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Growth of cocompact hyperbolic Coxeter groups and their rate

THESE

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" Le chemin parcouru est plus important que le but lui-même. "

citation bouddhique

Abstract

In this thesis we study the growth of Coxeter groups $G < I(\mathbb{H}^n)$ acting on hyperbolic n-space with compact fundamental polyhedron. For a fixed system of generators S, the growth series is given by

$$f_S(x) = \sum_{i \geq 0} a_i x^i,$$

where a_i is the number of elements in G which can be expressed as an S-word of minimal length i. Steinberg showed that the growth series is the power series of a rational function p/q in its convergence disk. The growth rate τ of G corresponds to the inverse of this convergence radius. By a result of Milnor, τ is strictly larger than 1.

A main theorem is a recursion formula for the coefficients a_i which depends on the presentation of G. This result is based on a detailed analysis of a certain complete form P/Q of $f_S(x)$, by following an idea of Chapovalov, Leites and Stekolshchik. An interesting application concerns the family of right-angled polytopes.

By restricting the study to the dimension $n = 4$, we present a theorem about the growth of the Lannér, Esselmann and Kaplinskaya groups, which are characterised by at most six generators. An adaptation of Parry's methods for $n = 2, 3$ allows to show that their growth rate is a Perron number. More precisely, except for one group, the function $f_S(x)$ possesses exactly four poles in $\mathbb{R}$. These ones are simple and appear in pairs of algebraic integers $\left(x_1, x_1^{-1}\right)$ and $\left(x_2, x_2^{-1}\right)$ with $\tau = x_1^{-1}$ and $0 < x_1 < x_2 < 1$. The non-real poles are distributed in a certain annulus around the unit circle.

Résumé

Dans cette thèse nous étudions la croissance des groupes de Coxeter $G < I(\mathbb{H}^n)$ qui opèrent dans le n-espace hyperbolique avec polyhèdre fondamental compact. En fixant un système générateur S pour G, la série d'accroissement est donnée par

$$f_S(x) = \sum_{i \geq 0} a_i x^i,$$

où a_i dénote le nombre d'éléments de G pouvant s'exprimer en un S-mot de longueur minimale i. Steinberg a prouvé que la série d'accroissement $f_S(x)$ est la série de Taylor d'une fonction rationnelle p/q dans son disque de convergence. Le taux d'accroissement τ de G correspond à l'inverse de ce rayon de convergence. Par un résultat de Milnor, τ est strictement supérieur à 1.

Un théorème principal est une formule de récursion pour les coefficients a_i qui dépend de la présentation de G. Ce résultat découle d'une analyse approfondie d'une certaine forme complétée P/Q de $f_S(x)$, en suivant une idée de Chapovalov, Leites et Stekolshchik. Une application intéressante concerne la famille des polytopes totalement rectangulaires.

En restreignant l'étude à la dimension $n = 4$, nous présentons un théorème sur la croissance des groupes de Lannér, Esselmann et Kaplinskaya, caractérisés par au plus six générateurs. Une adaptation des méthodes de Parry pour $n = 2, 3$ permet de montrer que leur taux d'accroissement est un nombre de Perron. Plus précisément, la fonction $f_S(x)$ possède dans $\mathbb{R}$ - à une exception près - exactement quatre pôles. Ceux-ci sont simples et apparaissent en paires d'entiers algébriques $\left(x_1, x_1^{-1}\right)$ et $\left(x_2, x_2^{-1}\right)$ avec $\tau = x_1^{-1}$ et $0 < x_1 < x_2 < 1$. Les pôles non-réels se trouvent dans un certain anneau autour du cercle unité.

Zusammenfassung

In dieser Dissertation untersuchen wir das Wachstum von Coxetergruppen $G < I(\mathbb{H}^n)$, die auf dem hyperbolischen n-Raum mit kompaktem Fundamentalpolyeder operieren. Für ein festes Erzeugendensystem S ist die Wachstumsreihe gegeben durch

$$f_S(x) = \sum_{i \geq 0} a_i x^i,$$

wobei a_i die Anzahl der Elemente von G ist, die durch ein S-Wort mit minimaler Länge i dargestellt werden können. Steinberg bewies, dass die Wachstumsreihe $f_S(x)$ die Taylorreihe einer rationalen Funktion p/q in ihrem Konvergenzkreis ist. Die Wachstumsrate τ von G entspricht dem Inversen dieses Konvergenzradius. Nach einem Resultat von Milnor ist τ strikt grösser als 1.

Ein Hauptsatz ist eine Rekursionsformel für die Koeffizienten a_i, welche von der Präsentation von G abhängt. Dieses Resultat basiert auf einem vertieften Studium einer bestimmten, vervollständigten Form P/Q von $f_S(x)$, die einer Idee von Chapovalov, Leites et Stekolshchik entspringt. Eine interessante Anwendung betrifft die Familie der total-rechtwinkligen Polytope.

Indem wir die Betrachtungen auf die Dimension $n = 4$ einschränken, gelangen wir zu einem Theorem über das Wachstum von Lannér-, Esselmann- und Kaplinskaya-Gruppen, welche durch höchstens sechs Spiegelungen erzeugt sind. Eine Anpassung der Methoden von Parry für $n = 2, 3$ erlaubt zu zeigen, dass ihre Wachstumsrate eine Perron-Zahl ist. Genauer ausgedrückt besitzt die Funktion $f_S(x)$ in $\mathbb{R}$ - bis auf eine Ausnahme - genau vier Pole. Diese sind einfach und treten auf in Paaren von algebraischen ganzen Zahlen $\left(x_1, x_1^{-1}\right)$ und $\left(x_2, x_2^{-1}\right)$ mit $\tau = x_1^{-1}$ und $0 < x_1 < x_2 < 1$. Die nicht-reellen Pole sind in einem gewissen Kreisring um den Einheitskreis verteilt.

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Contents

1 Introduction

Consider the hyperbolic space $\mathbb{H}^n$ and let G be a discrete group of isometries generated by finitely many reflections with respect to the facets of a compact polyhedron $P \subset \mathbb{H}^n$. The group G is called a *cocompact hyperbolic Coxeter group*, and its fundamental domain P is a Coxeter polytope. Let S be the set of the above reflections. The *growth* of G is described by means of the power series

$$f_S(x) = \sum_{i \geq 0} a_i x^i, \qquad (*)$$

where a_i denotes the number of elements of G which can be expressed as a S-word of minimal length i. It is well-known that $f_S(x)$ is the power series of a rational function p/q, with relatively prime integer polynomials p, q, in its convergence disk. The *growth rate* τ of G is given by the inverse of the convergence radius and equals an algebraic integer bigger than 1 by a result of Milnor.

In contrast to the spherical and euclidean cases, the growth of a hyperbolic Coxeter group G is of exponential type. Furthermore, a compact hyperbolic Coxeter polytope P is still simple, but combinatorially much more complicated than the spherical and euclidean ones. This is equivalent to saying that the structure of the maximal finite Coxeter subgroups of G is very rich. It also explains why hyperbolic Coxeter groups are far from being classified.

Despite these complications, there are some structural results in the planar and the spatial cases. By results of Floyd, Cannon, Wagreich and others, one has a closed formula for the growth series of a planar hyperbolic Coxeter group which allows one to characterise their growth rates in terms of Salem and Pisot numbers. In dimension 3, Cannon computed the growth series and rates of all cocompact Simplex groups, identifying the latter ones with Salem numbers.

The results above were extended by Parry who considered Coxeter groups acting cocompactly on $\mathbb{H}^2$ and $\mathbb{H}^3$. The ingredients of his nice, unifying proof will be reproduced in this work since it inspired our approach to attack growth problems in dimensions beyond 3.

We start by considering Coxeter groups in hyperbolic 4-space. Since we do not have a complete survey of them, we study the ones with at most seven generators. They form finite families comprising the Simplex, Esselmann, Kaplinskaya and Tumarkin groups. The computations of some of their growth series led us, among other things, to conjecture that their growth rate is always a *Perron number*. Such a number is a real algebraic integer $\alpha > 1$ all of whose conjugates are of absolute value strictly smaller than α. A rigorous proof of all our observations summarised in Conjecture 1 (see part 3.4) is very delicate. In a first step, we prove some partial results and postpone the verification of the precise pole description of $f_S(x)$ (see Theorem 3.7). In a second step, we extend the context and study the growth series of an arbitrary Coxeter group in $\mathbb{H}^n$ as follows.

Let G denote a cocompact Coxeter group acting with generating set S in $\mathbb{H}^n$. We associate to its growth series $f_S(x)$ a certain complete form by following an idea of Chapovalov, Leites and Stekolshchik (see part 4.1),

$$f_S(x) = \frac{P(x)}{Q(x)} = \frac{\prod_{i=1}^{m} [n_i]}{\sum_{i=0}^{N} b_i x^i}, \qquad (**)$$

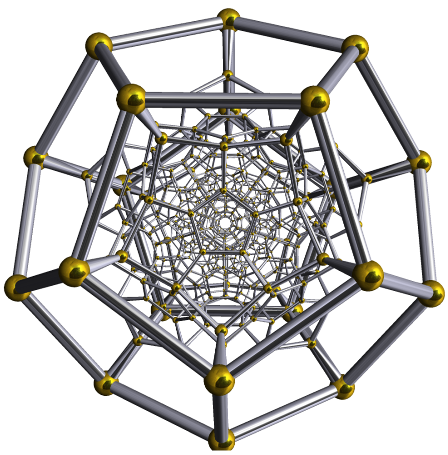
where $P(x)$ and $Q(x)$ are polynomials of equal degree over the integers, and where $[k] := 1 + x + \cdots + x^{k-1}$. The integers m and $n_1, \ldots, n_m \geq 2$ are related to the finite subgroups of G and their exponents. For the coefficients b_i, $i = 0, \ldots, N$, of the denominator polynomial Q, we derive a recursion formula which is presented in Theorem 4.10. In this way, for a given group G, we control $f_S(x) = P(x)/Q(x)$ in $(**)$ in a completely explicit manner. There are various applications of this recursion formula. Firstly, it allows to confirm Conjecture 1 as a whole and to verify it even for the family of Tumarkin groups. Secondly, we obtain a recursion formula for the coefficients a_i in the growth series $(*)$. Notice that the cardinalities a_i are usually very difficult to determine since they depend on the relations between the generators. Finally, we apply our recursion formula to the important family of right-angled hyperbolic Coxeter polytopes (see Corollary 4.11). For such a polytope in $\mathbb{H}^4$, having f_0 vertices and f_3 facets, the associated growth series is given by

$$\frac{(1+x)^4}{1 + (4 - f_3)x + (f_0 - 2f_3 + 6)x^2 + (4 - f_3)x^3 + x^4}.$$

As an example, Coxeter's 120-cell $P_\star(\pi/2) \subset \mathbb{H}^4$ has 600 vertices, 1200 edges, 720 pentagonal faces and 120 facets (see the picture below) and yields a growth series of the form

$$\frac{(1+x)^4}{1 - 116x + 336x^2 - 116x^3 + x^4},$$

which has exactly two pairs of positive simple poles.



A combinatorial picture of a 120-cell.

2 Preliminaries

In this section we present the general background of this work. Good references for it are [19], [31] and [37]. Let $\mathbb{X}^n$ denote one of the standard geometric n-spaces, that is, either the *sphere* $\mathbb{S}^n$, the *Euclidean* space $\mathbb{E}^n$ or the *hyperbolic* space $\mathbb{H}^n$. In this work we consider mostly $\mathbb{X}^n = \mathbb{H}^n$.

2.1 Standard geometric spaces

Embed $\mathbb{S}^n \subset \mathbb{E}^{n+1}$ and $\mathbb{H}^n \subset \mathbb{E}^{n,1}$, where $\mathbb{E}^{n,1}$ is Lorentz-Minkowski space of signature $(n,1)$. In this way each standard geometric n-space $\mathbb{X}^n$ is a subset of a real vector space $\mathbb{Y}^{n+1}$, equipped with the standard inner product $(\cdot,\cdot)_{\mathbb{Y}^{n+1}}$. Recall that $\mathbb{E}^{n,1}$ is equal to $\mathbb{R}^{n+1}$ endowed with the Lorentzian inner product (see [31, page 56]). More geometrically, we interpret $\mathbb{H}^n =: H^n$ as the upper shell of the *hyperboloid* in $\mathbb{R}^{n+1}$. Denote by $I(\mathbb{X}^n)$ the group of isometries of $\mathbb{X}^n$ which is a Lie group. More specifically $I(H^n)$ is isomorphic to the group of positive Lorentzian matrices $PO(n,1)$, while $I(\mathbb{S}^n)$ is isomorphic to the orthogonal group $O(n+1)$. As is well known, each isometry in $I(\mathbb{X}^n)$ can be written as a composition of a finite number of reflections through hyperplanes in $\mathbb{Y}^{n+1}$.

In the non-Euclidean case a hyperplane in $\mathbb{X}^n$ is the intersection of an n-subspace of $\mathbb{Y}^{n+1}$ with $\mathbb{X}^n$. Let H be a hyperplane in $\mathbb{X}^n$ and write H as orthogonal complement of a unit vector $e \in \mathbb{Y}^{n+1}$. In this way H separates $\mathbb{X}^n$ into two oriented closed *halfspaces* H^+ and H^-. A *convex n-polyhedron* $P \subset \mathbb{X}^n$ is the intersection of finitely many half-spaces having a non-empty interior (see [37, page 28]). That is, without loss of generality,

$$P = \bigcap_{i=1}^{m} H_i^- \subset \mathbb{X}^n. \tag{2.1}$$

It is always assumed that no half-space H_i^- contains the intersection of all the others. Each of the hyperplanes H_i is said to *bound* the polyhedron P and gives rise to a *facet* $F_i = H_i \bigcap P$. A convex polyhedron P is *acute-angled* if the interior angle formed by two intersecting distinct hyperplanes H_i and H_j is not greater than $\frac{\pi}{2}$.

By the Weyl-Minkowski theorem (see [37, page 24], for example), each *compact* convex polyhedron $P \subset \mathbb{X}^n$ is the convex hull of finitely many points or *vertices* in $\mathbb{X}^n$. We often call such a polyhedron a *polytope*. In case $\mathbb{X}^n = \mathbb{S}^n$ we assume moreover that a polytope doesn't have antipodal vertices in order to avoid degeneracy (compare with [37, page 104]). An n-polytope in $\mathbb{X}^n$ with precisely $n+1$ vertices is called a (geometric) *n-simplex*. In general, compact acute-angled polyhedra are *simple* (see [38, page 47]), that is, each face of codimension k is contained in exactly k facets. Hence, each vertex figure is a simplex of codimension 1. Acute-angled (compact or non-compact) polyhedra all of whose interior angles are submultiples of π are called *Coxeter polyhedra*. These polyhedra are central for this work.

We associate to each convex polyhedron $P = \bigcap_{i=1}^{m} H_i^- \subset \mathbb{X}^n$ a symmetric matrix $G(P) = (g_{ij})$, the *Gram matrix of* P, in the following way. The entries of $G(P)$ are defined by $g_{ii} = 1$ and $g_{ij} = -(e_i, e_j)_{\mathbb{Y}^{n+1}}$ for $i \neq j$. Geometrically g_{ij} is the negative of the cosine of the interior angle formed by H_i and H_j. In the hyperbolic case, g_{ij} is the negative of the hyperbolic cosine of the distance between H_i and H_j in case they don't intersect on P. It can be shown that $G(P)$ is positive definite (resp. positive semi-definite of rank n) if $P \subset \mathbb{S}^n$ (resp. $P \subset \mathbb{E}^n$). If $P \subset \mathbb{H}^n$, the Gram matrix $G(P)$ is of signature $(n, 1)$. Notice that the Gram matrix of a hyperbolic n-polyhedron can be of arbitrary order bigger than n. By the theorem of Perron-Frobenius for symmetric matrices (see [38, page 43]), every acute-angled spherical polytope is a simplex, while every acute-angled euclidean one is a product of simplices. There is no such description for hyperbolic polytopes.

2.2 Geometric Coxeter groups, their graphs, matrices and fundamental polyhedra

A discrete subgroup $G < I(\mathbb{X}^n)$ generated by finitely many reflections $s_1, \ldots, s_k$ is called a *discrete reflection group* or a *geometric Coxeter group* in $\mathbb{X}^n$ (see [37, page 200] and [19]). Each element s_i is a reflection with respect to a hyperplane or *mirror* H_i implying the following (finite order) relations :

$$s_i^2 = 1 \text{ and } (s_i s_j)^{m_{ij}} = 1, \text{ where } m_{ij} \in \mathbb{N}_{\geq 2} \tag{2.2}$$

if $H_i \bigcap H_j \neq \emptyset$. If two mirrors H_i, H_j are parallel (in the non-spherical case), or admit a common perpendicular in $\mathbb{H}^n$, then $s_i s_j$ is of infinite order. The images of all mirrors under the action of G give rise to a polyhedral decomposition of $\mathbb{X}^n$. The closure of each connected component is a Coxeter polyhedron and provides a fundamental domain for G (see [37, page 200, Proposition 1.4]). Vice versa, each Coxeter polyhedron $P \subset \mathbb{X}^n$ gives rise to a geometric Coxeter group G. In fact the reflections in the bounding hyperplanes of P serve as generators for G. If $P \subset \mathbb{H}^n$ ($\mathbb{E}^n$, $\mathbb{S}^n$, respectively), then G is called a *hyperbolic* (*euclidean*, *spherical*, respectively) Coxeter group.

A *Coxeter subgroup* of G is a subgroup of G generated by reflections $s_{i_1}, \ldots, s_{i_l}$ such that $i_j \in \{1, \ldots, k\}$ for $j = 1, \ldots, l$. In this thesis a subgroup of G is always understood as a Coxeter one.

It follows from the definition that each subgroup of a (geometric) Coxeter group is itself a (geometric) Coxeter group (cf. [3, Theorem 2]). Since finite Coxeter groups can be represented as spherical ones, a subgroup of a spherical Coxeter group is spherical as well. Subgroups of a euclidean Coxeter group are spherical or euclidean, with at least one euclidean subgroup. A *cocompact* hyperbolic Coxeter group, that is, a hyperbolic Coxeter group with compact fundamental domain, has spherical and possibly hyperbolic subgroups (cf. with [38, Theorem 4.1]). Notice that a cocompact hyperbolic simplex Coxeter group has only spherical subgroups.

Geometric Coxeter groups are often more conveniently described by a graph. Let G denote a geometric Coxeter group with fundamental polyhedron $P = \bigcap_{i=1}^m H_i^- \subset \mathbb{X}^n$ whose interior angles formed by H_i, H_j are of the form π/m_{ij}, $m_{ij} \in \mathbb{N}_{\geq 2}$. The *Coxeter graph* $\Gamma = \Gamma(G)$ consists of m nodes ν_i corresponding to the mirrors H_i and simultaneously to the reflections s_i. Each pair ν_i, ν_j is joined by an edge of weight m_{ij} if H_i, H_j intersect under the angle π/m_{ij}. If $m_{ij} = 3$, a frequent case, we do not put the label 3. If $m_{ij} = 2$ we omit putting an edge at all. If $\mathbb{X}^n = \mathbb{H}^n$, we join ν_i, ν_j by a dotted edge if H_i, H_j admit a common perpendicular segment l. The weight of this dotted edge (which is usually not indicated) corresponds to the hyperbolic distance of l. The case of parallel hyperplanes in $\mathbb{X}^n \neq \mathbb{S}^n$ will not be considered in the sequel.

It is known that a (geometric) Coxeter group G is irreducible (resp. reducible) if and only if its Coxeter graph Γ is connected (resp. disconnected), see (2.4) below. Furthermore, see [19, page 30], a reducible geometric Coxeter group is a direct product of irreducible ones, and a cocompact hyperbolic Coxeter group is always irreducible (cf [38]).

Let us introduce some further terminology. The graph of a hyperbolic, euclidean or spherical

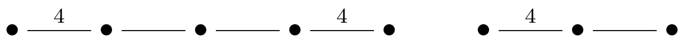
Coxeter group is called *hyperbolic*, *parabolic* or *elliptic*. The *graph of a Coxeter polyhedron* P is the Coxeter graph of the geometric Coxeter group G with fundamental domain P. We often do not distinguish between the geometric Coxeter group and its Coxeter polyhedron. The *Coxeter matrix* is given by the Gram matrix of P (and of G).

Example 2.1 **(a)** *For* $n \geq 1$,

A_n : (2.3)

is the graph of a spherical simplex Coxeter group which acts on $\mathbb{S}^{n-1}$. *Observe that the suffix* n *means the number of nodes.*

(b) *The disconnected graph*

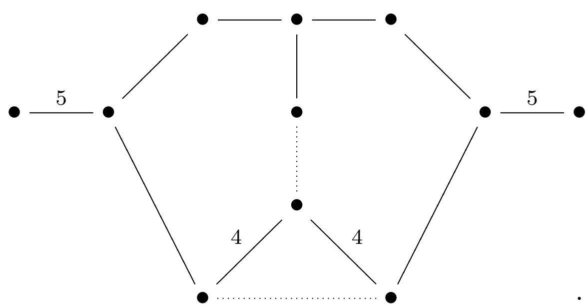


(2.4)

represents a euclidean Coxeter group which acts in $\mathbb{E}^7$, *since its Gram matrix is positive semi-definite of rank* 7. *Its irreducible components consist of a euclidean simplex group with graph* $\tilde{C}_4$ *and a spherical one with graph* B_3 *(see Tables* 1 *and* 2*).*

(c) *Let us now consider cocompact Coxeter groups* G *which act in* $\mathbb{H}^7$. *We know that each* G *possesses more than nine generators (cf [14] and [20]). Moreover Tumarkin [36, Lemma 4.2] proved that there is no Coxeter group generated by ten elements, wherefore each* G *possesses at least eleven generators. In fact, by a result of Felikson and Tumarkin [15, Theorem 3], there exists only one Coxeter group with eleven generators in* $\mathbb{H}^7$. *This group was discovered by Bugaenko [4]. Its graph is given by*

(2.5)



2.3 Lannér, Esselmann & Kaplinskaya groups

Spherical and Euclidean irreducible Coxeter groups were first classified by Coxeter in 1934 (see [10]). A list of the graphs of these groups is given in Tables 1 and 2. Observe that their classification is based on the computation of certain determinants (see [10]). The classification of hyperbolic Coxeter polyhedra, even in the compact case, is far from being complete. Poincaré [28] and Andreev [1] characterised all compact Coxeter polyhedra in dimensions 2 and 3. The compact hyperbolic Coxeter simplices were classified by Lannér in 1950 ; we call them and their associated groups *Lannér simplices* and *Lannér groups.* A Lannér group is characterised by the fact that all its subgroups are spherical (see [38, page 60]). However they exist only in dimensions $n \leq 4$. Kaplinskaya [20] and Esselmann [14] classified all the Coxeter polyhedra with $n+2$ facets, while those with $n+3$ facets were completely described by Tumarkin [36]. Observe that there is no complete classification of Coxeter polytopes with $n+k$ facets, for $k \geq 4$, up to now, and examples are known only for $n \leq 8$. Finally, by a fundamental result of Vinberg [39, Theorem 1], we know that Coxeter polytopes do not exist in dimensions $n \geq 30$.

Let us look now more carefully at compact Coxeter polyhedra with few facets in $\mathbb{H}^4$ which are in the focus of this work. By Table 3 there are only five Lannér 4-simplices. Kaplinskaya described the polytopes whose combinatorial type is a product of a segment and a 3-simplex, while Esselmann described those whose combinatorial type is a product of two 2-simplices. In this work such polytopes are termed *Kaplinskaya polyhedra*, respectively *Esselmann polyhedra.* Their associated groups are called *Kaplinskaya groups*, respectively *Esselmann groups.* It is a fact that there are no Coxeter polyhedra with six facets in $\mathbb{H}^4$ which are neither Kaplinskaya nor Esselmann. Let us point out (see [14]) that Esselmann polyhedra do not exist in dimensions bigger than 4.

2.4 Tables of geometric Coxeter groups with at most $n+2$ nodes in $\mathbb{X}^n$

In the following we present the lists of all cocompact irreducible geometric Coxeter groups in $\mathbb{X}^n$ with at most $n+2$ nodes. For details, see [19] or [37], [14] and [20]. Let us begin with the elliptic case and denote by Γ_n a connected positive definite Coxeter graph with n nodes. Γ_n describes an irreducible reflection group acting on $\mathbb{S}^{n-1}$ and with fundamental Coxeter n-simplex.

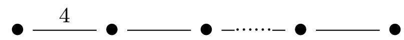
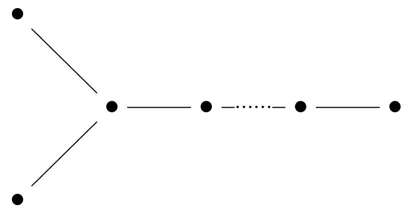
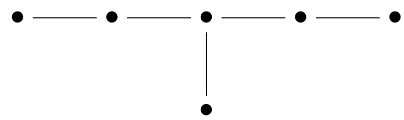
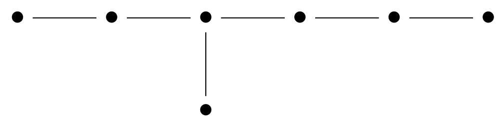
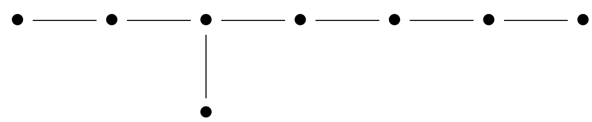
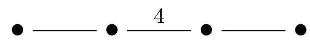
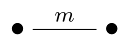
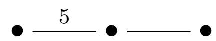
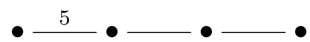
		Graph	Rank n
A_n	:		$n \geq 1$
B_n	:		$n \geq 2$
D_n	:		$n \geq 4$
E_6	:		$n = 6$
E_7	:		$n = 7$
E_8	:		$n = 8$
F_4	:		$n = 4$
$G_2^{(m)}$	:		$n = 2$
H_3	:		$n = 3$
H_4	:		$n = 4$

Table 1 : The irreducible elliptic graphs

For completeness, we present also all the irreducible parabolic graphs. We denote by $\tilde{\Gamma}_n$ a graph with $n+1$ nodes corresponding to a cocompact Coxeter group acting on $\mathbb{E}^n$ (and whose fundamental polyhedron is an n-simplex). The corresponding Gram matrix is positive semi-definite of rank n.

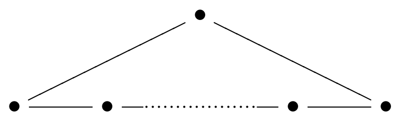
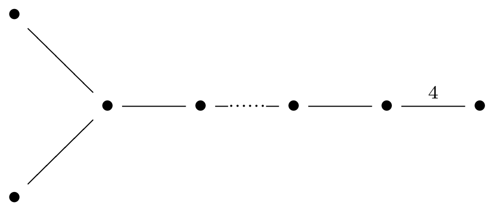
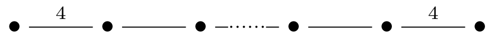
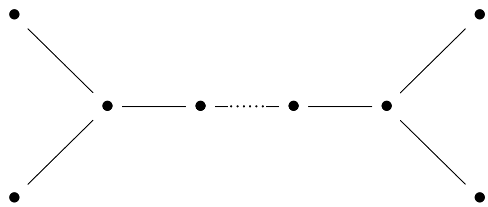
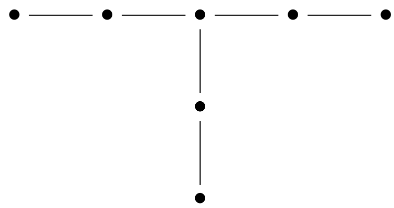
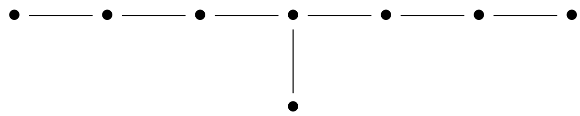
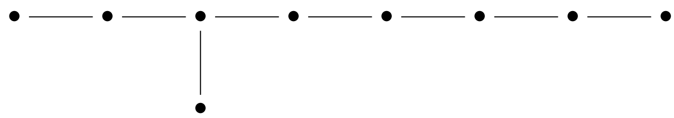
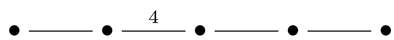
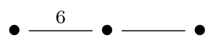
	Graph	Rank n
$\tilde{A}_n$:		$n \geq 2$
$\tilde{B}_n$:		$n \geq 3$
$\tilde{C}_n$:		$n \geq 2$
$\tilde{D}_n$:		$n \geq 4$
$\tilde{E}_6$:		$n = 6$
$\tilde{E}_7$:		$n = 7$
$\tilde{E}_8$:		$n = 8$
$\tilde{F}_4$:		$n = 4$
$\tilde{G}_2$:		$n = 2$

Table 2 : The irreducible parabolic graphs

The next table contains the graphs of all Lannér groups. Observe that these groups exist in $\mathbb{H}^n$ only for $n \leq 4$.

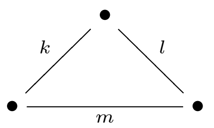
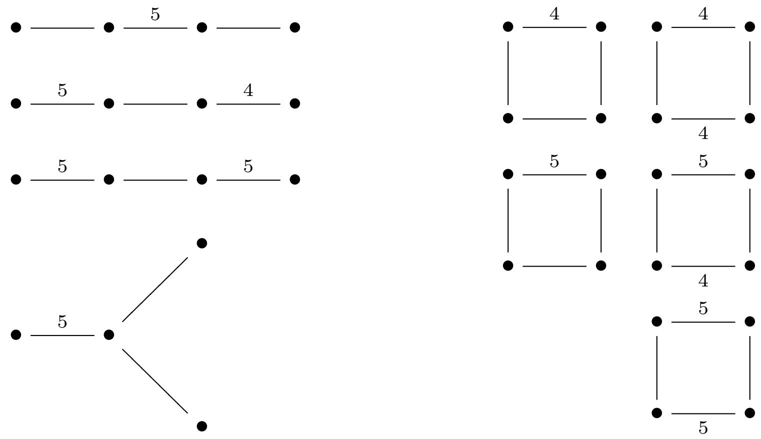
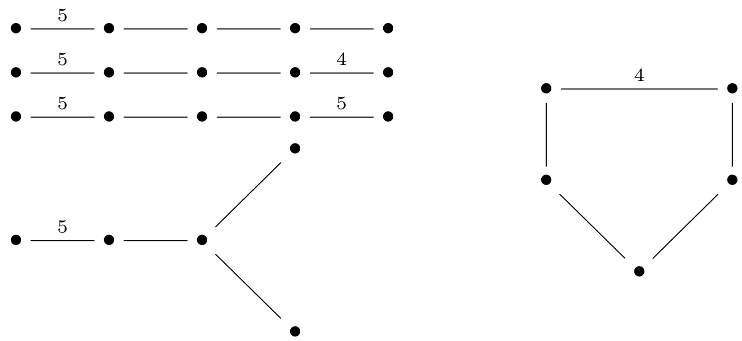
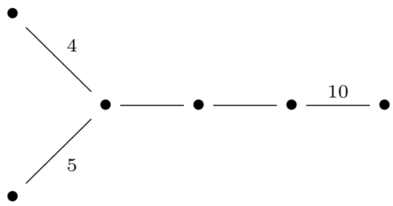
Graphs	Dimension n
 with $2 \leq k, l, m < \infty$ and $\frac{1}{k} + \frac{1}{l} + \frac{1}{m} < 1$	$n = 2$
	$n = 3$
	$n = 4$

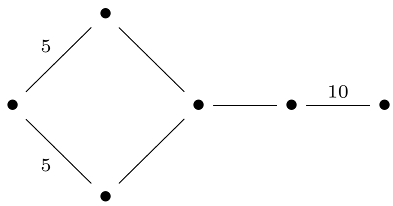
Table 3 : The Lannér graphs

Let us now present the Coxeter graphs of the Esselmann groups in $\mathbb{H}^4$ whose combinatorial type is a direct product of two Lannér triangles (see part 2.3).

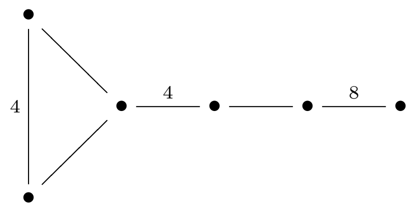
Γ_1 :



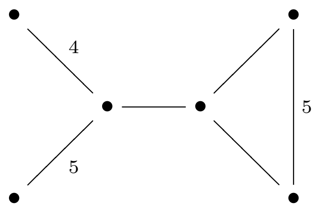
Γ_2 :



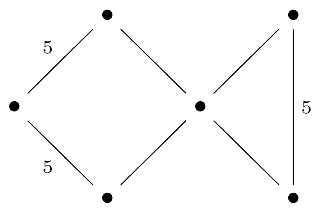
Γ_3 :



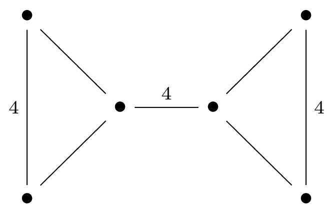
Γ_4 :



Γ_5 :



Γ_6 :



Γ_7 :

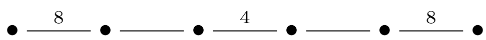


Table 4 : The Esselmann graphs

Finally we list all the Kaplinskaya groups in $\mathbb{H}^4$ whose graphs K_{ij} correspond to Coxeter polyhedra which arise by gluing those of the groups described by the graphs K_i and K_j (for more details, see [20]).

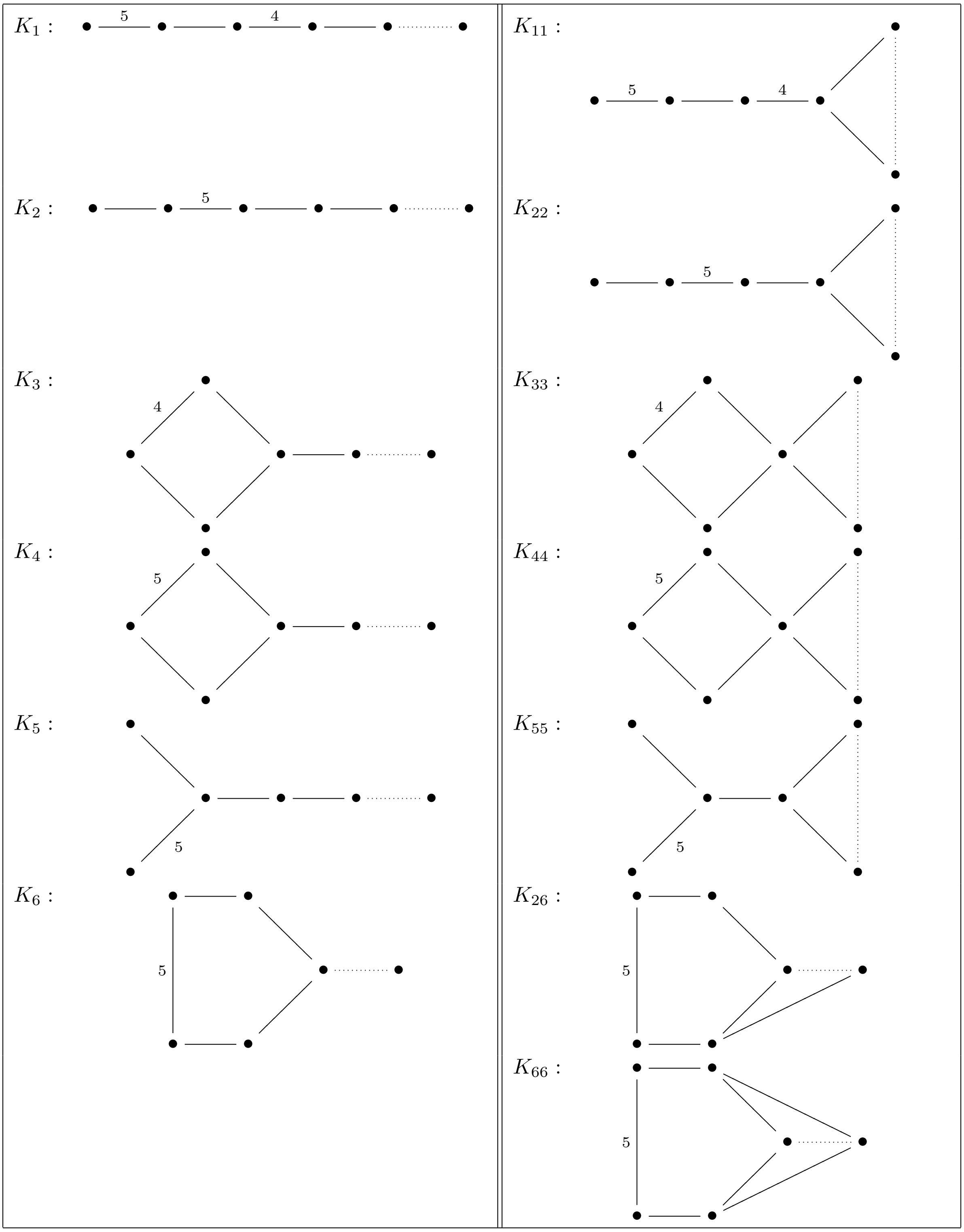


Table 5 : The Kaplinskaya graphs

3 Growth series of geometric Coxeter groups

This section is devoted to the growth of geometric Coxeter groups. Well known is the growth behaviour of the spherical and euclidean Coxeter groups as we shall see. This is in contrast to the hyperbolic context. In fact there are some results for Coxeter groups acting on $\mathbb{H}^2$ and $\mathbb{H}^3$, only, which are essentially due to Cannon [6], Floyd [16] and Parry [26]. All these aspects are presented in the first three parts. In the fourth part we propose some new results for Coxeter groups acting cocompactly and with few generators on $\mathbb{H}^4$. More precisely we shall study the growth of the Simplex, Esselmann and Kaplinskaya groups which have fundamental polytopes with at most six facets (see section 2). It turns out that a general approach as in the case of Coxeter polyhedra in $\mathbb{H}^2$ and $\mathbb{H}^3$ is not as yet realistic. In fact we treat each family in a similar but different way. In section 4 we consider cocompact hyperbolic Coxeter groups in *arbitrary* dimensions and provide, as a highlight, a recursion formula for the coefficients of the denominator polynomial of their growth series.

3.1 Growth - basic definitions and properties

Good references for this part are [19] and [40].

Let G denote a multiplicative group. A subset $S \subset G$ with $S \not\ni 1$ is a *generating set* of G if each element of G can be written as a finite product of elements in S. The elements of S are called *generators* of the group G. In this work we always suppose that S is finite and we put $S = \{s_1, s_2, \ldots, s_m\}$. We assume moreover that S also contains the inverses of its elements. In other words we have that if $s \in S$, then $s^{-1} \in S$. Let us remark that a generating set for a given group G is not unique : for example, the sets $S_1 = \{s\}$ and $S_2 = \{s^2, s^3\}$ both generate the cyclic group $C_6 = \langle s \mid s^6 = 1\rangle$. Thus our study of a given group G is based on a fixed generating set.

Let us consider a group G with fixed generating set $S = \{s_1, s_2, \ldots, s_m\}$. The *length function relative to S on G* is defined by

$$\begin{array}{rcccc} L_S & : & G & \longrightarrow & \mathbb{N} \\ & & g & \longmapsto & L_S(g), \end{array}$$

where $L_S(g)$ is the minimal number of generators in S needed to express g. By convention, $L_S(1_G) = 0$, and 1_G is the unique element of length 0. The *growth function relative to S on G* is given by

$$\begin{array}{rcccc} a & : & \mathbb{N} & \longrightarrow & \mathbb{N} \\ & & k & \longmapsto & a(k) =: a_k, \end{array} \tag{3.1}$$

where a_k is the number of elements $g \in G$ with $L_S(g) = k$. It follows that $a_0 = 1$ and $a_1 = |S|$. Let us now introduce one of the most important notions of this work.

Definition 3.1 The *(spherical) growth series f_S of G relative to S* is defined by the power series

$$\begin{array}{rcccc} f_S & : & \mathbb{C} & \longrightarrow & \mathbb{C} \\ & & x & \longmapsto & \sum\limits_{k\geq 0} a_k x^k. \end{array}$$

Properties 3.1 **(a)** $f_S(0) = 1$.

(b) *The Euler characteristic $\chi(G)$ of a Coxeter group G was studied by Serre and is related to homology and cohomology of groups (see [33]). Moreover Serre provided the following inductive formula.*

$$\chi(G) = (-1)^{|S|} \cdot \sum_{T \subsetneq S} (-1)^{|T|} \cdot \chi(G_T),$$

where G_T is a finite subgroup of G generated by $T \subsetneq S$. Furthermore, (see [33, Proposition 17])

$$\chi(G) = \frac{1}{f_S(1)}, \tag{3.2}$$

and (see [24], for example),

$$\frac{1}{f_S(1)} = \chi(G) \begin{cases} = \ |G|^{-1} & \text{, if } G \text{ is spherical,} \\ = \ 0 & \text{, if } G \text{ is euclidean,} \\ = \ 0 & \text{, if } G \text{ acts on } \mathbb{H}^{2l+1}, \\ < \ 0 & \text{, if } G \text{ acts on } \mathbb{H}^{4l+2}, \\ > \ 0 & \text{, if } G \text{ acts on } \mathbb{H}^{4l}. \end{cases}$$

Here is an example for a euclidean Coxeter group G with graph Γ_G containing the three components $\tilde{F}_4$, $G_2^{(7)}$ and A_5 (see Tables 1 and 2). More precisely we have

$$\Gamma_G : \bullet \!-\!\!-\!\!-\! \bullet \overset{4}{-\!\!-\!\!-} \bullet \!-\!\!-\!\!-\! \bullet \!-\!\!-\!\!-\! \bullet \qquad \bullet \overset{7}{-\!\!-\!\!-} \bullet \qquad \bullet \!-\!\!-\!\!-\! \bullet \!-\!\!-\!\!-\! \bullet \!-\!\!-\!\!-\! \bullet \!-\!\!-\!\!-\! \bullet\,.$$

The growth series of G is given by

$$f_S(x) = \frac{(\Phi_2(x))^8 \cdot (\Phi_4(x))^3 \cdot (\Phi_6(x))^3 \cdot (\Phi_3(x))^4 \cdot \Phi_8(x) \cdot \Phi_{12}(x)}{(\Phi_1(x))^4 \cdot \Phi_{11}(x)}, \tag{3.3}$$

where $\Phi_k(x)$ is the k-th cyclotomic polynomial evaluated at x. The power series $f_S(x)$ is in fact the Taylor series of the rational function of the right hand side of (3.3).

According to Definition 3.1 it is easy to see that if G is finite, then its growth series $f_S(x)$ is a polynomial with $f_S(1) = |G|$. In general, $f_S(x)$ is a rational function (see [35]). Its power series possesses a certain *radius of convergence* R. If G is infinite, its growth series is absolutly convergent for all $x \in \mathbb{C}$ with $|x| < R$. Note that the radius of convergence R is computable by Hadamard's formula

$$R = \frac{1}{\limsup\limits_{n\to\infty} \sqrt[n]{a_n}}.$$

By means of R we define the following important notion.

Definition 3.2 The *growth rate* of G is given by $\tau = R^{-1}$.

There exist already some results about the growth rate of a group G. By a result of Milnor [25, Theorem 2], the growth rate of a hyperbolic group is greater that 1. In fact this is a consequence of the following theorem.

Theorem 3.1 (Milnor) *If M is compact (connected) Riemannian manifold with all sectional curvatures strictly less than zero, then the growth function of the fundamental group $\pi_1(M)$ is at least exponential :*

$$a_k \geq \gamma^k$$

for some constant $\gamma > 1$.

It turns out that the growth rate of a Coxeter group G is the largest (in modulus) pole of f_S, since the growth series f_S has positive coefficients (see [12, § 17.1, page 322] and [27, equation (0.1)]).

Next we present *Steinberg's Formula* [35, pages 13-14] which provides an inductive procedure to compute the inverse of the growth series $f_S(x)$ in terms of the finite subgroups of G. More precisely,

$$\frac{1}{f_S(x^{-1})} = \sum_{G_T \in \mathcal{F}} \frac{(-1)^{|T|}}{f_T(x)}, \tag{3.4}$$

where $\mathcal{F} := \{T \subseteq S : G_T \text{ is finite}\}$. By (3.4), we only have to deal with the spherical subgroups of G which maybe reducible. It is an easy matter to verify the following product formula for the growth series of reducible groups (see [13], for example).

Lemma 3.2 *Let H be a direct product of the groups H_1 and H_2, all finitely generated. Then the growth series of H with respect to its generating set S is given by*

$$f_S(x) = f_{S_1}(x) \cdot f_{S_2}(x), \tag{3.5}$$

where f_{S_i} is the growth series of the group H_i related to the generating set S_i, for $i = 1, 2$, and $S = (S_1 \times \{1_{G_2}\}) \cup (\{1_{G_1}\} \times S_2)$.

The above lemma can obviously be generalised to a Coxeter group G with k components $G_1, \ldots, G_k$.

Despite the nice appearance of Steinberg's formula (3.4) its evaluation in concrete cases can lead to very long computations, especially when G is generated by a very large number of elements. We will see examples later.

Finally we present the fundamental notion of (anti-)reciprocity of and some results for the growth series of Coxeter groups. This will be a main feature in section 4.

A rational function $f(x)$ is *reciprocal* if $f(1/x) = f(x)$ and *antireciprocal* if $f(1/x) = -f(x)$. In the case of an infinite Coxeter group G with finite generating set S *all* of whose subgroups G_T, $T \subsetneq S$, are finite, Serre [33, page 112] showed that

$$f_S(x) = (-1)^{|S|+1} \cdot f_S(x^{-1}). \tag{3.6}$$

In other words this equation is valid for those euclidean groups all of whose subgroups are spherical, and for all Lannér groups. In [9], Charney and Davis extend this result as follows.

Theorem 3.3 *Let G be a Coxeter group acting cocompactly on $\mathbb{H}^n$. Denote by S its Coxeter generating set. Then its growth series $f_S(x)$ is reciprocal if n is even, while it is antireciprocal for n odd.*

3.2 Growth series of Coxeter groups acting on $\mathbb{S}^n$

In this section we present known results about the growth of spherical and euclidean Coxeter groups. This subject is classical. In fact Solomon [34] and Bott [19, page 179] developed formulas to compute the growth series of a spherical and of a euclidean Coxeter group. Let us remark that these formulas are more convenient to use than Steinberg's formula (3.4).

We start with the notion of *exponents* of a spherical Coxeter group G (see [34], for example). Suppose that G is generated by $S = \{s_1, \ldots, s_k\}$. An element $g = s_{\sigma(1)} \cdots s_{\sigma(k)} \in G$, $\sigma \in S_k$, is called a *Coxeter element*. One can show (see [19, page 74]) that all Coxeter elements are conjugate in G. Denote by h the order of a Coxeter element. h is called the *Coxeter number* of G. As $g = s_{\sigma(1)} \cdots s_{\sigma(k)}$ is a product of reflections, we can associate to g a matrix A, whose characteristic polynomial $P_A(x)$ possesses k roots on the unit circle. These can be written as $\left(e^{\frac{2\pi i}{h}}\right)^{m_j}$, where $m_j \in \mathbb{N}$, for all $j = 1, \ldots, k$, and $m_1 \le m_2 \le \ldots \le m_k$ such that $m_1 = 1$ and $m_k = h - 1$. These numbers $m_1, m_2, \ldots, m_k$ are called the *exponents* of G. They are listed in [11, page 141]. Given the exponents of G, the growth series of G can be computed by means of Solomon's formula [34, Corollary 2.3] according to

$$f_S(x) = \prod_{i=1}^{k} [m_i + 1], \tag{3.7}$$

where $[n] := 1 + x + x^2 + \ldots + x^{n-1}$, for all $n \in \mathbb{N}$. Sometimes we shall also use the notation $[n, m] := [n] \cdot [m]$ and so on. Observe that $f_S(x)$ is a polynomial of degree $m_1 + \ldots + m_k$.

The above formula allows to compute the growth series of any spherical Coxeter group. Below we attach the corresponding list in the irreducible case. Let us remark that Solomon's formula is very useful not only for spherical Coxeter groups, but also for euclidean and hyperbolic ones, as Steinberg's formula (3.4) takes into account their finite subgroups.

Graph	Exponents	Growth series $f_S(x)$
A_n	$1, 2, \dots, n-1, n$	$[2, 3, \dots, n, n+1]$
B_n	$1, 3, \dots, 2n-3, 2n-1$	$[2, 4, \dots, 2n-2, 2n]$
D_n	$1, 3, \dots, 2n-5, 2n-3, n-1$	$[2, 4, \dots, 2n-2] \cdot [n]$
$G_2^{(m)}$	$1, m-1$	$[2, m]$
F_4	$1, 5, 7, 11$	$[2, 6, 8, 12]$
E_6	$1, 4, 5, 7, 8, 11$	$[2, 5, 6, 8, 9, 12]$
E_7	$1, 5, 7, 9, 11, 13, 17$	$[2, 6, 8, 10, 12, 14, 18]$
E_8	$1, 7, 11, 13, 17, 19, 23, 29$	$[2, 8, 12, 14, 18, 20, 24, 30]$
H_3	$1, 5, 9$	$[2, 6, 10]$
H_4	$1, 11, 19, 29$	$[2, 12, 20, 30]$

Table 6 : The growth series of irreducible spherical Coxeter groups.

Let us recall (see part 3.1) that the growth series of a reducible Coxeter group is obtained by multiplying the growth series of its irreducible components. For example the growth series of the reducible spherical Coxeter group with graph

$$G_2^{(4)} \times G_2^{(5)} \quad : \quad \bullet \overset{4}{\text{———}} \bullet \qquad \bullet \overset{5}{\text{———}} \bullet$$

is given by

$$f_S(x) = [2]^2 \cdot [4] \cdot [5] = [2, 2, 4, 5].$$

Remark 3.1 *For completeness let us also deal with euclidean groups. Denote by* $\tilde{\Gamma}_n$ *the graph of a Coxeter group* G *acting on* $\mathbb{E}^n$*.* G *is a semidirect product of the spherical group* G' *with graph* Γ_n *and a certain group generated by reflections. For further details, see [37, page 202]. If* S *denotes the finite generating set for* G*, then by a result of Bott (see [19, page 179], for example), the growth series is given by*

$$f_S(x) = \frac{f_{S'}(x)}{(1-x)(1-x^{m_2})\cdots(1-x^{m_k})}, \tag{3.8}$$

where S' *is the generating set,* $f_{S'}(x)$ *is the growth series and* $1, m_2, \dots, m_k$ *are the exponents of the related spherical group* G' *(cf [37, Proposition 1.5]).*

We give explicitly the growth series of irreducible euclidean Coxeter groups in the following table.

Graph	Growth series $f_S(x)$
$\tilde{A}_1$	$\dfrac{1+x}{1-x}$
$\tilde{A}_n$	$\dfrac{[2,3,\ldots,n,n+1]}{(1-x)(1-x^2)\cdots(1-x^n)}$
$\tilde{B}_n$	$\dfrac{[2,4,\ldots,2n-2,2n]}{(1-x)(1-x^3)\cdots(1-x^{2n-1})}$
$\tilde{C}_n$	$\dfrac{[2,4,\ldots,2n-2,2n]}{(1-x)(1-x^3)\cdots(1-x^{2n-1})}$
$\tilde{D}_n$	$\begin{cases} \dfrac{[2,4,\ldots,2n-2]\cdot[n]}{(1-x)(1-x^3)\cdots(1-x^{2n-3})(1-z^{n-1})} & \text{if } n \text{ odd} \\ \dfrac{[2,4,\ldots,2n-2]\cdot[n-1]}{(1-x)(1-x^3)\cdots(1-x^{2n-3})(1-x^{n-1})} & \text{if } n \text{ even} \end{cases}$
$\tilde{E}_6$	$\dfrac{[2,5,6,8,9,12]}{(1-x)(1-x^4)(1-x^5)(1-x^7)(1-x^8)(1-x^{11})}$
$\tilde{E}_7$	$\dfrac{[2,6,8,10,12,14,18]}{(1-x)(1-x^5)(1-x^7)(1-x^9)(1-x^{11})(1-x^{13})(1-x^{17})}$
$\tilde{E}_8$	$\dfrac{[2,8,12,14,18,20,24,30]}{(1-x)(1-x^7)(1-x^{11})(1-x^{13})(1-x^{17})(1-x^{19})(1-x^{23})(1-x^{29})}$
$\tilde{F}_4$	$\dfrac{[2,6,8,12]}{(1-x)(1-x^5)(1-x^7)(1-x^{11})}$
$\tilde{G}_2$	$\dfrac{[2,m]}{(1-x)(1-x^{m-1})}$

Table 7 : The growth series of irreducible euclidean Coxeter groups.

3.3 Growth series of Coxeter groups acting in $\mathbb{H}^2$ and in $\mathbb{H}^3$

In low dimensions there are some general results about the growth of cocompact hyperbolic groups. More precisely there exists an explicit formula in the planar case, which is due to Floyd. For the spatial case the only result we have, beside some case-by-case computations, is of structural nature and due to Parry [26].

In this part we discuss the above mentioned results. The approach of Parry will be described quite in detail, as our methods and results in dimension 4 are inspired by it (see Theorem 3.7).

Let us start with Floyd's formula for a compact Coxeter polygon $P \subset \mathbb{H}^2$ with $N \geq 3$ vertices $v_1, \ldots, v_N$. Denote by $\frac{\pi}{a_k}$, where $a_k \in \mathbb{N}_{\geq 2}$ for $k = 1, \ldots, N$, its interior angles. Recall that such a polygon exists if and only if

$$\sum_{k=1}^{N} \frac{1}{a_k} < N - 2,$$

according to Poincaré's Theorem (see [13, pages 135-145], for example). Let G be the associated Coxeter group with generating set S. Then by means of Steinberg's formula (3.4), Floyd derived the following formula [16, page 478].

$$f_S(x) = \frac{[2] \cdot [a_1] \cdots [a_N]}{[2] \cdot [a_1] \cdots [a_N] - \sum\limits_{k=1}^{N} x \cdot [a_1] \cdots [a_{k-1}] \cdot [a_k - 1] \cdots [a_{k+1}] \cdots [a_N]}. \tag{3.9}$$

Example 3.1 *Let G be a hyperbolic triangle group given by the Coxeter graph*

$$\Gamma : \bullet \!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\bullet \overset{7}{-\!\!\!-\!\!\!-\!\!\!-} \bullet\,.$$

Let S denote the generating set of G. The growth series of G is given by (cf. Floyd's formula (3.9))

$$f_S(x) = \frac{[2, 2, 3, 7]}{1 + x - x^3 - x^4 - x^5 - x^6 - x^7 + x^9 + x^{10}}.$$

Observe that the denominator equals Lehmer's polynomial, namely the minimal polynomial of the smallest Salem number known up to now (see below). Besides G is the group with least co-area among all *discrete groups acting by isometries on $\mathbb{H}^2$.*

Let us now extend the context and consider $\mathbb{H}^3$.

In [26], Parry shows that the growth series of a cocompact Coxeter group acting in $\mathbb{H}^3$ (and in $\mathbb{H}^2$) is related to a *Salem number* $\alpha \in \mathbb{C}$, that is to say, $\alpha > 1$ is an algebraic integer all of whose conjugates have modulus at most equal to 1 and at least one is lying on the unit circle. Let us add that Parry generalises corresponding results of Cannon and Wagreich (see [7] and [6, §5, 6, 7]). More precisely we have the following theorem [26, Theorem 2.6 & Theorem 2.9].

Theorem 3.4 (Parry) *Let G be a geometric Coxeter group acting cocompactly on $\mathbb{H}^2$ or $\mathbb{H}^3$. Denote by S the set of its Coxeter generators. Then $f_S(x)$ can be expressed as a quotient of relatively prime polynomials with integer coefficients for which the denominator is a product of distinct irreducible cyclotomic polynomials and exactly one Salem polynomial.*

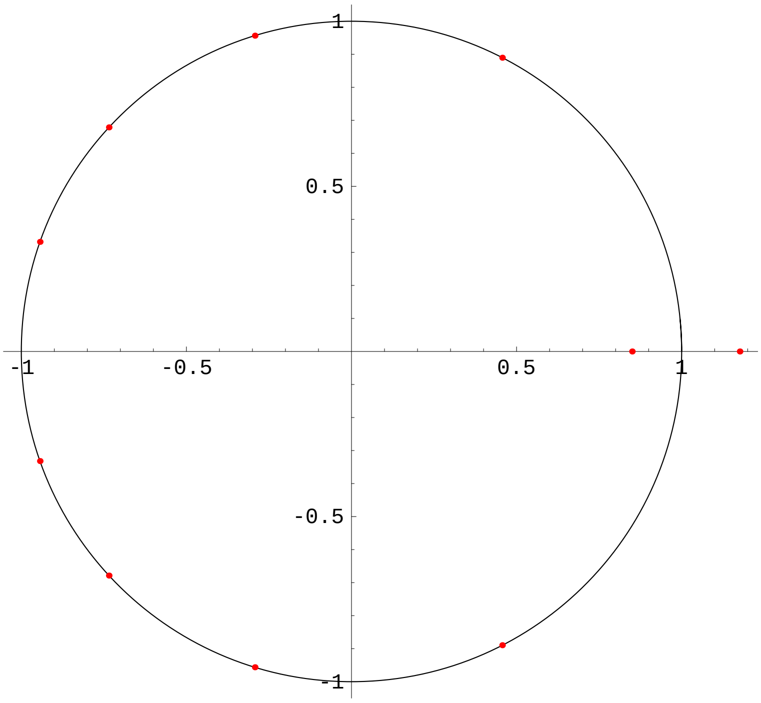


Figure 1: Roots of Lehmer's polynomial.

As for the proof, Parry's ideas consist in a study of the maximal subgroups of G associated to the vertices of a fundamental polyhedron of G and are based on the construction of a "nice" function adapted to these subgroups. Unfortunately we could not find similarly nice functions in dimensions $n \geq 4$ allowing conclusions about the distribution of the zeros in the complex plane (see part 3.4). The following detailed description of Parry's approach will be partially used later.

For $n = 2$ and 3 the proof is achieved by means of the following assertion (see [26, Theorem 1.7]).

Proposition 3.5 *Suppose given an antireciprocal rational function* $R(x) = \frac{P(x)}{Q(x)}$*, where* $P(x)$ *and* $Q(x)$ *are relatively prime monic polynomials with integer coefficients and equal degrees. Assume that* $R(x)$ *has a zero at a positive real number other than* 1 *and that all poles of* $R(x)$ *are simple and occur at roots of unity* ζ *with the essential factor of the residue at* ζ *positive if* $\zeta \neq 1$*. Then* $P(x)$ *is a product of distinct irreducible cyclotomic polynomials with exactly one Salem polynomial.*

By *essential factor of the residue at* ζ is meant the residue divided by ζ (see [26, Definition 1.3]).

The proof of Proposition 3.5 uses basically some properties of antireciprocal rational functions and an innocent but very useful lemma about Salem numbers (see [26, Proposition 1.2] and [26, Lemma 1.6]). In dimension 2 it is the following corollary to Proposition 3.5 which is of help (see [26, Corollary 1.8]).

Corollary 3.6 *Suppose given an integer $N \geq 2$. Let $c_2, \dots, c_N$ be nonnegative integers such that $\sum_{n=2}^{N} c_n \cdot \frac{n-1}{n} > 2$. Let $R(x)$ be the rational function*

$$R(x) = \frac{x+1}{x-1} + \sum_{n=2}^{N} c_n \cdot \frac{x - x^n}{(x-1) \cdot (x^n - 1)}. \tag{3.10}$$

Then $R(x)$ is an antireciprocal rational function, $R(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are relatively prime monic polynomials with integer coefficients and equal degrees, and $P(x)$ is a product of distinct irreducible cyclotomic polynomials with exactly one Salem polynomial.

After all these preparations we present now Parry's proof of Theorem 3.4.

P*roof of Theorem 3.4*
Let $G < I(\mathbb{H}^n)$, $n = 2, 3$, be a Coxeter group with generating set S and compact fundamental Coxeter domain P. The number of facets of P equals $|S|$. Let H denote a maximal finite Coxeter subgroup of G. For $n = 2$, H is dihedral generated by two reflections through the sides of P which meet under the angle $\frac{\pi}{k}$, say. For $n = 3$, each H corresponds to the stabiliser of a vertex of P. Since P is simple, H is a spherical triangle group.

Consider first the case $n = 2$, that is G is a planar hyperbolic Coxeter group. For $k = 2, \dots, N$, denote by c_k the number of vertices of the polygon P with interior angle $\frac{\pi}{k}$. Here, $N = N(P)$ is such that each angle of P is at least $\frac{\pi}{N}$. Let H be maximal in G and generated by the set $T \subset S$, say. Hence, $|T| = 2$. Let us assume that H is of order $2m$ for some positive integer m. Then its exponents are 1 and $m - 1$. By Solomon's formula (3.7),

$$f_T(x) = [2, m] = \frac{(x+1) \cdot (x^m - 1)}{x - 1}.$$

By means of (3.4) one deduces that

$$\begin{aligned}
\frac{1}{f_S(x^{-1})} &= 1 - \frac{|S|}{x+1} + \sum_{k=2}^{N} c_k \cdot \frac{x-1}{(x+1)\cdot(x^k-1)} \\
&= 1 + \sum_{k=2}^{N} c_k \cdot \left(\frac{x-1}{(x+1)\cdot(x^k-1)} - \frac{1}{x+1} \right) \\
&= 1 + \sum_{k=2}^{N} c_k \cdot \frac{x-x^k}{(x+1)\cdot(x^k-1)}.
\end{aligned}$$

Hence,

$$\frac{x+1}{(x-1) \cdot f_S(x^{-1})} = \frac{x+1}{x-1} + \sum_{k=2}^{N} c_k \cdot \frac{x - x^k}{(x-1) \cdot (x^k - 1)}. \tag{3.11}$$

In order to apply Lemma 3.6, divide P into $|S|$ triangles in a canonical way. By the angular defect formula one deduces that

$$\sum_{k=2}^{N} c_k \cdot \frac{\pi}{k} < (|S| - 2) \cdot \pi,$$

and

$$\sum_{k=2}^{N} c_k \cdot \frac{k-1}{k} > 2.$$

By Lemma 3.6 the claim of Theorem 3.4 for the planar case follows.
Let us remark that one can shorten the proof in the following way. The fundamental polygon P of G has exactly S vertices. Each stabiliser H is dihedral of order $2m$. Parry regrouped the terms in $1/f_S(x^{-1})$ accordingly by defining the following function associated to H.

$$g_H(x) := \frac{x - x^m}{(x+1)\cdot(x^m - 1)}. \tag{3.12}$$

Recall that the value of m depends on the group H. This is why we sometimes write $m(H)$ instead of m. By means of (3.4) one easily checks that

$$\frac{1}{f_S(x^{-1})} = 1 + \sum_{H \text{ dihedral}} g_H(x). \tag{3.13}$$

Then, by multiplying both sides of equation (3.13) by $\frac{x+1}{x-1}$ one gets

$$\frac{x+1}{(x-1)\cdot f_S(x^{-1})} = \frac{x+1}{x-1} + \sum_{H \text{ dihedral}} \frac{x - x^{m(H)}}{(x-1)\cdot(x^{m(H)}-1)}, \tag{3.14}$$

which is similar to (3.11).

Consider now the case $n = 3$. We proceed as above. Consider a maximal finite subgroup H of G with generating set T. Recall that H is a triangular Coxeter group and observe that G has at most $\binom{|S|}{3}$ of such subgroups. As in the planar case, one defines a function g associated to H in order to obtain a nice structural formula for the growth series $f_S(x)$ of G. In fact, Parry proposed (see [26, equation (2.11)])

$$g_H(x) := \frac{-1}{f_T(x)} + \frac{1}{2}\cdot\left(\frac{1}{f_{U_1}(x)} + \frac{1}{f_{U_2}(x)} + \frac{1}{f_{U_3}(x)}\right) - \frac{1}{2}\cdot\frac{1}{(1+x)}, \tag{3.15}$$

where U_1, U_2, U_3 are the three 2-element subsets of T. By means of (3.4) one deduces that

$$\frac{1}{f_S(x^{-1})} = \frac{x-1}{x+1} + \sum_{H \text{ triangular}} g_H(x). \tag{3.16}$$

Observe that (3.15) and (3.16) are the 3-dimensional analogs of (3.12) and (3.13) but appear in a more complicated way. Nevertheless, it is possible to derive a simpler shape also for $n = 3$. Since H is triangular with Coxeter graph $\bullet \overset{q}{\text{———}} \bullet \overset{r}{\text{———}} \bullet$, for $\frac{1}{q} + \frac{1}{r} > \frac{1}{2}$, its exponents $m_1 = 1$, m_2 and m_3 are of the form as in Table 8.

Graph	Type	q	r	m_1	m_2	m_3
$\bullet \qquad \bullet \overset{r}{\text{———}} \bullet$	$A_1 \times G_2^{(p)}, p \geq 2$	2	p	1	1	$p-1$
$\bullet \text{———} \bullet \text{———} \bullet$	A_3	3	3	1	2	3
$\bullet \overset{4}{\text{———}} \bullet \text{———} \bullet$	B_3	3	4	1	3	5
$\bullet \overset{5}{\text{———}} \bullet \text{———} \bullet$	H_3	3	5	1	5	9

Table 8 : Finite triangular Coxeter groups and their exponents

By Solomon's formula (3.7), (3.15) turns into

$$\begin{aligned} g_H(x) &= -\tfrac{(x-1)^3}{(x^{m_1+1}-1)\cdot(x^{m_2+1}-1)\cdot(x^{m_3+1}-1)} + \tfrac{1}{2}\left(\tfrac{(x-1)^2}{(x^2-1)\cdot(x^2-1)} + \tfrac{(x-1)^2}{(x^2-1)\cdot(x^q-1)} + \tfrac{(x-1)^2}{(x^2-1)\cdot(x^r-1)}\right) \\ &\quad -\tfrac{1}{2}\tfrac{1}{x+1} \\ &= -\tfrac{x-1}{2\cdot(1+x)} \cdot \left(\tfrac{2\cdot(x-1)}{(x^{m_2+1}-1)\cdot(x^{m_3+1}-1)} - \tfrac{1}{x^2-1} - \tfrac{1}{x^q-1} - \tfrac{1}{x^r-1} + \tfrac{1}{x-1}\right), \end{aligned}$$

which can be - miraculously - rewritten according to

$$g_H(x) = -\frac{1}{2}x(x-1)\frac{(x^{m_1}-1)\cdot(x^{m_2}-1)\cdot(x^{m_3}-1)}{(x^{m_1+1}-1)\cdot(x^{m_2+1}-1)\cdot(x^{m_3+1}-1)}. \tag{3.17}$$

Expression (3.17) is the nice closed formula which one was looking for (see (3.15)) ! By multiplying both sides of (3.16) by $\left(\frac{x+1}{x-1}\right)^2$, one obtains

$$\frac{(x+1)^2}{(x-1)^2 f_S(x^{-1})} = \frac{x+1}{x-1} + \sum_{H \text{ triangular}} \left(\frac{x+1}{x-1}\right)^2 g_H(x). \tag{3.18}$$

The factor $\left(\frac{x+1}{x-1}\right)^2$ forces the function (3.18) to have only simple poles. Finally, the function (3.18) satisfies the hypothesis of Proposition 3.5 which allows one to finish the proof of Theorem 3.4 for the 3-dimensional case. Let us add that the result of Milnor [25] is essential in the last step above. In fact, as G is hyperbolic, its growth rate is strictly greater than one, implying that the function (3.18) has a positive real zero different from 1. □

Example 3.2 *Consider the Lannér group G_L, with generating set S and graph*

$$\Gamma_L : \bullet \text{———} \bullet \overset{5}{\text{———}} \bullet \text{———} \bullet .$$

By means of Steinberg's formula (3.4) one easily computes

$$f_S(x) = \frac{[2,2,2,3]\cdot\left(1-2x+3x^2-3x^3+3x^4-2x^5+x^6\right)}{1-2x+x^2-x^4+2x^5-2x^6+x^7-x^9+2x^{10}-x^{11}}.$$

The poles of $f_S(x)$ are distributed in the complex plane as follows.

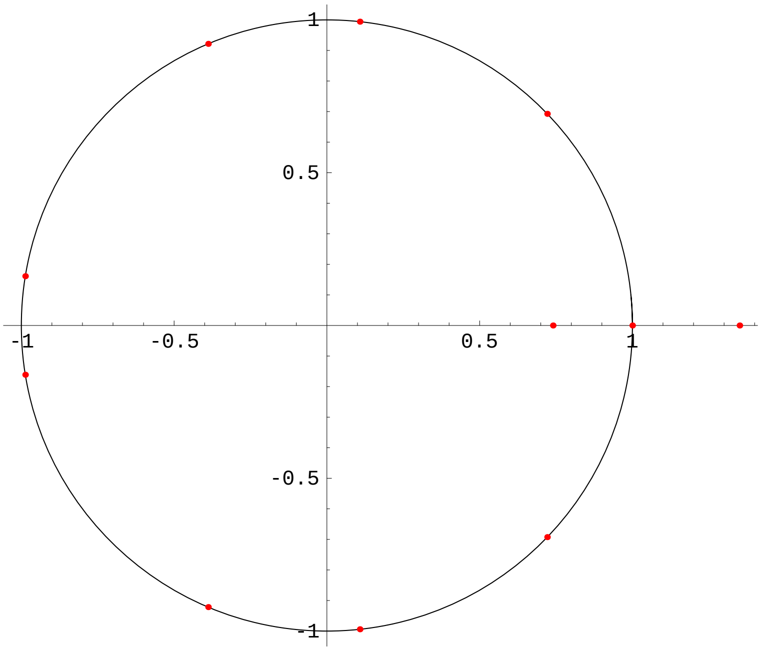


Figure 2: Poles of the growth series of a Lannér group with graph Γ_L.

At the end of this part let us add a general comment about the (non-)extendability to higher dimensions of the proof of Theorem 3.4.

Remark 3.2 *The basic idea of the above proof consists in associating to each vertex stabiliser H in G a help-function $g(x)$ and to regroup the different terms in Steinberg's formula such that a very convenient, tractable expression for the growth series $f_S(x)$ arises (see (3.14) and (3.18)). In dimensions $n \geq 4$ this becomes a difficult, if not impossible, task. For example, the variety of exponents appearing in connection with the maximal finite subgroups H of G is huge. Furthermore, there is not even a connection between the growth series of H and its 2-generator subgroups. In this sense the appearance of $\frac{1}{f_S(x^{-1})}$ as in (3.14) and (3.18) together with (3.17), and its algebraic properties for $n = 2, 3$, is very beautiful, but highly exceptional.*

3.4 On the growth of the Simplex, Esselmann and Kaplinskaya groups

In this part we study the growth of cocompact Coxeter groups in $I(\mathbb{H}^4)$ generated by at most six reflections. These are the Simplex, Esselmann and Kaplinskaya groups. Their growth properties are different from the ones in lower dimensions. In fact, some experiments led us to formulate the following rule.

Conjecture 1 *Let G be a Simplex, an Esselmann or a Kaplinskaya group with generating set S. Then, its growth series $f_S(x)$ can be expressed as a quotient of relatively prime, monic and palindromic polynomials of equal degree over the integers. Furthermore,*

(1) *the growth series $f_S(x)$ of G possesses four distinct positive real poles appearing in pairs (x_1, x_1^{-1}) and (x_2, x_2^{-1}) with $x_1 < x_2 < 1 < x_2^{-1} < x_1^{-1}$; these poles are simple.*

(2) *The growth rate $\tau = x_1^{-1}$ is a Perron number, that is, $\tau > 1$ is an algebraic integer all of whose conjugates are of absolute value strictly smaller than τ.*

(3) *The non-real poles of $f_S(x)$ are contained in an annulus of radii x_2, x_2^{-1} around the unit circle.*

(4) *The growth series $f_S(x)$ of the Kaplinskaya group G_{66} with graph K_{66} has four distinct negative and four distinct positive simple real poles. For $G \neq G_{66}$, $f_S(x)$ has no negative poles.*

Example 3.3 *Consider the Lannér group $G_L < I(\mathbb{H}^4)$, with generating set S and graph*

$$\Gamma_L : \bullet \overset{5}{\text{——}} \bullet \text{——} \bullet \text{——} \bullet \text{——} \bullet\,.$$

By means of Steinberg's formula (3.4),

$$f_S(x) = \frac{[2,12,20,30]}{\begin{array}{l}1 - x - x^7 + x^8 - x^9 + x^{10} - x^{11} + x^{14} - x^{15} + x^{16} - 2x^{17} + 2x^{18} - x^{19} + x^{20} \\ - x^{21} + x^{22} - x^{23} + 2x^{24} - 2x^{25} + 2x^{26} - 2x^{27} + 2x^{28} - x^{29} + x^{30} - x^{31} + 2x^{32} \\ - 2x^{33} + 2x^{34} - 2x^{35} + 2x^{36} - x^{37} + x^{38} - x^{39} + x^{40} - x^{41} + 2x^{42} - 2x^{43} + x^{44} \\ - x^{45} + x^{46} - x^{49} + x^{50} - x^{51} + x^{52} - x^{53} - x^{59} + x^{60}\end{array}}.$$

The poles of $f_S(x)$ are distributed in the complex plane as follows.

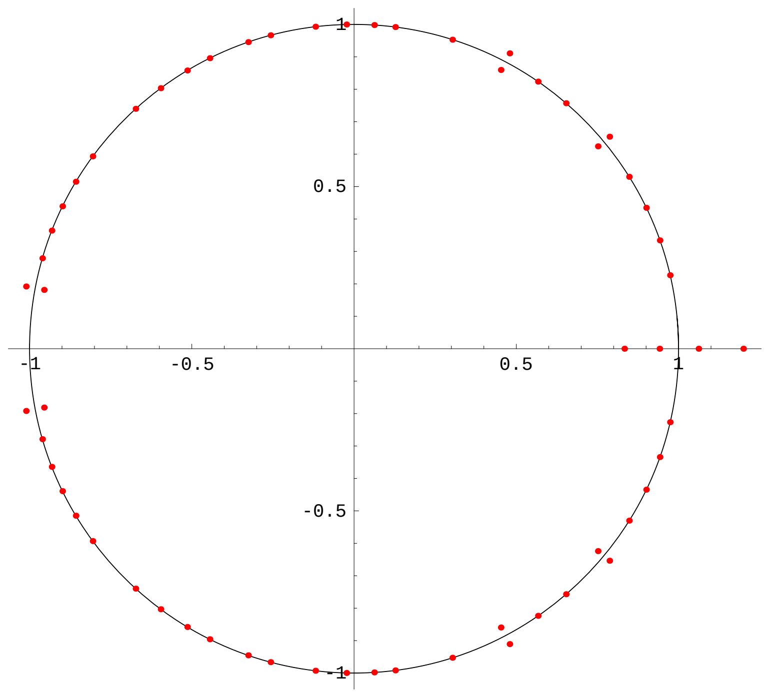


Figure 3: The poles of the growth series of the Lannér group G_L.

Let us mention that the value $\frac{1}{f_S(1)} = \chi(G_L) = \frac{1}{14400}$ *is proportional to the covolume* $\frac{\pi^2}{10800}$ *of* G_L*, a fact which follows from a result of Heckman [18]. Notice that this latter value is the minimal covolume of all cocompact arithmetic discrete groups in* $I(\mathbb{H}^4)$*, up to a factor* 2 *(cf. [2]).*

Let us return to our claim formulated in Conjecture 1.

By extending Parry's methods, developed for the dimensions 2 and 3, we are *not* able to prove Conjecture 1 *as a whole* (see Remark 3.2), but in the following restricted sense.

Theorem 3.7 *Let G be a Simplex, an Esselmann or a Kaplinskaya group with generating set S. Then, its growth series $f_S(x)$ can be expressed as a quotient of relatively prime, monic and palindromic polynomials of equal degree over the integers. Furthermore,*

(1)′ *the growth series $f_S(x)$ of G possesses at most two pairs of positive real poles.*

(2)′ *The growth rate τ is given by the largest real root and is a Perron number.*

(3)′ *The non-real poles of $f_S(x)$ are contained in an annulus of radii τ^{-1}, τ.*

(4)′ *The growth series of the Kaplinskaya group G_{66} with graph K_{66} has furthermore four distinct negative poles. For $G \neq G_{66}$, $f_S(x)$ has no negative poles.*

In the following we give a proof of Theorem 3.7 in the case of a Simplex group and illustrate this result by providing some instructive examples. Then, we briefly mention how to adapt the proof for the Esselmann and Kaplinskaya groups. In the case of Tumarkin groups, generated by seven reflexions, we will meet limitations of our approach. Some comments with an example will be added.

Proof of Theorem 3.7
Let G be a Simplex group with generating set S such that $|S| = 5$. Consider a maximal, and therefore spherical, subgroup $H := G_T$ of G with generating set $T \subsetneq S$. Generalising Parry's ideas, we define a help-function $h^S(x)$ associated to H, namely

$$h^S(x) = h_T^S(x) := -\frac{1}{x+1} + \frac{1}{3} \cdot \sum_U \frac{1}{f_U(x)} - \frac{1}{2} \cdot \sum_V \frac{1}{f_V(x)} + \frac{1}{f_T(x)}, \tag{3.19}$$

where U varies over the six 2-element subsets and V varies over the four 3-element subsets of T. The superscript S refers to the term " Simplex ". By means of Steinberg's formula (3.4) we easily check that (see also Theorem 3.3)

$$\frac{1}{f_S(x)} = 1 + \sum_{H<G \text{ maximal}} h^S(x). \tag{3.20}$$

Note that there exists only a very limited number of different maximal subgroups H in G, as the weights of the Coxeter graph of G are at most equal to 5. We list all possibilities in the following table.

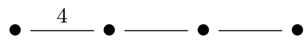
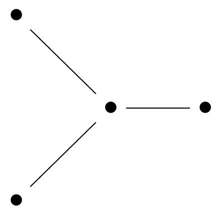
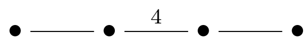
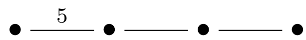
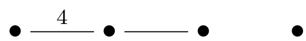
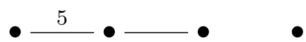
Symbol	Graph
A_4	
B_4	
D_4	
F_4	
H_4	
$A_3 \times A_1$	
$B_3 \times A_1$	
$H_3 \times A_1$	
$G_2^{(q)} \times G_2^{(r)}$	 for $3 \leq q \leq 5$ and $2 \leq r \leq 5$

Table 9 : Groups arising as maximal subgroups of a Simplex group.

In contrast to the lower dimensions, we did not succeed to derive a nice closed formula for (3.19), valid generally, as we explained in Remark 3.2. Despite of this, there exist some special cases, by taking into account the reducibility of H, so that we are able to derive general properties for $h^S(x)$. We say that a maximal subgroup H of G is *reducible of order* k if it contains precisely k irreducible components.

Lemma 3.8 *The help-function $h^S(x)$ (3.19) associated to a maximal subgroup H of a Simplex group G in $I(\mathbb{H}^4)$ can be written as the quotient*

$$h^S(x) = -x \cdot \frac{n(x)}{d(x)}, \tag{3.21}$$

where $n(x)$ and $d(x)$ are palindromic polynomials of even degrees over the integers. Moreover, $d(x)$ is cyclotomic with $\deg(d) = \deg(n) + 2$. Furthermore, $h^S(x)$ is negative for $x > 0$ and strictly decreasing on $(-1, 0)$.

P*roof*
Arrange the exponents of H so that $1 = m_1 \leq m_2 \leq m_3 \leq m_4$.

Suppose that H is reducible of order 3. Then its Coxeter graph Γ_H is given by

$$\Gamma_H \quad : \quad \bullet \qquad \bullet \qquad \bullet \overset{q}{\text{———}} \bullet \, ,$$

for an integer $q \geq 3$ with $m_4 = q - 1$. The help-function (3.19) associated to H equals

$$h^S(x) = -x \cdot \frac{1}{3 \cdot [2,2,2,q]} \cdot \{3 \cdot [q+1] + x \cdot [q-1] + x^2 \cdot [q-3]\}, \tag{3.22}$$

which shows that $h^S(x)$ is negative for $x > 0$. (3.22) transforms into

$$h^S(x) = \begin{cases} -x \cdot \frac{1}{3 \cdot [2,2,q]} \cdot \left(3 + \sum\limits_{i=0}^{k-1} x^{2i+1} + 4 \cdot \sum\limits_{i=1}^{k-1} x^{2i} + 3x^{2k}\right), & \text{for} \quad q = 2k+1 \\ -x \cdot \frac{1}{3 \cdot [2,2,2,q]} \cdot \left(3 + 4x + 5x^2 \cdot [q-3] + 4x^{q-1} + 3x^q\right), & \text{for} \quad q \text{ even.} \end{cases} \tag{3.23}$$

Formula (3.23) follows by an easy induction.

Suppose that H is reducible of order 2 such that both of its components are 2-generator subgroups. Then, its Coxeter graph Γ_H is given by

$$\Gamma_H \quad : \quad \bullet \overset{q}{\text{———}} \bullet \qquad \bullet \overset{r}{\text{———}} \bullet \, ,$$

for two integers $q, r \geq 3$ and yields the exponents $m_3 = q - 1$ and $m_4 = r - 1$. Without loss of generality assume that $q \leq r$. Thus there exists an integer $l \geq 0$ such that $r = q + l$. The help-function (3.19) associated to H becomes

$$h^S(x) = \frac{1}{3 \cdot [2,2,q,r]} \cdot P_l^q(x), \tag{3.24}$$

where

$$P_l^q(x) = 3 + [q] \cdot \{2x - 4 + [q] \cdot (1 - 3x)\} + x^q \cdot [l] \cdot \{x - 2 + [q] \cdot (1 - 3x)\}. \tag{3.25}$$

At the end of this section, in part 3.4.1 we shall prove in detail the following claim.

Claim 3.9 *The polynomial $P_l^q(x)$ (3.25) is a product of $-x$ and a palindromic polynomial such that the numerator of $h^S(x)$ in (3.24) is of odd degree.*

For completeness, a list of the six different help-functions $h^S(x)$ associated to finite groups with graphs $\Gamma_H : \bullet \overset{q}{\text{———}} \bullet \qquad \bullet \overset{r}{\text{———}} \bullet$ is appended (see Table 10 below).

Graph Γ_H	$h^S(x)$
$\bullet\!-\!\!-\!\bullet \quad \bullet\!-\!\!-\!\bullet$	$-x \cdot \dfrac{3+5x+5x^2+5x^3+3x^4}{3\cdot(1+2x+2x^2+x^3)^2}$
$\bullet\!-\!\!-\!\bullet \quad \bullet\overset{4}{-\!\!-}\bullet$	$-x \cdot \dfrac{3+2x+4x^2+2x^3+3x^4}{3\cdot[2]^2\cdot(1+x+2x^2+x^3+x^4)}$
$\bullet\!-\!\!-\!\bullet \quad \bullet\overset{5}{-\!\!-}\bullet$	$-x \cdot \dfrac{3+5x+6x^2+7x^3+6x^4+5x^5+3x^6}{3\cdot[2]^2\cdot(1+2x+3x^2+3x^3+3x^4+2x^5+x^6)}$
$\bullet\overset{4}{-\!\!-}\bullet \quad \bullet\overset{4}{-\!\!-}\bullet$	$-x \cdot \dfrac{3+5x+7x^2+7x^3+7x^4+5x^5+3x^6}{3\cdot[2]^4\cdot(1+x^2)^2}$
$\bullet\overset{4}{-\!\!-}\bullet \quad \bullet\overset{5}{-\!\!-}\bullet$	$-x \cdot \dfrac{3+2x+5x^2+3x^3+5x^4+2x^5+3x^6}{3\cdot[2]^2\cdot(1+x+2x^2+2x^3+2x^4+x^5+x^6)}$
$\bullet\overset{5}{-\!\!-}\bullet \quad \bullet\overset{5}{-\!\!-}\bullet$	$-x \cdot \dfrac{3+5x+7x^2+9x^3+9x^4+9x^5+7x^6+5x^7+3x^8}{3\cdot(1+2x+2x^2+2x^3+2x^4+x^5)^2}$

Table 10 : The help-functions associated to Γ_H.

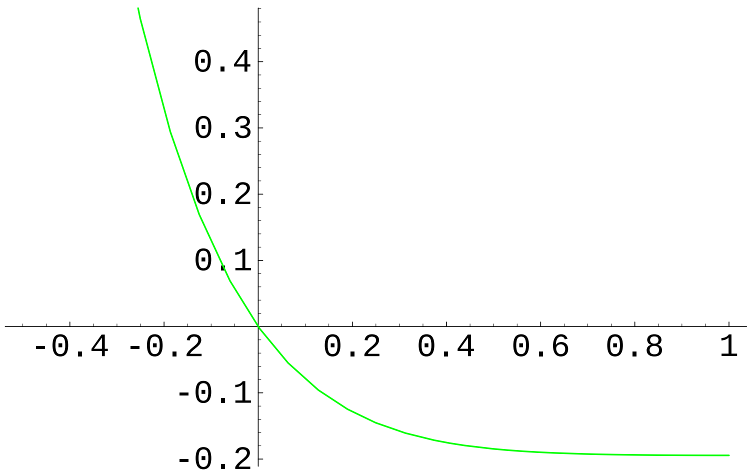


Figure 4: The help-function $h^S(x)$ associated to $A_2 \times G_2^{(5)}$ on $[-0.5, 1]$.

If H factors into a product of an irreducible 3-generator subgroup H' and A_1, the group H is necessarily of type $A_3 \times A_1$, $B_3 \times A_1$ or $H_3 \times A_1$ (see Table 8 in part 3.3). In the following table the relative help-functions are reproduced.

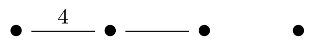
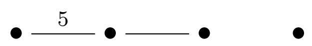
Graph of $H' \times A_1$	$h^S(x)$
	$-x \cdot \dfrac{6+4x+9x^2+4x^3+6x^4}{6 \cdot [2,2,3] \cdot (1+x^2)}$
	$-x \cdot \dfrac{\left(6+10x+15x^2+18x^3+20x^4+18x^5+15x^6 + 10x^7+6x^8\right)}{6 \cdot [2,2,2,2,3] \cdot (1+x^2) \cdot (1-x+x^2)}$
	$-x \cdot \dfrac{\left(6+10x+15x^2+20x^3+25x^4+28x^5+30x^6 + 30x^7+30x^8+28x^9+25x^{10}+20x^{11}+15x^{12} + 10x^{13}+6x^{14}\right)}{6 \cdot [2,2,2,2,3,5] \cdot (1-x+x^2) \cdot (1-x+x^2-x^3+x^4)}$

Table 11 : The help-functions associated to $A_3 \times A_1$, $B_3 \times A_1$ and $H_3 \times A_1$.

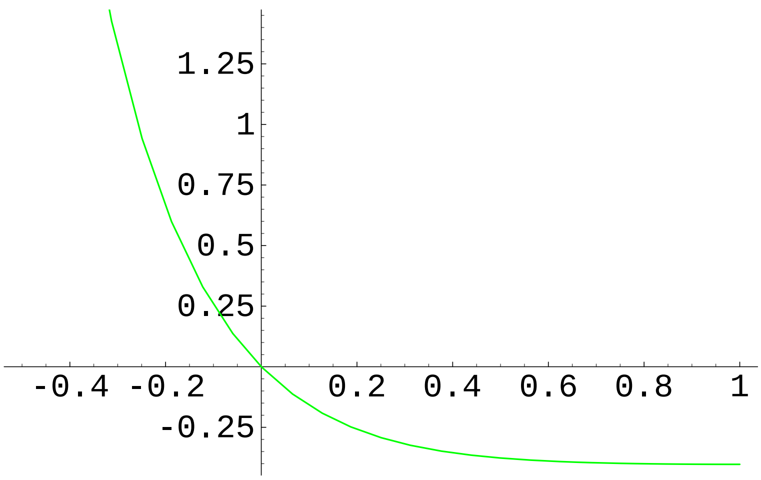


Figure 5: The help-function $h^S(x)$ associated to $A_1 \times A_3$ on $[-0.5, 1]$.

Finally suppose that H is irreducible. Then there are only five possible cases which, together with the associated help-functions, are given in the next table (see also Table 1 in part 2.4).

Graph	$h^S(x)$
A_4	$-x\cdot\dfrac{1-x+x^2}{[5]}$
B_4	$-x\cdot\dfrac{\big(6+12x+20x^2+29x^3+38x^4+46x^5+52x^6+52x^7+52x^8+46x^9 + 38x^{10}+29x^{11}+20x^{12}+12x^{13}+6x^{14}\big)}{6\cdot[2]^4\cdot(1+x^2)^2\cdot(1+x^2+2x^4+x^6+x^8)}$
D_4	$-x\cdot\dfrac{2+2x+3x^2+4x^3+4x^4+4x^5+3x^6+2x^7+2x^8}{2\cdot[2]^4\cdot(1+x^2)^2\cdot(x^2-x+1)}$
F_4	$-x\cdot\dfrac{\big(3+6x+10x^2+16x^3+22x^4+29x^5+36x^6+42x^7+48x^8+52x^9 + 55x^{10}+55x^{11}+55x^{12}+52x^{13}+48x^{14}+42x^{15}+36x^{16}+29x^{17} + 22x^{18}+16x^{19}+10x^{20}+6x^{21}+3x^{22}\big)}{3\cdot[2]^4\cdot(1+2x^2+2x^4+x^6)^2\cdot(1-x^2+2x^4-x^6+x^8)}$
H_4	$-x\cdot\dfrac{\big(6+12x+20x^2+31x^3+43x^4+60x^5+77x^6+97x^7+119x^8+145x^9 + 171x^{10}+199x^{11}+227x^{12}+255x^{13}+284x^{14}+313x^{15}+341x^{16} + 369x^{17}+397x^{18}+423x^{19}+449x^{20}+471x^{21}+491x^{22}+508x^{23} + 525x^{24}+537x^{25}+548x^{26}+556x^{27}+562x^{28}+562x^{29}+562x^{30} + 556x^{31}+548x^{32}+537x^{33}+525x^{34}+508x^{35}+491x^{36}+471x^{37} + 449x^{38}+423x^{39}+397x^{40}+369x^{41}+341x^{42}+313x^{43}+284x^{44} + 255x^{45}+227x^{46}+199x^{47}+171x^{48}+145x^{49}+119x^{50}+97x^{51} + 77x^{52}+60x^{53}+43x^{54}+31x^{55}+20x^{56}+12x^{57}+6x^{58}\big)}{6\cdot[2]^4\cdot[3]^2\cdot[5]^2\cdot(1+x^2)^2\cdot(1-x+x^2)^2\cdot(1-x^2+x^4) \cdot(1-x+x^2-x^3+x^4)^2\cdot(1-x^2+x^4-x^6+x^8) \cdot(1-x+x^3-x^4+x^5-x^7+x^8)\cdot(1+x-x^3-x^4-x^5+x^7+x^8)}$

Table 12 : The help-functions associated to irreducible maximal subgroups of G.

By means of (3.23), (3.24), Table 10, Table 11 and Table 12, the first part of Lemma 3.8 follows. In particular, the denominator $d(x)$ in $h^S(x)=-x\cdot n(x)/d(x)$ factors into cyclotomic polynomials with $\deg(d)=\deg(n)+2$.

As for the second part of Lemma 3.8, we have to study the behaviour of $h^S(x)$ on $(-1,0)$. If H is reducible of degree 3, it is possible to show, but very technical, that $h^S(x)$ is strictly decreasing on $(-1,0)$ (see part 3.4.1). If H is reducible of degree 2, we verify the claim by means of a symbolic Mathematica computation based on the formulas (3.24) and (3.25). For the remaining cases, with the help-functions as given in Table 11 and Table 12, we proceed in a similar way and achieve the proof of Lemma 3.8. $\square$

By means of Lemma 3.8, equation (3.20) transforms into

$$\frac{1}{f_S(x)} = 1 + \sum_{G_T \text{ maximal}} \left(-x \cdot \frac{n^T(x)}{d^T(x)} \right),$$

where $-x \cdot n^T(x)$ and $d^T(x)$ are the numerator and the denominator of the help-function $h^S(x)$ associated to G_T in G. Hence,

$$f_S(x) = \frac{\prod\limits_{G_T \text{ maximal}} d^T(x)}{\prod\limits_{G_T \text{ maximal}} d^T(x) - x \cdot \sum\limits_{G_T \text{ maximal}} \left(n^T(x) \cdot \prod\limits_{G_U \text{ maximal}, U \neq T} d^U(x) \right)}. \tag{3.26}$$

The numerator and the denominator in (3.26) are palindromic polynomials over the integers with equal (even) degree (see Lemma 3.8). This reproves the fact that $f_S(x)$ is a reciprocal function (cf. [9] and [33, page 112]). Now we use Proposition D.11 (see Appendix D).

Proposition D.11 *Let P be a palindromic polynomial over the integers with degree d. Then P can be factored into a product of a constant times linear (if d is odd), quadratic and quartic palindromic polynomials with real coefficients.*

It follows that both, numerator and denominator in (3.26), can be factored into products of a real constant, quadratic and quartic polynomials over $\mathbb{R}$. Up to cancellation of common factors, the rational function $f_S(x)$ in (3.26) is a quotient of monic polynomials over the integers (see also [35]) which are prime.

Let us now study the real poles of $f_S(x)$. As $f_S(x)$ is reciprocal, it is sufficient to consider only the interval $[-1, 1]$. Recall that $f_S(1) = \frac{1}{\chi(G)} > 0$ and $f_S(0) = 1$ (see part 3.1). We prove that $f_S(x)$ possesses a root in -1.

Lemma 3.10 *Let G denote a Simplex, an Esselmann or a Kaplinskaya group with generating set S. Then $f_S(-1) = 0$.*

P*roof*
A study of the combinatorics of G leads to the fact that it must contain at least one maximal spherical subgroup G_T, $T \subset S$, of type B_4, D_4, F_4 or H_4. Remark that the four exponents $m_1, \ldots, m_4$ of G_T are all odd (see Table 6 in part 3.2). Let k_i be an integer such that $m_i + 1 = 2k_i$, for $i = 1, \ldots, 4$. Thus the growth series of G_T is given by (see (3.7))

$$f_T(x) = \prod_{i=1}^{4} [m_i + 1] = \prod_{i=1}^{4} [2k_i] = \prod_{i=1}^{4} \left([2] \cdot \sum_{j=0}^{k_i - 1} x^{2j} \right) = [2]^4 \cdot \prod_{i=1}^{4} \left(\sum_{j=0}^{k_i - 1} x^{2j} \right),$$

Then Steinberg's formula (3.4) yields

$$\frac{1}{f_S(x)} = 1 - \frac{|S|}{[2]} + \frac{p_2(x)}{[2]^2 \cdot q_2(x)} + \frac{p_3(x)}{[2]^3 \cdot q_3(x)} + \frac{p_4(x)}{[2]^4 \cdot q_4(x)},$$

where $p_i(-1) \neq 0$ and $q_i(-1) \neq 0$, for $2 \leq i \leq 4$. Let $q(x) := \prod_{i=2}^{4} q_i(x)$. Then we obtain

$$f_S(x) = \frac{[2]^4 \cdot q(x)}{[2]^4 \cdot q(x) - |S| \cdot [2]^3 \cdot q(x) + p_2(x) \cdot [2]^2 \cdot \frac{q(x)}{q_2(x)} + p_3(x) \cdot [2] \cdot \frac{q(x)}{q_3(x)} + \frac{q(x)}{q_4(x)}},$$

hence $f_S(-1) = 0$. □

Remark 3.3 *Observe that Lemma 3.10 is also valid for cocompact Coxeter groups which act on $\mathbb{H}^3$ and for those acting on $\mathbb{H}^2$ with at least one maximal (dihedral) subgroup with exponents $m_1 = 1$ and m_2 odd. This follows easily from the equations (3.12) and (3.17) for the associated help-functions. However, a cocompact Coxeter group G acting in $\mathbb{H}^2$ with generating set S, all of whose dihedral subgroups have exponents $m_1 = 1$ and m_2 even, satisfies $f_S(-1) > 0$.*

Since $h^S(x)$ is strictly decreasing on $(-1, 0]$, by Lemma 3.8, the function $f_S(x)$, by (3.20), is strictly increasing on $(-1, 0]$. Then $f_S(x)$ is positive on $(-1, 0]$, as $f_S(-1) = 0$ (see Lemma 3.10). Moreover, as $f_S(x)$ is reciprocal, $f_S(x)$ is non-singular on $\mathbb{R}_-$. Let us now study its behaviour on the interval $I := [0, 1]$ which necessitates the use of a computer due to the complicated expressions for the components $h^S(x)$ (compare (3.12), (3.17) with (3.23), (3.24)).

By Theorem 3.1, we know that $f_S(x) = \sum_{i \geq 0} a_i x^i$ possesses a real pole in $(0, 1)$, which we denote by x_1. According to [12, § 17.1, page 322], x_1 is given by the convergence radius of $f_S(x)$. Its inverse $x_1^{-1} = \tau$ is the growth rate of G. This follows from the fact that the growth functions a_i are positive.

By exploiting the different shapes of $h^S(x)$, the following behaviour shows up.

Observation 1 *The help-function $h^S(x)$ behaves as follows.*

(a) Either, $h^S(x)$ is strictly decreasing on I,

(b) or, $h^S(x)$ possesses exactly one negative minimum M at a point $m \in (0, 1)$. For $0 \leq x < m$, $h^S(x)$ is strictly decreasing, and for $m < x \leq 1$, $h^S(x)$ is strictly increasing.

It is a remarkable fact that (b) happens only for irreducible finite subgroups H whose graphs are different from A_4, or for the reducible group H with graph $G_2^{(5)} \times G_2^{(5)}$.

Consider the *simplex function* related to (3.20)

$$H^S(x) := \sum_{k=1}^{5} h_k^S(x) = \frac{1}{f_S(x)} - 1, \tag{3.27}$$

where $h_k^S(x)$ are the help-functions associated to the maximal subgroups of G (see Table 9). Observe that $H^S(x)$ is negative on I by Lemma 3.8. By means of the above Observation, valid for each term in (3.27), a computer evaluation of the many different possible combinations based on Tables 10, 11, 12 allows to conclude that *the sum $H^S(x)$ is either strictly decreasing on I or possesses exactly one negative minimum in $x_M \in I$. In this latter case, $H^S(x)$ is* *strictly decreasing on* $(0, x_M)$*, while it is strictly increasing on* $(x_M, 1)$.

Since x_1 is a pole of $f_S(x)$ and $1/f_S(1) > 0$, $H^S(x_1) = -1$ and $H^S(1) > -1$. Therefore, $H^S(x)$ can not be strictly decreasing on I, but possesses exactly one negative minimum M. That is, there is a unique $x_M \in I$ such that $H^S(x_M) = M$. Obviously, $x_M \geq x_1$, since x_1 is the radius of convergence of $f_S(x)$. Summarising, we can deduce that $f_S(x)$ possesses exactly two simple poles in I if $x_M > x_1$, or it has a pole of (positive) even order in I if $x_M = x_1$. This finishes the proof of Theorem 3.7 for the Simplex groups. $\diamondsuit$

For the proof in the cases of the Esselmann groups and the Kaplinskaya groups, we will essentially proceed in the same way. Before we comment about their help-functions which depend on the combinatorics of their fundamental polytopes, we illustrate the result about Simplex groups by providing some examples.

Example 3.4 *Consider the Simplex group* G_1*, with generating set* S *and graph*

$$\Gamma_1 : \bullet \overset{5}{\text{———}} \bullet \text{———} \bullet \text{———} \bullet \overset{4}{\text{———}} \bullet\ .$$

The group G_1 *is associated to the Coxeter simplex of a compact (regular)* 120*-cell* $P \subset \mathbb{H}^4$ *of interior angles* $\pi/2$*. We shall study this polytope in Example 4.5.*

By means of Steinberg's formula (3.4), one derives

$$f_S(x) = \frac{[2,12,20,30]\cdot(1+x^4)}{\begin{aligned}&1 - x - x^3 + 2x^4 - 2x^5 + x^6 - 3x^7 + 3x^8 - 3x^9 + 3x^{10} - 5x^{11} + 5x^{12} - 5x^{13}\\ &+ 6x^{14} - 7x^{15} + 8x^{16} - 8x^{17} + 9x^{18} - 9x^{19} + 11x^{20} - 11x^{21} + 12x^{22} - 11x^{23}\\ &+ 14x^{24} - 13x^{25} + 14x^{26} - 13x^{27} + 16x^{28} - 14x^{29} + 15x^{30} - 14x^{31} + 17x^{32}\\ &- 14x^{33} + 15x^{34} - 14x^{35} + 16x^{36} - 13x^{37} + 14x^{38} - 13x^{39} + 14x^{40} - 11x^{41}\\ &+ 12x^{42} - 11x^{43} + 11x^{44} - 9x^{45} + 9x^{46} - 8x^{47} + 8x^{48} - 7x^{49} + 6x^{50} - 5x^{51}\\ &+ 5x^{52} - 5x^{53} + 3x^{54} - 3x^{55} + 3x^{56} - 3x^{57} + x^{58} - 2x^{59} + 2x^{60} - x^{61} - x^{63}\\ &+ x^{64}.\end{aligned}}$$

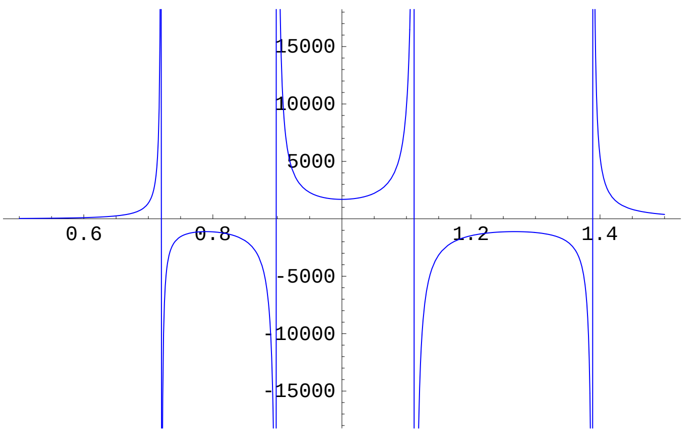


Figure 6: The growth series of G_1 on $[0.5, 1.5]$.

Note that the maximal subgroups of G_1 are characterised by the symbols B_4, H_4 , $H_3 \times A_1$, $B_3 \times A_1$ and $G_2^{(5)} \times G_2^{(4)}$. Their help-functions $h_1^S(x), \ldots, h_5^S(x)$ behave as follows on the positive axis.

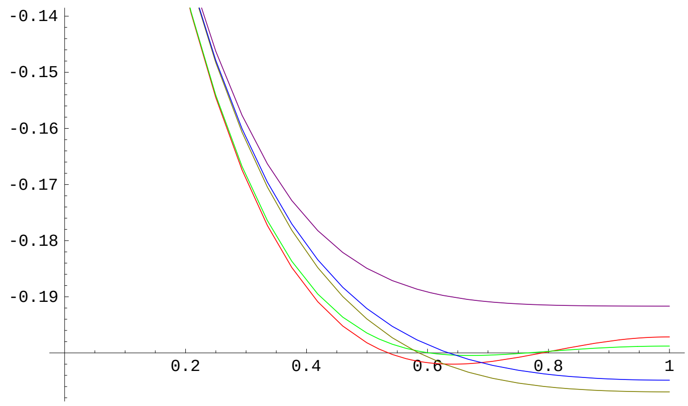


Figure 7: The help-functions $h_i^S(x)$, $1 \leq i \leq 5$, associated to the maximal subgroups of G_1.

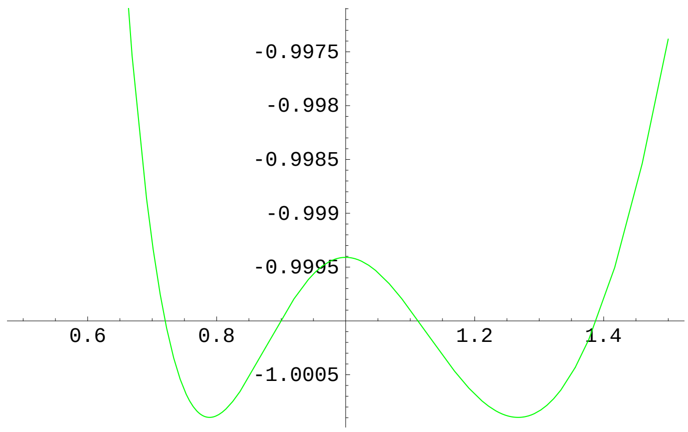
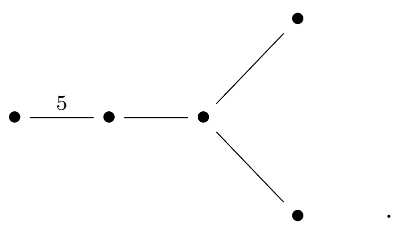


Figure 8: The function $H^S(x) = \sum_{i=1}^{5} h_i^S(x)$ of G_1.

Example 3.5 *Consider the Simplex group G_2, with generating set S and graph*

Γ_2 :



Here,

$$f_S(x) = \frac{[2,12,20,30]}{\begin{aligned}&1-x-x^3+x^4-x^5-x^7+x^8-x^9+x^{11}+2x^{12}-x^{13}+x^{14}-x^{15}+2x^{16}\\ &-x^{17}+x^{18}-x^{19}+3x^{20}-x^{21}+2x^{22}-x^{23}+3x^{24}-x^{25}+2x^{26}-x^{27}+3x^{28}\\ &-x^{29}+3x^{30}-x^{31}+3x^{32}-x^{33}+2x^{34}-x^{35}+3x^{36}-x^{37}+2x^{38}-x^{39}+3x^{40}\\ &-x^{41}+x^{42}-x^{43}+2x^{44}-x^{45}+x^{46}-x^{47}+2x^{48}-x^{49}-x^{51}+x^{52}-x^{53}\\ &-x^{55}+x^{56}-x^{57}-x^{59}+x^{60}.\end{aligned}}$$

The poles of $f_S(x)$ are distributed in the complex plane as follows.

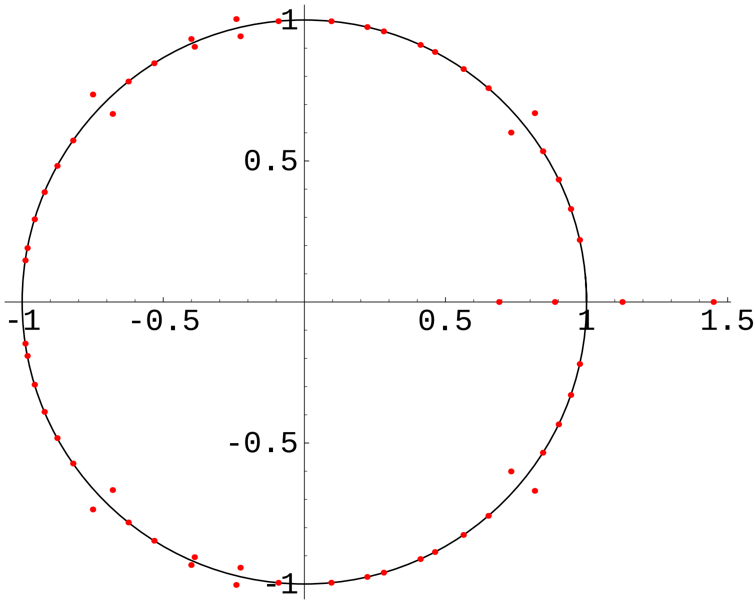


Figure 9: The poles of the growth series of G_2.

In the following we comment about the extendability of our proof from the Simplex groups to the Esselmann and Kaplinskaya groups. Recall that an Esselmann group G_E and a Kaplinskaya group G_K in $I(\mathbb{H}^4)$ are generated by six reflections. Their combinatorial structure is given as follows (see part 2.3 and Tables 4 and 5). An Esselmann polytope has the combinatorial type of a direct product of two triangles and possesses therefore precisely nine vertices. The Coxeter graph Γ_E of an Esselmann group G_E contains two disjoint Lannér diagrams, called L_1 and L_2, each of them with three nodes. Let H be one of the nine maximal finite subgroups of G_E and denote by T its generating set (see Table 13). Then we define the following help-function for H.

$$h^E(x) := h^S(x) + \frac{1}{3\cdot(1+x)} - \frac{1}{12}\sum_W \frac{1}{f_W(x)}, \tag{3.28}$$

where $h^S(x) = h^S_T(x)$ is the function (3.19) given by

$$h^S(x) = -\frac{1}{x+1} + \frac{1}{3}\cdot\sum_U \frac{1}{f_U(x)} - \frac{1}{2}\cdot\sum_V \frac{1}{f_V(x)} + \frac{1}{f_T(x)},$$

where U is a 2-element subset, V is a 3-element subset, and where W is a subset of T satisfying the following condition. The set W consists of four pairs of generators (s_p, s_q) such that the node in Γ_E corresponding to s_p belongs to L_1, while the node corresponding to s_q belongs to L_2.

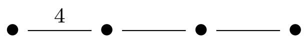
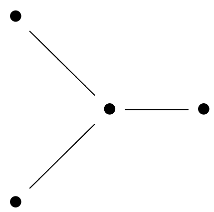
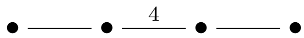
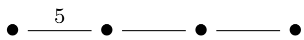
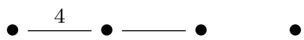
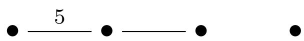
Symbol	Graph
B_4	
D_4	
F_4	
H_4	
$A_3 \times A_1$	
$B_3 \times A_1$	
$H_3 \times A_1$	
$G_2^{(q)} \times G_2^{(r)}$	 for $2 \leq q, r \leq 10$

Table 13 : Groups arising as maximal subgroups of an Esselmann group.

A Kaplinskaya polytope has the combinatorial type of a simplicial prism and possesses therefore precisely eight vertices. The Coxeter graph Γ_K of a Kaplinskaya group G_K contains a Lannér diagram L with four nodes which represents a tetrahedron P, and two additional nodes which represent the reflections through the top respectively the bottom of the simplicial prism $P \times [0, 1]$. Let H be one of the eight maximal finite subgroups of G_K and denote by T its generating set (see Table 14). Then we define the following help-function for H.

$$h^K(x) := h^S(x) + \frac{1}{4 \cdot (1 + x)} - \frac{1}{12} \sum_W \frac{1}{f_W(x)}, \tag{3.29}$$

where $h^S(x)$ is the function as above and where W is a subset of T containing three pairs (s_{L_1}, s_b), (s_{L_2}, s_b) and (s_{L_3}, s_b) such that s_{L_j} belongs to L, for $j = 1, 2, 3$, while $s_b \notin L$.
Now we proceed in the same spirit as in the case of Simplex groups. However, there are some particularities which have to dealt with carefully.

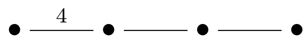
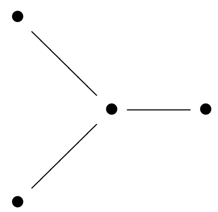
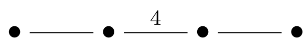
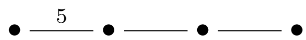
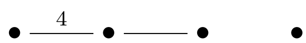
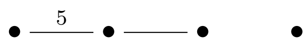
Symbol	Graph
A_4	
B_4	
D_4	
F_4	
H_4	
$A_3 \times A_1$	
$B_3 \times A_1$	
$H_3 \times A_1$	
$G_2^{(q)} \times G_2^{(r)}$	 for $2 \leq q \leq 3$ and $2 \leq r \leq 5$

Table 14 : Groups arising as maximal subgroups of a Kaplinskaya group.

In the Esselmann case, the help-function $h^E(x)$ related to the finite group with graph H_4 is not strictly decreasing on the interval $(-1, 0)$. This default is compensated by the other terms in $H^E(x)$ (see 3.27) so that the growth series of G_E does not have poles on the negative real axis. In the Kaplinskaya case, the functions $h^K(x)$ related to the finite groups with graph A_4 and H_4 are not strictly decreasing on $(-1, 0)$. Again, by considering $H^K(x)$, the growth series of G_K does not have poles on the negative real axis, with one exception : the growth series of a Kaplinskaya group having negative poles is uniquely associated to the group with graph K_{66} (see Example 3.8 and Table 5). $\square$

We provide now an example indicating to what extent the help-functions $h^S(x)$, $h^E(x)$ and $h^K(x)$ related to the same group (here with symbol B_4) differ.

Example 3.6 *Let G be a Simplex, an Esselmann or a Kaplinskaya group which contains at least one finite maximal subgroup H with Coxeter graph*

$$B_4 : \bullet \overset{4}{\text{———}} \bullet \text{———} \bullet \text{———} \bullet\ .$$

The help-functions $h^S_{B_4}(x)$, $h^E_{B_4}(x)$ and $h^K_{B_4}(x)$ are given by

$$h^S_{B_4}(x) = -x \cdot \frac{\begin{array}{l}(6+12x+20x^2+29x^3+38x^4+46x^5+52x^6+52x^7+52x^8+46x^9\\ \quad +38x^{10}+29x^{11}+20x^{12}+12x^{13}+6x^{14})\end{array}}{6 \cdot [2]^4 \cdot (1+x^2)^2 \cdot (1+x^2+2x^4+x^6+x^8)}$$

$$h^E_{B_4}(x) = -x \cdot \frac{\begin{array}{l}(8+15x+23x^2+32x^3+41x^4+49x^5+56x^6+52x^7+56x^8+49x^9\\ \quad +41x^{10}+32x^{11}+23x^{12}+15x^{13}+8x^{14})\end{array}}{12 \cdot [2,2,2,2,3] \cdot (1+x^2)^2 \cdot (1-x+x^2+x^4-x^5+x^6)}$$

$$h^K_{B_4}(x) = -x \cdot \frac{\begin{array}{l}(9+16x+25x^2+35x^3+45x^4+54x^5+61x^6+58x^7+61x^8+54x^9\\ \quad +45x^{10}+35x^{11}+25x^{12}+16x^{13}+9x^{14})\end{array}}{12 \cdot [2,2,2,2,3] \cdot (1+x^2)^2 \cdot (1+x^4) \cdot (1-x+x^2)}$$

These functions behave as follows.

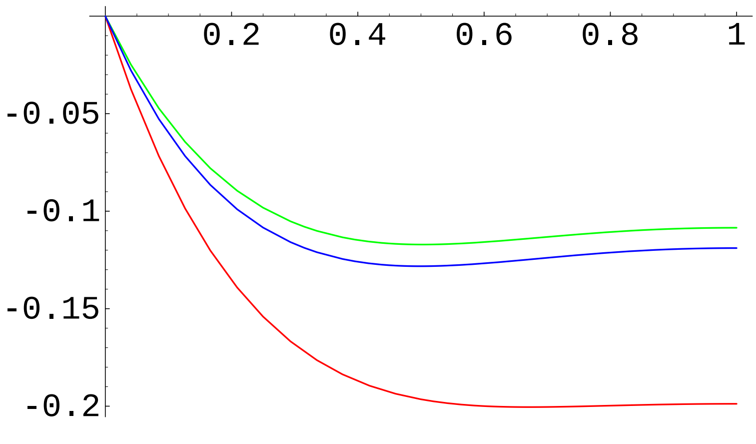
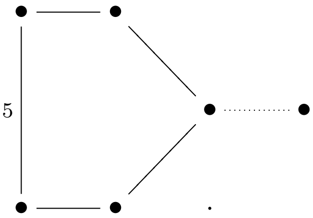


Figure 10: The graph of $h^S_{B_4}(x)$, $h^E_{B_4}(x)$ and $h^K_{B_4}(x)$ on $[0,1]$.

Let us now illustrate Theorem 3.7 by means of some examples about the Kaplinskaya groups.

Example 3.7 *Consider the Kaplinskaya group G_6, with generating set S and graph*

K_6 :



We compute

$$f_S(x) = \frac{[2,12,20,30]}{\begin{array}{l}1-2x-x^3-2x^7+2x^8-x^9+2x^{10}-x^{11}+2x^{12}+x^{13}+4x^{14}+4x^{16}-x^{17}\\ +6x^{18}+x^{19}+6x^{20}+x^{21}+6x^{22}+2x^{23}+8x^{24}+8x^{26}+x^{27}+8x^{28}+2x^{29}\\ +8x^{30}+2x^{31}+8x^{32}+x^{33}+8x^{34}+8x^{36}+2x^{37}+6x^{38}+x^{39}+6x^{40}+x^{41}\\ +6x^{42}-x^{43}+4x^{44}+4x^{46}+x^{47}+2x^{48}-x^{49}+2x^{50}-x^{51}+2x^{52}-2x^{53}\\ -x^{57}-2x^{59}+x^{60}\end{array}}$$

whose poles are distributed in the complex plane as follows.

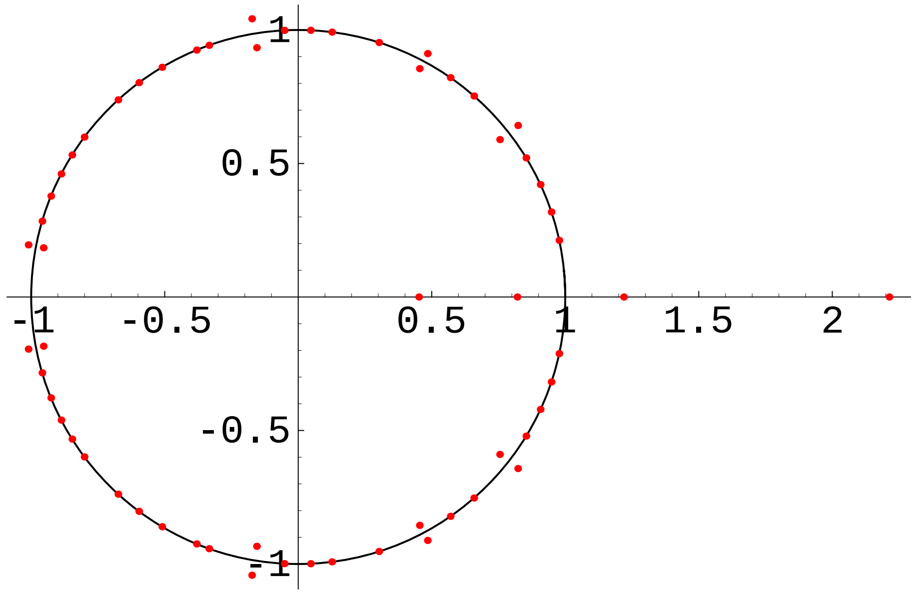
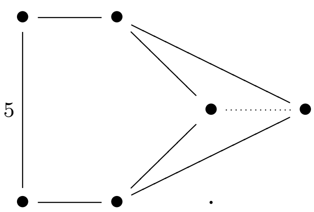


Figure 11: The poles of the growth series of the Kaplinskaya group G_6.

Example 3.8 *Consider the Kaplinskaya group G_{66}, with generating set S and graph*

K_{66} :



Here, we derive

$$f_S(x) = \frac{[2,12,20,30]}{\begin{array}{l}1-2x-2x^2-x^4+x^5-2x^6-x^7+x^8+x^9+x^{10}+x^{11}+x^{12}+5x^{13}+5x^{14}\\ +\,4x^{15}+3x^{16}+3x^{17}+7x^{18}+7x^{19}+7x^{20}+7x^{21}+7x^{22}+9x^{23}+10x^{24}+7x^{25}\\ +\,9x^{26}+8x^{27}+10x^{28}+10x^{29}+10x^{30}+10x^{31}+10x^{32}+8x^{33}+9x^{34}+7x^{35}\\ +\,10x^{36}+9x^{37}+7x^{38}+7x^{39}+7x^{40}+7x^{41}+7x^{42}+3x^{43}+3x^{44}+4x^{45}+5x^{46}\\ +\,5x^{47}+x^{48}+x^{49}+x^{50}+x^{51}+x^{52}-x^{53}-2x^{54}+x^{55}-x^{56}-2x^{58}-2x^{59}\\ +\,x^{60}\end{array}}$$

whose poles are distributed in the complex plane as follows.

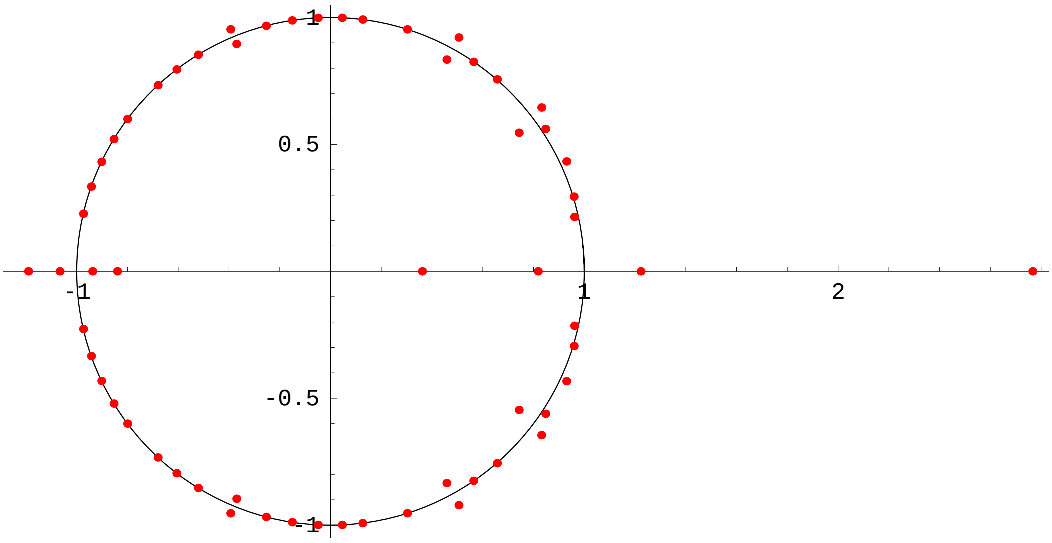
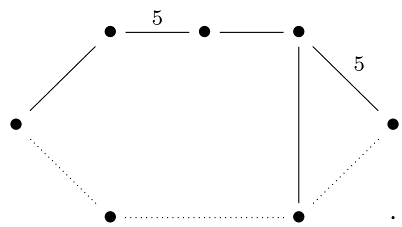


Figure 12: The poles of the growth series of the Kaplinskaya group G_{66}.

Let us extend our setting and consider the family of 4-dimensional compact Tumarkin polytopes which are characterised by seven facets (see [36]). Denote by H a maximal finite subgroup of such a Tumarkin group G_{Tu}. The theoretical variety of these subgroups can not be restricted by combinatorial conditions (for $G = G_E$ and G_K, we have a fundamental polytope which is product of a k-simplex with an l-simplex). When trying to adapt the proof of Theorem 3.7 to G_{Tu}, one is forced to make a fastidious and long case-by-case study. This is the reason why we abandoned this task. However, for all the forty Tumarkin groups (see Appendix B), one checks, by means of a computer, that their growth series satisfy Conjecture 1 !

Example 3.9 *Consider a 4-dimensional Tumarkin group G_{Tu}, with generating set S and graph*



By means of Steinberg's formula (3.4) one computes

$$f_S(x) = \frac{[2,12,20,30]}{\begin{aligned}&1-3x+x^2-2x^3+x^4-2x^5+3x^6-4x^7+3x^8-x^9+3x^{10}-3x^{11}+7x^{12}\\&-3x^{13}+7x^{14}-x^{15}+7x^{16}-3x^{17}+11x^{18}-3x^{19}+11x^{20}-x^{21}+11x^{22}\\&-2x^{23}+15x^{24}-4x^{25}+13x^{26}+13x^{28}-3x^{29}+17x^{30}-3x^{31}+13x^{32}+13x^{34}\\&-4x^{35}+15x^{36}-2x^{37}+11x^{38}-x^{39}+11x^{40}-3x^{41}+11x^{42}-3x^{43}+7x^{44}\\&-x^{45}+7x^{46}-3x^{47}+7x^{48}-3x^{49}+3x^{50}-x^{51}+3x^{52}-4x^{53}+3x^{54}-2x^{55}\\&+x^{56}-2x^{57}+x^{58}-3x^{59}+x^{60}.\end{aligned}}$$

The poles of $f_S(x)$ are distributed in the complex plane as follows.

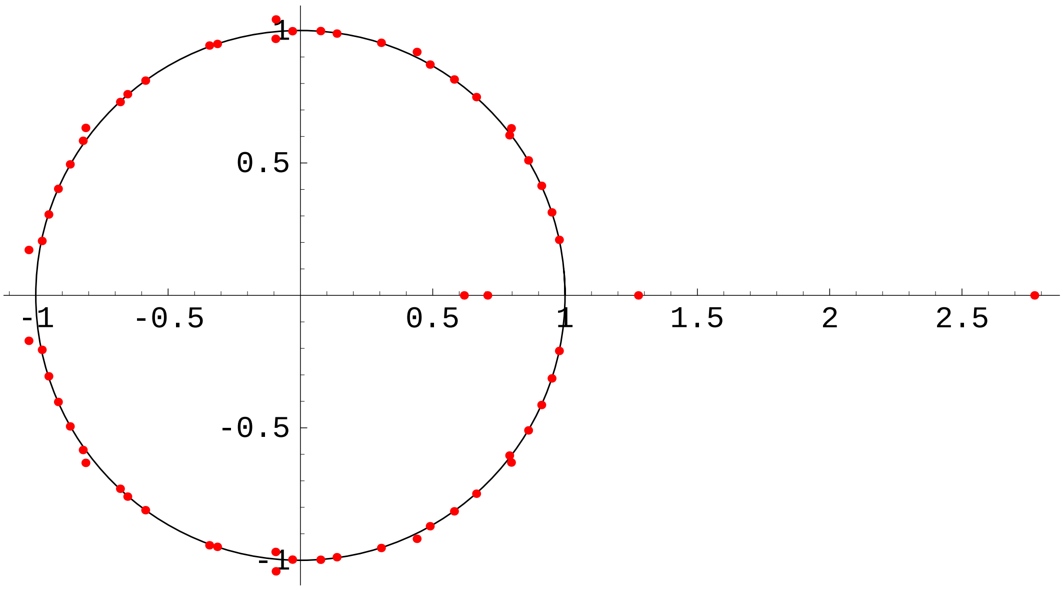


Figure 13: The poles of the growth series of G_{Tu}.

Remark 3.4 *It is an interesting fact that the growth series of all known cocompact Coxeter groups in $I(\mathbb{H}^4)$ possess exactly four different poles $0 < x_1 < x_2 < 1 < x_2^{-1} < x_1^{-1}$ in $\mathbb{R}$; they are moreover simple (for Coxeter garlands, see [41]). This holds apart from the single exception of the Kaplinskaya group G_{66} described in Example 3.8. This group is distinguished by the fact that its maximal finite subgroups are given by the symbols A_4 and H_4, only, so that the help-function is not strictly decreasing on $(-1, 0)$.*

Before we end this part, let us briefly explain how we will achieve the proof of Conjecture 1.

The method which we adapt from Parry's [26] in order to prove Theorem 3.7 does not allow to determine the multiplicities of the real poles of the growth series associated to a Simplex, Esselmann or Kaplinskaya group (see Conjecture 1). We tried to solve this problem by exploiting several different ideas. However, we always got to the conclusion that we need to control explicitly the coefficients of the denominator polynomial $Q(x)$ of the growth series (see [23]). In section 4 we will present a recursion formula which provides an effective algorithm to compute all coefficients of $Q(x)$, and this for any cocompact Coxeter group in $I(\mathbb{H}^n)$, $n \geq 2$.

3.4.1 Complement on help-functions for Simplex groups ; technical proofs

In this part we present some technical proofs needed to achieve the proof of Lemma 3.8 in part 3.4. Let G be a Lannér group with five generators and associated growth function $f_S(x)$. For a maximal finite subgroup H of G with Coxeter graph

$$\Gamma : \bullet \overset{q}{\text{———}} \bullet \qquad \bullet \overset{r}{\text{———}} \bullet$$

where $3 \leq q, r \leq 5$ with $l := r - q$, consider its help-function (3.24)

$$h^S(x) = \frac{1}{3 \cdot [2,2,q,r]} \cdot P_l^q(x),$$

where, by (3.25),

$$P_l^q(x) = 3 + [q] \cdot \{2x - 4 + [q] \cdot (1 - 3x)\} + x^q \cdot [l] \cdot \{x - 2 + [q] \cdot (1 - 3x)\}.$$

We will show that the numerator of $h^S(x)$ is a product of $-x$ and a palindromic polynomial such that the numerator of $h^S(x)$ is of odd degree. Furthermore $h^S(x)$ will be shown to be strictly decreasing on $(-1, 0)$.

The numerator $P_l^q(x)$ of $h^S(x)$

Consider the polynomial

$$P_l^q(x) = 3 + [q] \cdot \{2x - 4 + [q] \cdot (1 - 3x)\} + x^q \cdot [l] \cdot \{x - 2 + [q] \cdot (1 - 3x)\}$$

and rewrite it by using $[0] = 0$ so that

$$P_l^q(x) = P_0^q(x) + x^q \cdot [l] \cdot \{x - 2 + [q] \cdot (1 - 3x)\}.$$

Furthermore, consider the polynomial

$$Q_l^q(x) = x^q \cdot [l] \cdot \{x - 2 + [q] \cdot (1 - 3x)\}.$$

Hence,

$$P_l^q(x) = P_0^q(x) + Q_l^q(x). \tag{3.30}$$

In the following we rewrite $P_0^q(x)$ and $Q_l^q(x)$, which allows to prove the claim.

Lemma 3.11 *The polynomial $P_0^q(x)$ is palindromic and of the form*

$$P_0^q(x) = -x \cdot \left(\sum_{i=1}^{q-1} (2i+1) \cdot x^{i-1} + (2q-1) \cdot x^{q-1} + \sum_{i=q+1}^{2q-1} (4q - 2i + 1) \cdot x^{i-1} \right). \tag{3.31}$$

P*roof*
Put

$$R_0^q(x) := -x \cdot \left(\sum_{i=1}^{q-1} (2i+1) \cdot x^{i-1} + (2q-1) \cdot x^{q-1} + \sum_{i=q+1}^{2q-1} (4q - 2i + 1) \cdot x^{i-1} \right).$$

We have to show that $P_0^q \equiv R_0^q$. This will be done by induction with respect to q. It is easy to verify that $P_0^3 = R_0^3$. Since

$$P_0^{q+1}(x) = P_0^q(x) + 2 \cdot x^q \cdot [q] \cdot (1-3x) + x^q \cdot (2x-4) + x^{2q} \cdot (1-3x),$$

we derive

$$\begin{aligned} P_0^{q+1}(x) &= R_0^q(x) + 2 \cdot x^q \cdot [q] \cdot (1-3x) + x^q \cdot (2x-4) + x^{2q} \cdot (1-3x) \\ &= -\sum_{i=1}^{q-1}(2i+1) \cdot x^i - (2q-1) \cdot x^q - \sum_{i=q+1}^{2q-1}(4q-2i+1) \cdot x^i \\ &\quad + 2 \cdot x^q \cdot [q] \cdot (1-3x) + x^q \cdot (2x-4) + x^{2q} \cdot (1-3x). \end{aligned}$$

After rearranging these terms we obtain

$$\begin{aligned} P_0^{q+1}(x) &= -\sum_{i=1}^{q-1}(2i+1) \cdot x^i - (2q+1) \cdot x^q - (2q+1) \cdot x^{q+1} - \sum_{i=q+2}^{2q-1}(4q-2i+5) \cdot x^i \\ &\quad - 5x^{2q} - 3x^{2q+1} \\ &= -\sum_{i=1}^{q}(2i+1) \cdot x^i - (2q+1)x^{q+1} - \sum_{i=q+2}^{2q+1}(4q-2i+5) \cdot x^i \\ &= -x \cdot \left(\sum_{i=1}^{q}(2i+1) \cdot x^{i-1} + (2q+1) \cdot x^q + \sum_{i=q+2}^{2q+1}(4q-2i+5) \cdot x^{i-1} \right), \end{aligned}$$

which equals $R_0^{q+1}(x)$. This finishes the proof. □

Lemma 3.12 *For integers $q \geq 3$ and $l \geq 1$,*

$$\begin{aligned} Q_l^q(x) &= -\, x^q - \sum_{j=q+1}^{q+l-1}(2j-2q)x^j + (1-2l)x^{q+l} - \sum_{j=q+l+1}^{2q-1} 2lx^j - (1+2l)x^{2q} \\ &\quad - \sum_{j=2q+1}^{2q+l-2}(2j-4q-2l-1)x^j - 3x^{2q+l-1}. \end{aligned}$$

P*roof*
The proof consists of easy but lengthy manipulations of $Q_l^q(x)$, based on the identity given by Lemma D.5,

$$[l,q] = \sum_{j=1}^{l} j \cdot x^{j-1} + \sum_{j=l}^{q-1} l \cdot x^j + \sum_{j=0}^{l-2}(l-1-j) \cdot x^{q+1}.$$

□

Thus, Lemma 3.11 and Lemma 3.12 yield together with (3.30)

$$\begin{aligned} P_l^q(x) &= -\sum_{j=1}^{q-1}(2j+1)x^j - 2qx^q - \sum_{j=q+1}^{q+l-1}(2q+1)x^j - 2qx^{q+l} \\ &\quad - \sum_{j=q+l+1}^{2q-1}(4q-2j+2l+1)x^j - (2l+1)x^{2q} \\ &\quad - \sum_{j=2q+1}^{2q+l-2}(2j-4q-2l-1)x^j - 3x^{2q+l-1}. \end{aligned}$$

Hence,

$$\begin{aligned} P_l^q(x) &= -x \cdot \Bigg(\sum_{j=0}^{q-2} (2j+1)x^j + 2qx^{q-1} + \sum_{j=q}^{q+l-2} (2q+1)x^j - 2qx^{q+l-1} \\ &\quad + \sum_{j=q+l}^{2q-2} (4q-2j+2l+1)x^j + (2l+1)x^{2q-1} \\ &\quad + \sum_{j=2q}^{2q+l-3} (2j-4q-2l-1)x^j + 3x^{2q+l-2} \Bigg), \end{aligned}$$

which implies that $P_l^q(x)$ is palindromic.

From this it follows that the numerator of $h^S(x)$ is of odd degree. In fact, if l is even, the degree of $P_l^q(x)$ is odd. If l is odd, the palindromic polynomial $P_l^q(x)$ can be factored according to $P_l^q(x) = -x \cdot [2] \cdot r(x)$, where $r(x)$ is a palindromic polynomial of even degree by Lemma D.7. In this way, (3.24) becomes

$$h^S(x) = -x \cdot \frac{r(x)}{3 \cdot [2, q, q+l]}$$

with a numerator of odd degree.

The monotonicity of $h^S(x)$

Let H be an arbitrary spherical Coxeter group with four generators such that H is reducible of order 3 with Coxeter graph

$$\Gamma_H : \bullet \overset{q}{\text{———}} \bullet \qquad \bullet \qquad \bullet \qquad , q \geq 3.$$

We show that the help-function $h^S(x)$ (3.19) associated to H is strictly decreasing on $(-1, 0)$, by considering the cases $q = 2k+1$ and q even.

For $q = 2k+1$, by (3.23),

$$h^S(x) = -x \cdot \frac{1}{3 \cdot [2, 2, 2k+1]} \cdot \left(3 + \sum_{i=0}^{k-1} x^{2i+1} + 4 \cdot \sum_{i=1}^{k-1} x^{2i} + 3x^{2k} \right).$$

By Lemma D.2 and Lemma D.3, the denominator is strictly increasing on $(-1, 0)$. Hence, by means of Lemma D.1, it remains to prove that the numerator is strictly decreasing on $(-1, 0)$.

Lemma 3.13 *The function*

$$N_k(x) = 3 \cdot [2, 2, 2k+1] \cdot h^S(x) = -x \cdot \left(3 + \sum_{i=0}^{k-1} x^{2i+1} + 4 \cdot \sum_{i=1}^{k-1} x^{2i} + 3x^{2k} \right),$$

is strictly decreasing on $(-1, 0)$.

P*roof*
Rewrite

$$N_k(x) = -x \cdot (3+x) - 3 \cdot x^{2k+1} - (4+x) \cdot \sum_{i=1}^{k-1} x^{2i+1}$$

and take its derivative in order to obtain

$$\begin{aligned} N_k'(x) &= -3 - 2x - 3 \cdot (2k+1) \cdot x^{2k} - \sum_{i=1}^{k-1} (2i+2) \cdot x^{2i+1} - 4 \cdot \sum_{i=1}^{k-1} (2i+1) \cdot x^{2i} \\ &=: I_k^1(x) + I_k^2(x) + I_k^3(x). \end{aligned}$$

The term $I_k^1(x) = -3 - 2x - 3 \cdot (2k+1) \cdot x^{2k}$ is strictly negative on $(-1,0)$. It remains to show that $I_k^2(x) + I_k^3(x)$ is strictly negative on $(-1,0)$, too. Let us proceed by induction with respect to k. If $k = 2$, we compute $I_2^2(x) + I_2^3(x) = -4x^2 \cdot (3-x)$, which is negative for $x \in (-1,0)$. Let us now assume that $I_k^2(x) + I_k^3(x)$ is strictly negative for $x \in (-1,0)$. Then,

$$\begin{aligned} I_{k+1}^2(x) + I_{k+1}^3(x) &= I_k^2(x) - 4 \cdot (2k+1) \cdot x^{2k} + I_k^3(x) - 2 \cdot (k+1) \cdot x^{2k+1} \\ &= I_k^2(x) + I_k^3(x) - x^{2k} \cdot (8k + 4 - 2kx - 2x) < 0, \end{aligned}$$

as claimed. Thus, $N_k'(x)$ is strictly negative for $x \in (-1,0)$. □

For $q = 2k$ even, the function $h^S(x)$ as given by (3.23) equals

$$h^S(x) = -x \cdot \frac{1}{3 \cdot [2,2,2,2k]} \cdot \left(3 + 4x + 5x^2 \cdot [2k-3] + 4x^{2k-1} + 3x^{2k}\right).$$

As above, it is sufficient to show that the numerator of $h^S(x)$ is strictly decreasing on $(-1,0)$.

Lemma 3.14 *The function*

$$M_k(x) := x \cdot \left(3 + 4x + 5x^2 \cdot [2k-3] + 4x^{2k-1} + 3x^{2k}\right)$$

is strictly increasing on $(-1,0)$.

P*roof*
By induction, one shows that

$$\begin{aligned} M_k'(x) &= (1+x) \cdot (3 + 5x + 2x^2) + x^{2k-2} \cdot \big((6k+3) \cdot x^2 + (8k-2) \cdot x + 5k - 2\big) \\ &\quad + x^2 \cdot (1+x)^2 \cdot \left(2 \cdot \sum_{i=0}^{k-3} x^{2i+1} + \sum_{i=0}^{k-3} (8+5i) \cdot x^{2i}\right). \end{aligned}$$

It follows easily that $M_k'(x)$ is strictly positive on $(-1,0)$. □

4 The recursion formula for the complete form of growth

Let G denote a cocompact Coxeter group acting in $\mathbb{H}^n$. We assume that G is generated by the elements of S and has growth series $f_S(x)$. Let us recall that $f_S(x)$ can be written as a quotient

$$f_S(x) = \frac{p(x)}{q(x)}, \tag{4.1}$$

where $p(x)$ and $q(x)$ are relatively prime polynomials over the integers.

In this section we present a recursion formula by means of which we are able to compute the coefficients of the denominator of the *complete form* of $f_S(x)$. This complete form will be designed in such a way that $f_S(x)$ is a rational function with a numerator factoring into terms of type $[k]$, only. This nice form will be crucial in the proof of Theorem 4.10.

This section is organised as follows. We start with the presentation of a new description of $f_S(x)$, called the *complete form*. Then we explain in detail the method leading to our recursion formula. An important and nice application concerns the family of *right-angled* Coxeter groups. At the end of the section we discuss two general consequences of our formula. The first one consists of a description of the coefficients a_i of the growth series $f_S(x) = \sum a_i x^i$ of G (see Definition 3.1), while the second one allows to complete our study of the Coxeter groups acting in $\mathbb{H}^4$ with at most six generators (see part 3.4).

4.1 The complete form of the growth series

The notion of complete form of $f_S(x)$ associated to (4.1) has first been defined by Chapovalov, Leites and Stekolshchik (see [8, paragraph 5.4.2]).

We know by Steinberg's formula (3.4) that the growth series $f_S(x)$ of any Coxeter group is a rational function. Moreover, according to Theorem 3.3, $f_S(x)$ is reciprocal if n is even, while it is antireciprocal if n is odd. Here $p(x)$ corresponds obviously to the denominator (up to the sign) of the sum

$$\sum_{T\in\mathcal{F}} \frac{(-1)^{|T|}}{f_T(x)}, \tag{4.2}$$

where $\mathcal{F} = \{T \subsetneq S : G_T \text{ is finite}\}$. In [8, equation (5.37)], the following natural description of $p(x)$ is introduced. The least common multiple

$$\mathrm{Virg}(S) := \mathrm{LCM}\{f_T(x) : T \in \mathcal{F}\} \tag{4.3}$$

is called the *virgin form*. It is obvious that $p(x)$ divides $\mathrm{Virg}(S)$. Although $\mathrm{Virg}(S)$ is always a product of polynomials $[k]$ (see part 3.2), certain factorisation properties of the latter yield factors of the form $1+x^l$. Since $[l](1+x^l) = [2l]$, we extend, for each factor $1+x^l$ in $\mathrm{Virg}(S)$, numerator and denominator of $f_S(x)$ by $[l]$. This new form of $\mathrm{Virg}(S)$ is called the *extended form* and denoted by

$$\mathrm{Ext}(S). \tag{4.4}$$

Obviously, when no factor $1+x^l$ shows up, we have $\mathrm{Virg}(S) = \mathrm{Ext}(S)$. Let

$$P(x) := \mathrm{Ext}(S)$$

denote the extended form of the numerator $p(x)$, and let $Q(x)$ be the extended form of the denominator $q(x)$. Then, the growth series $f_S(x)$ can be written as a rational function $P(x)/Q(x)$ which is called its *complete form.* Let us point out that $P(x)$ and $Q(x)$ are in general no more relatively prime. The next example illustrates the above procedure.

Example 4.1 *Consider the Lannér group G_L with graph*

$$\Gamma_L : \bullet \overset{5}{\text{———}} \bullet \text{———} \bullet \text{———} \bullet \overset{4}{\text{———}} \bullet \,,$$

its natural generating set S and growth series $f_S(x) = p(x)/q(x)$. By means of (4.2) and (4.3), one computes

$$\mathrm{Virg}(S) = [2, 12, 20, 30] \cdot (1 + x^4). \tag{4.5}$$

By (3.4), one calculates

$$f_S(x) = \frac{[2, 12, 20, 30] \cdot (1 + x^4)}{\begin{array}{l}\big(1 - x - x^3 + 2x^4 - 2x^5 + x^6 - 3x^7 + 3x^8 - 3x^9 + 3x^{10} - 5x^{11} + 5x^{12} - 5x^{13} \\ + 6x^{14} - 7x^{15} + 8x^{16} - 8x^{17} + 9x^{18} - 9x^{19} + 11x^{20} - 11x^{21} + 12x^{22} - 11x^{23} \\ + 14x^{24} - 13x^{25} + 14x^{26} - 13x^{27} + 16x^{28} - 14x^{29} + 15x^{30} - 14x^{31} + 17x^{32} \\ - 14x^{33} + 15x^{34} - 14x^{35} + 16x^{36} - 13x^{37} + 14x^{38} - 13x^{39} + 14x^{40} - 11x^{41} \\ + 12x^{42} - 11x^{43} + 11x^{44} - 9x^{45} + 9x^{46} - 8x^{47} + 8x^{48} - 7x^{49} + 6x^{50} - 5x^{51} \\ + 5x^{52} - 5x^{53} + 3x^{54} - 3x^{55} + 3x^{56} - 3x^{57} + x^{58} - 2x^{59} + 2x^{60} - x^{61} - x^{63} \\ + x^{64}\big)\,,\end{array}}$$

which gives rise to the extended form of $p(x)$ by multiplying (4.5) with the polynomial $[4]$*. Thus*

$$P(x) = \mathrm{Ext}(S) = [2, 8, 12, 20, 30] = [2, 12, 20, 30](1 + x^4)[4].$$

Hence, the complete form is given by

$$f_S(x) = \frac{[2, 8, 12, 20, 30]}{\begin{array}{l}\big(1 - x^3 - x^5 - 2x^7 - x^8 - 2x^9 - 2x^{11} - 2x^{13} + x^{14} - x^{15} + 2x^{16} - x^{17} + 2x^{18} \\ + 3x^{20} + 3x^{22} + x^{23} + 4x^{24} + 2x^{25} + 4x^{26} + 2x^{27} + 4x^{28} + 3x^{29} + 4x^{30} + 3x^{31} \\ + 4x^{32} + 4x^{33} + 4x^{34} + 4x^{35} + 3x^{36} + 4x^{37} + 3x^{38} + 4x^{39} + 2x^{40} + 4x^{41} + 2x^{42} \\ + 4x^{43} + x^{44} + 3x^{45} + 3x^{47} + 2x^{49} - x^{50} + 2x^{51} - x^{52} + x^{53} - 2x^{54} - 2x^{56} \\ - 2x^{58} - x^{59} - 2x^{60} - x^{62} - x^{64} + x^{67}\big)\,.\end{array}}$$

An important feature of putting a growth series into its complete form is that the number of its real poles and their location in the complex plane are not changed. In fact the extension of the denominator $q(x)$ arises by multiplying it with *cyclotomic* polynomial(s) of the form $[l]$ for an integer $l \geq 2$.

4.2 Some technical tools

Before we analyse and describe recursively the coefficients of $Q(x)$, we need some preparations.

Lemma 4.1 *Let $f(x)$ be a non-vanishing complex function such that $f(0) = 1$. Then, the k-th derivative of $1/f$ is given by*

$$\left(\frac{1}{f}\right)^{(k)}(0) = -\sum_{j=1}^{k} \binom{k}{j} \cdot f^{(j)}(0) \cdot \left(\frac{1}{f}\right)^{(k-j)}(0), \tag{4.6}$$

for each $k \geq 1$.

P*roof*
Let us apply Leibniz' formula to $\frac{1}{f(x)} \cdot f(x)$, that is,

$$0 = \left(\frac{1}{f} \cdot f\right)^{(k)}(x) = \sum_{j=0}^{k} \binom{k}{j} \cdot f^{(j)}(x) \cdot \left(\frac{1}{f}\right)^{(k-j)}(x).$$

Thus,

$$f(x) \cdot \left(\frac{1}{f}\right)^{(k)}(x) = -\sum_{j=1}^{k} \binom{k}{j} \cdot f^{(j)}(x) \cdot \left(\frac{1}{f}\right)^{(k-j)}(x).$$

Evaluate this last expression at $x = 0$. As $f(0) = 1$, we obtain

$$\left(\frac{1}{f}\right)^{(k)}(0) = -\sum_{j=1}^{k} \binom{k}{j} \cdot f^{(j)}(0) \cdot \left(\frac{1}{f}\right)^{(k-j)}(0).$$

□

Next, we study the function

$$g_m(x) := \prod_{i=1}^{m} [n_i], \tag{4.7}$$

where $n_1, \ldots, n_m \geq 2$ are integers. Our aim is to present a formula for the k-th derivative, evaluated at 0, of the function $g_m(x)$ wherefore we express $g_m(x)$ in a more convenient, additive way. Recall that

$$[k] = 1 + x + \ldots + x^{k-1}, \text{ for } k \geq 1, \tag{4.8}$$

and, by convention, $[0] := 0$.

Lemma 4.2 *Let k and l be integer numbers such that $k \geq 2$ and $l \geq 0$. Then*

$$[k]^{(l)}(0) = \begin{cases} l!, & \text{in case } l < k \\ 0, & \text{in case } l \geq k. \end{cases} \tag{4.9}$$

P*roof*
As $[k]=\sum\limits_{i=0}^{k-1} x^i$ is a polynomial of degree $k-1$, $[k]^{(l)}\equiv 0$, for $l\geq k$. Let us now suppose that $l<k$. Then,

$$[k]^{(l)}=\sum_{i=l}^{k-1}\frac{i!}{(i-l)!}\cdot x^{i-l}.$$

Evaluating this last equation at $x=0$ finishes the proof. $\square$

The next auxiliary lemma is based on a well-known combinatorial trick.

Lemma 4.3 *Let $J_m=\{n_1,\ldots,n_m\}$ be a set of integer numbers bigger than 1. Then,*

$$\sum_{\emptyset\subsetneq J\subsetneq J_m}(-1)^{|J|+1}=1+(-1)^m. \tag{4.10}$$

P*roof*
Let us first remark that

$$\sum_{\emptyset\subsetneq J\subsetneq J_m}(-1)^{|J|+1}=\sum_{\emptyset\subseteq J\subseteq J_m}(-1)^{|J|+1}-(-1)-(-1)^{m+1},$$

and that

$$\sum_{\emptyset\subseteq J\subseteq J_m}(-1)^{|J|+1}=-\sum_{\emptyset\subseteq J\subseteq J_m}(-1)^{|J|}=-\sum_{k=0}^{m}(-1)^k\cdot\binom{m}{k}.$$

Newton's binomial formula leads to

$$\sum_{k=0}^{m}(-1)^k\cdot\binom{m}{k}=(1+(-1))^m=0.$$

In combination, we finally get

$$\sum_{\emptyset\subsetneq J\subsetneq J_m}(-1)^{|J|+1}=1+(-1)^m.$$

$\square$

Now we need some terminology. Consider a subset $J\subset J_m$ and write $J=\{n_{j_1},\ldots,n_{j_{|J|}}\}$ with $j_{|J|}\leq m$. Let k be a positive integer such that $k\leq m$. For convenience define

$$\Sigma_k(J_m)=\sum_{\emptyset\subseteq J\subseteq J_k}(-1)^{|J|}\cdot\left[\sum_{i=1}^{m}n_i-(n_{j_1}+\ldots+n_{j_{|J|}})\right] \tag{4.11}$$

and

$$\Sigma_k^*(J_m)=\sum_{\emptyset\subsetneq J\subsetneq J_k}(-1)^{|J|}\cdot\left[\sum_{i=1}^{m}n_i-(n_{j_1}+\ldots+n_{j_{|J|}})\right], \tag{4.12}$$

where $[l]=1+x+\ldots+x^{l-1}$, as usual. It follows that

$$\Sigma_m(J_m)=\Sigma_m^*(J_m)+\left[\sum_{i=1}^{m}n_i\right]. \tag{4.13}$$

Lemma 4.4 *Let $g_m(x)$ be as in (4.7). Then*

$$(x-1)^{m-1} \cdot g_m(x) = \left[\sum_{i=1}^{m} n_i\right] + \Sigma_m^*(J_m), \tag{4.14}$$

for every integer $m \geq 2$.

P*roof*
The proof is by induction over $m \geq 2$. Let us first assume that $m = 2$. We have to show

$$(x-1) \cdot g_2(x) = [n_1 + n_2] - [n_1] - [n_2]. \tag{4.15}$$

Since $[k] = \frac{x^k - 1}{x-1}$, for $k \geq 2$, by (4.8), we get

$$\begin{aligned} (x-1) \cdot [n_1] \cdot [n_2] &= (x-1) \cdot \tfrac{x^{n_1}-1}{x-1} \cdot \tfrac{x^{n_2}-1}{x-1} \\ &= \tfrac{x^{n_1+n_2} - x^{n_1} - x^{n_2} + 1}{x-1} \\ &= \tfrac{\left(x^{n_1+n_2}-1\right) - (x^{n_1}-1) - (x^{n_2}-1)}{x-1} \\ &= [n_1 + n_2] - [n_1] - [n_2], \end{aligned}$$

which proves (4.15). Let us now assume that the claim (4.14) is valid for every integer $l \leq m$. We have to show that

$$(x-1)^m \cdot g_{m+1}(x) = \left[\sum_{i=1}^{m+1} n_i\right] + \Sigma_{m+1}^*(J_{m+1}). \tag{4.16}$$

For the left hand side of (4.16) the computation (4.15) together with (4.14) and Lemma 4.3 yields

$$\begin{aligned} (x-1)^m \cdot g_{m+1}(x) &= (x-1) \cdot [n_{m+1}] \cdot (x-1)^{m-1} \cdot g_m(x) \\ &= (x-1) \cdot [n_{m+1}] \cdot \left\{ \left[\sum_{i=1}^{m} n_i\right] + \Sigma_m^*(J_m) \right\} \\ &= (x-1) \cdot [n_{m+1}] \cdot \left[\sum_{i=1}^{m} n_i\right] + (x-1) \cdot [n_{m+1}] \cdot \Sigma_m^*(J_m) \\ &= \left[\sum_{i=1}^{m+1} n_i\right] - [n_{m+1}] - \left[\sum_{i=1}^{m} n_i\right] - \left(\sum_{\emptyset \subsetneq J \subsetneq J_m} (-1)^{|J|} \right) \cdot [n_{m+1}] \\ &\quad - \Sigma_m^*(J_m) + \Sigma_m^*(J_{m+1}) \\ &= \left[\sum_{i=1}^{m+1} n_i\right] - \left[\sum_{i=1}^{m} n_i\right] + (-1)^m \cdot [n_{m+1}] - \Sigma_m^*(J_m) + \Sigma_m^*(J_{m+1}). \end{aligned}$$

In view of (4.16) it remains therefore to show that

$$\Sigma_{m+1}^*(J_{m+1}) = -\left[\sum_{i=1}^{m} n_i\right] + (-1)^m \cdot [n_{m+1}] - \Sigma_m^*(J_m) + \Sigma_m^*(J_{m+1}). \tag{4.17}$$

The definitions (4.11) and (4.12) yield

$$\begin{aligned} \Sigma_m(J_{m+1}) &= \Sigma_{m+1}(J_{m+1}) + \sum_{\emptyset \subseteq J \subseteq J_m} (-1)^{|J|} \cdot \left[\sum_{i=1}^{m+1} n_i - (n_{j_1} + \ldots + n_{j_{|J|}} + n_{m+1})\right] \\ &= \Sigma_{m+1}(J_{m+1}) + \Sigma_m(J_m), \end{aligned}$$

hence

$$\Sigma_{m+1}(J_{m+1}) = \Sigma_m(J_{m+1}) - \Sigma_m(J_m).$$

Moreover, (4.13) leads to

$$\Sigma_{m+1}(J_{m+1}) = \Sigma_m(J_{m+1}) - \Sigma_m^*(J_m) - \left[\sum_{i=1}^{m} n_i\right]. \tag{4.18}$$

Introducing (4.18) in (see (4.13))

$$\Sigma_{m+1}^*(J_{m+1}) = \Sigma_{m+1}(J_{m+1}) - \left[\sum_{i=1}^{m+1} n_i\right]$$

yields to

$$\begin{aligned}
\Sigma_{m+1}^*(J_{m+1}) &= -\left[\sum_{i=1}^{m+1} n_i\right] - \left[\sum_{i=1}^{m} n_i\right] + \Sigma_m(J_{m+1}) - \Sigma_m^*(J_m) \\
&= -\left[\sum_{i=1}^{m} n_i\right] + (-1)^m \cdot [n_{m+1}] - \Sigma_m^*(J_m) + \Sigma_m^*(J_{m+1}),
\end{aligned}$$

which corresponds to (4.17) and finishes the proof. □

Recall by (4.7) that

$$g_m(x) = \prod_{i=1}^{m} [n_i]$$

which is of degree

$$\deg(g_m(x)) = \sum_{i=1}^{m} n_i - m. \tag{4.19}$$

Before formulating the next result, consider a finite set of numbers $T = \{t_1, \ldots, t_r\}$ and put

$$\Sigma(T) := \sum_{i=1}^{r} t_i. \tag{4.20}$$

Proposition 4.5 *Let $k \geq 1$ be an integer such that $k \leq \deg(g_m(x))$. Then the k-th derivative of $g_m(x)$ at 0 is given recursively by*

$$\begin{aligned}
g_m^{(k)}(0) &= (-1)^{m+1} \cdot k! \\
&\quad \cdot \left(1 + \sum_{\emptyset \subsetneq J \subsetneq J_m} (-1)^{|J|} \cdot \#\left\{(m-|J|) - \text{tuples } T : \Sigma(T) > k\right\}\right) \\
&\quad + \sum_{i=1}^{k} \binom{k}{i} \cdot (-1)^{-1-i} \cdot \prod_{h=1}^{i} (m-h) \cdot g_m^{(k-i)}(0).
\end{aligned} \tag{4.21}$$

If $k > \deg(g_m(x))$, then $g_m^{(k)}(x) \equiv 0$.

P*roof*
The claim is trivial if $k > \deg(g_m(x))$. Assume that $k \leq \deg(g_m(x))$. By Lemma 4.4,

$$(x-1)^{m-1} \cdot g_m(x) = \left[\sum_{i=1}^{m} n_i\right] + \Sigma_m^*(J_m).$$

Let $\varphi(x) = (x-1)^{m-1} \cdot g_m(x)$. By Leibniz' formula,

$$\varphi^{(k)}(0) = \sum_{j=0}^{k} \binom{k}{j} \cdot h^{(j)}(0) \cdot g_m^{(k-j)}(0),$$

where

$$h(x) := (x-1)^{m-1}.$$

Since $h^{(j)}(0) = (-1)^{m-1-j} \cdot \prod\limits_{h=1}^{j}(m-h)$, for $j \geq 1$, we get

$$\varphi^{(k)}(0) = (-1)^{m-1} \cdot g_m^{(k)}(0) + \sum_{j=1}^{k} \binom{k}{j} \cdot (-1)^{m-1-j} \cdot \prod_{h=1}^{j}(m-h) \cdot g_m^{(k-j)}(0),$$

for $k \geq 1$, and finally,

$$g_m^{(k)}(0) = (-1)^{m+1} \cdot \varphi^{(k)}(0) + \sum_{j=1}^{k} \binom{k}{j} \cdot (-1)^{-1-j} \cdot \prod_{h=1}^{j}(m-h) \cdot g_m^{(k-j)}(0), \tag{4.22}$$

for $k \geq 1$. Let us now determine $\varphi^{(k)}(0)$. By definition we have

$$\varphi(x) = \left[\sum_{i=1}^{m} n_i\right] + \Sigma_m^*(J_m),$$

so that taking derivatives and using Lemma 4.2 we obtain

$$\varphi^{(l)}(0) = l! + \sum_{\emptyset \subsetneq J \subsetneq J_m} (-1)^{|J|} \cdot l! \cdot \#\left\{(m-|J|) - \text{tuples } T : \Sigma(T) > l\right\},$$

for an arbitrary integer $l < \sum\limits_{i=1}^{m} n_i$. Thus, as $k < \sum\limits_{i=1}^{m} n_i$ by (4.19), we conclude that

$$\varphi^{(k)}(0) = k! \cdot \left(1 + \sum_{\emptyset \subsetneq J \subsetneq J_m} (-1)^{|J|} \cdot \#\left\{(m-|J|) - \text{tuples } T : \Sigma(T) > k\right\}\right), \tag{4.23}$$

and by (4.22) and (4.23),

$$\begin{aligned} g_m^{(k)}(0) \;&=\; (-1)^{m+1} \cdot k! \cdot \left(1 + \sum_{\emptyset \subsetneq J \subsetneq J_m} (-1)^{|J|} \cdot \#\left\{(m-|J|) - \text{tuples } T : \Sigma(T) > k\right\}\right) \\ &\quad + \sum_{j=1}^{k} \tbinom{k}{j} \cdot (-1)^{-1-j} \cdot \prod_{h=1}^{j}(m-h) \cdot g_m^{(k-j)}(0). \end{aligned}$$

Otherwise, it is obvious that $\varphi^{(l)}(0) = 0$, for $l \geq \sum_{i=1}^{m} n_i$. $\Box$

Despite the quite daunting form of (4.21) we get simple formulas for low order derivatives of $g_m(x)$, that is, for $k \leq 3$. We present them in the next corollary where we need the number

$$N_l := \#\{n_i \in J_m : n_i > l\}, \tag{4.24}$$

for an integer l.

Corollary 4.6 *Let $g_m(x)$ be the function (4.7). Then,*

$$\begin{array}{lrcl} (1) & g'_m(0) & = & m \quad , \\ (2) & g''_m(0) & = & m \cdot (m-1) + 2 \cdot N_2 \quad , \\ (3) & g^{(3)}_m(0) & = & m \cdot (m-1) \cdot (m-2) + 6 \cdot N_3 + 6 \cdot (m-1) \cdot N_2 \quad . \end{array}$$

P*roof*
Let us first remark that a direct computation yields easily (1). Therefore, consider (2). By means of (4.21), we get

$$\begin{aligned} g''_m(0) \;=\; & (-1)^{m+1} \cdot 2 \cdot \left(1 + \sum_{\emptyset \subsetneq J \subsetneq J_m} (-1)^{|J|} \cdot \# \left\{(m-|J|)\,\text{–tuples } T : \Sigma(T) > 2\right\}\right) \\ & +2 \cdot (m-1) \cdot g'_m(0) - (m-1) \cdot (m-2) \cdot g_m(0). \end{aligned}$$

Let us now consider a j-tuple $J := \{n_{i_1}, \ldots, n_{i_j}\}$. By hypothesis, $\Sigma(J) > 2$, except if $j = 1$ and $J = \{n_{i_1} = 2\}$. Then, by partitionning the summation over all $\emptyset \subsetneq J \subsetneq J_m$ into partial sums with respect to $|J| < m-1$ and $|J| = m-1$, we derive the equality

$$\sum_{\emptyset \subsetneq J \subsetneq J_m} (-1)^{|J|} \cdot \# \left\{(m-|J|)\,\text{–tuples } T : \Sigma(T) > 2\right\} = \sum_{i=1}^{m-2} (-1)^j \cdot \binom{m}{m-j} + (-1)^{m-1} N_2.$$

By means of

$$\sum_{i=1}^{m-2} (-1)^j \cdot \binom{m}{m-j} = (-1)^m \cdot (m-1) - 1,$$

we get

$$\begin{aligned} g''_m(0) \;&=\; (-1)^{m+1} \cdot 2 \cdot \left(1 + (-1)^m \cdot (m-1) - 1 + (-1)^{m-1} \cdot N_2\right) \\ &\quad + 2 \cdot (m-1) \cdot g'_m(0) - (m-1) \cdot (m-2) \cdot g_m(0) \\ &=\; m \cdot (m-1) + 2 \cdot N_2. \end{aligned}$$

The proof of (3) is similar. $\Box$

Let us briefly explain why we can't get similar nice formulas for higher derivatives. Working along the lines of the previous proof one sees that associated counting functions involve also numbers of k-tuples such as $T = \{2, 2\}$ in the case of the fourth derivative. Such a phenomenon leads to expressions which are non-linear polynomials in N_k. This is why we prefer to use the additive form (4.21) for the derivatives $g^{(k)}_m(x)$, when $k \geq 4$.

4.3 The main result

Let G denote a cocompact hyperbolic Coxeter group with generating set S. Its growth series is denoted by $f_S(x)$ and assumed to be in its *complete* form $P(x)/Q(x)$ (see part 4.1). More precisely, by (4.4),

$$P(x) = \prod_{i=1}^{m} [n_i], \tag{4.25}$$

for some integers $n_i \geq 2$. Moreover, $P(0) = 1$ according to (4.8). The denominator $Q(x)$ is given by the polynomial

$$Q(x) = \sum_{i=0}^{N} b_i x^i, \tag{4.26}$$

where $b_0 = 1$. Note that $P(x)$ and $Q(x)$ are of equal degree N, so that $N = \sum\limits_{i=1}^{m} n_i - m$. It is moreover clear that $Q(x)$ is palindromic if n is even, while it is antipalindromic if n is odd. Besides it satisfies

$$Q^k(0) = k! \cdot b_k. \tag{4.27}$$

Hence, the growth series of G is given by

$$f_S(x) = \frac{\prod\limits_{i=1}^{m} [n_i]}{\sum\limits_{i=0}^{N} b_i x^i}, \tag{4.28}$$

with $b_0 = b_N$ and some b_i are negative. By Steinberg's formula (3.4),

$$\frac{1}{f_S(x)} = (-1)^n \cdot \sum_{T \in \mathcal{F}} \frac{(-1)^{|T|}}{f_T(x)}, \tag{4.29}$$

where $\mathcal{F} = \{T \subseteq S : G_T \text{ is finite}\}$, as usual.

The aim of this part is a recursion formula for the coefficients b_i of the denominator $Q(x)$. The basic idea is to compare the derivatives in 0 of the inverse $1/f_S(x)$ as given by Steinberg's formula (4.29) and by its complete form (4.28). The first coefficients are easy to obtain. We present them in the sequel. Let us mention that the coefficient b_1 has first been described in [8, Theorem 5.4.3], but by a different method. In the proof of [8] there is furthermore a little flaw concerning the (non-)reciprocity of $f_S(x)$ when differentiating and evaluating its inverse at $x = 0$.

Lemma 4.7 *Let G be a Coxeter group acting cocompactly on $\mathbb{H}^n$ with $f_S(x)$ given in its complete form (4.28) with respect to the generating set S. Then, the coefficient b_1 is given by*

$$b_1 = m - |S|. \tag{4.30}$$

P*roof*
Let us recall the original definition 3.1 as given by

$$f_S(x) = \sum_{i \geq 0} a_i x^i = 1 + |S| \cdot x + \sum_{i \geq 2} a_i x^i, \tag{4.31}$$

where the $a_i > 0$, $i \geq 2$, are certain cardinalities. For example, $a_1 = S$. Then, as $\sum\limits_{i \geq 0} a_i x^i = \frac{P(x)}{Q(x)}$, we have

$$\left(\sum_{i=0}^{N} b_i x^i \right) \cdot \left(1 + |S| \cdot x + a_2 x^2 + \ldots \right) = \prod_{i=1}^{m} [n_i]. \tag{4.32}$$

A comparison between the coefficients of both sides of (4.32) leads to

$$m = |S| + b_1.$$

□

The nice short form of (4.30) is due to the fact that $a_1 = |S|$. The other coefficients a_i can't be described is a similar fashion since they depend on the relations connecting the generators in S. However, some closed formulas for b_2 and b_3 will be presented. Recall some notations of part 4.2. Let $J_m = \{n_1, \ldots, n_m\}$ denote a set of integers $n_1, \ldots, n_m \geq 2$. Besides, for an integer l, we defined N_l as the number of integers $n_i \in J_m$, such that $n_i > l$ (see (4.24)).

Let G_T be a spherical Coxeter group with generating set T arising as a subgroup of G. According to Solomon's formula (3.7), its growth series is given by

$$f_T(x) = \prod_{i=1}^{|T|} [c_i], \qquad c_i := 1 + m_i,$$

where $m_1, \ldots, m_{|T|}$ are the exponents of G_T. Let $C(T) := \{c_1, \ldots, c_{|T|}\}$ and define the number

$$C_l := \#\{c_i : c_i > l\}. \tag{4.33}$$

It will be convenient to consider the set

$$\mathcal{F}' := \{T \subsetneq S : 2 \leq |T| < |S| \text{ and } G_T \text{ is finite}\}. \tag{4.34}$$

We are now ready to present formulas for the coefficients b_2 and b_3. The proof of the following Proposition serves as warm-up for the general case culminating in Theorem 4.10.

Proposition 4.8 *Let G be a cocompact Coxeter group acting on $\mathbb{H}^n$ with generating set S and growth series $f_S(x) = P(x)/Q(x)$ in complete form (4.28). Then the coefficients b_2 and b_3 satisfy*

$$\begin{aligned}
(1)\quad 2\cdot b_2 \;&=\; (-1)^{n+1}\cdot 2\cdot |S| + (-1)^n \cdot \Big(\sum_{T\in\mathcal{F}'} (-1)^{|T|}\cdot |T|\cdot (|T|+1)\Big)\\
&\quad + (-1)^{n+1}\cdot 2\cdot \Big(\sum_{T\in\mathcal{F}'} (-1)^{|T|}\cdot C_2\Big)\\
&\quad - m\cdot(m+1) + 2\cdot N_2 + 2\cdot m\cdot b_1 \quad ,\\[2ex]
(2)\quad 6\cdot b_3 \;&=\; (-1)^n\cdot 6\cdot |S| + (-1)^{n+1}\cdot \Big(\sum_{T\in\mathcal{F}'} (-1)^{|T|}\cdot |T|\cdot(|T|+1)\cdot(|T|+2)\Big)\\
&\quad + (-1)^n\cdot 6\cdot \Big(\sum_{T\in\mathcal{F}'} (-1)^{|T|}\cdot(-C_3 + (|T|+1)\cdot C_2)\Big)\\
&\quad + m\cdot(m+1)\cdot(m+2)\\
&\quad + 6\cdot N_3 - 6\cdot(m+1)\cdot N_2\\
&\quad + 3\cdot(2N_2 - m(m+1))\cdot b_1 + 6\cdot m\cdot b_2 \quad .
\end{aligned} \tag{4.35}$$

P*roof*
By means of (4.29) and (3.7), we easily compute that

$$\begin{aligned}
\Big(\tfrac{1}{f_S}\Big)''(0) \;&=\; (-1)^n\cdot\Big(-2\cdot|S| + \sum_{T\in\mathcal{F}'} (-1)^{|T|}\cdot\Big(-\tfrac{f_T''(x)\cdot(f_T(x))^2 - 2\cdot f_T(x)\cdot(f_T'(x))^2}{(f_T(x))^4}\Big)\Big|_{x=0}\Big)\\
&=\; (-1)^n\cdot\Big(-2\cdot|S| + \sum_{T\in\mathcal{F}'} (-1)^{|T|}\cdot\big(2\cdot(f_T'(0))^2 - f_T''(0)\big)\Big).
\end{aligned}$$

Then Corollary 4.6 leads to

$$\Big(\frac{1}{f_S}\Big)''(0) = (-1)^n\cdot\Big(-2\cdot|S| + \sum_{T\in\mathcal{F}'} (-1)^{|T|}\cdot\{|T|\cdot(|T|+1) - 2\cdot C_2\}\Big). \tag{4.36}$$

By (4.28), we have furthermore that

$$\Big(\frac{1}{f_S}\Big)''(0) = \Big(\frac{Q}{P}\Big)''(0).$$

Differentiating two times the quotient Q/P and using Corollary 4.6 and (4.27) yield

$$\Big(\frac{Q}{P}\Big)''(0) = m\cdot(m+1) - 2\cdot N_2 + 2\cdot b_2 - 2\cdot m\cdot b_1. \tag{4.37}$$

It remains then to compare (4.36) and (4.37) to obtain the desired formula. We proceed in a similar way for b_3. □

By Proposition 4.8, we also have identical formulas for the coefficients b_{N-2}, b_{N-1} and b_N up to a sign, according to whether $Q(x)$ is palindromic or antipalindromic.

Remark 4.1 *In case G possesses more finite subgroups than hyperbolic ones, the formulas in Proposition 4.8 aren't convenient to work with. In order to bypass this disadvantage one* *shows that*

$$\sum_{T\in\mathcal{F}'}(-1)^{|T|}\cdot(|T|)_k=\sum_{l=0}^{k}(-1)^{k-l}\cdot s(k,l)\cdot j_l(|S|)-\sum_{T\in\mathcal{F}_\infty}(-1)^{|T|}\cdot(|T|)_k-k!\cdot|S|, \quad (4.38)$$

where $\mathcal{F}_\infty := \{T \subseteq S : |G_T| = \infty\}$*. Furthermore,* $(x)_k = x\cdot(x+1)\cdot\ldots\cdot(x+k)$ *is the Pochhammer symbol,* $s(k,l)$ *is a Stirling number of first kind, and the function* $j_l(x)$ *is defined by*

$$j_l(n) := \sum_{p=0}^{n}\binom{n}{p}\cdot(-1)^p\cdot p^l,$$

for an integer l*. As for the proof of (4.38), we only mention that the study of the related sum* $\sum_{T\in\mathcal{F}'}(-1)^{|T|}|T|^k$ *involves the well-known relation between the Pochhammer symbol* $(x)_k$ *and the Stirling numbers* $s(k,l)$

$$(x)_k=\sum_{k=0}^{n}(-1)^{n-k}\cdot s(n,k)\cdot x^k.$$

Let us point out that (4.38) shows that the formulas for b_2 *and* b_3 *in Proposition 4.8 depend only on the numbers* $|S|$ *and* $|\mathcal{F}_\infty|$*.*

In case G is a Lannér group in $\mathbb{H}^n$, the formulas of Lemma 4.8 simplify as follows.

Corollary 4.9 *Let* G *be a Lannér group with* $|S| = n+1$ *and growth series* $f_S(x) = P(x)/Q(x)$*. Then, the coefficients* b_1*,* b_2 *and* b_3 *satisfy the following rules :*

$$\begin{array}{lrcl}
(1) & b_1 & = & -(n+1)+m \quad , \\[2ex]
(2) & 2\cdot b_2 & = & (n+1)\cdot(n+2) \\
 & & & +\ (-1)^{n+1}\cdot 2\cdot\left(\sum_{T\in\mathcal{F}'}(-1)^{|T|}\cdot C_2\right) \\
 & & & -\ m\cdot(m+1)+2\cdot N_2+2\cdot m\cdot b_1 \quad , \\[2ex]
(3) & 6\cdot b_3 & = & -(n+1)\cdot(n+2)\cdot(n+3) \\
 & & & +\ (-1)^n\cdot 6\cdot\left(\sum_{T\in\mathcal{F}'}(-1)^{|T|}\cdot(-C_3+(|T|+1)\cdot C_2)\right) \\
 & & & +\ m\cdot(m+1)\cdot(m+2) \\
 & & & +\ 6\cdot N_3-6\cdot(m+1)\cdot N_2 \\
 & & & +\ 3\cdot(2N_2-m(m+1))\cdot b_1+6\cdot m\cdot b_2 \quad .
\end{array}$$

P*roof*
Since all subgroups of G are finite, $G_T \in \mathcal{F}'$ for every $T \subsetneq S$. Let us consider an integer k such that $0 \le k \le 2$. Then it is easy to see that

$$(-1)^n\cdot\left(\sum_{T\in\mathcal{F}'}(-1)^{|T|}\cdot|T|\cdot(|T|+1)\cdot\dots\cdot(|T|+k)\right)=(-1)^n\cdot(k+1)!\cdot(n+1)+\prod_{i=1}^{k+1}(n+i).$$

Apply now Proposition 4.8. □

Example 4.2 *Let G be a Lannér group given by the Coxeter graph*

$$\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet\overset{5}{-\!\!\!-\!\!\!-}\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet$$

acting on $\mathbb{H}^3$. The complete form of $f_S(x)$ is given by

$$f_S(x) = \frac{[2,6,10]}{1 - x - x^4 + x^{11} + x^{14} - x^{15}}. \tag{4.39}$$

It is easy to check that the coefficients $b_0 = -b_{15}$, $b_1 = -b_{14}$, $b_4 = -b_{11}$ satisfy the conclusion of Corollary 4.9.

In what follows we present the general method to derive a recursion formula for the coefficients b_k of the denominator $Q(x)$ in (4.26). We use the same notations and terminology as introduced for Proposition 4.8.

Let us consider an integer $k \geq 1$. By means of (4.29) we obtain

$$\left(\frac{1}{f_S}\right)^{(k)}(0) = (-1)^{n+k+1}\cdot k!\cdot|S| + (-1)^n \cdot \sum_{T\in\mathcal{F}'}(-1)^{|T|}\cdot\left(\frac{1}{f_T}\right)^{(k)}(0), \tag{4.40}$$

where $\mathcal{F}' = \{T \subseteq S : |T| \geq 2 \text{ and } G_T \text{ is finite}\}$. On the other hand, the complete form of $f_S(x)$ as given by (4.28), (4.27) leads to

$$\left(\frac{1}{f_S}\right)^{(k)}(0) = \left(\frac{1}{P}\right)^{(k)}(0) + \sum_{j=1}^{k-1}\left(\binom{k}{j}\cdot j!\cdot b_j\cdot\left(\frac{1}{P}\right)^{(k-j)}(0)\right) + k!\cdot b_k. \tag{4.41}$$

By comparison of (4.40) and (4.41) one derives a first formula for the coefficient b_k, $k \geq 1$, as follows.

$$\begin{aligned} k!\cdot b_k \;=\; & (-1)^{n+k+1}\cdot k!\cdot|S| \;+\; (-1)^n\cdot\sum_{T\in\mathcal{F}'}\left((-1)^{|T|}\cdot\left(\tfrac{1}{f_T}\right)^{(k)}(0)\right) \\ & -\left(\tfrac{1}{P}\right)^{(k)}(0) \;-\; \sum_{j=1}^{k-1}\left(\tbinom{k}{j}\cdot j!\cdot b_j\cdot\left(\tfrac{1}{P}\right)^{(k-j)}(0)\right) \end{aligned}$$

Thus,

$$\boxed{\begin{aligned} & k!\cdot b_k = (-1)^{n+k+1}\cdot k!\cdot|S| + (-1)^n\cdot P_k^T - P_k^m + B_k, \quad \text{for } k\geq 1, \\ & \text{where} \\ & \quad P_k^m := \left(\tfrac{1}{P}\right)^{(k)}(0) \\ & \quad P_k^T := \sum_{T\in\mathcal{F}'}\left((-1)^{|T|}\cdot\left(\tfrac{1}{f_T}\right)^{(k)}(0)\right) \\ & \quad B_k := -\sum_{j=1}^{k-1}\left(\tbinom{k}{j}\cdot j!\cdot b_j\cdot\left(\tfrac{1}{P}\right)^{(k-j)}(0)\right). \end{aligned}} \tag{4.42}$$

Let us now study in detail the different terms in (4.42) by using the results of part 4.2.

By Lemma 4.1 and Lemma 4.5, we have

$$P_k^m = -\sum_{j=1}^{k} \binom{k}{j} \cdot P^{(j)}(0) \cdot P_{k-j}^m \text{ and}$$

$$\begin{aligned} P_k^m \quad = \quad & -\sum_{j=1}^{k} \tbinom{k}{j} \cdot \big((-1)^{m+1} \cdot j! \\ & \cdot \left(1 + \sum_{\emptyset \subsetneq J \subsetneq M^m} (-1)^{|J|} \cdot \# \left\{(m-|J|)\,\text{-tuples } Y : \Sigma(Y) > j\right\}\right) \\ & + \sum_{i=1}^{j} \tbinom{j}{i} \cdot (-1)^{-1-i} \cdot \prod_{h=1}^{i} (m-h) \cdot P^{(j-i)}(0) \Bigg) \cdot P_{k-j}^m. \end{aligned} \tag{4.43}$$

Similarly, we get a recursion formula for P_k^T in the form

$$P_k^T = \sum_{T \in \mathcal{F}'} (-1)^{|T|+1} \cdot \left(\sum_{j=1}^{k} \binom{k}{j} \cdot f_T^{(j)}(0) \cdot \left(\frac{1}{f_T}\right)^{(k-j)} (0) \right).$$

Thus,

$$\begin{aligned} P_k^T \quad = \quad & \sum_{T \in \mathcal{F}'} (-1)^{|T|+1} \cdot \Bigg(\sum_{j=1}^{k} \tbinom{k}{j} \cdot \big\{(-1)^{|T|+1} \cdot j! \\ & \cdot \left(1 + \sum_{\emptyset \subsetneq J \subsetneq C^{|T|}} (-1)^{|J|} \cdot \# \left\{(|T|-|J|)\,\text{-tuples } Y : \Sigma(Y) > j\right\}\right) \\ & + \sum_{i=1}^{j} \tbinom{j}{i} \cdot (-1)^{-1-i} \cdot \prod_{h=1}^{i} (|T|-h) \cdot f_T^{(j-i)}(0) \Bigg\} \cdot \left(\tfrac{1}{f_T}\right)^{(k-j)} (0) \Bigg). \end{aligned} \tag{4.44}$$

Finally, for B_k, we easily derive the relation

$$B_k = -k \cdot P_{k-1}^m \cdot b_1 - \sum_{j=2}^{k-2} \left(\frac{k!}{(k-j)!} \cdot P_{k-j}^m \cdot b_j \right) + k! \cdot m \cdot b_{k-1}. \tag{4.45}$$

Then, by plugging (4.43), (4.44) and (4.45) into (4.42), we obtain the following recursion.

$$
\begin{aligned}
k! \cdot b_k \quad = \quad & (-1)^{n+k+1} \cdot k! \cdot |S| \\
& + (-1)^n \cdot \left(\sum_{T \in \mathcal{F}'} (-1)^{|T|+1} \cdot \left(\sum_{j=1}^{k} \binom{k}{j} \cdot \Big\{ (-1)^{|T|+1} \cdot j! \right.\right. \\
& \quad \cdot \left(1 + \sum_{\emptyset \subsetneq J \subsetneq C^{|T|}} (-1)^{|J|} \cdot \# \left\{ (|T| - |J|) - \text{tuples } Y : \Sigma(Y) > j \right\} \right) \\
& \quad \left.\left. + \sum_{i=1}^{j} \binom{j}{i} \cdot (-1)^{-1-i} \cdot \prod_{h=1}^{i} (|T| - h) \cdot f_T^{(j-i)}(0) \Big\} \right) \right. \\
& \quad \left. \cdot \left(\frac{1}{f_T} \right)^{(k-j)} (0) \right) \\
& + \sum_{j=1}^{k} \binom{k}{j} \cdot \Big((-1)^{m+1} \cdot j! \\
& \quad \cdot \left(1 + \sum_{\emptyset \subsetneq J \subsetneq M^m} (-1)^{|J|} \cdot \# \left\{ (m - |J|) - \text{tuples } Y : \Sigma(Y) > j \right\} \right) \\
& \quad \left. + \sum_{i=1}^{j} \binom{j}{i} \cdot (-1)^{-1-i} \cdot \prod_{h=1}^{i} (m - h) \cdot P^{(j-i)}(0) \right) \cdot P_{k-j}^m \\
& - k \cdot P_{k-1}^m \cdot b_1 - \sum_{j=2}^{k-2} \left(\frac{k!}{(k-j)!} \cdot P_{k-j}^m \cdot b_j \right) + k! \cdot m \cdot b_{k-1}, \quad \text{for } k \geq 1.
\end{aligned}
$$

Our main result summarises the results for b_0, b_1, b_2, b_3 and for b_k, $k \geq 4$.

Theorem 4.10 *Let G be a Coxeter group with generating set S which acts cocompactly on $\mathbb{H}^n$. Assume that $f_S(x) = P(x)/Q(x)$ is given in its complete form with $Q(x) = \sum_{k=0}^{N} b_k x^k$ (4.28). Then,*

$$
\begin{aligned}
\bullet\quad b_0 \quad &= \quad 1 \\
\bullet\quad b_1 \quad &= \quad m - |S| \\
\bullet\quad 2 \cdot b_2 \quad &= \quad (-1)^{n+1} \cdot 2 \cdot |S| + (-1)^n \cdot \left(\sum_{T \in \mathcal{F}'} (-1)^{|T|} \cdot |T| \cdot (|T|+1) \right) \\
&\quad + (-1)^{n+1} \cdot 2 \cdot \left(\sum_{T \in \mathcal{F}'} (-1)^{|T|} \cdot C_2 \right) \\
&\quad - m \cdot (m+1) + 2 \cdot N_2 + 2 \cdot m \cdot b_1 \\
\bullet\quad 6 \cdot b_3 \quad &= \quad (-1)^n \cdot 6 \cdot |S| + (-1)^{n+1} \cdot \left(\sum_{T \in \mathcal{F}'} (-1)^{|T|} \cdot |T| \cdot (|T|+1) \cdot (|T|+2) \right) \\
&\quad + (-1)^n \cdot 6 \cdot \left(\sum_{T \in \mathcal{F}'} (-1)^{|T|} \cdot (-C_3 + (|T|+1) \cdot C_2) \right) \\
&\quad + m \cdot (m+1) \cdot (m+2) \\
&\quad + 6 \cdot N_3 - 6 \cdot (m+1) \cdot N_2 \\
&\quad + 3 \cdot (2N_2 - m(m+1)) \cdot b_1 + 6 \cdot m \cdot b_2 \\
\bullet\quad k! \cdot b_k \quad &= \quad (-1)^{n+k+1} \cdot k! \cdot |S| \\
&\quad + (-1)^n \cdot \Bigg(\sum_{T \in \mathcal{F}'} (-1)^{|T|+1} \cdot \Bigg(\sum_{j=1}^{k} \binom{k}{j} \cdot \Big\{ (-1)^{|T|+1} \cdot j! \\
&\qquad \cdot \left(1 + \sum_{\emptyset \subsetneq J \subsetneq C^{|T|}} (-1)^{|J|} \cdot \# \{ (|T| - |J|) - \text{tuples } Y : \Sigma(Y) > j \} \right) \\
&\qquad + \sum_{i=1}^{j} \binom{j}{i} \cdot (-1)^{-1-i} \cdot \prod_{h=1}^{i} (|T| - h) \cdot f_T^{(j-i)}(0) \Big\} \Bigg) \\
&\qquad \cdot \left(\frac{1}{f_T} \right)^{(k-j)} (0) \Bigg) \\
&\quad + \sum_{j=1}^{k} \binom{k}{j} \cdot \Big((-1)^{m+1} \cdot j! \\
&\qquad \cdot \left(1 + \sum_{\emptyset \subsetneq J \subsetneq M^m} (-1)^{|J|} \cdot \# \{ (m - |J|) - \text{tuples } Y : \Sigma(Y) > j \} \right) \\
&\qquad + \sum_{i=1}^{j} \binom{j}{i} \cdot (-1)^{-1-i} \cdot \prod_{h=1}^{i} (m - h) \cdot P^{(j-i)}(0) \Big) \cdot P^m_{k-j} \\
&\quad - k \cdot P^m_{k-1} \cdot b_1 - \sum_{j=2}^{k-2} \left(\frac{k!}{(k-j)!} \cdot P^m_{k-j} \cdot b_j \right) + k! \cdot m \cdot b_{k-1}, \qquad \text{for } k \geq 4.
\end{aligned}
$$

It is obvious that the formula in Theorem 4.10 depends strongly on the finite subgroups of the group G. Thus one has first to study in detail the combinatorics of G before applying Theorem 4.10. Let us add that the use of a computer is helpful. The next example illustrates this.

Example 4.3 *Consider the Esselmann group G_7, with generating set S and graph (cf. Table 4 in part 2.4)*

$$\Gamma_7 : \bullet \overset{8}{\text{——}} \bullet \text{——} \bullet \overset{4}{\text{——}} \bullet \text{——} \bullet \overset{8}{\text{——}} \bullet\,.$$

The finite subgroups G_T of G and their exponents are listed in the following table.

$\lvert T\rvert$	Group G_T	Multiplicity	Exponents
4	$\bullet \text{——} \bullet \overset{4}{\text{——}} \bullet \text{——} \bullet$	1	$1, 5, 7, 11$
	$\bullet \overset{4}{\text{——}} \bullet \text{——} \bullet \qquad \bullet$	2	$1, 1, 3, 5$
	$\bullet \overset{8}{\text{——}} \bullet \qquad \bullet \overset{8}{\text{——}} \bullet$	1	$1, 1, 7, 7$
	$\bullet \overset{8}{\text{——}} \bullet \qquad \bullet \text{——} \bullet$	2	$1, 1, 2, 7$
	$\bullet \overset{8}{\text{——}} \bullet \qquad \bullet \qquad \bullet$	2	$1, 1, 1, 7$
	$\bullet \overset{4}{\text{——}} \bullet \qquad \bullet \qquad \bullet$	1	$1, 1, 1, 3$
3	$\bullet \overset{4}{\text{——}} \bullet \text{——} \bullet$	2	$1, 3, 5$
	$\bullet \overset{8}{\text{——}} \bullet \qquad \bullet$	6	$1, 1, 7$
	$\bullet \overset{4}{\text{——}} \bullet \qquad \bullet$	2	$1, 1, 3$
	$\bullet \text{——} \bullet \qquad \bullet$	4	$1, 1, 2$
	$\bullet \qquad \bullet \qquad \bullet$	4	$1, 1, 1$
2	$\bullet \overset{8}{\text{——}} \bullet$	2	$1, 7$
	$\bullet \overset{4}{\text{——}} \bullet$	1	$1, 3$
	$\bullet \text{——} \bullet$	2	$1, 2$
	$\bullet \qquad \bullet$	10	$1, 1$
1	$\bullet$	6	1

Table 15 : The finite subgroups of G_7

It follows that the extended form of the numerator of $f_S(x)$ is given by

$$\mathrm{Ext}(S) = [2, 6, 8, 8, 12]$$

which is of degree 31 *and yields $m = 5$ (cf. (4.25)). Hence, by (4.28),*

$$f_S(x) = \frac{[2, 6, 8, 8, 12]}{Q(x)},$$

where

$$Q(x) = \sum_{i=0}^{31} b_i x^i.$$

Let us now determine the coefficients b_i, $i = 0, \ldots, 31$. As G_7 acts cocompactly on $\mathbb{H}^4$, $Q(x)$ is palindromic so that it is sufficient to study only the coefficients b_i for $i = 0, \ldots 15$. By means of Theorem 4.10 we get

$$\begin{array}{lcll}
\bullet\ b_0 & = & 1 & \\
\bullet\ b_1 & = & -1 & \\
\bullet\ 2 \cdot b_2 & = & -2 \cdot 6 + (2 \cdot 3 \cdot 15 - 3 \cdot 4 \cdot 18 + 4 \cdot 5 \cdot 9) - 2 \cdot (3 - 12 + 14) & \\
 & & -5 \cdot 6 + 2 \cdot 4 + 2 \cdot 5 \cdot (-1) = 0 & \text{, that is,} \\
 b_2 & = & 0 & \\
\bullet\ 6 \cdot b_3 & = & 6 \cdot 6 - (2 \cdot 3 \cdot 4 \cdot 15 - 3 \cdot 4 \cdot 5 \cdot 18 + 4 \cdot 5 \cdot 6 \cdot 9) & \\
 & & + 6 \cdot ((-2 + 3 \cdot 3) - (-8 + 4 \cdot 12) + (-11 + 5 \cdot 14)) & \\
 & & + 5 \cdot 6 \cdot 7 + 6 \cdot 4 - 6 \cdot 6 \cdot 4 + 3 \cdot (2 \cdot 4 - 5 \cdot 6) \cdot (-1) + 0 = -12 & \text{, that is,} \\
 b_3 & = & -2 &
\end{array}$$

The determination of the remaining coefficients is done with the aid of a computer and based on (4.42). We join the Mathematica documentation below.

Finally, we obtain that

$$\begin{aligned}
Q(x) \quad = \quad & 1 - x - 2x^3 - 4x^5 - 4x^7 + 2x^8 - 2x^9 + 5x^{10} + x^{11} + 7x^{12} + 5x^{13} + 8x^{14} + 6x^{15} \\
& + 6x^{16} + 8x^{17} + 5x^{18} + 7x^{19} + x^{20} + 5x^{21} - 2x^{22} + 2x^{23} - 4x^{24} - 4x^{26} - 2x^{28} \\
& - x^{30} + x^{31}.
\end{aligned}$$

It follows that $Q(x)$ possesses exactly two (inversive) pairs of real zeros ; they are simple and positive. Besides, we check that its non-real roots are contained in the annulus whose interior and exterior radii are given by the two real roots of intermediate size.

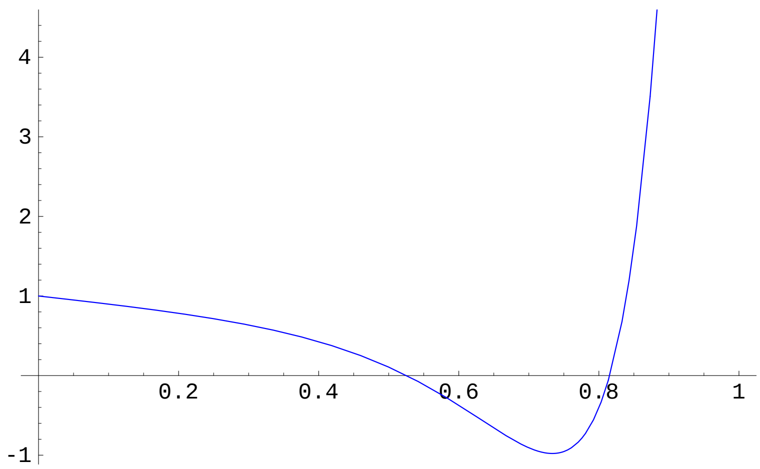


Figure 14: The denominator $Q(x)$ of the growth series of G_7 for $x > 0$.

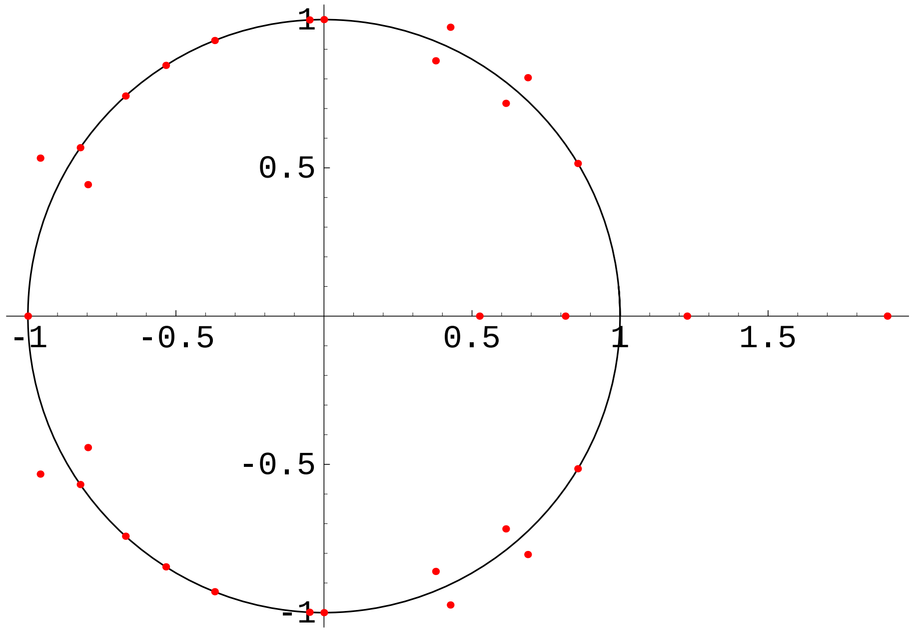


Figure 15: The poles of the growth series $f_S(x)$, $x \in \mathbb{C}$, of G_7.

Esselmann group G7

In this file we compute the numerator Q[x] of the growth function of the Esselmann group given by the graph

a[t_] := Sum[x^1, {1, 0, t - 1}]

Expand[a[2] * a[6] * a[8] * a[8] * a[12]]

$1+5x+14x^2+30x^3+55x^4+91x^5+139x^6+199x^7+269x^8+345x^9+423x^{10}+499x^{11}+568x^{12}+624x^{13}+663x^{14}+683x^{15}+683x^{16}+663x^{17}+624x^{18}+568x^{19}+499x^{20}+423x^{21}+345x^{22}+269x^{23}+199x^{24}+139x^{25}+91x^{26}+55x^{27}+30x^{28}+14x^{29}+5x^{30}+x^{31}$

P[x] = extended form of the numerator

P[x_] := $1+5x+14x^2+30x^3+55x^4+91x^5+139x^6+199x^7+269x^8+345x^9+423x^{10}+499x^{11}+568x^{12}+624x^{13}+663x^{14}+683x^{15}+683x^{16}+663x^{17}+624x^{18}+568x^{19}+499x^{20}+423x^{21}+345x^{22}+269x^{23}+199x^{24}+139x^{25}+91x^{26}+55x^{27}+30x^{28}+14x^{29}+5x^{30}+x^{31}$

m = 5

5

s = number of generators

s = 6

6

n = dimension of the hyperbolic space

n = 4

4

b[0] = 1

1

We separate the formula in 3 parts.

---

1) 1 st part :

f[x] = alternate sum of "1 over" the growth series of the finite subgroups of G7.

```
f[x_] := (2 / (a[2] * a[8])) + (1 / (a[2] * a[4])) + (2 / (a[2] * a[3])) +
   (10 / (a[2] * a[2])) - (2 / (a[2] * a[4] * a[6])) - (6 / (a[2] * a[2] * a[8])) -
   (4 / (a[2] * a[2] * a[3])) - (2 / (a[2] * a[2] * a[4])) - (4 / (a[2] * a[2] * a[2])) +
   (2 / (a[2] * a[2] * a[3] * a[8])) + (1 / (a[2] * a[2] * a[8] * a[8])) +
   (2 / (a[2] * a[2] * a[4] * a[6])) + (2 / (a[2] * a[2] * a[2] * a[8])) +
   (1 / (a[2] * a[6] * a[8] * a[12])) + (1 / (a[2] * a[2] * a[2] * a[4]))
```


Simplify[f[x]]

$\left(6+12x+28x^2+42x^3+73x^4+98x^5+146x^6+176x^7+231x^8+252x^9+309x^{10}+318x^{11}+360x^{12}+338x^{13}+363x^{14}+326x^{15}+330x^{16}+270x^{17}+255x^{18}+192x^{19}+171x^{20}+116x^{21}+92x^{22}+50x^{23}+37x^{24}+18x^{25}+10x^{26}\right)/\left((1+x)^4(1-x^2+x^4)(1+2x^2+3x^4+3x^6+2x^8+x^{10})^2\right)$

g[x_] := $\left(6+12x+28x^2+42x^3+73x^4+98x^5+146x^6+176x^7+231x^8+252x^9+309x^{10}+318x^{11}+360x^{12}+338x^{13}+363x^{14}+326x^{15}+330x^{16}+270x^{17}+255x^{18}+192x^{19}+171x^{20}+116x^{21}+92x^{22}+50x^{23}+37x^{24}+18x^{25}+10x^{26}\right)/\left((1+x)^4(1-x^2+x^4)(1+2x^2+3x^4+3x^6+2x^8+x^{10})^2\right)$

h[k_] := Derivative[k][g][0]

---

2) p_k^m (2 nd part) :

Simplify[1 / (P[x])]

$$\frac{1}{(1+x)^5(1+x^2)^3(1-x^2+x^4)(1+x^2+2x^4+x^6+x^8)^2}$$

p[x_] := $\dfrac{1}{(1+x)^5(1+x^2)^3(1-x^2+x^4)(1+x^2+2x^4+x^6+x^8)^2}$

---

3) B_k (3 rd part) : for k ≥ 1

B[k_] := -Sum[Binomial[k, j] * (j !) * b[j] * (Derivative[k - j][p][0]), {j, 1, k - 1}]

B[k]

$-\sum_{j=1}^{-1+k} b[j]\,\mathrm{Binomial}[k,j]\,j!\,p^{(-j+k)}[0]$

Let c[k] denote k ! * b[k], for k ≥ 1 .Then we have

c[k_] := (((-1) ^ (n + k + 1)) * (k !) * s) + (((-1) ^ n) * h[k]) - Derivative[k][p][0] + B[k]

As c[k] denotes k ! * b[k] we have

b[k_] := (1 / (k !)) * c[k]

Then we compute the coefficients b[1], ..., b[15].

```
{b[1], b[2], b[3], b[4], b[5], b[6], b[7],
 b[8], b[9], b[10], b[11], b[12], b[13], b[14], b[15]}
```

{-1, 0, -2, 0, -4, 0, -4, 2, -2, 5, 1, 7, 5, 8, 6}

4.4 Application for right-angled Coxeter groups

Let us consider a hyperbolic Coxeter group G with presentation (cf. (2.2))

$$G = \langle \{s_1, \ldots, s_k\} \mid (s_i s_j)^{m_{ij}} = 1 \rangle.$$

Then G is called *right-angled* if and only if $m_{ij} \in \{1, 2, \infty\}$. The terminology is justified by the fact that a fundamental polyhedron $P \subset \mathbb{H}^n$ has all interior angles equal to $\pi/2$. Notice that each subgroup of G and all l-faces, $2 \leq l \leq n-1$, of P are right-angled. By results of Vinberg [39], there exist no cocompact right-angled Coxeter groups in $\mathbb{H}^n$ for $n \geq 5$. For $n = 2$, right-angled Coxeter polygons are realisable as long as they have at least five vertices. For $n = 3$, the (compact) right-angled dodecahedron is the one with the minimal number of facets (and vertices). A beautiful example in $\mathbb{H}^4$ is the compact (regular) 120-cell of interior angle $\pi/2$ whose symmetry group is generated by the reflections of the Lannér group

$$\bullet \overset{5}{\text{——}} \bullet \text{——} \bullet \text{——} \bullet \overset{4}{\text{——}} \bullet\ .$$

For further details about right-angled Coxeter groups we refer to [29].

Let G be a cocompact right-angled Coxeter group, with generating set S, and which acts on $\mathbb{H}^n$. Hence, $n \leq 4$. As usual $f_S(x) = P(x)/Q(x)$ denotes the growth series of G in its complete form (see part 4.1).

Let G_T denote a finite subgroup of G which is generated by the set $T \subset S$. Since G_T is right-angled, its growth series $f_T(x)$ is given by (3.7) as follows.

$$f_T(x) = [2]^{|T|}$$

implying that

$$f_S(x) = \frac{[2]^n}{Q(x)}, \text{ with } Q(x) = \sum_{i=0}^{n} b_i x^i. \tag{4.46}$$

In fact, the numerator in its virgin form equals $[2]^n$ since the maximal (right-angled) subgroups in G are of rank n.
Since $n \leq 4$ and since $Q(x)$ is palindromic or antipalindromic, there are at most three non-zero coefficients b_0, b_1, b_2. In the next corollary we describe these coefficients by using the same notations as in Theorem 4.10 .

Corollary 4.11 *Let G be a right-angled hyperbolic Coxeter group, with generating set S, which acts cocompactly on $\mathbb{H}^n$, $n \leq 4$. We assume that its growth series $f_S(x)$ is given in its complete form (4.46). Then the coefficients b_i of its denominator satisfy the following rules :*

- $b_0 = 1$,
- $b_1 = n - |S|$,
- $2 \cdot b_2 = (-1)^{n+1} \cdot 2 \cdot |S| + (-1)^n \cdot \sum_{T \in \mathcal{F}'} (-1)^{|T|} \cdot |T| \cdot (|T| + 1)$
  $- \ n(n+1) + 2nb_1$.

P*roof*
As G is right-angled, we obviously have that $N_2 = 0$ (see (4.24)) and $C_2 = 0$ (see (4.33)). Insert this into the formulas of Theorem 4.10. □

Example 4.4 *Let G_H be a planar right-angled Coxeter group whose generating set S consists of the reflections through the edges of a compact (regular) hexagon P_H, all of whose interior angles equal $\frac{\pi}{2}$. By Poincaré's Theorem (see [13, page 135], for example), $P_H \subset \mathbb{H}^2$ exists and has area $A = \pi$, as given by the well-known defect formula.*

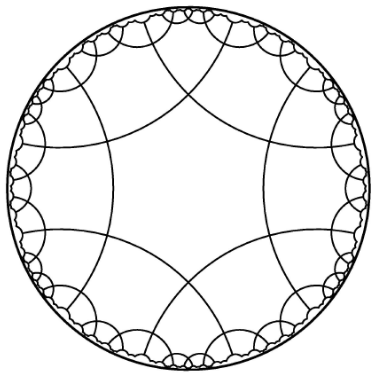


Figure 16: A regular hyperbolic hexagon.

The growth series of G_H is given by (see (4.46))

$$f_S(x) = \frac{[2]^2}{b_0 + b_1 x + b_0 x^2},$$

which, by means of Corollary 4.11, turns into

$$f_S(x) = \frac{[2]^2}{1 - 4x + x^2}.$$

Finally observe that we can confirm (see [18])

$$\frac{1}{f_S(1)} = -\frac{1}{2} = -\frac{2A}{\mathrm{area}(\mathbb{S}^2)}.$$

Let us present a formula for b_2 different from the one presented in Corollary 4.11. To this end, consider an arbitrary geometric convex n-polytope $P \subset \mathbb{X}^n$. Its *f-vector* $f = f(P)$ is defined by (see [17, page 130])

$$f := (f_0, f_1, \ldots, f_{n-1})\,, \tag{4.47}$$

where f_i denotes the number of i-faces of P. The components of f are related by Euler's formula (cf. [17, page 131], for example) according to

$$\sum_{i=0}^{n-1} (-1)^i \cdot f_i = 1 - (-1)^n. \tag{4.48}$$

Let us remark that Euler's formula is a particular case of the famous Dehn-Sommerville equations (cf. [17, page 146], for example). In the sequel we need these equations only for $n = 4$.

Proposition 4.12 *Let G be a right-angled Coxeter group, with generating set S, acting cocompactly on $\mathbb{H}^4$ with fundamental polytope P. Let $f_S(x)$ denote the growth series of G in its complete form. Then,*

$$\boxed{f_S(x) = \frac{[2]^4}{1 + (4 - f_3)x + (f_0 - 2 \cdot f_3 + 6)x^2 + (4 - f_3)x^3 + x^4}}$$

P*roof*
It is sufficient to show that $b_2 = f_0 - 2 \cdot f_3 + 6$. The Dehn-Sommerville equations in this case provide an important link between the subgroup structure of G and the combinatorial f-vector of P. As P is simple (see part 2.1), we have that $4f_0 = 2f_1$. Moreover, $f_3 = |S|$, and by Euler's formula (4.48), $f_2 = f_0 + |S|$. Furthermore, the number of finite subgroups G_T of G with $|T| = l$ equals f_{4-l}, for $l = 1, \ldots, 4$. Hence, by Corollary 4.11,

$$\sum_{T \in \mathcal{F}'} (-1)^{|T|} \cdot |T| \cdot (|T| + 1) = 6f_2 - 12f_1 + 20f_0. \tag{4.49}$$

Thus, (4.49) becomes

$$\sum_{T \in \mathcal{F}'} (-1)^{|T|} \cdot |T| \cdot (|T| + 1) = 6 \cdot |S| + 2 \cdot f_0.$$

Introducing this last equality in Corollary 4.11 leads to

$$2 \cdot b_2 = 2 \cdot |S| + 6 \cdot |S| + 2 \cdot f_0 - 20 + 8 \cdot b_1. \tag{4.50}$$

Since $b_1 = 4 - |S|$ (see Corollary 4.11), (4.50) transforms into

$$2 \cdot b_2 = 2 \cdot f_0 - 4 \cdot |S| + 12.$$

□

Remark 4.2 *In [12, Example 17.4.3], an analogous formula for the 3-dimensional case is presented. More precisely, the growth series of a cocompact right-angled Coxeter group in $I(\mathbb{H}^3)$ is given by*

$$f_S(x) = \frac{[2]^3}{1 - (f_0 - 3)x + (f_0 - 3)x^2 - x^3},$$

which, for $f_0 \geq 6$, has the three positive real roots 1, τ , τ^{-1} where

$$\tau = \frac{(f_0 - 4) + \sqrt{(f_0 - 4)^2 - 4}}{2}.$$

Example 4.5 *Let G_{120} be the Coxeter group generated by the 120 reflections with respect to the facets of a right-angled (compact) 120-cell $P \subset \mathbb{H}^4$. The polyhedron P has f-vector*

$$f = (600, 1200, 720, 120)$$

and is the 4-dimensional analog of a right-angled dodecahedron D. In fact, all facets of P are isometric to D. By means of (4.46) and Corollary 4.12, the growth series of G_{120} with respect to the set S of the above reflections is given by

$$f_S(x) = \frac{[2]^4}{1 - 116x + 366x^2 - 116x^3 + x^4}.$$

One deduces that $f_S(x)$ possesses exactly two pairs of real poles, which are simple and positive.

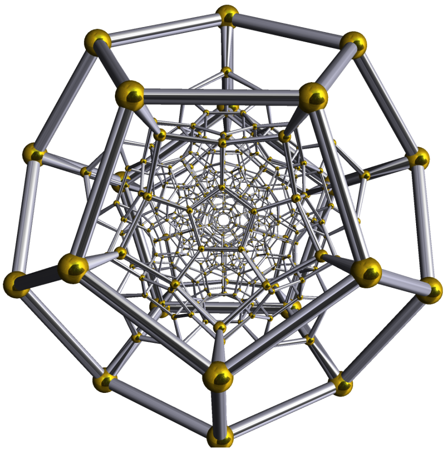


Figure 17: A combinatorial picture of a 120-cell.

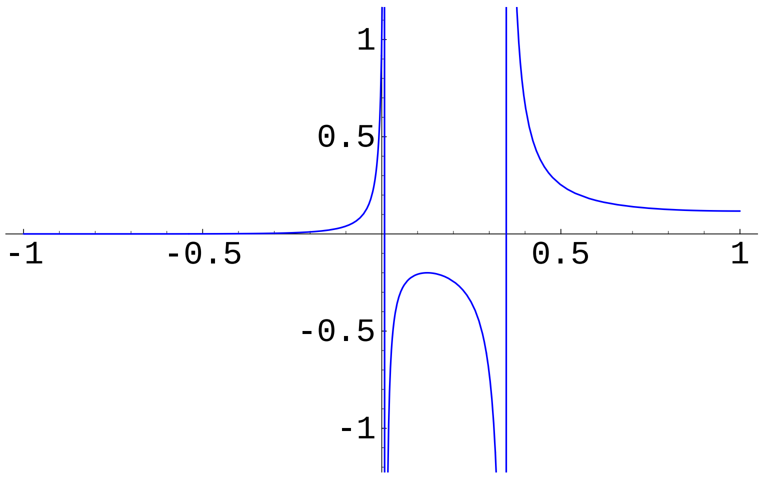


Figure 18: The growth series of G_{120} restricted to $[-1, 1]$.

It is an interesting question whether the growth series $f_S(x)$ of any cocompact right-angled Coxeter group G in $\mathbb{H}^4$ has always exactly two pairs of positive simple poles. For this, one has to study the discriminant behaviour of the denominator $1 + (4 - f_3)x + (f_0 - 2 \cdot f_3 + 6)x^2 + (4 - f_3)x^3 + x^4$ in Proposition 4.12 with respect to the f-vector (f_0, f_1, f_2, f_3) of a fundamental polytope P of G. We hope to come back to this question in [21]. Nevertheless, we formulate a first characterisation as follows (see also Remark 4.2).

Corollary 4.13 *Let G be a right-angled Coxeter group, with generating set S, acting cocompactly on $\mathbb{H}^4$ with fundamental polytope P. Denote by $f = (f_0, f_1, f_2, f_3)$ the f-vector of P. If $f_3^2 > 4f_0$, then $f_S(x)$ has the four real distinct poles*

$$\begin{array}{rclcrcl} x_1 &=& \frac{1}{4}\left(-a+\sqrt{\delta}+\sqrt{\gamma-2a\sqrt{\delta}}\right) &;& x_1^{-1} &=& \frac{1}{4}\left(-a+\sqrt{\delta}-\sqrt{\gamma-2a\sqrt{\delta}}\right) \\ x_2 &=& \frac{1}{4}\left(-a-\sqrt{\delta}+\sqrt{\gamma+2a\sqrt{\delta}}\right) &;& x_2^{-1} &=& \frac{1}{4}\left(-a-\sqrt{\delta}-\sqrt{\gamma+2a\sqrt{\delta}}\right), \end{array}$$

where

$$\begin{array}{rcl} a &=& 4 - f_3 \\ \delta &=& f_3^2 - 4f_0 \\ \gamma &=& -2af_3 - 4f_0. \end{array}$$

P*roof*
Consider the denominator $Q(x) = 1 + (4 - f_3)x + (f_0 - 2f_3 + 6)x^2 + (4 - f_3)x^3 + x^4$ in the complete form of $f_S(x)$ (cf Proposition 4.12). The polynomial $Q(x)$ is quartic over the integers with discriminant (see [32, *Discriminants*])

$$\Delta = f_0 \cdot (16 + f_0 - 4 \cdot f_3) \cdot (f_3^2 - 4 \cdot f_0)^2.$$

Since P is simple, $f_0 \geq 5f_3$. The condition $f_3^2 > 4f_0$ implies that $\Delta > 0$ and that $Q(x)$ has only simple, real roots. It is a standard matter to determine the explicit form of these roots. In fact, by applying the transformation $x = X + \frac{1}{X}$ to the quartic polynomial $Q(x)$, which does not change the discriminant, one obtains a reduced cubic $\tilde{Q}(x)$ with classical formulas for its roots (see [32, *Classical Formulas*]). $\square$

Consider next the growth series of a right-angled Coxeter group G as given by Definition 3.1, that is,

$$f_S(x) = \sum_{i \geq 0} a_i x^i = 1 + |S| \cdot x + \sum_{i \geq 2} a_i x^i, \tag{4.51}$$

where $a_i > 0$ is the number of words of length i in G. We present a recursion formula for the coefficients a_i in terms of b_j as follows.

Theorem 4.14 *Let G be a cocompact right-angled Coxeter group with generating set S in*

$I(\mathbb{H}^n)$. Then, the coefficients a_i of $f_S(x)$ (4.51) satisfy the recursion formula

$$(1) \quad a_0 = 1 \quad , \quad a_1 = |S|$$

$$(2) \quad a_k = \begin{cases} \binom{n}{k} - \sum_{j=1}^{k} a_{k-j}b_j, & \text{for } 2 \le k \le n, \\ -\sum_{j=1}^{n} a_{k-j}b_j, & \text{for } k > n \end{cases} ,$$

where the coefficients b_i, $i = 0, \ldots, n$, are given in Corollary 4.11 with $b_i := 0$, for $i > n$.

P*roof*
According to (4.28) and (4.51) we have

$$\frac{[2]^n}{\sum_{j=0}^{n} b_j x^j} = \sum_{i \ge 0} a_i x^i,$$

and

$$(1+x)^n = \sum_{k \ge 0} \left(\sum_{i+j=k} a_i b_j \right) x^k = \sum_{k \ge 0} \left(\sum_{j=0}^{k} a_{k-j} b_j \right) x^k.$$

Apply Newton's formula in order to obtain

$$\sum_{k=0}^{n} \binom{n}{k} x^k = \sum_{k \ge 0} \left(\sum_{j=0}^{k} a_{k-j} b_j \right) x^k.$$

Hence, for a non-negative integer k, we have

$$\binom{n}{k} = \sum_{j=0}^{k} a_{k-j} b_j,$$

from which we easily derive the first part of formula (2). For the second part of (2) and $k > n$, $b_k = 0$ and

$$0 = \sum_{j=0}^{k} a_{k-j} b_j = \sum_{j=0}^{n} a_{k-j} b_j + \sum_{j=n+1}^{k} a_{k-j} b_j = \sum_{j=0}^{n} a_{k-j} b_j.$$

□

Example 4.6 *Consider the hexagonal planar group G_H introduced in Example 4.4 with growth series*

$$f_S(x) = \frac{[2]^2}{1 - 4x + x^2},$$

which we will express as a power series. It is evident that $a_0 = 1$ *and* $a_1 = 6$. *In order to determine the remaining coefficients we use the help of a computer ; the Mathematica documentation is attached below. Here are the first few coefficients.*

- $a_2 = 24$
- $a_3 = 90$
- $a_4 = 336$
- $a_5 = 1'254$
- $a_6 = 4'680$
- $a_7 = 17'466$
- $a_8 = 65'184$
- $a_9 = 243'270$.

```
b[0] = 1
```

1

```
b[1] = -4
```

-4

```
b[2] = 1
```

1

```
B[i_] := b[i] * (KroneckerDelta[0 - i] + KroneckerDelta[1 - i] + KroneckerDelta[2 - i])
```

```
a[0] = 1
```

1

For the a[k] such that k ≤ 2 :

```
a[k_] := Binomial[2, k] - Sum[a[k - j] * B[j], {j, 1, k}]
```

```
a[1]
```

6

```
a[2]
```

24

For the a[k] such that k > 2 :

```
c[k_] := -Sum[a[k - j] * B[j], {j, 1, k}]
```

```
x = 2; While[(x = x + 1) < 16, Print[c[x]]]
```

90

336

1254

4680

17466

65184

243270

907896

3388314

12645360

47193126

176127144

657315450

4.5 Further consequences of the recursion formula

In this part we present two further major consequences of the recursion formula in Theorem 4.10. The main result is a recursion formula for the coefficients a_i of the growth series $\sum_{i\geq 0} a_i x^i$ of any cocompact hyperbolic Coxeter group. The second result is based on Theorem 4.10 and allows to prove Conjecture 1 (see part 3.4).

Firstly consider a cocompact Coxeter group G in $I(\mathbb{H}^n)$, with generating set S, and with growth series $f_S(x)$ represented by

$$f_S(x) = 1 + |S| \cdot x + \sum_{i\geq 2} a_i x^i, \tag{4.52}$$

where the $a_i > 0$ are certain cardinalities. The complete form

$$f_S(x) = \frac{P(x)}{Q(x)} \tag{4.53}$$

consists of integer polynomials $P(x)$ and $Q(x)$ of equal degree. The numerator is given by (see (4.25))

$$P(x) = \prod_{i=1}^{m} [n_i] = g_m(x)$$

and of degree $d_P = \sum\limits_{i=1}^{m} n_i - m$. In what follows, we write $P(x)$ as

$$P(x) = \sum_{i=0}^{d_P} c_i x^i. \tag{4.54}$$

The denominator $Q(x)$ is given by (see (4.26))

$$Q(x) = \sum_{i=0}^{d_P} b_i x^i$$

so that

$$\left(1 + |S| \cdot x + \sum_{i\geq 2} a_i x^i\right) \cdot \left(\sum_{j=0}^{d_P} b_j x^j\right) = \sum_{l=0}^{d_P} c_l x^l.$$

Hence,

$$\sum_{k\geq 0} \left(\sum_{i=0}^{k} a_i b_{k-i}\right) x^k = \sum_{l=0}^{d_P} c_l x^l.$$

Theorem 4.15 *Let G be a Coxeter group acting cocompactly on $\mathbb{H}^n$ with generating set S. Then, the coefficients of its growth series*

$$f_S(x) = 1 + |S| \cdot x + \sum_{i\geq 2} a_i x^i$$

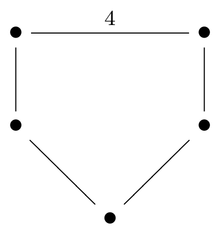
satisfy the following rules.

$$(1) \quad a_0 = 1 \quad , \quad a_1 = |S|$$

$$(2) \quad a_k = \begin{cases} c_k - \sum\limits_{j=1}^{k} a_{k-j} b_j, & \text{for } 2 \leq k \leq d_P, \\ -\sum\limits_{j=1}^{d_P} a_{k-j} b_j, & \text{for } k > d_P, \end{cases}$$

where the coefficients c_k are given by (4.54).

Example 4.7 *Consider the Lannér group G with graph (cf. Table 3 in part 2.4)*



and growth series

$$f_S(x) = \sum_{i \geq 0} a_i x^i = \frac{P(x)}{Q(x)}. \tag{4.55}$$

Obviously, $a_0 = 1$ and $a_1 = 5$. The complete form of $f_S(x)$ can be determined according to (cf. Example 4.3 or [8, Table 7.5])

$$f_S(x) = \frac{[2,5,6,8,12]}{\begin{array}{l}\big(1 - x^2 - x^3 - x^4 - 2x^5 - 2x^6 - x^7 + x^8 + x^9 + 2x^{10} + 2x^{11} + 3x^{12} \\ + 2x^{13} + 3x^{14} + 2x^{15} + 3x^{16} + 2x^{17} + 2x^{18} + x^{19} + x^{20} - x^{21} - 2x^{22} \\ - 2x^{23} - x^{24} - x^{25} - x^{26} + x^{28}\big)\end{array}}, \tag{4.56}$$

which leads, together with (4.55) and (4.56), to

$$[2,5,6,8,12] = \left(\sum_{i \geq 0} a_i x^i \right) \cdot Q(x).$$

Comparing coefficients allows to determine a_i. Theorem 4.15, implemented into a Mathematica program, yields the following values.

- $a_0 = 1$
- $a_1 = 5$
- $a_2 = 15$
- $a_3 = 36$
- $a_4 = 76$
- $a_5 = 148$
- $a_6 = 273$
- $a_7 = 486$
- $a_8 = 843$
- $a_9 = 1'435$
- $a_{10} = 2'410$
- $a_{11} = 4'009$
- $a_{12} = 6'623$
- $a_{13} = 10'887$
- $a_{14} = 17'833$
- $a_{15} = 29'135$
- $a_{16} = 47'511$
- $a_{17} = 77'372$
- $a_{18} = 125'879$
- $a_{19} = 204'652$
- $a_{20} = 332'551$
- $a_{21} = 540'183$
- $a_{22} = 877'221$
- $a_{23} = 1'424'278$
- $a_{24} = 2'312'177$
- $a_{25} = 3'753'224$
- $a_{26} = 6'091'955$
- $a_{27} = 9'887'499$
- $a_{28} = 16'047'226$
- $a_{29} = 26'043'662$
- $a_{30} = 42'266'383$
- $a_{31} = 68'593'441$

Thus,

$$\begin{aligned} f_S(x) \quad = \quad & 1 + 5x + 15x^2 + 36x^3 + 76x^4 + 148x^5 + 273x^6 + 486x^7 + 843x^8 + 1435x^9 \\ & + 2410x^{10} + 4009x^{11} + 6623x^{12} + 10887x^{13} + 17833x^{14} + 29135x^{15} + 47511x^{16} \\ & + 77372x^{17} + 125879x^{18} + 204652x^{19} + 332551x^{20} + 540183x^{21} + 877221x^{22} \\ & + 1424278x^{23} + 2312177x^{24} + 3753224x^{25} + 6091955x^{26} + 9887499x^{27} \\ & + 16047226x^{28} + 26043662x^{29} + 42266383x^{30} + 68593441x^{31} + \dots \end{aligned}$$

```
In[12]:= ClearAll[p, a, b, c, P]

In[13]:= p[n_] := Sum[x^k, {k, 0, n - 1}]

In[14]:= Expand[p[2] * p[5] * p[6] * p[8] * p[12]]
```

$$Out[14]= 1 + 5x + 14x^2 + 30x^3 + 55x^4 + 90x^5 + 134x^6 + 185x^7 + 240x^8 + 295x^9 + 346x^{10} + 390x^{11} + 424x^{12} + 445x^{13} + 452x^{14} + 445x^{15} + 424x^{16} + 390x^{17} + 346x^{18} + 295x^{19} + 240x^{20} + 185x^{21} + 134x^{22} + 90x^{23} + 55x^{24} + 30x^{25} + 14x^{26} + 5x^{27} + x^{28}$$

The numerator of the growth series is given by

```
In[15]:= P[x_] := 1 + 5 x + 14 x^2 + 30 x^3 + 55 x^4 + 90 x^5 + 134 x^6 + 185 x^7 + 240 x^8 + 295 x^9 +
            346 x^10 + 390 x^11 + 424 x^12 + 445 x^13 + 452 x^14 + 445 x^15 + 424 x^16 + 390 x^17 + 346 x^18 +
            295 x^19 + 240 x^20 + 185 x^21 + 134 x^22 + 90 x^23 + 55 x^24 + 30 x^25 + 14 x^26 + 5 x^27 + x^28
```

Q[x] denotes the denominator of the growth series. Let b[i] be the coefficient of Q[x] related to x^i. Then we have

```
In[16]:= {b[0] = 1, b[1] = 0, b[2] = -1, b[3] = -1, b[4] = -1, b[5] = -2, b[6] = -2,
          b[7] = -1, b[8] = 1, b[9] = 1, b[10] = 2, b[11] = 2, b[12] = 3, b[13] = 2, b[14] = 3,
          b[15] = 2, b[16] = 3, b[17] = 2, b[18] = 2, b[19] = 1, b[20] = 1, b[21] = -1,
          b[22] = -2, b[23] = -2, b[24] = -1, b[25] = -1, b[26] = -1, b[27] = 0, b[28] = 1}

Out[16]= {1, 0, -1, -1, -1, -2, -2, -1, 1, 1, 2, 2,
          3, 2, 3, 2, 3, 2, 2, 1, 1, -1, -2, -2, -1, -1, -1, 0, 1}

In[17]:= a[0] = 1

Out[17]= 1
```

We first determine the coefficients a[i] such that i ≤ degree of P[x]. Let us recall that the degree of P[x] equals 28.

```
In[32]:= a[k_] := Coefficient[P[x], x^k] - Sum[a[k - j] * b[j], {j, 1, k}]

In[46]:= {a[1], a[2], a[3], a[4], a[5], a[6], a[7], a[8],
          a[9], a[10], a[11], a[12], a[13], a[14], a[15], a[16]}

Out[46]= {5, 15, 36, 76, 148, 273, 486, 843, 1435, 2410, 4009, 6623, 10887, 17833, 29135, 47511}

In[48]:= {a[17], a[18], a[19], a[20], a[21], a[22], a[23], a[24]}
```

$$Out[48]= \{77372, 125879, 204652, 332551, 540183, 877221, 1.42428\times10^6, 2.31218\times10^6\}$$

```
In[49]:= {a[25], a[26], a[27], a[28]}
```

$$Out[49]= \{3.75322\times10^6, 6.09196\times10^6, 9.8875\times10^6, 1.60472\times10^7\}$$

Let us now determine (some of) the coefficients a[i] such that i > degree of P[x]. In the first part we computed that


```
In[50]:= {a[1] = 5, a[2] = 15, a[3] = 36, a[4] = 76, a[5] = 148, a[6] = 273,
          a[7] = 486, a[8] = 843, a[9] = 1435, a[10] = 2410, a[11] = 4009, a[12] = 6623,
          a[13] = 10887, a[14] = 17833, a[15] = 29135, a[16] = 47511, a[17] = 77372,
          a[18] = 125879, a[19] = 204652, a[20] = 332551, a[21] = 540183, a[22] = 877221,
          a[23] = 1.42428 * 10^6, a[24] = 2.31218 * 10^6, a[25] = 3.75322 * 10^6,
          a[26] = 6.09196 * 10^6, a[27] = 9.8875 * 10^6, a[28] = 1.60472 * 10^7}
```

$$Out[50]= \{5, 15, 36, 76, 148, 273, 486, 843, 1435, 2410, 4009, 6623, 10887, 17833, 29135, 47511, 77372, 125879, 204652, 332551, 540183, 877221, 1.42428\times10^6, 2.31218\times10^6, 3.75322\times10^6, 6.09196\times10^6, 9.8875\times10^6, 1.60472\times10^7\}$$

```
In[51]:= a[k_] := -Sum[a[k - j] * b[j], {j, 1, 28}]

In[52]:= k = 28; While[(k = k + 1) < 40, Print[a[k]]]
```

2.60437×10^7

4.22664×10^7

6.85934×10^7

1.11318×10^8

1.80653×10^8

2.93172×10^8

4.75772×10^8

7.721×10^8

1.25299×10^9

2.03339×10^9

3.29984×10^9

Secondly, consider the growth series $f_S(x)$ of Coxeter groups $G = \langle S \rangle$ acting cocompactly on $\mathbb{H}^4$ with at most six generators. In part 3.4, Theorem 3.7, we described as precisely as possible the growth behaviour. Furthermore, at the end of part 3.4, we pointed out that detailed knowledge about the denominator coefficients is required in order to understand the pole distribution of $f_S(x)$. By Theorem 4.10, we dispose of a convenient recursive tool in order to determine explicitly the denominator of $f_S(x)$ and its poles on the (positive) real axis and in the complex plane. In this way one can check that the positive real poles of $f_S(x)$, for a given G, appear in two inversive pairs and that the non-real poles are distributed in a certain thin annulus around the unit circle.
Let us point out that all these properties remain valid for the Tumarkin groups.

In the same proof, one deduces that the growth rate τ of all these Coxeter groups is a *Perron number*, that is, a real algebraic integer such that all its conjugates are of absolute value less than τ. In this context, recall that in dimensions 2 and 3, the growth rate of any cocompact hyperbolic Coxeter group is a Salem number (see Theorem 3.4). We summarise our results as follows.

Theorem 4.16 *Let G be a Simplex, an Esselmann, a Kaplinskaya or a Tumarkin group acting cocompactly on $\mathbb{H}^4$ with generating set S. Then,*

(1) *the growth series $f_S(x)$ of G possesses four distinct positive real poles appearing in pairs (x_1, x_1^{-1}) and (x_2, x_2^{-1}) with $x_1 < x_2 < 1 < x_2^{-1} < x_1^{-1}$; these poles are simple.*

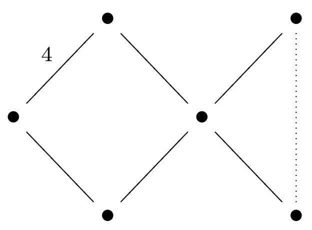
(2) *The growth rate $\tau = x_1^{-1}$ is a Perron number.*

(3) *The non-real poles of $f_S(x)$ are contained in an annulus of radii x_2, x_2^{-1}.*

(4) *The growth series $f_S(x)$ of the Kaplinskaya group with graph K_{66} has four distinct negative and four distinct positive simple real poles ; for $G \neq G_{66}$, $f_S(x)$ has no negative pole.*

Example 4.8 *Consider the Kaplinskaya group G_{44} given by the graph*

K_{44} :



and with generating set S. The growth series of G_{44}, in its complete form, is given by

$$f_S(x) = \frac{[2,4,5,6,8]}{\begin{array}{l}(1 - x - 2x^2 - 3x^3 - 2x^4 - 2x^5 + x^6 + 2x^7 + 7x^8 + 6x^9 + 8x^{10} + 6x^{11} + 7x^{12} \\ +\ 2x^{13} + x^{14} - 2x^{15} - 2x^{16} - 3x^{17} - 2x^{18} - x^{19} + x^{20})\end{array}}$$

and has the simple poles

$$x_1 = 0.406794\ldots \quad , \quad x_2 = 0.787865\ldots \quad , \quad x_2^{-1} = 1.26925\ldots \quad , \quad \tau = x_1^{-1} = 2.45825\ldots$$

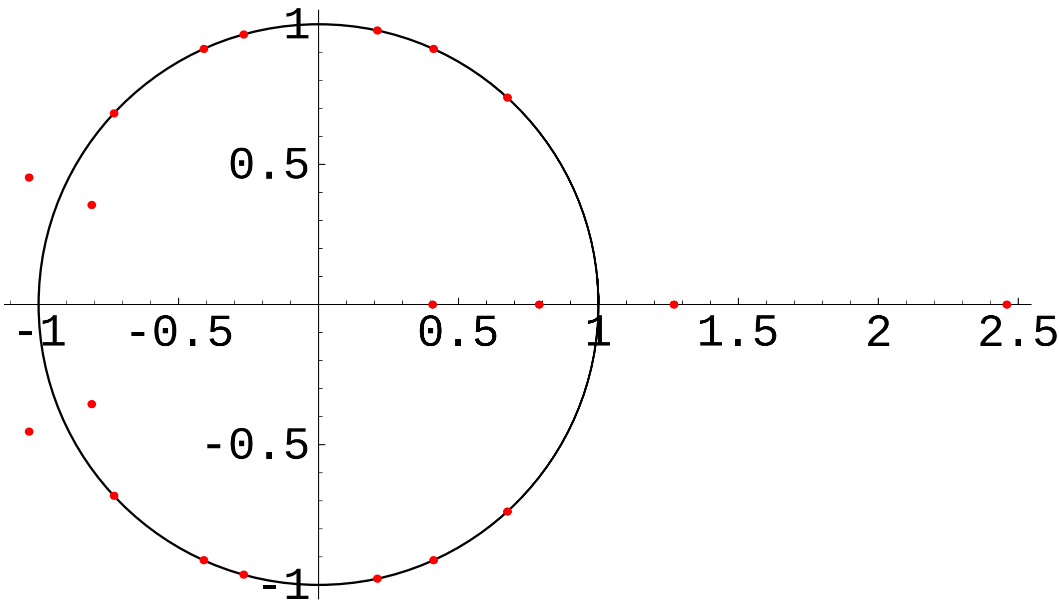


Figure 19: The poles of the growth series of G_{44}.

For the non-real (and real) poles, Theorem 4.16 allows to draw the picture in Figure 19.

5 Outlook

In section 4 we analysed in a systematic way growth properties of Coxeter groups acting cocompactly on $\mathbb{H}^n$. In the sequel, we look at the higher dimensional cases $n \geq 5$ and comment about experimental evidence as well as the *cofinite* case of non-cocompact hyperbolic Coxeter groups with finite volume orbit spaces.

Hyperbolic Coxeter groups in higher dimensions are not classified at all. There are still some families, namely the Kaplinskaya groups [20] and the Tumarkin groups [36] consisting each of a handful of examples, only. Furthermore, in dimension 7, there is a nice, arithmetical example G_B discovered by Bugaenko. Their study leads to a simple Conjecture (see below) about the pole distribution of the growth series of any cocompact Coxeter group in $\mathbb{H}^n$, $n > 3$.

All Coxeter graphs of this section are given in Appendix C.

Let G be a Coxeter group acting on $\mathbb{H}^n$ with finite generating set S and growth series $f_S(x)$. For n odd, by Properties 3.1,

$$\frac{1}{f_S(1)} = \chi(G) = 0,$$

so that $f_S(x)$ has a pole at $x = 1$.

For $n = 5$, we computed the growth series of all Kaplinskaya groups and Tumarkin groups and observed that their growth series possess, beside the pole at 1, exactly two (inversive) pairs of positive poles. Moreover, their non-real poles are contained in an annulus whose radii are given by the intermediate real poles.

For $n = 6$, there are precisely three Tumarkin groups. Kaplinskaya groups, however, do not exist anymore in dimensions beyond 5. Again, we compute the respective growth series and remark that they possess exactly six simple real poles arising in three (inversive) pairs (x_1, x_1^{-1}), (x_2, x_2^{-1}) and (x_3, x_3^{-1}) with the property $0 < x_1 < x_2 < x_3 < 1$. Moreover, the non-real poles are located in an annulus of radii x_2 and x_2^{-1}.

For $n \geq 7$, the situation simplifies drastically since there is only one Tumarkin group left (see Appendix C) which acts on $\mathbb{H}^8$ with eleven generators. This group, as an *arithmetic* singleton, was discovered first by Bugaenko [5]. As anticipated, it turns out that its growth series has exactly four (inversive) pairs of poles $\big(x_1, x_1^{-1}\big)$, $\big(x_2, x_2^{-1}\big)$, $\big(x_3, x_3^{-1}\big)$ and $\big(x_4, x_4^{-1}\big)$ with the property $0 < x_1 < x_2 < x_3 < x_4 < 1$. The non-real poles are contained in the interior of the annulus bounded by concentric circles of radii x_3 and x_3^{-1}. These last computations have been performed by T. Zehrt (unpublished).

The example G_B of Bugaenko, mentionned above and in (2.5) at the end of part 2.2, is the unique cocompact Coxeter group with eleven generators acting on $\mathbb{H}^7$. The poles of its growth series comprise 1 and three (inversive) pairs (x_1, x_1^{-1}), (x_2, x_2^{-1}) and (x_3, x_3^{-1}) with $0 < x_1 < x_2 < x_3 < 1$. Its non-real poles are contained in the annulus of radii x_3 and x_3^{-1}. A picture of the pole distribution in the complex plane is shown below.

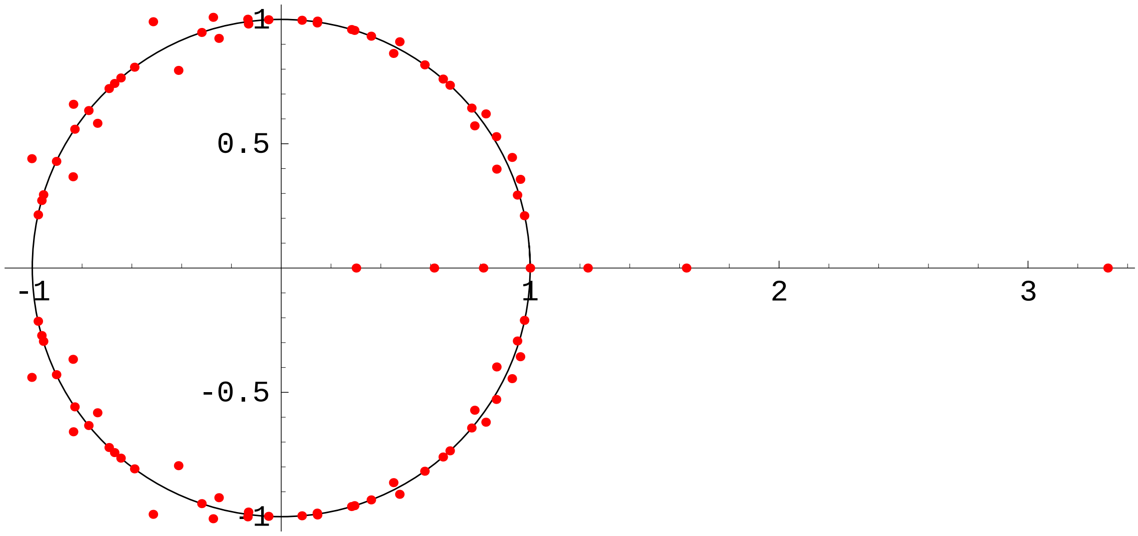


Figure 20: The poles of the growth series of G_B.

Example 5.1 *Consider the Kaplinskaya group G_K acting on $\mathbb{H}^5$ with generating set S and graph*

$$\Gamma_K : \bullet \overset{5}{\text{———}} \bullet \text{———} \bullet \text{———} \bullet \text{———} \bullet \text{———} \bullet \cdots\cdots \bullet \, .$$

By means of Steinberg's formula (3.4), one computes

$$f_S(x) = \frac{[2,2,12,20,30]}{\begin{aligned}&1 - 2x + x^3 - x^7 + x^8 + x^{10} - 2x^{11} + x^{12} + x^{13} - x^{15} + x^{16} - 2x^{17} + 3x^{18} - x^{19}\\ &- x^{21} + 2x^{22} - x^{23} + x^{24} - 2x^{25} + 2x^{26} - 2x^{27} + 3x^{28} - 2x^{29} + 2x^{32} - 3x^{33}\\ &+ 2x^{34} - 2x^{35} + 2x^{36} - x^{37} + x^{38} - 2x^{39} + x^{40} + x^{42} - 3x^{43} + 2x^{44} - x^{45} + x^{46}\\ &- x^{48} - x^{49} + 2x^{50} - x^{51} - x^{53} + x^{54} - x^{58} + 2x^{60} - x^{61}.\end{aligned}}$$

Then $f_S(x)$ behaves as follows for $x \in \mathbb{R}$.

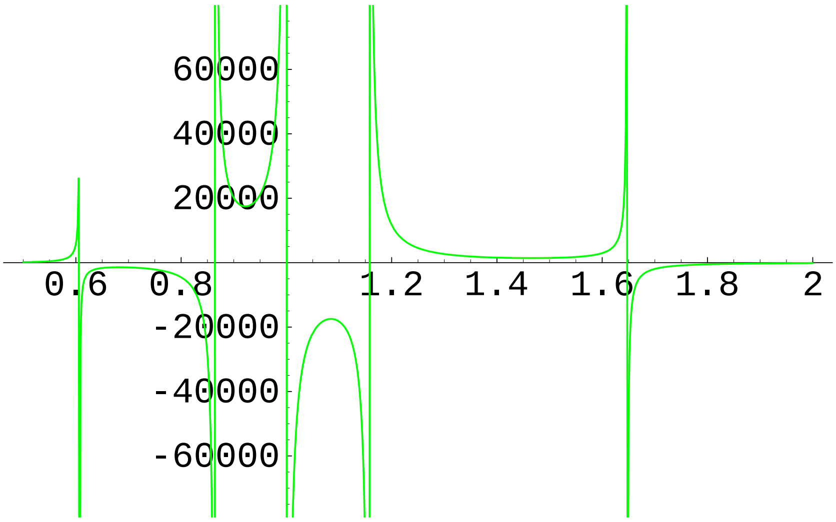


Figure 21: The growth series of G_K restricted to $\mathbb{R}_+$.

We would like to add that the example $G_K \in I(\mathbb{H}^5)$ is conjectured to be of minimal covolume among all *discrete groups in $I(\mathbb{H}^5)$.*

Based on these experimental data, which cover all *known* examples of cocompact Coxeter groups up to dimension 8, apart from garland or similar constructions, it is natural to formulate the following conjecture.

Conjecture 2 *Let G be a cocompact Coxeter group in $I(\mathbb{H}^n)$ with generating set S and growth series $f_S(x)$. Then,*

(a) *for n even, $f_S(x)$ has precisely $\frac{n}{2}$ pairs of poles (x_i, x_i^{-1}) with $0 < x_1 < \ldots < x_{\frac{n}{2}} < 1$.*

(b) *for n odd, $f_S(x)$ has precisely the pole 1 and $\frac{n-1}{2}$ pairs of poles (x_i, x_i^{-1}) with $0 < x_1 < \ldots < x_{\frac{n-1}{2}} < 1$.*

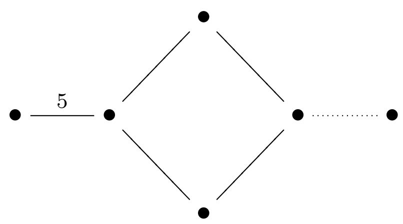
In both cases, the poles are simple, and the non-real poles of $f_S(x)$ are contained in the annulus of radii $x_\star$ and $x_\star^{-1}$ for some $\star \in \{1, \ldots, [\frac{n}{2}]\}$.

At the end of this work, we add some comments about a possible generalisation of Theorem 4.10 to the non-cocompact case. Let G be a *cofinite* hyperbolic Coxeter group, that is, the quotient space $\mathbb{H}^n/G$ is non-compact, but of finite volume. Such cofinite groups exist at most up to dimension 995, a result due to Vinberg, Prokhorov and Khovanskij (see [30] and [22]). However, examples are known only up to dimension 21. In contrast to cocompact groups, a cofinite one contains at least one euclidean subgroup. Suppose that G has finite generating set S and growth series $f_S(x)$. For the computation of $f_S(x)$, the presence of infinite subgroups is of no influence (see Steinberg's formula (3.4)). As a consequence, the complete form of $f_S(x)$ is not affected by the presence of euclidean subgroups as well. Since all this is at the basis of the proof, one is tempted to try to generalise Theorem 4.10 to the cofinite case right away. But, there is a delicate point : the growth series is not reciprocal (or antireciprocal) !

The following example illustrates the differences between the cocompact and the cofinite case.

Example 5.2 *Consider the cofinite Kaplinskaya group G_K, with generating set S and graph*

Γ_K :



acting on $\mathbb{H}^4$. Observe that Γ_K contains the euclidean subgraph $\tilde{A}_3$. By means of Steinberg's formula (3.4), one gets

$$f_S(x) = \frac{[2,12,20,30]}{\begin{aligned}&1 - 2x - 2x^3 + 2x^4 - 3x^5 + 3x^6 - 5x^7 + 7x^8 - 7x^9 + 9x^{10} - 9x^{11} + 14x^{12}\\ &- 11x^{13} + 18x^{14} - 14x^{15} + 22x^{16} - 17x^{17} + 26x^{18} - 19x^{19} + 30x^{20} - 21x^{21}\\ &+ 34x^{22} - 23x^{23} + 36x^{24} - 25x^{25} + 39x^{26} - 26x^{27} + 39x^{28} - 26x^{29} + 41x^{30}\\ &- 26x^{31} + 40x^{32} - 26x^{33} + 38x^{34} - 25x^{35} + 37x^{36} - 23x^{37} + 33x^{38} - 21x^{39}\\ &+ 31x^{40} - 19x^{41} + 26x^{42} - 17x^{43} + 22x^{44} - 14x^{45} + 18x^{46} - 11x^{47} + 14x^{48}\\ &- 9x^{49} + 10x^{50} - 7x^{51} + 6x^{52} - 5x^{53} + 4x^{54} - 3x^{55} + x^{56} - 2x^{57} + x^{58} - 2x^{59}.\end{aligned}}$$

Here, the numerator and the denominator of $f_S(x)$ are of different degrees (cf. also [8, Chapter 5.4]). The poles of $f_S(x)$ are distributed in the complex plane as follows.

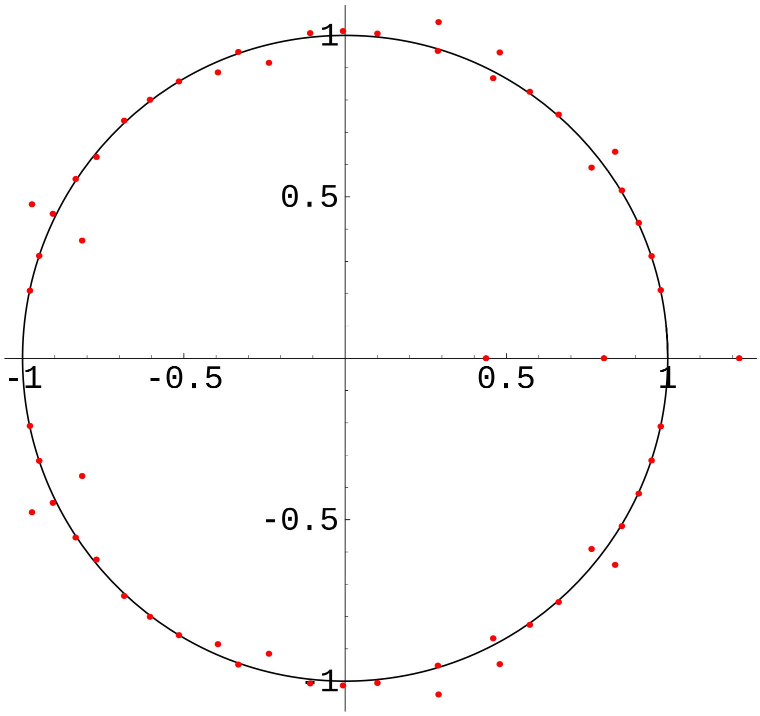


Figure 22: The poles of the growth series of G_K.

One sees that $f_S(x)$ possesses only three real poles, two of them form an inversive pair (x_2, x_2^{-1}). For the non-real poles, they do not come in 4-tuples anymore, but lie still in an annulus of radii x_2 and x_2^{-1} around the unit circle.

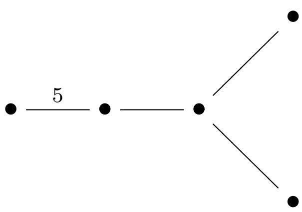
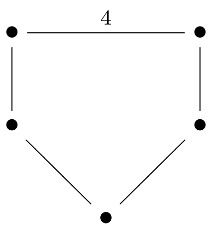
A The growth series of the Simplex, Esselmann and Kaplinskaya groups

In this appendix we present explicitly the complete forms of the growth series of all cocompact Coxeter groups in $I(\mathbb{H}^4)$ generated by at most six reflections, namely the Simplex, Esselmann and Kaplinskaya groups (see [6], [8], for example). Let us recall that the complete form is not necessarily a reduced form.

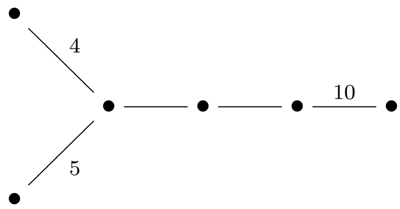
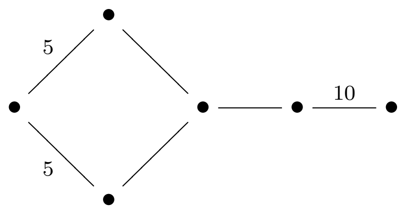
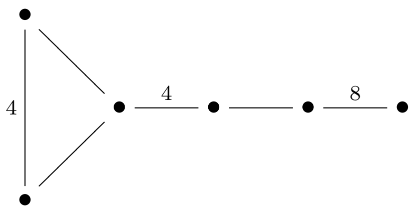
Growth series of the Simplex groups

Graph	$f_S(x)$ in complete form
	$\dfrac{[2,12,20,30]}{\begin{array}{l}1 - x - x^7 + x^8 - x^9 + x^{10} - x^{11} + x^{14} - x^{15} + x^{16}\\ - 2x^{17} + 2x^{18} - x^{19} + x^{20} - x^{21} + x^{22} - x^{23} + 2x^{24}\\ - 2x^{25} + 2x^{26} - 2x^{27} + 2x^{28} - x^{29} + x^{30} - x^{31} + 2x^{32}\\ - 2x^{33} + 2x^{34} - 2x^{35} + 2x^{36} - x^{37} + x^{38} - x^{39} + x^{40}\\ - x^{41} + 2x^{42} - 2x^{43} + x^{44} - x^{45} + x^{46} - x^{49} + x^{50}\\ - x^{51} + x^{52} - x^{53} - x^{59} + x^{60}\end{array}}$
	$\dfrac{[2,8,12,20,30]}{\begin{array}{l}1 - x - x^3 + 2x^4 - 2x^5 + x^6 - 3x^7 + 3x^8 - 3x^9\\ + 3x^{10} - 5x^{11} + 5x^{12} - 5x^{13} + 6x^{14} - 7x^{15} + 8x^{16}\\ - 8x^{17} + 9x^{18} - 9x^{19} + 11x^{20} - 11x^{21} + 12x^{22}\\ - 11x^{23} + 14x^{24} - 13x^{25} + 14x^{26} - 13x^{27} + 16x^{28}\\ - 14x^{29} + 15x^{30} - 14x^{31} + 17x^{32} - 14x^{33} + 15x^{34}\\ - 14x^{35} + 16x^{36} - 13x^{37} + 14x^{38} - 13x^{39} + 14x^{40}\\ - 11x^{41} + 12x^{42} - 11x^{43} + 11x^{44} - 9x^{45} + 9x^{46}\\ - 8x^{47} + 8x^{48} - 7x^{49} + 6x^{50} - 5x^{51} + 5x^{52} - 5x^{53}\\ + 3x^{54} - 3x^{55} + 3x^{56} - 3x^{57} + x^{58} - 2x^{59} + 2x^{60}\\ - x^{61} - x^{63} + x^{64}\end{array}}$

continues on the following page

Graph	$f_S(x)$ in complete form
	$\frac{[2,12,20,30]}{1 - x - x^3 - x^7 - x^9 + 2x^{10} - 2x^{11} + 2x^{12} - 2x^{13} + 2x^{14} + 2x^{16} - 2x^{17} + 2x^{18} - 2x^{19} + 6x^{20} - 3x^{21} + 4x^{22} - 3x^{23} + 4x^{24} + 4x^{26} - 3x^{27} + 4x^{28} - 3x^{29} + 8x^{30} - 3x^{31} + 4x^{32} - 3x^{33} + 4x^{34} + 4x^{36} - 3x^{37} + 4x^{38} - 3x^{39} + 6x^{40} - 2x^{41} + 2x^{42} - 2x^{43} + 2x^{44} + 2x^{46} - 2x^{47} + 2x^{48} - 2x^{49} + 2x^{50} - x^{51} - x^{53} - x^{57} - x^{59} + x^{60}}$
	$\frac{[2,12,20,30]}{1 - x - x^3 + x^4 - x^5 - x^7 + x^8 - x^9 + x^{11} + 2x^{12} - x^{13} + x^{14} - x^{15} + 2x^{16} - x^{17} + x^{18} - x^{19} + 3x^{20} - x^{21} + 2x^{22} - x^{23} + 3x^{24} - x^{25} + 2x^{26} - x^{27} + 3x^{28} - x^{29} + 3x^{30} - x^{31} + 3x^{32} - x^{33} + 2x^{34} - x^{35} + 3x^{36} - x^{37} + 2x^{38} - x^{39} + 3x^{40} - x^{41} + x^{42} - x^{43} + 2x^{44} - x^{45} + x^{46} - x^{47} + 2x^{48} - x^{49} - x^{51} + x^{52} - x^{53} - x^{55} + x^{56} - x^{57} - x^{59} + x^{60}}$
	$\frac{[2,5,6,8,12]}{1 - x^2 - x^3 - x^4 - 2x^5 - 2x^6 - x^7 + x^8 + x^9 + 2x^{10} + 2x^{11} + 3x^{12} + 2x^{13} + 3x^{14} + 2x^{15} + 3x^{16} + 2x^{17} + 2x^{18} + x^{19} + x^{20} - x^{21} - 2x^{22} - 2x^{23} - x^{24} - x^{25} - x^{26} + x^{28}}$

Growth series of the Esselmann groups

Graph	$f_S(x)$ in complete form
	$\dfrac{[2,8,12,20,30]}{1 - x - 3x^3 - 4x^5 + x^6 - 6x^7 + 2x^8 - 6x^9 + 6x^{10} - 5x^{11} + 10x^{12} - 4x^{13} + 14x^{14} + 19x^{16} + 3x^{17} + 22x^{18} + 6x^{19} + 27x^{20} + 11x^{21} + 30x^{22} + 14x^{23} + 32x^{24} + 20x^{25} + 35x^{26} + 24x^{27} + 34x^{28} + 26x^{29} + 36x^{30} + 31x^{31} + 35x^{32} + 32x^{33} + 32x^{34} + 35x^{35} + 31x^{36} + 36x^{37} + 26x^{38} + 34x^{39} + 24x^{40} + 35x^{41} + 20x^{42} + 32x^{43} + 14x^{44} + 30x^{45} + 11x^{46} + 27x^{47} + 6x^{48} + 22x^{49} + 3x^{50} + 19x^{51} + 14x^{53} - 4x^{54} + 10x^{55} - 5x^{56} + 6x^{57} - 6x^{58} + 2x^{59} - 6x^{60} + x^{61} - 4x^{62} - 3x^{64} - x^{66} + x^{67}}$
	$\dfrac{[2,12,20,30]}{1 - 2x - 2x^3 + x^4 - x^5 + x^6 - 2x^7 + 3x^8 - 2x^9 + 5x^{10} - 3x^{11} + 8x^{12} - 3x^{13} + 8x^{14} - x^{15} + 11x^{16} - 3x^{17} + 11x^{18} - 3x^{19} + 18x^{20} - 4x^{21} + 16x^{22} - 4x^{23} + 18x^{24} - x^{25} + 18x^{26} - 4x^{27} + 19x^{28} - 4x^{29} + 23x^{30} - 4x^{31} + 19x^{32} - 4x^{33} + 18x^{34} - x^{35} + 18x^{36} - 4x^{37} + 16x^{38} - 4x^{39} + 18x^{40} - 3x^{41} + 11x^{42} - 3x^{43} + 11x^{44} - x^{45} + 8x^{46} - 3x^{47} + 8x^{48} - 3x^{49} + 5x^{50} - 2x^{51} + 3x^{52} - 2x^{53} + x^{54} - x^{55} + x^{56} - 2x^{57} - 2x^{59} + x^{60}}$
	$\dfrac{[2,6,8,12]}{1 - 2x - x^3 + x^4 - 3x^5 + 3x^6 - 2x^7 + 6x^8 - 4x^9 + 6x^{10} - 3x^{11} + 7x^{12} - 3x^{13} + 6x^{14} - 4x^{15} + 6x^{16} - 2x^{17} + 3x^{18} - 3x^{19} + x^{20} - x^{21} - 2x^{23} + x^{24}}$

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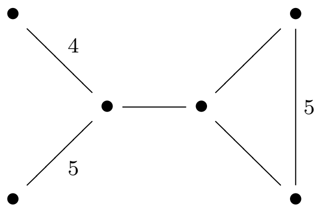
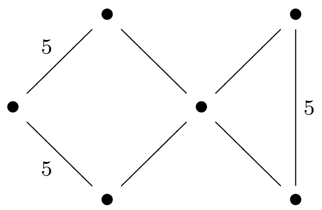
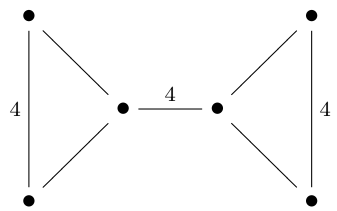
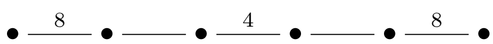
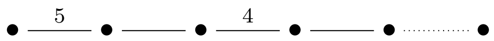
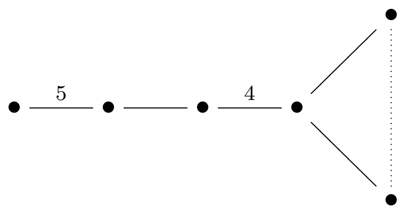
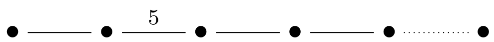
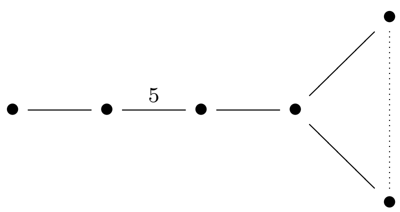
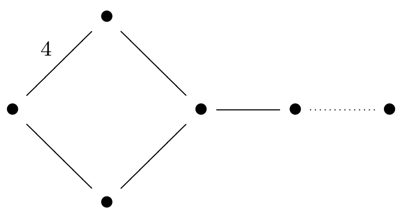
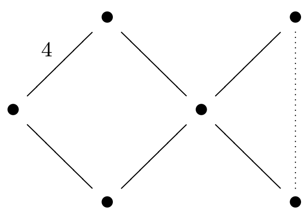
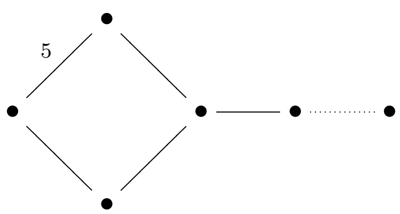
Graph	$f_S(x)$ in complete form
	$\dfrac{[2,8,12,20,30]}{1-x-x^2-3x^3-2x^4-2x^5-2x^6-3x^7-2x^8+2x^{10}+4x^{11}+7x^{12}+8x^{13}+13x^{14}+18x^{15}+21x^{16}+24x^{17}+26x^{18}+30x^{19}+36x^{20}+39x^{21}+42x^{22}+43x^{23}+48x^{24}+53x^{25}+56x^{26}+58x^{27}+56x^{28}+58x^{29}+64x^{30}+65x^{31}+65x^{32}+63x^{33}+63x^{34}+65x^{35}+65x^{36}+64x^{37}+58x^{38}+56x^{39}+58x^{40}+56x^{41}+53x^{42}+48x^{43}+43x^{44}+42x^{45}+39x^{46}+36x^{47}+30x^{48}+26x^{49}+24x^{50}+21x^{51}+18x^{52}+13x^{53}+8x^{54}+7x^{55}+4x^{56}+2x^{57}-2x^{59}-3x^{60}-2x^{61}-2x^{62}-2x^{63}-3x^{64}-x^{65}-x^{66}+x^{67}}$
	$\dfrac{[2,12,20,30]}{1-2x-x^2-2x^3+x^4+x^5-x^6+2x^8+x^9+4x^{10}+x^{11}+9x^{12}+2x^{13}+8x^{14}+8x^{15}+10x^{16}+6x^{17}+9x^{18}+7x^{19}+22x^{20}+7x^{21}+16x^{22}+8x^{23}+19x^{24}+15x^{25}+17x^{26}+10x^{27}+19x^{28}+10x^{29}+28x^{30}+10x^{31}+19x^{32}+10x^{33}+17x^{34}+15x^{35}+19x^{36}+8x^{37}+16x^{38}+7x^{39}+22x^{40}+7x^{41}+9x^{42}+6x^{43}+10x^{44}+8x^{45}+8x^{46}+2x^{47}+9x^{48}+x^{49}+4x^{50}+x^{51}+2x^{52}-x^{54}+x^{55}+x^{56}-2x^{57}-x^{58}-2x^{59}+x^{60}}$
	$\dfrac{[2,6,8,12]}{1-2x-x^2-2x^5+x^6+2x^7+4x^8+3x^{10}+2x^{11}+6x^{12}+2x^{13}+3x^{14}+4x^{16}+2x^{17}+x^{18}-2x^{19}-x^{22}-2x^{23}+x^{24}}$
	$\dfrac{[2,6,8,8,12]}{1-2x-2x^3-4x^5-4x^7+2x^8-2x^9+5x^{10}+x^{11}+7x^{12}+5x^{13}+8x^{14}+6x^{15}+6x^{16}+8x^{17}+5x^{18}+7x^{19}+x^{20}+5x^{21}-2x^{22}+2x^{23}-4x^{24}-4x^{26}-2x^{28}-x^{30}+x^{31}}$

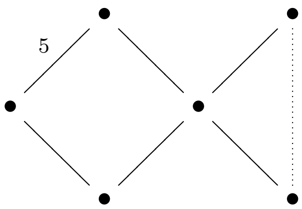
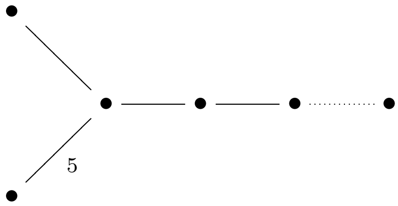
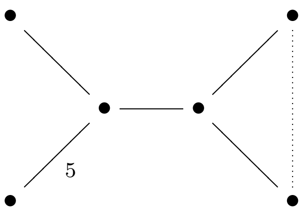
Growth series of the Kaplinskaya groups

Graph	$f_S(x)$ in complete form
	$\frac{[4,6,8,10,12]}{1 - x + x^2 - 3x^3 - 5x^5 - 6x^7 + x^8 - 5x^9 + 3x^{10} - 2x^{11} + 6x^{12} + 3x^{13} + 10x^{14} + 8x^{15} + 12x^{16} + 11x^{17} + 11x^{18} + 12x^{19} + 8x^{20} + 10x^{21} + 3x^{22} + 6x^{23} - 2x^{24} + 3x^{25} - 5x^{26} + x^{27} - 6x^{28} - 5x^{30} - 3x^{32} + x^{33} - x^{34} + x^{35}}$
	$\frac{[4,6,8,10,12]}{1 - x - 3x^3 - 3x^4 - 5x^5 - 4x^6 - 6x^7 - 2x^8 - 4x^9 + 2x^{10} + x^{11} + 8x^{12} + 9x^{13} + 16x^{14} + 17x^{15} + 21x^{16} + 21x^{17} + 21x^{18} + 21x^{19} + 17x^{20} + 16x^{21} + 9x^{22} + 8x^{23} + x^{24} + 2x^{25} - 4x^{26} - 2x^{27} - 6x^{28} - 4x^{29} - 5x^{30} - 3x^{31} - 3x^{32} - x^{34} + x^{35}}$
	$\frac{[2,12,20,30]}{1 - 2x + x^2 - x^3 - x^5 + 2x^6 - 3x^7 + 2x^8 - x^9 + x^{10} - 2x^{11} + 4x^{12} - 3x^{13} + 3x^{14} - x^{15} + 3x^{16} - 3x^{17} + 6x^{18} - 4x^{19} + 5x^{20} - x^{21} + 4x^{22} - 3x^{23} + 8x^{24} - 5x^{25} + 6x^{26} - x^{27} + 5x^{28} - 4x^{29} + 9x^{30} - 4x^{31} + 5x^{32} - x^{33} + 6x^{34} - 5x^{35} + 8x^{36} - 3x^{37} + 4x^{38} - x^{39} + 5x^{40} - 4x^{41} + 6x^{42} - 3x^{43} + 3x^{44} - x^{45} + 3x^{46} - 3x^{47} + 4x^{48} - 2x^{49} + x^{50} - x^{51} + 2x^{52} - 3x^{53} + 2x^{54} - x^{55} - x^{57} + x^{58} - 2x^{59} + x^{60}}$

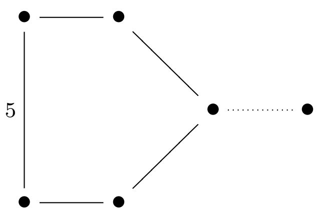
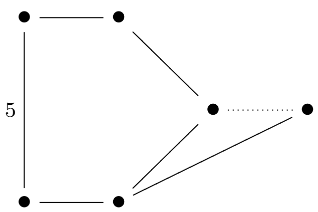
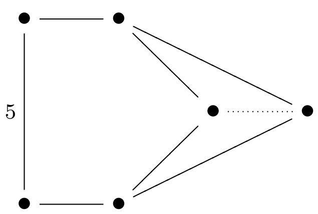
continues on the following page

Graph	$f_S(x)$ in complete form
	$\frac{[2,12,20,30]}{1-2x-x^4-x^5+2x^6-3x^7+x^8+x^9-x^{10}-x^{11} + 5x^{12}-3x^{13}+3x^{14}+2x^{15}+x^{16}-x^{17}+7x^{18} - 3x^{19}+5x^{20}+3x^{21}+3x^{22}-x^{23}+10x^{24}-3x^{25} + 5x^{26}+4x^{27}+4x^{28}-2x^{29}+12x^{30}-2x^{31}+4x^{32} + 4x^{33}+5x^{34}-3x^{35}+10x^{36}-x^{37}+3x^{38}+3x^{39} + 5x^{40}-3x^{41}+7x^{42}-x^{43}+x^{44}+2x^{45}+3x^{46} - 3x^{47}+5x^{48}-x^{49}-x^{50}+x^{51}+x^{52}-3x^{53}+2x^{54} - x^{55}-x^{56}-2x^{59}+x^{60}}$
	$\frac{[2,4,5,6,8]}{1-x-x^2-2x^3-x^4-2x^5+x^6+x^7+4x^8+3x^9 + 5x^{10}+3x^{11}+4x^{12}+x^{13}+x^{14}-2x^{15}-x^{16}-2x^{17} - x^{18}-x^{19}+x^{20}}$
	$\frac{[2,4,5,6,8]}{1-x-2x^2-3x^3-2x^4-2x^5+x^6+2x^7+7x^8+6x^9 + 8x^{10}+6x^{11}+7x^{12}+2x^{13}+x^{14}-2x^{15}-2x^{16} - 3x^{17}-2x^{18}-x^{19}+x^{20}}$
	$\frac{[2,12,20,30]}{1-2x-x^3+x^4-x^5+x^6-2x^7+3x^8-2x^9+3x^{10} - 2x^{11}+5x^{12}-x^{13}+5x^{14}--2x^{15}+8x^{16}-3x^{17} + 8x^{18}-2x^{19}+10x^{20}-2x^{21}+10x^{22}-2x^{23}+12x^{24} - 3x^{25}+12x^{26}-3x^{27}+13x^{28}-2x^{29}+12x^{30}-2x^{31} + 13x^{32}-3x^{33}+12x^{34}-3x^{35}+12x^{36}-2x^{37} + 10x^{38}-2x^{39}+10x^{40}-2x^{41}+8x^{42}-3x^{43}+8x^{44} - 2x^{45}+5x^{46}-x^{47}+5x^{48}-2x^{49}+3x^{50}-2x^{51} + 3x^{52}-2x^{53}+x^{54}-x^{55}+x^{56}-x^{57}-2x^{59}+x^{60}}$

continues on the following page

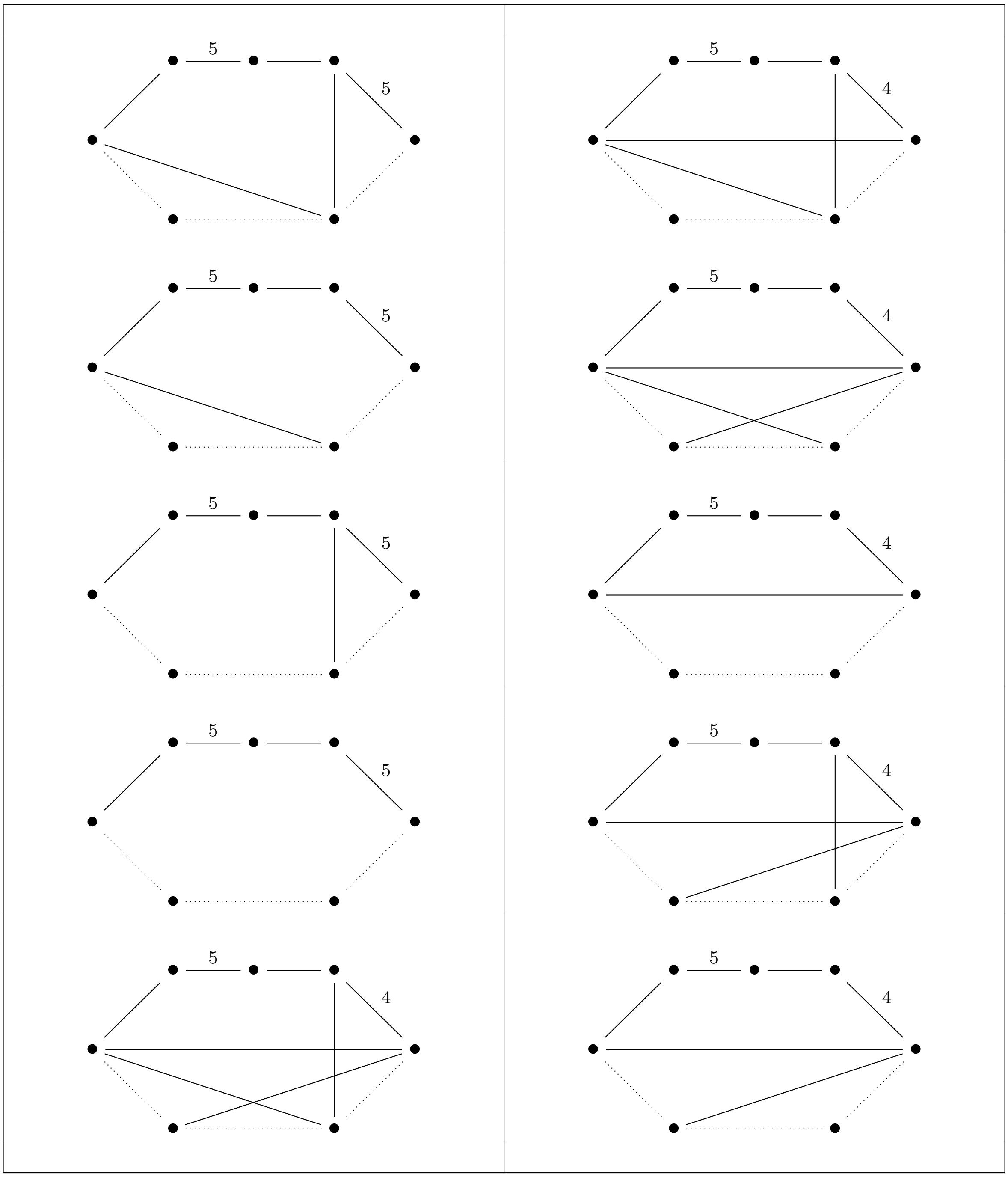
Graph	$f_S(x)$ in complete form
	$[2,12,20,30]$ $1 - 2x - x^2 - x^3 + x^4 - 2x^7 + 4x^8 - x^9 + 3x^{10} - x^{11} + 7x^{12} + x^{13} + 7x^{14} + 11x^{16} - x^{17} + 11x^{18} + x^{19} + 15x^{20} + x^{21} + 14x^{22} + 2x^{23} + 18x^{24} + 17x^{26} + x^{27} + 19x^{28} + 2x^{29} + 18x^{30} + 2x^{31} + 19x^{32} + x^{33} + 17x^{34} + 18x^{36} + 2x^{37} + 14x^{38} + x^{39} + 15x^{40} + x^{41} + 11x^{42} - x^{43} + 11x^{44} + 7x^{46} + x^{47} + 7x^{48} - x^{49} + 3x^{50} - x^{51} + 4x^{52} - 2x^{53} + x^{56} - x^{57} - x^{58} - 2x^{59} + x^{60}$
	$[2,12,20,30]$ $1 - 2x + x^2 - 2x^3 + 2x^4 - 3x^5 + 4x^6 - 6x^7 + 7x^8 - 8x^9 + 10x^{10} - 11x^{11} + 13x^{12} - 13x^{13} + 17x^{14} - 17x^{15} + 21x^{16} - 21x^{17} + 25x^{18} - 23x^{19} + 28x^{20} - 26x^{21} + 31x^{22} - 28x^{23} + 34x^{24} - 31x^{25} + 36x^{26} - 32x^{27} + 37x^{28} - 32x^{29} + 37x^{30} - 32x^{31} + 37x^{32} - 32x^{33} + 36x^{34} - 31x^{35} + 34x^{36} - 28x^{37} + 31x^{38} - 26x^{39} + 28x^{40} - 23x^{41} + 25x^{42} - 21x^{43} + 21x^{44} - 17x^{45} + 17x^{46} - 13x^{47} + 13x^{48} - 11x^{49} + 10x^{50} - 8x^{51} + 7x^{52} - 6x^{53} + 4x^{54} - 3x^{55} + 2x^{56} - 2x^{57} + x^{58} - 2x^{59} + x^{60}$
	$[2,12,20,30]$ $1 - 2x - x^3 - 2x^7 + 2x^8 - x^9 + 2x^{10} - x^{11} + 2x^{12} + x^{13} + 4x^{14} + 4x^{16} - x^{17} + 6x^{18} + x^{19} + 6x^{20} + x^{21} + 6x^{22} + 2x^{23} + 8x^{24} + 8x^{26} + x^{27} + 8x^{28} + 2x^{29} + 8x^{30} + 2x^{31} + 8x^{32} + x^{33} + 8x^{34} + 8x^{36} + 2x^{37} + 6x^{38} + x^{39} + 6x^{40} + x^{41} + 6x^{42} - x^{43} + 4x^{44} + 4x^{46} + x^{47} + 2x^{48} - x^{49} + 2x^{50} - x^{51} + 2x^{52} - 2x^{53} - x^{57} - 2x^{59} + x^{60}$

continues on the following page

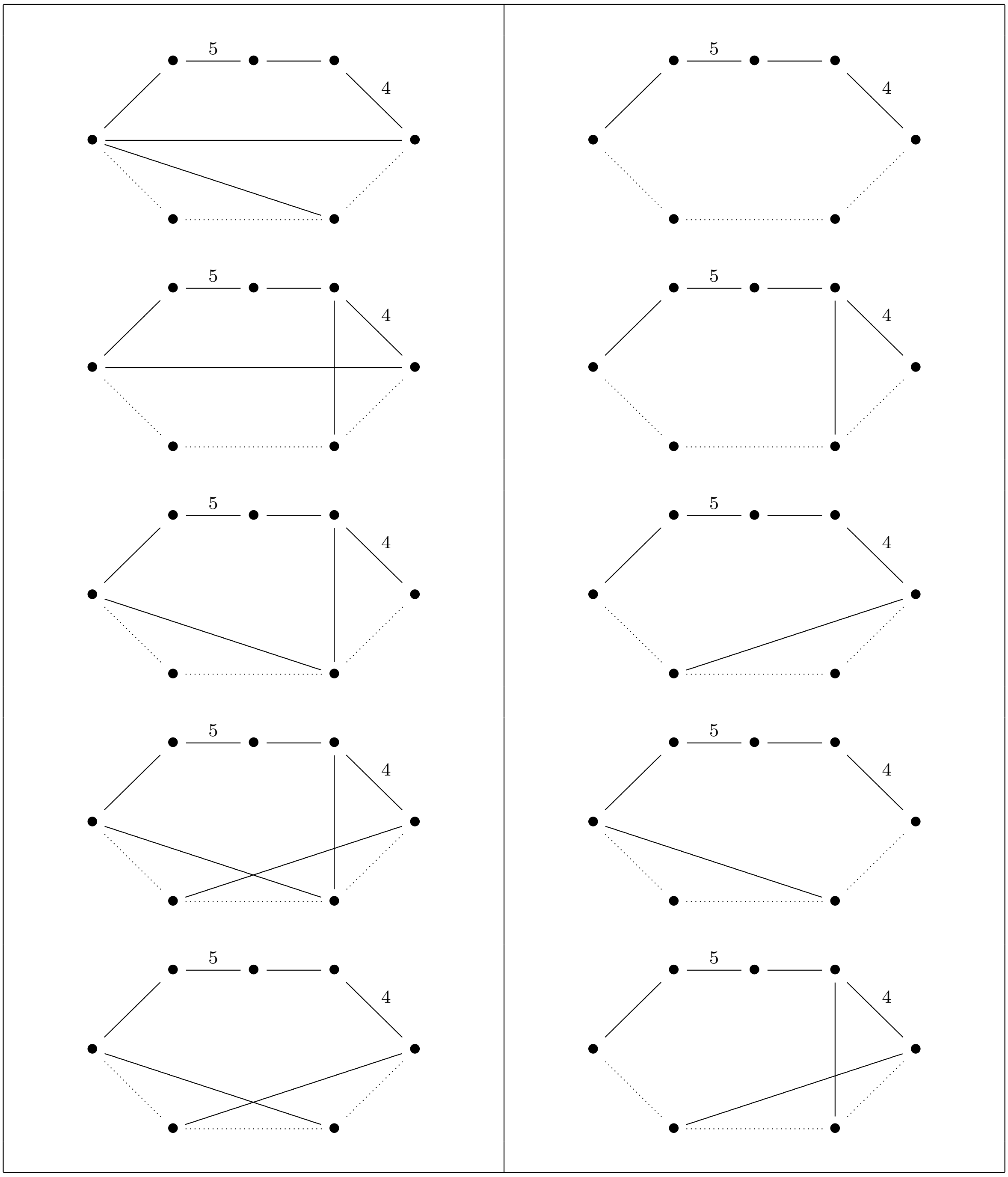
Graph	$f_S(x)$ in complete form
	$\dfrac{[2,12,20,30]}{\begin{array}{l} 1-2x-x^{3}-2x^{7}+2x^{8}-x^{9}+2x^{10}-x^{11}+2x^{12}+x^{13} \\ +4x^{14}+4x^{16}-x^{17}+6x^{18}+x^{19}+6x^{20}+x^{21}+6x^{22} \\ +2x^{23}+8x^{24}+8x^{26}+x^{27}+8x^{28}+2x^{29}+8x^{30}+2x^{31} \\ +8x^{32}+x^{33}+8x^{34}+8x^{36}+2x^{37}+6x^{38}+x^{39}+6x^{40} \\ +x^{41}+6x^{42}-x^{43}+4x^{44}+4x^{46}+x^{47}+2x^{48}-x^{49} \\ +2x^{50}-x^{51}+2x^{52}-2x^{53}-x^{57}-2x^{59}+x^{60} \end{array}}$
	$\dfrac{[2,12,20,30]}{\begin{array}{l} 1-2x-x^{2}-x^{4}-2x^{7}+x^{8}+x^{9}+3x^{12}+x^{13}+4x^{14} \\ +3x^{15}+2x^{16}+x^{17}+7x^{18}+2x^{19}+6x^{20}+5x^{21}+5x^{22} \\ +4x^{23}+10x^{24}+2x^{25}+7x^{26}+6x^{27}+7x^{28}+4x^{29}+11x^{30} \\ +4x^{31}+7x^{32}+6x^{33}+7x^{34}+2x^{35}+10x^{36}+4x^{37}+5x^{38} \\ +5x^{39}+6x^{40}+2x^{41}+7x^{42}+x^{43}+2x^{44}+3x^{45}+4x^{46} \\ +x^{47}+3x^{48}+x^{51}+x^{52}-2x^{53}-x^{56}-x^{58}-2x^{59}+x^{60} \end{array}}$
	$\dfrac{[2,12,20,30]}{\begin{array}{l} 1-2x-2x^{2}-x^{4}+x^{5}-2x^{6}-x^{7}+x^{8}+x^{9}+x^{10}+x^{11} \\ +x^{12}+5x^{13}+5x^{14}+4x^{15}+3x^{16}+3x^{17}+7x^{18}+7x^{19} \\ +7x^{20}+7x^{21}+7x^{22}+9x^{23}+10x^{24}+7x^{25}+9x^{26}+8x^{27} \\ +10x^{28}+10x^{29}+10x^{30}+10x^{31}+10x^{32}+8x^{33}+9x^{34} \\ +7x^{35}+10x^{36}+9x^{37}+7x^{38}+7x^{39}+7x^{40}+7x^{41}+7x^{42} \\ +3x^{43}+3x^{44}+4x^{45}+5x^{46}+5x^{47}+x^{48}+x^{49}+x^{50} \\ +x^{51}+x^{52}-x^{53}-2x^{54}+x^{55}-x^{56}-2x^{58}-2x^{59}+x^{60} \end{array}}$

B The 4-dimensional Tumarkin polytopes

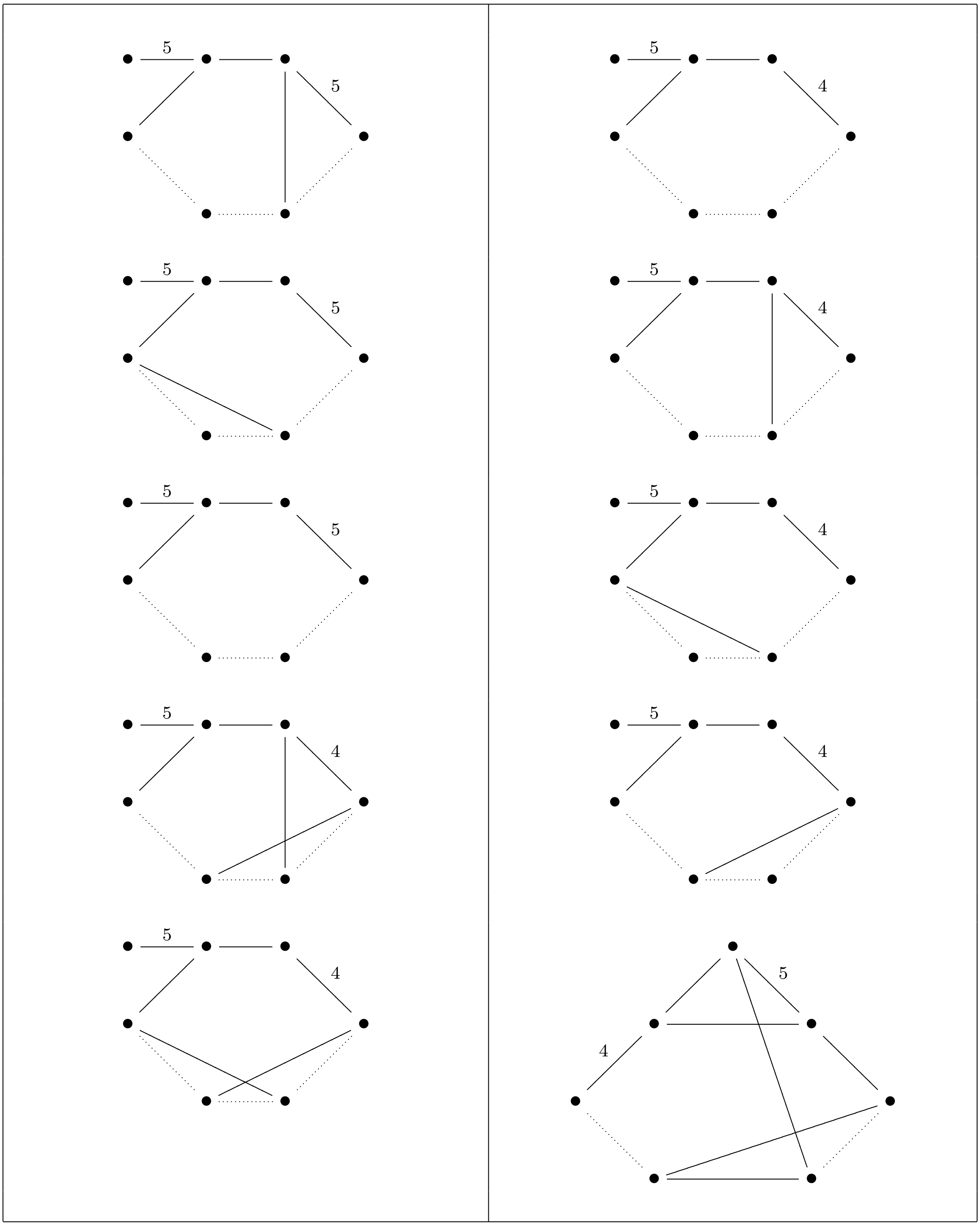
In this appendix we list the Coxeter graphs of the 4-dimensional (compact) Tumarkin polytopes.



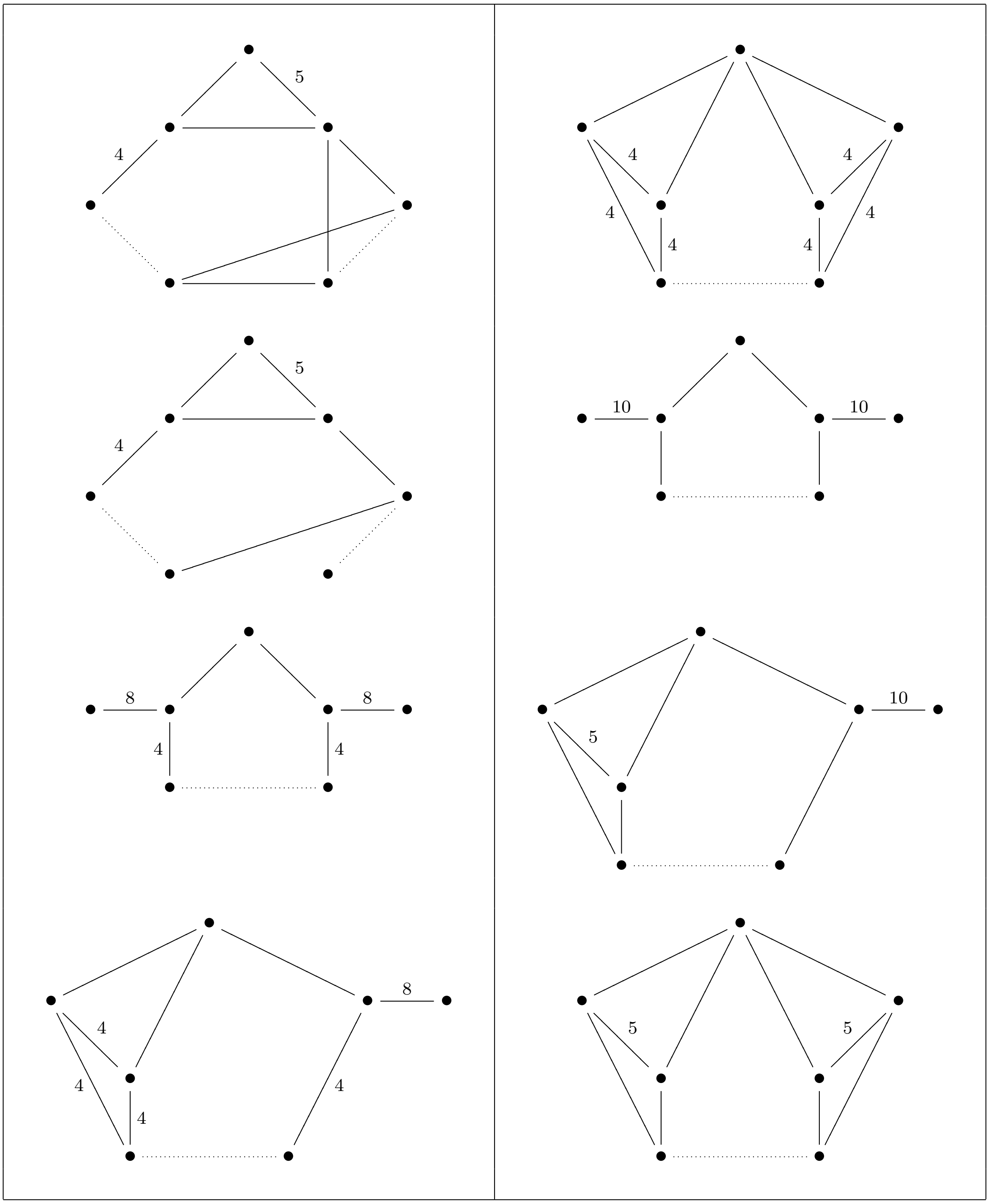
continues on the following page



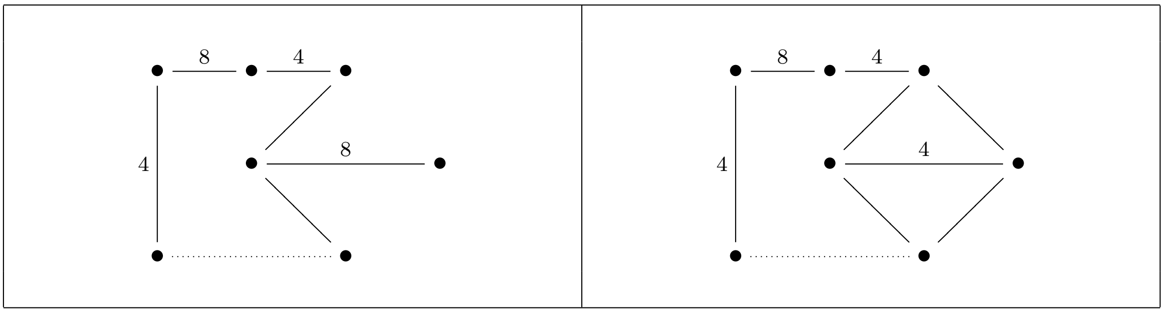
continues on the following page



continues on the following page



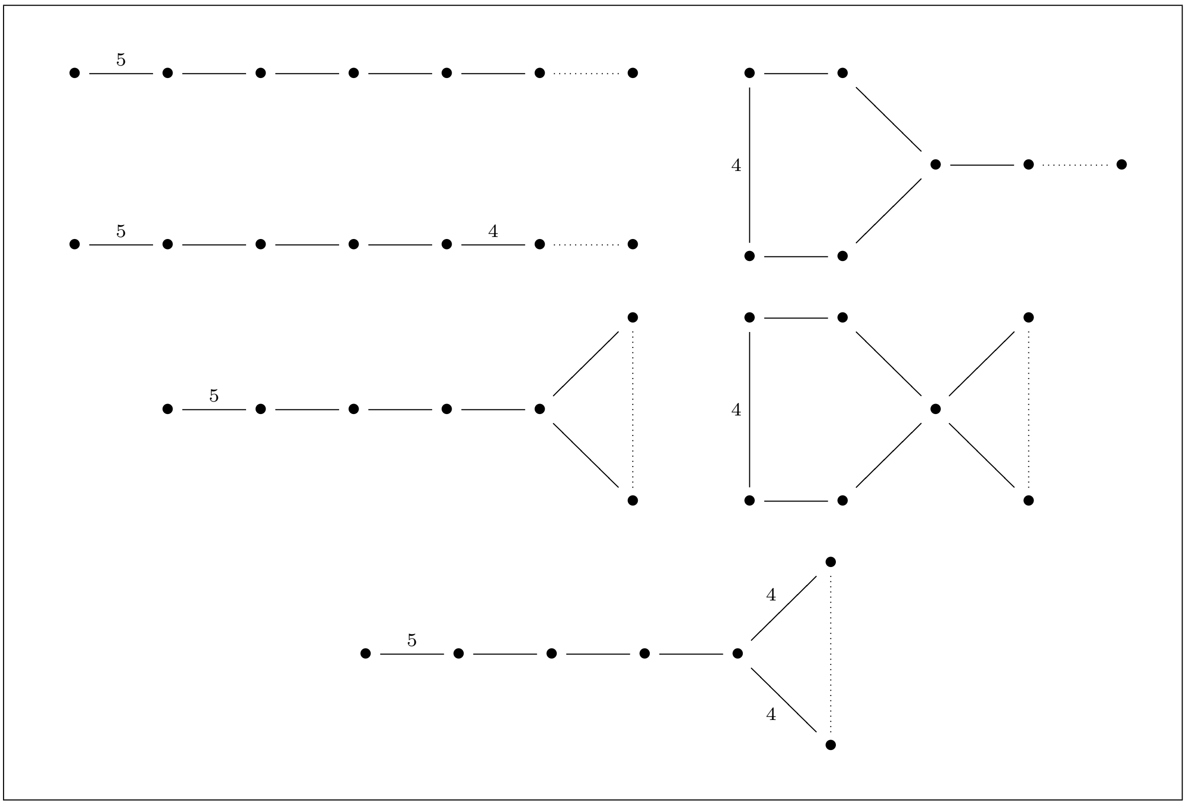
continues on the following page


8
4
4
8
8
4
4
4

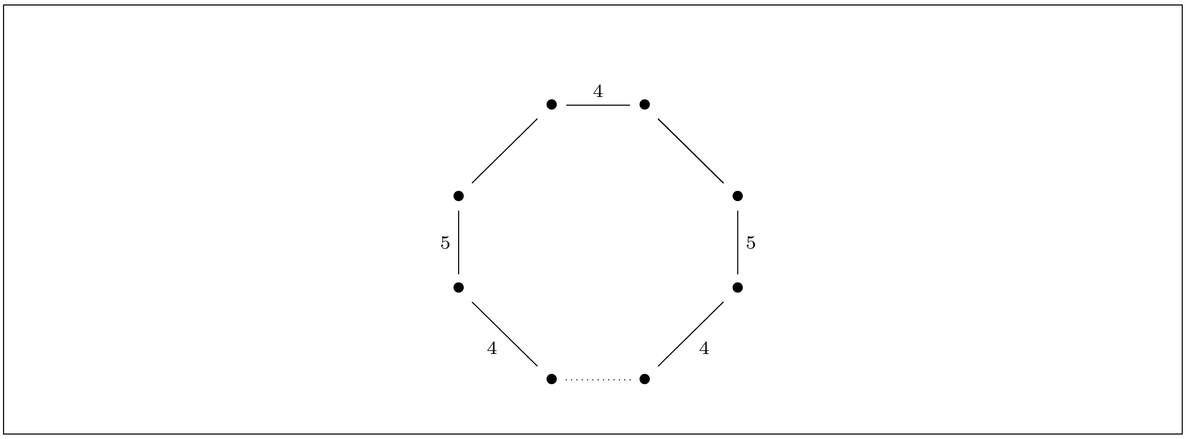
C Compact Coxeter polytopes in dimensions beyond 4

In this appendix we list the graphs of all known compact Coxeter polytopes in $\mathbb{H}^n$ for $n > 4$, up to garland or similar constructions.

The 5-dimensional Kaplinskaya polytopes



The 5-dimensional Tumarkin polytopes



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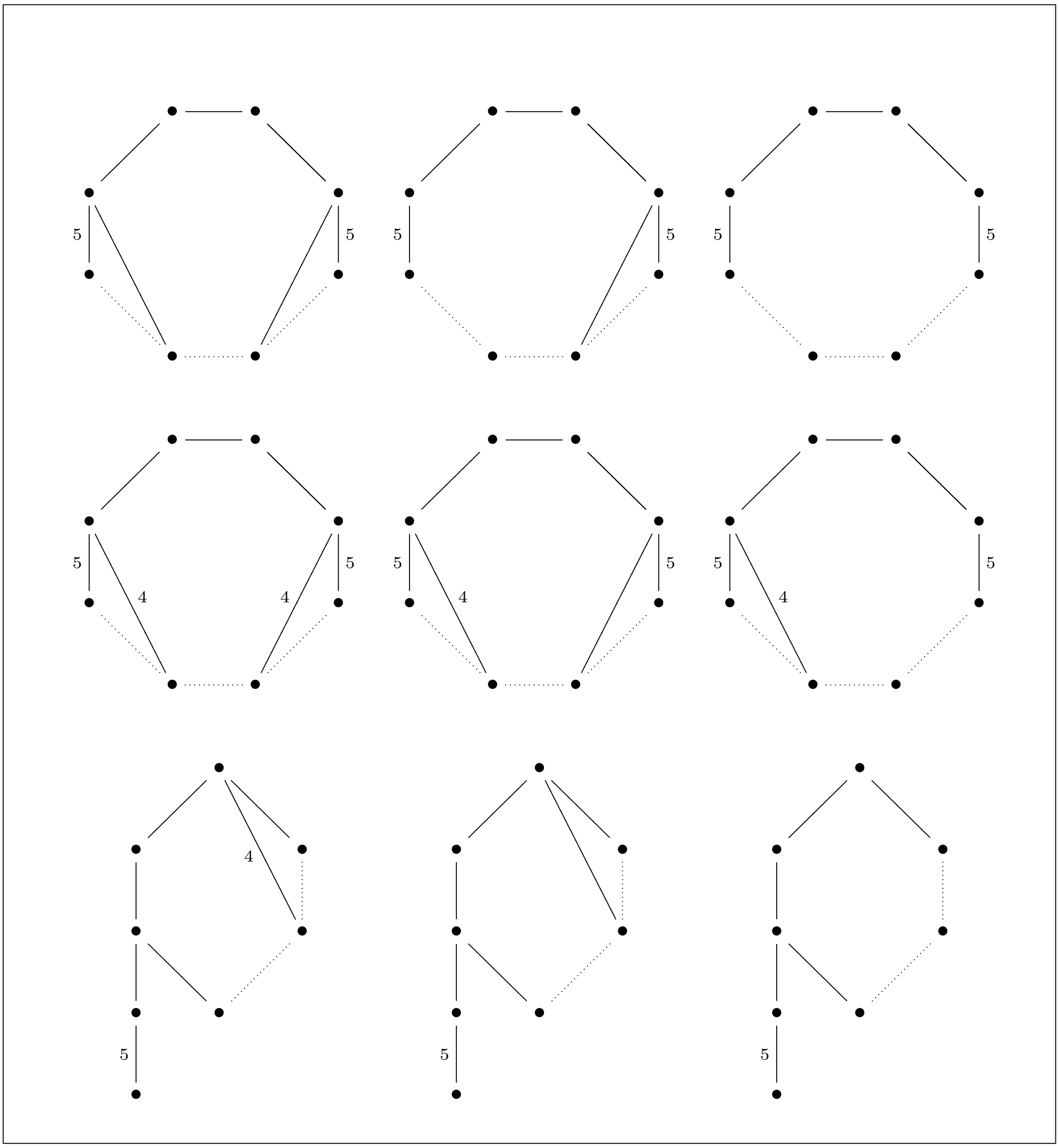

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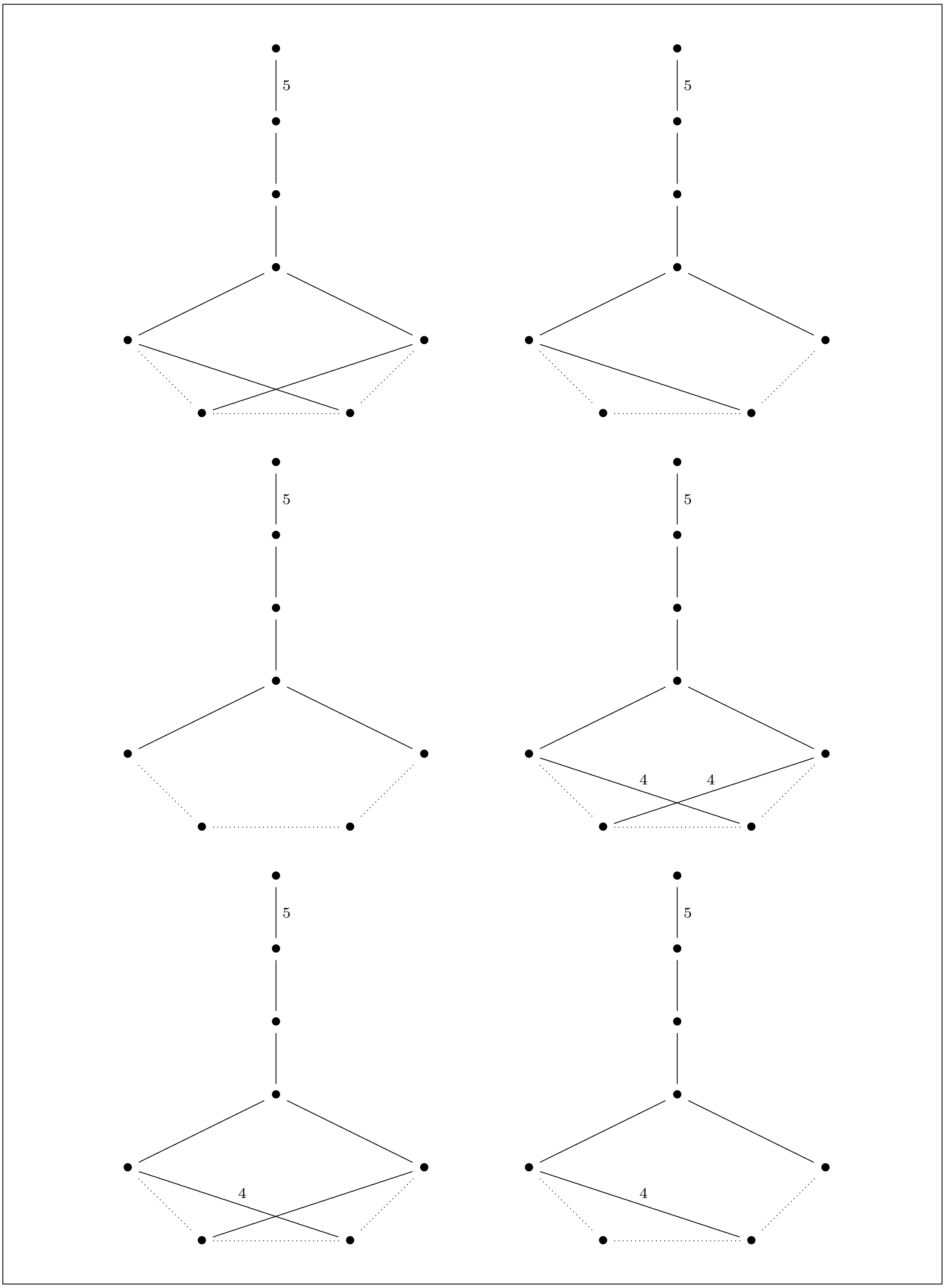
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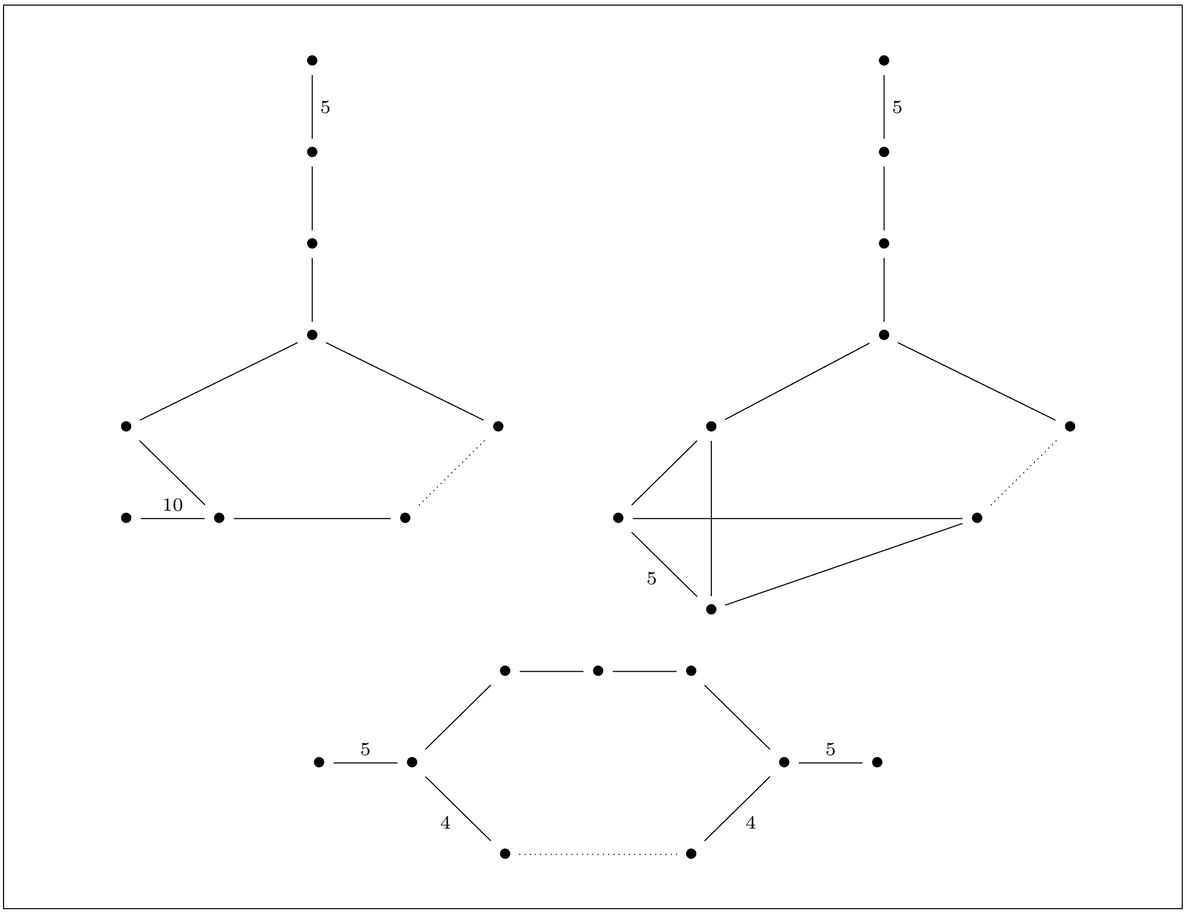
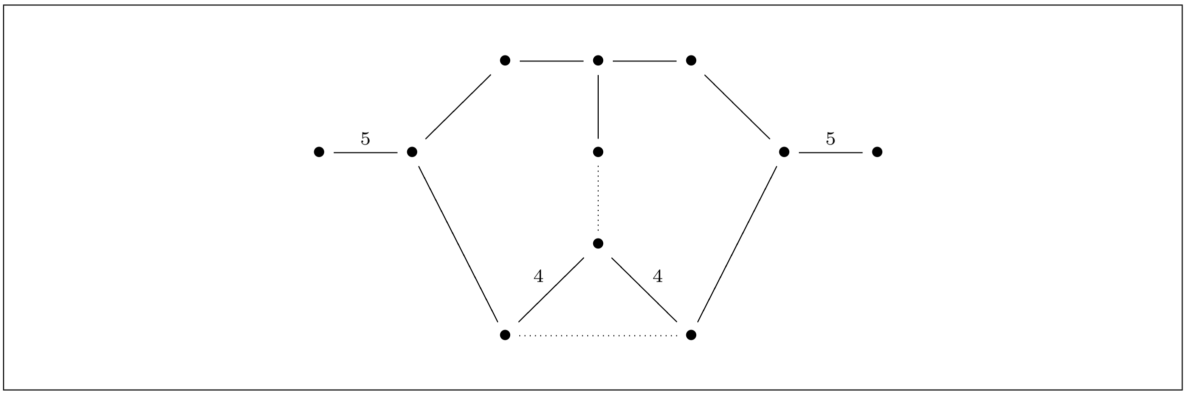
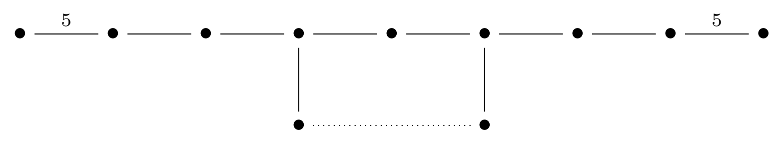
The 6-dimensional Tumarkin polytopes



The 7-dimensional Bugaenko polytope



The 8-dimensional Bugaenko polytope



D Polynomials

This appendix is a collection of results about polynomials which we use throughout our work. We are especially interested in palindromic polynomials. We begin with a very simple observation which is used in part 3.4.1.

Lemma D.1 *Let J be an arbitrary interval in $\mathbb{R}$. Let $P, Q \in \mathbb{R}[x]$ such that $P(x) \geq 0$ and $Q(x) > 0$ for $x \in J$. Assume that P is strictly decreasing and Q is strictly increasing on J. Then, the quotient P/Q is strictly decreasing over J.*

The polynomial $[k]$

Consider the polynomial (4.8),

$$[k] := 1 + x + \ldots + x^{k-1}.$$

The next lemmas describe the behaviour of this function.

Lemma D.2 *The polynomial $[2k]$ is strictly increasing on $\mathbb{R}$.*

P*roof*
If $x \geq 0$, the claim is trivial. Let us now assume that $x < 0$. Since

$$[2k] = \frac{x^{2k} - 1}{x - 1},$$

we easily obtain

$$\begin{aligned} [2k]' &= \frac{2kx^{2k-1} \cdot (x-1) - (x^{2k} - 1)}{(x-1)^2} \\ &= \frac{2kx^{2k} - x^{2k} - 2kx^{2k-1} + 1}{(x-1)^2} \\ &= \frac{1 + (2k-1)x^{2k} - 2kx^{2k-1}}{(x-1)^2} \\ &= \frac{1 + (2k-1)x^{2k-1}(x-1) - x^{2k-1}}{(x-1)^2}, \end{aligned}$$

which is strictly positive for $x < 0$. □

Lemma D.3 *The polynomial $[2, 2k+1]$ is strictly increasing on $\mathbb{R}$.*

P*roof*
The proof of this lemma is due to Aleš Janka. By means of an induction we proof that

$$[2, 2k+1]' = 2 \cdot [2] \cdot \sum_{i=1}^{k-1} i \cdot x^{2 \cdot (i-1)} + \underbrace{2k \cdot x^{2k-2} + 2k \cdot x^{2k-1} + (2k+1) \cdot x^{2k}}_{=: q_k(x)}.$$

The expression $2 \cdot [2] \cdot \sum_{i=1}^{k-1} i \cdot x^{2\cdot(i-1)}$ is obviously strictly positive on $\mathbb{R}$. Moreover, we easily check that $q_k(x)$ is strictly positive on $\mathbb{R}^*$. Then $[2, 2k+1]'$ is strictly positive on $\mathbb{R}$. Hence $[2, 2k+1]$ is strictly creasing on $\mathbb{R}$. □

Lemma D.4 *The function* $[2k+1]$ *is strictly positive on* $(-1, 0)$.

Consider now the product of two polynomials $[a]$, $[b]$.

Lemma D.5 *Let* a *and* b *be integers such that* $2 \leq a < b$. *Then,*

$$[a, b] = \sum_{j=1}^{a} j \cdot x^{j-1} + \sum_{j=a}^{b-1} a \cdot x^j + \sum_{j=0}^{a-2} (a-1-j) \cdot x^{b+j}. \tag{D.1}$$

P*roof*

We proceed by induction with respect to a. For $a = 2$, we easily compute that

$$[2, b] = (1+x) \cdot \sum_{i=0}^{b-1} x^i = \sum_{i=0}^{b-1} x^i + \sum_{i=0}^{b-1} x^{i+1} = 1 + \sum_{j=1}^{b-1} 2x^j + x^b = 1 + 2x + \sum_{j=2}^{b-1} 2x^j + x^b.$$

Let us now deal with $a > 2$. Since $[a+1] = [a] + x^a$, we obtain

$$[a+1, b] = [a, b] + x^a \cdot [b] = [a, b] + x^a + x^{a+1} + \ldots + x^{a+b-2} + x^{a+b-1}.$$

Then, by means of the induction hypothesis, we get

$$[a+1, b] = \sum_{j=1}^{a} j \cdot x^{j-1} + \sum_{j=a}^{b-1} (a+1) \cdot x^j + \sum_{j=0}^{a-2} (a-j) \cdot x^{b+j} + x^{a+b-1}.$$

□

Observe that (D.1) can be rewritten in

$$[a, b] = \sum_{j=1}^{a} j \cdot x^{j-1} + \sum_{j=a}^{b-1} a \cdot x^j + \sum_{j=b}^{a+b-2} (a+b-1-j) x^j. \tag{D.2}$$

The next result is very similar to Lemma D.5.

Lemma D.6 *Let* $a \geq 2$ *be an integer. Then,*

$$[a, a] = \sum_{j=1}^{a} j \cdot x^{j-1} + \sum_{j=a}^{2a-2} (2a-1-j) \cdot x^j. \tag{D.3}$$

Palindromic and antipalindromic polynomials

In this subsection we present some relations between palindromic and antipalindromic polynomials. Recall that a polynomial $p(x)$ of degree d is said to be *palindromic* if and only if $p(x) = x^d \cdot p\left(\frac{1}{x}\right)$, for all x, while it is *antipalindromic* if $p(x) = -x^d \cdot p\left(\frac{1}{x}\right)$, for all x. Equivalently we have that $p_i = p_{d-i}$, for $i = 0, \dots d$, if $p(x)$ is palindromic, while $p_i = -p_{d-i}$, for $i = 0, \dots d$, if $p(x)$ is antipalindromic.

In this paragraph, all polynomials are defined over the real numbers.

Lemma D.7 *Let P be a palindromic polynomial of degree $2k+1$. Then, $P(x) = (x+1)\cdot Q(x)$, where Q is a palindromic polynomial of degree $2k$.*

P*roof*
As P is palindromic of odd degree, we obviously have $P(-1) = 0$, and

$$P(x) = (x+1) \cdot Q(x),$$

where $Q(x)$ is a polynomial of degree $2k$. We prove now that $Q(x)$ is palindromic, that is $Q(x) = x^{2k} \cdot Q\left(\frac{1}{x}\right)$. Let $x \neq -1$. Then,

$$Q(x) = \frac{P(x)}{x+1} = \frac{1}{x+1} \cdot x^{2k+1} \cdot P\left(\frac{1}{x}\right) = \frac{1}{x+1} \cdot x^{2k+1} \cdot \frac{x+1}{x} \cdot Q\left(\frac{1}{x}\right) = x^{2k} \cdot Q\left(\frac{1}{x}\right).$$

Finally, it is clear that $(-1)^{2k} \cdot Q(-1) = Q(-1)$. □

Lemma D.8 *Let P be an antipalindromic polynomial of degree $2k$. Then $P(x) = (x+1) \cdot Q(x)$, where $Q(x)$ is an antipalindromic polynomial of degree $2k-1$.*

P*roof*
Let $p_0, p_1, \dots, p_{2k}$ denote the coefficients of P. Since P is antipalindromic, $p_k = 0$. Thus,

$$P(x) = \sum_{i=0}^{k-1} p_i \cdot \left(1 - x^{2\cdot(k-i)}\right).$$

Hence, -1 is a root of P, and

$$P(x) = (x+1) \cdot Q(x), \tag{D.4}$$

where $Q(x)$ is a polynomial of degree $2k-1$. Let $q_0, \dots, q_{2k-1}$ denote the coefficients of $Q(x)$. We determine them by solving the system of $2k+1$ equations given by (D.4). It remains to check that $Q(x)$ is antipalindromic which is a simple matter. □

In a similar fashion, the following result is verified.

Lemma D.9 *Let P be an antipalindromic polynomial of degree $2k+1$. Then, $P(x) = (x-1) \cdot Q(x)$, where $Q(x)$ is a palindromic polynomial of degree $2k$.*

Corollary D.10 *Let P be an antipalindromic polynomial of degree $2k \geq 2$. Then, $P(x) = (x+1) \cdot (x-1) \cdot Q(x)$, where $Q(x)$ is a palindromic polynomial of degree $2k-2$.*

P*roof*
This result is an easy consequence of Lemma D.8 and Lemma D.9. □

Factorisation of palindromic polynomials

This paragraph is devoted to the proof of the following result.

Theorem D.11 *Let P be a palindromic polynomial of degree d over the integers. Then, P can be factored into a product of a constant times linear (if d is odd), quadratic and quartic palindromic polynomials with* real *coefficients.*

Before proving Theorem D.11 we mention the well-known fundamental Theorem of Algebra together with some implications.

Theorem D.12 (Gauss) *Every non-constant polynomial with complex coefficients possesses at least one complex root.*

Corollary D.13 *Let $Q \in \mathbb{C}[x]$ be a polynomial of degree d. Then, there exist $a, \lambda_1, \ldots, \lambda_d \in \mathbb{C}$ such that*

$$Q(x) = a \cdot (x - \lambda_1) \cdots (x - \lambda_n) \ .$$

Corollary D.14 *A polynomial over the real numbers possesses an even number of non-real roots.*

The proof of Corollary D.14 provides the following useful lemma.

Lemma D.15 *Let $Q \in \mathbb{R}[x]$ be a polynomial with root $z \in \mathbb{C}$. Then, $Q(\bar{z}) = 0$.*

In the following, we look at *palindromic* polynomials.

Lemma D.16 *Let $Q \in \mathbb{R}[x]$ be a palindromic polynomial of degree $d \geq 2$. Then, its roots occur in pairs of the form (α, α^{-1}).*

P*roof*
Let $\alpha \neq 0$ be a root of $Q(x)$. Then $Q(\alpha) = 0$. As $Q(x) = x^d \cdot Q(1/x)$, it follows that $Q(\alpha^{-1}) = 0$. □

Lemma D.17 *Let $Q \in \mathbb{R}[x]$ be a palindromic polynomial of degree $d \geq 4$. Then, its non-real roots of absolute value different from 1 occur in 4-tuples $(\beta, \beta^{-1}, \bar{\beta}, \bar{\beta}^{-1})$, while the ones on the unit circle occur in pairs (γ, γ^{-1}).*

P*roof*
Let $\beta \in \mathbb{C} \setminus \mathbb{R}$ such that $|\beta| \neq 1$ and $Q(\beta) = 0$. Then, β^{-1} satisfies $Q(\beta^{-1}) = 0$ by Lemma D.16. By means of Lemma D.15, $\bar{\beta}$ and $\bar{\beta}^{-1}$ are also roots of Q. Observe that if $|\gamma| = 1$, then $\bar{\gamma} = \gamma^{-1}$ and $\bar{\gamma}^{-1} = \gamma$. This finishes the proof. □

We are now ready to prove Theorem D.11.

P*roof of Theorem D.11*
Let us first assume that P is of degree $d = 2k$ with coefficients $p_1, \ldots, p_{2k} \in \mathbb{Z}$. Suppose, without loss of generality, that $p_0 \neq 0$. Then,

$$P(x) = p_0 \cdot \sum_{i=0}^{2k} \frac{p_i}{p_0} x^i \ .$$

Let $\alpha_1, {\alpha_1}^{-1}, \ldots, \alpha_l, {\alpha_l}^{-1}$ denote the real roots of P, while $\beta_1, {\beta_1}^{-1}, \bar{\beta_1}, \bar{\beta_1}^{-1}, \ldots, \beta_m, {\beta_m}^{-1}$, $\bar{\beta_m}, \bar{\beta_m}^{-1}$ denote the non-real roots of absolute value different from 1, and $\gamma_1, \gamma^{-1}, \ldots, \gamma_n, {\gamma_n}^{-1}$ denote its non-real roots on the unit circle. Thus,

$$P(x) = p_0 \cdot \prod_{i=1}^{l} (x - \alpha_i)(x - {\alpha_i}^{-1}) \prod_{i=1}^{m} (x - \beta_i)(x - {\beta_i}^{-1})(x - \bar{\beta}_i)(x - \bar{\beta}_i^{-1}) \prod_{i=1}^{n} (x - \gamma_i)(x - \bar{\gamma_i}),$$

and

$$P(x) = p_0 \cdot \prod_{i=1}^{l} \left(x^2 - \frac{{\alpha_i}^2 + 1}{\alpha_i} + 1 \right) \prod_{i=1}^{m} (x^4 - A_i x^3 + B_i x^2 - A_i x + 1) \prod_{i=1}^{n} (x^2 - 2\mathrm{Re}(\gamma_i) x + 1)$$

where $A_i = 2\left(\mathrm{Re}({\beta_i}^{-1}) + \mathrm{Re}(\beta_i)\right)$ and $B_i = |\beta_i|^{-2} + 4\mathrm{Re}(\beta_i)\mathrm{Re}({\beta_i}^{-1}) + |\beta_i|^2$.

Finally, assume that P is of degree $2k + 1$. Then, Lemma D.7 implies that

$$P(x) = (x + 1) \cdot Q(x),$$

where Q is a palindromic polynomial over the real of degree $2k$. Now apply the preceeding arguments to Q. $\square$

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