=0}^{m-1} \sum_{i=1}^{\mu^{2 d}} \sigma_{d}\left(\Omega_{(i)}^{2 d}\right) \frac{1}{g_{i}^{2 d}}+2 \sum_{k=0}^{m}(-1)^{k} a_{2 k+1} a^{2 m-2 k}(P),
\end{aligned}
$$

where we use the definitions of the invariants $\sigma_{d}$ in Corollary 11.3.3. Now we will consider the last summand and get

$$
\begin{aligned}
\sum_{k=0}^{m}(-1)^{k} a_{2 k+1} a^{2 m-2 k}(P)= & (-1)^{m} a_{2 m+1}\left(a_{i n f}^{0}(P)+a_{o r d}^{0}(P)\right)+a_{1} \\
& +\sum_{k=1}^{m-1}(-1)^{k} a_{2 k+1} a^{2 m-2 k}(P) \\
= & (-1)^{m} a_{2 m+1}\left(a_{i n f}^{0}(P)+a_{o r d}^{0}(P)\right)+a_{1} \\
& +\sum_{k=1}^{m-1}(-1)^{m-k} a_{2 m-2 k+1} a^{2 k}(P) \\
= & \frac{1}{2}+(-1)^{m} a_{2 m+1} a_{i n f}^{0}(P)+(-1)^{m} a_{2 m+1}\left(\sum_{i=1}^{\mu^{0}} l_{i}^{0}\right) \\
& +\sum_{k=1}^{m-1}(-1)^{m-k} a_{2 m-2 k+1}\left(\sum_{i=1}^{\mu^{2 k}} l_{i}^{2 k}\right),
\end{aligned}
$$

where we have reversed the order of summation in the second step. In the third step we have used the equations 1. If we combine both formulas we get

$$
\begin{aligned}
2 K^{m} c_{2 m}^{-1} \operatorname{vol}_{\mathbb{X} 2 m}(P)= & \sum_{d=0}^{m-1} \sum_{i=1}^{\mu^{2 d}} \sigma_{d}\left(\Omega_{(i)}^{2 d}\right) \frac{1}{g_{i}^{2 d}}+1+(-1)^{m} 2 a_{2 m+1} a_{i n f}^{0}(P) \\
& +(-1)^{m} 2 a_{2 m+1} \sum_{i=1}^{\mu^{0}} l_{i}^{0}+\sum_{d=1}^{m-1}(-1)^{m-d} 2 a_{2 m-2 d+1}\left(\sum_{i=1}^{\mu^{2 d}} l_{i}^{2 d}\right) \\
= & 1+(-1)^{m} 2 a_{2 m+1} a_{i n f}^{0}(P) \\
& +(-1)^{m} 2 a_{2 m+1} \sum_{i=1}^{\mu^{0}} l_{i}^{0}+\sum_{i=1}^{\mu^{0}} \sigma_{0}\left(\Omega_{(i)}^{0}\right) \frac{1}{g_{i}^{0}} \\
& +\sum_{d=1}^{m-1} \sum_{i=1}^{\mu^{2 d}} \sigma_{d}\left(\Omega_{(i)}^{2 d}\right) \frac{1}{g_{i}^{2 d}}+\sum_{d=1}^{m-1}(-1)^{m-d} 2 a_{2 m-2 d+1}\left(\sum_{i=1}^{\mu^{2 d}} l_{i}^{2 d}\right) \\
= & \sum_{d=0}^{m-1} \sum_{i=1}^{\mu^{2 d}}\left((-1)^{m-d} 2 a_{2 m-2 d+1} l_{i}^{2 d}+\sigma_{d}\left(\Omega_{(i)}^{2 d}\right) \frac{1}{g_{i}^{2 d}}\right) \\
& +1+(-1)^{m} 2 a_{2 m+1} a_{i n f}^{0}(P) .
\end{aligned}
$$

Corollary 13.3.1 Let $P$ be a 2-dimensional (trivially simple) normal fundamental polytope in $\mathbb{X}^{2}=\mathbb{S}^{2}$ or $\mathbb{H}^{2}$. Then

$$
2 K c_{2}^{-1} \operatorname{vol}_{\mathbb{X}^{2}}(P)=\sum_{i=1}^{\mu^{0}}\left(\frac{1}{g_{i}^{0}}-\frac{1}{2} l_{i}^{0}\right)+1-\frac{1}{2} a_{i n f}^{0}(P) .
$$

Corollary 13.3.2 Let $P$ be a 4-dimensional simple normal fundamental polytope in $\mathbb{X}^{4}=\mathbb{S}^{4}$ or $\mathbb{H}^{4}$. Then

$$
2 c_{4}^{-1} \operatorname{vol}_{\mathbb{X}^{4}}(P)=\sum_{i=1}^{\mu^{0}}\left(\frac{1}{g_{i}^{0}}+l_{i}^{0}\right)+\sum_{i=1}^{\mu^{2}}\left[\left(1-\frac{1}{2} a^{0}\left(\Omega_{(i)}^{2}\right)\right) \frac{1}{g_{i}^{2}}-\frac{1}{2} l_{i}^{2}\right]+1+a_{\text {inf }}^{0}(P) .
$$

Proof: We have

$$
\begin{aligned}
& \sigma_{0}\left(\Omega_{(i)}^{0}\right)=1 \\
& \sigma_{1}\left(\Omega_{(i)}^{2}\right)=1-\frac{1}{2} a^{2}\left(\Omega_{(i)}^{2}\right)
\end{aligned}
$$

and the corollary follows immediately.
Example 13.3.1 We will compute the volumes of the normal fundamental simplices $P=P(\Gamma)$ with schemes

for $k=3,4,5$, where $\Gamma$ is the corresponding reflection group. We have $a_{\text {inf }}^{0}(P)=0, a^{0}\left(\Omega_{(i)}^{2}\right)=$ $3, \mu^{0}=5, \mu^{2}=10, l_{i}^{0}=1$ for $i=1, \ldots, 5$ and $l_{i}^{2}=1$ for $i=1, \ldots, 10$. To compute all the numbers $g_{i}^{0}$ and $g_{i}^{2}$ we pick out all elliptic subschemes $\sigma$ of order 4 and 2 with their multiplicities $\sharp(\sigma)$ and determine the orders of the corresponding (finite) groups.

| $\sigma$ | $\sharp(\sigma)$ | $k$ | Order |
| :---: | :---: | :---: | :---: |
|  | 1 |  | 14400 |
| - $\square \longrightarrow \square^{5}$ | 1 |  | 240 |
| $\bigcirc \stackrel{5}{\square} \bigcirc$ | 1 | 3 | 60 |
|  |  | 4 5 | 80 100 |
|  | 1 | 3 | 48 |
|  |  | 4 5 | 96 240 |
|  | 1 | 3 | 120 |
|  |  | 4 | 384 |
|  |  | 5 | 14400 |
| $\bigcirc \xrightarrow{5}$ | 1 |  | 10 |
| $\bigcirc$ | 2 |  | 6 |
| $\bigcirc \circ$ | 6 |  | 4 |
| $\bigcirc \xrightarrow{k}$ | 1 | 3 | 6 |
|  |  | 4 | 8 |
|  |  |  | 10 |

Hence we get

$$
\operatorname{vol}_{\mathbb{H}^{4}}(P)=\left\{\begin{array}{lll}
\frac{1}{10800} \pi^{2} & , & k=3 \\
\frac{17}{21600} \pi^{2} & , & k=4 \\
\frac{13}{5400} \pi^{2} & , & k=5
\end{array} .\right.
$$

## 14 Examples: Volumes of Polytopes

In this section we will apply the reduction formulas to compute the volume of some 4 -dimensional polytopes. The main problem of this computation is the decoding of the combinatorial structure of the polytopes. For polytopes with known Gram matrix or graph (and with few facets) we can use the GIA or the SIA, described in the Sections 5.3 and 6. For polytopes with many facets we use our geometrical imagination.
A nice tool to describe the combinatorics of a 4-dimensional polytope is a so-called Schlegel diagram. A Schlegel diagram can be constructed in the following way. We choose a point $p$ outside an $n$-dimensional polytope $P$ and in some way "near" an arbitrary facet $F$. Then we project the whole polytope from the point $p$ into the facet $F$. We get an $(n-1)$-dimensional polytopal complex which is called a Schlegel diagram of $P$. A Schlegel diagram determines the complete combinatorics of $P$ (compare [ Z ], section 5).
Let $P$ be a 4-dimensional Coxeter polytope in $\mathbb{X}^{4}=\mathbb{S}^{4}$ or $\mathbb{H}^{4}$ with scheme $S=S(P)$ and Gram matrix $G=G(P)$. The group $\Gamma$ generated by the reflections in the facets of $P$ is discrete and $P$ is a fundamental polytope for $\Gamma$. So we have $\operatorname{covol}(\Gamma)=\operatorname{vol}_{\mathbb{X}^{4}}(P)$ and for the volume of $P$ we get with Corollary 11.3.2

$$
\operatorname{vol}_{\mathbb{X}^{4}}(P)=\frac{\pi^{2}}{3}\left[\kappa^{4}(P)-2 \sum_{P^{2} \in \Omega^{2}(P)}\left(a^{0}\left(P^{2}\right)-2\right) \alpha_{1}\left(P^{2}\right)+4 \sum_{P^{0} \in \Omega^{0}(P)} \alpha_{3}\left(P^{0}\right)\right]
$$

with

$$
\kappa^{4}(P):=4-2\left(a^{0}(P)+a^{2}(P)\right)+\sum_{P^{2} \in \Omega^{2}(P)} a^{0}\left(P^{2}\right) .
$$

### 14.1 Volumes of the Tumarkin Pyramids

We want to compute the volumes of the 4-dimensional hyperbolic Coxeter polytopes, classified by P. Tumarkin [T]. From the combinatorial point of view each of these polytopes $P$ is a pyramid over a 3 -dimensional simplicial prism. All combinatorial datas can be read off from the Schlegel


Figure 14: Pyramid over a Prism
diagram (see Figure 14) and we find: $a^{0}(P)=7, a^{1}(P)=15, a^{2}(P)=14$ and $a^{3}(P)=6$. The set of 2-dimensional faces consists of 11 triangles and 3 rectangles; so we get $\kappa^{4}(P)=7$. If we decode the connection between angles and combinatorics we get the results stated in the Appendix (compare Section 16.6). For the volume of the 3-dimensional angles use the Appendix 16.4.

### 14.2 Volumes of the Kaplinskaja Prisms

We want to compute the volumes of the 4-dimensional hyperbolic Coxeter polytopes, classified by Kaplinskaja [Kapl]. From the combinatorial point of view each of these polytopes $P$ is a prism over a 3-dimensional simplex. All combinatorial datas can be read off from the Schlegel diagram


Figure 15: Prism over a Simplex
(see Figure 15) and we find: $a^{0}(P)=6, a^{1}(P)=16, a^{2}(P)=14$ and $a^{3}(P)=6$. We have two simplicial facets and four facets which are prisms over a triangle. The set of 2 -dimensional faces consists of 8 triangles and 6 rectangles; so we get $\kappa^{4}(P)=8$. If we decode the connection between angles and combinatorics we get the results stated in the Appendix (compare Section 16.6).

### 14.3 Volumes of the Esselmann Polytopes

We want to compute the volumes of the compact hyperbolic Coxeter polytopes with 6 facets in $\mathbb{H}^{4}$, classified by F. Esselmann [Es]. From the combinatorial point of view each of these polytopes $P$ is a product of two 2-dimensional simplices and so is simple. Furthermore, all


Figure 16: The Product of two 2-simplices
combinatorial datas can be read off from the Schlegel diagram (see Figure 16) and we find: $a^{0}(P)=9, a^{1}(P)=18, a^{2}(P)=15$ and $a^{3}(P)=6$. All facets are simplicial prisms and the set of 2 -dimensional faces consists of 6 triangles and 9 rectangles; so we get $\kappa^{4}(P)=10$. If we decode the connection between angles and combinatorics we get the results stated in the Appendix (compare Section 16.6).

### 14.4 Volumes of Regular Polytopes

Let $R=R_{p_{1} p_{2} p_{3}}$ be a regular polytope in $\mathbb{X}^{4}=\mathbb{S}^{4}$ or $\mathbb{X}^{4}$ with the Schläfli symbol $\left\{p_{1}, p_{2}, p_{3}\right\}$. This means that $p_{i}=p_{i}(R)$ is the number of $i$-dimensional faces of $R$ containing an $(i-2)$ dimensional face $R^{i-2}$ and contained in an $(i+1)$-dimensional face $R^{i+1}$ for all $i \geq 1$. Let $\alpha_{1}=\alpha_{1}\left(R^{2}\right)$ and $\alpha_{3}=\alpha_{3}\left(R^{0}\right)$ be the 1- and 3-dimensional angle of $R$. There are only six combinatorially different regular polytopes in $\mathbb{X}^{4}$. We describe them in Table 2 where

$$
\begin{aligned}
\kappa^{4}(R) & =4-2\left(a^{0}(R)+a^{2}(R)\right)+\sum_{R^{2} \in \Omega^{2}(R)} a^{0}\left(R^{2}\right) \\
& =4+\left(p_{1}-2\right) a^{2}(R)-2 a^{0}(R) .
\end{aligned}
$$

| Name | Notation | $p_{1}$ | $p_{2}$ | $p_{3}$ | $a^{0}$ | $a^{1}$ | $a^{2}$ | $a^{3}$ | $\kappa^{4}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 5 -cell | $R_{333}$ | 3 | 3 | 3 | 5 | 10 | 10 | 5 | 4 |
| 8 -cell | $R_{433}$ | 4 | 3 | 3 | 16 | 32 | 24 | 8 | 20 |
| 16 -cell | $R_{334}$ | 3 | 3 | 4 | 8 | 24 | 32 | 16 | 20 |
| 24 -cell | $R_{343}$ | 3 | 4 | 3 | 24 | 96 | 96 | 24 | 52 |
| 120 -cell | $R_{533}$ | 5 | 3 | 3 | 600 | 1200 | 720 | 120 | 964 |
| 600 -cell | $R_{335}$ | 3 | 3 | 5 | 120 | 720 | 1200 | 600 | 964 |

Table 2: Regular Polytopes

The general volume formula reduces to

$$
\operatorname{vol}_{\mathbb{X}^{4}}(R)=\frac{\pi^{2}}{3}\left[4+a^{2}(R)\left(p_{1}-2\right)\left(1-2 \alpha_{1}\right)-2 a^{0}(R)\left(1-2 \alpha_{3}\right)\right] .
$$

If $R$ is an ideal regular $a^{3}(R)$-cell in $\mathbb{H}^{4}$, which implies that all 3 -angles $\alpha_{3}$ are zero, the volume formula simplifies to

$$
\operatorname{vol}_{\mathbb{X}^{4}}(R)=\frac{\pi^{2}}{3}\left[4+a^{2}(R)\left(p_{1}-2\right)\left(1-2 \alpha_{1}\right)-2 a^{0}(R)\right] .
$$

For the dihedral angles in the Euclidean and in the ideal hyperbolic case, as well as for the volumes of the ideal hyperbolic regular polytopes see the results in Table 3.

| Polytope | $\alpha_{1}$ |  | Ideal |
| :---: | :---: | :---: | :---: |
| $R_{333}$ | $\frac{1}{2 \pi} \arccos \left(\frac{1}{3}\right)$ | $\arccos \left(\frac{1}{4}\right)$ | $\frac{2}{3} \pi\left(2 \pi-5 \arccos \left(\frac{1}{3}\right)\right)$ |
| $R_{433}$ | $\frac{1}{2 \pi} \arccos \left(\frac{1}{3}\right)$ | $\frac{1}{4}$ | $\frac{4}{3} \pi\left(5 \pi-4 \arccos \left(\frac{1}{3}\right)\right)$ |
| $R_{334}$ | $\frac{1}{\pi} \arccos \left(\frac{1}{\sqrt{3}}\right)$ | $\frac{1}{3}$ | $\frac{4}{3} \pi\left(5 \pi-16 \arccos \left(\frac{1}{\sqrt{3}}\right)\right)$ |
| $R_{343}$ | $\frac{1}{4}$ | $\frac{1}{3}$ | $\frac{4}{3} \pi^{2}$ |
| $R_{533}$ | $\frac{1}{2 \pi} \arccos \left(\frac{1}{3}\right)$ | $\frac{2}{5}$ | $\frac{4}{3} \pi\left(241 \pi-180 \arccos \left(\frac{1}{3}\right)\right)$ |
| $R_{335}$ | $\frac{1}{2}-\frac{1}{2 \pi} \arccos \left(\frac{\sqrt{5}}{3}\right)$ | $1-\frac{1}{2 \pi} \arccos \left(\frac{3 \sqrt{5}+1}{8}\right)$ | $\frac{8}{3} \pi\left(83 \pi-75 \arcsin \left(\frac{\sqrt{5}}{3}\right)\right)$ |

Table 3: Hyperbolic Regular Polytopes

### 14.5 A Fundamental Polytope for $\operatorname{PSL}\left(2, \mathbb{Z}\left[i_{1}, i_{2}\right]\right)$

Let $\mathbb{H}^{4}$ be the hyperbolic space in the upper half-space model

$$
I U^{4}=\left\{x_{0}+x_{1} i_{1}+x_{2} i_{2}+x_{3} i_{3}: x_{i}>0 \text { for all } i=0, \ldots, 3\right\} .
$$

We follow C. Maclachlan, P.L. Waterman and N.J.Wielenberg [MWW] and construct a polytope $P$ as the convex hull of the following set of points

$$
\begin{aligned}
\mathbf{S}= & \left\{i_{3}, \frac{1}{2}+\frac{\sqrt{3}}{2} i_{3}, \frac{1}{2}+\frac{1}{2} i_{1}+\frac{\sqrt{2}}{2} i_{3}, \frac{1}{2} i_{1}+\frac{\sqrt{3}}{2} i_{3}, \frac{1}{2} i_{2}+\frac{\sqrt{3}}{2} i_{3}, \frac{1}{2}+\frac{1}{2} i_{2}+\frac{\sqrt{2}}{2} i_{3},\right. \\
& \left.\frac{1}{2}+\frac{1}{2} i_{1}+\frac{1}{2} i_{2}+\frac{1}{2} i_{3}, \frac{1}{2} i_{1}+\frac{1}{2} i_{2}+\frac{\sqrt{2}}{2} i_{3}, \infty\right\} .
\end{aligned}
$$

Now $\operatorname{conv}(\mathbf{S}-\infty)$ is a cube (see Figure 17), lying on the hyperplane $H_{3}=\left\{x_{0}+x_{1} i_{1}+x_{2} i_{2}+x_{3} i_{3}\right.$ : $\left.\sum x_{i}^{2}=1 ; x_{3}>0\right\}$ and $P:=\operatorname{conv}(\mathbf{S})$ can be viewed as a cone with basis $\operatorname{conv}(\mathbf{S}-\infty)$ and center $\infty$. The scheme of $P$ is

which we denote by $\left[\infty, 3, \begin{array}{c}3, \infty \\ 3, \infty\end{array}\right]$. We obtain $a_{\text {ord }}^{0}(P)=8, a_{\text {inf }}^{0}(P)=1, a^{1}(P)=20, a^{2}(P)=18$ and $a^{3}(P)=7$. The set of 2-dimensional faces of $P$ splits into a set of 12 triangles with dihedral


Figure 17: $\operatorname{conv}(\mathbf{S}-\infty)$
angles $1 / 4$, a set of 3 rectangles with dihedral angles $1 / 4$ and a set of 3 rectangles with dihedral angles $1 / 6$. Furthermore we have $\sum_{i=1}^{a^{2}(P)} a^{0}\left(P_{i}^{2}\right)=60$ and $\kappa^{4}(P)=10$.
The vertex at infinity of $P$ has a Euclidean vertex figure of type $[\infty, 2, \infty, 2, \infty]$ (cube). In the set of the 8 ordinary vertices of $P$ we find the following types of vertex figures: one vertex figure of type $[2,2,2]$ which represents an angle of $1 / 16$, three vertex figures of type [2, 2, 3] (angle $1 / 24$ ), three vertex figures of type $[2,3,3]$ (angle $1 / 48$ ), and one vertex figure of type $\left[3,3^{1,1}\right]$ (angle 1/192) (compare 16.4). Therefore we obtain

$$
\begin{aligned}
\operatorname{vol}_{\mathbb{H}^{4}}(P) & =\frac{\pi^{2}}{3}\left[10-2\left(12 \frac{1}{4}+6 \frac{1}{4}+6 \frac{1}{6}\right)+4\left(\frac{1}{16}+3 \frac{1}{24}+3 \frac{1}{48}+\frac{1}{192}\right)\right] \\
& =\frac{1}{144} \pi^{2} .
\end{aligned}
$$

From the results of N.W. Johnson and A.I. Weiss [JW] we know that the group $\left[\infty, 3,3, \begin{array}{l}3, \infty \\ 3, \infty\end{array}\right]^{+}$of orientation preserving elements in $\left[\infty, 3, \begin{array}{c}3, \infty \\ 3, \infty\end{array}\right]$ is isomorphic to the group $\operatorname{PSL}\left(2, \mathbb{Z}\left[i_{1}, i_{2}\right]\right)$. It follows that

$$
\begin{aligned}
\operatorname{covol}\left(P S L\left(2, \mathbb{Z}\left[i_{1}, i_{2}\right]\right)\right) & =2 \operatorname{covol}\left(\left[\infty, 3, \begin{array}{l}
3, \infty \\
3, \infty
\end{array}\right]\right) \\
& =\frac{1}{72} \pi^{2}
\end{aligned}
$$

### 14.6 The Ivanšić Polytope

The following construction is copied quite directly from the paper of D. Ivanšić $[\mathrm{I}]$. Consider the planes that bound the rectangular box $R \subset \mathbb{R}^{3}, R=[-2,2] \times[-2,2] \times[-2 \sqrt{2}, 2 \sqrt{2}]$. Add


Figure 18: The Boundary Spheres
to them the 12 spheres of radius $\sqrt{2}$ with centers $( \pm 1, \pm 1, j 2 \sqrt{2})$ for $j=-1,0,1$ and the 18 spheres of the same radius with centers $(j, k, \pm \sqrt{2})$ for $j, k=-2,0,2$ (Figure 18).
Each of the 6 planes that comprise the boundary of the rectangular box and each of the 30 spheres determine a hyperplane in $\mathbb{H}^{4}=\left\{(x, y, z, t) \in \mathbb{R}^{4}: t>0\right\}$ that divides $\mathbb{H}^{4}$ into two half-spaces. For the spheres we choose the half-spaces whose boundary at infinity is unbounded in $\mathbb{R}^{3}$, for the planes the half-spaces so that the intersection of their boundaries at infinity is the rectangular box $R$. The polytope $P$ is defined as the intersection of those half-spaces. It has the following combinatorial data: $a_{\text {inf }}^{0}(P)=36, a_{\text {ord }}^{0}(P)=48, a^{1}(P)=216, a^{2}(P)=168$ and $a^{3}(P)=36$.
The set of 2 -dimensional faces of $P$ splits into a set of 16 triangles with dihedral angles $1 / 8$, a set of 8 rectangles with dihedral angles $1 / 8$, a set of 24 triangles with dihedral angle $1 / 4$, a set of 112 rectangles with dihedral angles $1 / 4$ and a set of 6 hexagons with dihedral angles $1 / 4$. Furthermore we have $\sum_{i=1}^{a^{2}(P)} a^{0}\left(P_{i}^{2}\right)=648$ and $\kappa^{4}(P)=148$.
In the set of the 48 ordinary vertex figures of $P$ we have 16 vertex figures of type [2, 2, 2] which represent an angle of $1 / 32$ and 32 vertex figures of type [4, 2, 2] also representing an angle of $1 / 32$ (compare 16.4). Thus we get for the volume of $P$

$$
\begin{aligned}
\operatorname{vol}_{\mathbb{H}^{4}}(P) & =\frac{\pi^{2}}{3}\left[148-2\left(16 \frac{1}{8}+16 \frac{1}{8}+124 \frac{1}{4}+24 \frac{1}{4}+32 \frac{1}{4}\right)+4\left(16 \frac{1}{16}+32 \frac{1}{32}\right)\right] \\
& =\frac{8}{3} \pi^{2} .
\end{aligned}
$$

Furthermore, we have the following types of 3 -faces (with the notations from the paper [ I$]$ ):
For instance, the hexagonal 2-face in the 3 -face $C_{1}$ is the intersection of $C_{1}$ with $Y_{1}^{\prime}$.


Figure 19: The 3-dimensional Faces of $P$

## 15 Examples: Volumes of Orbifolds

Let $M=\mathbb{X}^{2 m} / \Gamma$ be a geometric orbifold of dimension $2 m$ with $\mathbb{X}^{2 m}=\mathbb{S}^{2 m}$ or $\mathbb{H}^{2 m}$ such that there exists a normal fundamental polytope $P$ for the discrete group $\Gamma$. Then we have by Theorem 13.2.2

$$
\begin{aligned}
\operatorname{vol}_{\mathbb{X}^{2 m}}(M) & =\operatorname{covol}_{\mathbb{X} 2 m}(\Gamma)=\operatorname{vol}_{\mathbb{X} 2 m}(P) \\
& =\sum_{d=0}^{m} \sum_{i=1}^{\mu^{2 d}} \sigma^{2 d}\left(\Omega_{(i)}^{2 d}\right) \frac{1}{g_{i}^{2 d}} .
\end{aligned}
$$

In the special case where $M=\mathbb{H}^{4} / \Gamma$ is a 4 -dimensional hyperbolic manifold with a normal fundamental polytope $P$ for the discrete group $\Gamma$ we have with Corollary 13.2.3 resp. 13.2.1

$$
\begin{aligned}
\operatorname{vol}_{\mathbb{H}^{4}}(M) & =\operatorname{vol}_{\mathbb{T}^{4}}(P) \\
& =\frac{1}{3} \pi^{2}\left[4-2\left(a^{0}(P)+a^{2}(P)\right)+2\left(\mu^{0}+\mu^{2}\right)-2 \sum_{i=1}^{\mu^{2}} a^{0}\left(\Omega_{(i)}^{2}\right)\left(1-\frac{1}{2} l_{i}^{2}\right)\right] \\
& =\frac{4}{3} \pi^{2} \sum_{d=0}^{n}(-1)^{d} \mu^{d} .
\end{aligned}
$$

### 15.1 The 24-cell Manifold

Let $M=\mathbb{H}^{4} / \Gamma$ be the ideal 24-cell manifold. Then $M$ can be constructed from the ideal regular hyperbolic 24 -cell $P$ with dihedral angles $1 / 4$ by pairing opposite sides. Of course, we could use the result from Section 14.4, where we have computed the volume of this polytope. However, we will go the way described above. Each facet of $P$ is a octahedron and each 2-dimensional face is a triangle, so

$$
a^{0}\left(\Omega_{(i)}^{2}\right)=3
$$

for all $i=1, \ldots, \mu^{2}$. For the details of the constuction see $[\mathrm{R}]$ (page 509). Of course, $P$ is a fundamental polytope for $\Gamma$. The group $\Gamma$ is torsionfree and so all stabilizers are trivial. We have $a^{0}(P)=a_{\text {inf }}^{0}(P)=24, a^{2}(P)=96, \mu^{0}=0$ (all vertices are at infinity), $\mu^{2}=96$ and $l_{i}^{2}=4$ for all $i=1, \ldots, \mu^{2}$. So we get

$$
\operatorname{vol}_{\mathbb{H}^{4}}(M)=\frac{4}{3} \pi^{2} .
$$

Furthermore, we remark that $\chi(M)=1$.

### 15.2 The Davis Manifold

Let $M=\mathbb{H}^{4} / \Gamma$ be the (compact) Davis manifold. Then $M$ can be constructed from the regular hyperbolic 120 -cell $P$ with dihedral angles $1 / 5$ by pairing opposite sides. Each facet of $P$ is a dodecahedron and each 2-dimensional face is a pentagon, so

$$
a^{0}\left(\Omega_{(i)}^{2}\right)=5
$$

for all $i=1, \ldots, \mu^{2}$. For the details of the constuction see $[\mathrm{D}]$ or $[\mathrm{R}]$ (page 505). Clearly, $P$ is a fundamental polytope for the torsionfree group $\Gamma$. We have $a^{0}(P)=a_{\text {ord }}^{0}(P)=600$, $a^{2}(P)=720, \mu^{0}=1$ (all vertices are ordinary and will be paired), $\mu^{2}=144$ and $l_{1}^{0}=600$ and $l_{i}^{2}=5$ for all $i=1, \ldots, \mu^{2}$. So we get

$$
\operatorname{vol}_{\mathbb{H}^{4}}(M)=\frac{104}{3} \pi^{2}=\frac{4}{3} \pi^{2} \chi(M),
$$

where $\chi(M)=26$ is the Euler-Poincaré characteristic of the Davis manifold.

### 15.3 Ivanšić's Manifolds

Let $M_{i}=\mathbb{H}^{4} / \Gamma_{i}(i=1,2)$ be the 4-dimensional manifolds, described by D. Ivanšić [I]. Both $M_{1}$ and $M_{2}$ can be constructed by pairing sides of the Ivanšić polytope (compare Section 14.6) and so in principle we know the volume of these manifolds. However, we will go another way. From the constructions (compare [I], Lemma 5.1) we can read off the numbers

$$
\begin{aligned}
\mu^{0} & =2 \\
\mu^{1}=17+5 & =22 \\
\mu^{2}=36+3 & =39 \\
\mu^{3} & =18 \\
\mu^{4} & =1 .
\end{aligned}
$$

We remark that the two manifolds differ in the number of cusps (we have $m^{0}=7$ for $M_{1}$ and $m^{0}=8$ for $M_{2}$ ). Hence we can use Corollary 13.2.1 and get

$$
\begin{aligned}
\operatorname{vol}_{\mathbb{H}^{4}}(M) & =\operatorname{vol}_{\mathbb{H}^{4}}(P) \\
& =\frac{4}{3} \pi^{2} \sum_{d=0}^{4}(-1)^{d} \mu^{d} \\
& =\frac{8}{3} \pi^{2} .
\end{aligned}
$$

| $m$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $q_{m}$ | 1 | $\frac{1}{2}$ | $-\frac{1}{8}$ | $\frac{1}{16}$ | $-\frac{5}{128}$ | $\frac{7}{256}$ | $-\frac{21}{1024}$ | $\frac{33}{2048}$ | $-\frac{429}{32768}$ | $\frac{715}{65536}$ | $-\frac{2431}{262144}$ | $\frac{4199}{524288}$ | $-\frac{29393}{4194304}$ | $\frac{52003}{8388608}$ | $-\frac{185725}{33554432}$ |
| $B_{m}$ | 1 | $-\frac{1}{2}$ | $\frac{1}{6}$ | 0 | $-\frac{1}{30}$ | 0 | $\frac{1}{42}$ | 0 | $-\frac{1}{30}$ | 0 | $\frac{5}{66}$ | 0 | $-\frac{691}{2730}$ | 0 | $\frac{7}{6}$ |
| $a_{m}$ | - | $\frac{1}{2}$ | - | $\frac{1}{4}$ | - | $\frac{1}{2}$ | - | $\frac{17}{8}$ | - | $\frac{31}{2}$ | - | $\frac{691}{4}$ | - | $\frac{5461}{2}$ | - |
| $G_{m}$ | - | 1 | - | 1 | - | 3 | - | 17 | - | 155 | - | 2073 | - | 38227 | - |
| $T_{m}$ | - | 1 | - | 2 | - | 16 | - | 272 | - | 7936 | - | 353792 | - | 22368256 | - |
| $E_{m}$ | 1 | - | 1 | - | 5 | - | 61 | - | 1385 | - | 50521 | - | 2702765 | - | 199360981 |
| $Z(m)$ | - | 1 | 2 | 4 | 10 | 32 | 122 | 544 | 2770 | 15872 | 101042 | 707584 | 5405530 | 44736512 | 398721962 |



|  | 1. step |  | 2. step |  | 3. step |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\omega_{2 m-1}$ |  | $\omega_{2 m-1}$ |  | $\omega_{2 m-1}$ |  | $\omega_{2 m-1}$ |
| $-\omega_{2 m-2}$ |  |  |  |  |  |  |
| $\omega_{2 m-3}$ | $-\frac{1}{2} 2\binom{2}{1} \omega_{2 m-3}=$ | $-\omega_{2 m-3}$ |  | $-\omega_{2 m-3}$ |  | $-\omega_{2 m-3}$ |
| $-\omega_{2 m-4}$ | $+\frac{1}{2} 2^{2}\binom{3}{1} \omega_{2 m-4}=$ | $5 \omega_{2 m-4}$ |  |  |  |  |
| $\omega_{2 m-5}$ | $-\frac{1}{2} 2^{3}\binom{4}{1} \omega_{2 m-5}=$ | $-15 \omega_{2 m-5}$ | $+\frac{5}{2} 2\binom{4}{3} \omega_{2 m-5}=$ | ${ }_{5} \omega_{2 m-5}$ |  | $5 \omega_{2 m-5}$ |
| $-\omega_{2 m-6}$ | $+\frac{1}{2} 2^{4}\binom{5}{1} \omega_{2 m-6}=$ | $39 \omega_{2 m-6}$ | $-\frac{5}{2} 2^{2}\binom{5}{3} \omega_{2 m-6}=$ | ${ }^{-61} \omega_{2 m-6}$ |  |  |
| $\omega_{2 m-7}$ | $-\frac{1}{2} 2^{5}\binom{6}{1} \omega_{2 m-7}=$ | $-95 \omega_{2 m-7}$ | $+\frac{5}{2} 2^{3}\binom{6}{3} \omega_{2 m-7}=$ | $305 \omega_{2 m-7}$ | $-\frac{61}{2} 2\binom{6}{5} \omega_{2 m-7}=$ | $-61 \omega_{2 m-7}$ |
|  |  |  |  |  |  |  |

### 16.4 Coxeter Simplices in $\mathbb{S}^{3}$

| Scheme | Denoting | Volume | Value | Order of the groupe |
| :---: | :---: | :---: | :---: | :---: |
| $k \circ$ - | $[k, 2, l]$ | $\frac{1}{2 k l} \pi^{2}$ |  | 4 kl |
| $5-$ - 0 | $[5,3,3]$ | $\frac{1}{7200} \pi^{2}$ | 0.00137078 | 14400 |
| $\bigcirc-\bigcirc$ | $[3,4,3]$ | $\frac{1}{576} \pi^{2}$ | 0.01713473 | 1152 |
| $4-\mathrm{O}-\mathrm{O}$ | $[4,3,3]$ | $\frac{1}{192} \pi^{2}$ | 0.05140419 | 384 |
| - $-\bigcirc$ | [2, 3, 5] | $\frac{1}{120} \pi^{2}$ | 0.08224670 | 240 |
| - - - - | [3, 3, 3] | $\frac{1}{60} \pi^{2}$ | 0.16449341 | 120 |
| - $-\longrightarrow$ | [2, 3, 4] | $\frac{1}{48} \pi^{2}$ | 0.20561676 | 96 |
| $\bigcirc \mathrm{O}-\mathrm{O}$ | [2, 3, 3] | $\frac{1}{24} \pi^{2}$ | 0.41123352 | 48 |
|  | $\left[3,3^{1,1}\right]$ | $\frac{1}{24} \pi^{2}$ | 0.41123352 | 192 |

### 16.5 Coxeter Polytopes in $\mathbb{E}^{3}$

| Scheme | Denoting | Type |
| :---: | :---: | :---: |
| $\bigcirc \bigcirc \bigcirc$ | $[\infty, 2, \infty, 2, \infty]$ | cube |
| $\bigcirc \bigcirc \bigcirc$ | $[\infty, 2,4,4]$ | prism |
| $\infty$ ○ $-\ldots$ | [ $\infty, 2,3,6]$ | prism |
| $\circ \infty$ | $\left[\infty, 2,3{ }^{[3]}\right]$ | prism |
| $\bigcirc \xrightarrow{4} \mathrm{O}-$ | $[4,3,4]$ | simplex |
|  | $\left[4,3^{1,1}\right]$ | simplex |
|  | $\left[\left(3^{4}\right)\right]$ | simplex |

### 16.6 Coxeter Polytopes in $\mathbb{H}^{4}$

Tumarkin Pyramids (Non-Compact, $\kappa^{4}=7$ )

| Scheme | Denoting | $k$ | $l$ | Volume | Value |
| :--- | :--- | :--- | :--- | :--- | :--- |
|  | $[4,4,(k, \infty, l)]$ | 2 | 3 | $\frac{1}{144} \pi^{2}$ | 0.0685389195 |



| Scheme | Denoting | $k$ | $l$ | Volume | Value |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | $[(3,4,4,3),(k, \infty, l)]$ | 2 | 3 | $\frac{1}{72} \pi^{2}$ | 0.1370778390 |
|  | 2 | 4 | $\frac{1}{36} \pi^{2}$ | 0.2741556779 |  |
|  | 3 | 3 | $\frac{1}{36} \pi^{2}$ | 0.2741556779 |  |
|  | 3 | 4 | $\frac{1}{24} \pi^{2}$ | 0.4112335169 |  |
|  | 4 | 4 | $\frac{1}{18} \pi^{2}$ | 0.5483113558 |  |
|  |  |  |  |  |  |

Kaplinskaja prisms (Compact, $\kappa^{4}=8$ )

| Scheme | Denoting | Volume | Value |
| :---: | :---: | :---: | :---: |
| $0 \cdot \ldots . .0-$ - - - 5 - 0 | [... $3,3,5,3]$ | $\frac{41}{10800} \pi^{2}$ | 0.037467943 |
| $0 \cdots . . .0-0-4$. | [... $3,4,3,5]$ | $\frac{17}{4320} \pi^{2}$ | 0.038838721 |
|  | $[. ., 3,3,5]$ | $\frac{61}{10800} \pi^{2}$ | 0.0557449878 |
|  | [.., $(3,3,4,3,3)]$ | $\frac{61}{5400} \pi^{2}$ | 0.1114899757 |
|  | $[. ., 3,(3,4,3,3)]$ | $\frac{7}{480} \pi^{2}$ | 0.1439317309 |
|  | $[. ., 3,(3,5,3,3)]$ | $\frac{211}{10800} \pi^{2}$ | 0.1928228268 |

Kaplinskaja prisms (Non-compact, $\kappa^{4}=8$ )

| Scheme | Denoting | Volume | Value |
| :---: | :---: | :---: | :---: |
| $[. ., 3,(3,3,3,4)]$ | $\frac{53}{4320} \pi^{2}$ | 0.1210854244 |  |
|  | $[\ldots, 3,(3,4,3,4)]$ | $\frac{1}{54} \pi^{2}$ | 0.1827704519 |

Esselmann Polytopes (compact, $\kappa^{4}=10$ )

| Scheme | Denoting | Volume | Value |
| :---: | :---: | :---: | :---: |
| $\bigcirc-8$ - $0-4$ | [8, 3, 4, 3, 8] | $\frac{11}{1728} \pi^{2}$ | 0.062827343 |
|  | $[4,5,3,30]$ | $\frac{221}{21600} \pi^{2}$ | 0.10098067 |
|  | $[8,3,4,(3,4,3)]$ | $\frac{11}{864} \pi^{2}$ | 0.12565469 |
|  | $[4,5,3,5,3)]$ | $\frac{221}{10800} \pi^{2}$ | 0.20196135 |
|  | $[(3,5,5,3), 3,10]$ | $\frac{221}{10800} \pi^{2}$ | 0.20196135 |
|  | $[(3,4,3), 4,(3,4,3)]$ | $\frac{11}{432} \pi^{2}$ | 0.25130937 |
|  | $[(3,5,5,3),(3,5,3)]$ | $\frac{221}{5400} \pi^{2}$ | 0.4039227 |

## 17 Notations

| $\mathbb{S}^{n}$ | $n$-dimensional spherical space |
| :---: | :---: |
| $\mathbb{E}^{n}$ | $n$-dimensional Euclidean space |
| $\mathbb{H}^{n}$ | $n$-dimensional hyperbolic space |
| $\mathbb{X}^{n}$ | one of the spaces $\mathbb{S}^{n}, \mathbb{E}^{n}$ or $\mathbb{H}^{n}$ |
| $H^{n}$ | vector space model of $\mathbb{H}^{n}$ |
| $\mathbb{B}^{n}$ | ball model of $\mathbb{H}^{n}$ |
| $\mathbb{I N}^{n}$ | projective disc model of $\mathbb{H}^{n}$ |
| $U^{n}$ | upper half-space model of $\mathbb{H}^{n}$ |
| conv | convex hull of a pointset |
| $P$ | polytope |
| $\mathrm{P}^{\mathbf{n}}$ | set of $n$-dimensional polytopes in $\mathbb{X}^{n}$ |
| $\mathbf{P}_{\sim}^{\sim}$ | set of combinatorial equivalence classes |
| $C l(P)$ | set of polytopes, combinatorial equivalent to $P$ |
| $\Pi$ | polytopal complex |
| $\Pi^{\mathbf{n}}$ | set of $n$-dimensional polytopal complexes in $\mathbb{X}^{n}$ |
| $F(P)$ | face poset of $P$ |
| $\mathcal{F}(P)$ | complex of the polytope $P$ |
| $\overline{\mathcal{F}}(P)$ | extended complex of the polytope $P$ |
| $G(P)$ | Gram matrix of $P$ |
| $M(P)$ | incidence matrix of $P$ |
| $L\left(P^{k}\right)$ | face figure or link in $P^{k}\left(P^{k}\right.$ face of $\left.P\right)$ |
| $S\left(P^{k}\right)$ | scheme of $L\left(P^{k}\right)$ |
| $T$ | simplex |
| C | cone |
| W | cube |
| $W^{*}$ | dual cube (cross polytope) |
| $R_{p_{1} p_{2} p_{3}}$ | regular polytope in $\mathbb{X}^{4}$ with Schläfli symbol $\left\{p_{1} p_{2} p_{3}\right\}$ |
| $\Omega^{d}()$ | set of $d$-dimensional ordinary faces |
| $\Upsilon^{0}()$ | set of vertices at infinity |
| $\#$ | number of elements of a set |
| $a^{d}$ | $\sharp \Omega^{d}()$ number of $d$-dimensional faces |
| $a_{o r d}^{0}$ | $\sharp \Omega^{0}()$ number of ordinary vertices |
| $a_{\text {inf }}^{0}$ | $\sharp \Upsilon^{0}()$ number of vertices at infinity |
| $\chi_{c}$ | combinatorial Euler-Poincaré characteristic |
| $\chi_{g}$ | geometrical Euler-Poincaré characteristic |
| $\alpha_{n-k-1}\left(P^{k} \mid P\right)$ | $(n-k-1)$-dimensional angle of $P$ with apex $P^{k}$ for $k=0, \ldots, n$ |
| $\beta_{n-k-1}\left(P^{k} \mid P\right)$ | $:=\frac{1}{2}-\alpha_{n-k-1}\left(P^{k} \mid P\right)$ for $k=0, \ldots, n$ |
| $\alpha_{n-k-1}^{\Pi}\left(P^{k}\right)$ | complex angle with apex $P^{k}$ |
| $\omega_{n-k-1}(P)$ | $:=\sum_{P^{k} \in \Omega^{k}(P)} \alpha_{n-k-1}\left(P^{k} \mid P\right)$ for $k=0, \ldots, n$ |
| $\omega_{n}(P)$ | $:=c_{n}^{-1} \operatorname{vol}(P)$ |
| $W(P)$ | $\sum_{k=0}^{n}(-1)^{k} \omega_{n-k-1}(P)$ |


| $\mathcal{G}^{d}()$ | $d$-dimensional skeleton |
| :---: | :---: |
| $\mathcal{D}()$ | polytopal decomposition |
| $\mathcal{K}(P)$ | (canonical) decomposition of $P$ in cones |
| $\mathcal{S}(C)$ | decomposition of a cone $C$ in simplices |
| $\mathcal{B}()$ | barycentric decomposition |
| $b$ ( ) | barycenter |
| $\Omega_{l}^{k}(\Pi, \mathcal{D}(\Pi))$ | $k$-dimensional elements in $\mathcal{D}$ in the $l$-skeleton of $\Pi$ |
| $Z_{l}^{k}(\Pi, \mathcal{D}(\Pi))$ | number of elements in $\Omega_{l}^{k}(\Pi, \mathcal{D}(\Pi))$ |
| $z\left(k, P^{l}, \mathcal{D}(\Pi)\right)$ | $:=Z_{l}^{k}\left(P^{l}, \mathcal{D}(\Pi) \cap P^{l}\right)$ for all $P^{l} \in \Omega^{l}(\Pi)$ |
| $B_{l}^{k}(\Pi)$ | $:=Z_{l}^{k}(\Pi, \mathcal{B}(\Pi))$ |
| $b\left(k, P^{l}\right)$ | $:=B_{l}^{k}\left(P^{l}\right)$ |
| $b(k, l)$ | $:=b\left(k, T^{l}\right)$ where $T^{l}$ is a $l$-simplex |
| $\operatorname{Iso}\left(\mathbb{X}^{n}\right)$ | group of isometries of $\mathbb{X}^{n}$ |
| $\Gamma$ | discrete subgroup of $\operatorname{Iso}\left(\mathbb{X}^{n}\right)$ |
| $\operatorname{Stab}(K, \Gamma)=\Gamma_{K}$ | $\{\gamma \in \Gamma: \gamma K=K\}$ |
| $\begin{aligned} & \operatorname{Stab}_{p}(K, \Gamma)=\Gamma_{K}^{\prime} \\ & \Phi \end{aligned}$ | $\{\gamma \in \Gamma: \gamma k=k$ for all $k \in K\}$ tesselation |
| $\partial \Pi$ | boundary complex of $\Pi$ |
| $\|\Pi\|$ | underlying topological space |
| ri | relative interior |
| cl | closure |
| rb | relative boundary |
| $K$ | sectional curvature |
| $c_{n}(\epsilon)$ | volume of the $n$-sphere of radius $\epsilon$ |
| $c_{n}$ | $:=c_{n}(1)$ |
| $S_{n}^{m}$ | Stirling number of the second kind |
| $q_{m}$ | square-root number |
| $B_{m}$ | Bernoulli number |
| $T_{m}$ | tangent number |
| $E_{m}$ | Euler number |
| $a_{m}$ | $2^{-m} T_{m}$ |
| $Z(m)$ | number of zick-zack permutations of $m$ elements |

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