Department of Mathematics
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Hyperbolic isometries in (in-)finite dimensions and discrete reflection groups: theory and computations

THESIS

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Ever tried. Ever failed. No
matter. Try Again. Fail again.
Fail better.

Samuel Beckett

Abstract

Hyperbolic Coxeter groups, which arise as isometry groups generated by reflections in the facets of polyhedra in the hyperbolic space $\mathbb{H}^n$, give rise to numerous questions many of which are still unresolved. Among these questions, we can mention:

(Q1) Creation of cofinite groups
What are the tools to create new hyperbolic Coxeter groups of finite covolume?

(Q2) Computations of invariants
Is it possible to efficiently compute the invariants of a given hyperbolic Coxeter group (Euler characteristic, growth series, growth rate) and of its associated polyhedron (volume, compactness, f-vector)?

(Q3) Classification
What methods do we have to classify groups up to commensurability?

Regarding question **(Q1)**, we present our implementation of Vinberg's algorithm (see [Vin72]), which can be used to find the presentation of the reflection group associated to a quadratic form. Our work consists of a computer program called `AlVin`, which is designed to carry out the algorithm. We then generalize an approach, due to Allcock (see [All06]), which allows to build an infinite sequence of index two subgroups in a given Coxeter group (not necessarily hyperbolic or even geometric).

Our contribution to question (**Q2**) comes in the form of `CoxIter`, a computer program whose aim is to compute the invariants of a given hyperbolic Coxeter group. An article describing `CoxIter` was published in [Gug15].

Concerning question (**Q3**) the commensurability classification of a substantial class of hyperbolic Coxeter groups was studied in a joint work with Matthieu Jacquemet and Ruth Kellerhals and results were published in [GJK16] and [GJKar]. In the present thesis, we explain in details a method, first described by Maclachlan, which allows to decide the (in)commensurability of two *arithmetic* hyperbolic Coxeter groups. This presentation is our contribution to [GJK16] and [GJKar].

When studying reflections and isometries of finite-dimensional hyperbolic spaces, a natural step towards generalization is the consideration of isometries of *infinite* dimensional hyperbolic spaces. At the end of this thesis, we show that an important class of these isometries can be obtained via Clifford matrices.

Résumé

Les groupes de Coxeter hyperboliques, qui apparaîssent comme des groupes d'isométries engendrés par des réflexions dans les côtés d'un polyèdre de l'espace hyperbolique $\mathbb{H}^n$, donnent lieu à de nombreuses questions dont beaucoup sont encore non-résolues. Parmi celles-ci, on peut mentionner :

(Q1) Création de groupes de covolume fini
De quelle manière peut-on créer de nouveaux groupes de Coxeter hyperboliques de covolume fini ?

(Q2) Calcul d'invariants
Peut-on calculer de manière efficace les invariants du groupe (caractéristique d'Euler, série de croissance, taux de croissance) et ceux du polyèdre associé (volume, compacité, f-vecteur) ?

(Q3) Classification
Quelles méthodes a-t-on à disposition pour classifier des groupes à commensurabilité près ?

Concernant la question **(Q1)**, nous présentons notre implémentation de l'algorithme de Vinberg (voir [Vin72]), qui permet de déterminer la présentation du groupe des réflexions associé à une forme quadratique, sous forme d'un programme informatique appelé `AlVin`. Ensuite, nous généralisons une approche d'Allcock (voir [All06]) qui permet de construire une suite infinie de sous-groupes d'indices 2 dans un groupe de Coxeter donné (pas nécessairement hyperbolique).

Notre contribution à la question (**Q2**) est apportée par `CoxIter`, un programme informatique qui a pour but de calculer les invariants d'un groupe de Coxeter hyperbolique donné. Un article concernant `CoxIter` a été publié dans [Gug15].

Concernant la question (**Q3**), la classification à commensurabilité près d'une classe importante de groupes de Coxeter hyperboliques a été traitée dans un projet commun avec Matthieu Jacquemet et Ruth Kellerhals et les résultats ont été publiés dans [GJK16] et [GJKar]. Nous proposons une présentation détaillée d'une méthode, exposée par Maclachlan dans [Mac11], qui permet de décider de la commensurabilité de deux groupes de Coxeter hyperboliques arithmétiques. Cette présentation constitue essentiellement notre contribution à [GJK16] et [GJKar].

Lors de l'étude des réflexions et des isométries de l'espace hyperbolique $\mathbb{H}^n$, un pas naturel dans la généralisation est la considération d'isométries d'un espace hyperbolique de dimension infinie. A la fin de cette thèse, nous montrons qu'une classe importante de ces isométries peuvent être obtenues via des matrices de Clifford.

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Merci à tous !

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Table of notations

Symbol	Meaning	Page
$\hat{\otimes}$	Tensor product of Hilbert spaces	25
$\Omega(K)$	Places of a number field	14
$A\Gamma$	Invariant quaternion algebra	57
$\mathrm{Br}\, K$	Brauer group	17
$c(f)$	Witt invariant	61
$\mathrm{Cl}(V, f)$	Clifford algebra associated to f	161
δ^i_j	Kronecker symbol	
f_S, f_Γ	Growth series	40
$\mathrm{Gal}(K, k)$	Galois group of the Galois extension $K\|k$	9
$\mathscr{H}$	Hilbert space	23
$\hat{\mathscr{H}}$	$\mathscr{H} \cup \{\infty\}$	
$\mathcal{H}^n$	Hyperboloid model of the hyperbolic space	31
$\mathcal{H}^{\mathfrak{n}}$	Hyperboloid model of the infinite dimensional hyperbolic space	31
ℓ^2	The sequence space	24
$L(\mathcal{H}_1; \mathcal{H}_2)$	Continuous linear maps between two Hilbert spaces	24
$K(\Gamma)$	Invariant trace field	56
$\mathrm{Möb}(\mathscr{H})$	Möbius group of a Hilbert space	28
$\mathrm{Möb}^*(\mathscr{H})$	Group of finite composition of reflections in spheres	26
$\mathbb{N}$	Positive integers	
$\mathbb{N}_0$	Non-negative integers	
$\mathcal{O}_K$	Ring of integers	9
$\mathbb{P}$	Set of prime numbers	
$\mathrm{Ram}\, B$	Ramification of a quaternion algebra	19
$s(f)$	Hasse invariant	61
$\mathrm{Sim}(\mathscr{H})$	Group of similarities	26

Continued on next page

Symbol	Meaning	Page
τ	Growth rate	41
$\mathcal{U}^n$	Upper half-space model of the hyperbolic space	31
$\chi(\Gamma)$	Euler characteristic	46
$\mathbb{X}^n$	$\mathbb{S}^n$, $\mathbb{E}^n$ or $\mathbb{H}^n$	
X^n	S^n, E^n or $\mathcal{H}^n$	
$\hat{V}$	$V \cup \{\infty\}$	
$\hat{V}_{\text{ext}}$	$K \oplus V \cup \{\infty\}$	166

CHAPTER 1

Introduction

Let X^n denote the unit n-sphere S^n, the Euclidean n-space E^n or the vector space model $\mathcal{H}^n$ of the hyperbolic n-space $\mathbb{H}^n$. Any polyhedron $P \subset X^n$ gives rise to a subgroup $\Gamma := \Gamma(P)$ of the isometry group $\operatorname{Isom} X^n$ generated by the reflections in the facets, or sides, of P. If all dihedral angles between adjacent sides of P are of the form $\frac{\pi}{k}$ for some $k \in \{2, 3, \dots\} \cup \{\infty\}$, then P is a fundamental cell for Γ and is called a *Coxeter polyhedron*. In this case, Γ is a *Coxeter* group, or *geometric Coxeter group*, and we call it *spherical*, *Euclidean*, or *hyperbolic*, depending on X^n.

Although finite Coxeter groups -which correspond to the spherical case- and the Euclidean ones have been fully classified (see [Cox35] and [Bou68a]), we are far from a classification in the hyperbolic setting. Moreover, we do not have many examples of *cofinite* hyperbolic Coxeter groups when n is bigger than 12 and none when $n > 21$. We are therefore interested in the following three questions:

(Q1) How can we create new cofinite hyperbolic Coxeter groups?

(Q2) Is there an efficient way to compute invariants of a hyperbolic Coxeter group Γ (Euler characteristic, growth series and growth rate) and of its associated polyhedron (finiteness, compactness, volume, f-vector)?

(Q3) What are the methods that can be used in order to classify hyperbolic Coxeter groups up to commensurability?

Vinberg presented a method which gives a partial answer to question **(Q1)**: in [Vin72], he described an algorithm whose goal is to find the group of reflections in $\mathrm{PO}(n, 1)$ associated to a given quadratic form of signature $(n, 1)$. Many authors used Vinberg's algorithm with ad hoc methods for a quadratic form of the type $\langle -\alpha, 1, \dots, 1\rangle$. However, since the computations are tedious to carry out, there are not many examples for other quadratic forms. In chapter 6, we present our computer program `AlVin`, which is a general implementation of the algorithm. We give the necessary theoretical background, details about computational aspects and explain how we used `AlVin` to find *new* polyhedra.

In chapter 7, we generalize Allcock's approach (see [All06]) in order to construct infinite sequences $\{\Gamma_n\}_{n\geq 0}$ of index two subgroups in a Coxeter group Γ (which is not assumed to be hyperbolic, or even geometric). We analyse the ranks of the groups Γ_n and, in the case where Γ is a geometric Coxeter group, we compute the f-vector of the associated polyhedra $P(\Gamma_n)$. When the group Γ is hyperbolic, we also describe the growth series of the groups Γ_n in terms of the growth series of Γ.

Our contribution to question **(Q2)** consists of `CoxIter`, a computer program which yields certain invariants of a given hyperbolic Coxeter group. The input of `CoxIter` is the presentation of the group and the output contains the following: cocompactness and cofiniteness, arithmeticity, f-vector of the associated polyhedron, Euler characteristic, dimension, growth series and growth rate. An article describing `CoxIter` was published in [Gug15].

Concerning question (**Q3**) the commensurability classification of an important class of hyperbolic Coxeter groups was studied in a joint work with Matthieu Jacquemet and Ruth Kellerhals ([GJK16] and [GJKar]). In our joint work, we classified up to commensurability the family of 200 Coxeter pyramids groups first described by Tumarkin (see [Tum04]). In chapter 4, we give an overview of different methods which can be used to decide the (in-)commensurability of hyperbolic Coxeter groups. We then focus on the classification in the *arithmetic* setting, which was our contribution to this joint work. We give a detailed presentation of a method, first described by Maclachlan in [Mac11], that can be used to compute a *complete* set of invariants for a given cofinite arithmetic hyperbolic Coxeter group. These computations, which are done in the Brauer group of the defining field using quaternion algebras, generalize the invariant trace field and invariant quaternion algebra which appeared in [MR03]. We also present the computations for some new compact polyhedra found using `AlVin`.

When studying reflections and isometries of the hyperbolic space $\mathbb{H}^n$, a natural step towards generalization, is the investigation of isometries of infinite-dimensional hyperbolic spaces. The two well-known group isomorphisms

$$\mathrm{Isom}^+\,\mathbb{H}^2 = \mathrm{PSL}(2;\mathbb{R}), \quad \mathrm{Isom}^+\,\mathbb{H}^3 = \mathrm{PSL}(2;\mathbb{C})$$

are the first instances of the appearance of Clifford matrices: Ahlfors and Waterman ([Ahl85] and [Wat93]) showed that the group $\mathrm{Isom}^+\,\mathbb{H}^n$ can be described using two-by-two matrices with coefficients in a Clifford algebra. Other authors, such as Frunză and Li, extended this idea to the infinite-dimensional hyperbolic space $\mathbb{H}^{\aleph_0}$ modelled on the (separable) sequence space ℓ^2 (see [Fru91] and [Li11]). In chapter 8, we show that a similar result holds for any infinite-dimensional hyperbolic space $\mathbb{H}^\infty$: we are able to describe an important subgroup of the isometry group $\mathrm{Isom}\,\mathbb{H}^\infty$ using Clifford matrices (see Theorem 8.3.3). Our approach, which does not rely on a specific representation of the underlying Hilbert space $\mathscr{H}$, allows to establish a connection to the group Möb^* of all *finite* compositions of reflections in generalized spheres of $\mathscr{H}$ preserving the upper half-space $\mathcal{U}_\mathscr{H}$ (see Proposition 8.2.17). This group was discussed in Das's PhD thesis [Das12]. We also discuss some further questions which can be addressed using our approach.

At the end of the thesis, we present several appendices which contain some invariants of Kaplinskaya's infinite families of compact Coxeter prisms in $\mathbb{H}^3$.

We also give the commensurability invariants of arithmetic hyperbolic Coxeter polytopes in $\mathbb{H}^n$ with $n+3$ facets and one non-simple vertex, which were classified by [Rob15]. Finally, we give two `Mathematica`® codes which were used in this thesis.

CHAPTER 2

Algebraic background

In this chapter, we provide basic material which will be used during this work. We present both classical theoretical results together with algorithmic and computational aspects.

The first two parts are dedicated to field extensions and number fields. These notions will be particularly used when dealing with arithmetic groups and their classification up to commensurability (see Section 4.3). We also present results about prime elements and factorization in number fields, which will be useful to set the computational background for the Vinberg algorithm (see Chapter 6). Most of the material presented here is covered in the book [Gri07].

In sections 2.4 and 2.5, we are mostly interested in quaternion algebras and their classification up to isomorphism. This last point is the key to the classification of arithmetic hyperbolic Coxeter groups, since the invariant of the commensurability class consists almost entirely of the isomorphism class of a quaternion algebra over a number field (see Section 4.3.1.3). References for quaternion algebras are [GS06], [Lam05], and [Vig80] for some technical results.

In the before last part, we give technical results about roots of polynomials which will be used to determine the (non-)arithmeticity of some groups and to determine the asymptotic behaviour of the growth rate of some families of groups (see sections 4.4.1 and 3.6 for example).

Finally, we present basic properties and constructions related to Hilbert spaces. They will be used in the chapter dedicated to isometries of the infinite-dimensional hyperbolic space. Most of the materiel can be found in the books [Lax02] and [Con85].

2.1 Field extensions

Let k be a field. A *field extension* of k is a field K such that $k \subset K$. Such an extension is often denoted by $K|k$. The *degree* of K over k, denoted by $[K : k]$, is the dimension of K as a k-vector space. An element $\alpha \in K$ is *algebraic* over k if there exists a polynomial $f \in k[t]$ such that $f(\alpha) = 0$. Among all the *monic* polynomials with coefficients in k which vanish at α, the one with the smallest degree is called the *minimal polynomial* of α and is denoted $\min(\alpha, k)$. The *simple extension* $k[x]/\min(\alpha, k)$ is then denoted by $k(\alpha)$, or $k[\alpha]$.

If all the elements of K are algebraic over k, we say that K *is algebraic over*

k. We will consider only algebraic field extensions. Of course, any extension of finite degree is algebraic.

Definition 2.1.1 (Algebraically closed field)
A field k is *algebraically closed* if every non-constant polynomial with coefficients in k has at least one root (and thus all its roots) in k.

Definition 2.1.2 (Algebraic closure)
Let k be a field. We say that an *algebraic* extension of k is an *algebraic closure* of k if it is algebraically closed.

Proposition 2.1.3
Two algebraic closures F_1 and F_2 of a field k are k-isomorphic (i.e. there exists a field isomorphism $\tau : F_1 \longrightarrow F_2$ such that $\tau\big|_k = \mathrm{id}_k$). Moreover, Zorn's lemma implies the existence of at least one algebraic closure of any field.

Remark 2.1.4
Because of the previous proposition, we will often speak about *the* algebraic closure of k and denote it by $\overline{k}$.

Definition 2.1.5 (Splitting field)
Let k be a field and let $\mathcal{F}$ be a family of non-constant polynomials with coefficients in k. A *splitting field* for $\mathcal{F}$ is an algebraic extension $k_{\mathcal{F}}$ of k which satisfies the following two properties:

1. Every polynomial $f \in \mathcal{F}$ splits as a product of factors of degree 1 in $k_{\mathcal{F}}$.
2. $k_{\mathcal{F}}$ is generated by all the roots of the polynomials of $\mathcal{F}$ (i.e. it is the smallest algebraic extension of k enjoying property 1.).

In fact, one can easily show that two splitting fields of a family $\mathcal{F}$ are isomorphic. Also, a splitting field $k_{\mathcal{F}}$ can be constructed as the subfield of $\overline{k}$ generated by all the roots of all polynomials of $\mathcal{F}$. Therefore, we will speak about *the* splitting field of $\mathcal{F}$.

Definition 2.1.6 (Separable element, separable extension)
Le $K|k$ be a field extension. An element $\alpha \in K$ is *separable* if all the roots of its minimal polynomial, in a chosen algebraic closure $\overline{k}$ of k containing K, are distinct. The extension $K|k$ is *separable* if all the elements of K are separable.

Example 2.1.7
If the characteristic of k is zero, then all its algebraic extensions are separable. Any algebraic extension of a finite field is separable.

Consider an algebraic field extension $K|k$ and fix an algebraic closure $\overline{k}$ of k which contains K. We are interested in field homomorphisms $\sigma : K \longrightarrow \overline{k}$ which act by identity on k. Such maps are called *k-homomorphisms*, *k-embeddings*, or just *embeddings*, when there is no ambiguity on k. If $K = k(\alpha)$ is a simple extension, then any k-embedding sends α to another root of $\min(\alpha, k)$. Conversely, a root of $\min(\alpha, k)$ determines a k-homomorphism of $k(\alpha)$ into $\overline{k}$. In particular, the number of such embeddings is given by the number of different roots of $\min(\alpha, k)$. Note that if an extension $K|k$ is separable, then the number of k-embeddings $\sigma : K \longrightarrow \overline{k}$ is equal to the degree $[K : k]$.

We remark that for a k-embedding σ, we may have $\sigma(K) \not\subset K$. This issue is discussed and settled in the next proposition and definition.

Proposition 2.1.8
Let $K|k$ be an algebraic extension and fix an algebraic closure $\overline{k}$ of k containing K, i.e. $k \subset K \subset \overline{k}$. Then, the following are equivalent:

- *K is the splitting field of a family of polynomials with coefficients in k.*
- *For every k-embedding $\sigma : K \longrightarrow \overline{k}$, we have $\sigma(K) \subset K$.*
- *For every k-embedding $\sigma : K \longrightarrow \overline{k}$, we have $\sigma(K) = K$.*
- *Every irreducible polynomial with coefficients in k which has one root in K has all its roots in K.*

Definition 2.1.9 (Normal field extension)
An algebraic field extension $K|k$ which satisfies one of the equivalent conditions of the previous proposition is called a *normal extension*.

Definition 2.1.10 (Galois extension, Galois group)
An algebraic field extension $K|k$ is called a *Galois extension* if it is both normal and separable. In this setting, the *Galois group* is defined to be the group of all k-homomorphisms $\sigma : K \longrightarrow K$. It is a group of order $[K : k]$ and is denoted by $\mathrm{Gal}(K, k)$.

Example 2.1.11
Let $p \in \mathbb{P}$ be an odd prime number and consider the primitive pth root of unity $\zeta = e^{\frac{2\pi i}{p}}$. The extension $\mathbb{Q}(\zeta)|\mathbb{Q}$ is a Galois extension of degree $p - 1$ whose Galois group is cyclic. Inside this extension sits a totally real number field of degree $\frac{p-1}{2}$ generated by $\cos \frac{2\pi}{p}$.

2.2 Number fields

A number field is a finite extension of $\mathbb{Q}$. For a number field K of degree n, there exist precisely n embeddings $\sigma : K \longrightarrow \mathbb{C}$. If an embedding σ is such that $\sigma(K) \subset \mathbb{R}$, it is called a *real* embedding. Otherwise, we call σ a *complex* embedding. Since the complex embeddings come in conjugate pairs, then $n = r+2s$, where r denotes the number of real embeddings and s the number of pairs of complex embeddings. The pair (r, s) is called the *signature* of the number field. If K has no complex embedding we say that it is a *totally real number field.*

2.2.1 The ring of integers

Let K be a number field. We say that an element $\alpha \in K$ is an *integer* of K, if there exists a *monic* polynomial $f \in \mathbb{Z}[x]$ such that $f(\alpha) = 0$. This is equivalent to the fact that the minimal polynomial $\min(\alpha, \mathbb{Q})$ has coefficients in $\mathbb{Z}$. It can be shown that the set of integers is a ring called the *ring of integers* and denoted by $\mathcal{O}_K$. Although the elements of $\mathcal{O}_K$ may fail to possess some basic arithmetic properties (for example, we may not have a unique decomposition into a product of prime elements) the ring of integers has good properties concerning the factorization of its ideals (see Theorem 2.2.4).

The multiplicative group of invertible elements $\mathcal{O}_K^*$ modulo its torsion, which consists of roots of unity, is a group of finite rank. This is made precise by the following theorem.

Theorem 2.2.1 (Dirichlet's unit theorem)
Let K be a number field and let (r, s) be its signature. Then, we have

$$\mathcal{O}_K^* \cong \mu(K) \times \mathbb{Z}^{r+s-1},$$

where $\mu(K)$ is the finite cyclic group of the roots of unity in $\mathcal{O}_K$.

Definition 2.2.2 (Fundamental unit, fundamental system of units)
If the group $\mathcal{O}_K^*$ has rank 1, i.e. $r + s - 1 = 1$, then a generator is called a *fundamental unit*. More generally, the set of the $r + s - 1$ generators of the non-torsion part of $\mathcal{O}_K^*$ is called a *fundamental system of units*.

Example 2.2.3
Let d be a positive square-free integer. For the quadratic field $K = \mathbb{Q}[\sqrt{d}]$, Dirichlet's unit theorem implies that there exists $\eta \in \mathcal{O}_K^*$ such that

$$\mathcal{O}_K^* = \{\pm\, \eta^m : m \in \mathbb{Z}\}.$$

The element η can be found by solving a Pell type equation.

2.2.1.1 Decomposition of prime ideals in a number field

If $\mathcal{P}$ is a prime ideal of $\mathcal{O}_K$, then $\mathcal{P} \cap \mathbb{Z}$ is a prime ideal of $\mathbb{Z}$, that is $\mathcal{P} \cap \mathbb{Z} = \langle p \rangle$, for some prime number $p \in \mathbb{P}$. In this setting we say that $\mathcal{P}$ *is above* $\langle p \rangle$ (or above p). If there exists $\pi \in \mathcal{O}_K$ such that $\mathcal{P} = \langle \pi \rangle$, we say that π *is above* p.

Theorem 2.2.4 ([Coh93, Theorem 4.8.3])
If $p \in \mathbb{P}$ is a prime number, then there exist positive integers e_i such that

$$p\mathcal{O}_K = \prod_{i=1}^{g} \mathcal{P}_i^{e_i},$$

where the $\mathcal{P}_i$ are all the prime ideals above $\langle p \rangle$.

Definition 2.2.5 (Ramification index, residual degree)
In the setting of Theorem 2.2.4, the integer e_i corresponding to $\mathcal{P}_i$, which is also written $e(\mathcal{P}_i/p)$, is called the *ramification index*. The (finite) degree of the field extension

$$f_i = f(\mathcal{P}_i/p) = \left[\mathcal{O}_K/\mathcal{P}_i : \mathbb{Z}/p\mathbb{Z}\right]$$

is called the *residual degree*.

We have the following relation between ramification indices and residual degrees (see [Coh93, Theorem 4.8.5]):

$$\sum_{i=1}^{g} e_i \cdot f_i = [K : \mathbb{Q}].$$

If K is a Galois extension (for example when K is a quadratic field or when K is the maximal real subfield of the pth cyclotomic field), then all the e_i, respectively all the f_i, are equal (see [Coh93, Theorem 4.8.6]). This motivates the following definition.

Definition 2.2.6
Let K be a Galois number field (i.e. K is a number field and $K|\mathbb{Q}$ is a Galois extension), $p \in \mathbb{P}$ be a prime and let $p\mathcal{O}_K = \prod_{i=1}^{g} \mathcal{P}_i^{e_i}$ be its decomposition. We say that:

- p is *inert* if $g = 1$ and $e_1 = 1$ (meaning that $p\mathcal{O}_K$ is a prime ideal).
- p *splits completely* if $g = n$ (and thus $e_i = f_i = 1$ for all i).
- p is *ramified* if there exists i such that $e_i \geq 2$.
- p is *completely ramified* if $g = 1$ and $f_1 = 1$. Hence, $p\mathcal{O}_K$ is the nth power of a prime ideal $\mathcal{P}$.
- p is *unramified* otherwise.

The primes which ramify are exactly the rational primes which divide the discriminant of the number field (see [Coh93, Theorem 4.8.8]). In particular, in a real quadratic field $\mathbb{Q}[\sqrt{d}]$, these are the primes which divide d and 2 if $d \not\equiv 1 \mod 4$.

In some cases, we have an effective way to compute the decomposition of an ideal, as explained in the next theorem.

Theorem 2.2.7 ([Coh93, 4.8.13])
Let Θ be a primitive element of the number field K, that is $K = \mathbb{Q}[\Theta]$, and suppose that $\mathcal{O}_K = \mathbb{Z}[\Theta]$. Let $T(x) \in \mathbb{Z}[x]$ be the minimal polynomial of Θ. Then, for any $p \in \mathbb{P}$, we can compute the decomposition of $p\mathcal{O}_K$ as follows:

- *Compute the decomposition in irreducible factors of $T(x)$ in $\mathbb{F}_p[x]$:*

$$T(x) \equiv \prod_{i=1}^{g} T_i(x)^{e_i}, \quad (mod\ p).$$

- *Let $\mathcal{P}_i = \langle p, T_i(\Theta) \rangle = p\mathcal{O}_K + T_i(\Theta)\mathcal{O}_K$.*

Then, we have

$$p\mathcal{O}_K = \prod_{i=1}^{g} \mathcal{P}_i^{e_i}.$$

Moreover, the residual degree $f_i = f(\mathcal{P}_i/p)$ is equal to the degree of T_i.

Remarks 2.2.8

- The assumption $\mathcal{O}_K = \mathbb{Z}[\Theta]$ can be removed but then Theorem 2.2.7 only holds for primes p which do not divide $\left[\mathcal{O}_K : \mathbb{Z}[\Theta]\right]$. When K is a quadratic extension or one of our real cyclotomic fields, the assumption is satisfied.
- There exist efficient algorithms to factorize polynomials over finite fields. These are for example implemented in `PARI`. More information can be found in [Coh93, Section 3.4].

2.2.2 Computing the GCD

For two rational integers $a, b \in \mathbb{Z}$, there exist essentially two methods to compute their greatest common divisor $\gcd(a, b)$.

The first possibility is to compute the prime decomposition of the two elements and to take the common factors. Although this is pretty inefficient from an algorithmic point of view, it can be generalized to any number field as soon as we are able to decompose rational prime numbers in $\mathcal{O}_K$ (which can present some difficulty, for example in cyclotomic fields, as it will be explained in Section 6.5.3.1).

For the second possibility, we use the fact that $\gcd(a, b) = \gcd(a, b \bmod a)$. Hence, we can consider the following algorithm:

Algorithm 1 gcd()

```
while b ≠ 0 do
    t ← b
    b ← a mod b
    a ← t
end
return a
```

The approach in Algorithm 1 is based on the fact that we can perform a Euclidean division. This can be generalized as follows.

Definition 2.2.9 (Euclidean ring)
Let R be an integral domain and let $f : R \setminus \{0\} \longrightarrow \mathbb{N}_0$. Then, f is to said to be an *Euclidean function*, if the following properties are satisfied:

1. For every pair $a, b \in R$ with $b \neq 0$, there exist $q, r \in R$ such that $a = qb + r$ and either $r = 0$ or $f(r) < f(b)$.

2. For every pair a, b of non-zero elements of R, then $f(a) \leq f(ab)$.

The domain R is *Euclidean* if it admits (at least) one such function.
If $R = \mathcal{O}_K$ for some number field K, we will say that K is *Euclidean* if R is Euclidean. Moreover, if the function f can be taken to be the absolute value of the usual norm, then K is said to be *norm-Euclidean*.

Examples 2.2.10

- A real quadratic field $K = \mathbb{Q}[\sqrt{d}]$ is norm-Euclidean if and only if d is one of the following values: 2, 3, 5, 6, 7, 11, 13, 17, 19, 21, $29, 33, 37, 41, 57, 73$. Note that there also exist Euclidean quadratic fields which are not norm-Euclidean (for example $d = 69$, as shown in [Cla94]).

- Let $m \in 3, 4, 5, 7, 8, 9, 11, 12, 15, 20$ and let ζ_n be a primitive nth root of unity. Then, $\mathbb{Z}[\zeta_m]$ is norm-Euclidean (see [Len75]).

Hence, we could, in theory, use a Euclidean function of a Euclidean number field K in order to compute the gcd of two elements a and b. However, in practice, we don't know how to find the algebraic integers q and r such that $a = qb + r$, even when the field K is norm-Euclidean.

Notice that when $K = \mathbb{Q}[\sqrt{d}]$ with $d = 2, 3$, then the problem becomes easy. We first perform the division $\frac{a}{b} = \tilde{x} + \tilde{y}\sqrt{d}$ in K and let $q = x + y\sqrt{d}$, where x and y are the nearest integers to $\tilde{x}$ and $\tilde{y}$, respectively. Since $N\left(\frac{a}{b} - q\right) < 1$, then $r := q \cdot b$ has the required property.

Other algorithms It is worth to mention two other algorithms which have been designed to compute gcd. The first one uses reduction of quadratic forms and applies only to quadratic number fields and is explained in [AF06]. The second one concerns arbitrary number fields and was developed by Wikström (see [Wik05]) but the bounds for the computations are too big to be implemented [Wik15].

2.2.3 Trace, norm and factorization of elements

We consider the *norm* and the *trace* of K given as follows:

$$\begin{aligned} N : K^* &\longrightarrow \mathbb{C} \\ \alpha &\longmapsto \prod_\sigma \sigma(\alpha) \end{aligned} \qquad\qquad \begin{aligned} \mathrm{Tr} : K &\longrightarrow \mathbb{C} \\ \alpha &\longmapsto \sum_\sigma \sigma(\alpha), \end{aligned}$$

where σ runs through the Galois embeddings of K into $\mathbb{C}$. These two homomorphisms enjoy the following two properties:

- For a Galois embedding $\sigma : K \longrightarrow \mathbb{C}$ and $\alpha \in K$, both $N(\alpha)$ and $\mathrm{Tr}(\alpha)$ are invariant under σ. In particular, we have $N(\alpha), \mathrm{Tr}(\alpha) \in \mathbb{Q}$.
- The image of an element of $\mathcal{O}_K$ lies in $\mathbb{Z}$ (when K is a quadratic extension of $\mathbb{Q}$, the converse is also true: if $\alpha \in \mathcal{O}_{\mathbb{Q}[\sqrt{d}]}$ is such that $N(\alpha), \mathrm{Tr}(\alpha) \in \mathbb{Z}$, then $\alpha \in \mathcal{O}_K$).

Moreover, the multiplicative property of N implies the following facts:

- An element $\alpha \in \mathcal{O}_K$ is a unit if and only if $N(\alpha) = \pm 1$.
- If $\alpha \in \mathcal{O}_K$ is such that $N(\alpha)$ is a rational prime, then α is prime in $\mathcal{O}_K$.
- A prime $p \in \mathbb{P}$ is either prime in $\mathcal{O}_K$ or splits in a product of at most $[K^{\mathrm{nc}} : \mathbb{Q}]$ primes of $\mathcal{O}_K$, where K^{nc} is the normal closure of K.

Proposition 2.2.11
Suppose that $\mathcal{O}_K$ is a unique factorization domain (UFD) and let $\pi \in \mathcal{O}_K$ be a prime. There exists a unique rational prime $p \in \mathbb{P}$ such that $\pi \mid p$.

Proof. Since $\pi \mid N(\pi) \in \mathbb{Z}$, the set of positive rational integers which are divisible by π is not empty. The least element of this set has to be a rational prime, as required. □

Corollary 2.2.12
When $\mathcal{O}_K$ is a UFD, in order to find all prime elements of $\mathcal{O}_K$, it is sufficient to find the factorization of all rational primes in $\mathcal{O}_K$.

Therefore, we have a procedure to find the prime decomposition of an element $\alpha \in \mathcal{O}_K$:

1. For each rational prime $p \in \mathbb{P}$, find the decomposition of p in $\mathcal{O}_K$.
2. Compute the prime decomposition of $N(\alpha) \in \mathbb{Z}$ in the integers.
3. For each $p \mid N(\alpha)$, decompose the prime p in a product $\pi_1 \cdot \ldots \cdot \pi_r$ of prime elements of $\mathcal{O}_K$. For each factor π_i, determine the maximal power of π_i which divides α.

2.2.4 Places of a number field

Let K be a number field. Recall that a *place* of K is an equivalence class of absolute values[1]: two non-trivial absolute values $|\cdot|_1, |\cdot|_2 : K \longrightarrow \mathbb{R}$ are *equivalent* if there exists some number $e \in \mathbb{R}$ such that $|x|_1 = |x|_2^e$ for all $x \in K$. We can easily create two kind of places:

Infinite places Any real Galois embedding $\sigma : K \longrightarrow \mathbb{R}$ yields a place by composition with the usual absolute value. Similarly, any complex Galois embedding $\sigma : K \longrightarrow \mathbb{C}$ gives rise to a place by composition with the modulus. These places are called *infinite places*.
Note that in our setting, where the number fields will often supposed to be totally real, we will only get real embeddings.

Finite places Let $\mathcal{P}$ be a prime ideal of $\mathcal{O}_K$. This defines a valuation on $\mathcal{O}_K$ as follows:

$$\eta_{\mathcal{P}} : \mathcal{O}_K \longrightarrow \mathbb{Z} \cup \{\infty\}, \quad \eta_{\mathcal{P}}(x) = \sup\{r \in \mathbb{N}_0 : x \in \mathcal{P}^r\}.$$

This valuation can be extended to K by setting $\eta_{\mathcal{P}}(x/y) = \eta_{\mathcal{P}}(x) - \eta_{\mathcal{P}}(y)$. Now, we pick any $0 < \lambda < 1$ we define an the associated absolute value

$$|\cdot|_{\mathcal{P}} : K \longrightarrow \mathbb{R}, \quad |x|_{\mathcal{P}} = \lambda^{\eta_{\mathcal{P}}(x)}.$$

Note that the place associated to this absolute value is independent of the choice of λ. The places defined in this way are called *finite places*.

Using Ostrowski's theorem and theorems about extensions of absolute values, one gets the following standard result.

Theorem 2.2.13
Let K be a number field. The two constructions explained above give all the places on K.

We will denote by $\Omega(K)$ (respectively $\Omega_\infty(K)$ and $\Omega_f(K)$) the set of all places (respectively infinite places and finite places) of K. If $v \in \Omega(K)$ is a place, we denote by K_v the completion of K with respect to v. For a quaternion algebra B over K, we write B_v for $B \otimes_K K_v$, which is a quaternion algebra over K_v. When the place v comes from a prime ideal $\mathcal{P}$ of $\mathcal{O}_K$, we will sometimes write $K_{\mathcal{P}}$ instead of K_v and $B_{\mathcal{P}}$ instead of B_v.

2.2.5 Special elements in algebraic number fields

Definition 2.2.14 (Perron number)
A *real* algebraic integer $\lambda \in \mathbb{C}$ is a *Perron number* if $\lambda > 1$ and if all its real conjugates have an absolute value strictly smaller than λ.

Definition 2.2.15 (Pisot number)
A *real* algebraic integer $\lambda \in \mathbb{C}$ is a *Pisot–Vijayaraghavan number*, or sometimes just a *Pisot number*, if $\lambda > 1$ and if all of its Galois conjugates have absolute value strictly less than 1.

[1]Some authors use the word *(multiplicative) valuation* for what we call *absolute value*. This why a place is often denoted by v.

Definition 2.2.16 (Salem number)
A *real* algebraic integer $\lambda \in \mathbb{C}$ with $\lambda > 1$ is a *Salem number* if it satisfies the following properties:

- $\deg \lambda \geq 4$;
- λ^{-1} is a Galois conjugate of λ;
- all conjugates of λ except λ and λ^{-1} lie on the unit circle S^1, where $S^1 = \{z \in \mathbb{C} : |z| = 1\}$.

Remark 2.2.17
Some authors also consider Salem numbers of degree 2. We will adopt this point of view.

If λ is a Salem number, then the degree of its minimal polynomial is even. Moreover, this polynomial is self-reciprocal (or palindromic). We have the following alternative definition for Salem numbers.

Proposition 2.2.18 ([Sal63, page 26])
Let $\lambda \in \mathbb{C}$ be a real algebraic integer greater than 1. Then, λ is a Salem number if and only if the following condition is satisfied: every Galois conjugate of λ lies inside the unit disk and at least one of its conjugates lies on the unit circle.

2.2.6 Maximal real subfield of the cyclotomic field

We give in this section basic properties of the maximal real subfield of the cyclotomic field associated to a prime number. These fields will be considered in Chapter 6 about the Vinberg algorithm.

Let $q \in \mathbb{P}$ be an odd prime number and let $\mu = \mu_q$ be a primitive qth root of unity. The field $\mathbb{Q}[\mu]$ is a Galois extension of degree $q-1$ of $\mathbb{Q}$. The Galois group is $\left(\mathbb{Z}/_{q\mathbb{Z}}\right)^*$ which acts on $\mathbb{Q}[\mu]$ via

$$\sigma_l : \mathbb{Q}[\mu] \longrightarrow \mathbb{Q}[\mu], \quad \mu^a \longmapsto \mu^{l \cdot a}.$$

Let $\lambda = \mu + \mu^{-1} = \mu + \bar{\mu}$, so that $\lambda = 2\cos\frac{2\pi}{q}$ if $\mu = e^{\frac{2\pi i}{q}}$, and let $K = \mathbb{Q}[\lambda]$. We show by induction that $\mu^k + \mu^{-k} \in K$. If $k = 2m$, then we have

$$\begin{aligned}\lambda^k &= \sum_{i=0}^{k} \binom{k}{i} \mu^{k-2i} \\ &= \left(\mu^k + \mu^{-k}\right) + \sum_{i=1}^{m-1} \binom{k}{i} \left(\mu^{2i-k} + \mu^{k-2i}\right) + \binom{k}{m}.\end{aligned}$$

Thus, by induction hypothesis $\mu^k + \mu^{-k} \in K$. Similarly, if $k = 2m+1$, we find

$$\lambda^k = \left(\mu^k + \mu^{-k}\right) + \sum_{i=1}^{m} \binom{i}{k} \left(\mu^{2i-k} + \mu^{k-2i}\right).$$

Since all the Galois conjugates of λ lie in K, then K is a Galois extension of $\mathbb{Q}$. Moreover, we note that the degree $\left[K : \mathbb{Q}\right] = (q-1)/2$. Indeed we see that the only elements $\sigma_l \in \mathrm{Gal}(\mathbb{Q}[\mu], \mathbb{Q})$ that fix K pointwise are σ_1 and σ_{q-1} (or we

can remark that $(x+\mu)\big(x+\mu^{-1}\big) = x^2+\lambda\cdot x+1$ is the minimal polynomial of μ over K).
Since σ_{q-1}, which acts on the powers of μ by complex conjugation, fixes K we have $K \subset \mathbb{R}$. On the other hand, if we let $\alpha \in \mathbb{Q}[\mu] \cap \mathbb{R}$, then we can write $\alpha = \sum_{i=0}^{q-2} a_i \mu^i$ for some $a_i \in \mathbb{Q}$ and since $\alpha \in \mathbb{R}$, then $\sigma_{q-1}(\alpha) = \alpha$, which implies $\alpha_1 = \alpha_{q-1}, \alpha_2 = \alpha_{q-2}, \ldots$. Hence, we can write

$$\alpha = a_0 + \sum_{i=1}^{(q-1)/2} a_i \cdot \big(\mu^i + \mu^{-i}\big)$$

and thus $\alpha \in K$. Therefore, K is the maximal real subfield of $\mathbb{Q}[\mu]$.
It is known that the ring of integers of $\mathbb{Q}[\mu]$ is $\mathbb{Z}[\mu]$ which implies that $\mathbb{Z}[\lambda] \subset \mathcal{O}_K$. For the reverse inclusion, we use, as above, that $\mathbb{Z}[\mu] \cap \mathbb{R} \subset \mathbb{Z}[\lambda]$. The next proposition summarizes theses facts.

Proposition 2.2.19
Let $q \in \mathbb{P}$ be an odd prime number and let $K = \mathbb{Q}\big[\cos\frac{2\pi}{q}\big]$. Then, we have the following:

- *K is a Galois extension of $\mathbb{Q}$ of degree $(q-1)/2$.*
- *K is the maximal real subfield of the qth cyclotomic field.*
- *The ring of integers $\mathcal{O}_K$ of K is $\mathbb{Z}\big[\mu+\mu^{-1}\big]$.*
- *$\mathbb{Q}[\mu]$ is a CM-field.*
- *λ and all its conjugates $\lambda_i := \mu^i + \mu^{-i}$ form a $\mathbb{Z}$-basis of $\mathcal{O}_K$.*
- *$\sigma_j(\lambda_i) = \lambda_{i\cdot j}$.*

For an odd prime number q, it is known that $\mathcal{O}_{\mathbb{Q}[\mu]}$ is a principal ideal domain (PID) if and only if $q \in \{3, 5, 7, 11, 13, 17, 19\}$ (see [Was82, Theorem 11.1]). Moreover, if $\mathcal{O}_{\mathbb{Q}[\mu]}$ is a PID, then so is $\mathcal{O}_{\mathbb{Q}[\cos 2\pi/q]}$ (see [Was82, Theorem 4.10]). Since we get the fields $\mathbb{Q}$ and $\mathbb{Q}[\sqrt{5}]$ for $q = 3$ and $q = 5$ respectively, we will assume that $q \in \{7, 11, 13, 17, 19\}$.

The minimal polynomials of the $\mu + \mu^{-1}$ are the following:

q	Minimal polynomial
7	$x^3 + x^2 - 2x - 1$
11	$x^5 + x^4 - 4x^3 - 3x^2 + 3x + 1$
13	$x^6 + x^5 - 5x^4 - 4x^3 + 6x^2 + 3x - 1$
17	$x^8 + x^7 - 7x^6 - 6x^5 + 15x^4 + 10x^3 - 10x^2 - 4x + 1$
19	$x^9 + x^8 - 8x^7 - 7x^6 + 21x^5 + 15x^4 - 20x^3 - 10x^2 + 5x + 1$

We also notice that the discriminant of K is $q^{(q-3)/2}$. In particular, q is the only prime which ramifies in K.

Invertible elements of $\mathcal{O}_K$ We consider, as above, $K = \mathbb{Q}\big[\cos\frac{2\pi}{q}\big]$, where $q \in \{7, 11, 13, 19\}$. The elements of the multiplicative group

$$C = \langle \pm\mu, 1-\mu^a : 1 < a \leq q-1 \rangle \cap \mathcal{O}^*_{\mathbb{Q}[\mu]}$$

are called *cyclotomic units*. In general (i.e. when K is the maximal real subfield of the q^mth cyclotomic field for a prime q), the group $C \cap \mathcal{O}^*_K$ is of finite index in $\mathcal{O}^*_K$. However, in our case, since $m = 1$ and since $q = 7, 11, 13, 19$, we have $C \cap \mathcal{O}^*_K = \mathcal{O}^*_K$ (see [Was82, Theorem 8.2]). Moreover, we know a nice generating set for $\mathcal{O}^*_K$ (see [Was82, Lemma 8.1]):

$$\mathcal{O}^*_K = \langle -1, \mu^{(1-a)/2} \cdot \frac{1-\mu^a}{1-\mu} : 1 < a \leq \frac{q-1}{2} \rangle$$

Hence, we find that a generating set for $\mathcal{O}^*_K$ is given the following set;

$$\{-1, \lambda_{\frac{q-1}{2}}\} \cup \Big\{ -\lambda_i - \lambda_{i+1} - \ldots - \lambda_{(q-1)/2} : i = 2, \ldots, \frac{q-3}{2} \Big\}.$$

2.3 Some other results of number theory

Theorem 2.3.1
For every $\varepsilon > 0$, there exists $N(\varepsilon) \in \mathbb{N}$ such that for every $n \geq N(\varepsilon)$ there is a prime p such that $n < p < (1+\varepsilon)n$. Moreover, we have $N(1) = 4$ and $N\big(\frac{1}{5}\big) = 25$.

Proof. The first part of the result is proved in [HW08, 22.19, p. 494]. The case $\varepsilon = 1$ corresponds to Bertrand's postulate, proved in 1852 by Chebyshev, while $\varepsilon = \frac{1}{5}$ is proved in [Nag52]. □

2.4 The Brauer group

Computations of commensurability invariants of arithmetic hyperbolic Coxeter groups take place inside the so called *Brauer group*. We give here the definition and a few examples. For more details, the reader can refer to [Lam05] and [GS06].

Let K be a field and let A be a finite-dimensional central simple algebra over K (the center of A is K and A has no proper non-trivial two-sided ideal). By Wedderburn's theorem, there exists a unique (up to isomorphism) division algebra D over K and a unique integer n such that $A \cong \mathrm{Mat}(n; D)$. This allows to define an equivalence relation on the set of isomorphism classes of central simple algebras over K: two algebras $A \cong \mathrm{Mat}(n; D)$ and $A' \cong \mathrm{Mat}(n'; D')$ are said to be *Brauer equivalent* if and only if $D \cong D'$. The quotient set is endowed with the structure of an abelian group as follows:

$$[A] \cdot [B] = [A \otimes_K B].$$

We remark that the neutral element is the class of $\mathrm{Mat}(\cdot; K)$ and $[A]^{-1} = \big[A^{\mathrm{op}}\big]$, where A^{op} denotes the opposite algebra of A, that is $a \cdot_{\mathrm{op}} b = b \cdot a$. Note that we will often write $A \cdot B$ instead of $[A] \cdot [B]$.
The Brauer group of K is denoted by $\mathrm{Br}\, K$. All its elements are of finite order and the 2-torsion is generated by quaternions algebras (see [Mer82]).

Examples 2.4.1 • If K is an algebraically closed field and if A is a simple K-algebra of finite dimension, then there exists $n \in \mathbb{N}_0$ such that $A \cong \mathrm{Mat}(n; K)$. In particular, we have $\mathrm{Br}\, K = 1$.

- $\mathrm{Br}\,\mathbb{R} = \{1, \mathbb{H}\}$, where $\mathbb{H} = (-1, -1)_{\mathbb{R}}$ denotes the quaternions of Hamilton (see below).
- $\mathrm{Br}\, K = 1$ for every finite field K.

2.5 Quaternion algebras

Let K be a field of characteristic different of two. A *quaternion algebra* over K is a four dimensional central simple algebra over K. Since the characteristic of K is different of two, there exist a K-basis $\{1, i, j, k\}$ of A and two non-zero elements a and b of K such that the multiplication in A is given by the following rules:

$$i^2 = a, \quad j^2 = b, \quad ij = -ji = k.$$

We then write $A = (a, b)_K$ or just (a, b) if there is no confusion about the base field. We sometimes call $(a, b)_K$ the *Hilbert symbol* of the quaternion algebra. This is kind of unfortunate because we also have the *Hilbert symbol* of a field K, which is the function $K^* \times K^* \longrightarrow \{-1, 1\}$ defined as follows:

$$(a, b) = \begin{cases} 1 & \text{if } ax^2 + by^2 - z^2 = 0 \text{ has a non-trivial solution in } K^3, \\ -1 & \text{otherwise.} \end{cases}$$

We will use this function later when speaking about the ramification of rational quaternion algebras.
For an element $q = x + yi + zj + tk$, with $x, y, z, t \in K$, the standard involution $\overline{q} = x - yi - zj - tk$ gives rise to the norm

$$N : A \longrightarrow K, \quad q \longmapsto N(q) = q \cdot \overline{q} = x^2 - ay^2 - bz^2 + abt^2.$$

Since an element $q \in A$ is invertible if and only if $N(q) \neq 0$, we have the following proposition.

Proposition 2.5.1 ([Lam05, Chapter III, Theorem 2.7])
For a quaternion algebra $A = (a, b)_K$, the following are equivalent:

(i) A is a division algebra;

(ii) the norm $N : A \longrightarrow K$ has no non-trivial zero;

(iii) the equation $aX^2 + bY^2 = 1$ has no solution in $K \times K$;

(iv) the equation $aX^2 + bY^2 - Z^2 = 0$ has only the trivial solution.

Moreover, if A is not a division algebra, then $A \cong \mathrm{Mat}(2; K)$.

Therefore, deciding whether a given quaternion algebra is a division algebra or not reduces to a purely number theoretical question. We will come back to this question later.

Proposition 2.5.2
For every $a, b, c \in K^*$, we have the following isomorphisms of quaternion algebras:

$$\begin{aligned}
&(a,b) \cong (b,a), \quad (a,c^2 b) \cong (a,b), \quad (a,a) \cong (a,-1)\\
&(a,1) \cong (a,-a) \cong (a,1-a) \cong (1,1) \cong 1\\
&(a,b)\cdot(a,c) \cong (a,bc)\cdot \mathrm{Mat}(2;K), \quad (a,b)^2 \cong \mathrm{Mat}(4;K).
\end{aligned}$$

We note that the last two relations can be rewritten in the Brauer group as follows

$$(a,bc) = (a,b)\cdot(a,c), \quad (a,b)^2 = 1.$$

Proposition 2.5.3 ([Vig80, chapitre I, Théorème 2.9; chapitre III, Section 3])
If K is a number field, and if B_1 and B_2 are quaternion algebras over K, there exists a quaternion algebra B such that $B_1 \cdot B_2 = B$ in $\mathrm{Br}\, K$.

2.5.1 Isomorphism classes of quaternion algebras

We will see below (see Section 4.3.1.2) that the question about the commensurability of two arithmetic Coxeter subgroups of $\mathrm{Isom}\,\mathbb{H}^n$ reduces almost to deciding whether to quaternions algebras are isomorphic. Hence, we investigate in this section the isomorphism classes of quaternion algebras.

First, it is worth to mention that the isomorphism classes of quaternion algebras are *not* determined by Hilbert symbols (for example, we have $(5,3)_{\mathbb{Q}} \cong (-10,33)_{\mathbb{Q}}$). However, we will see that there is an efficient way to produce a set which completely describes the quaternion algebra: *the ramification set.*

The fact that $B_v := B \otimes_K K_v$ is either a division algebra or a matrix algebra (see Proposition 2.5.1) motivates the following definition.

Definition 2.5.4 (Ramification of a quaternion algebra)
Let B be a quaternion algebra defined over a number field K. The *ramification set* of B, denoted $\mathrm{Ram}\, B$, is defined as follows:

$$\mathrm{Ram}\, B = \big\{ v \in \Omega(K) : B_v := B \otimes_K K_v \text{ is a division algebra} \big\}.$$

We will also write

$$\mathrm{Ram}_f\, B = \mathrm{Ram}\, B \cap \Omega_f(K), \quad \mathrm{Ram}_\infty\, B = \mathrm{Ram}\, B \cap \Omega_\infty(K).$$

Theorem 2.5.5 ([Vig80, Chapter III, Theorem 3.1])
Let B be a quaternion algebra defined over a number field K. The ramification set of B is a finite set of even cardinality. Conversely, if $R \subset \Omega(K)$ is a finite set of even cardinality, there exists, up to isomorphism, a unique quaternion algebra B' over K such that $\mathrm{Ram}\, B' = R$.

Remark 2.5.6
Using (iv) of Proposition 2.5.1 it is easy to compute the infinite ramification $\mathrm{Ram}_\infty\, B$ of a quaternion algebra $B = (a,b)_K$. Indeed, if $\sigma : K \longrightarrow \mathbb{R}$ is a Galois embedding and if v is the corresponding absolute value, then $B_v \cong \big(\sigma(a), \sigma(b)\big)$. Thus, $v \in \mathrm{Ram}_\infty(a,b)_K$ if and only if $\sigma(a) < 0$ and $\sigma(b) < 0$.

Remark 2.5.7
When $K = \mathbb{Q}$, the previous theorem comes from classical results such as the Hasse-Minkowski principle (since two quaternion algebras are isomorphic if the quadratic spaces induced by their norms are isomorphic) and Hilbert's reciprocity law.

Finally, let us mention a result which helps for computations.

Proposition 2.5.8 ([Vig80, Page 78])
Let B_1 and B_2 be two quaternion algebras over a number field K and let B be such that $B_1 \cdot B_2 = B \in \operatorname{Br} K$ (see Proposition 2.5.3). Then, we have

$$\operatorname{Ram} B = \big(\operatorname{Ram} B_1 \cup \operatorname{Ram} B_2\big) \setminus \big(\operatorname{Ram} B_1 \cap \operatorname{Ram} B_2\big).$$

When dealing with arithmetic groups of odd dimension, we will need to compute the ramification of quaternion algebras over a quadratic extension of a number field.

Proposition 2.5.9
Let K be a number field and let $L = K(\sqrt{\delta})$ be a quadratic extension of K. Let also B be a quaternion algebra over K and $A := B \otimes_K L$. Then, the ramification sets at finite places of A and B are related as follows:

$$\operatorname{Ram}_f A = \{\mathfrak{P}_1, \mathfrak{P}_1', \ldots, \mathfrak{P}_r, \mathfrak{P}_r'\},$$

where each pair $\mathfrak{P}_i, \mathfrak{P}_i'$ is a pair of prime ideals of $\mathcal{O}_L$ which lie above a prime ideal $\mathcal{P}_i$ of $\mathcal{O}_K$ such that B is ramified at $\mathcal{P}_i$ and $\mathcal{P}_i$ splits completely (see Definition 2.2.6).

Proof. Let $\mathfrak{P}$ be a prime ideal of $\mathcal{O}_L$ and let $\mathcal{P} := \mathfrak{P} \cap \mathcal{O}_K$. We also consider the completions $L_{\mathfrak{P}}$ (respectively $K_{\mathcal{P}}$) of L (respectively K) with respect to the valuation defined by $\mathfrak{P}$ (respectively $\mathcal{P}$). We first note that $A_{\mathfrak{P}} \cong B_{\mathcal{P}} \otimes_{K_{\mathcal{P}}} L_{\mathfrak{P}}$. Indeed, we have

$$A_{\mathfrak{P}} \cong A \otimes_L L_{\mathfrak{P}} = (B \otimes_K L) \otimes_L L_{\mathfrak{P}} \cong B \otimes_K L_{\mathfrak{P}} \cong B_{\mathcal{P}} \otimes_{K_{\mathcal{P}}} L_{\mathfrak{P}}.$$

By classical results, we then have three possibilities for the ideal $\mathcal{P}\mathcal{O}_L$:

Inert case The ideal $\mathcal{P}\mathcal{O}_L$ is prime, meaning that $\mathcal{P}\mathcal{O}_L = \mathfrak{P}$.

Ramified case We have $\mathcal{P}\mathcal{O}_L = \mathfrak{P}^2$.

Split case There exists another prime ideal $\mathfrak{P}'$ above $\mathcal{P}$ such that $\mathcal{P}\mathcal{O}_L = \mathfrak{P}\mathfrak{P}'$.

In the first two cases, $[L_{\mathfrak{P}} : K_{\mathcal{P}}] = 2$ (see, for example, [CF67, Chapter I, §5, Proposition 3]) which implies that $A_{\mathfrak{P}}$ is a matrix algebra (see [Vig80, Chapitre II, théorème 1.3]). In the last case, we have $K_{\mathcal{P}} \cong L_{\mathfrak{P}}$ (again by [CF67]) and thus $A_{\mathfrak{P}} \cong B_{\mathcal{P}}$. Therefore, A is ramified at $\mathfrak{P}$ if and only if B is ramified at $\mathcal{P}$ if and only if A is ramified at $\mathfrak{P}'$, as required. □

Computing the ramification set when $K = \mathbb{Q}$ We consider in this section a quaternion algebra $B = (a, b)$ over $\mathbb{Q}$ and we explain how to compute its ramification. In this setting, the finite places are the p-adic valuations and there is exactly one infinite place, denoted by $|\cdot|_\infty$, corresponding to the usual absolute value. By virtue of Proposition 2.5.1(iv), it is clear that B is ramified at ∞ if and only if $a < 0$ and $b < 0$. Moreover, using propositions 2.5.2 and 2.5.8, we see that we only have to compute the ramification set for quaternion algebras which have one of the following form

$$(-1, q), \quad (-p, q), \quad (p, q), \quad \forall p, q \in \mathbb{P}.$$

Finally, the identity

$$(p, q) = (-p, q) \cdot (-1, q) \in \operatorname{Br} \mathbb{Q},$$

implies that the only first two cases are sufficient.

Proposition 2.5.10
We have $\operatorname{Ram}(-1, 2) = \emptyset$ *and* $\operatorname{Ram}(-1, -2) = \{2, \infty\}$.

If q is a prime number different from two, then we have the following ramification sets:

	$q \equiv 1$ *mod* 8	$q \equiv 3$ *mod* 8	$q \equiv 5$ *mod* 8	$q \equiv 7$ *mod* 8
$(-1, q)$	$\emptyset$	$\{2, q\}$	$\emptyset$	$\{2, q\}$
$(-1, -q)$	$\{2, \infty\}$	$\{q, \infty\}$	$\{2, \infty\}$	$\{q, \infty\}$
$(2, -q)$	$\emptyset$	$\{2, q\}$	$\{2, q\}$	$\emptyset$
$(-2, q)$	$\emptyset$	$\emptyset$	$\{2, q\}$	$\{2, q\}$

Finally, let $q_1, q_2 \in \mathbb{P} \setminus \{2\}$ be two distinct prime numbers. The ramification set of the quaternion algebra $(-q_1, q_2)$ is as follows:

	$q_2 \equiv 1$ *(mod 4)*	$q_2 \equiv 3$ *(mod 4)*
$q_1 \equiv 1$ *(mod 4)*	$\{q_1, q_2\}$ *if* $\left(\frac{q_1}{q_2}\right) = -1$ $\emptyset$ *otherwise*	$\{2, q_1\}$ *if* $\left(\frac{q_1}{q_2}\right) = -1$ $\{2, q_2\}$ *otherwise*
$q_1 \equiv 3$ *(mod 4)*	$\{q_1, q_2\}$ *if* $\left(\frac{q_1}{q_2}\right) = -1$ $\emptyset$ *otherwise*	$\{q_1, q_2\}$ *if* $\left(\frac{q_1}{q_2}\right) = 1$ $\emptyset$ *otherwise*

where $\left(\frac{a}{b}\right)$ denotes the Legendre symbol of a and b.

Proof. For $x, y \in \mathbb{Q}$ we have the Hilbert symbol $(x, y)_p \in \{-1, 1\}$ (see, for example, [Cas78]) and for a diagonal quadratic form $f = \langle a_1, \ldots, a_n \rangle$, we define:

$$c(f)_p = \prod_{i<j} (a_i, a_j)_p, \quad c(B)_p = c\big(\langle 1, -a, -b, ab \rangle\big)_p.$$

Using a lemma (see [Cas78, Lemme 2.6, page 59]), we find:

$$p \in \operatorname{Ram}(B) \Leftrightarrow c(B)_p = \begin{cases} 1 & p = 2, \infty \\ -1 & p \text{ odd.} \end{cases}$$

Therefore, to find the ramification set it is sufficient to compute the Hilbert symbols $c(B)_p$. To compute these symbols, we can use [Ser96, Part I, Chapter III, Theorem 1]. We note that if $p \in \operatorname{Ram} B$, then we must have $p = 2$ or $p \mid a$ or $p \mid b$.
We compute the Hilbert symbols:

$$\begin{aligned} c\big((-1,2)\big)_2 &= (-1,-2)_2 = -1, & c\big((-1,-2)\big)_2 &= (2,-1)_2 = 1,\\ c\big((-1,q)\big)_2 &= (-q,-q)_2 = (-1)^{\varepsilon(-q)}, & c\big((-1,q)\big)_q &= (-q,-q)_q = (-1)^{\varepsilon(q)},\\ c\big((-1,-q)\big)_2 &= (q,q)_2 = (-1)^{\varepsilon(q)}, & c\big((-1,-q)\big)_q &= (q,q)_q = (-1)^{\varepsilon(q)},\\ c\big((2,-q)\big)_q &= (-1)^{\omega(q)}, & c\big((-2,q)\big)_q &= (-1)^{\varepsilon(q)+\omega(q)}, \end{aligned}$$

$$\begin{aligned} c\big((2,-q)\big)_2 &= (q,-1)_2 \cdot (-2,-q)_2 = (-1)^{1+\omega(-q)},\\ c\big((-2,q)\big)_2 &= (2,-1)_2 \cdot (-q,-2)_2 = (-1)^{\varepsilon(-q)+\omega(-q)}, \end{aligned}$$

where

$$\varepsilon(n) = \begin{cases} 0 & \text{if } n \equiv 1 \pmod 4 \\ 1 & \text{if } n \equiv 3 \pmod 4 \end{cases}, \qquad \omega(n) = \begin{cases} 0 & \text{if } n \equiv \pm 1 \pmod 8 \\ 1 & \text{if } n \equiv \pm 3 \pmod 8. \end{cases}$$

Finally, if $q_1, q_2 \in \mathbb{P} \setminus \{2\}$ are two distinct prime numbers, we have

$$\begin{aligned} c\big((-q_1,q_2)\big)_2 &= (q_1,-1)_2 \cdot (-q_2,-q_1)_2 = (-1)^{\varepsilon(q_1)+\varepsilon(-q_1)\cdot\varepsilon(-q_2)}\\ c\big((-q_1,q_2)\big)_{q_1} &= (q_1,-1)_{q_1} \cdot (-q_2,-q_1)_{q_1} = \left(\frac{q_2}{q_1}\right)\\ c\big((-q_1,q_2)\big)_{q_2} &= (q_1,-1)_{q_2} \cdot (-q_2,-q_1)_{q_2} = (-1)^{\varepsilon(q_2)} \cdot \left(\frac{q_1}{q_2}\right), \end{aligned}$$

where $\left(\frac{a}{b}\right)$ denotes the Legendre symbol of a and b. □

2.6 Roots of polynomials

Definition 2.6.1 (Number of sign changes)
Let $\{a_1, \dots, a_n\}$ be an ordered sequence of real numbers. We say that a *sign change* occurs between a_k and a_l if $a_k \cdot a_l < 0$ and if either $l = k+1$, or $l > k+1$ and $a_{k+1} = \ldots = a_{l-1} = 0$. The *number of sign changes* of the sequence is the total number of sign changes.

Theorem 2.6.2 (Descartes' rule of signs)
Let $p(x) = a_0 + a_1 x + \ldots + a_n x^n$ be a real polynomial with $a_n \neq 0$. Let σ be number of sign changes in the sequence $\{a_0, , \dots, a_{n-1}, a_n\}$. Then, p has at most σ positive real zeros. Moreover, if r denotes the number of positive real zeros of p, then $\sigma - r$ is even.

Proof. See [Hen74, Theorem 6.2d]. □

Definition 2.6.3 (Sturm's sequence)
Let $p \in \mathbb{R}[x]$ be a square-free (i.e. without square factors) polynomial and construct the following sequence of polynomials:

- $p_0(x) = p(x)$;
- $p_1(x) = p'(x)$;
- for $k \geq 2$, we define p_k as the opposite of the remainder of the polynomial division of p_{k-2} by p_{k-1}, i.e. $p_{k-2} = p_{k-1} \cdot q_k - p_k$, with $\deg p_k < \deg p_{k-1}$;

The last polynomial p_m of the sequence is the first constant polynomial. Then, we call $\{p_0, p_1, \ldots, p_m\}$ a *Sturm sequence* for p.

Remarks 2.6.4
- The constant polynomial p_m is non-zero if and only if p is square-free.
- The previous definition is in fact a particular case of a Sturm sequence.

Theorem 2.6.5 (Sturm's theorem)
Let $p \in \mathbb{R}[x]$ be a square-free polynomial and let $\{p_0, p_1, \ldots, p_m\}$ be a Sturm sequence for p. For a real number $\lambda \in \mathbb{R}$, we denote by $\sigma(\lambda)$ the number of sign changes in the sequence $\{p_0(\lambda), \ldots, p_m(\lambda)\}$. Let $\alpha < \beta$ be two real numbers which are not roots of p. Then, the number of real zeros of p in the interval $[a, b]$ is $\sigma(\alpha) - \sigma(\beta)$.

Proof. See [Hen74, Section 6.3]. □

Remark 2.6.6
If we drop the assumption $p(\alpha) \cdot p(\beta) \neq 0$, then the result is the following: the number of real zeros in the interval $(\alpha, \beta]$ is given by $\sigma(\alpha) - \sigma(\beta)$.

Example 2.6.7
A Sturm sequence of the polynomial $p(x) = -4 + x - x^2 + x^3$ is given by $\{-4 + x - x^2 + x^3, 1 - 2x + 3x^2, \frac{35}{9} - \frac{4}{9}x, -\frac{3411}{16}\}$. We compute the sequences of signs:

- for $\alpha = 0$: $\{-, +, +, -\}$;
- for $\beta = 1$: $\{+, +, +, -\}$.

Therefore, p has $2 - 1 = 1$ real zero between 0 and 2.

Theorem 2.6.8
Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0$ be a real polynomial with $a_n \neq 0$. Then, all the real roots of p lie in the interval $(-\rho, \rho)$, where $\rho = 1 + \frac{1}{|a_n|} \max_{0 \leq k \leq n-1} |a_k|$.

Proof. See [RS02, Theorem 8.1.7]. □

2.7 Hilbert spaces

Definition 2.7.1 (Hilbert space)
A *Hilbert space* is a real inner product space $(\mathscr{H}, \langle -, - \rangle)$ (not necessarily of finite dimension) such that $\mathscr{H}$ is complete with respect to the metric induced by the inner product.

Definition 2.7.2 (Orthonormal basis of a Hilbert space)
Let $(\mathscr{H}, \langle -, - \rangle)$ be a Hilbert space and let $\mathcal{B} = \{x_i\} \subset \mathscr{H}$ be a collection of vectors of $\mathscr{H}$. We say that $\mathcal{B}$ is an *orthonormal Hilbert basis* of $\mathscr{H}$ if the following are satisfied:

1. For every $x_i, x_j \in \mathcal{B}$, we have $\langle x_i, x_j \rangle = \delta_i^j$.
2. The closure of the linear span of $\mathcal{B}$ is the whole space $\mathscr{H}$.

Theorem 2.7.3 ([Lax02, Chapter 6, Theorem 9])
Every Hilbert space has an orthonormal Hilbert basis.

Theorem 2.7.4 ([Con85, Proposition 4.14])
Any two orthonormal bases of a Hilbert space have the same cardinality.

Definition 2.7.5 (Dimension of a Hilbert space)
The *Hilbert dimension*, or just *dimension*, of a Hilbert space is the cardinality of one its orthonormal Hilbert bases.

Remark 2.7.6
In general, an orthonormal Hilbert basis is different from an orthonormal basis, and the Hilbert dimension is different from the dimension of the underlying vector space (see example 2.7.8). However, when the dimension is finite, these notions coincide. We will use the terms *algebraic basis* or *Hamel basis* to specify a basis of the underlying vector space.

Notation 2.7.7
For two Hilbert spaces $\mathscr{H}_1, \mathscr{H}_2$ we denote by $L(\mathscr{H}_1; \mathscr{H}_2)$ the vector space of continuous $\mathbb{R}$-linear maps from $\mathscr{H}_1$ to $\mathscr{H}_2$.

Example 2.7.8 (The sequence space ℓ^2)
Let ℓ^2 be the set of sequences of real numbers $(x_n)_{n\in\mathbb{N}}$ such that $\sum_{n\in\mathbb{N}} x_n^2 < \infty$. It is easy to see that the form

$$\begin{aligned} \ell^2 \times \ell^2 &\longrightarrow \mathbb{R} \\ (x, y) &\longmapsto \langle x, y \rangle = \sum_{n\in\mathbb{N}} x_n y_n \end{aligned}$$

is an inner product which turns ℓ^2 into a Hilbert space. The set $\{e_m\}_{m\in\mathbb{N}} \subset \ell^2$ of sequences such that $(e_m)_n = \delta_n^m$ is an orthonormal Hilbert basis of ℓ^2 but it obviously is not an orthonormal algebraic basis. In fact, the Hilbert dimension of ℓ^2 is $\aleph_0$ while the dimension of the underlying vector space is $2^{\aleph_0}$.

Definition 2.7.9 (Separable Hilbert space)
A Hilbert space is called *separable* if its Hilbert dimension is $\aleph_0$.

Remark 2.7.10
Up to isomorphism, ℓ^2 is the only separable Hilbert space.

Proposition 2.7.11 ([Lax02, Chapter 6, Theorem 3])
Let $\mathscr{H}$ be a Hilbert space and V be a closed linear subspace of $\mathscr{H}$. Then, we have the following:

- *$V^\perp$ is a closed linear subspace;*
- *we have the decomposition $\mathscr{H} = V \oplus V^\perp$.*

Corollary 2.7.12
If V is a finite-dimensional linear subspace of $\mathscr{H}$, then $\mathscr{H} = V \oplus V^\perp$ and both V and $V^\perp$ are Hilbert spaces.

Definition 2.7.13 (Tensor product of Hilbert spaces)
Let $(\mathscr{H}_i, \langle -,- \rangle_i)$, $1 \leq i \leq r$, be a finite family of Hilbert spaces and consider the usual *tensor product* (or *algebraic tensor product*) $\mathscr{H} := \mathscr{H}_1 \otimes \ldots \otimes \mathscr{H}_r$. We have a natural bilinear form defined on simple tensors

$$\big(e_{i_1} \otimes \ldots \otimes e_{i_r}, e'_{i_1} \otimes \ldots \otimes e'_{i_r}\big) \longmapsto \prod_{j=1}^{r} \langle e_{i_j}, e'_{i_j} \rangle, \quad e_{i_j}, e'_{i_j} \in \mathscr{H}_j, \forall 1 \leq j \leq r,$$

which extends by linearity to an inner product on $\mathscr{H}$ (see [Wei80, Section 3.4]). The completion of $\mathscr{H}$ with respect to the norm defined by this inner product is a Hilbert space which is denoted by $\mathscr{H}_1 \hat{\otimes} \ldots \hat{\otimes} \mathscr{H}_r$.

Proposition 2.7.14
Let $\mathscr{H}, \mathscr{H}'$ be two Hilbert spaces with their orthonormal Hilbert bases $\{e_i\}$ and $\{e'_j\}$. Then, the collection $\{e_i \otimes e'_j\}$ is an orthonormal Hilbert basis of $\mathscr{H} \hat{\otimes} \mathscr{H}'$.

Proof. See [Wei80, Theorem 3.12]. □

2.7.1 Geometry in Hilbert spaces

Let $(\mathscr{H}, \langle -,- \rangle)$ be a Hilbert space. A unit vector $a \in \mathscr{H}$ and a scalar $t \in \mathbb{R}$ define a hyperplane

$$P(a,t) = \big\{x \in \mathscr{H} : \langle a, x \rangle = t\big\},$$

which in turn gives rise to the reflection with respect to $P(a,t)$ given by

$$\begin{aligned} \tau : \mathscr{H} &\longrightarrow \mathscr{H} \\ x &\longmapsto x + 2(t - \langle a, x \rangle) \cdot a. \end{aligned}$$

The reflection extends naturally to a bijection of $\hat{\mathscr{H}} := \mathscr{H} \cup \{\infty\}$ by setting $\tau(\infty) = \infty$. In a similar way, a vector $a \in \mathscr{H}$ and a positive real number r define the sphere $S(a,r)$ of radius r centered at a in the usual way, which in turn leads to the inversion with respect to $S(a,r)$,

$$\begin{aligned} \sigma : \hat{\mathscr{H}} &\longrightarrow \hat{\mathscr{H}} \\ x &\longmapsto a + \left(\frac{r}{d(x,a)}\right)^2 \cdot (x - a), \end{aligned} \tag{2.1}$$

with the convention that $\sigma(a) = \infty$ and $\sigma(\infty) = a$.

Definition 2.7.15 (Generalized sphere)
A *generalized sphere* in $\hat{\mathscr{H}}$ is either an extended hyperplane $\hat{H} = H \cup \{\infty\}$, where H is a hyperplane of $\mathscr{H}$, or a sphere as above. If we want to emphasize on the parameters, we will write $\Sigma(a,r)$ for $\hat{P}(a,r) = P(a,r) \cup \{\infty\}$ or for $S(a,r)$.

Definition 2.7.16 (Reflection in a generalized sphere)
A *reflection* in a generalized sphere Σ is a reflection with respect to Σ if Σ is an extended hyperplane and an inversion in Σ if Σ is a sphere.

Definition 2.7.17
The group of transformations whose elements are *finite* products of reflections in generalized spheres is denoted by $\text{Möb}^*(\mathscr{H})$.

Definition 2.7.18 (Similarity)
Let $y \in \mathscr{H}$, $T \in O(\mathscr{H})$ and $\lambda \in \mathbb{R}_+^*$. The bijection of $\mathscr{H}$ to itself given by

$$x \longmapsto \lambda \cdot T(x) + y,$$

is called a *similarity*. The group of all similarities is denoted by $\text{Sim}(\mathscr{H})$; it contains $O(\mathscr{H})$ as a subgroup.

Topology of $\hat{\mathscr{H}}$ The collection of open subsets of $\mathscr{H}$ together with sets of the form $\{\infty\} \cup (\mathscr{H} \setminus F)$, where F is a *bounded* subset of $\mathscr{H}$, defines a topology on $\hat{\mathscr{H}}$. Equipped with this topology, reflections with respect to generalized spheres are homeomorphisms of $\hat{\mathscr{H}}$ to itself.

2.8 Riemannian manifolds

We briefly present here the basic definitions which lead to the concept of a Riemannian manifold. A standard reference for this topic is [Kli95]. For this section, we fix a Hilbert space $\mathscr{H}$ (note that $\mathscr{H}$ can be finite or infinite dimensional, separable or not).

Definition 2.8.1 (Topological manifold)
Let M be a topological space. We say that M is a *topological manifold modelled on $\mathscr{H}$*, or just *topological manifold* if there is no ambiguity on $\mathscr{H}$, if M is locally homeomorphic to $\mathscr{H}$.

Remarks 2.8.2
- Some authors require the underlying topological space of a topological manifold to be separable and Hausdorff. In this case, the Hilbert space $\mathscr{H}$ is separable and thus $\mathscr{H} \cong \ell^2$.
- A topological manifold modelled on a Hilbert space is sometimes called a *Hilbert manifold.*

Definition 2.8.3 (Differentiable map)
Let $U_1 \subset \mathscr{H}_1$ and $U_2 \subset \mathscr{H}_2$ be two open subsets of two Hilbert spaces $\mathscr{H}_1, \mathscr{H}_2$ and let $f : U_1 \longrightarrow U_2$ be a continuous map. We say that f is *differentiable* at $u_0 \in U_1$ if there exists $df_{u_0} \in L(\mathscr{H}_1; \mathscr{H}_2)$ such that

$$f(u) - f(u_0) - df_{u_0}(u - u_0) = o(|u - u_0|).$$

The map is called *differentiable of class* C^1 if it is differentiable at every $u_0 \in U_1$ and if the map $u \longmapsto df_u$ is continuous. A map of class $\mathcal{C}^k$ is defined in a similar way by induction. If f is of class C^k for every $k \in \mathbb{N}$, we say that f is *differentiable*, or *smooth.*

Definition 2.8.4 (Diffeomorphism)
A map $f : U_1 \longrightarrow U_2$ between two open subsets of two Hilbert spaces is a *diffeomorphism* if it is differentiable, bijective and if its inverse is also differentiable.

Definition 2.8.5 (Differentiable atlas)
Let M be a topological manifold modelled on $\mathscr{H}$. A *differentiable atlas* for M is a collection $(\varphi_i, U_i)_{i\in I}$ of *charts* which enjoys the following properties:

- Each U_i is an open set of M and the family $\{U_i\}_{i\in I}$ is a covering of M.
- $\varphi_i : U_i \longrightarrow \varphi_i(U_i)$ is a homeomorphism of U_i onto an open subset $\varphi_i(U_i)$ of $\mathscr{H}$.
- For every $i, j \in I$, the *transition map*

$$\varphi_{i,j} : \varphi_i(U_i \cap U_j) \longrightarrow \varphi_j(U_i \cap U_j), \quad \varphi_{i,j} = \varphi_j \circ \varphi_i^{-1},$$

  is a *diffeomorphism*.

Definition 2.8.6 (Equivalent atlases)
Two atlases of a topological manifold M are called *equivalent* if their union is an atlas of M.

Definition 2.8.7 (Differentiable structure)
A *differentiable structure* on a topological manifold is an equivalence class of differentiable atlases.

Definition 2.8.8 (Differentiable manifold)
A *differentiable manifold* is a topological manifold modelled on a Hilbert space together with a differentiable structure.

For a given differentiable manifold M and a point $p \in M$, one can define the tangent space T_pM at p as in the finite-dimensional case. This gives rise to the tangent bundle TM and leads to the definition of *vector field*. All the details are presented in [Kli95].

Definition 2.8.9 (Riemannian metric)
A *Riemannian metric* g on a differentiable manifold M is a family of inner products

$$g_p : T_pM \times T_pM \longrightarrow \mathbb{R},$$

such that for every pair of differentiable vector fields X, Y on M, the map

$$\begin{aligned} M &\longrightarrow \mathbb{R} \\ p &\longmapsto g_p(X_p, Y_p), \end{aligned}$$

is differentiable.

Definition 2.8.10 (Riemannian manifold)
A *Riemannian manifold* is a pair (M, g), where M is a differentiable manifold and g is a Riemannian metric on M.

Remark 2.8.11
We will often consider the Riemannian metric of a given Riemannian manifold as implicitly given and write $\langle v, w\rangle$ for two vectors $v, w \in T_pM$ instead of $g_p(v, w)$.

Definition 2.8.12 (Conformal map)
Let $f : M \longrightarrow N$ be a diffeomorphism between two Riemannian manifolds. The map f is called *conformal*, if there exists a differentiable map $\alpha : M \longrightarrow \mathbb{R}_+^*$ such that

$$\langle df_p(v), df_p(w)\rangle = \alpha(p)^2 \cdot \langle v, w\rangle, \quad \forall p \in M, \forall v, w \in T_pM.$$

Remark 2.8.13
The above definition means that a conformal map should preserve angles between curves meeting at a given point.

Example 2.8.14
Reflections in generalized spheres and similarities are conformal maps.

In fact, the converse is also true: similarities, eventually composed with a sphere inversion, are the only conformal maps of a real Hilbert space.

Theorem 2.8.15 (Liouville's theorem)
Let $U \subset \mathscr{H}$ be a connected open subset of a Hilbert space of dimension at least 3 and let $f : U \longrightarrow \mathscr{H}$ be a conformal map. Then, one of the following cases holds:

- *There exist $\lambda > 0$, $y \in \mathscr{H}$ and $T \in O(\mathscr{H})$ such that*
$$f(z) = \lambda \cdot T(z) + y, \quad \forall z \in U.$$
- *There exist $\lambda > 0$, $x, y \in \mathscr{H}$ and $T \in O(\mathscr{H})$ such that*
$$f(z) = \lambda \cdot T\big(\iota_x(z)\big) + y, \quad \forall z \in U,$$
where ι_x is the inversion with respect to the sphere $S(x,1)$, as given by equation (2.1) of page 25.

Proof. See [Huf76]. □

The previous theorem implies that every conformal map $f : U \longrightarrow \mathscr{H}$, where U is an open connected subset of $\mathscr{H}$, can be extended in a unique way to a homeomorphism $\hat{f} : \hat{\mathscr{H}} \longrightarrow \hat{\mathscr{H}}$, where $\hat{\mathscr{H}} := \mathscr{H} \cup \{\infty\}$.

Definition 2.8.16 (Möbius transformation, Möbius group)
The map $\hat{f}$ is called a *Möbius transformation*. The group of all Möbius transformations of a Hilbert space $\mathscr{H}$ is called the *Möbius group* of $\mathscr{H}$ and is denoted by $\text{Möb}(\mathscr{H})$.

Remark 2.8.17
When the dimension of the Hilbert space is finite, a Möbius transformation is the composition of a finite number of reflections in generalized spheres (see [Rat06, §4.3] for example). However, when the dimension of the space is infinite, some elements of $\text{Möb}(\mathscr{H})$ cannot be written as a finite composition of inversions. Indeed, a map of type $f(z) = \lambda \cdot T\big(\iota_x(z)\big) + y$ or $f(z) = \lambda \cdot T(z) + y$ can be written as a finite product of reflections in generalized spheres if and only if the space of fixed points of T has finite codimension. In other words, $\text{Möb}^*(\mathscr{H})$ (see Definition 2.7.17) is a proper subgroup of $\text{Möb}(\mathscr{H})$ if $\mathscr{H}$ is infinite-dimensional.

CHAPTER 3

Hyperbolic space, Coxeter groups and Coxeter polyhedra

In this chapter, we present the theoretical background related to hyperbolic space, reflection groups and polyhedra. In the first section, we introduce different models for the three simply connected complete Riemannian manifolds $\mathbb{X}^n$ of constant sectional curvature $+1$, 0, and -1. Concerning the hyperbolic space, we also explain how the classical models can be extended to the infinite-dimensional setting. The finite-dimensional case is treated in details in [Rat06] while its infinite-dimensional counterpart is presented in [Das12][1]. Finally, we also present well-known general facts about isometries of the hyperbolic space, especially in the upper half-space model.

Concerning Coxeter groups, we first give the abstract definition before restricting ourselves to *geometric* Coxeter groups, that is, Coxeter groups which are realized as discrete subgroups generated by finitely many reflections in hyperplanes of $\mathbb{X}^n$.

Finally, we present various invariants of hyperbolic Coxeter groups (growth series and growth rate, Euler characteristic, cocompactness and cofiniteness, f-vector, arithmeticity) and explain how we can compute these invariants. All these computations are implemented in my computer program `CoxIter`, which is presented in Chapter 5.

3.1 Three geometries and their models

It is well known that there exist only three simply connected complete Riemannian manifolds of constant sectional curvature of dimension $n \geq 2$: the spheres $\mathbb{S}^n$, the Euclidean spaces $\mathbb{E}^n$ and the hyperbolic spaces $\mathbb{H}^n$. Up to a rescaling of the metric, we can suppose that the curvatures are respectively $+1$, 0 and -1. We will write $\mathbb{X}$, or $\mathbb{X}^n$ if we want to emphasize on the dimension, for one of the three spaces.

For $\varepsilon = +1, 0, -1$, we consider $\mathbb{R}^{n+1}$ equipped with the bilinear form given

[1] Although only the *separable* case is treated in [Das12], the results we need also work for any infinite-dimensional hyperbolic space (i.e. not necessarily separable).

by

$$\begin{aligned}\langle -,-\rangle_\varepsilon : \mathbb{R}^{n+1}\times\mathbb{R}^{n+1} &\longrightarrow \mathbb{R}\\ (x,y) &\longmapsto \sum_{i=1}^{n} x_i y_i + \varepsilon\cdot x_{n+1}y_{n+1}.\end{aligned}$$

Recall that the form $\langle -,-\rangle_{-1}$ is often called *Lorentzian form*. We can now define the models for our geometries. The *n-dimensional sphere* $\mathbb{S}^n$ is defined to be the set

$$\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : \langle x,x\rangle_1 = 1\},$$

together with the distance function $d = d_{\mathbb{S}^n}$ given by

$$\cos d(x,y) = \langle x,y\rangle_1, \quad \forall x,y \in \mathbb{S}^n.$$

The *n-Euclidean space* $\mathbb{E}^n$ can be identified with the set

$$\mathbb{E}^n = \{x \in \mathbb{R}^{n+1} : x_{n+1} = 0\},$$

endowed with the metric given by

$$d(x,y) = d_{\mathbb{E}^n}(x,y) = \sqrt{\langle x,y\rangle_0}, \quad \forall x,y \in \mathbb{R}^{n+1}.$$

The *hyperboloid model*, or *vector space model*, of the *hyperbolic n-space* $\mathbb{H}^n$ arises as the set

$$\mathcal{H}^n = \{x \in \mathbb{R}^{n+1} : \langle x,x\rangle_{-1} = -1, x_{n+1} > 0\}, \tag{3.1}$$

together with the distance defined by the relation

$$d(x,y) = d_{\mathcal{H}^n}(x,y) = \operatorname{arcosh}(-\langle x,y\rangle_{-1}), \quad \forall x,y \in \mathcal{H}^n.$$

The volume element is given by

$$\frac{dx_1\cdot\ldots\cdot dx_n}{\sqrt{1+x_1^2+\ldots+x_n^2}},$$

as shown in [Rat06, Theorem 3.4.1]. The *boundary* of $\mathcal{H}^n$ can be identified with the set

$$\partial\mathcal{H}^n = \big\{x \in \mathbb{R}^{n+1} : \langle x,x\rangle_{-1} = 0, \sum_{i=1}^{n+1} x_i^2 = 1, x_{n+1} \geq 0\big\},$$

and we let $\overline{\mathcal{H}^n} := \partial\mathcal{H}^n \cup \mathcal{H}^n$. A *hyperplane* of $\mathcal{H}^n$ is given by the intersection of the orthogonal complement (with respect to the Lorentzian product) of a vector v of Lorentzian norm 1 and $\mathcal{H}^n$. We denote such a hyperplane by H_v. Remark that such a hyperplane splits $\mathcal{H}^n$ into two half-spaces $H_v^+ := \big\{x \in \mathbb{H}^n : \langle v,x\rangle \geq 0\big\}$ and $H_v^- := \big\{x \in \mathbb{H}^n : \langle v,x\rangle \leq 0\big\}$ whose intersection is H_v. The relative behaviour of two distinct hyperplanes H_v and H_w can be described by means of the Lorentzian product of v and w (see [Rat06, Theorems 3.2.6, 3.2.7 and 3.2.9] or [Vin85, Section 1.1]):

- The hyperplanes intersect if and only if $|\langle v,w\rangle| < 1$. In this case, the acute dihedral angle in $\big(0,\frac{\pi}{2}\big]$ between them is given by $\arccos(|\langle v,w\rangle|)$.

- The hyperplanes are *parallel* if and only if $|\langle v, w\rangle| = 1$. In this case their dihedral angle is 0.

- The hyperplanes are *ultraparallel* if and only if $|\langle v, w\rangle| > 1$. In this setting, the two hyperplanes admit a common perpendicular of length $\operatorname{arcosh}(|\langle v, w|\rangle)$.

3.2 Models of the hyperbolic space

In this section, we present two other models of the hyperbolic space, the *upper half-space model* and the *Poincaré ball model*, together with a different version of the hyperboloid model. We adopt an approach that allows us to define models which are both finite and infinite-dimensional. We use the notation $\mathbb{H}^{\mathfrak{n}}$ to denote any of the three models $\mathcal{U}^{\mathfrak{n}}$, $\mathcal{H}^{\mathfrak{n}}$ and $\mathcal{B}^{\mathfrak{n}}$. We also describe the isometries of the upper half-space model.

We consider a Hilbert space $(\mathscr{H}, \langle -, -\rangle_{\mathscr{H}})$ of Hilbert dimension $\mathfrak{n}$ (see Definition 2.7.5) and some unit vector $u \in \mathscr{H}$. By Corollary 2.7.12, we have a decomposition of $\mathscr{H}$ as a direct sum of Hilbert space, that is $\mathscr{H} = \langle u\rangle^{\perp} \oplus \langle u\rangle$ and this decomposition comes with the projection $\pi_{\langle u\rangle^{\perp}}$ of $\mathscr{H}$ onto $\langle u\rangle^{\perp}$ and the functional l_u defined as follows:

$$\begin{aligned} l_u : \mathscr{H} &\longrightarrow \mathbb{R} \\ x &\longmapsto \langle x, u\rangle_{\mathscr{H}}. \end{aligned}$$

3.2.1 The upper half-space model

We consider the set

$$\mathcal{U}^{\mathfrak{n}} = \mathcal{U}_{\mathscr{H}} = \{x \in \mathscr{H} : l_u(x) > 0\},$$

together with the distance function given by

$$d(x, y) = d_{\mathcal{U}^{\mathfrak{n}}}(x, y) = \operatorname{arcosh}\left(1 + \frac{d_{\mathscr{H}}(x, y)}{2 \cdot l_u(x) \cdot l_u(y)}\right), \quad \forall x, y \in \mathcal{U}^{\mathfrak{n}}.$$

We call $\mathcal{U}^{\mathfrak{n}}$ the *upper half-space model* of the hyperbolic $\mathfrak{n}$-space. The *boundary* $\mathcal{U}^{\mathfrak{n}}$ is given by

$$\partial\mathcal{U}^{\mathfrak{n}} = \big\{x \in \mathscr{H} : l_u(x) = 0\big\} \cup \{\infty\}.$$

Remarks 3.2.1

- The upper half-space model is suitable to get isometries from 2×2 Clifford matrices. We will come back to this question in Chapter 8.

- If $\mathscr{H} = \mathbb{R}^n$ and $u = e_n$, the above construction gives the usual upper half-space model U^n (see [Rat06, §4.6]).

- When $\mathscr{H} = \ell^2$ (see Example 2.7.8), we usually take $u = (1, 0, \ldots)$.

- If the dimension of $\mathscr{H}$ is infinite, we will often discard the cardinal $\mathfrak{n}$ and only write $\mathcal{U}^{\infty}$.

3.2.2 The hyperboloid model

We define on the product $\mathbb{R} \times \mathscr{H}$ the bilinear form

$$B\big((\lambda, x), (\mu, y)\big) = \langle x, y\rangle_{\mathscr{H}} - \lambda\mu,$$

and we consider its associated quadratic form q. Then

$$\mathcal{H}^{\mathfrak{n}} = \mathcal{H}_{\mathscr{H}} = \big\{(\lambda, x) \in \mathbb{R} \times \mathscr{H} : q(\lambda, x) = \|x\|^2_{\mathscr{H}} - \lambda^2 = -1, \lambda > 0\big\},$$

together with the distance function defined by

$$d = d_{\mathcal{H}^{\mathfrak{n}}} = \operatorname{arcosh}\big(-B\big((\lambda, x), (\mu, y)\big)\big), \quad \forall (\lambda, x), (\mu, y) \in \mathcal{H}^{\mathfrak{n}}.$$

We call $\mathcal{H}^{\mathfrak{n}}$ the *hyperboloid model*, or the *vector space model*, of the hyperbolic $\mathfrak{n}$-space in $\mathbb{R} \times \mathscr{H}$.

Remarks 3.2.2
- The hyperboloid model is suitable to define and treat hyperplanes in terms of linear subspaces.
- If $\mathscr{H} = \mathbb{R}^n$, the above construction gives the classical hyperboloid model $\mathcal{H}^n$ presented above (see equation (3.1) on page 63 and [Rat06, §3.2]).
- As for the upper half-space model, if the dimension of $\mathscr{H}$ is infinite, we will often discard the cardinal $\mathfrak{n}$ and write only $\mathcal{H}^\infty$.
- Another standard way to construct $\mathcal{H}^{\mathfrak{n}}$ is to look at the quotient $C/_{\mathbb{R}^*}$, where C is the cone $\big\{(\lambda, x) \in \mathcal{H}^{\mathfrak{n}} : q(\lambda, x) < 0\big\}$. Then, the distance d on C given by
$$\cosh^2 d(x, y) = \frac{B(x, y)^2}{q(x) \cdot q(y)}, \quad \forall x, y \in C,$$
induces a distance on $C/_{\mathbb{R}^*}$.
- The space $\mathbb{R} \times \mathscr{H}$, together with the quadratic form $(\lambda, x) \longmapsto \|x\|^2 - \lambda^2$, is a quadratic space of index 1 (see [BIM05] for the definition). Moreover, the space $\mathcal{H}^{\mathfrak{n}}$ is a geodesic CAT(-1) space (see [BIM05, Proposition 4.2]).

3.2.3 Poincaré ball model

We consider the set

$$\mathcal{B}^{\mathfrak{n}} = \mathcal{B}_{\mathscr{H}} = \big\{x \in \mathscr{H} : \|x\|_{\mathscr{H}} < 1\big\},$$

together with the distance function given by

$$d_{\mathcal{B}^{\mathfrak{n}}}(x, y) = \operatorname{arcosh}\left(1 + 2 \cdot \frac{d_{\mathscr{H}}(x, y)}{\big(1 - \|x\|^2_{\mathscr{H}}\big) \cdot \big(1 - \|y\|^2_{\mathscr{H}}\big)}\right), \quad \forall x, y \in \mathcal{B}^{\mathfrak{n}}.$$

We call $\mathcal{B}^{\mathfrak{n}}$ the *Poincaré ball model* of the hyperbolic $\mathfrak{n}$-space. Its boundary is given by

$$\partial\mathcal{B}^{\mathfrak{n}} = \{x \in \mathscr{H} : \|x\|_{\mathscr{H}} = 1\}.$$

Remarks 3.2.3
- When the dimension of $\mathscr{H}$ is finite, the Poincaré ball model is suitable to visualize non-compact polyhedra.
- If $\mathscr{H} = \mathbb{R}^n$, the above construction gives the usual Poincaré ball model (see [Rat06, §4.5]).
- If the dimension of $\mathscr{H}$ is infinite, we will often discard the cardinal $\mathfrak{n}$ and write only $\mathcal{B}^\infty$.

3.2.4 Models of the hyperbolic space as Riemannian manifolds

As in the finite dimensional case, the models $\mathcal{U}^{\mathfrak{n}}$ and $\mathcal{H}^{\mathfrak{n}}$ of the hyperbolic space can be viewed as Riemannian manifolds (see Definition 2.8.10). The Riemannian metric at a point $p \in \mathbb{H}^{\mathfrak{n}}$ are given in the next table.

Model	Riemannian metric
$\mathcal{U}^{\mathfrak{n}}$	$g_p(v,w) = \dfrac{\langle v,w\rangle_{\mathscr{H}}}{l_u(p)^2}, \quad \forall v,w \in T_p\mathcal{U}^{\mathfrak{n}} \cong \mathscr{H}$
$\mathcal{H}^{\mathfrak{n}}$	$g_p\big((\lambda,x),(\mu,y)\big) = -\lambda\cdot\mu + \langle x,y\rangle_{\mathscr{H}}, \quad \forall (\lambda,x),(\mu,y) \in T_p\mathcal{H}^{\mathfrak{n}}$
$\mathcal{B}^{\mathfrak{n}}$	$g_p(v,w) = \dfrac{4\cdot\langle v,w\rangle_{\mathscr{H}}}{\big(1-\lVert x\rVert_{\mathscr{H}}^2\big)^2}, \quad \forall v,w \in T_p\mathcal{B}^{\mathfrak{n}} \cong \mathscr{H}$

Note that the tangent space $T_p\mathcal{H}^{\mathfrak{n}}$ at a point $p = (\nu, z) \in \mathcal{H}^{\mathfrak{n}}$ is given by

$$T_p\mathcal{H}^{\mathfrak{n}} = \big\{(\lambda,x) \in \mathbb{R}\times\mathscr{H} : B\big((\nu,z),(\lambda,x)\big)\big) = \langle z,x\rangle_{\mathscr{H}} - \nu\lambda = 0\big\}.$$

3.2.5 Isometries of the upper half-space

Reflections in generalized spheres and similarities (see Section 2.7.1) of a codimension 1 subspace in $\mathscr{H}$ can be used to define isometries of the hyperbolic space as follows. We consider the decomposition $\mathscr{H} = \langle u\rangle^{\perp} \oplus \langle u\rangle$, where $u \in \mathscr{H}$ is a unit vector, and the associated upper half-space model $\mathcal{U} = \mathcal{U}_{\mathscr{H}} = \{x \in \mathscr{H} : l_u(x) > 0\}$. Now, if $\Sigma(a,r)$ is a generalized sphere in the Hilbert space $\langle u\rangle^{\perp}$ with associated reflection σ, then $\Sigma(a+0,r)$ is a generalized sphere of $\mathscr{H}$ with associated reflection $\tilde{\sigma}$ in a natural way. On the other hand, if

$$\eta : \langle u\rangle^{\perp} \longrightarrow \langle u\rangle^{\perp}, \quad x \longmapsto \lambda\cdot T(x) + y$$

is a similarity (see Definition 2.7.18) of $\langle u\rangle^{\perp}$, then we have an induced similarity $\tilde{\eta}$ of $\mathscr{H}$ given by

$$\tilde{\eta} : \mathscr{H} \longrightarrow \mathscr{H}, \quad x \longmapsto \lambda\cdot\big(T(\pi_{\langle u\rangle^{\perp}}(x)) + \pi_{\langle u\rangle}(x)\big) + y,$$

where π_V denotes the projection onto the subspace V of $\mathscr{H}$. It is easily shown that these induced maps are isometries of the space $\mathcal{U}^{\mathfrak{n}}$.

Definition 3.2.4 (Poincaré extension)
For a reflection σ in a generalized sphere of $\langle u\rangle^{\perp}$ (respectively a similarity $\eta \in \mathrm{Sim}(\langle u\rangle^{\perp})$), the induced bijection $\tilde{\sigma}$ (respectively $\tilde{\eta}$) of $\mathscr{H}$ is called the *Poincaré extension* of σ (respectively η).

Remark 3.2.5
In a similar way, if V is any subspace of $\mathscr{H}$ and if φ is an element of $\mathrm{M\ddot{o}b}(V)$, then φ can be extended via Poincaré extension to an element of $\mathrm{M\ddot{o}b}(\mathscr{H})$.

The following result provides an important characterisation of isometries of the upper half-space.

Theorem 3.2.6
Let $g : \mathcal{U}^{\mathrm{n}} \longrightarrow \mathcal{U}^{\mathrm{n}}$ be a diffeomorphism. Then, the following are equivalent:

1. *g is a conformal isomorphism of $\mathcal{U}^{\mathrm{n}}$.*
2. *g extends to a Möbius transformation $\hat{g} \in \mathrm{Möb}(\mathscr{H})$ which preserves $\mathcal{U}^{\mathrm{n}}$.*
3. *g preserves the Riemannian metric of $\mathcal{U}^{\mathrm{n}}$.*
4. *g is an isometry of $\mathcal{U}^{\mathrm{n}}$, that is,*

$$d_{\mathcal{U}^{\mathrm{n}}}\big(g(x), g(y)\big) = d_{\mathcal{U}^{\mathrm{n}}}(x, y), \quad \forall x, y \in \mathcal{U}^{\mathrm{n}}.$$

Proof. The proof given in [Das12, Theorem 2.15] also works in the non-separable case. □

Remark 3.2.7
To avoid cumbersome notations, we will often use the same notation for an isometry $g : \mathcal{U}^{\mathrm{n}} \longrightarrow \mathcal{U}^{\mathrm{n}}$ and its Möbius extension $\hat{g} : \mathscr{H} \longrightarrow \mathscr{H}$.

If V is any closed subspace of $\langle u\rangle^{\perp}$ and if g is a similarity of V, then the Poincaré extension $\hat{g} : \mathscr{H} \longrightarrow \mathscr{H}$ will preserve the upper half-space $\mathcal{U}^{\mathrm{n}}$, which means that the restriction of $\hat{g}$ to $\mathcal{U}^{\mathrm{n}}$ is an isometry. Reciprocally, if g is an isometry of $\mathcal{U}^{\mathrm{n}}$, then the extension $\hat{g} : \mathscr{H} \longrightarrow \mathscr{H}$ preserves $\partial\mathcal{U}^{\mathrm{n}} = \langle u\rangle^{\perp} \cup \{\infty\}$. Therefore, $\hat{g}\big|_{\partial\mathcal{U}^{\mathrm{n}}}$ is an element of $\mathrm{Möb}(\partial\mathcal{U}^{\mathrm{n}})$ whose Poincaré extension is $\hat{g}$. In other words, we have the following corollary.

Corollary 3.2.8
The group of isometries $\mathrm{Isom}(\mathcal{U}^{\mathrm{n}})$ is isomorphic to the group $\mathrm{Möb}(\langle u\rangle^{\perp})$.

We have the following analogous result for the group $\mathrm{Möb}^*$.

Corollary 3.2.9
If $g : \mathcal{U}^{\mathrm{n}} \longrightarrow \mathcal{U}^{\mathrm{n}}$ is an isometry of $\mathcal{U}^{\mathrm{n}}$ whose Möbius extension $\hat{g}$ can be written as a composition of a finite number of reflections in generalized spheres (i.e. we have $\hat{g} \in \mathrm{Möb}^*(\mathscr{H})$), then there exists a finite-dimensional *subspace V of $\langle u\rangle^{\perp}$ and an element $\varphi \in \mathrm{Möb}(V)$ such that the Poincaré extension of φ to $\mathscr{H}$ is equal to $\hat{g}$.*

Remark 3.2.10
We will see in Chapter 8 that the isometries arising as a *finite* composition of reflections in generalized spheres are *exactly* the isometries coming from Clifford matrices.

3.2.5.1 Classification of isometries

Any isometry of $\mathbb{H}^{\mathrm{n}}$ can be represented as an isometry of the ball model $g : \mathcal{B}^{\mathrm{n}} \longrightarrow \mathcal{B}^{\mathrm{n}}$ and this isometry can be extended in a unique way to a continuous bijective map $g : \overline{\mathcal{B}}^{\mathrm{n}} \longrightarrow \overline{\mathcal{B}}^{\mathrm{n}}$. This extension has (at least) one fixed point by the Brouwer fixed point theorem (or the Schauder fixed point theorem in the infinite-dimensional setting). Then, the isometry g is said to be

1. *elliptic* if g has a fixed point in $\mathcal{B}^{\mathrm{n}}$;
2. *parabolic* if g has exactly one fixed point in $\partial\mathcal{B}^{\mathrm{n}}$ and no fixed point in $\mathcal{B}^{\mathrm{n}}$;

3. *loxodromic*, if g has exactly two fixed points in $\partial\mathcal{B}^{\mathrm{n}}$ and no fixed point in $\mathcal{B}^{\mathrm{n}}$.

We notice that this definition is independent of the conjugacy class of g. Moreover, the same classification can be done independently of the model. Therefore, we will call an isometry $g : \mathbb{H}^{\mathrm{n}} \longrightarrow \mathbb{H}^{\mathrm{n}}$

1. *elliptic* if g has a fixed point in $\mathbb{H}^{\mathrm{n}}$;

2. *parabolic* if g has exactly one fixed point in $\partial\mathbb{H}^{\mathrm{n}}$ and no fixed point in $\mathbb{H}^{\mathrm{n}}$;

3. *loxodromic*, or *hyperbolic*, if g has exactly two fixed points in $\partial\mathbb{H}^{\mathrm{n}}$ and no fixed point in $\mathbb{H}^{\mathrm{n}}$.

There is some work to be done to show that these three possibilities are mutually exclusive and that any isometry of $\mathbb{H}^{\mathrm{n}}$ falls into precisely one of this three cases; we refer to [Das12, §2.7] for the proof and some details. Finally, we have the following explicit characterization of isometries.

Proposition 3.2.11 ([Das12, Proposition 2.30])
Let $g \in \operatorname{Isom}\mathbb{H}^{\mathrm{n}}$ be an isometry of the hyperbolic space $\mathbb{H}^{\mathrm{n}}$. Then, g is conjugate to exactly one of the following:

- *An isometry of $\mathcal{B}^{\mathrm{n}}$ coming from an isometry T of $O(\mathscr{H})$. This is the* elliptic *case.*

- *An isometry of $\mathcal{U}^{\mathrm{n}}$ coming from a Möbius transformation $x \mapsto T(x) + y$, with $y \neq 0$ and $T \in O(\mathscr{H})$. This corresponds to the* parabolic *case.*

- *An isometry of $\mathcal{U}^{\mathrm{n}}$ coming from a Möbius transformation $x \mapsto \lambda T(x)$, with $T \in O(\mathscr{H})$ and $\lambda > 0$, $\lambda \neq 1$. This corresponds to the* loxodromic *case.*

Proposition 3.2.12
Let g be a parabolic isometry of the hyperbolic space $\mathcal{U}^{\mathrm{n}}$ and suppose that g is conjugate to the map $\varphi : x \longmapsto T(x) + y$, with $T \in O(\mathscr{H})$ (see Proposition 3.2.11) and some $y \in \mathscr{H}$. If the space $\operatorname{Fix} T$ of fixed points of T has finite codimension (or equivalently if T can be written as a finite composition of reflections), then one can suppose that $y \in \operatorname{Fix} T$.

Proof. Since T is continuous, the set $\operatorname{Fix} T$ is a closed subspace of $\mathscr{H}$. Thus, Proposition 2.7.11 implies that we have the decomposition $\mathscr{H} = \operatorname{Fix} T \oplus C$, with $C := (\operatorname{Fix} T)^{\perp}$. Now, we can write $y = f + c$ with $f \in \operatorname{Fix} T$ and $c \in C$. Moreover, since the restriction of $T - \mathrm{id}$ to C is injective, and since C is finite dimensional, then there exists $c' \in C$ such that $(T - \mathrm{id})(c') = c$. As a consequence, the conjugate $\tau \circ \varphi \circ \tau^{-1}$, with $\tau(x) = x + c'$, sends x to $T(x) + f$, as required. □

3.3 Abstract Coxeter groups

Definition 3.3.1 (Coxeter group)
A *Coxeter group* is a finitely presented group generated by elements $s_1, \ldots, s_d$ and such that

$$\langle s_1, \ldots, s_d : (s_i\, s_j)^{m_{ij}} = 1 \rangle,$$

where $m_{ij} = 1$ if and only if $i = j$, and $m_{ij} = m_{ji} \in \{2, 3, \ldots\} \cup \{\infty\}$ if $i \neq j$. By $m_{ij} = \infty$, we mean that there is no relation between s_i and s_j. If Γ is such a group and $S := \{s_1, \ldots, s_d\}$ is the set of generators, we will refer to the group as (Γ, S), or just Γ if there is no ambiguity on S. When it is convenient, we use the notation $m(s_i, s_j) := m_{ij}$.
If S' is a subset of S, we denote by $\Gamma_{S'}$ the subgroup of Γ generated by the elements of S'.

Definition 3.3.2 (Coxeter graph of a Coxeter group)
An easy way to represent a Coxeter group $\big(\Gamma, S = \{s_1, \ldots, s_d\}\big)$ is by its *Coxeter diagram*, or *Coxeter graph*. The Coxeter diagram of Γ is the graph whose vertices correspond to the elements of S. Moreover, there is an edge between the vertices s_i and s_j if and only if $m(s_i, s_j) \geq 3$. We label such an edge with $m(s_i, s_j)$ if $m(s_i, s_j) \geq 4$. Sometimes, instead of labelling the edge, we use a double edge if $m(s_i, s_j) = 4$ and a bold (or heavy) edge if $m(s_i, s_j) = \infty$.

Examples 3.3.3
- It is well known that the symmetric group S_n is isomorphic to the Coxeter group A_{n-1} (see Figure 3.1): the ith node of A_{n-1} corresponds to the transposition $(i, i+1)$ of S_n.
- The dihedral group of $2m$ elements is isomorphic to $G_2^{(m)}$ (see Figure 3.1).
- If P is a polyhedron (see Definition 3.4.1) in the affine space $\mathbb{R}^n$ whose dihedral angles are integer submultiples of π and if $\Gamma < \operatorname{Isom} \mathbb{R}^n$ is the (discrete) group generated by the reflections in the facets of P, then Γ is a Coxeter group.

Definition 3.3.4 (Irreducible Coxeter group)
A Coxeter group is said to be *irreducible* if its Coxeter diagram is connected.

It is easy to see that a Coxeter group is the product of the groups corresponding to the connected components of its Coxeter diagram. Moreover, the finite irreducible Coxeter groups were classified by Coxeter [Cox35] by means of their Coxeter diagrams. The classification is presented in Figure 3.1.

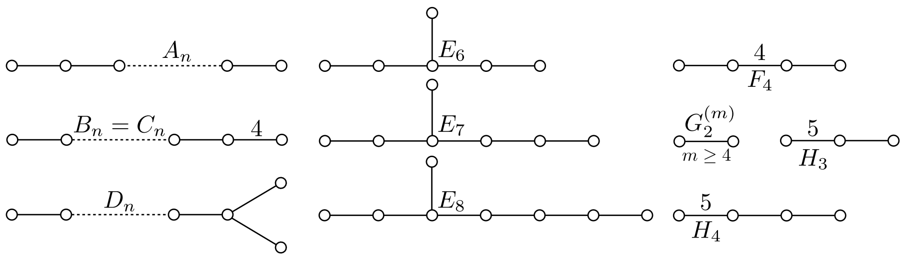


Figure 3.1 – Irreducible finite Coxeter groups

3.4 Geometric Coxeter groups

In this section, we present the connection between abstract Coxeter groups and discrete reflection groups in the facets (sides of codimension 1) of convex finite-sided polyhedra. The main reference is [Rat06, §7.1]. In what follows, X denotes the unit n-sphere S^n, the Euclidean n-space E^n or $\mathcal{H}^n$.

A vector $v \in E^n$ and a real number $t \in \mathbb{R}$ determine an (affine) *hyperplane* $H_{v,t} := \{x \in E^n : \langle x, v\rangle = t\}$, denoted H_v if $t = 0$, and such a hyperplane splits E^n in two *half-spaces* $H^+_{v,t} := \big\{x \in E^n : \langle v, x\rangle \geq t\big\}$ and $H^-_{v,t} := \big\{x \in E^n : \langle v, x\rangle \leq t\big\}$ whose intersection is $H_{v,t}$. A *hyperplane* in S^n is the intersection of a hyperplane H_v of E^{n+1} with S^n. Similarly, we saw in Section 3.1 that a hyperplane of $\mathcal{H}^n$ is the intersection of a hyperplane of E^n with $\mathcal{H}^n$.

Definition 3.4.1 (Polyhedron)
A *polyhedron* P in X is a subset of X which is the intersection of finitely many half-spaces, each one bounded by a hyperplane. Moreover, we require that P has non-empty interior and that no half-space contains the intersection of the others.

Definition 3.4.2 (Reflection with respect to a hyperplane)
Let H_v be a hyperplane of X. The reflection $r_v = r_{H_v}$ associated to the hyperplane H_v is defined as follows (recall that we choose v with norm 1):

$$\begin{aligned} r_{H_v} : X &\longrightarrow X \\ x &\longmapsto x - 2\langle x, v\rangle v. \end{aligned}$$

Definition 3.4.3 (Reflection group associated to a polyhedron)
Let $P = \bigcap_{i=1}^{d} H^-_{v_i}$ be a polyhedron in X and $\Gamma \leq \operatorname{Isom} X$. We say that Γ is the *reflection group* associated to P if Γ is generated by the reflections in the facets of P, that is $\Gamma = \langle r_{H_{v_1}}, \dots, r_{H_{v_d}}\rangle$.

Definition 3.4.4 (Cell of a group generated by reflections)
Let $\Gamma < \operatorname{Isom} X$ be a discrete group generated by reflections. The hyperplanes corresponding to all reflections of Γ decompose X into convex polyhedra called *cells* of Γ.

Remark 3.4.5
Of course, Γ acts transitively on the set of its cells. Moreover, each cell is the closure of a fundamental region for Γ.

Let $\Gamma \leq \operatorname{Isom} X$ be a discrete reflection group with respect to a polyhedron $P \subset X$ (i.e. P is a cell for Γ, Γ is generated by reflections in the sides of P and Γ is discrete). Then, all the dihedral angles of P are submultiples of π (i.e. 0 or of the form π/k where $k \in \mathbb{N}$, $k \geq 2$). For any facet S_i of P, we let H_{v_i} be the hyperplane containing S_i such that $P \subset H^-_{v_i}$. Now, if the dihedral angle between two adjacent facets S_i and S_j of P is $\pi/\theta(S_i, S_j)$, then the order of $r_{v_i} \circ r_{v_j}$ in Γ is $\theta(S_i, S_j)$ (see [Rat06, Theorem 7.1.2]). Suppose now that P is of finite volume and has facets $\mathcal{S} = \{S_1, \dots, S_d\}$. Then, by [Rat06, Theorem 7.1.4] we have

$$\langle r_1, \dots, r_d | r_i^2 = 1, (r_i \cdot r_j)^{\theta(S_i,S_j)} = 1\rangle \cong \Gamma.$$

This motivates the following definition.

Definition 3.4.6 (Geometric Coxeter group)
A Coxeter group Γ is a *geometric Coxeter group* if there exists a polyhedron $P \subset X$, where $X = S^n, E^n, \mathcal{H}^n$, such that Γ is the associated reflection group of P.

Definition 3.4.7 (Gram matrix)
Let $\Gamma < \operatorname{Isom} X$ be a geometric Coxeter group and let $P = \bigcap_{i=1}^{r} H_{v_i}^-$ be its corresponding polyhedron. The *Gram matrix* of the polyhedron $P \subset X$, or the *Gram matrix* of the geometric Coxeter group Γ, is the matrix $G = G(P) = G(\Gamma) \in \operatorname{Mat}(r; \mathbb{R})$ defined as $G = \big(\langle v_i, v_j \rangle\big)_{1 \leq i,j \leq r}$.

If we are given the matrix $G = G(\Gamma)$ of a geometric Coxeter group Γ, we can find the space on which Γ acts as follows. The matrix $G = G(\Gamma)$ induces a quadratic form called the *Tits form* on $\mathbb{R}^d$, where $d = |S|$, via $x \longmapsto x^t\, G\, x$. If the matrix G is indecomposable (meaning that we cannot transform G to a block diagonal matrix with permutations of the rows and columns of G) or, equivalently, if the graph of the group is connected, then we get information about the group by looking at the signature (n, p, q) of the quadratic form, where n (respectively p, q) is the number of positive (respectively negative, zero) eigenvalues of G. For positive n, we are interested in the following cases:

$p = q = 0$ (the quadratic form is positive definite) In this case, the group is finite (see the classification of irreducible finite Coxeter groups in Figure 3.1). It can be shown that Γ can be realized as a discrete group of isometries of the n-sphere S^n. Thus, Γ is called *spherical*.

$p = 0, q > 0$ (the quadratic form is positive semidefinite) The group can be realized as a discrete subgroup of $\operatorname{Isom} E^n$. Hence, it is said to be *affine* (or *Euclidean* or *parabolic*). The classification of irreducible affine Coxeter groups is given in Figure 3.2 (see, for example, [Bou68a, Chapter 6, §4.3, Theorem 4]).

$p = 1, q \geq 0$ The group can be realized as a discrete subgroup of $\operatorname{Isom} \mathcal{H}^n$ (see details in Section 3.5). Thus, it is called *hyperbolic*.

Now, if the matrix G is decomposable to a diagonal block matrix with blocks $G_1, \ldots, G_l$, we say that the group is *spherical* (respectively *affine*) if each block G_i is *spherical* (respectively *affine*). If the matrix G has signature $(n, 1, q)$ for some n and q, we say that the group is *hyperbolic*.
If $\mathcal{G}$ is the Coxeter diagram of Γ, then a subdiagram of $\mathcal{G}$ is called *spherical* (respectively *affine*, *hyperbolic*) if the corresponding subgroup of Γ is *spherical* (respectively *affine*, *hyperbolic*).

3.5 Hyperbolic Coxeter groups and hyperbolic Coxeter polyhedra

In this section, we present some concepts related to hyperbolic Coxeter groups and polyhedra. Let $P \subset \mathcal{H}^n$ be a polyhedron. Hence, there exist vectors $v_1, \ldots, v_r$ of Lorentzian norm 1 and hyperplanes $H_i = \langle v_i \rangle^{\perp}$ such that $P = \bigcap_{i=1}^{r} H_i^-$, where H_i^- is the half-space delimited by H_i and given by $H_i^- := \{x \in \mathcal{H}^n : \langle x, v_i \rangle \leq 0\}$.

Remark 3.5.1
Unlike in the Euclidean space, there exist unbounded hyperbolic Coxeter polyhedra of finite volume. Such a polyhedron is the convex hull of a finite set of points $x_1, \ldots, x_k \in \overline{\mathcal{H}^n}$ with at least one $x_i \in \partial\mathcal{H}^n$. Such a vertex is called a

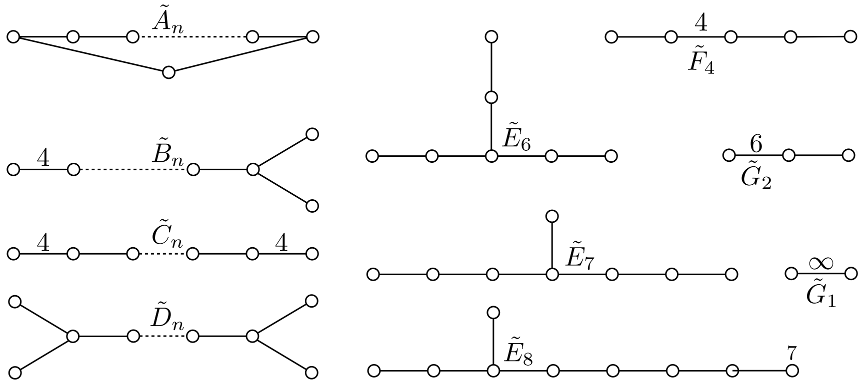


Figure 3.2 – Irreducible affine Coxeter groups

vertex at infinity, or an *ideal vertex*, of the polyhedron (see [Rat06, 6.4] for more details).

Definition 3.5.2 (Coxeter graph)
Suppose now that P is a hyperbolic Coxeter polyhedron, i.e. the dihedral angles are either 0 or submultiples of π. The *Coxeter diagram*, or *Coxeter graph*, of P is the graph whose vertices s_i correspond to the hyperplanes H_i. For two hyperplanes H_i and H_j, the vertices s_i and s_j are joined as follows:

- A *dotted line* if H_i and H_j are ultraparallel, sometimes labelled by $l_{ij} = d_{\mathcal{H}}(H_i, H_j) > 0$.
- A line labelled with ∞ if H_i and H_j are parallel (we use sometimes a bold, or heavy, edge with no labelling).
- A line with label $m_{ij} \geq 3$ if the dihedral angle is $\frac{\pi}{m_{ij}}$; the line is simple if $m_{ij} = 3$ and is labelled by m_{ij} if $m_{ij} > 3$. Notice that sometimes, instead of labelling the edge, we use a double edge if $m_{ij} = 4$.

Remark 3.5.3
The previous definition implies that the Coxeter graph of a hyperbolic Coxeter group encodes more information than the Coxeter graph of the underlying abstract Coxeter group.

Note that we don't distinguish between the Gram matrix of a hyperbolic Coxeter polyhedron, the corresponding Coxeter diagram and the associated hyperbolic Coxeter group. We will make an exception for the *rank*, as explained in the following definition.

Definition 3.5.4 (Rank)
Let $P \subset X$ be a polyhedron and Γ its associated Coxeter group. The *rank* is the number of reflection generators (which correspond to the number of facets of the polyhedron P) of Γ. Let $\mathcal{G}$ be a Coxeter graph which is either spherical or Euclidean. We will define the *rank* of $\mathcal{G}$ as the *rank* of the associated Gram matrix. For example, a spherical Coxeter graph with n vertices has rank n while an *irreducible* Euclidean graph with n vertices has rank $n-1$.

There is the following natural question: given a Coxeter graph, is it possible to find a hyperbolic Coxeter polyhedron whose graph is the given one? The answer is yes if the Gram matrix of the graph has the correct signature, as explained in the next theorem.

Theorem 3.5.5 ([Vin85, Theorem 2.1])
Let $G = (G_{ij})$ be an indecomposable symmetric real matrix of signature $(n, 1, k)$ such that

- $G_{ii} = 1$ *for every* i*;*
- $G_{ij} \leq 0$ *for every* $i \neq j$.

Then, there exists a convex polyhedron P in $\mathcal{H}^n$ whose Gram matrix is equal to G. Moreover, P is unique up to isometry. We will refer to P as the polyhedron associated to *G.*

Therefore, given a hyperbolic Coxeter group Γ we can speak of geometrical properties of its associated polyhedron $P \subset \mathcal{H}^n$. If the polyhedron P is compact, then the group Γ is called *cocompact*. Note that there is a criterion to decide if Γ is cocompact or not (see Section 3.8). In a similar way, if P is of finite volume (with respect to the hyperbolic metric), we say that Γ is *cofinite* or of *finite covolume*. In this case, the *covolume* of Γ is the volume of P. As before, there is a nice criterion to decide whether the group is of finite covolume or not (see Section 3.8).

It has been shown that hyperbolic Coxeter groups of finite covolume do not exist in dimensions above 995 (see [Pro87, Theorem C]) but examples of such groups are known only up to dimension 21 (see [Bor87, Example 5]). In the cocompact case, it is known that such groups do not exist if the dimension is greater than 29 (see [Vin81, Theorem 1]). However, examples of such groups are known only up to dimension 8 (see [Bug92] for the arithmetic cocompact hyperbolic Coxeter group in dimension 8 and [Per09] for a list of cocompact groups in dimensions 5 to 8).

3.6 Growth series and growth rate

We present here basic information about the growth series and the growth rate of a finitely generated group. A more detailed account of these notions can be found in [Per09] and [Kel13].

Notation 3.6.1
Let Γ be a finitely generated group with generating set S. We denote by l_S the *length function* of Γ with respect to S: for an element $g \in \Gamma$, we have

$$l_S(g) = \min \left\{ k \in \mathbb{N}_0 : \exists g_1, \ldots, g_k \in S \cup S^{-1} \text{ such that } g = g_1 \cdot \ldots \cdot g_k \right\}.$$

Definition 3.6.2 (Growth series)
Let Γ be a finitely generated group with generating set S. The *growth series* of Γ is the formal power series $f_{(\Gamma,S)} = f_S(x) = \sum_{g \in \Gamma} x^{l_S(g)}$. This series is also called the *Poincaré series* of Γ.

When Γ is a geometric Coxeter group, the growth series is a rational function (see [Ste68, 1.27]). Moreover, Steinberg's formula (see [Ste68, 1.29]) allows to compute this rational function using only the growth series of the finite subgroups as follows:

$$\frac{1}{f_S(x^{-1})} = \sum_{T\in\mathcal{F}} \frac{(-1)^{|T|}}{f_T(x)}, \tag{3.2}$$

where $\mathcal{F} = \{T \subset S : \Gamma_T \text{ is finite}\}$. If $(\Gamma_1, S_1), (\Gamma_2, S_2)$ are two finitely generated groups, then the growth series of $\Gamma_1 \times \Gamma_2$ with respect to the generating set $S := S_1 \times \{e_2\} \cup \{e_1\} \times S_2$ is given by

$$f_{(\Gamma_1\times\Gamma_2,S)}(x) = f_{(\Gamma_1,S_1)}(x) \cdot f_{(\Gamma_2,S_2)}(x).$$

A convenient way to encode the series of the finite group is the following *symbol*.

Definition 3.6.3 (Symbol)
For $k \in \mathbb{N}$, we write $[k](x)$, or sometimes just $[k]$, for the polynomial $1+x+\ldots+x^{k-1}$. Moreover, for integers $k_1, \ldots, k_r \in \mathbb{N}$, we adopt the following convention: $[k_1, \ldots, k_r] := [k_1] \cdot \ldots \cdot [k_r]$. We will call $[k]$ a *symbol*.

With this notation, the growth series of the finite irreducible Coxeter groups are presented in Table 3.1 (see [CM72, Table 10, page 141]).

Group	Growth series
A_n	$[2, 3, \ldots, n, n+1]$
B_n	$[2, 4, \ldots, 2n-2, 2n]$
D_n	$[2, 4, \ldots, 2n-2] \cdot [n]$
$G_2^{(m)}$	$[2, m]$
F_4	$[2, 6, 8, 12]$
E_6	$[2, 5, 6, 8, 9, 12]$
E_7	$[2, 6, 8, 10, 12, 14, 18]$
E_8	$[2, 8, 12, 14, 18, 20, 24, 30]$
H_3	$[2, 6, 10]$
H_4	$[2, 12, 20, 30]$

Table 3.1 – Growth series of the finite irreducible Coxeter groups

Using Steinberg's formula, we can write the growth series f_S as the rational function $f_S(x) = \frac{p(x)}{q(x)}$, where $p(x)$, $q(x) \in \mathbb{Z}[x]$ are two coprime polynomials. The series has a certain radius of convergence $R \in \mathbb{R}$ which is smaller than 1 if Γ is a hyperbolic group of finite covolume. Moreover, R is equal to the smallest positive root of $q(x)$.

Definition 3.6.4 (Growth rate)
Let (Γ, S) be a geometric Coxeter group and let R be the radius of convergence of its growth series $f_S(x)$. Then, the *growth rate* of Γ is $\tau := R^{-1}$.

Remark 3.6.5
By Steinberg's formula, the growth rate is an algebraic integer.

Algebraic properties of the growth rate of hyperbolic Coxeter groups have attracted much attention in recent years although we don't have results which are independent of the dimension. For cocompact Coxeter groups Γ in dimension 2 and 3, it has been shown that the growth rate τ is a Salem number (see [CW92] and [Par93]). In [Flo92], Floyd showed that the growth rate of any cofinite, non-cocompact, planar, two dimensional hyperbolic Coxeter group is a Pisot number. Finally, Kellerhals and Perren conjectured that the growth rate of any cocompact (or even cofinite) hyperbolic Coxeter group is a Perron number. We will come back to this question later (see Section 5.3.5).

Before computing the growth rate for an infinite family of prisms, we need the following lemma.

Lemma 3.6.6
For $m \in \mathbb{N}$, consider the following polynomial

$$\begin{aligned} f_m(x) = &-1 + x + 2x^3 + x^4 + 2x^5 + x^6 + x^7 \\ &- x^{1+m} - x^{2+m} - 2x^{3+m} - x^{4+m} - 2x^{5+m} - x^{7+m} + x^{8+m}. \end{aligned}$$

When m is even, f'_m has exactly three real zeros. Moreover, there is one zero in each of the following intervals: $(-1, 0)$, $(0, 1)$ and $(1, 2)$.

Proof. Since the degree of f'_m is unbounded, we will proceed by induction on m. The relative behaviour of f'_{2m} and f'_{2m+2} is depicted in Figure 3.3. The first

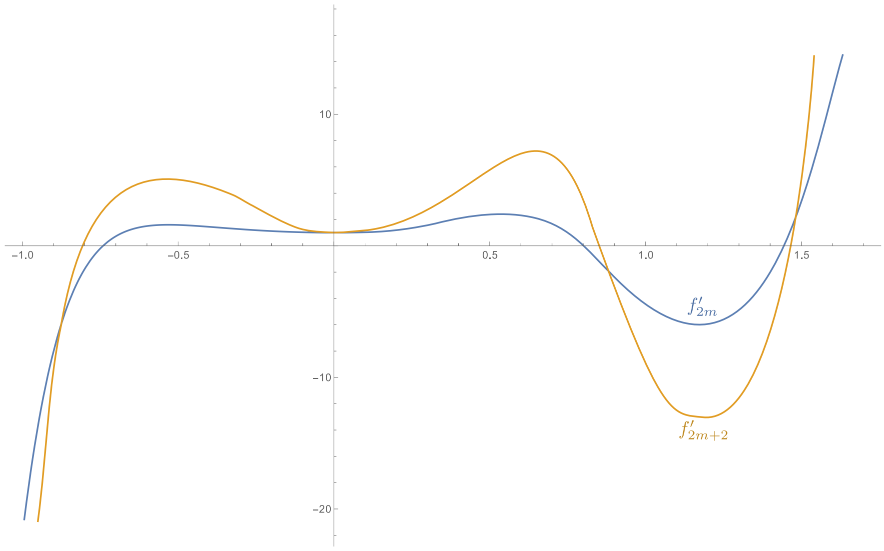


Figure 3.3 – Relative behaviour of f'_{2m} and f'_{2m+2}

step is to show that for $m \geq 1$, the function $g_m := f'_{2m+2} - f'_{2m}$ has only three zeros, all of them in the interval $(-1, 2)$. First, we remark that by Theorem 2.6.8, all the real roots of g_m lie in the interval $(-2, 2)$. Now, if we let

$$\begin{aligned} h_m(x) = &1 + 2m + 2x(1+m) + x^2(3+2m) - 2x^5(3+m) - x^6(7+2m) \\ &- 2x^7(4+m) - x^8(9+2m) + 2x^9(5+m), \end{aligned}$$

then we have $g_m(x) = x^{2m} \cdot h_m(x)$. Using `Mathematica`® (see Section B.1.1), we can compute a Sturm sequence (see Definition 2.6.3) $\{p_0 = h_m, p_1 = h'_m, \dots, p_9\}$ for each m. Moreover, since we have $p_0(-2) = -6647 - 1338m$, $p_0(-1) = -10$, $p_0(0) = 1 + 2m$, $p_0(1) = -14$ and $p_0(2) = 1169 + 78m$, we can use Sturm's theorem (see Theorem 2.6.5) to count the zeros of $h_m(x)$ in the intervals $(-1, 0)$, $(0, 1)$ and $(1, 2)$. We compute the signs of $p_i(\alpha_i)$ for $\alpha_i = -2, -1, 0, 1, 2$ and the number $\sigma(\alpha_i)$ of sign changes:

α_i	0	1	2	3	4	5	6	7	8	9	$\sigma(\alpha_i)$
-2	$-$	$+$	$-$	$+$	$+$	$+$	$-$	$+$	$+$	$-$	6
-1	$-$	$+$	$-$	$+$	$+$	$+$	$-$	$-$	$+$	$-$	6
0	$+$	$+$	$-$	$+$	$+$	$+$	$-$	$+$	$+$	$-$	5
1	$-$	$-$	$-$	$+$	$+$	$-$	$+$	$+$	$-$	$-$	4
2	$+$	$+$	$+$	$-$	$-$	$-$	$+$	$+$	$-$	$-$	3

Since the number of zeros in (α_i, α_{i+1}) is given by $\sigma(\alpha_i) - \sigma(\alpha_{i+1})$, the first step is done. We compute some values of f'_{2m} and f'_{2m+2}:

$$\begin{aligned}
&f'_{2m+2}(-1) = -2(11 + 5m) < f'_{2m}(-1) = -2(6 + 5m) < 0\\
&f'_{2m}(0) = 1\\
&f'_{2m+2}(1) = -2(1 + 7m) < f'_{2m}(1) = -2(7m - 6) < 0\\
&f'_{2m+2}(2) = 4^{m+1}(26m + 381) + 857 > f'_{2m}(2) = 4^m(26m + 355) + 857.
\end{aligned} \tag{3.3}$$

We found numerically that the roots of f'_2 are -0.740856, 0.880032 and 1.25209. Suppose now that f'_{2m} has the prescribed zeros (i.e. f'_{2m} has exactly three real zeros and there is exactly one in each of the three intervals $(-1, 0)$, $(0, 1)$ and $(1, 2)$). Since f'_{2+2m} has (at least) one change of sign in each of the interval $(-1, 0)$, $(0, 1)$ and $(1, 2)$ (see equations (3.3)), then f'_{2m+2} has at least the prescribed zeros. Now, since $f'_{2m+2} < f'_{2m}$ on $(-\infty, -1)$, f'_{2m+2} has no zero on this interval. To prove that f'_{2m+2} has no zero on $(2, \infty)$ we use the fact that $f'_{2m+2}(x) > f'_{2m}(x) > 0$ when $x \in (2, \infty)$. If f'_{2m+2} has a second zero on $(-1, 0)$, then so must have f'_{2m} (since g_m has exactly one zero on $(-1, 0)$) and that is not possible by the induction hypothesis. Hence, f'_{2m+2} has exactly one zero on $(-1, 0)$. We proceed in a similar way for the intervals $(0, 1)$ and $(1, 2)$. □

Example 3.6.7
We want to compute the growth series and the growth rate of the family of 3-dimensional compact Coxeter prisms Γ_m whose graphs $\mathcal{G}_m$ are given in Figure 3.4 (see [Kap74]). More precisely, we show that the sequence of growth

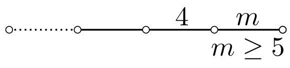


Figure 3.4 – A family of compact Coxeter prisms in $\mathcal{H}^3$

rates $\{\tau_m\}$ of Γ_m (which are Salem numbers since the groups are cocompact) converges to the Pisot number $\tau_\infty \cong 1.90648$ which is the only real root of the polynomial $x^7 - x^6 - 2x^4 - x^3 - 2x^2 - x - 1$.

We see that each graph $\mathcal{G}_m$ contains the following irreducible finite subgraphs:

$5 \times A_1$	$1 \times A_2$	$1 \times G_2^{(4)}$
$1 \times G_2^{(m)}$	$1 \times B_3$	

Hence, the spherical subgraphs of $\mathcal{G}_m$ are the following:

$5 \times A_1$	$1 \times G_2^{(m)}$	$1 \times G_2^{(4)}$	$1 \times A_2$
$6 \times A_1 \times A_1$	$2 \times A_1 \times G_2^{(m)}$	$1 \times A_1 \times G_2^{(4)}$	$1 \times A_1 \times A_2$
$1 \times A_1 \times A_1 \times A_1$	$1 \times B_3$		

Using Steinberg's formula (see equation (3.2) page 41 and Table 3.1 page 41), we can compute the growth series of the Γ_m as follows:

$$\begin{aligned}\frac{1}{f_{\Gamma_m}(x)} &= 1 + \frac{-5x}{[2]} + \frac{x^m}{[2,m]} + \frac{x^4}{[2,4]} + \frac{x^3}{[2,3]} \\ &+ \frac{6x^2}{[2,2]} + \frac{-2x^{m+1}}{[2,2,m]} + \frac{-x^5}{[2,2,4]} + \frac{-1x^4}{[2,2,3]} \\ &+ \frac{-x^3}{[2,2,2]} + \frac{-x^9}{[2,4,6]}.\end{aligned}$$

Hence, we find

$$f_{\Gamma_m}(x) = -\frac{(x+1)^3 \cdot (x^2+1) \cdot (x^2-x+1) \cdot (x^2+x+1) \cdot (x^m-1)}{(-1+x) \cdot \tilde{q}(x)},$$

where

$$\begin{aligned}\tilde{q}(x) = &-x^{m+1} - x^{m+2} - 2x^{m+3} - x^{m+4} - 2x^{m+5} - x^{m+7} + x^{m+8} \\ &+ x^7 + x^6 + 2x^5 + x^4 + 2x^3 + x - 1.\end{aligned}$$

The polynomial $\tilde{q}(x)$ is divisible by $x-1$ and if $m > 5$ the quotient can be written as follows

$$q_m(x) = 1 - 2x^3 - 3x^4 - 5x^5 - 6x^6 - 7\sum_{i=7}^{m} x^i - 6x^{m+1} - 5x^{m+2} - 3x^{m+3} - 2x^{m+4} + x^{m+7},$$

which means

$$f_{\Gamma_m}(x) = -\frac{(x+1)^3 \cdot (x^2+1) \cdot (x^2-x+1) \cdot (x^2+x+1) \cdot (x^m-1)}{(-1+x)^2 \cdot q_m(x)}, \quad m \geq 6.$$

Since the polynomial $q_m(x)$ is palindromic, it implies that the growth rate τ_m is the biggest positive real root of q_m. We remark that in general the polynomial $q_m(x)$ is not irreducible: for example, $q_m(x)$ is divisible by $x+1$ when m is even. However, we can show for small values of m that $q_m(x)$ can be factored in a product of cyclotomic polynomials and a polynomial which only has two positive real roots (since $q_m(x)$ is palindromic, these two roots are of course inverse of each other); the only real root bigger than 1 is thus the growth rate. The graphs

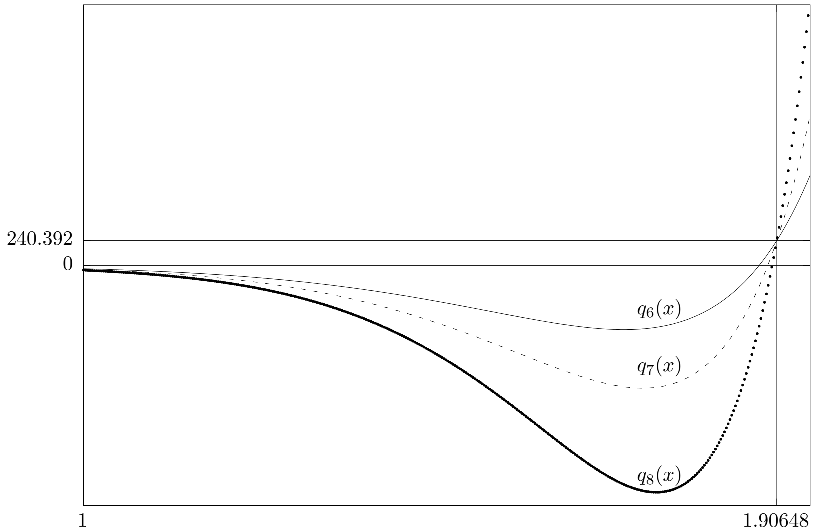


Figure 3.5 – Factors of the denominator of the growth series of Kaplinskaya prisms for $m = 6, 7, 8$

of the polynomials q_6, q_7 and q_8 are depicted in Figure 3.5. The picture suggests that $q_{m+1}(x) \leq q_m(x) \leq q_6(\alpha) \approx 240.392$ on $[1, \alpha]$ for some $\alpha \cong 1.90648$ and then $q_{m+1}(x) > q_m(x)$ on (α, ∞). If this is true, then we will have that $\tau_m < \alpha$, and $\{\tau_m\}$ is an increasing sequence. We compute the difference

$$\begin{aligned} q_{m+1}(x) - q_m(x) &= -x^{m+1} - x^{m+2} - 2x^{m+3} - x^{m+4} - 2x^{m+5} - x^{m+7} + x^{m+8} \\ &= x^{m+1} \cdot (x^7 - x^6 - 2x^4 - x^3 - 2x^2 - x - 1). \end{aligned}$$

The polynomial $x^7 - x^6 - 2x^4 - x^3 - 2x^2 - x - 1$ has only one real root $\beta \cong 1.90648$, is negative on $(-\infty, \beta)$ and positive on (β, ∞). Hence, we see using induction that the q_m have no real root bigger than β. Therefore, τ_m is an increasing sequence bounded above by β, which means that sequence converges to a number $\tilde{\beta} \leq \beta$.

We present first some analytical facts that strongly support the fact that $\tilde{\beta} = \beta$. Then, we conclude using a geometrical argument.

First notice that the equality $\tilde{\beta} = \beta$ is equivalent to the fact that the subsequence τ_{2m} converges to β. We have:

1. Using Lemma 3.6.6, since $f_m(x) = (x-1) \cdot q_m(x)$, we see that each q_{2m} has a unique minima bigger than 1 at α_{2m} (and we have $1 < \alpha_{2m} < \tau_{2m}$). Then, each q_{2m} is strictly increasing on (α_{2m}, ∞).

2. Moreover, we can check numerically that

$$q'_{2m}(\beta) \cong 1.21697\,(389.322 + 190.168 \cdot 1.90648^m) \overset{m\to\infty}{\longrightarrow} \infty.$$

The last two points strongly support our claim.

We now present the geometrical argument. The polyhedra associated to the groups Γ_m are examples of polyhedra with a ridge of type $\langle 2, 2, m, 2, 2\rangle$

(see [Kol12, Section 3]) and they converge to the non-compact polyhedron P_∞ associated to the group Γ_∞ given by the Coxeter symbol $[\infty, 3, 4, \infty]$. Now, [Kol12, Theorem 5] implies that the sequence τ_m converges to the growth rate τ_∞ of Γ_∞. This group has a growth series given by

$$f_{\Gamma_\infty}(x) = \frac{(x+1)^3 \cdot \left(x^2+1\right) \cdot \left(x^2-x+1\right) \cdot \left(x^2+x+1\right)}{(x-1) \cdot (x^7+x^6+2x^5+x^4+2x^3+x-1)}.$$

Now, the polynomial $x^7+x^6+2x^5+x^4+2x^3+x-1$ is irreducible and has reciprocal polynomial $x^7-x^6-2x^4-x^3-2x^2-x-1$ which implies $\tau_m \overset{m\to\infty}{\longrightarrow} \tau_\infty$, as required.

3.7 Euler characteristic, f-vector and volume

In what follows, unless stated otherwise, $\Gamma < \operatorname{Isom}\mathbb{H}^n$ denotes a hyperbolic Coxeter group with finite set of natural generators S. Let P be its associated fundamental convex polyhedron.

The next proposition gives the key tool which is used by `CoxIter` (see Chapter 5) to compute the orbifold Euler characteristic $\chi(\Gamma)$ of $\mathbb{H}^n/\Gamma$.

Proposition 3.7.1 ([KP11, (1.2) and (1.3)])
Let (Γ, S) be an abstract Coxeter group and let $\mathcal{F} = \{T \subset S : \Gamma_T \text{ is finite}\}$. We have

$$\chi(\Gamma) = \sum_{T\in\mathcal{F}} \frac{(-1)^{|T|}}{f_T(1)},$$

where f_T is the growth series of the group $\Gamma_T \leq \Gamma$ generated by T (see Definition 3.3.1).

In order to compute $\chi(\Gamma)$, we see by using the classification of finite Coxeter groups (see Figure 3.1) that it is sufficient to know the value $f_T(1)$ for all irreducible finite Coxeter groups. With the growth series (see Table 3.1), we find the values given in Table 3.2.

Group	$f_S(1)$
A_m	$(m+1)!$
B_m	$2^m \cdot m!$
D_m	$2^{m-1} \cdot m!$
G_m	$2m$
F_4	1152

Group	$f_S(1)$
E_6	51840
E_7	2903040
E_8	696729600
H_3	120
H_4	14400

Table 3.2 – Orders of finite Coxeter groups

The next result relates the Euler characteristic of a hyperbolic Coxeter group $\Gamma < \operatorname{Isom}\mathbb{H}^n$ to its covolume when n is even.

Proposition 3.7.2 ([KP11, (1.4)])
When n is even, we have:

$$\text{covolume}(\Gamma) = (-1)^{n/2} \cdot \frac{\pi^{n/2} \cdot 2^n \cdot (n/2)!}{n!} \cdot \chi(\Gamma).$$

Remark 3.7.3
When n is odd, we have $\chi(\Gamma) = 0$.

We are also interested in the combinatorial properties of the Coxeter polyhedron P associated to Γ.

Definition 3.7.4 (f-vector)
The vector $(f_0, \dots, f_{n-1}, 1) \in \mathbb{Z}^{n+1}$, where f_i is the number of faces of dimension i of P, is called the *f-vector* of P (or f-vector of Γ)

To compute the f-vector, we will use the following results:

Theorem 3.7.5 ([Vin85, Theorem 3.1])
Let $\mathcal{G}$ be the Coxeter diagram of Γ. There is a bijective correspondence between spherical subdiagrams of rank k of $\mathcal{G}$ and faces of codimension k of P.

Theorem 3.7.6 ([Vin85, Theorem 3.2])
Let $\mathcal{G}$ be the Coxeter diagram of Γ. There is a bijective correspondence between parabolic subdiagrams of rank $n-1$ of $\mathcal{G}$ and vertices at infinity of P.

As a test for the output of the program we also use the following classical result of Euler-Schläfli.

Proposition 3.7.7 ([Poi93])
For a polyhedron in $\mathbb{X}^n$, we have the following equality:

$$\sum_{i=0}^{n-1} (-1)^i \cdot f_i = 1 - (-1)^n.$$

3.8 Compactness and finite volume criterion

Since a polyhedron P is compact if and only if it is the convex hull of a finite number of vertices in $\mathbb{H}^n$ (also called *ordinary* vertices), we have the following result.

Proposition 3.8.1 ([Vin85, Proposition 4.2])
The polyhedron P is compact if and only if the following conditions are satisfied:

- *P contains at least one vertex (i.e. face of dimension 0) in $\mathbb{H}^n$.*
- *For every vertex of P and every edge of P emanating from it there is precisely one other vertex of P on that edge.*

Since a polyhedron P is of finite volume if and only if it is the convex hull of a finite number of vertices in $\overline{\mathbb{H}^n}$, we have the following result.

Proposition 3.8.2 ([Vin85, Proposition 4.2])
The polyhedron P has finite volume if and only if the following conditions are satisfied:

- *P contains at least one vertex (ordinary or at infinity).*
- *For every vertex (ordinary or at infinity) of P and every edge of P emanating from it there is another vertex of P (ordinary or at infinity) on that edge.*

Using theorems 3.7.5 and 3.7.6 we deduce the following two criteria.

Proposition 3.8.3 (Cocompactness criterion in `CoxIter`)
Let $\Gamma < \operatorname{Isom}\mathbb{H}^n$ be a Coxeter group and $\mathcal{G}$ be its Coxeter diagram. The group Γ is cocompact if and only if the following conditions hold:

- *$\mathcal{G}$ contains at least one spherical subdiagram of rank n.*
- *Each spherical subdiagram of rank $n-1$ of $\mathcal{G}$ can be extended in exactly two ways to a spherical subdiagram of rank n of $\mathcal{G}$.*

Proposition 3.8.4 (Finite covolume criterion in `CoxIter`)
Let $\Gamma < \operatorname{Isom}\mathbb{H}^n$ be a Coxeter group and $\mathcal{G}$ be its Coxeter diagram. The group Γ has finite covolume if and only if the following conditions hold:

- *$\mathcal{G}$ contains at least one spherical subdiagram of rank n or one parabolic subdiagram of rank $n-1$.*
- *Each spherical subdiagram of rank $n-1$ of $\mathcal{G}$ can be extended in exactly two ways to one of the following type of subdiagrams of $\mathcal{G}$:*
  - *a spherical diagram of rank n;*
  - *a parabolic diagram of rank $n-1$.*

3.9 Arithmetic groups

3.9.1 Definition

Before we give the precise definition of an arithmetic group (of the simplest type), we start with an example. Let $f_n(x) = -{x_0}^2 + {x_1}^2 + \ldots + {x_n}^2$ be the standard Lorentzian quadratic form and consider the cone $C = \{x \in \mathbb{R}^{n+1} : f_n(x) < 0\}$. We denote by $O(f_n, \mathbb{Z}) \subset \mathrm{GL}(n+1, \mathbb{R})$ the group of linear transformations of $\mathbb{R}^{n+1}$ with coefficients in $\mathbb{Z}$ which preserve the quadratic form f_n. The index two subgroup $O^+(f_n, \mathbb{Z})$ consisting of elements of $O(f_n, \mathbb{Z})$ which preserve each of the two connected components of the cone C is the prototype of an arithmetic group. Moreover, it is well known that this discrete group has finite covolume in $O^+(n,1)$, which is related to hyperbolic isometries via $\mathrm{SO}^+(n,1) = \operatorname{Isom}^+\mathbb{H}^n$. Notice that the condition that the transformation has coefficients in $\mathbb{Z}$ is equivalent to ask the preservation of the standard $\mathbb{Z}$-lattice in $\mathbb{R}^{n+1}$. Now, let K be any totally real number field, $\mathcal{O}_K$ its ring of integers, and let V be an $(n+1)$-dimensional vector space over K endowed with an *admissible* quadratic form f of signature $(n,1)$, that is: all conjugates f^σ of f by the non-trivial Galois embeddings $\sigma : K \longrightarrow \mathbb{R}$ are positive definite. The cone $C_f = \{x \in V \otimes \mathbb{R} : f(x) < 0\}$ has two connected components $C_f^\pm$ and gives rise to the vector space model $C_f^+/_{\mathbb{R}^*}$ of the hyperbolic n-space $\mathbb{H}^n$ (see Remarks 3.2.2). We now let

$$O(f) = \big\{T \in \mathrm{GL}(V \otimes \mathbb{R}) : f \circ T(x) = f(x),\ \forall x \in V \otimes \mathbb{R}\big\}.$$

Finally, for a full $\mathcal{O}_K$-lattice L in V, we consider the group

$$O(f, L) := \big\{T \in O(f)_K : T(C_f^+) = C_f^+,\ T(L) = L\big\}.$$

It is a discrete subgroup of $\operatorname{Isom}\mathbb{H}^n$ of finite covolume (see [Bor62]).

Definition 3.9.1 (Arithmetic group (of the simplest type))
A discrete group $\Gamma < \operatorname{Isom}\mathbb{H}^n$ is an *arithmetic group of the simplest type* if there exist K, f and L as above such that Γ is commensurable to $O(f,L)$. In this setting, we say that Γ is *defined over* K and that f is the *quadratic form associated* to Γ.

Remarks 3.9.2 1. In particular, an arithmetic group is cofinite.

2. If such a group is non-cocompact, then it must be defined over $\mathbb{Q}$ (see [Vin88, Chapter 6]).

3. If Γ is defined over $\mathbb{Q}$, then it is non-cocompact if and only if the lattice is isotropic[2] (see also [GP87, Section 2.3]). In particular, Γ is non-cocompact if $n \geq 4$ (this follows from [Cas78, Chapter 4, Lemma 2.7] and the Hasse-Minkowski theorem).

4. There is a slightly more general definition of discrete arithmetic subgroups of $\operatorname{Isom}\mathbb{H}^n$. However, when a discrete subgroup of $\operatorname{Isom}\mathbb{H}^n$ is arithmetic *and* contains reflections, then it is of the simplest type (see [Vin67, Lemma 7]). In other words, an arithmetic hyperbolic Coxeter group is not only commensurable to some $O(f,L)$ but actually *contained* in some $O(f,L)$. Since we focus in this work on Coxeter groups, we will refer to *arithmetic of groups of the simplest type* as *arithmetic groups.*

5. If $\mathcal{O}_K$ is a PID, then the condition that the transformations T preserve a lattice can be replaced by the condition that T has coefficients in $\mathcal{O}_K$. Indeed, in this setting, the lattice L is a free $\mathcal{O}_K$-module of rank $n+1$ with basis $v_1, \ldots, v_{n+1}$. Now, the transformation $X : e_i \longmapsto v_i$ induces an isomorphism

$$\begin{aligned} O(f,L) &\xrightarrow{\cong} O\big(g, \mathcal{O}_K^{n+1}\big) \\ \phi &\longmapsto X^{-1} \circ \phi \circ X, \end{aligned}$$

where $g := f \circ X^{-1}$.

Reflective quadratic forms Consider, as above, an admissible quadratic form f of signature $(n,1)$ given by

$$f(x_0, \ldots, x_n) = \sum_{i,j} a_{i,j} \cdot x_i x_j, \quad a_{i,j} \in \mathcal{O}_K,$$

with respect to some basis $\{v_0, \ldots, v_n\}$, and consider $L = \operatorname{span}_{\mathcal{O}_K}\{v_0, \ldots, v_n\}$. As said above, the group $O(f,L)$ is a discrete subgroup of $\operatorname{Isom}\mathbb{H}^n$ of finite covolume. Moreover, we have the decomposition $O(f,L) = \Gamma \rtimes H$, where Γ is the subgroup of $O(f,L)$ generated by all the reflections in $O(f,L)$ and H is a subgroup of the symmetry group of a cell of Γ.

Definition 3.9.3 (Reflective quadratic form)
The quadratic form f is *reflective* if the group Γ in the decomposition $O(f,L) = \Gamma \rtimes H$ has finite index in $O(f,L)$. We note that it is equivalent to the fact that P is finite sided and of finite covolume.

[2]In this setting, it means that there exists $v \in V \setminus \{0\}$ such that $f(v) = 0$.

Example 3.9.4
The Lorentzian quadratic form $\langle -1, 1, \ldots, 1 \rangle$ is reflective if and only if $n \leq 19$ (see [Vin72] and [KV78])

The next result allows to deal with the reflectivity of families of quadratic forms.

Proposition 3.9.5
Let f be an admissible quadratic form of signature $(n, 1)$ with $n \geq 3$. If the quadratic form $f \oplus \langle 1 \rangle$ is reflective, then so is f.

Proof. See [Bug90, Corollary 2]. □

We will come back to the question of reflectivity in the chapter 6), which dedicated to Vinberg's algorithm.

3.9.2 Criterion for arithmeticity

In [Vin67] and [Vin88], Vinberg presented a criterion to decide whether a discrete subgroup of $\operatorname{Isom} \mathbb{H}^n$ is arithmetic or not. In order to present the criterion, we need the following definition.

Definition 3.9.6 (Cycle in a matrix)
Let $A := (A_{i,j}) \in \operatorname{Mat}(n; K)$ be a square matrix with coefficients in a field K. A *cycle* of length k, or k*-cycle*, in A is a product $A_{i_1,i_2} \cdot A_{i_2,i_3} \cdot \ldots \cdot A_{i_{k-1},i_k} \cdot A_{i_k,i_1}$. Such a cycle is denoted by $A_{(i_1,\ldots,i_k)}$. If the i_j are all distinct, the cycle is called *irreducible*.

Theorem 3.9.7 (Arithmeticity criterion)
Let Γ be a cofinite discrete subgroup of $\operatorname{Isom} \mathbb{H}^n$ generated by reflections and let G be its Gram matrix. Let $\tilde{K}$ be the field generated by the entries of G and let $K \subset \tilde{K}$ be the field generated by the (irreducible) cycles in $2G$. Then, Γ is arithmetic if and only if:

(i) *$\tilde{K}$ is a totally real number field;*

(ii) *for each Galois embedding $\sigma : \tilde{K} \longrightarrow \mathbb{R}$ which is not the identity on K, the matrix G^σ is positive semi-definite;*

(iii) *the cycles in $2G$ are algebraic integers in K.*

In this case, the field of definition, or defining field, of Γ is K.

Proof. See [Vin88, Chapter 6, §3, Theorem 3.1]. □

Remark 3.9.8
Observe that each cycle is a product of irreducible cycles. In particular, it is sufficient to check condition (*iii*) only on irreducible cycles.

For a discrete subgroup of $\operatorname{Isom} \mathbb{H}^n$ generated by reflections, there is a weaker property called *quasi-arithmeticity*[3]. The precise definition of a *quasi-arithmetic* group, together with the proof of the next theorem, can be found in [Vin67].

[3] In fact, both arithmetic and quasi-arithmetic group can be defined in the more general context of Lie groups and algebraic groups. More information can be found in [Vin67]

Theorem 3.9.9 (Quasi-arithmeticity criterion)
Let Γ *be a cofinite discrete subgroup of* $\operatorname{Isom}\mathbb{H}^n$ *generated by reflections and let* G *be its Gram matrix. Let* $\tilde{K}$ *be the field generated by the entries of* G *and let* $K \subset \tilde{K}$ *be the field generated by the (irreducible) cycles in* $2G$*. Then, the group* Γ *is quasi-arithmetic if and only if the two following conditions are satisfied:*

(i) $\tilde{K}$ *is a totally real number field;*

(ii) for each Galois embedding $\sigma : \tilde{K} \longrightarrow \mathbb{R}$ *which is not the identity on* K*, the matrix* G^σ *is positive semi-definite.*

Moreover, a quasi-arithmetic group is arithmetic if and only if all the elements $2G_{i,j}$ *are algebraic integers. In this case, the field of definition of the arithmetic group* Γ *is* K*.*

Proof. See [Vin67, Theorem 2]. □

Remark 3.9.10
A matrix is positive semi-definite if and only if all its principal submatrices are positive semi-definite. In particular, a necessary condition for the group Γ to be (quasi-)arithmetic is that all conjugates of all principal minors of G are non-negative (for the Galois embeddings which are not trivial on $\tilde{K}$). Or, equivalently, the necessary condition is that all principal minors of G^σ, for any Galois embedding σ which is not the identity on $\tilde{K}$, are non-negative.

For a non-cocompact (quasi-)arithmetic group, the field of definition K has to be $\mathbb{Q}$. Hence, the condition (ii) of the two previous theorems is automatically satisfied. In particular, the criterion can be simplified as follows.

Theorem 3.9.11 ([Vin88])
Let Γ *be a non-cocompact hyperbolic Coxeter group of finite covolume and let* G *be its Gram matrix. Then,* Γ *is arithmetic if and only if all the cycles of the matrix* $2 \cdot G$ *are rational integers.*

Corollary 3.9.12
Let $\Gamma = (W, S)$ *be a* non-cocompact *hyperbolic Coxeter group of finite covolume and let* $G' = 2 \cdot G$*, where* G *is its Gram matrix. If the Coxeter graph of* Γ *contains no dotted edges, then we have the following result:*
Γ *is arithmetic if and only if the two following conditions are satisfied*

(i) For every $s, t \in S$*, we have* $m(s,t) \in \{\infty, 2, 3, 4, 6\}$*.*

(ii) Every irreducible cycle $G'_{(i_1,\dots,i_k)}$ *of length at least* 3 *in* G' *lies in* $\mathbb{Z}$*.*

In particular, a necessary condition for Γ *to be arithmetic is that the matrix* G *has coefficients in* $\mathbb{Q}[\sqrt{2}, \sqrt{3}]$*.*

Definition 3.9.13 (Cycle in a graph)
Let $\mathcal{G}$ be an undirected graph. A *cycle*, or *closed walk*, in $\mathcal{G}$ is a sequence $(v_{i_1}, \dots, v_{i_m}, v_{i_1})$ of adjacent vertices of $\mathcal{G}$. We say that the cycle is *simple* if all the v_{i_j} are different.

From now on, we suppose that $\Gamma = (W, S)$ is a non-cocompact hyperbolic Coxeter group of finite covolume which satisfies condition (i) of Corollary 3.9.12

and we let $G' = 2 \cdot G$. If $s \in S$ is a leaf of the Coxeter graph (meaning that $m(s,t) = 2$ for every $t \in S \setminus \{s\}$ except for one vertex), then it is clear that s cannot be a member of a non-zero irreducible cycle. Therefore, we can forget this vertex for the test. Also, any edge which is not part of a closed simple cycle can be dropped without changing the result of the test. Applying this idea recursively until the graph does not change any more, in a process we will call *recursively deleting separating edges*, lead to a simpler graph we can test. For example, the two Coxeter graphs of Figure 3.6 are equivalent for the arithmeticity criterion.

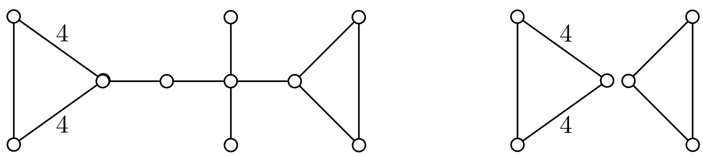


Figure 3.6 – Two equivalent graphs for the arithmeticity criterion

Now, if $G'_{(i_1,\ldots,i_k)}$ is an irreducible k-cycle, then it corresponds to a simple cycle in the Coxeter graph (see Definition 3.9.13) if and only if it is non-zero. Thus, it is sufficient to consider simple cycles in the Coxeter graph. This can be summarized in the next proposition.

Proposition 3.9.14 (Arithmetic criterion in CoxIter ([Gug15]))
Let $\Gamma = \big(W, S = \{s_1, \ldots, s_d\}\big)$ be a non-cocompact hyperbolic Coxeter group of finite covolume and let $\mathcal{G}$ be its Coxeter graph. We suppose that $\mathcal{G}$ contains no dotted line. Then, Γ is arithmetic if and only if the two following conditions are satisfied

1. *For every $s, t \in S$, we have $m(s,t) \in \{\infty, 2, 3, 4, 6\}$.*
2. *For every simple cycle $(s_{i_1}, \ldots, s_{i_k}, s_{i_1})$ in $\mathcal{G}$, the product*

$$2^k \cdot \prod_{j=1}^{k-1} \cos \frac{\pi}{m(s_{i_j}, s_{i_{j+1}})} \cdot \cos \frac{\pi}{m(s_{i_k}, s_{i_1})}$$

is an integer. Moreover, it is sufficient to test this condition in the graph obtained by recursively deleting separating edges.

Remark 3.9.15
If the graph $\mathcal{G}$ contains dotted lines, we cannot decide the arithmeticity of the group only by looking at the weights $m(s,t)$ in the graph. In order to extend Proposition 3.9.14, we need to know the all values in the Gram matrix (G_{ij}). If there is a dotted line between vertices i and j in $\mathcal{G}$, a necessary condition for arithmeticity is that $4 \cdot G_{ij}^2 \in \mathbb{Z}$.

Remark 3.9.16
For cocompact groups, the application of the criterion can be tricky. Examples can be found in Section 4.4.1 which are based on Kaplinskaya's 3-dimensional prisms.

CHAPTER 4

Commensurability

The aim of this chapter is to introduce the notion and to study the concept of commensurability as well as to present a few methods which can be used to decide the (non-)commensurability of two hyperbolic Coxeter groups. These tools are extensively used in a joint work with Matthieu Jacquemet and Ruth Kellerhals, where we classified a family of hyperbolic Coxeter pyramids with $n+2$ facets (see [GJK16]).

The first two sections are dedicated to the basic material and to an overview of some of the methods used in our article. The details for the analytical, geometric and combinatorial methods can be found in [GJK16]. Then, a systematic treatment of the arithmetic case will be presented, which was my main contribution in our joint work. In particular, we explain in detail a method, first described by Maclachlan in [Mac11], to decide the commensurability for arithmetic groups of hyperbolic isometries. We also illustrate this method for new arithmetic cocompact hyperbolic Coxeter groups in $\operatorname{Isom}\mathbb{H}^4$ and for the infinite families of 3-dimensional prisms found by Kaplinskaya. Finally, we compute the invariants for arithmetic hyperbolic Coxeter groups with underlying quadratic forms $\langle -p, 1, \ldots, 1\rangle$ and $\langle -1, p, 1, \ldots, 1\rangle$.

Before we start with the first definitions, let us make some general comments. First, since only very few families of hyperbolic Coxeter groups are entirely characterized, there are only very few families classified up to commensurability. Among them, we can cite:

1. Takeuchi gave a characterization of arithmetic Fuchsian groups in [Tak75] (see also [Tak77a] for the particular case of triangle groups). He then classified arithmetic triangle groups up to commensurability in [Tak77b].

2. The cofinite hyperbolic Coxeter n-simplex groups (i.e. groups associated to Coxeter polyhedra in $\mathbb{H}^n$ bounded by $n+1$ hyperplanes), which exist up to dimension 9, have been classified up to commensurability in [Joh+02].

3. The hyperbolic Coxeter groups in $\operatorname{Isom}\mathbb{H}^n$, $n \geq 3$, whose fundamental polyhedra are pyramids over a product of two simplices of positive dimensions, have been classified by Tumarkin in [Tum04]. These 200 pyramid Coxeter groups, which exists up to dimension 17, are classified up to commensurability in [GJK16] (see also theorems 4.0.1 and 4.0.2).

The classification presented in [Joh+02] relies heavily on the fact the Gram matrices of the Coxeter n-simplices are *invertible*, which is not true anymore when the groups are of higher rank. This is why, in [GJK16], we had to use and to develop a lot of different techniques.

Using the same techniques as for the above three families, the classification for cofinite non-cocompact Coxeter groups in $\operatorname{Isom}\mathbb{H}^n$ with $n+2$ generators seems within reach. However, passing to families with $n+k$ generators for arbitrary k seems unrealistic at the moment.

Our classification of Tumarkin's pyramid Coxeter groups can be summarized in the following two theorems.

Theorem 4.0.1 (See [GJK16])
Among the 200 *pyramid Coxeter groups given by Tumarkin in [Tum04],* 162 *of them are arithmetic. Moreover, from dimension* 7 *to dimension* 17*, with one exception in dimension* 10*, all the considered groups are arithmetic. The following table summarizes the commensurability classes and contains, for each of them, the Coxeter symbol of a representative as well as the number of groups in the class.*

n	$\mathcal{A}_n^1 \div \alpha_n^1$	$\mathcal{A}_n^2 \div \alpha_n^2$	$\mathcal{A}_n^3 \div \alpha_n^3$	$\mathcal{A}_n^4 \div \alpha_n^4$
3	$[(3,\infty,3),(4,\infty,4)]$ 4	$[(3,\infty,3),(6,\infty,6)]$ 4	$[(3,\infty,3),(3,\infty,3)]$ 6	
4	$[6,3,3,3,\infty]$ 4	$[4,4,3,3,\infty]$ 20		
5	$[(3,4^2,3),(3,4^2,3)]$ 20	$[3^{[4]},3,(3,\infty,3)]$ 4	$[(3,4^2,3),3,3^{[3]}]$ 6	$[3^{[3]},3^2,3^{[3]}]$ 3
6	$[3^{[5]},3,(3,\infty,3)]$ 2	$[3^{[4]},3^2,3^{[3]}]$ 4	$[3^{[4]},3,(3,4^2,3)]$ 18	
7	$[3^{[5]},3^2,3^{[3]}]$ 2	$[3^{1,1},3^{1,2},(3,\infty,3)]$ 4	$[3^{[6]},3,(3,\infty,3)]$ 8	$[3^{[4]},3^2,3^{[4]}]$ 12
8	$[3^{2,2},3^3,(3,\infty,3)]$ 16			
9	$[3^{2,2},3^4,3^{[3]}]$ 10			
10	$[3^{2,1},3^6,(3,\infty,3)]$ 4			
11	$[3^{2,1},3^7,(3,\infty,3)]$ 2	$[3^{2,1},3^6,(3,4^2,3)]$ 3		
12	$[3^{2,1},3^6,3^{[4]}]$ 2			
13	$[3^{2,1},3^8,3^{1,1,1}]$ 3			

Continued on next page

n	$\mathcal{A}_n^1 \div \alpha_n^1$	$\mathcal{A}_n^2 \div \alpha_n^2$	$\mathcal{A}_n^3 \div \alpha_n^3$	$\mathcal{A}_n^4 \div \alpha_n^4$
17	$[3^{2,1}, 3^{12}, 3^{1,2}]$ 1			

Theorem 4.0.2 (See [GJK16])
Among the 200 *pyramid Coxeter groups given by Tumarkin in [Tum04],* 38 *of them are* not *arithmetic. The following table summarizes the commensurability classes and contains, for each of them, the Coxeter symbol of a representative of the class. The header of the columns gives the number of elements in the class.*

n	$\lvert\mathcal{N}_n\rvert = 1$	$\lvert\mathcal{N}_n\rvert = 2$	$\lvert\mathcal{N}_n\rvert = 3$	$\lvert\mathcal{N}_n\rvert = 4$
3	$[(3,\infty,4),(3,\infty,4)]$	$[\infty,3,(3,\infty,k)]$ $k=4,5,6$		$[\infty,3,5,\infty]$
		$[\infty,3,(l,\infty,m)]$ $4 \le l < m \le 6$		
		$[\infty,4,(3,\infty,4)]$		
4		$[6,3^2,(k,\infty,l)]$ $3 \le k < l \le 5$	$[4^2,3,(3,\infty,4)]$	$[6,3^2,5,\infty]$
5		$[4,3^{2,1},(3,\infty,4)]$		
6			$[3,4,3^3,(3,\infty,4)]$	
10	$[3^{2,1},3^6,(3,\infty,4)]$			

4.1 Generalities

Definition 4.1.1 (Commensurable groups)
Let Γ be a group and let Γ_1 and Γ_2 be two subgroups of Γ. We say that Γ_1 and Γ_2 are *commensurable* if and only if the intersection $\Gamma_1 \cap \Gamma_2$ has finite index both in Γ_1 and Γ_2. They are said to be *commensurable in the wide sense* if Γ_1 is commensurable to a conjugate of Γ_2. In this work, we will always use the term commensurable to designate commensurable in the wide sense.

Definition 4.1.2 (Commensurable topological spaces)
Two topological spaces X_1 and X_2 are said to be *commensurable* if they admit homeomorphic finite-sheeted coverings.

Proposition 4.1.3
Two discrete subgroups Γ_1, Γ_2 *of* $\operatorname{Isom}\mathbb{H}^n$ *are commensurable if and only if the two orbifolds* $\mathbb{H}^n/_{\Gamma_1}$ *and* $\mathbb{H}^n/_{\Gamma_2}$ *are commensurable.*

The previous proposition implies in particular the following facts.

Proposition 4.1.4
The ratio of the volumes of the associated polyhedra of two commensurable discrete hyperbolic Coxeter groups is a rational number. Moreover, for subgroups of $\operatorname{Isom}\mathbb{H}^n$*, the following properties are stable by commensurability:*

- *being discrete;*
- *being cofinite;*
- *begin cocompact;*
- *being arithmetic.*

The converse of the previous proposition is not true. In particular, there exist incommensurable cofinite hyperbolic Coxeter groups of the same volume.

4.2 Different methods to test commensurability

In this section, we consider only cofinite hyperbolic Coxeter groups.

4.2.1 Subgroup relations

The first trivial and most natural example of commensurability relation is the one of a finite index subgroup: every finite index subgroup is commensurable with the bigger group. Some index two subgroups are easy to find by looking at the Coxeter graph. It is the case, for example, when a vertex of the graph is connected to precisely one other vertex by a even weight. In this setting, the group contain an index two subgroup as shown in Figure 4.1 (see Proposition 7.1.9, with $I = \{s\}$). Note that if $m_{s,t} = \infty$ in the situation depicted in Figure

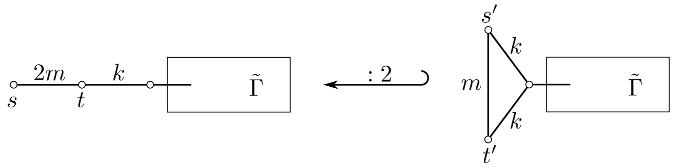


Figure 4.1 – An index two subgroup

4.1, then $m_{s',t'} = \infty$ as well (and the same holds when s and t are connected by a dotted edge). More information and examples about index two subgroups of Coxeter groups can be found in Chapter 7.

4.2.2 Invariant trace field and invariant quaternion algebra

Let Γ be a Kleinian group, that is, Γ is a cofinite discrete subgroup of $\operatorname{Isom}^+ \mathbb{H}^3$, which means that every element $\gamma \in \Gamma$ is orientation preserving and has a representative $\gamma \in \mathrm{SL}(2;\mathbb{C})$. Let $\Gamma^{(2)}$ be the subgroup of Γ generated by squares of elements of Γ.

Definition 4.2.1 (Invariant trace field)
The field $\mathbb{Q}\big(\operatorname{Tr}\gamma : \gamma \in \Gamma^{(2)}\big)$ generated by the traces of representatives of elements in $\mathrm{SL}(2;\mathbb{C})$ is called the *invariant trace field* and is denoted by $K(\Gamma)$.

Proposition 4.2.2
The invariant trace field $K(\Gamma)$ is a finite extension of $\mathbb{Q}$ and is an invariant of the commensurability class of Γ.

Proof. See Theorems 3.1.2 and 3.3.4 of [MR03]. □

Let $A\Gamma$ be the subring of $\mathrm{Mat}(2;\mathbb{C})$ generated by $K(\Gamma)$ and by the representatives $\gamma \in \mathrm{SL}(2;\mathbb{C})$ of the elements of $\Gamma^{(2)}$.

Proposition 4.2.3
The ring $A\Gamma$ is a quaternion algebra over $K(\Gamma)$ and is an invariant of the commensurability class of Γ. Moreover, if Γ is non-cocompact, then the quaternion algebra is trivial, that is $A\Gamma \cong \mathrm{Mat}(2;K(\Gamma))$.

Proof. See Theorem 3.2.1, Corollary 3.3.5 and Theorem 3.3.8 of [MR03]. □

Definition 4.2.4 (Invariant quaternion algebra)
The quaternion algebra $A\Gamma$ is called the *invariant quaternion algebra*.

Remarks 4.2.5

- In Section 4.3.2, we will see an effective way to compute the invariant trace field and the invariant quaternion algebra.
- For 3-dimensional cofinite *arithmetic* Coxeter groups, the pair $(K(\Gamma), A\Gamma)$ is a complete invariant for the commensurability class.

4.2.3 Covolumes

As explained above, in order for two groups $\Gamma_1, \Gamma_2 < \mathrm{Isom}\,\mathbb{H}^n$ to be commensurable, one must have $\frac{\mathrm{covol}\,\Gamma_1}{\mathrm{covol}\,\Gamma_2} \in \mathbb{Q}$. Of course, because of the Gauss-Bonnet theorem, this condition is not interesting when the dimension n is even since the covolume is a rational multiple of the volume of the n-sphere. More precisely, for n even and a cofinite Coxeter group $\Gamma < \mathrm{Isom}\,\mathbb{H}^n$, we have

$$\mathrm{covolume}(\Gamma) = (-1)^{n/2} \cdot \frac{\pi^{n/2} \cdot 2^n \cdot (n/2)!}{n!} \cdot \chi(\Gamma).$$

However, for $n = 3$ and $n = 5$ volume considerations can help to exclude the commensurability. As an example, consider the two arithmetic hyperbolic Coxeter pyramids depicted in Figure 4.2. These two groups have the same invariant trace field, $K(\Gamma) = \mathbb{Q}[\sqrt{2}, i]$, and invariant quaternion algebra (which is trivial, since the groups are non-cocompact, as indicated in Proposition 4.2.3). Moreover, one can show (see [GJK16]) that the volumes are given by

$$\mathrm{covol}\,\Gamma_1 = Л\left(\frac{\pi}{4}\right), \quad \mathrm{covol}\,\Gamma_2 = \frac{1}{8}\,Л\left(\frac{\pi}{6}\right) + Л\left(\frac{5\pi}{24}\right) - Л\left(\frac{\pi}{24}\right),$$

where Л is the *Lobachevsky function*, defined by

$$Л(x) = -\int_0^x \log|2\sin t|\mathrm{d}t, \quad x \in \mathbb{R}.$$

Then, one can show that the quotient $\frac{\mathrm{covol}\,\Gamma_1}{\mathrm{covol}\,\Gamma_2}$ is irrational (see [GJK16, pp 158-159]), which implies that Γ_1 is not commensurable to Γ_2.

$\Gamma_1 = [\infty, 3, (3, \infty, 4)]$ $\Gamma_2 = [\infty, 4, (3, \infty, 4)]$

Figure 4.2 – Three dimensional pyramids

4.3 Arithmetical aspects

In [Mac11], Maclachlan described an effective way to decide the commensurability of two discrete *arithmetic* subgroups of $\mathrm{Isom}\,\mathbb{H}^n$. The key point is that the classification of arithmetic hyperbolic groups is equivalent to the classification of their underlying quadratic forms up to similarity. In turn, the classification of these quadratic forms can be achieved using a complete set of invariants based on quaternion algebras.

In this section, we present the result of Maclachlan and a few examples. We first describe the general ideas behind the classification and then explain how to compute the complete set of invariants of an arithmetic hyperbolic Coxeter groups. We illustrate the method with examples in dimension 4 and 6 and then consider some infinite families described by Kaplinskaya. Other examples can be found in [GJK16], where we classified 168 arithmetic pyramid groups.

4.3.1 Classification

Let $\Gamma_1, \Gamma_2 < \mathrm{Isom}\,\mathbb{H}^n$ be two arithmetic Coxeter groups defined over the number fields K_1 and K_2 respectively and let f_1 and f_2 be their underlying quadratic forms. A result of Gromov and Piatetski-Shapiro (see [GP87, Section 2.6]) implies the following result:

$$\Gamma_1 \sim \Gamma_2 \Leftrightarrow K_1 = K_2 \text{ and } f_1 \sim f_2,$$

where $f_1 \sim f_2$ means that the two quadratic forms are *similar* (meaning that there exists $c \in K_1 = K_2$ such that $c \cdot f_1$ is isomorphic to f_2). Therefore, deciding whether Γ_1 is commensurable to Γ_2 can be done in two steps. First, we have to find the defining fields and the underlying quadratic forms (see [Vin67, Part II, Theorem 2]). Secondly, we have to detect the similarity of the forms. The main references for this section are [Mac11] and [Vin67].

Remark 4.3.1
An immediate consequence of the result of Gromov and Piatetski-Shapiro is the following: when n is odd, a necessary condition for Γ_1 and Γ_2 to be commensurable is that $K_1 = K_2$ and that the quotient $\det f_1 / \det f_2$ is a square in K_1. Indeed, if the forms are similar it means that there exists $\lambda \in K_1^*$ such that $f_1 \cong \lambda \cdot f_2$ and thus $\det f_1 = \lambda^{n+1} \cdot \det f_2$.

We will present the two points separately and illustrate them with an example: the goal is to decide whether the two subgroups $\Gamma_1^6, \Gamma_2^6 < \mathrm{Isom}\,\mathbb{H}^6$ presented in Figure 4.3 are commensurable or not. We first remark that these two groups are arithmetic by virtue of Proposition 3.9.14:

- The Gram matrix of the first one has entries in $\mathbb{Z}\left[\frac{1}{2}\right]$.
- The graph of the second group contains only weights $3, 4$ and 6 and no cycles.

The group Γ_1^6 is the simplex group $\overline{P}_6$ and has covolume $\frac{13}{1360800} \cdot \pi^3$ (see Chapter 5 or [Joh+99]). The group Γ_2^6 is one of the pyramids given by Tumarkin in [Tum04] and has covolume $\frac{13}{604800} \cdot \pi^3$ (see Chapter 5); the ratio of their volumes is $\frac{4}{9}$. Since both groups are non-cocompact, the fields of definition are $\mathbb{Q}$.

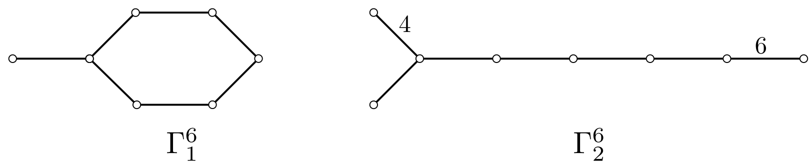


Figure 4.3 – Two arithmetic Coxeter subgroups of $\operatorname{Isom}\mathbb{H}^6$

4.3.1.1 Finding the defining field and the underlying quadratic form

Let $\Gamma < \operatorname{Isom}\mathbb{H}^n$ be an arithmetic Coxeter group of rank d. We want to find the defining field K and the underlying quadratic form f of Γ. We consider the associated fundamental polyhedron $P(\Gamma)$ and the outward-pointing normal unit vectors $e_i \in E^{n,1}$ of the bounding hyperplanes H_i of $P(\Gamma)$, which means that $P(\Gamma) = \bigcap_{i=1}^d H_{e_i}^-$. Finally, we denote by $G(\Gamma) = (a_{i,j}) \in \operatorname{Mat}(\mathbb{R}; d)$ the Gram matrix of Γ, meaning that $a_{i,j} = \langle e_i, e_j \rangle$, for the Lorentzian product.
Using Vinberg's arithmeticity criterion (see Theorem 3.9.7), we can determine the defining field K of Γ (recall that in this setting K is a totally real number field) by computing cycles in $2 \cdot G(\Gamma)$. Now, consider the K-subvector space V of $\mathbb{R}^{n+1}$ spanned by all the vectors

$$v_{i_1,\ldots,i_k} = 2^k \cdot a_{1,i_1} \cdot a_{i_1,i_2} \cdot \ldots \cdot a_{i_{k-1},i_k} e_{i_k}, \quad \{i_1, \ldots, i_k\} \subset \{1, \ldots, d\}. \tag{4.1}$$

The restriction of the Lorentzian form f_0 to V is an admissible quadratic space of dimension $n+1$ over K. Moreover, Γ is commensurable to $O(f, L)$, where L is the lattice spanned over K by a basis $v_1, \ldots, v_{n+1}$ of V and f is the quadratic form of signature $(n, 1)$ defined by the symmetric matrix whose entries are the Lorentzian products $\langle v_i, v_j \rangle$.

Hence, we have an explicit method to find the defining field K and the underlying quadratic form f of an arithmetic hyperbolic Coxeter group Γ.

Remark 4.3.2
The task of finding the outward-pointing normal vectors of the polyhedron $P(\Gamma)$ is difficult: one has to find the vectors $e_1, \ldots, e_d$ of $E^{n,1}$ such that $\langle e_i, e_j \rangle = a_{i,j}$. This basically is equivalent to solve a system of $\frac{d(d+1)}{2}$ quadratic equations with $d \cdot (n+1)$ unknowns. Assuming that $e_1 = (0, \ldots, 0, 1)$ and that the first vectors contain a lot of zeroes, we can guess the remaining vectors using software such as `Mathematica`®. Another possibility is to compute sufficiently good approximations of the vectors e_i using numerical methods and then use the LLL algorithm to find the exact components. This approach is described in [AMR09] for hyperbolic Coxeter groups in $\operatorname{Isom}\mathbb{H}^3$. However, it seems that extending this method to groups of higher ranks would be difficult.

The normal vectors of the polyhedra $P(\Gamma_1^6)$ and $P(\Gamma_2^6)$ associated to the groups Γ_1^6 and Γ_2^6 (see Figure 4.3) are presented in Figure 4.4. We know that the normal vectors of the simplex Γ_1^6 are linearly independent and they all appear on the right hand side of (4.1), up to a rescaling by a rational number. Therefore, we can take $e_1, \dots, e_7$ as a $\mathbb{Q}$-basis of the space V. Hence, in this case, the underlying quadratic form is the form induced by the Gram matrix of Γ_1^6. In the case of the group Γ_2^6, we find the following vectors:

$$e_1, \dots, e_8, \sqrt{2} \cdot e_2, \dots, \sqrt{2} \cdot e_7, \sqrt{6} \cdot e_8.$$

We see that the vectors $v_1 := e_1, v_2 := \sqrt{2} \cdot e_2, \dots, v_7 := \sqrt{2} \cdot e_7$ are linearly independent over $\mathbb{Q}$. We thus find the matrix of the quadratic form

$$\big(\langle v_i, v_j \rangle\big)_{i,j} = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

Finally, we consider the diagonal forms of these two quadratic forms (this can be done using a software such as `SageMath` or by hand):

$$f_1^6 = \langle 1, 3, 6, 10, 15, 21, -21 \rangle$$
$$f_2^6 = \langle 1, 1, 2, 2, 2, 6, -1 \rangle.$$

Here, $q = \langle a_1, \dots, a_m \rangle$, with $a_i \in K$, denotes the diagonal quadratic form with coefficients $a_1, \dots, a_m$, that is q is the quadratic form on K^m defined by

$$q(x_1, \dots, x_m) = \sum_{i=1}^{m} a_i \cdot x_i^2, \quad \forall (x_1, \dots, x_m) \in K^m.$$

4.3.1.2 Constructing the invariants

Before we present the construction of the invariants, we briefly introduce its ingredients. The reader only interested in computations can skip this section.

For a quadratic space (V, f) we will consider the associated Clifford algebra $\mathrm{Cl}(V, f)$ and its even subalgebra $\mathrm{Cl}_0(V, f)$ (see Chapter 8 for more information). While Maclachlan requires that $v^2 = f(v)$ in $\mathrm{Cl}(V, q)$, we assume in Chapter 8 that $v^2 = -f(v)$. This does not change the theory but simplifies our computations.

Since the commensurability classes of arithmetic hyperbolic Coxeter groups are determined by similarity classes of quadratic forms (see [GP87, §2.6]), we define, for a totally real number field[1] K, the following map:

$$\begin{aligned} \Theta : \mathrm{SimQF}(K, n+1) &\longrightarrow \mathrm{IsomAlg}(K, 2^n) \\ \big[(V, f)\big] &\longmapsto \mathrm{Cl}_0(V, f), \end{aligned}$$

[1] Recall that the field of definition of an arithmetic hyperbolic Coxeter group is a totally real number field.

$e_1 = (1,0,0,0,0,0,0)$

$e_2 = \Big(-\frac{1}{2}, \frac{1}{6}\big(6-\sqrt{6}\big), \frac{1}{8}\big(-5+3\sqrt{6}\big),$
$\frac{1}{24}\big(-15\sqrt{2}+22\sqrt{3}\big), \frac{1}{24}\big(-9+11\sqrt{6}\big),$
$\frac{1}{2}, \frac{1}{12}\big(15-11\sqrt{6}\big)\Big)$

$e_3 = (0,-1,2,2\sqrt{2},2,0,-4)$

$e_4 = \Big(0,0,-\frac{11}{8},-\frac{3}{4\sqrt{2}},-\frac{5}{8},0,\frac{5}{4}\Big)$

$e_5 = (0,0,0,-\sqrt{2},0,0,1)$

$e_6 = \Big(0,0,0,0,-1,\frac{1}{2},\frac{1}{2}\Big)$

$e_7 = (0,0,0,0,0,-1,0)$

(a) Normal vectors of $P(\Gamma_1^6)$

$e_1 = (1,0,0,0,0,0,0)$

$e_2 = \Big(0,-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0,0,0,0\Big)$

$e_3 = \Big(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0,0,0,0,0\Big)$

$e_4 = \Big(0,-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},-\frac{\sqrt{3}}{2},-\frac{1}{2},0,-1\Big)$

$e_5 = \Big(0,0,0,\frac{\sqrt{3}}{2},-\frac{1}{2},0,0\Big)$

$e_6 = \Big(0,0,0,0,1,\frac{1}{2},\frac{1}{2}\Big)$

$e_7 = (0,0,0,0,0,-1,0)$

$e_8 = \Big(0,0,0,-\frac{1}{2},-\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2},-\frac{\sqrt{3}}{2}\Big)$

(b) Normal vectors of $P(\Gamma_2^6)$

Figure 4.4 – Outward-pointing normal vectors of the Coxeter polyhedra associated to the groups $\Gamma_1^6, \Gamma_2^6 < \operatorname{Isom}\mathbb{H}^6$

where $\operatorname{SimQF}(K, n+1)$ is the set of similarity classes of quadratic spaces over K of dimension $n+1$ and $\operatorname{IsomAlg}(K, 2^n)$ is the set of isomorphism classes of algebras over K of dimension 2^n. If we restrict the map Θ to the quadratic forms f of signature $(n,1)$ whose conjugates f^σ under non-trivial Galois embeddings $\sigma : K \longrightarrow \mathbb{R}$ are positive definite, then we get an injective map (see [Mac11, Theorems 6.1 and 6.2]). Hence, we have to be able to decide whether the even parts of the Clifford algebras (see Chapter 8) associated to the underlying quadratic forms are isomorphic as algebras. Luckily for us, there is an efficient way to decide that using the Witt invariants and the Hasse invariants.

Definition 4.3.3 (Witt invariant)
Let (V,f) be a quadratic space over a field K. The *Witt invariant* of (V,f), denoted by $c(V,f)$, or only $c(f)$, is the element of $\operatorname{Br} K$ defined as follows:

$$c(f) = c(V,f) := \begin{cases} \big[\operatorname{Cl}_0(V,f)\big] & \text{if } \dim V \text{ is odd,} \\ \big[\operatorname{Cl}(V,f)\big] & \text{if } \dim V \text{ is even.} \end{cases}$$

Definition 4.3.4 (Hasse invariant)
The *Hasse invariant* of a diagonal quadratic form $\langle a_1,\dots,a_n\rangle$ over K is the element of the Brauer group defined as follows:

$$s(\langle a_1,\dots,a_n\rangle) = \bigotimes_{i<j} (a_i,a_j)_K.$$

It can be shown (see [Lam05, Chapter V, Proposition 3.18]) that two isomorphic diagonal quadratic forms have the same Hasse invariant. Therefore, we can define the Hasse invariant of quadratic space (V,f) over K, or a quadratic form f over K, to be the Hasse invariant of any diagonalization of f. It will be denoted by $s(V,f)$ or just $s(f)$.

Proposition 4.3.5 ([Lam05, Proposition 3.20])
For a quadratic space (V, f), the Hasse invariant and the Witt invariant are related as follows:

$$c(f) = \begin{cases} s(f) & \dim V \equiv 1, 2 \ (mod\ 8) \\ s(f) \cdot (-1, -\det f) & \dim V \equiv 3, 4 \ (mod\ 8) \\ s(f) \cdot (-1, -1) & \dim V \equiv 5, 6 \ (mod\ 8) \\ s(f) \cdot (-1, \det f) & \dim V \equiv 7, 8 \ (mod\ 8) \end{cases}$$

Remark 4.3.6
Recall from Proposition 2.5.3 that if K is a number field and if (a_i, b_i), $i = 1, 2$, are two quaternion algebras over K, then there exists a quaternion algebra (x, y) over K such that the following equality holds in the Brauer group $\mathrm{Br}(K)$:

$$(a_1, b_1) \cdot (a_2, b_2) = (x, y).$$

In particular, it implies that we can choose a representative of $s(f)$ and $c(f)$ which is a quaternion algebra. However, in some cases, finding such a quaternion algebra may be difficult.

The even case We suppose that n is even. Let Γ be an arithmetic group defined over a totally real number field K with underlying quadratic form f. Consider the Witt invariant $c(f)$ and a quaternion algebra B which represents $c(f)$. Then, we have

$$\Theta(f) = \mathrm{Cl}_0(f) \cong \mathrm{Mat}\left(2^{(n-2)/2}; B\right),$$

where the last isomorphism comes from [Mac11, Theorem 7.1]. Hence, $\Theta(f)$ is a central simple algebra over K whose class in the Brauer group is B (and thus it is completely determined by B). In particular, we get the following theorem.

Theorem 4.3.7 ([Mac11, Theorem 7.2])
When n is even, the commensurability class of an arithmetic group of the simplest type in $\mathrm{Isom}\,\mathbb{H}^n$ is completely determined by the isomorphism class of a quaternion algebra which represents the Witt invariant of its underlying quadratic form.

The odd case When n is odd, the situation is more complicated. We consider, as before, Γ, f, $c(f)$ and B. We also consider the signed determinant $\delta = (-1)^{n(n+1)/2} \cdot \det f$ of f. Since f is admissible, $\det f = \delta \cdot (-1)^{n(n+1)/2}$ is negative but all is non-trivial conjugates are positive. We distinguish now two cases (details can be found in [Mac11]):

- δ is a square in K^*:
  Then, δ cannot have any non-trivial conjugates which implies $K = \mathbb{Q}$. Moreover, we must have $n \equiv 1 \pmod 4$. We finally have $\mathrm{Cl}_0(f) \cong \mathrm{Mat}\left(2^{(n-3)/2}; B\right)^2$.

- δ is not a square in K^*:
  In this case, we have $\mathrm{Cl}_0(f) \cong \mathrm{Mat}\left(2^{(n-3)/2}; B \otimes_K K(\sqrt{\delta})\right)$.

About the ramification at infinite places As we saw above, commensurability classes are in fact determined by the ramification of certain quaternion algebra (either the Witt invariant or the Witt invariant over a quadratic extension of the base field). However, the ramification at infinite places is completely independent of the Witt invariant. Indeed, using admissibility of the quadratic form f, propositions 2.5.1 and 4.3.5 and the fact that $\sigma(\det f) > 0$ for every $\sigma : K \longrightarrow \mathbb{R}$ with $\sigma \neq \mathrm{id}$, we find

$$\mathrm{Ram}_\infty c(f) = \begin{cases} \emptyset & \dim V \equiv 1, 2 \pmod 8 \\ \Omega_\infty(K) \setminus \{\mathrm{id}\} & \dim V \equiv 3, 4 \pmod 8 \\ \Omega_\infty(K) & \dim V \equiv 5, 6 \pmod 8 \\ \{\mathrm{id}\} & \dim V \equiv 7, 8 \pmod 8 \end{cases} \tag{4.2}$$

4.3.1.3 Computing the invariants

We explain in this section how to compute the invariants which completely determine the commensurability class of an arithmetic hyperbolic *Coxeter* group.

Let $\Gamma < \mathrm{Isom}\,\mathbb{H}^n$ be an arithmetic Coxeter group defined over the totally real number field K and let f be the underlying quadratic form (see Section 4.3.1.1). As above, let $\delta = (-1)^{n(n+1)/2} \cdot \det f$ be the signed determinant of f. From a diagonalized form of f compute the Hasse invariant $s(f)$ of f (see Definition 4.3.4) and the Witt invariant $c(f)$ (see Proposition 4.3.5). Finally, choose a representative B of $c(f)$ which is a quaternion algebra (see remarks below). Then, a complete invariant for the commensurability class of Γ is given in Table 4.3.

n	Complete invariant
even	$\{K, \mathrm{Ram}_f\, B\}$
odd	δ is a square in K^*: $\{\mathbb{Q}, \mathrm{Ram}_f\,(B)\}$
	δ not a square in K^*: $\{K, \delta, \mathrm{Ram}_f\,(B \otimes_K K(\sqrt{\delta}))\}$

Table 4.3 – Complete invariant for an arithmetic hyperbolic Coxeter group

Remarks 4.3.8
- In certain cases, finding a representative of $c(f)$ which is a quaternion algebra can be tricky. However, this step is not really important as we are only interested in the ramification of $c(f)$ (see Proposition 2.5.8).
- When n is odd and when δ is not a square in K^*, one can use Proposition 2.5.9 to compute the ramification of $B \otimes_K K(\sqrt{\delta})$ (as a quaternion algebra over $K(\sqrt{\delta})$).

If $K = \mathbb{Q}$ and if the associated diagonal quadratic form f is known, then `AlVin` (see Chapter 6) can be used to compute the invariant. For example, to compute the invariant of a group associated to the quadratic form $f_1^6 = \langle 1, 3, 6, 10, 15, 21, -21\rangle$ of the example given in Section 4.3.1.1, one can do

```
./alvin -maxv 1 -iqf -qf 1,3,6,10,15,21,-21
```

The "-maxv 1" option indicates not to compute normal vectors and the "-iqf" parameter (for **i**nvariant of the **q**uadratic **f**orm) asks the program to display the invariant. Then, the output is the following:

```
Commensurability invariant:
  {Q,{infinity,3}}
```

Therefore, the commensurability invariant of the group Γ_1^6 is $\big\{\mathbb{Q}, \{\infty, 3\}\big\}$.

4.3.1.4 Finishing the analysis for the examples in dimension 6

We come back to our two examples in dimension 6 defined over $\mathbb{Q}$ and with quadratic forms

$$\begin{aligned} f_1^6 &= \langle 1, 3, 6, 10, 15, 21, -21\rangle \\ f_2^6 &= \langle 1, 1, 2, 2, 2, 6, -1\rangle. \end{aligned}$$

We first compute the Hasse invariants $s(f_i^6)$ for $i = 1, 2$. Recall that for a diagonal quadratic form $f = \langle a_1, \dots, a_m\rangle$, we have $s(f) = \bigotimes_{i<j}(a_i, a_j) \in \operatorname{Br} K$. Using the properties of Proposition 2.5.2, we compute:

$$\begin{aligned} s(f_1^6) &= (3,6)\cdot(3,10)\cdot(3,15)\cdot\overbrace{(3,21)\cdot(3,-21)}^{(3,-1)} \\ &\quad\cdot(6,10)\cdot(6,15)\cdot(6,-1) \\ &\quad\cdot(10,15)\cdot(10,-1) \\ &\quad\cdot(15,-1) \\ &= (5,-1). \end{aligned}$$

Similarly, we find $s(f_2^6) = 1$. Using Proposition 4.3.5 we find $c(f_1^6) = (5,-1)\cdot(-1,-3) = (-1,-15)$ and $c(f_2^6) = 1\cdot(-1,-3)$. By Proposition 2.5.10 we obtain the following ramification sets:

$$\operatorname{Ram}(-1,-3) = \{3,\infty\}, \quad \operatorname{Ram}(-1,5) = \emptyset.$$

Finally, using Proposition 2.5.8 we get $\operatorname{Ram}_f c(f_1^6) = \{3,\infty\} = \operatorname{Ram}_f c(f_2^6)$, which implies that the groups Γ_1^6 and Γ_2^6 are commensurable.

4.3.1.5 New cocompact examples worked out in detail

We study the commensurability of three cocompact arithmetic Coxeter subgroups Γ_i^4, $i = 1, 2, 3$, of $\operatorname{Isom}\mathbb{H}^4$. These groups correspond to maximal subgroups generated by reflections in the groups of units $O(f_i, \mathcal{O}_K{}^5)$ of three (reflective) quadratic forms f_i with base field $K = \mathbb{Q}[\sqrt{5}]$ and they were found using Vinberg's algorithm (see Chapter 6). For simplicity, we write $\Theta = \frac{1+\sqrt{5}}{2}$ for the generator of the ring of integers of $\mathcal{O}_K = \mathbb{Z}[\Theta]$. The first group is the simplex group $[5, 3, 3, 4]$ and is associated to the quadratic form $f_1 = \langle -\Theta, 1, 1, 1, 1\rangle$ (see [Bug84]) while the last two correspond to *new* Coxeter polyhedra $P(\Gamma_i^4)$, $i = 2, 3$, I discovered (see Section 6.8.3) and are associated to the quadratic forms $f_2 = \langle -\Theta, 1, 1, 1, 2+\Theta\rangle$ and $f_3 = \langle -\Theta, 1, 1, 2+\Theta, 2+\Theta\rangle$ respectively. The Coxeter graphs of the first two groups are presented in Figure 4.5.

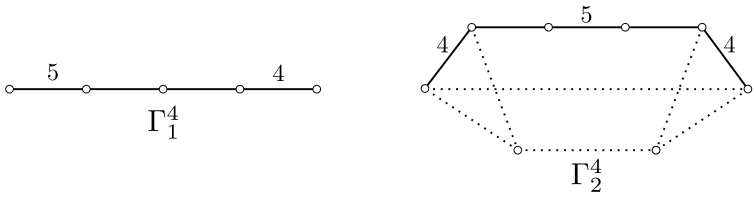


Figure 4.5 – $\Gamma_1^4, \Gamma_2^4 < \operatorname{Isom} \mathbb{H}^4$

The Coxeter graph of the third group is more complicated: it has 16 vertices and 72 edges. We depict in Figure 4.6 the non-dotted edges of the graph and the dotted edges can be computed using the normal vectors given in Figure 4.7. The invariants of the three groups are presented in Figure 4.8 page 67.

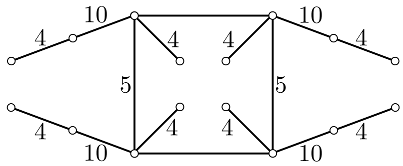


Figure 4.6 – $\Gamma_3^4 < \operatorname{Isom} \mathbb{H}^4$

$$
\begin{aligned}
e_1 &= (0,-1,1,0,0), & e_4 &= (0,0,0,0,-1),\\
e_2 &= (0,0,-1,0,0), & e_5 &= (-1+1\cdot\Theta, 1\cdot\Theta, 0,0,0),\\
e_3 &= (0,0,0,-1,1), & e_6 &= (1,0,0,1,0),
\end{aligned}
$$

$$
\begin{aligned}
e_7 &= (1+3\cdot\Theta, 2+1\cdot\Theta, 0, 1+1\cdot\Theta, 1+1\cdot\Theta),\\
e_8 &= (2+3\cdot\Theta, 1+2\cdot\Theta, 2\cdot\Theta, 1+1\cdot\Theta, 1+1\cdot\Theta),\\
e_9 &= (4+7\cdot\Theta, 3+4\cdot\Theta, 3+4\cdot\Theta, 2+3\cdot\Theta, 2\cdot\Theta),\\
e_{10} &= (2+5\cdot\Theta, 1+2\cdot\Theta, 1+2\cdot\Theta, 1+2\cdot\Theta, 1+2\cdot\Theta),\\
e_{11} &= (9+12\cdot\Theta, 4+7\cdot\Theta, 4+7\cdot\Theta, 3+6\cdot\Theta, 2+4\cdot\Theta),\\
e_{12} &= (10+15\cdot\Theta, 4+7\cdot\Theta, 4+7\cdot\Theta, 4+7\cdot\Theta, 3+6\cdot\Theta),\\
e_{13} &= (11+18\cdot\Theta, 7+11\cdot\Theta, 7+11\cdot\Theta, 4+7\cdot\Theta, 4+5\cdot\Theta),\\
e_{14} &= (7+10\cdot\Theta, 4+6\cdot\Theta, 4+6\cdot\Theta, 3+4\cdot\Theta, 2+3\cdot\Theta),\\
e_{15} &= (7+11\cdot\Theta, 4+7\cdot\Theta, 4+5\cdot\Theta, 3+5\cdot\Theta, 2+3\cdot\Theta),\\
e_{16} &= (15+25\cdot\Theta, 10+15\cdot\Theta, 7+11\cdot\Theta, 7+11\cdot\Theta, 5+7\cdot\Theta).
\end{aligned}
$$

Figure 4.7 – Normal unit vectors for $P(\Gamma_3^4)$

Since the dimension of the space is even, the commensurability class is completely determined by the ramification set of the Witt invariant $c(f_i)$, $i = 1, 2, 3$, as indicated in Theorem 4.3.7. Direct computations (see Figure 4.8) show that the Witt invariants are given by

$$c(f_1) = (-1,-1), \quad c(f_2) = (-\Theta, 2+\Theta)\cdot(-1,-1), \quad c(f_3) = (-1, -2-\Theta).$$

Therefore, in order to decide about the commensurability of our groups, we have to compute the finite ramification of the quaternion algebras $(-1,-1)_K$, $(-\Theta, 2+\Theta)_K$ and $(-1, 2+\Theta)_K$.

Using Proposition 2.5.9 we see that $\mathrm{Ram}_f(-1,-1) = \emptyset$. Since $N_{K/Q}(-\Theta) = -1$, then $-\Theta$ is invertible, while $N_{K/Q}(2+\Theta) = 5$ implies that $2+\Theta$ is a prime element of $\mathcal{O}_K$ lying over 5. For a quaternion algebra $(a,b)_K$, a necessary condition for a prime ideal $\mathcal{P} \subset \mathcal{O}_K$ to belong to the ramification set is that $\mathcal{P} \mid \langle 2ab \rangle = (2ab)\mathcal{O}_K$. In our case, since 2 remains prime in $\mathcal{O}_K$, the only candidates are 2 and $\sqrt{5}$. Moreover, since the ramification set has even cardinality (see Theorem 2.5.5) and since $|\mathrm{Ram}_\infty B_i| = 2$ (see equation (4.2) page 63), then either $\mathrm{Ram}_f B_i = \emptyset$ or $\mathrm{Ram}_f B_i = \{2, \sqrt{5}\}$. To compute the ramification, we will use the following theorem.

Theorem 4.3.9 ([AMR09, Theorem 16])
Let K be a number field. Let $\mathcal{P}$ be a prime ideal of $\mathcal{O}_K$ and let $a, b \in \mathcal{O}_K$ be such that the valuations of these elements satisfy $\eta_\mathcal{P}(a), \eta_\mathcal{P}(b) \in \{0,1\}$. We define an integer m as follows:

- *if $\mathcal{P} \mid \langle 2 \rangle$, $m = 2\eta_\mathcal{P}(2) + 3$;*
- *if $\mathcal{P} \nmid \langle 2 \rangle$, then $m = 1$ if $\eta_\mathcal{P}(a) = \eta_\mathcal{P}(b) = 0$ and $m = 3$ otherwise.*

We also let S be a finite set of representatives for the ring $\mathcal{O}_K/\mathcal{P}^m$.
Then, $\mathcal{P} \notin \mathrm{Ram}_f(a,b)_K$, i.e. (a,b) splits at $\mathcal{P}$, if and only if there exists a triple $(X,Y,Z) \in S^3$ such that the two following conditions are satisfied:

- *$aX^2 + bY^2 - Z^2 = 0$ or $m \leq \eta_\mathcal{P}(aX^2 + bY^2 - Z^2)$;*
- *$\eta_\mathcal{P}(X) = 0$ or $\eta_\mathcal{P}(Y) = 0$ or $\eta_\mathcal{P}(Z) = 0$.*

Remark 4.3.10
Since $(ac^2, b) \cong (a,b)$, the condition $\eta_\mathcal{P}(a), \eta_\mathcal{P}(b) \in \{0,1\}$ is not a restriction.

We start with the group Γ_2^4. We take $\mathcal{P} = \langle \sqrt{5} \rangle$ and $a = -\Theta$, $b = 2+\Theta$, which gives $\eta_\mathcal{P}(a) = 0$ and $\eta_\mathcal{P}(b) = 1$ and thus $m = 3$. The quotient $\mathcal{O}_K/\mathcal{P}^3$ has 125 elements which can be described as follows (see [EIR11, Theorem 1]):

$$\mathcal{O}_K/\mathcal{P}^3 = \left\{ (x + y\sqrt{5})\mathcal{P}^3 : 0 \leq x < 25, 0 \leq y < 5 \right\}.$$

Using a computer, we can check by means of Theorem 4.3.9 that $\sqrt{5}$ belongs to $\mathrm{Ram}_f(-\Theta, 2+\Theta)$ and thus $\sqrt{5} \in \mathrm{Ram}_f B_2$. Therefore, we have $\mathrm{Ram}_f B_2 = \{2, \sqrt{5}\}$. On the other hand, since the equation $(2+\Theta)X^2 - Y^2 - Z^2 = 0$ is satisfied with $X = Y = -1+\Theta$ and $Z = 1$ (and thus $\eta_\mathcal{P}(X) = \eta_\mathcal{P}(Y) = \eta_\mathcal{P}(Z) = 0$), then $\sqrt{5} \notin \mathrm{Ram}_f(-1, 2+\Theta)$ which implies that $\mathrm{Ram}_f B_3 = \emptyset$. Hence, the simplex group Γ_1^4 is commensurable to the group Γ_3^4 but *not* to Γ_2^4.

4.3.2 Case $n = 3$ (arithmetic and non-arithmetic)

When $n = 3$, we have two interesting features concerning (in-)commensurable groups in $\mathrm{Isom}\,\mathbb{H}^3$:

- the computation of the invariants is easier;
- some of the results are true even when the groups are not arithmetic.

invariant	Γ_1^4	Γ_2^4	Γ_3^4
f_i	$\langle -\Theta, 1, 1, 1, 1\rangle$	$\langle -\Theta, 1, 1, 1, 2+\Theta\rangle$	$\langle -\Theta, 1, 1, 2+\Theta, 2+\Theta\rangle$
f-vector	$(5, 10, 10, 5, 1)$	$(14, 28, 22, 8, 1)$	$(48, 96, 64, 16, 1)$
covol.	$\frac{17}{21600}\cdot\pi^2$	$\frac{1}{60}\cdot\pi^2$	$\frac{221}{900}\cdot\pi^2$
$s(f_i)$	1	$(-\Theta, 2+\Theta)$	$(-1, 2+\Theta)$
$c(f_i)$	$(-1,-1)$	$(-\Theta, 2+\Theta)\cdot(-1,-1)$	$(-1, -2-\Theta)$
$\mathrm{Ram}_f\, B_i$	$\emptyset$	$\{2, \sqrt{5}\}$	$\emptyset$

Figure 4.8 – Invariants of the three 4-dimensional cocompact groups Γ_i^4

A standard reference for the three dimensional case is [MR03]. Let $\Gamma < \operatorname{Isom}\mathbb{H}^3$ be a cofinite Coxeter group (not necessarily arithmetic) of rank d. As above, we denote by $G(\Gamma) = (a_{i,j})$ the Gram matrix of Γ and by e_i the normal vectors unit of the bounding hyperplanes H_{e_i} of the fundamental polyhedron $P(\Gamma)$ of Γ. We let K be the field generated by the cycles in $G(\Gamma)$:

$$K = \mathbb{Q}\big(a_{i_1,i_2}\cdot\ldots\cdot a_{i_{k-1},i_k}\cdot a_{i_k,i_1} : i_j \in \{1,\ldots,d\}\big).$$

We also consider the K-vector space $M(\Gamma)$ spanned by the vectors

$$a_{1,i_1}\cdot a_{i_1,i_2}\cdot\ldots\cdot a_{i_{k-1},i_k}\cdot e_{i_k}, \quad i_j \in \{1,\ldots,d\}.$$

This space, endowed with the restriction of the Lorentzian form, is a quadratic space of signature $(3,1)$. If we let δ be the determinant of this quadratic space we then have the following results.

Proposition 4.3.11 ([MR98, Theorem 3.1])
The field $K(\sqrt{\delta})$ is equal to the invariant trace field.

Let $\langle -a_1, a_2, a_3, a_4\rangle$, $a_i > 0$, be a diagonal form of the quadratic form corresponding to $M(P)$.

Proposition 4.3.12 ([MR98, Theorem 3.1])
The invariant quaternion algebra $A\Gamma$ is given by $(a_1\cdot a_2, a_1\cdot a_3)_{K(\Gamma)}$.

Theorem 4.3.13 ([MR03, Theorem 3.3.8])
If Γ is non-cocompact, then $A\Gamma = 1$.

Remark 4.3.14
If Γ is arithmetic, then the invariant trace field and the invariant quaternion algebra correspond to what we get using the Witt invariant (and thus form a *complete* set of invariants). Indeed, since we are only interested in the similarity class of the quadratic form, we can suppose that it can be written in the form $\langle 1, \tilde{b}, \tilde{c}, -\tilde{d}\rangle$, with $\tilde{b}, \tilde{c}, \tilde{d} > 0$. We then let $\delta = -bcd$ be the determinant of the form which gives

$$\langle 1, \tilde{b}, \tilde{c}, \tilde{d}\rangle \cong \langle 1, -\delta\tilde{c}\tilde{d}, -\delta\tilde{b}\tilde{d}, \delta\tilde{b}\tilde{c}\rangle = \langle 1, -\delta c, -\delta d, \delta cd\rangle,$$

with $c := \tilde{c}\tilde{d}$, $d := \tilde{b}\tilde{d}$. Then, a direct computation shows that the Witt invariant of this last quadratic form is $(c,d) = (\tilde{c}\tilde{d}, \tilde{b}\tilde{d})$.

4.4 Examples

4.4.1 Kaplinskaya's 3-dimensional families

In this section, we present the computations of the commensurability invariants for some families of Coxeter prisms in $\mathbb{H}^3$ given by Kaplinskaya in [Kap74]. The summary of the invariants for these families can be found in the appendices A.1.1 and A.1.2. The infinite families of Kaplinskaya are of special interest since they are the among the very few known infinite families of hyperbolic Coxeter groups not arising as sequences of doublings (see Chapter 7 for examples of such sequences).

Example 4.4.1
We continue to study the family of 3-dimensional cocompact groups Γ_m shown in Figure 4.9 (see Example 3.6.7 for the growth series and the growth rates of these groups). Our aim is to decide which groups of the family are (quasi-)arithmetic and to compute the invariant trace field and the invariant quaternion algebra. The Gram matrix of such a group is given by G_m and has the following form

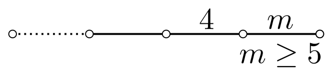


Figure 4.9 – A family of 3-dimensional compact Coxeter prisms

$$G_m = \begin{pmatrix} 1 & -\frac{\sqrt{3\cos\left(\frac{2\pi}{m}\right)+1}}{2\sqrt{\cos\left(\frac{2\pi}{m}\right)}} & 0 & 0 & 0 \\ -\frac{\sqrt{3\cos\left(\frac{2\pi}{m}\right)+1}}{2\sqrt{\cos\left(\frac{2\pi}{m}\right)}} & 1 & -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & 1 & -\cos\left(\frac{\pi}{m}\right) \\ 0 & 0 & 0 & -\cos\left(\frac{\pi}{m}\right) & 1 \end{pmatrix}.$$

In order for Γ_m to be (quasi-)arithmetic, it is necessary that all the entries of G_m are totally real algebraic numbers. Since $\cos\frac{\pi}{m}$ is always a totally real algebraic number we will see for which value of m there is at least one conjugate of $\alpha_m := \frac{3\cos\left(\frac{2\pi}{m}\right)+1}{\cos\left(\frac{2\pi}{m}\right)}$ which is negative. Thus, if there exists an integer $k < m$ which is coprime with m and such that the image $\frac{3\cos\left(\frac{2k\pi}{m}\right)+1}{\cos\left(\frac{2k\pi}{m}\right)}$ of α_m by the Galois automorphism corresponding to k is negative, then Γ_m is not quasi-arithmetic. This happens when k lies between $\frac{1}{4}m$ and $\arccos\left(-\frac{1}{3}\right)\cdot\frac{1}{2\pi}\cdot m \cong 0.304087\cdot m$. Therefore, we see using Theorem 2.3.1 with $\varepsilon = \frac{1}{5}$ that Γ_m is not quasi-arithmetic if $m \geq 100$. Direct computations show that the only possibilities are $m \in \{5, 6, 8, 9, 12, 13, 14, 16, 20, 21, 22, 28, 30, 36, 54\}$. Since the minimal polynomial of $\left(2\cdot G_m\right)_{1,2} = -\sqrt{\alpha_m}$ is not monic when $m \in \{12, 20, 28, 36\}$, then Γ_m cannot be arithmetic for these values.

Now, the field K generated by the cycles in $2G_m$ is $\mathbb{Q}\left[\cos\frac{2\pi}{m}\right]$ and the field $\tilde{K}$ generated by the entries of G_m is $\mathbb{Q}\left[\cos\frac{\pi}{m}, \sqrt{\alpha_m}, \sqrt{2}\right]$. For $m \in \{5, 6, 8, 9, 12, 13, 14, 16, 20, 21, 22, 28, 30, 36, 54\}$, the group Γ_m will be quasi-arithmetic if and only if the matrix G_m^σ is positive semi-definite for every Galois embedding σ of $\tilde{K}$ which is not the identity on K.

For $m = 5$, we have $\cos\frac{\pi}{5} = \frac{1}{4}\left(1+\sqrt{5}\right)$, and the minimal polynomial of $-\frac{1}{2}\sqrt{\alpha_5} = -\frac{\sqrt{3\cos\left(\frac{2\pi}{m}\right)+1}}{2\sqrt{\cos\left(\frac{2\pi}{m}\right)}} = -\frac{1}{2}\sqrt{4+\sqrt{5}}$ is equal to $11 - 32x^2 + 16x^4$, whose roots are

$$-\frac{1}{2}\sqrt{4-\sqrt{5}}, \frac{1}{2}\sqrt{4-\sqrt{5}}, -\frac{1}{2}\sqrt{4+\sqrt{5}}, \frac{1}{2}\sqrt{4+\sqrt{5}}.$$

Now, a Galois embedding $\sigma : \tilde{K} \longrightarrow \mathbb{R}$ cannot send $-\frac{1}{2}\sqrt{4+\sqrt{5}}$ to one of the two last roots for otherwise it would act trivially on $K = \mathbb{Q}[\sqrt{5}]$. Therefore, we have to consider the four Galois embeddings corresponding to

$$-\frac{1}{2}\sqrt{4+\sqrt{5}} \longmapsto \pm\frac{1}{2}\sqrt{4-\sqrt{5}}, \quad \sqrt{2} \longmapsto \pm\sqrt{2},$$

and $\cos\frac{\pi}{5} = \frac{1}{4}\left(1+\sqrt{5}\right)$ to $\frac{1}{4}\left(1-\sqrt{5}\right)$. The four matrices G_5^σ are positive semidefinite, as required.

For $m = 6$, we have $K = \mathbb{Q}$ and there is nothing to check.

For $m = 8$, the degree of $\tilde{K}$ over $\mathbb{Q}$ is 8 and the four embeddings do not act by identity on $K = \mathbb{Q}[\sqrt{2}]$. For each of these embeddings, the conjugate of G_8 is positive semi-definite.

For $m = 12$, the degree of $\tilde{K}$ over $\mathbb{Q}$ is 16 and eight embeddings do not act by identity on $K = \mathbb{Q}[\sqrt{3}]$. For each of these embeddings, the conjugate of G_{12} is positive semi-definite.

For the remaining m, we can show that Γ_m is not quasi-arithmetic (see Appendix B.1.2 for details).

In summary, Γ_m is arithmetic if $m \in \{5, 6, 8\}$, quasi-arithmetic if $m = 12$ and not quasi-arithmetic for any other value of m. We now want to compute the invariant trace field and invariant quaternion algebra. The normal vectors $\{e_1, \ldots, e_5\}$ of the associated polyhedron P_m are given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{\sqrt{3\cos\left(\frac{2\pi}{m}\right)+1}}{2\sqrt{\cos\left(\frac{2\pi}{m}\right)}} & 0 & 0 & -\frac{1}{2}\sqrt{\cos\left(\frac{2\pi}{m}\right)^{-1} - 1} \\ 0 & 0 & \frac{1}{\sqrt{1-\cos\left(\frac{2\pi}{m}\right)}} & -\frac{\sqrt{\cot^2\left(\frac{\pi}{m}\right)-1}}{\sqrt{2}} \\ 0 & -\cos\left(\frac{\pi}{m}\right) & -\sin\left(\frac{\pi}{m}\right) & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

With the notation of Section 4.3.2, a basis for the space $M(\Gamma_m)$ is given by

$$e_1, \quad \sqrt{\alpha_m}\cdot e_2, \quad \sqrt{\alpha_m}\cdot e_3, \quad \sqrt{\frac{\alpha_m}{2}}\cdot e_4,$$

and the Gram matrix of these vectors is

$$M_m = \begin{pmatrix} 1 & -\frac{\alpha_m}{2} & 0 & 0 \\ -\frac{\alpha_m}{2} & \alpha_m & -\frac{\alpha_m}{2} & 0 \\ 0 & -\frac{\alpha_m}{2} & \alpha_m & -\frac{\alpha_m}{2} \\ 0 & 0 & -\frac{\alpha_m}{2} & \frac{\alpha_m}{2} \end{pmatrix}.$$

Now, if we let $P_m := \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{\alpha_m}{2} & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, then $P_m \cdot M_m \cdot {P_m}^t$ is the diagonal matrix with entries equivalent (i.e. equal up to a square factor) to

$$-\left(1 + \sec\frac{2\pi}{m}\right)\alpha_m, 1, 2\alpha_m, 2\alpha_m.$$

This implies that the invariant trace field $K(\Gamma)$ is given by

$$K(\Gamma) = \mathbb{Q}\left[\cos\frac{2\pi}{m}, \sqrt{-\left(3\cos\frac{2\pi}{m} + 1\right)\left(\cos\frac{2\pi}{m} + 1\right)}\right].$$

Finally, the invariant quaternion algebra is

$$A\Gamma = \left(-2\left(1 + \sec\frac{2\pi}{m}\right), -2\left(1 + \sec\frac{2\pi}{m}\right)\right)_{K(\Gamma)}.$$

These results enable us to quickly decide the (in)commensurability of two specific groups of the sequence. Moreover, let us notice that besides a few values of m, the degree of the field extension $[K(\Gamma) : \mathbb{Q}]$ is $\varphi(m)$. In particular, the sequence contains an infinite number of incommensurable groups.

Example 4.4.2
We present the main steps for the computations of some invariants of the family of prisms Γ_m given in Figure 4.10. The methods and results used will be quite similar to the ones used in the previous example. We have

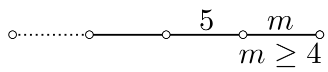


Figure 4.10 – A family of 3-dimensional compact Coxeter prisms

$\tilde{K} = \mathbb{Q}\left[\cos\frac{\pi}{m}, \sqrt{5}, \sqrt{\alpha_m}\right]$, where $\alpha_m = \frac{3\cos\left(\frac{2\pi}{m}\right)+\sqrt{5}}{4\cos\left(\frac{2\pi}{m}\right)+\sqrt{5}-1}$, and $K = \mathbb{Q}\left[\cos\frac{2\pi}{m}, \sqrt{5}\right]$. The fifth minor of the Gram matrix (i.e. the determinant of submatrix of the Gram matrix obtained by deleting the last row and last column) is equivalent (i.e. equal up to a square factor) to $-\frac{3+\sqrt{5}}{4\cos\left(\frac{2\pi}{m}\right)+\sqrt{5}-1}$ which has, for k coprime with m, Galois conjugates

$$a_{m,k} = -\frac{3+\sqrt{5}}{4\cos\left(\frac{2k\pi}{m}\right)+\sqrt{5}-1} \quad \text{and} \quad b_{m,k} = -\frac{3-\sqrt{5}}{4\cos\left(\frac{2k\pi}{m}\right)-\sqrt{5}-1}$$

(depending on the value of m, $\cos\left(\frac{2k\pi}{m}\right)$ and $\sqrt{5}$ may be dependant over $\mathbb{Q}$). If $m > 300$, then there exists a prime number p such that $\frac{m}{12} < p < \frac{m}{12} \cdot \frac{6}{5}$ and thus both $a_{m,p}$ and $b_{m,p}$ are negative. Therefore, these groups can be (quasi-)arithmetic only if $m \leq 300$. Now, if $11 < m \leq 300$, we can check that it is always possible to find $k < m$ coprime with m such that both $a_{m,k}$ and $b_{m,k}$ are negative. For $m = 8, 9$, the number $\sqrt{\alpha_m}$ is not totally real. Hence, we still have to consider the cases $m = 4, 5, 6, 7, 10$.

For $m=4$, we have $\tilde{K}=\mathbb{Q}\left[\sqrt{2},\sqrt{5},\sqrt{5+\sqrt{5}}\right]$ and G_4 is positive semi-definite for the four embeddings which do not fix $K=\mathbb{Q}\left[\sqrt{5}\right]$. For $m=5,6,10$, similar computations give the same results. For $m=7$, the Galois embedding which sends $\cos\frac{\pi}{7}$ to the single negative root of $1-4x-4x^2+8x^3=\min\left(\cos\frac{\pi}{7},\mathbb{Q}\right)$ and $\sqrt{5}$ to itself gives a negative first minor. Therefore, the group is not quasi-arithmetic.

In summary, the group is arithmetic if $m=4,6$ and quasi-arithmetic if $m=5,10$.

The normal vectors $\{e_1,\ldots,e_5\}$ of the polyhedron P_m are given by the line vectors of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -\sqrt{\alpha_m} & 0 & 0 & \frac{1}{\sqrt{\frac{1}{2}\left(3+\sqrt{5}\right)\csc^2\frac{\pi}{m}-4}} \\ 0 & 0 & -\frac{1}{4}\left(1+\sqrt{5}\right)\csc\left(\frac{\pi}{m}\right) & \frac{1}{4}\sqrt{2\left(3+\sqrt{5}\right)\csc^2\frac{\pi}{m}-16} \\ 0 & -\cos\frac{\pi}{m} & \sin\frac{\pi}{m} & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

and a basis for the space $M(\Gamma_m)$ is given by

$$e_1, \quad \sqrt{\alpha_m}\cdot e_2, \quad \sqrt{\alpha_m}\cdot e_3, \quad (1+\sqrt{5})\sqrt{\alpha_m}\cdot e_4.$$

If we denote by M_m the Gram matrix of these vectors, and if we let P_m be the matrix

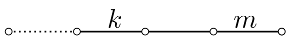
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ \alpha_m & 1 & 1+\frac{1}{\sqrt{5}} & \frac{1}{20}\left(5+\sqrt{5}\right) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2}\left(3+\sqrt{5}\right) & 1 \end{pmatrix},$$

then $P_m\cdot M_m\cdot {P_m}^t$ is the diagonal matrix with entries

$$-\frac{2\left(5+2\sqrt{5}\right)\cos^2\frac{\pi}{m}}{\left(4\cos\left(\frac{2\pi}{m}\right)+\sqrt{5}-1\right)}\alpha_m, 1, \alpha_m, 2\left(5+\sqrt{5}\right)\alpha_m$$

and with signed determinant $\delta=-\frac{\left(7+3\sqrt{5}\right)\cos^2\frac{\pi}{m}}{4\cos\left(\frac{2\pi}{m}\right)+\sqrt{5}-1}\alpha_m$. From this, it is easy to check the (in)commensurability of two given members of this family.

Example 4.4.3
We present the main steps for the computation of some invariants of the family of Coxeter prisms $\Gamma_{k,m}$ given in Figure 4.11. As before, by considering the fifth



$m\geq 7 \qquad k=3,4,5$

Figure 4.11 – A family of 3-dimensional compact Coxeter prisms

minor of the Gram matrix, we see that $\Gamma_{k,m}$ can be quasi-arithmetic only if $m\in\{7,8,9,10,11,12,14,16,18,24,30\}$.

For $k=3$, we see that $\Gamma_{3,m}$ is arithmetic if and only if $m=7,8,9,10,14$ and quasi-arithmetic when $m=12,18,24,30$. This is coherent with [Vin67].

For $k=4$, we see that $\Gamma_{4,m}$ is arithmetic if and only if $m=7,8,10,12$ and quasi-arithmetic when $m=18$.

The other invariants of the family can be found in Appendix A.1.1.

4.4.2 Forms $\langle -p, 1, \ldots, 1\rangle$ and $\langle -1, 1, \ldots, 1, p\rangle$

The aim of this section is to give the commensurability invariants of the two quadratic forms $\langle -p, 1, \ldots, 1\rangle$ and $\langle -1, 1, \ldots, 1, p\rangle$. These forms are of particular interest because they are the first natural candidates for the Vinberg algorithm (see Chapter 6) after the standard quadratic form of signature $(n, 1)$. The investigation of the reflectivity of these quadratic forms for primes $p < 30$ is presented in Section 6.8.1.3. The tables 4.4 and 4.5 present the commensurability invariants for groups related to the two quadratic forms. In view of the tables, we note that groups corresponding to the forms $\langle -p, 1, \ldots, 1\rangle$ and $\langle -1, 1, \ldots, 1, p\rangle$ are *not* commensurable if and only if:

- $n \equiv 0, 2, 4, 6\ (8)$ and $p \equiv 3\ (4)$;
- $n \equiv 3, 7\ (8)$ and $p \equiv 7\ (8)$.

The first column of the tables contains the value of n modulo 8. We first note that $s(\langle -p, 1, \ldots, 1\rangle) = 1$ and $s(\langle -1, p, \ldots, 1\rangle) = (-1, p)$.

n	$c(V)$	$\mathrm{Ram}_f\, c(V)$	Invariant
0	1	$\emptyset$	$\{\mathbb{Q}, \emptyset\}$
1	1	$\emptyset$	$\{\mathbb{Q}, p, \emptyset\}$
2	$(-1, p)$	$\begin{cases} \emptyset & p = 2, p \equiv 1\ (4) \\ \{2, p\} & p \equiv 3\ (4) \end{cases}$	$\begin{cases} \{\mathbb{Q}, \emptyset\} & p = 2, p \equiv 1\ (4) \\ \{\mathbb{Q}, \{2, p\}\} & p \equiv 3\ (4) \end{cases}$
3	$(-1, p)$	$\begin{cases} \emptyset & p = 2, p \equiv 1\ (4) \\ \{2, p\} & p \equiv 3\ (4) \end{cases}$	$\begin{cases} \{\mathbb{Q}, -p, \{2\}\} & p \equiv 7\ (8) \\ \{\mathbb{Q}, -p, \emptyset\} & p \not\equiv 7\ (8) \end{cases}$
4	$(-1, -1)$	$\{2\}$	$\{\mathbb{Q}, \{2\}\}$
5	$(-1, -1)$	$\{2\}$	$\begin{cases} \{\mathbb{Q}, p, \{2\}\} & p \equiv 1\ (8) \\ \{\mathbb{Q}, p, \emptyset\} & p \not\equiv 1\ (8) \end{cases}$
6	$(-1, -p)$	$\begin{cases} \{p\} & p \equiv 3\ (4) \\ \{2\} & p = 2, p \equiv 1\ (4) \end{cases}$	$\begin{cases} \{\mathbb{Q}, \{p\}\} & p \equiv 3\ (4) \\ \{\mathbb{Q}, \{2\}\} & p = 2, p \equiv 1\ (4) \end{cases}$
7	$(-1, -p)$	$\begin{cases} \{p\} & p \equiv 3\ (4) \\ \{2\} & p = 2, p \equiv 1\ (4) \end{cases}$	$\{\mathbb{Q}, -p, \emptyset\}$

Table 4.4 – Commensurability invariants for groups related to the quadratic form $\langle -p, 1, \ldots, 1\rangle$

n	$c(V)$	$\mathrm{Ram}_f\, c(V)$	Invariant
0	$(-1,p)$	$\begin{cases}\emptyset & p=2, p\equiv 1\ (4)\\ \{2,p\} & p\equiv 3\ (4)\end{cases}$	$\begin{cases}\{\mathbb{Q},\emptyset\} & p=2, p\equiv 1\ (4)\\ \{\mathbb{Q},\{2,p\}\} & p\equiv 3\ (4)\end{cases}$
1	$(-1,p)$	$\begin{cases}\emptyset & p=2, p\equiv 1\ (4)\\ \{2,p\} & p\equiv 3\ (4)\end{cases}$	$\{\mathbb{Q},p,\emptyset\}$
2	1	$\emptyset$	$\{\mathbb{Q},\emptyset\}$
3	1	$\emptyset$	$\{\mathbb{Q},-p,\emptyset\}$
4	$(-1,-p)$	$\begin{cases}\{p\} & p\equiv 3\ (4)\\ \{2\} & p=2, p\equiv 1\ (4)\end{cases}$	$\begin{cases}\{\mathbb{Q},\{p\}\} & p\equiv 3\ (4)\\ \{\mathbb{Q},\{2\}\} & p=2, p\equiv 1\ (4)\end{cases}$
5	$(-1,-p)$	$\begin{cases}\{p\} & p\equiv 3\ (4)\\ \{2\} & p=2, p\equiv 1\ (4)\end{cases}$	$\begin{cases}\{\mathbb{Q},p,\{2\}\} & p\equiv 1\ (8)\\ \{\mathbb{Q},p,\emptyset\} & p\not\equiv 1\ (8)\end{cases}$
6	$(-1,-1)$	$\{2\}$	$\{2\}$
7	$(-1,-1)$	$\{2\}$	$\begin{cases}\{\mathbb{Q},-p,\emptyset\} & p\not\equiv 7\ (8)\\ \{\mathbb{Q},-p,\{2\}\} & p\equiv 7\ (8)\end{cases}$

Table 4.5 – Commensurability invariants for groups related to the quadratic form $\langle -1,p,1,\dots,1\rangle$

CHAPTER 5

CoxIter

The aim of this chapter is to present `CoxIter`, a computer program which we developed, whose purpose is to compute the invariants of hyperbolic Coxeter groups (for the theory, see sections 3.6 to 3.9). Most of the material presented in this chapter was published in [Gug15]. The additional material consists of the following:

- Section 5.1 containing general information about the program.
- Some details and figures in Section 5.2, explaining the implementations of the algorithms.
- All information relating to the growth series and to the growth rate (in particular in Section 5.2 and the part about the conjecture of Kellerhals and Perren).

Consider the Coxeter subgroup $\Gamma^{(14,1)}$ of $\operatorname{Isom}\mathbb{H}^{14}$ whose graph $\mathcal{G}$ is given in Figure 5.1. If one wants to compute the covolume (or, equivalently, the Euler characteristic), the growth series and growth rate, the first step is to find the number of connected spherical subgraphs $\mathcal{G}$. A careful (and painful) analysis of the graph shows that there exist 182 of them. The list of spherical subgraphs of $\mathcal{G}$, which correspond to all faces of the polyhedron associated to Γ (except the vertices at infinity), is too big to be determined by hand (there are 96001 of them).

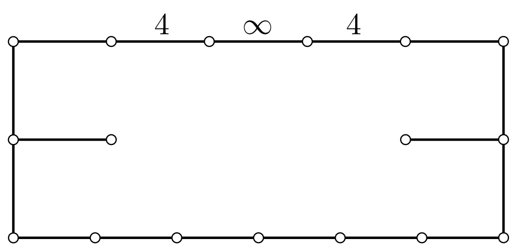


Figure 5.1 – The hyperbolic Coxeter group $\Gamma^{(14,1)} < \operatorname{Isom}\mathbb{H}^{14}$

The input of the program `CoxIter` is the graph of some hyperbolic Coxeter group (see for example Section 5.4.1 for our group $\Gamma^{(14,1)}$) together with some options. The output of `CoxIter` consists then of the following elements:

- Cocompactness
  Whether the group is cocompact or not.
- Cofiniteness
  Whether the group has finite covolume or not.
- Arithmeticity
  If the group is non-cocompact and has no dotted edge, we can decide if the group is arithmetic or not. If the group is non-cocompact and the graph has dotted edges, it is necessary to know the weights of the dotted edges (i.e. the length of the common perpendicular of the corresponding hyperplanes) to decide whether the group is arithmetic or not. In this setting `CoxIter` prints the tests to decide the arithmeticity. If the group is cocompact, it is much more complicated to decide about the arithmeticity (see examples 4.4.1, 4.4.2 and 4.4.3) and the program does not handle this case.
- f-vector
- Euler characteristic
- Signature
  If the graph contains no dotted edge or if the weight of the dotted edges are given, the signature is computed numerically[1] using `PARI`.
- Dimension
  For the cocompactness and cofiniteness tests, it is necessary to know the dimension of the hyperbolic space where the corresponding polyhedron lives. If this dimension is not provided, then the program will guess it[2].
- Growth series
  The rational expansion $\frac{p(x)}{q(x)}$ of the growth series is given, where $p(x)$ is a product of cyclotomic polynomials.
- Growth rate
  An approximation of the growth rate is given. It is also checked if it is a Perron, a Pisot, or a Salem number.

For the group $\Gamma^{(14,1)}$, the output of `CoxIter` is the following (the denominator of the growth series, which is a polynomial of degree 246 is not written here for the sake of space):

```
Reading file: ../graphs/14-vinb85.coxiter
    Number of vertices: 17
    Dimension: 14
    Vertices: 1, 2, 3, 4, ..., 17
    Field generated by the entries of the Gram matrix: Q[
        sqrt(2)]
File read
```

[1] First, we compute numerical approximations of the roots and then we decide which are negative, zero and positive.

[2] More precisely, it will assume that the dimension correspond to the maximal rank of a spherical or Euclidean subgraph.

```
Finding connected subgraphs......
Finding graphs products......
Computations......
    Computation time: 4.09233s

Information
    Cocompact: no
    Finite covolume: yes
    Arithmetic: yes
    f-vector: (94, 704, 2695, 6825, 12579, 17633, 19215,
        16425, 11009, 5733, 2275, 665, 135, 17, 1)
    Number of vertices at infinity: 5
    Alternating sum of the components of the f-vector: 0
    Euler characteristic: -87757/289236647411712000
    Covolume: pi^7 * 87757/30535933084360704000
    Signature (numerically): 14,1,2

Growth series:
f(x) = C(2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 3, 3, 3,
    3, 3, 3, 3, 4, 4, 4, 4, 4, 4, 4, 5, 5, 5, 6, 6, 6, 6, 6,
    6, 7, 7, 8, 8, 8, 9, 9, 10, 10, 10, 11, 12, 12, 12, 13,
    14, 14, 15, 16, 18, 18, 20, 22, 24, 26, 28, 30)/(...)

Growth rate: 2.3365920620831499457950462497773311312
    Perron number: yes
    Pisot number: no
    Salem number: no
```

In the next section, we provide general information about the program. In the second section, we explain how the well-know classical results can be implemented in algorithms. The two following parts are dedicated to the use of CoxIter: we use the program to compute the invariants of a few Coxeter graphs in higher dimensions, including recently found groups of Vinberg in dimension 18. In the last part, we explain how the program was tested.

5.1 General information about the program

The program CoxIter is written in C++ and is free/open source. More precisely, it is published under a free license, the GPLv3 (the GNU General Public License v3) and can be used freely in various projects[3]. CoxIter and its documentation (to build and use the program) is available on the website https://coxiter.rgug.ch and on GitHub: https://github.com/rgugliel/CoxIter

In addition to the main program, there are two other possibilities to use CoxIter:

- GAP package
  I created a GAP package called CoxIterGAP which gives the possibility to use CoxIter inside of GAP. For more information, see Section 5.3.6

[3]More information about the license can be found here: https://www.gnu.org/licenses/gpl.html A short guide is also available here: https://www.gnu.org/licenses/quick-guide-gplv3.html

- `CoxIterWeb`
  A web version of `CoxIter`, which can be used via any web-browser, can be downloaded and installed on any server. The source code is on my GitHub account: `https://github.com/rgugliel/CoxIterWeb`
  Alternatively, a demo version can be found here: `https://coxiter.rafaelguglielmetti.ch/` Hence, you can use `CoxIter` from anywhere for small groups (generated by at most 20 reflections).

5.1.1 Versions

The first version of the program, `CoxIter` 1.0.0, was published in [Gug15]. After that, more versions were released:

- 1.0.1
  Correction of a small bug.
- 1.0.2
  Correction of a bug causing incorrect answer in the cofiniteness test.
- 1.1
  This version added the following features:
  - growth series and growth rate;
  - test for the growth rate (Perron, Pisot and Salem numbers);
  - numerical computation of the signature;
  - more formats for the output (LaTeX, `PARI`);
  - guess the dimension if it is not specified;
  - speed-up of some computations.
- 1.1.1
  This function brought the following features:
  - ability to read a file directly;
  - ability to export the graph;
  - `GAP` format for the output;
  - display some help at start-up if no file is given.
- 1.2
  This function brought the following features:
  - correction of a bug;
  - fully functional Windows and OSX versions.

5.1.2 External libraries used

`CoxIter` is developed in `C++11`. Moreover, it requires the following external libraries:

GMP (or GNU Multiple Precision Arithmetic Library) is a free library for arbitrary precision arithmetic with integers.

OpenMP This API allows `CoxIter` to parallelize some parts of the computations and speed-up the execution. It is not mandatory.

PARI We use the `C` library `PARI` to compute the signature of the group, the growth rate and to determine whether it is a Perron, Pisot or Salem numbers. It is not mandatory but if it is not available, then `CoxIter` will compute only the growth series and not the growth rate (and won't be able to compute the signature).

PCRE The PCRE library (PCRE - Perl Compatible Regular Expressions) is used to parse the user input (parameters of the program and `CoxIter` files).

5.1.3 Design description

We briefly explain the main steps of the working flow of `CoxIter`. Depending on the given parameters, some steps may be skipped.

- Reading the parameters
  When the program is launched, it first lists all the given parameters (the file to read and the options for the computations). This is done in the `App::bReadMainParameters` function.

- Reading the graph
  The Coxeter graph is read from the file (`CoxIter::bReadGraphFromFile`) and some initializations are made (memory allocation, field generated by the entries of the Gram matrix, some vertices may be removed).

- First outputs
  Before the computations, we may have to print the first informations: Gram matrix, Coxeter matrix, writing the graph in a file, drawing of the Coxeter graph.

- Finding connected spherical and Euclidean subgraphs
  We find all connected spherical and Euclidean subgraphs of the given Coxeter graph (see Section 5.2.1 for more information).

- Partial non-cofiniteness test
  A partial non-cofiniteness test[4] has been implemented, mostly for its use inside the Vinberg algorithm (see Section 6.3.3.1 for more information), and can be thousand times faster than the complete cofiniteness test.

- Graph products
  All the possible products (spherical and Euclidean) are computed and counted. If `CoxIter` has to perform the cocompactness and cofiniteness tests, some of them are stored (for each component of the product we keep the type of the graph and the list of the vertices).

- Cofiniteness test

- Euler characteristic and f-vector

[4]If the output of the test is "true", then it is certain that the group is not cofinite. If the answer is "false", then we cannot say anything and have to perform the complete cofinite test.

- Compactness and arithmeticity tests
- Other computations
  The following are computed: growth series, growth rate (and Perron, Pisot and Salem tests) and signature.
- Output
  The results of the computations and tests are printed.

5.1.3.1 Different files

We list all the files in the project and their main features.

app Main file for the execution of the program.

arithmeticity Function to recursively delete separating edges the queues of the graph (see Proposition 3.9.14) and to test the arithmeticity.

coxiter All global functions to do the input/output, manage the graph and most of the computations.

graph To describe one graph: list of its vertices, adjacent vertices of the graph, test of inclusion.

graphs.list Allows to store and browse all graphs of one type (spherical or Euclidean).

graph.list.iterator Allows to go through the list of graphs in a sequential order.

graphs.list.n All graphs of a given rank.

graphs.product & **graphs.product.set** Store one product of graphs.

growthrate The compute the growth rate and do the Perron, Pisot and Salem tests (the `PARI` library is needed).

index2 To create the index two subgroup (see Proposition 7.1.9).

signature Numerical computation of the signature.

lib/maths_tools Some mathematical functions: integer square root, list of divisors, prime factors and prime decomposition, ...

lib/numbers/number_rational To handle computations with rational numbers.

lib/polynomials Divisibility and division of polynomials, product by symbols[5] and list of cyclotomic polynomials.

5.2 Algorithms

Let (Γ, S) be a hyperbolic Coxeter group of rank r. In what follows, we use the notations of the program. In particular, the r vertices of the Coxeter graph are labelled $0, \ldots, r-1$ instead of $1, \ldots, r$.

[5] Recall that, for a positive integer k, the *symbol* $[k]$ is the polynomial $1 + x + \ldots + x^{k-1}$.

5.2.1 Euler characteristic and f-vector

We present here the main steps for the computation of the Euler characteristic χ (see Section 3.7) and the f-vector. These steps are the following:

(1) We find all paths (simple walks with all edges labelled with a 3) starting from every vertex, that is, every A_m. Note that a single vertex is such a path.

(2) We extend these paths in order to find the finite and affine irreducible Coxeter subgroups.

(3) We compute all the possible products of groups.

(4) We count these products with their orders and multiplicities. The Euler characteristic is then computed with Proposition 3.7.1 (by taking into account the orders of finite irreducible spherical Coxeter groups in Table 3.2).

Note that using (1) – (3), and by theorems 3.7.5 and 3.7.6, we can compute the f-vector of Γ.

First, we use a *depth-first search* algorithm to explore the graph from every vertex and to find any subgraph of type A_m (function CoxIter::DFS). For each such A_m, we try to extend it to other connected spherical and Euclidean graphs (function CoxIter::addGraphsFromPath).
Once the connected graphs are found, we compute all the possible products. Counting them with their multiplicities gives the f-vector and the Euler characteristic.

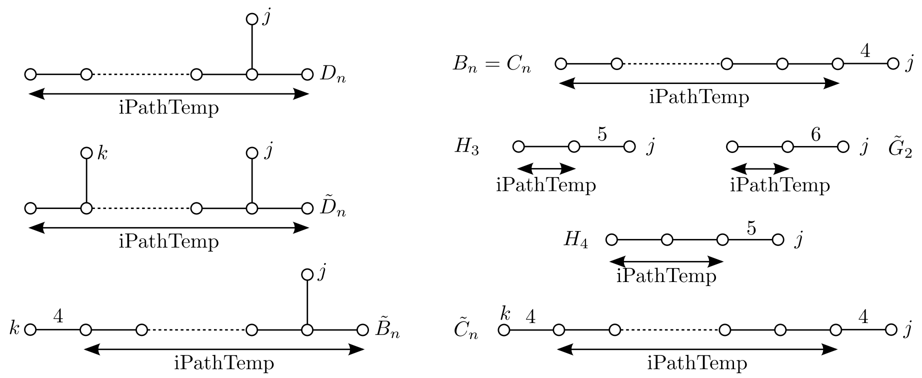


Figure 5.2 – Extending paths to graphs

5.2.2 Growth series and growth rate

We want to compute the growth series f_Γ of Γ. Using Steinberg's formula, we can write

$$\frac{1}{f_\Gamma(x)} = \sum_{T \in \mathcal{F}} \frac{(-1)^{|T|}}{f_T(x^{-1})},$$

where $\mathcal{F} = \big\{T \subset S : |\Gamma_T| < \infty\big\}$. If $T \subset S$ corresponds to the finite subgroup Γ_T, then we can write $\Gamma_T = \prod_{i=1}^{l} \Gamma_i$, where each Γ_i is a finite and irreducible

Input: void
Output: void
Result: Find connected spherical and parabolic graphs from iPath

```
for every vertex v in iPath do
    v is added to iPathTemp
    Add an A_n, corresponding to iPathTemp
    If the two ends of iPathTemp can be connected, there is a Ã_n
    if iPathTemp has at least 4 vertices then // Look for: D_n, B̃_n, D̃_n
        for every vertex j do
            if j is a neighbour of the next to last vertex of iPath (and only
             of this one) then
                Add D_n
                Look for a B̃_n
                Look for a D̃_n
            end
            Look for a B̃_3
        end
    end
    if iPathTemp has between 5 and 8 vertices then
        Look for: E_6, E_7, E_8, Ẽ_6, Ẽ_7 and Ẽ_8
    end
    if iPathTemp has at least two vertices then
        for each vertex j of iPathTemp do
            if we can find B_n, F_4 and H_n, G̃_2, C̃_n or F̃_4 with j (see
             Figure 5.2) then
                if m(v,j) < 6 then
                    B_n or H_4
                else
                    G̃_2
                end
            end
            if m(v,j) = 4 then
                Try to find a C̃_n
            end
            if we have a B_3 or a B_4 then
                Try to extend it to F_4 or F̃_4 with B3ToF4_B4ToTF4
            end
        end
    end
end
```

Function addGraphsFromPath

subgroup with generating set S_i. We then have $\dfrac{(-1)^{|T|}}{f_T(x^{-1})} = \prod_{i=1}^{l} \dfrac{(-1)^{|S_i|}}{f_{S_i}(x^{-1})}$ and each $f_{S_i}(x)$ is a product of symbols (see Definition 3.6.3), as indicated in Table 3.1. We can rewrite each fraction $\dfrac{(-1)^{|S_i|}}{f_{S_i}(x^{-1})}$ as $\dfrac{(-1)^{|S_i|} \cdot x^{k_{S_i}}}{f_{S_i}(x)}$, where the power k_{S_i} is as in Figure 5.3.

Group	Power k_{S_i}
A_n	$\frac{n(n+1)}{2}$
B_n	n^2
D_n	$n(n-1)$
$G_2^{(m)}$	m
F_4	24
E_6	36
E_7	63
E_8	120
H_3	15
H_4	60

Figure 5.3 – Exponents for the growth series of finite irreducible Coxeter groups

Based on these computations, we get $f_\Gamma(x) = \frac{\tilde{p}(x)}{\tilde{q}(x)}$, where $\tilde{p}(x)$ is a product of symbols, say $\tilde{p}(x) = [n_1] \cdot \ldots \cdot [n_s]$. For a positive integer n, we have $[n] \cdot (x-1) = \prod_{d|n} \Phi_d$, where Φ_d denotes the dth cyclotomic polynomial. Therefore, we can split each $[n_i]$ into a product of irreducible factors. Finally, we get $f_\Gamma(x) = \frac{p(x)}{q(x)}$, with p and q coprime and p is a product of cyclotomic polynomials.

Growth rate As explained in Section 3.6, the growth rate τ of $\Gamma = (\Gamma, S)$ is given by R^{-1}, where R is the radius of convergence of the series $f_S(x)$. This radius is equal the smallest positive root of the polynomial $q(x)$ and must be smaller than 1, if Γ is cofinite. For these computations (and to check whether τ is a Perron, a Pisot or a Salem number) we use the `PARI` library. We will mostly use the two following functions:

- roots(f, prec)
  This function computes numerical approximations of the roots of the polynomial f with a precision given by prec.
- sturm(f) and sturmpart(f, a, b)
  These two functions compute the number of real roots[6] of the polynomial f. The latter one only computes the number of roots which lie in the interval $]a, b]$.

[6]For the computations, `PARI` uses Sturm's theorem (see Theorem 2.6.5 and Example 3.6.7).

For the computations, we fix a small number $\varepsilon > 0$ (typically, $\varepsilon = 10^{-50}$) which should be big compared to the precision used to compute the roots (typically we do the computations with 57 decimals on 32 bits computers and 115 decimals on 64 bits computers). For each root, we decide that the root is real if its imaginary part is smaller than ε. Notice that since we know the exact number of real roots, approximation errors are not a problem: if we detect too many real roots, then we increase the precision and choose a smaller ε and do again the computations. We then can pick the smallest positive root R and its inverse τ (again, we do not have to worry about approximation errors: if two roots r_1 and r_2 are such that $|r_1 - r_2| < \varepsilon$, then we can increase the precision and choose a smaller ε). We also pick the corresponding irreducible factor $\tilde{g}(x)$ of $q(x)$ and we let $g(x) = x^{\deg \tilde{g}} \cdot g\big(x^{-1}\big)$. Observe that g is monic and irreducible which mean that g is the minimal polynomial of the algebraic integer τ.

Properties of the growth rate Once we get the growth rate and its minimal polynomial we try to determine whether it is a Perron, a Salem or a Pisot number.

To decide whether τ is a Perron number or not, we only have to check if every negative root r of g satisfies $|r| < \tau$. We then have the following three possibilities:

- For every such root r we have $|r| + \varepsilon < \tau$
  Then we know that τ is a Perron number.
- There exists r such that $|r| - \varepsilon < \tau < |r| + \varepsilon$
  We have to increase the precision in order to conclude.
- Otherwise
  τ is not a Perron number.

As pointed out above, checking whether λ is a Pisot number is easy and, as usual, we display a warning if we cannot decide (up to ε) whether a root have absolute value smaller than one or not.

We now come to the last point: deciding whether τ is a Salem number or not. A necessary condition for τ to be a Salem number is that its minimal polynomial g is a self-reciprocal polynomial with $\deg g$ even. If it is the case, it remains to verify that $(\deg g - 2)$ roots of g lie on S^1, the unit circle. We use the Möbius transformation $h(t) = \frac{t-i}{t+i}$ of $\mathbb{C}$ to send $\mathbb{R} \cup \{\infty\}$ to S^1 and we consider $\tilde{g}(t) := g \circ h(t)$ and the polynomial $G(t) := (t+i)^{\deg g} \cdot \tilde{g}(t)$. Since g is irreducible, we have $|R_g \cap S^1| = |R_{\tilde{g}} \cap \mathbb{R}| = |R_G \cap \mathbb{R}|$, where R_p denotes the set of complex roots of the polynomial (or rational function) p. We remark that since g is self-reciprocal, then $G(t)$ is a real polynomial and we can use the sturm function of `PARI` to compute the number of its real roots. For computational purpose, we give an explicit description of $G(t)$. For that, we will need the following two identities:

$$(t+i)^{2l} + (t-i)^{2l} = 2 \cdot \sum_{k=0}^{l} \binom{2l}{2k} \cdot t^{2k} \cdot (-1)^{l-k},$$

$$(t+i)^{2l+1} + (t-i)^{2l+1} = 2 \cdot \sum_{k=0}^{l} \binom{2l+1}{2k+1} \cdot t^{2k+1} \cdot (-1)^{l-k}.$$

We write $G(t) = \sum_{j=0}^{m} a_j t^j$ and we distinguish two cases. When m is even, we have

$$\begin{aligned} G(t) &= a_{m/2} \cdot (t^2+1)^{\frac{m}{2}} + \sum_{j=0}^{\frac{m}{2}-1} a_j \cdot \big((z-i)^j (z+i)^{m-j} + (z-i)^{m-j}(z+i)^j\big) \\ &= a_{m/2} \cdot (t^2+1)^{\frac{m}{2}} + \sum_{j=0}^{\frac{m}{2}-1} a_j (z^2+1)^j \cdot \big((z+i)^{m-2j} + (z-i)^{m-2j}\big) \\ &= a_{m/2} \cdot (t^2+1)^{\frac{m}{2}} + 2\sum_{j=0}^{\frac{m}{2}-1} a_j (z^2+1)^j \cdot \sum_{k=0}^{\frac{m}{2}-j} \binom{m-2j}{2k} t^{2k} \cdot (-1)^{\frac{m}{2}-j-k}. \end{aligned}$$

When m is odd, we find

$$G(t) = 2 \cdot \sum_{j=0}^{\frac{m-1}{2}} a_j (z^2+1)^j \cdot \sum_{k=0}^{\frac{m-1}{2}-j} \binom{m-2j}{2k+1} \cdot t^{2k+1} \cdot (-1)^{\frac{m-1}{2}-j-k}.$$

5.2.3 Arithmeticity

To check the arithmeticity of a given Coxeter group, we use Proposition 3.9.14. The code for the test is in the `Arithmeticity` class.

First, we check if the graph contains no dotted edge, if it is non-cocompact and if all the edge labels $m(s,t)$ lie in the set $\{\infty, 2, 3, 4, 6\}$. These verifications are done in the beginning of the `Arithmeticity::test` function. At this point, we know that the coefficients of the matrix $2 \cdot G$ can take the values $\{0, -1, -2, 2, -\sqrt{2}, -\sqrt{3}\}$. We then use multiple calls to the function `Arithmeticity::collapseQueues` to recursively delete separating edges of the graph (regardless of the labels which appear in these paths) to reduce the computation time of the determination of the cycles.
For each vertex v, we look for simple cycles passing through v (note that we only look at cycles which contain vertices j with $v \leq j$ to avoid multiple counting of the graphs). This is done by calling `Arithmeticity::findCycles(v,v)`. For each cycle, we check whether the number of edges labeled with 4 and 6 respectively along the cycle is even (labels 3 and ∞ are not an obstacle to arithmeticity at this point).

As explained in Remark 3.9.15, if the graph contains dotted lines, we need to determine all the relevant entries of the Gram matrix in order to decide about the arithmeticity of the group. In this case, `CoxIter` will indicate what are the conditions for the group to be arithmetic. The program would give conditions in the following way:

```
l1m3: weight of the dotted line between hyperplanes 2 and 4
l0m5: weight of the dotted line between hyperplanes 1 and 6

The group is arithmetic if and only if all the following
    values lie in Z:
4 * l1m3^2
4 * l0m5^2
2^3 * l1m3 * l0m5
```

Input: iRoot: starting vertex for the search
iFrom: previous vertex (or iRoot if it is a non-recursive call of the function)
Output: void
Result: Look for simple cycles passing through iPath[0]

```
Add the vertex iRoot to the path
for every neighbour i of iRoot with i ≥ iPath[0] do
    if the edge (i,iFrom) was not visited then
        if i == iPath[0] then
            This is a simple cycle. Use testCycle to test it.
        end
        else if i is not a vertex of iPath then
            Mark the edge (iRoot,i) as visited
            Call findCycles(i,iRoot)
    end
end
```

Function findCycles

5.2.4 Linearization of a (sparse) symmetric matrix

It is possible to give the values of the dotted lines (i.e. the corresponding entries of the Gram matrix) to `CoxIter`. These values will be used for the output (for example when printing the Gram matrix), the results of the arithmeticity test or for the computation of the signature. Since the Gram matrix is symmetric and because these values are not always given and are rarely used, we want to store them in a vector instead. We start with an $r \times r$ matrix with rows and columns labelled $0, \dots, r-1$ and we want to induce a bijection

$$\iota : \{0, \dots, r-1\}^2 \longrightarrow \left\{0, \dots, \frac{r(r+1)}{2} - 1\right\}.$$

We will enumerate the elements of the first row $0, \dots, r-1$, the ones of the second row $r, \dots, 2r-2$, etc. Now, if the pair (i, j), where $0 \leq i < j \leq r-1$, designates an element of the matrix, its index $\iota(i, j)$ in the vector is given by

$$\iota(i, j) = \frac{i(2r-1-i)}{2} + j.$$

Conversely, if $k \in \left\{0, \dots, \frac{r(r+1)}{2} - 1\right\}$ is the index of an element of the vector, then the corresponding entry of the matrix is on the row

$$\text{row}(k) = \text{floor}\frac{(2r+1) - \sqrt{(2r+1)^2 - 8k}}{2},$$

and column

$$\text{col}(k) = k - \frac{\text{row}(k) \cdot \big(2r - 1 - \text{row}(k)\big)}{2}$$

Finally, note that with this process, we can then use a `map[int, String]` to store only the existing values.

5.3 Using CoxIter - Some examples

In this section we present some worked examples in dimensions 13, 16, 17, 18 and 19. In dimension 18 we compute the invariants for three groups given by Vinberg in a recent article [Vin15]. The complete documentation of the program can be found online at `https://coxiter.rgug.ch`.

5.3.1 Two arithmetic groups of Vinberg

We start with the arithmetic Coxeter groups $\Gamma_i < \operatorname{Isom}\mathbb{H}^i$, $i = 16, 17$, with Coxeter graphs Σ_{16} and Σ_{17} and ranks 20 and 22 (see Figure 5.4 and [Vin85]).

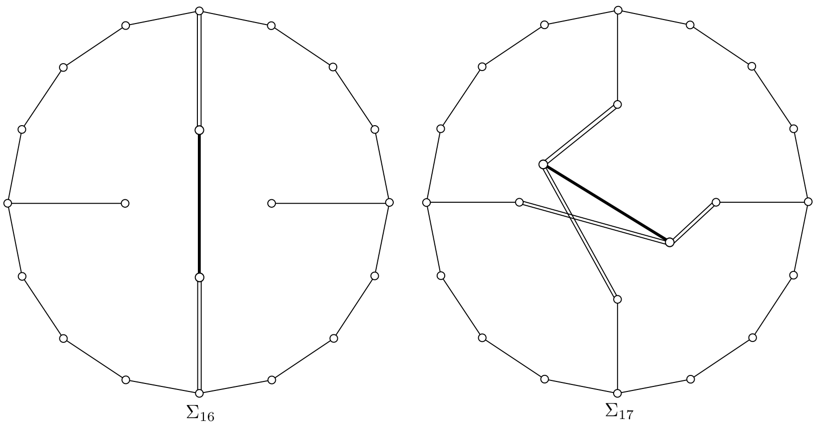


Figure 5.4 – Σ_{16} and Γ_{17}

The program is called with parameters "-full" (we want to get all the invariants) and the output is given in Table 5.1. The growth series are not given in the table because of the space needed. Moreover, the running time is decomposed in two: the first part measure the time needed for all the computations except the growth rate while the second part concerns only the computation of the growth rate.

We see that the values match the theoretical results. For the Euler characteristic, see [RT97, Theorem 22] up to a correction factor 2^k, where k is the number of symmetry axis of the graph (see [RT13, Section 6]). In our case, $k = 2$.

5.3.2 An arithmetic group of McLeod

In his paper [Mcl11], McLeod constructs the maximal reflection groups in the automorphism groups of the quadratic forms $-3x_0 + x_1^2 + \ldots + x_n^2$ for $2 \leq n \leq 13$ using Vinberg's algorithm. In Figure 5.5 we present the graph of the group for $n = 13$ and its invariants are given in Table 5.2.

5.3.3 A free product with amalgamation in dimension 18

In [Vin15], Vinberg explains how to construct a non-arithmetic group Γ in dimension 18 as a mixture (in the sense of Gromov–Piatetski-Shapiro) of two

Invariant	Σ_{16}	Σ_{17}
Cocompact	no	no
Finite covolume	yes	yes
Arithmetic	yes	yes
f-vector	(325, 2804, 11914, 33164, 67410, 105462, 130646, 130062, 104670, 68042, 35490, 14658, 4690, 1122, 189, 20, 1)	(807, 7586, 33960, 98184, 206120, 332982, 427584, 444428, 377232, 262050, 148500, 68076, 24884, 7089, 1518, 230, 22, 1)
Euler charact.	$\frac{642332179}{2360171042879569920000}$	0
Growth rate τ	2.70239585907681598724	3.26561859868532761547
Is τ Perron	yes	yes
Running time	0.6s + 10.5s	2.7s + 15s

Table 5.1 – Output of `CoxIter` for Σ_{16} and Σ_{17}

Invariant	Value
Cocompact	no
Finite covolume	yes
f-vector	(413, 2964, 10238, 22761, 36024, 42265, 37380, 25005, 12556, 4641, 1218, 213, 22, 1)
Euler characteristic	0
Growth rate τ	7.0978514419211704695935654214851191548
Is τ Perron	yes
Running time	0.18s + 7s

Table 5.2 – Output of `CoxIter` for the reflection group corresponding to the automorphism group of the quadratic form $-3x_0 + x_1^2 + \ldots + x_{13}^2$ (see Figure 5.5)

non-commensurable arithmetic hyperbolic Coxeter groups Γ_1 and Γ_2. Geometrically, the two associated polyhedra P_1 and P_2 of the groups $\Gamma_1, \Gamma_2 < \operatorname{Isom} \mathbb{H}^{18}$ have an isometric facet P_0. In this setting, the associated polyhedron P of Γ is the gluing of P_1 and P_2 along their common facet P_0. From an algebraic point of view, the group Γ is the free product with amalgamation given by $\Gamma = \Delta_1 \star_{\Gamma_0} \Delta_2$, where Δ_i is the subgroup of Γ_i generated by all the generators of Γ_i except the one corresponding to the hyperplane containing the facet P_0.

We use `CoxIter` to compute the invariants of the groups Γ_1, Γ_2 and Γ and we check that the covolume of Γ is indeed the sum of the covolumes of the two components. We also compute the growth series of Δ_1 and Δ_2 and we check that $\frac{1}{f_\Gamma} = \frac{1}{f_{\Delta_1}} + \frac{1}{f_{\Delta_2}} - \frac{1}{f_{\Gamma_0}}$ (see [Alo91, Theorem 2]).

Remark 5.3.1
The polyhedron P obtained by glueing (or, equivalently the free product with amalgamation $\Gamma = \Delta_1 \star_{\Gamma_0} \Delta_2$) is not necessarily Coxeter. However, if the facet P_0 is perpendicular to every adjacent facet, then the result will be of Coxeter type.

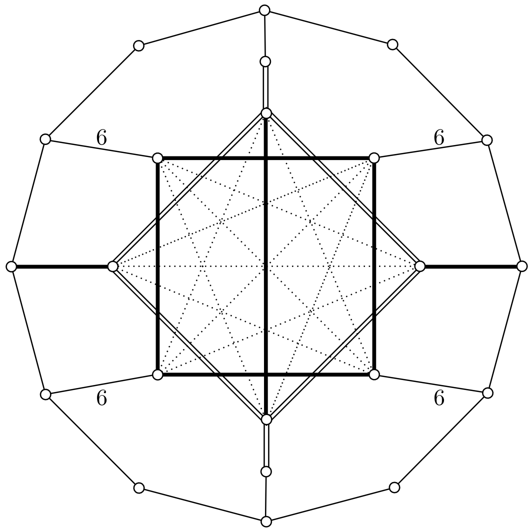


Figure 5.5 – Group of reflections related to automorphism group of the quadratic form $-3x_0 + x_1^2 + \ldots + x_{13}^2$

5.3.3.1 First component of the product

Information for the construction of the first component Γ_1 of the product are given in [KV78] and [Vin15]. The generators $\{s_1, \ldots, s_{37}\}$ of Γ_1 are given by the roots of a certain integral lattice L_1 (in fact, L_1 is the unique odd unimodular quadratic lattice of signature $(18, 1)$).

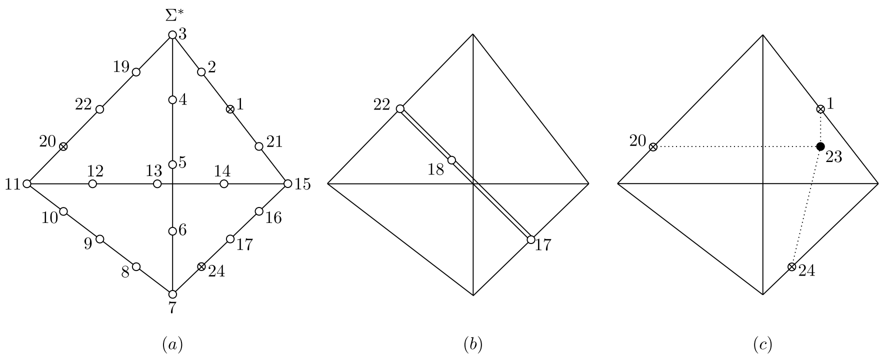


Figure 5.6 – Construction of $\Gamma_1 < \operatorname{Isom} \mathbb{H}^{18}$ - First component of the product

We use the terminology and present the results of [KV78]. There are 22 long roots (i.e. roots of norm 2) and 15 short roots (i.e. roots of norm 1). The action of the symmetry group of the graph (which is isomorphic to S_4, see below) splits the short roots into two orbits of size 3 and 12. The 3 roots are called roots of the *first kind* while the 12 other elements are called roots of the *second kind*. The diagram Σ^* of the long roots is presented in Figure 5.6(a). If we consider the 6 "long edges" (or "tetrahedral edges") of Figure 5.6(b), then each pair of opposite edges gives rise to a root of the first kind. Hence, we get the three vertices of the first kind $18, 25, 26$ connected respectively to vertices $(17, 22)$, $(1, 9)$ and $(5, 13)$

by a double edge. The figure 5.6(c) depicts the connection of one vertex of the second kind with the graph Σ^*. We then let the symmetry group of Σ^* (see below) act on the triple of vertices $(1, 20, 24)$ and find 11 other triples, corresponding to the 11 others vertices of the second kind. The 12 triples of vertices are the following: $(1, 20, 24)$, $(1, 6, 12)$, $(5, 16, 20)$, $(5, 10, 21)$, $(9, 16, 19)$, $(9, 4, 14)$, $(13, 2, 8)$, $(13, 19, 24)$, $(17, 2, 10)$, $(17, 4, 12)$, $(22, 8, 21)$, $(22, 6, 14)$. Now, vertices of the first kind are connected among themselves by lines labelled with an ∞ while vertices of the second kind are connected among themselves by dotted edges. Each vertex of the first kind is connected to each vertex of the second kind either by an ∞ or by a dotted edge according to the rule explained in [KV78, Section 2]. We finally found a Coxeter graph $\Sigma(\Gamma_1)$ with 37 vertices and 171 edges. When given to `CoxIter`, it produces the output given in Table 5.3.

Invariant	Value
Cocompact	no
Finite covolume	yes
f-vector	(3839, 37842, 177812, 540624, 1197240, 2050008, 2807602, 3135528, 2883540, 2189924, 1369854, 700352, 288801, 94113, 23497, 4282, 525, 37, 1)
Euler characteristic	$-\frac{109638854849}{22028263066875985920000}$
Volume	$\frac{109638854849}{1482580623111880900608000000} \cdot \pi^9$
Growth rate τ	16.976062922983291305757316760341665593
Is τ Perron	yes
Running time	55s + 20s

Table 5.3 – Output of `CoxIter` for Γ_1

Remark 5.3.2
Using `AlVin` (see Chapter 6) it is also possible to compute directly the 37 normal vectors of polyhedron P_1 and the presentation of its reflection group Γ_1. The computations take approximatively 2-3 minutes on a desktop computer.

Lemma 5.3.3 ([KV78])
The symmetry group of the graph $\Sigma(\Gamma_1)$ is isomorphic to S_4.

Proof. To determine the symmetry group of the graph of the group Γ_1 we first compute the group of symmetries $\operatorname{sym}\Sigma^*$ of the graph Σ^*, presented in Figure 5.6. It is easy to see that any automorphism $\sigma \in \operatorname{sym}\Sigma^*$ is completely determined by the images of the middle-points of two adjacent long edges (for example the images of the vertices 22 and 1). There are 6 possibilities for the images of 22 and four possibilities for the vertex 1, corresponding to the four adjacent long edges of the image of 22. Hence, the group $\operatorname{sym}\Sigma^*$ has order 24. We consider the automorphisms

$$\sigma_1 = (22\ 9)\,(1\ 17)$$
$$\sigma_2 = (1\ 5)\,(9\ 13)$$
$$\sigma_3 = (22\ 1)\,(9\ 17)$$

and we see that $\sigma_1 \circ \sigma_2$ has order 3, $\sigma_2 \circ \sigma_3$ has order 3 and $\sigma_1 \circ \sigma_3$ has order 2. Therefore, $\operatorname{sym} \Sigma^* = S_4$.
By a lemma of [KV78], every element of $\operatorname{sym} \Sigma^*$ can be extended to an element of $\operatorname{sym} \Sigma$, where Σ denotes the graph of the group Γ_1. Conversely, since $\Sigma \setminus \Sigma^*$ contains two complete graphs K_{12} and K_3 (here, K_l denotes the complete graph with l vertices and $\binom{l}{2}$ edges), any element of $\operatorname{sym} \Sigma$ must preserve the subgraph Σ^*. We thus have a surjective restriction map $\operatorname{res} : \operatorname{sym} \Sigma \longrightarrow \operatorname{sym} \Sigma^*$. Since this map is also injective we have $\operatorname{sym} \Sigma = S_4$. $\square$

5.3.3.2 Second component of the product

The lattice for the second group is $L_2 = L_0 \oplus \mathbb{Z}e \subset \mathbb{R}^{18,1}$, where:

- L_0 is the unique even unimodular quadratic lattice of signature $(17, 1)$. Its maximal subgroup generated by reflections is Γ_0, the hyperbolic Coxeter group whose associated fundamental polyhedron is P_0, which is the common facet of P_1 and P_2.
- e is a long root of square norm 2.

The graph of the group $\Gamma_2 < \operatorname{Isom} \mathbb{H}^{18}$ is presented in Figure 5.7 and its resulting invariants in Table 5.4.

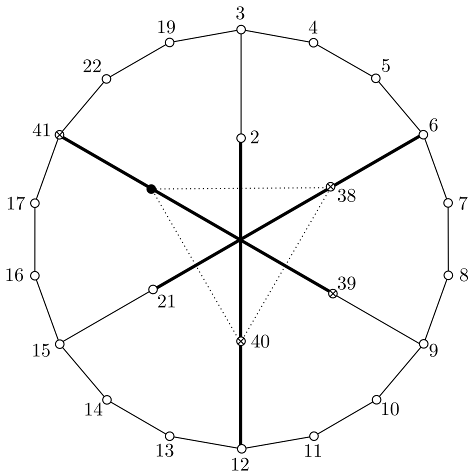


Figure 5.7 – $\Gamma_2 < \operatorname{Isom} \mathbb{H}^{18}$ - Second component of the product

5.3.3.3 Amalgamated product

As described above, we want to construct the amalgamated product of the groups Δ_1 and Δ_2. We see that they have a common hyperbolic subgroup Γ_0 of signature $(17, 1, 1)$ (see Figure 5.8). The hyperplane in $\mathbb{H}^{18}$ which contains the associated polyhedron P_0 is the one corresponding to the black dot in the graphs of Γ_1 and Γ_2.

A presentation for the product $\Gamma = \Delta_1 \star_{\Gamma_0} \Delta_2$ is obtained as follows.

- Start with the presentation of $\Gamma_1 = \langle s_1, \ldots, s_{37} | (s_i \cdot s_j)^{m_{ij}} \rangle$.

Invariant	Value
Cocompact	no
Finite covolume	yes
f-vector	(535, 5160, 24876, 79590, 188352, 348012, 517247, 628599, 629544, 520631, 354651, 197676, 89148, 31977, 8892, 1843, 267, 24, 1)
Euler characteristic	$-\frac{109638854849}{2260099790661476155392000}$
Volume	$\frac{109638854849}{152112771931278980402380800000} \cdot \pi^9$
Growth rate τ	4.4381437732572741094606253388221388290
Is τ Perron	yes
Running time	3.2s + 20s

Table 5.4 – Output of `CoxIter` for Γ_2

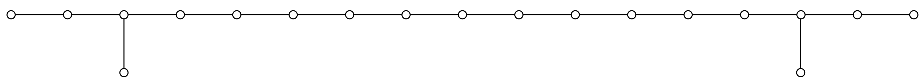


Figure 5.8 – $\Gamma_0 < \text{Isom}\,\mathbb{H}^{17}$

- Remove the vertex 23, corresponding to the hyperplanes containing P_0.
- Add four generators $s_{38}, s_{39}, s_{40}, s_{41}$ corresponding to the four crossed vertices of Γ_2 (vertices $38, 39, 40, 41$ in Figure 5.7).
- The relations between these four new generators and the generators s_2, $s_3, \ldots, s_{17}, s_{19}, s_{21}, s_{22}$ are according to Figure 5.7.
- There is no relation between any of the generators s_{38}, s_{39}, s_{40}, s_{41} and any of s_1, s_{18}, s_{20}, $s_{24}, \ldots, s_{37}$, meaning that the corresponding m_{ij} are ∞. Note that at this point we do not know if the corresponding hyperplanes are parallel or ultraparallel (or, equivalently, if in the graph we have dotted or bold edges). This means that we cannot fully determine the f-vector (i.e. the number f_0 of vertices is unknown). However, this does not influence the computation for the Euler characteristic.

The output of `CoxIter` for the product Γ is given in Table 5.5. As expected, we find the equality $\chi(\Gamma) = \chi(\Gamma_1) + \chi(\Gamma_2)$. Moreover, using Proposition 3.7.7, we find that the number f_0 of vertices of the polyhedron is 4212.

Using `CoxIter`, we also check that $\frac{1}{f_\Gamma} = \frac{1}{f_{\Delta_1}} + \frac{1}{f_{\Delta_2}} - \frac{1}{f_{\Gamma_0}}$ (the growth series are too big to be written here).

5.3.4 An arithmetic group in dimension 19

Kaplinskaya and Vinberg described in [KV78] the construction of an arithmetic hyperbolic Coxeter group Γ_{19} in $\text{Isom}\,\mathbb{H}^{19}$ related to the standard quadratic form $-x_0^2 + x_1^2 + \ldots + x_{19}^2$. This construction is similar to the first component of the free amalgamated product in Section 5.3.3.1.

We use in this section the terminology of [KV78]. There are 25 long roots (depicted in the diagram Σ^* of Figure 5.9), 5 roots of the first kind (such as the

Invariant	Value
f-vector	(4212, 41464, 195047, 594510, 1321044, 2271012, 3123717, 3503919, 3235974, 2467015, 1548147, 793376, 327561, 106710, 26575, 4814, 583, 40, 1)
Euler characteristic	$-\frac{8661469533071}{17385383005088278118400000}$
Volume	$\frac{8661469533071}{117009824562522292617216000000} \cdot \pi^9$
Growth rate τ	20.082709777287151523318262297132352789
Is τ Perron	yes
Running time	74s + 29s

Table 5.5 – Output of `CoxIter` for Γ

vertex 19 in Figure 5.9(b)) and 20 roots of the second kind (such as the vertex 24 in Figure 5.9(c)). In order to explain how to connect the vertices of the first and second kind to the graph Σ^* we first compute the automorphism group of the graph Σ^*.

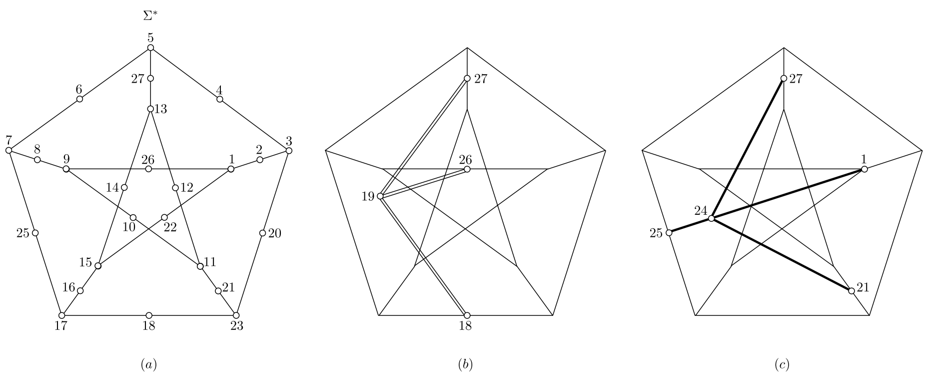


Figure 5.9 – Construction of $\Gamma_{19} < \operatorname{Isom}\mathbb{H}^{19}$

Lemma 5.3.4 ([KV78])
The automorphism group of Σ^* is isomorphic to S_5.

Proof. Using the same method as in Lemma 5.3.3 we find the four generators of sym Σ^* in order to conclude:

$$\sigma_1 = (27\ 4)\,(14\ 20)\,(16\ 18)\,(2\ 12)\,(21\ 22)\,(26\ 10)$$
$$\sigma_2 = (4\ 6)\,(8\ 20)\,(2\ 25)\,(10\ 21)\,(16\ 22)\,(18\ 26)$$
$$\sigma_3 = (6\ 25)\,(4\ 16)\,(14\ 20)\,(2\ 22)\,(12\ 21)\,(18\ 27)$$
$$\sigma_4 = (2\ 20)\,(18\ 26)\,(8\ 25)\,(21\ 22)\,(12\ 14)\,(10\ 16).$$

Each element σ_i has order two, each of the three products $\sigma_i \circ \sigma_{i+1}$ has order 3 and every other product $\sigma_i \circ \sigma_j$ with $i \neq j$ has order two. Therefore, sym Σ^* has a subgroup isomorphic to S_5 and thus sym $\Sigma^* \cong S_5$. □

5.3.4.1 Vertices of the first and second kind

In order to determine the vertices of the first kind and their connections to the graph Σ^* we consider the triple of vertices $(27, 26, 18)$ (see Figure 5.9(b)) and look at its images under action of $\operatorname{sym}\Sigma^* \cong S_5$. We find 5 triples which correspond to 5 vertices of the first kind:

$$(27, 26, 18), \quad (2, 12, 25), \quad (21, 22, 6), \quad (16, 10, 4), \quad (8, 14, 20).$$

The vertices of the first kind are connected among themselves by bold edges.

To determine the vertices of the second kind and their connections to the graph Σ^* we consider the quadruple of vertices $(27, 1, 21, 25)$ (see Figure 5.9(c)) and look at its images under the action of $\operatorname{sym}\Sigma^*$. We find 20 quadruples which correspond to the 20 vertices of the second kind:

$$\begin{array}{lllll} (1,21,25,27), & (8,22,23,27), & (4,8,15,21), & (4,11,22,25), & (2,10,17,27) \\ (9,16,20,27), & (4,9,14,18), & (4,12,17,26), & (2,8,13,18), & (13,20,25,26) \\ (3,10,14,25), & (3,8,12,16), & (1,6,12,18), & (6,14,23,26), & (2,6,11,16) \\ (6,10,15,20), & (2,7,14,21), & (7,12,20,22), & (5,16,21,26), & (5,10,18,22). \end{array}$$

The vertices of the second kind are connected among themselves by dotted edges.

A vertex of the first kind and a vertex of the second kind are connected by a double edge or a dotted edge according to the rule described in [KV78, Section 2, page 196].

Remark 5.3.5
As in the case of the first component of the product of Section 5.3.3.1, it is also possible to use `AlVin` (see Chapter 6) to compute directly the complete presentation of the group.

5.3.4.2 Summary and computations

Finally, the connections between the 50 vertices of the group Γ_{19} are as follows:

- 30 simple edges in Σ^*;
- a complete graph with 5 vertices (vertices of the first kind, bold edges);
- 15 double edges (vertices of the first kind and Σ^*);
- a complete graph with 20 vertices (vertices of the second kind, dotted edges);
- 80 bold edges (vertices of the second kind and Σ^*);
- 100 edges between vertices of the first and second kind.

The output of `CoxIter` for the group Γ_{19} is presented in Table 5.6.

Invariant	Value
Cocompact	no
Finite covolume	yes
f-vector	(27841, 292340, 1429615, 4465955, 10081519, 17518035, 24310230, 27542850, 25791030, 20062168, 12956240, 6908365, 3009960, 1054645, 290315, 60660, 9125, 905, 50, 1)
Growth rate τ	27.90472928162717219034651
Is τ Perron	yes

Table 5.6 – Output of `CoxIter` for Γ_{19}

5.3.5 A conjecture of Kellerhals and Perren

Kellerhals and Perren conjectured that the growth rate of a (cofinite) hyperbolic Coxeter group is always a Perron number. For cofinite hyperbolic Coxeter groups in dimension 2, the claim has been proved. More recently, Yukita announced that he proved the case $n = 3$ (see [Yuk16b] and [Yuk16a]). We used `CoxIter` to check the conjecture on large number of groups (see Table 5.7).

Reference	Groups
[FTZ07]	13 compact simplicial prisms in $\mathbb{H}^4$
[Im 85]	134 doubly truncated orthoschemes
[Joh+99]	all simplices
[Per09, Appendix C]	all groups
[Rob15]	all groups
[Tum04]	all 200 groups
[Tum04]	all the derived non-cofinite simplices
Vinberg algorithm	Γ_1^n, $2 \leq n \leq 18$
Vinberg algorithm	Γ_3^n, $2 \leq n \leq 13$
Vinberg algorithm	Γ_5^n, $2 \leq n \leq 8$
Doublings	see Section 7.2.1

Using the procedure described in Section 7.2, more than 3000 thousands groups were created on which the conjecture was tested.

Table 5.7 – Groups for which the conjecture of Kellerhals and Perren was checked

5.3.6 The GAP package `CoxIterGAP`

We present here a short example to illustrate the use of the package `CoxIterGAP`. More examples and installation instructions are available on the webpage of the package: `https://github.com/rgugliel/CoxIterGAP`

The goal is to compute the invariants of the cocompact hyperbolic Coxeter group $\Gamma < \text{Isom}\,\mathbb{H}^8$ found by Bugaenko (see [Bug92] and Figure 5.10). First, we have to load the package and give the presentation of the group Γ inside `GAP`. This can be achieved with the following commands:

```
LoadPackage("coxiter");
ci8 := CreateCoxIterFromCoxeterGraph( [[1,[2,5]], [2,[3,3]],
     [3,[4,3]], [4,[5,3],[10,3]], [5,[6,3]],
    [6,[7,3],[11,3]], [7,[8,3]], [8,[9,5]], [10,[11,1]]], 8);
```

At this point, we have a variable, called `ci8`, which contains the `CoxIter` object.

The first argument of the function `CreateCoxIterFromCoxeterGraph` is a description of the Coxeter graph of Γ: we create a list containing the neighbours of every vertex, together with the weights (as in `CoxIter`, we use "0" for a bold edge and "1" for a dotted edge). The second parameter is the dimension of the space (0 can be specified if we don't know the dimension).

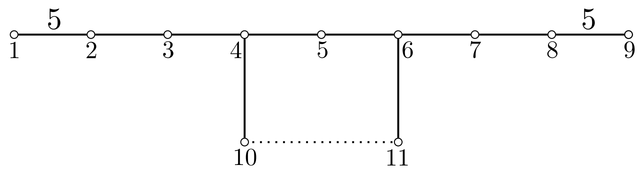


Figure 5.10 – Bugaenko's 8-dimensional compact hyperbolic Coxeter polyhedron

Then, we can call certain specific functions in order to compute the invariants. The following list presents the available functions and the value they return:

- `FVector`: f-vector
- `Cocompact`: 1 (yes), 0 (no), -1 (unable to decide)
- `Cofinite`: 1 (yes), 0 (no), -1 (unable to decide)
- `EulerCharacteristic`: rational number
- `GrowthSeries`: list $[f, g]$ of numerator and denominator of the rational function

For our example, the GAP session could be:

```
gap> FVector(ci8);
[ 41, 164, 316, 374, 294, 156, 54, 11, 1 ]
gap> Cocompact(ci8);
1
gap> Cofinite(ci8);
1
gap> EulerCharacteristic(ci8);
24187/8709120000
gap> g := GrowthSeries(ci8);;
gap> Value(g[2],1)/Value(g[1],1) - EulerCharacteristic(ci8);
0
```

5.4 Encoding a graph

To give a Coxeter graph to `CoxIter`, it is sufficient to create a text file to describe the graph. The file contains the following:

- Number of vertices and dimension
  The first line contains the number of vertices of the graph and the dimension n such that the associated polyhedron lives in $\mathbb{H}^n$. Remark that the dimension is optional: it is needed only for the compactness and cofiniteness tests and `CoxIter` will try to guess it if it is not given.

- Name of the vertices
  By default, the vertices are labelled $1, 2, \ldots, r$. It is however possible to specify other labels. In order to do this, the second line has to be of the form

  ```
  vertices labels: label1 label2 label3 ...
  ```

  Each label can contains letters, digits, -, _ but not space.

- Edges of the graph
  Then, we have to add one line to describe each edge of the graph. The syntax is the following:

  ```
  vertex1 vertex2 weight
  ```

  Remarks:

  - 0 is used to specify the weight infinity and 1 is used to specify a dotted edge.
  - The weight 2 doesn't have to be specified.

For example, the 2-dimensional hyperbolic Coxeter group whose graph is given in Figure 5.11 could be described for `CoxIter` in the two following equivalent ways:

```
4 2
1 2 1
2 3 4
3 4 1
```

```
4 2
vertices labels: s1 s2 s3 t
s1 s2 1
s2 s3 4
s3 t 1
```


Figure 5.11 – The group of units of the quadratic form $\langle -6, 1, 1\rangle$

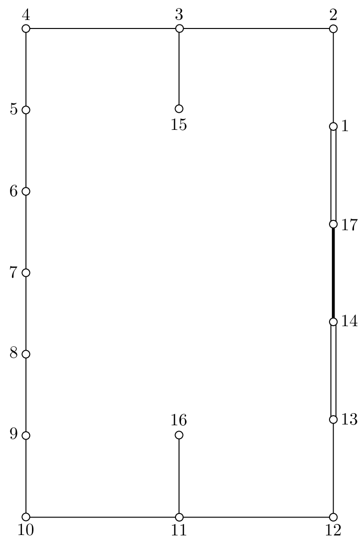
Note that it is possible (although not mandatory) to indicate the weight of a dotted line after a #. If all these weights are specified, then `CoxIter` will be able to numerically compute the signature as well as improve some outputs. The example of Figure 5.11 then becomes:

```
4 2
1 2 1 # -sqrt(2)
2 3 4
3 4 1 # -sqrt(3/2)
```

5.4.1 Example of the introduction of this chapter

The output for the group presented in the introduction of this chapter is the group $\Gamma^{(14,1)}$ generated by reflections in the group of units of the Lorentzian quadratic form $-x_0^2 + x_1^2 + \ldots + x_{14}^2$ of signature $(14, 1)$ described in [Vin72]. We describe the group Γ by means of the text file "14-vinb85.coxiter" as shown in Figure 5.12.

```
17 14
1 2 3
2 3 3
3 4 3
4 5 3
5 6 3
6 7 3
7 8 3
8 9 3
9 10 3
10 11 3
11 12 3
12 13 3
13 14 4
14 17 0
17 1 4
11 16 3
3 15 3
```



Content of the text file Corresponding Coxeter diagram

Figure 5.12 – 14-vinb85.coxiter

Remarks:

- The first line indicates that the Coxeter group Γ has 17 generators and that it is a subgroup of $\mathrm{Isom}\,\mathbb{H}^{14}$.
- Each of the remaining 17 lines describes one edge of the Coxeter diagram: the first two numbers are the labels of the generators and the third one is the label of the edge (a "0" indicates a bold edge).

5.5 Program testing and some values

To test the accuracy of the program, we ran it on a collection of (around 1200) groups for which some of the invariants were known. These graphs can be found in the graphs/ folder of the source code. Except for a few graphs, the name of each file goes as follows:

dimension – reference_page number – name or number of the graph.

The reference is as given in Table 5.8.

5.5.1 Euler characteristic

For d a positive square-free integer, we consider the reflection subgroup Γ_d^n in the automorphism group $O(f_d^n, \mathbb{Z}^{n+1})^+$ of the quadratic form $-dx_0^2 + x_1^2 + \ldots + x_n^2$ (see Section 3.9). When d is odd, the Euler characteristic of Γ_d^n can easily be computed (see [RT13, Corollary 4]). For other groups, we give the reference where another computation of the Euler characteristic can be found.

All the groups of Table 5.9 were tested.

5.5.2 Growth series and growth rate

The growth series and growth rate of the following simplices (see [Joh+99]) were computed and compared with the values given in [Ter15]: $\overline{BH}_3$, $\overline{J}_3$, $\overline{DH}_3$, $\widehat{AB}_3$, $\overline{H}_4$, $\overline{DH}_4$, $\overline{BH}_4$, $\widehat{AF}_4$, $\overline{P}_3$, $\overline{R}_3$, $\overline{V}_3$, $\overline{O}_3$, $\overline{R}_4$, $\overline{S}_4$, $\overline{O}_4$, $\overline{P}_4$, $\overline{M}_4$, $\overline{U}_5$, $\overline{X}_5$, $\overline{O}_5$, $\overline{S}_5$, $\overline{Q}_5$, $\overline{L}_5$, $\overline{P}_5$, $\overline{S}_6$, $\overline{Q}_6$, $\overline{P}_6$, $\overline{S}_7$, $\overline{Q}_7$, $\overline{T}_7$, $\overline{P}_7$, $\overline{T}_8$, $\overline{S}_8$, $\overline{Q}_8$, $\overline{T}_9$, $\overline{Q}_9$, $\overline{S}_9$.

The growth series of all cocompact Coxeter groups in $\operatorname{Isom} \mathbb{H}^4$ generated by at most 6 reflections (simplices and groups found by Esselmann and Kaplinskaja) were compared with the results found in [Per09].

We checked that $\frac{1}{f_\Gamma} = \frac{1}{f_{\Delta_1}} + \frac{1}{f_{\Delta_2}} - \frac{1}{f_{\Gamma_0}}$ for the free product with amalgamation in dimension 18 (see Section 5.3.3).

For each group of Table 5.9, we checked that the denominator of the growth function vanishes when n is odd and that $\frac{1}{f_\Gamma(1)} = \chi_\Gamma$ when n is even.

The following errors in the literature were found:

Group	Paper
$\overline{BH}_4$	[Per09]
$\overline{DH}_4$	[Per09]
$[8, 3, 4, 3, 8]$	[Per09]

5.5.3 f-vector

The f-vectors of the groups of Table 5.10 were tested. The alternating sum of the components of the f-vectors of the groups [Tum04], [Joh+99], [Per09, Appendix C] and [Vin85] were tested (see Proposition 3.7.7).

5.5.4 Cocompactness and finite covolume criterion

The groups of Table 5.11 were tested for cocompactness and the ones of Table 5.12 were tested for the finite covolume.

5.5.5 Arithmeticity

The groups of Table 5.13 were tested.

5.5.6 Some more complicated Coxeter graphs

Using groups mentioned in this section, we created more than 3000 thousands groups (small sequences of index two subgroups, as explained in Section 7.2) and these groups allowed us to test `CoxIter` and the conjecture of Kellerhals and Perren on more complicated graphs (the number of nodes ranges from 7 to 775).

5.6 Tables

Reference in the paper	Name of the file
[Ess96]	ess96
[FTZ07]	felikson2007
[Joh+99]	jkrt
[Im 85]	imhof85
[Mcl11]	mcl11
[Per09]	per09
[Rob15]	roberts15
[Tum04]	tum04

Table 5.8 – Correspondence between bibliography and file names

Reference	Group	Theoretical result
[Joh+99], [Tum04], [Vin85], [Im 85], [Per09, Appendix C], [Rob15]	all groups of odd dimension	Remark 3.7.3
[FTZ07]	13 compact simplicial prisms in $\mathbb{H}^4$, page 117	[FTZ07]
[Joh+99]	14 groups in Isom $\mathbb{H}^4$	[Joh+99]
[Joh+99]	3 groups in Isom $\mathbb{H}^6$	[Joh+99]
[Per09, Appendix C]	Bugaenko $P_6 \subset \mathbb{H}^6$, Bugaenko $P_8 \subset \mathbb{H}^8$	[Kel14]
[Joh+99]	4 groups in Isom $\mathbb{H}^8$	[Joh+99]
[Vin85]	Σ_{10} and a subgroup	[RT97, Theorem 22]
[Hil07]	P_1^{10}	[Hil07]
Vinberg algorithm	Γ_1^n, $2 \leq n \leq 19$	[RT97, Theorem 22]
Vinberg algorithm	Γ_3^n, $2 \leq n \leq 13$	[RT13, Corollary 4]
Vinberg algorithm	Γ_5^n, $2 \leq n \leq 8$	[RT13, Corollary 4]

Table 5.9 – Euler characteristic of some groups

Group	Theoretical result
all groups of [Joh+99]	simplices
all pyramids in $\mathbb{H}^3$ of [Tum04]	pyramids
Bugaenko P_6	[Kel14]
Bugaenko P_8	[Zeh09]
Birectified 5-simplex	`http://en.wikipedia.org/wiki/Rectified_5-simplexes`

Table 5.10 – f-vector of some groups

Reference	Cocompact	Groups
[Ess96]	yes	groups in Isom $\mathbb{H}^4$
[Per09, Appendix C]	yes	all groups
[FTZ07]	yes	13 compact simplicial prisms in $\mathbb{H}^4$, page 117
[Joh+02]	yes	$\overline{BH}_3$, $\bar{J}_3$, $\overline{DH}_3$, $\widehat{AB}_3$, $\bar{K}_3$, $\widehat{AH}_3$ $\widehat{BB}_3$, $\widehat{BH}_3$, $\widehat{HH}_3$, $\bar{H}_4$, $\overline{BH}_4$, $\overline{DH}_4$, $\bar{K}_4$, $\widehat{AF}_4$
[Rob15]	no	all groups
[Tum04]	no	all groups
[Tum04]	no	all groups obtained by removing a polar
[Vin85]	no	$\left\{\Sigma_n : 10 \leq n \leq 17\right\}$
[Joh+02]	no	$\bar{V}_3$, $\bar{R}_3$, $\bar{P}_3$, $\overline{BV}_3$, $\bar{O}_3$, $\bar{Y}_3$, $\overline{HV}_3$, $\overline{BP}_3$, $\overline{DV}_3$, $\bar{N}_3$, $\bar{Z}_3$, $\widehat{BR}_3$, $\overline{HP}_3$, $\widehat{AV}_3$, $\overline{DP}_3$, $\bar{M}_3$, $\overline{VP}_3$, $\widehat{BV}_3$, $\widehat{CR}_3$, $\widehat{HV}_3$, $\widehat{VV}_3$, $\widehat{RR}_3$, $\widehat{PP}_3$, $\bar{S}_4$, $\bar{R}_4$, $\bar{P}_4$, $\bar{O}_4$, $\bar{N}_4$, $\bar{M}_4$, $\overline{BP}_4$, $\widehat{FR}_4$, $\overline{DP}_4$, $\bar{U}_5$, $\bar{S}_5$, $\bar{X}_5$, $\bar{Q}_5$, $\bar{R}_5$, $\bar{P}_5$, $\bar{O}_5$, $\bar{N}_5$, $\widehat{AU}_5$, $\bar{M}_5$, $\bar{L}_5$, $\widehat{UR}_5$, $\bar{S}_6$, $\bar{Q}_6$, $\bar{P}_6$, $\bar{T}_7$, $\bar{S}_7$, $\bar{Q}_7$, $\bar{P}_7$, $\bar{T}_8$, $\bar{S}_8$, $\bar{Q}_8$, $\bar{P}_8$

Table 5.11 – Cocompactness

Reference	Finite covolume	Groups
[FTZ07]	yes	13 compact simplicial prisms in $\mathbb{H}^4$, page 117
[Im 85]	yes	all groups
[Joh+99]	yes	all groups
[Per09, Appendix C]	yes	all groups
[Rob15]	yes	all groups
[Tum04]	yes	all groups
[Vin85]	yes	$\{\Sigma_n : 10 \leq n \leq 19\}$
[Tum04]	no	every polar was removed to create 387 non-cofinite groups

Table 5.12 – Finite covolume

Reference	Arithmetic	groups
[Joh+99]	yes (see [Joh+02])	$\bar{S}_6$, $\bar{Q}_6$, $\bar{P}_6$, $\bar{T}_7$, $\bar{S}_7$, $\bar{Q}_7$, $\bar{P}_7$, $\bar{T}_8$, $\bar{S}_8$, $\bar{Q}_8$, $\bar{P}_8$
[Vin85]	yes	$\{\Sigma_n : 10 \leq n \leq 17\}$
[Tum04]		all groups

Table 5.13 – Arithmeticity

CHAPTER 6

The Vinberg algorithm and AlVin

In this chapter, we present Vinberg's algorithm and study it from an algorithmic point of view. The goal is to explain the key ingredients which are used in `AlVin` (for Algorithm of Vinberg), a `C++` program which we design to carry on the computations of the algorithm for diagonal quadratic forms with $K = \mathbb{Q}$ (non-cocompact case) and with $K = \mathbb{Q}[\sqrt{d}], \mathbb{Q}\big[\cos\frac{2\pi}{7}\big]$ (cocompact cases). Since we also want the program to be able in some cases to decide the non-reflectivity of a quadratic form, we present two methods for achieving this goal.

In the first section, we give an overview of the algorithm and present an exhaustive list of computations done with the algorithms by other authors. In the second section, we present in details the theoretical background needed to understand the algorithm.

In sections 6.3 to 6.5, we give more information about the implementations into the computer program `AlVin`: we explain how the solutions of the equations can be parametrized for each of our three fields. Finally, we show in details how to use `AlVin` and we present some results: new polyhedra (compacts and non-compacts, in dimensions 2 to 9), classification of diagonal quadratic form of signature $(3, 1)$ with small coefficients, non-reflectivity of some forms, etc.

The program, together with its documentation, can be found here: `https://github.com/rgugliel/AlVin`

6.1 Introduction

In [Vin72], Vinberg presented an algorithm to create hyperbolic Coxeter groups[1] in the arithmetic context. The starting point is a totally real number field K, together with an admissible (we will be more precise below) quadratic form f of signature $(n, 1)$ which is given, with respect to some basis $\{v_0, \dots, v_n\}$ of K^{n+1} by

$$f(x_0, \dots, x_n) = \sum_{i,j} a_{i,j} \cdot x_i\, x_j,$$

[1]Strictly speaking, the algorithm is more general: its goal is to find, for any discrete subgroup Θ of $\operatorname{Isom}\mathbb{H}^n$, the maximal subgroup of Θ generated by reflections. However, if we don't take Θ to be the group of units of some quadratic form, then finding the reflections is much more difficult.

where $a_{i,j} \in \mathcal{O}_K$. We also consider a lattice L in K^{n+1} (i.e. L is the $\mathcal{O}_K$-span of $n+1$ linearly independent vectors, where $\mathcal{O}_K$ denotes the ring of integers of K).

We now consider the group $\Theta_{f,L}$ of isomorphisms of the quadratic form f, which preserves the lattice L and the two sheets of the hyperboloid defined by f (see Section 3.9.1) and we know that $\Theta_{f,L}$ is a discrete group of finite covolume (again, see Section 3.9.1). We recall from Section 3.9, that we have the decomposition $\Theta_{f,L} = \Gamma_{f,L} \rtimes H$, as a semi-direct product, where $\Gamma_{f,L}$ is the subgroup of $\Theta_{f,L}$ generated by all the reflections in $\Theta_{f,L}$ and H is a subgroup of the symmetry group of a cell P of $\Gamma_{f,L}$. Vinberg's algorithm describes a way to find successively the normal vectors of the polyhedron[2] P (and of course a presentation of the group $\Gamma_{f,L}$):

1. First, we start with a set of n vectors $\mathcal{V} = \{e_1, \ldots, e_n\}$ whose corresponding hyperplanes go through a specified point (for example the point p_0 corresponding to the vector v_0, which is chosen such that $f(v_0) < 0$). These vectors give rise to a polyhedron $\bigcap_{e_i \in \mathcal{V}} H^-_{e_i}$ of infinite volume.
2. We look for a vector e_{n+1} which stabilizes the lattice spanned by the $v_0, \ldots, v_n$ and which satisfies some relations with the previous found vectors (again, we will be more precise below) and we add e_{n+1} to our collection $\mathcal{V}$. If the polyhedron $\bigcap_{e_i \in \mathcal{V}} H^-_{e_i}$ is of finite volume, we stop.
3. As long as the polyhedron defined by the vectors in $\mathcal{V}$ is not of finite volume, we continue to look for another vector and we add it to the collection $\mathcal{V}$.

At this point, it is worth to mention that we cannot say a priori if $\Gamma_{f,L}$ is of finite index in $\Theta_{f,L}$ or, what is equivalent, if the covolume of $\Gamma_{f,L}$ is finite (in this setting, we say that the form f is *reflective*). In terms of the algorithm it means the following: if the polyhedron $\bigcap_{e_i \in \mathcal{V}} H^-_{e_i}$ is not of finite volume, there is no way to know whether we have to continue to add vectors to the list $\mathcal{V}$ or whether P is not finite sided. If the algorithm stops, then we have found a cell P of the group $\Gamma_{f,L}$ (see [Vin72, Proposition 4]).

Remark 6.1.1
Although the choice of the base point p_0 (corresponding to the vector v_0) is irrelevant, considering only diagonal quadratic forms (and to the lattice spanned over the ring of integers by a diagonalizing basis) is a limitation. Indeed, even if every quadratic form can be transformed into a diagonal form over its defining field, it is not true that we can do the same with a transformation with coefficients in the ring of integers. For example, the 8 dimensional compact Coxeter polyhedron found by Bugaenko (see [Bug92]) is based on the lattice $[-(\sqrt{5}+1)] \oplus E_8$, which does not correspond to a diagonal quadratic form over $\mathcal{O}_{\mathbb{Q}[\sqrt{5}]}$.

Remark 6.1.2
In this work, we will always take f to be a diagonal quadratic form with respect

[2] The word *polyhedron* is used here in a wide sense since we do not know whether it is finite sided or not and whether if it has finite volume. We will allow this generalization for the whole chapter.

to a basis $\{v_0, \ldots, v_n\}$ and take L to be the lattice spanned over $\mathcal{O}_K$ by the v_i (this choice regarding the lattice is not a restriction, as indicated in Remark 3.9.2). In this setting, we will write Θ_f for $\Theta_{f,L}$ and Γ_f for $\Gamma_{f,L}$. Moreover, the group Θ_f is then called the *group of units* of f and we will sometimes refer to Γ_f (respectively P) as the reflection group (respectively the polyhedron) associated to f.

Before our work, the algorithm has mostly been used for diagonal quadratic forms of the type $f_\alpha^n = \langle -\alpha, 1, \ldots, 1 \rangle$ (see Section 6.6 for more complicated and new examples):

- $\alpha = 1$ (standard Lorentzian form)
  In [Vin72], Vinberg found groups for $2 \le n \le 17$. Together with Kaplinskaya, they applied the algorithm for $n = 18, 19$ and proved that the form f_1^n is not reflective if $n > 19$ (see [KV78]).

- $\alpha = 2$
  The form is reflective for $2 \le n \le 14$ (see [Vin72]).

- $\alpha = 3$
  McLeod studied in [Mcl11] the form for $\alpha = 3$ and showed that it is reflective for $2 \le n \le 13$. In his thesis (see [Mcl13]) he also performed some other applications of the algorithm, with some errors (see Remark 6.2.3 and sections 6.8.1.3 and 6.8.3.3).

- $\alpha = 5$
  The form is reflective for $2 \le n \le 8$ (see [Mar12]).

- $\alpha = 7, 11, 13, 17, 19, 23$
  Mark showed in [Mar12] that the forms with $\alpha = 7, 17$ are reflective only if $n = 2, 3$; the forms $\alpha = 13, 19, 23$ are reflective only if $n = 2$ and the form with $\alpha = 11$ is reflective if $n = 2, 3, 4$.

- $\alpha = \frac{1+\sqrt{5}}{2}$
  The form is reflective for $2 \le n \le 7$ (see [Bug84]).

- $\alpha = 1 + \sqrt{2}$
  The form is reflective for $2 \le n \le 6$ (see [Bug90]).

- $\alpha = 2\cos\frac{2\pi}{7}$
  The form is reflective for $n = 2, 3, 4$ (see [Bug92]).

In [Gro08], Grosek applied the algorithm to the forms $\langle -1, 1, m \rangle$, with $m = 1, 2, \ldots, 14, 16, 18, 20$ and to the forms $\langle -1, m, m \rangle$ for $m = 1, 2, \ldots, 12$.

In [Sch89], R. Scharlau classified all reflective quadratic forms of signature $(3, 1)$ such that the group Γ_f is non-cocompact (meaning that f is defined over $\mathbb{Z}$ and is isotropic, as indicated in Remark 3.9.2)[3]. There are 49 of such forms. Whenever it is possible, we will give the corresponding diagonal quadratic form (see Section 6.8.1.1).

[3]Note that the forms given by Scharlau are not in diagonal form.

6.2 Theoretical part

Let K be a totally real number field such that the ring of integers $\mathcal{O}_K$ of K is a unique factorization domain[4]. Let V be a K-vector space of dimension $n+1$ with basis $\{v_0, \ldots, v_n\}$ and let L be the lattice generated by $\{v_0, \ldots, v_n\}$, that is L is the $\mathcal{O}_K$-span of the basis $\{v_i\}$. As in Section 3.9, we consider a quadratic form f of signature $(n, 1)$ which is diagonal with respect to the basis $\{v_0, \ldots, v_n\}$: $f = \langle -\alpha_0, \alpha_1, \ldots, \alpha_n \rangle$, where each $\alpha_i \in \mathcal{O}_K$ is positive and does not have any square factor. We suppose moreover that f is *admissible*, which means that f^σ is positive definite for every non-identity Galois embedding $\sigma : K \longrightarrow \mathbb{R}$. In this setting, we will write Θ_f for the group $O(f, L)$. Finally, we suppose that if $1 \leq i < j \leq n$ are such that $\alpha_i \neq \alpha_j$ then the ratio α_i/α_j is not the square of an invertible element in $\mathcal{O}_K$: in other words, we fix once and for all one representative of each class of the quotient $\mathcal{O}_K/(\mathcal{O}_K^*)^2$.

Given such a quadratic form f, we also consider its associated bilinear form

$$(x, y) := \frac{1}{2}\big(f(x+y) - f(x) - f(y)\big) = -\alpha_0 \cdot x_0 y_0 + \sum_{i=1}^{n} \alpha_i \cdot x_i y_i,$$

where the x_i (respectively y_i) denote the components of x (respectively y) with respect to the basis $\{v_0, \ldots, v_n\}$. Any vector e gives rise to a reflection

$$\begin{aligned} R_e : V &\longrightarrow V \\ x &\longmapsto x - 2\frac{(x, e)}{(e, e)} e. \end{aligned}$$

By a *root of* f, or simply a *root*, we mean a vector e in V such that the reflection R_e with respect to the hyperplane H_e orthogonal to e preserves the lattice generated by $v_0, \ldots, v_n$. If e is such a root, then we can suppose that the coefficients $k_0, \ldots, k_n$ of e with respect to the basis $\{v_0, \ldots, v_n\}$ lie in $\mathcal{O}_K$ and don't have any common factor. Indeed, since K is the field of fractions of $\mathcal{O}_K$, each component k_i can be written $\frac{a_i}{b_i}$. Then, the vector $m \cdot e$, where m is the least common multiple $b_0, b_1, \ldots, b_n$, which exists since $\mathcal{O}_K$ is supposed to be a unique factorization domain, satisfies $R_{m \cdot e} = R_e$ and has coefficients in $\mathcal{O}_K$. Moreover, if the gcd g of the k_i is not 1, then can we can consider $e' = \frac{e}{g}$ in order to get an *irreducible*, or *primitive*, root.

Finally, notice that preservation of the lattice is equivalent to the *crystallographic condition*

$$(e, e) \mid 2\,\alpha_i\,k_i, \quad \forall\ 0 \leq i \leq n. \tag{6.1}$$

Indeed, applying R_e to the basis $\{v_i\}$, we see that the preservation of the lattice is equivalent to

$$-2\frac{\alpha_i\,k_i}{(e, e)} k_j, \quad \forall i \neq j.$$

Since the the root e is supposed to be primitive (i.e. the k_i don't have any common factor), then we must have $(e, e) \mid 2\,\alpha_i\,k_i$ for every i, as required.

[4]This assumption, although not theoretically necessary, is needed for computations in the ring of the integers $\mathcal{O}_K$

Remark 6.2.1
When working with the algorithm, we will use the vector space model $\mathcal{H}^n$ induced by the quadratic form f: we consider the cone $C_f = \{x \in K^{n+1} \otimes \mathbb{R} : f(x) < 0\}$ and its two connected components $C_f^{\pm}$ and we let $\mathcal{H}^n$ be $C_f^{+}/\mathbb{R}^*$.

6.2.1 Description of the algorithm

We fix an admissible quadratic form f and we let $\Theta_f = O(f, L)$ and Γ_f as above.

The first step is to fix a point $p_0 \in \mathcal{H}^n$. For the sake of simplicity, we will always consider p_0 corresponding to the basis vector v_0. The subgroup Γ_0 of the discrete group Γ_f generated by the reflections which fix p_0 is a finite group of rank k at most n. We consider one cell P_0 of Γ_0 (it is a polyhedral cone emanating from p_0) and its normal vectors $e_1, \dots, e_n$ (see Section 6.2.3 about finding these vectors and the reason why $k = n$ when p_0 corresponds to v_0). Among all the cells of Γ_f, we choose the unique cell P which is the same as P_0 in a sufficiently small neighbourhood of p_0.

At this point, we have $P_0 = \bigcap_{i=1}^n H_{e_i}^-$ and we want to construct a (hopefully finite) sequence $e_1, \dots, e_n, e_{n+1}, \dots$ such that $R_{e_i} \in \Gamma_f$ and $P := \bigcap_i H_{e_i}^-$ has finite volume. We proceed inductively. Suppose we have the finite sequence $e_1, \dots, e_r$, $r \geq n$. Among all the vectors e such that $R_e \in \Gamma_f$, we choose e_{r+1} such that:

- $(e_{r+1}, e_i) \leq 0$ for all $1 \leq i \leq r$;
- the distance $d(p_0, H_{e_{r+1}})$ is minimal.

Moreover, we choose the orientation of e_{r+1} such that $p_0 \in H_{e_{r+1}}^-$ (i.e. we want p_0 to lie inside P). As long as the polyhedron given by the e_i is of infinite volume, we continue to look for new vectors[5].

A result of Vinberg (see [Vin72, Proposition 4]) ensures that the vectors e_i indeed give rise to the required polyhedron P which is a cell of the group Γ_f.

Remarks 6.2.2
- The distance $d(p_0, H_{e_{r+1}})$ is given by

$$\sinh^2 d(p_0, H_{e_{r+1}}) = -\frac{(e_{r+1}, v_0)^2}{(e_{r+1}, e_{r+1}) \cdot (v_0, v_0)}.$$

  Hence, since $(v_0, v_0) = -\alpha_0 < 0$, minimizing this distance is equivalent to minimizing the value $\frac{(e_{r+1}, v_0)^2}{(e_{r+1}, e_{r+1})}$. With our assumption, this in turn reduces to minimizing the value $\frac{k_0^2}{(e_{r+1}, e_{r+1})}$, where k_0 is the first component of e_{r+1}.

- The condition $p_0 \in H_{e_{r+1}}^-$ is equivalent to $(v_0, e_{r+1}) \leq 0$.

6.2.2 Possible values for (e, e)

Let e be a root and suppose that $(e, e) \in \mathcal{O}_K$ is not invertible. Let $\pi \in \mathcal{O}_K$ be a prime factor of (e, e). Since the coefficients of e don't have any common

[5] As we will see later, the more we go on, the more complicated the computations become.

factor by assumption, there exists $0 \leq j \leq n$ such that $\pi \mid 2 \cdot \alpha_j$. If we denote by $\mathcal{P}$ the set of prime elements of $\mathcal{O}_K$ which divide at least one coefficient of the quadratic form f and by $\rho_1, \ldots, \rho_k \in \mathcal{O}_K$ the prime factors of 2, then we see that (e, e) can be written as follows:

$$(e, e) = u \cdot \rho_1^{0,1} \cdot \ldots \cdot \rho_k^{0,1} \cdot \prod_{i=1}^{r} \pi_i, \quad \pi_i \in \mathcal{P}, \quad u \in \mathcal{O}_K^*, \tag{6.2}$$

where $x^{0,1}$ means x^0 or x^1. Moreover, since the coefficients of f don't have any square factor, we can suppose that that the π_i are distinct. Finally, since f is admissible, all the conjugates of (e, e) must be positive.

Finally, equation (6.2) implies that in order to find the possible norms (e, e) it is sufficient to find the primes dividing the coefficients of the quadratic form, the decomposition of 2 and the invertible elements of $\mathcal{O}_K$ (see Theorem 2.2.1) and to list all the possible products, omitting the ones which have some negative conjugate.

Remark 6.2.3
Our description of the possible values of (e, e) is in contradiction with [Mcl13, Lemma 3.1.5], where only a smaller number of values is considered. This will lead us to some differences in the set of normal vectors of certain polytopes.

6.2.3 First set of vectors

First, we consider the group Γ_0 generated by reflections in Γ stabilizing p_0. If e is a root such that $R_e \in \Gamma_0$, then the angle between e and any v_i is 0, $\frac{\pi}{2}$ or $\frac{\pi}{4}$ (see [Bug92, Lemma 2.2]). Now, if there exists $1 \leq i \leq n$ such that $\measuredangle(e, v_i) = \frac{\pi}{4}$, then there exists precisely another $j \neq i$ such that $\measuredangle(e, v_j) = \frac{\pi}{4}$. Hence, we have

$$e = k_i v_i + k_j v_j.$$

The crystallographic condition (6.1) implies that $k_i, k_j \in \mathcal{O}_K^*$ and thus the quotient $\frac{\alpha_i}{\alpha_j}$ is a square in $\mathcal{O}_K^*$. By hypothesis on f, we thus have $\alpha_i = \alpha_j$ and finally $k_i = \pm k_j$. Therefore, the reflections in Γ_0 are precisely the following

$$R_{\pm v_i}, 1 \leq i \leq n, \qquad R_{\pm v_i \pm v_j}, 1 \leq i < j \leq n, \text{ when } \alpha_i = \alpha_j.$$

Suppose now that $f = \langle -\alpha_0, \alpha_1', \ldots, \alpha_1', \ldots, \alpha_r', \ldots, \alpha_r' \rangle$, where the α_i' are distinct, and that the coefficient α_i' appears n_i times. We can now use the well-known classification of irreducible finite Coxeter groups to see that

$$\Gamma_0 \cong \prod_{i=1}^{r} C_{n_i},$$

with the convention that $C_1 = A_1$ and $C_2 = G_2^4$ (see also Figure 3.1, page 36). Therefore, we form the initial set of vectors as follows:

$$\begin{array}{ll} -v_1 + v_2, \ldots, -v_{n_1-1} + v_{n_1}, -v_{n_1} & n_1 \text{ vectors} \\ -v_{n_1+1} + v_{n_1+2}, \ldots, -v_{n_1+n_2-1} + v_{n_1+n_2}, -v_{n_1+n_2} & n_2 \text{ vectors} \\ \vdots & \vdots \\ -v_{n_1+\ldots+n_{r-1}+1} + v_{n_1+\ldots+n_{r-1}+2}, \ldots, -v_{n-1} + v_n, -v_n & n_r \text{ vectors} \end{array}$$

Hence, we have $n_1 + \ldots + n_r = n$ vectors and if we label them $e_1, \ldots, e_n$, then we have $P_0 = \bigcap_{i=1}^{n} H_{e_i}^-$, as required.

6.2.4 Solving the norm equation

We want to find another root $e = (k_0, \ldots, k_n)$. Since the first component and the norm are prescribed by the fraction $\frac{(e,v_0)^2}{(e,e)}$ which we have to minimize, we only need to find the n remaining components. We do that inductively and by a "brute-force" search: if the components $k_0, \ldots, k_{j-1}$ are chosen, then we have

$$\sum_{i \geq j} \alpha_i k_i^2 = (e,e) + \alpha_0 k_0^2 - \sum_{i=1}^{j-1} \alpha_i k_i^2.$$

If $\sigma : K \longrightarrow \mathbb{R}$ denotes any non-trivial Galois embedding, then the fact that f^σ is positive definite implies

$$\sigma\left(\alpha_j k_j^2 + \sum_{i>j} \alpha_i k_i^2\right) = \sigma\left((e,e) + \alpha_0 k_0^2 - \sum_{i=1}^{j-1} \alpha_i k_i^2\right) \geq 0,$$

which easily yields the condition

$$|\sigma(k_j)| \leq \sqrt{\sigma\left(\frac{(e,e) + \alpha_0 k_0^2 - \sum_{i=1}^{j-1} \alpha_i k_i^2}{\alpha_j}\right)}.$$

If we let $S_j = (e,e) + \alpha_0 k_0^2 - \sum_{i=1}^{j-1} \alpha_i k_i^2$, then we get the system

$$\begin{cases} 0 \leq k_j \leq \sqrt{\frac{S_j}{\alpha_j}} \\ -\sqrt{\sigma\left(\frac{S_j}{\alpha_j}\right)} \leq \sigma(k_j) \leq \sqrt{\sigma\left(\frac{S_j}{\alpha_j}\right)}, \quad \forall \sigma \neq \mathrm{id}\,. \end{cases} \tag{6.3}$$

For each possible value of k_j, we check if the crystallographic condition (6.1) is satisfied, and if it is the case, we try to find possible values for k_{j+1}.

Remark 6.2.4 (i) Let $e' = (k'_0, \ldots, k'_n)$ be a root of the quadratic form f. Since the coefficients $\alpha_1, \ldots, \alpha_n$ of the quadratic form are positive, the "partial product" between e' and the candidate e

$$\begin{aligned} \mathcal{O}_K^* &\longrightarrow \mathcal{O}_K^* \\ k_{j+1} &\longmapsto \sum_{i=0}^{j+1} \alpha_i k'_i k_i \end{aligned}$$

is increasing. Therefore, the condition $(e, e_i) \leq 0$ for each found root e_i gives a stronger condition in the first inequality of the system (6.3).

(ii) When $k = \mathbb{Q}$, the crystallographic condition (6.1) allows us to replace 0 by $\frac{(e,e)}{\texttt{gcd}\left((e,e), 2\alpha_j\right)}$ in the first inequality of the system (6.3).

6.2.5 A first example

Let $f = \langle -1, 2, 6, 6 \rangle$. Using Section 6.2.3, we see that $\Gamma_0 \cong A_1 \times G_2^{(4)}$, and that the normal vectors of P_0 are

$$e_1 = (0, -1, 0, 0), \quad e_2 = (0, 0, -1, 1), \quad e_3 = (0, 0, 0, -1).$$

For a root e, the possible values for (e,e) are $1,2,3,4,6,12$. Moreover, we have the conditions

$$(e, v_0) < 0, \quad (e, e_i) \leq 0, \ \forall 1 \leq i \leq 3. \tag{6.4}$$

If the components of e with respect to the basis $v_0, \dots, v_3$ are denoted by k_0, k_1, k_2, k_3, the conditions (6.4) give

$$k_0 > 0, \quad k_1 \geq 0, \quad 0 \leq k_3 \leq k_2.$$

The vector e_4 should thus satisfy these conditions and minimize the value $\frac{k_0^2}{(e,e)}$. For $k_0 = 1$ and $(e,e) = 12, 6, 4, 3, 2$ we don't find any vector. For $(e,e) = 1$, so that $\frac{k_0^2}{(e,e)} = 1$, we find the vector $e_4 = (1, 1, 0, 0)$. The value 1 of the quotient can also be obtained with the values $k_0 = 2$ and $(e,e) = 4$ which gives the vector $e_5 = (2, 1, 1, 0)$ (we also check that $(e_4, e_5) \leq 0$). The polyhedron $P_5 = \bigcap_{i=1}^5 H_{e_i}^-$ defined by these 5 vectors has infinite volume. For $k_0 = 2$ and $(e,e) = 2$ we find a vector which is not compatible with e_5. Finally, we find the vector $e_6 = (3, 0, 1, 1)$ for $k_0 = 3$ and $(e,e) = 3$. The polyhedron P defined by these 6 vectors is of finite volume and is moreover compact (see Figure 6.1 for its Coxeter graph). This Napier cycle appears in [Im 90].

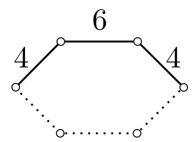


Figure 6.1 – Algorithm for the form $\langle -1, 2, 6, 6 \rangle$

6.2.6 Non-reflectivity of the quadratic forms

As mentioned above, the major issue with the algorithm is that there is no way to know a priori if the algorithm will stop or not. Also, since there is no bound on the rank of hyperbolic Coxeter groups and on the combinatorial complexity of hyperbolic Coxeter polyhedra (see the polyhedra in Section 6.8.1.6 and Chapter 7), it is not clear at all when it is reasonable to stop the algorithm and try to prove that the form is not reflective. Among all the methods that can be used to prove the non-reflectivity of a quadratic form, we will focus on two of them. It is worth to mention that these methods are "ad hoc" and tedious to use: there is probably no way to have a unique method to prove the non-reflectivity of a given non-reflective form. However, in some cases, `AlVin` can be used to handle these forms.

For the next two sections, we suppose that we found by the algorithm the vectors $e_1, \dots, e_r$ with the algorithm and that the polyhedron $P_r := \bigcap_{i=1}^r H_{e_i}^-$ has not finite volume. We also let $\mathcal{E}_r$ be the Coxeter graph induced by the vectors $e_1, \dots, e_r$.

6.2.6.1 First method: the finite volume condition

The aim of the first method, which is presented by McLeod in [Mcl11], is to show that it is not possible to find another root e_{r+1} of the quadratic form

f such that $P_r \cap H^-_{e_{r+1}}$ is of finite volume or such that there exist other roots $e_{r+2}, \dots, e_{r+d}$ with the property that $P_r \cap H^-_{e_{r+1}} \cap \bigcap_{i=2}^d H^-_{e_{r+i}}$ is of finite volume.

Since P_r has infinite volume, and in view of Proposition 6.3.1, it is possible[6] that $\mathcal{E}_r$ contains a Euclidean graph $\mathcal{E}'$ which cannot be extended to a Euclidean subgraph of rank $n-1$ of $\mathcal{E}_r$. In order for f to be reflective, we have to find (at least) one new root $e_{r+1} = (k_0, \dots, k_n)$ such that in the new Coxeter graph $\mathcal{E}_{r+1}$, the graph $\mathcal{E}'$ can be extended correctly. Suppose that the vectors corresponding to the vertices of $\mathcal{E}'$ are $e_{i_1}, \dots, e_{i_k}, e_{i_{k+1}}, \dots, e_{i_m}$, with $i_j < i_{j+1}$ for every $1 \le j \le m$ and $i_k \le n < i_{k+1}$. Since a connected Euclidean graph cannot be extended to another connected Euclidean graph, the new vector e_{r+1} has to be perpendicular to the vectors $e_{i_1}, \dots, e_{i_k}$. Because each e_i is of the form $-v_l$ or $-v_l + v_{l+1}$ for some l by Section 6.2.3, this gives strong conditions on the coefficients k_i of e_{r+1}.

Example 6.2.5
We consider the quadratic form $f = \langle -11, 1, 1, 1, 1, 1\rangle$. Using the algorithm we find the first 8 vectors:

$$\begin{aligned} &e_1 = (0,-1,1,0,0,0), && e_2 = (0,0,-1,1,0,0), && e_3 = (0,0,0,-1,1,0), \\ &e_4 = (0,0,0,0,-1,1), && e_5 = (0,0,0,0,0,-1), && e_6 = (3,11,0,0,0,0), \\ &e_7 = (1,2,2,2,1,0), && e_8 = (1,3,1,1,1,1). \end{aligned}$$

The corresponding Coxeter graph $\mathcal{E}_8$ is presented in Figure 6.2. The subgraph

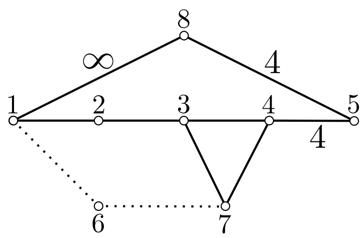


Figure 6.2 – Graph $\mathcal{E}_8$ - 8 first vectors for the form $\langle -11, 1, 1, 1, 1, 1\rangle$

$\tilde{A}_1 \times \tilde{A}_2$, associated to e_1, e_8 and e_3, e_4, e_7, cannot be extended to a graph of rank 4. The new vector $e_9 = (x_0, \dots, x_5)$ should be perpendicular to e_1, e_3, e_4, e_7 and e_8. Therefore, we have $x_1 = x_2$ and $x_3 = x_4 = x_5$. Finally, we have to solve the system

$$\begin{cases} (e_9, e_9) \in \{1, 2, 11, 22\} \\ (e_9, e_7) = 0 \\ (e_9, e_8) = 0 \end{cases} \Leftrightarrow \begin{cases} -11x_0^2 + 2x_1^2 + 3x_3^2 \in \{1, 2, 11, 22\} \\ -11x_0 + 4x_1 + 3x_3 = 0 \\ -11x_0 + 4x_1 + 3x_3 = 0 \end{cases}$$

with the additional constraints $x_0 > 0$, $x_1 \ge x_3$. Since the system has no solution, the quadratic form $\langle -11, 1, 1, 1, 1, 1\rangle$ is not reflective.

Remark 6.2.6
Depending on the value of the norm of the root, we may get extra constraints from the crystallographic condition (6.1). In the above example, if $-11x_0^2 + 2x_1^2 + 3x_3^2 = 11$ or 22, then we must have $11 \mid x_2$ and $11 \mid x_3$.

[6] In fact, tests show that it is almost always the case.

Remark 6.2.7
This first approach only works well when working over $\mathbb{Q}$: for other fields, the resolution of the system of equations is quite difficult. Moreover, programs such as `Mathematica`® don't handle equations solving in number fields. One can write every element in a integral basis and write a system with more equations but then the computations become too complicated.

6.2.6.2 Second method: symmetries of the polyhedron

Recall that we have the decomposition $\Theta_f = \Gamma_f \rtimes H$, where H is a subgroup of the symmetry group $\operatorname{sym} P$ of the fundamental polyhedron P. Here, we will try directly to show that P is not finite sided by showing that its symmetry group is infinite. The steps are the following:

1. Create symmetries $\eta_1, \ldots, \eta_m$ of P_r which extend to integral symmetries of P (see Lemma 6.2.8).
2. Show that the group $\Delta := \langle \eta_1, \ldots, \eta_m \rangle \leq \operatorname{sym} P$ has no fixed point in $\mathcal{H}^n$ (see lemmas 6.2.10 and 6.2.11). Usually, two or three involutions will be sufficient.

We will provide an example of this method in Section 6.6.5.

Lemma 6.2.8 (See also [Bug92, Lemma 3.3])
Let $\mathcal{G}$ be a subgraph of the Coxeter graph of P, such that:

- *The $\mathbb{R}$-span of the vectors corresponding to the vertices of $\mathcal{G}$ has dimension $n+1$.*
- *The graph $\mathcal{G}$ contains a spherical or Euclidean subgraph of maximal rank.*

Then, any symmetry of $\mathcal{G}$ extends to a symmetry of P.

Remark 6.2.9
In contrast to what the original lemma [Bug92, Lemma 3.3] says, the above lemma only gives us a linear transformation of the space which preserves P. We still have to check that this involution preserves the quadratic form and the lattice spanned over $\mathcal{O}_K$ by the canonical basis.

Lemma 6.2.10 (See [Bug92, Lemma 3.1])
Let H be a discrete subgroup of $\operatorname{Isom} \mathcal{H}^n$. Then, H is infinite if and only if there exists a subgroup Δ of H with $\operatorname{Fix}(\Delta) \cap \mathcal{H}^n = \emptyset$, where $\operatorname{Fix}(\Delta)$ denotes the set of vectors fixed by every transformation on Δ.

Lemma 6.2.11 (See [Bug92, Lemma 3.2])
Let η be an involutive transformation of an arbitrary real vector space V. Then, $\operatorname{Fix} \eta$ is generated by the vectors $w_i + \eta(w_i)$, where $\{w_i\}$ is a $\mathbb{R}$-basis of V.

Remark 6.2.12
This method becomes more difficult to apply when the dimension of $\mathcal{H}^n$ increases because we have to find symmetric subgraphs with more and more vertices. Also, when the set of possible values for the norm of a root (which grows exponentially with the number of prime number dividing at least one coefficient of the quadratic form) is big, many symmetries of subgraphs do not preserve the

quadratic form. In both cases, it is necessary to compute more normal vectors before trying this method. However, one advantage of this second approach compared to the first one is to be independent[7] of the base field K.

6.3 Towards the implementations

6.3.1 Global description of the implementation of the algorithm

Here are the main steps of the implementation of the algorithm for a given quadratic form:

- We check that the quadratic form is admissible, has appropriate signature and satisfies the condition on the ratios of its coefficients (see Section 6.2).
- We find a first set of n vectors, corresponding to roots of the lattice whose reflections generate the stabilizer Γ_0 of v_0 (see Section 6.2.3).
- We find the possible values for the norm of a root e (see Section 6.2.2).
- For given integers $m < M$ we find all the possible fractions $m < \frac{k_0}{(e,e)} \leq M$, where $e = (k_0, \dots, k_n)$ is a root. We start with $m = 0$, $M = 1$ and we continue until the algorithm terminates (or when the maximal number of vectors is achieved). We choose the fraction which has minimal value.
- For each fraction $\frac{k_0^2}{(e,e)}$ we try to find the remaining components $k_1, \dots, k_n$ of the root e. If we succeed, we add the vector e to the list and we check if the polyhedron which is bounded by the hyperplanes perpendicular to the vectors is of finite volume. If it is not the case, we try to find another root from the next fraction.

6.3.2 Couples $(k_0, (e, e))$

Let $m < M$ be two positive integers and let $e = (k_0, \dots, k_n)$ and $\varepsilon = (e, e)$. We want to determine all the fractions such that $m < \frac{k_0^2}{\varepsilon} \leq M$. Since the quadratic form f^σ is positive definite for all non-trivial Galois embeddings $\sigma : K \longrightarrow \mathbb{R}$, we have $\sigma\big(\varepsilon + \alpha_0 \cdot {k_0}^2\big) > 0$, with $\sigma(\alpha_0) < 0$, which gives the equations

$$\begin{cases} m \cdot \varepsilon < {k_0}^2 \leq M \cdot \varepsilon \\ |\sigma(k_0)| \leq \sqrt{\sigma\left(\frac{\varepsilon}{-\alpha_0}\right)}, \quad \forall \sigma \neq \mathrm{id}\,. \end{cases} \tag{6.5}$$

We will describe the solutions of this system when $K = \mathbb{Q}[\sqrt{d}]$ and $K = \mathbb{Q}[\cos\frac{2\pi}{7}]$ (see sections 6.5.2.3 and 6.5.3.3).

[7]This is not completely true since arithmetic computations in $\mathbb{Q}[\cos\frac{2\pi}{7}]$ and $\mathbb{Q}[\sqrt{d}]$ are slower than in $\mathbb{Q}$ but the difference is acceptable.

6.3.3 Some optimizations

Most of the computation time is dedicated to one of the two following tasks:

1. Solving the norm equation.
2. Once a candidate has been found, checking whether the polyhedron is of finite volume.

6.3.3.1 Testing the volume finiteness

Besides the finite volume criterion (see Proposition 3.8.4), we have a partial criterion from [Vin72, Proposition 1].

Proposition 6.3.1
Let $\Gamma < \operatorname{Isom} \mathbb{H}^n$ be a Coxeter group and let $\mathcal{G}$ be the Coxeter graph of Γ. If Γ is of finite covolume, then for each Euclidean subgraph $\mathcal{E}$ of $\mathcal{G}$ there exists a Euclidean subgraph $\mathcal{E}'$ of $\mathcal{G}$ of rank $n-1$ (see Definition 3.5.4) such that $\mathcal{E} \subset \mathcal{E}'$.

If the condition of the Proposition 6.3.1 is satisfied, then it is possible that Γ is of finite covolume. Since the number of Euclidean subgraphs is usually much smaller than the number of spherical subgraphs, the test is faster than the complete criterion (more than 10000 faster for big graphs). If the partial test succeeds, then we try again with the complete test.

6.4 General information about the program

The program `AlVin` is written in `C++` and is free/open source. More precisely, it is published under a free license, the GPLv3 (the GNU General Public License v3) and can be used freely in various projects[8]. `AlVin` and its documentation (to build and use the program) is available on the website `https://alvin.rgug.ch` and on GitHub: `https://github.com/rgugliel/AlVin`

6.4.1 External libraries used

`AlVin` requires the following external libraries:

CoxIter It is used as a library to check at each step if the polyhedron is of finite volume or not. When the form is reflective, we use `CoxIter` to compute the invariants of the final polyhedron.

Eigen It is a `C++` template library for linear algebra which is used for the non-reflective test.

Gaol This library is used to perform interval arithmetic.

GMP (or GNU Multiple Precision Arithmetic Library) is a free library for arbitrary precision arithmetic with integers.

[8]More information about the license can be found here: `https://www.gnu.org/licenses/gpl.html` A short guide is also available here: `https://www.gnu.org/licenses/quick-guide-gplv3.html`

igraph It is a `C/C++` package designed to manipulate graphs and do network analysis used for the non-reflective test.

OpenMP Via `CoxIter` and for the non-reflective test.

PARI We use the `C` library `PARI` to factor the minimal polynomial of $2\cos\frac{2\pi}{7}$ over $\mathbb{F}_p[x]$. This library is also used through `CoxIter` to compute the growth rate of the final Coxeter group.

PCRE The PCRE library (PCRE - Perl Compatible Regular Expressions) is used to parse the user input.

6.4.2 Design description

We briefly explain the main steps of the working flow of `AlVin`.

- Reading the parameters
  When the program is launched, it first lists all the given parameters (the file to read and the options for the computations). This is done in the `App::bReadMainParameters` function.

- First tests and initializations
  We test that the form is admissible, remove the square factors of the coefficients and check whether the gcd of the coefficients is invertible. Then, we compute the possible values for the norm of a root (`AlVin::AlVin`, `AlVin::initializations` and `AlVin::findPossibleNorms2`).

- First set of vectors
  We list the first n vectors, as explained in Section 6.2.3 (see function `AlVin::findFirstVectors`).

- Computing the vectors
  We go through pairs $(k_0,(e,e))$ by increasing value of $\frac{k_0^2}{(e,e)}$ (`AlVin::Run`). Once a vector is found, we check whether the polyhedron has finite volume or not. We continue as long as the polyhedron does not have finite volume or the number of vectors is less than the value "-maxv".

- Non-reflective test
  If the algorithm did not succeed and if it was specified in the parameters, we try to determine if the form is not reflective (`InfiniteNSymetries`).

- Displaying final information
  Finally, we display the information about the execution.

6.5 Implementations

In this section, we provide some details about computations in the different implementations of the algorithm. Mostly, this consists of the following:

- decompositions into prime numbers;

- parametrization of the solutions of systems (6.3) and (6.5) on pages 113 and 117.

6.5.1 Rational integers

6.5.1.1 Operations

For a real number $x \in \mathbb{R}$, we have the respective functions floor(x) and ceil(x) which return respectively the largest integer not greater than x and smallest integer not less than x.
For a positive real number x, we define the integral square root SQRTinf(x) of x and the supremal integral square root SQRTsup(x) of x to be the integers such that

$$\text{SQRTinf}(x) \leq \sqrt{x} < \text{SQRTinf}(x) + 1, \quad \text{SQRTsup}(x) - 1 < \sqrt{x} \leq \text{SQRTsup}(x).$$

Note that we have the following equality for every positive integer $n \in \mathbb{N}$

$$\text{SQRTsup}(n) = \text{SQRTinf}(n-1) + 1.$$

6.5.1.2 Computations

The solutions of systems (6.3) and (6.5) are then parametrized as follows:

$$\begin{cases} \text{SQRTsup}(m \cdot \varepsilon) \leq k_0 \leq \text{SQRTsup}(M \cdot \varepsilon) \\ 0 \leq k_j \leq \text{SQRTinf}\left(\text{ceil}\, \frac{S_j}{\alpha_j}\right). \end{cases}$$

6.5.1.3 Some optimizations

Solving the norm equation Suppose that all but two components of a root have been found. Then, we have to solve the equation

$$\alpha_{n-1} k_{n-1}^2 + \alpha_n k_n^2 = S_{n-2}, \quad k_{n-1}, k_n \in \mathbb{N}_0.$$

In order for this equation to have a solution, the following quadratic form must be isotropic over $\mathbb{Z}$:

$$\alpha_{n-1} k_{n-1}^2 + \alpha_n k_n^2 - S_{n-2} k^2. \tag{6.6}$$

To check whether this quadratic form is isotropic or not, we use the following result, due to Legendre.

Theorem 6.5.1 (Legendre, see [Lam05, Chapter IV, exercise 19])
Let $a, b, c \in \mathbb{N}$ be three square-free and pairwise coprime integers. Then, the ternary quadratic form $q(x, y, z) = ax^2 + by^2 - cz^2$ is isotropic if and only if $-bc$ is a square modulo a, $-ac$ is a square modulo b and $-ab$ is a square modulo c.

In order to use the previous result, we have the following useful lemma.

Lemma 6.5.2
Let a, b, c be three positive square-free integers and let d be a divisor of a and b. Then, the equation $ax^2 + by^2 - cz^2 = 0$ has a non-trivial solution if and only if the equation $\frac{a}{d}x^2 + \frac{b}{d}y^2 - (cd)z^2 = 0$ has a non-trivial solution.

We consider equation (6.6) and use the reduction of Lemma 6.5.2 to get an equation $ax^2 + by^2 - cz^2 = 0$ as in Theorem 6.5.1. Let $p \in \mathbb{P}$ be a prime factor of c such that $p^2 \nmid c$. If $-ab$ is not a square mod p, then the equation does not admit a non-trivial solution. By implementing the Legendre symbol and doing this test with a few primes (for example $p = 3, 5, 7, 11, 13$) we can reduce the computation time by 5-10%.

Also, since the sum of two squares is always 0 or 1 mod 4, if $\alpha_{n-1} \equiv \alpha_n \equiv 1$ mod n and $S_{n-2} \equiv 3$, then the equation cannot have a solution.

With these two simple tricks we reduce the computation time by 20-25%.

6.5.2 Quadratic integers

In this section, we consider the quadratic field $K = \mathbb{Q}[\sqrt{d}]$, with $d \in \mathbb{N}$ square-free. We will restrict (both for the theoretical part and for the program) to the case where K is a principal ideal domain. Moreover, for the computations, we will assume that $d < 100$. In this case, we must have $d \in \{2, 3, 5, 6, 7, 11, 13, 14, 17, 19, 21, 22, 23, 29, 31, 33, 37, 38, 41, 43, 46, 47, 51, 53, 57, 58, 59, 61, 62, 67, 69, 71, 73, 74, 77, 78, 79, 82, 83, 85, 86, 89, 91, 93, 94, 95, 97\}$ (see [Neu99, Chapter I, §6, page 37]). In this setting, an integral basis for the ring of integers $\mathcal{O}_K$ is $\{1, \Theta\}$, with $\Theta = \sqrt{d}$ if $d \equiv 2, 3 \bmod 4$ and $\Theta = \frac{1+\sqrt{d}}{2}$ if $d \equiv 1 \bmod 4$.

In this section, we write $\overline{x}$ for the non-trivial Galois conjugate of an element $x \in K$.

6.5.2.1 Operations

Unlike the rational case, computations with quadratic fields are non-trivial. First, the decomposition of integers relies on solutions of the Pell equations. Also, computations of the `gcd` have to be done using the decomposition into prime numbers (and not with the Euclidean algorithm). This leads of course to less efficient computations. Finally, all the equations have to be solved for the two components of the integers.

Decomposition of rational primes As explained in Section 2.2.1, each prime number $p \in \mathbb{P}$ either remains prime in $\mathcal{O}_K$ or splits into a product of two primes. The following theorem describes which primes split.

Theorem 6.5.3 ([NZM08, Theorem 9.28])
When $\mathcal{O}_{\sqrt{d}} := \mathcal{O}_{\mathbb{Q}[\sqrt{d}]}$ is a unique factorization domain, then:

1. *If $p \in \mathbb{P}$ is odd and $p \nmid d$, then p is a product of two primes $\pi_1, \pi_2 \in \mathcal{O}_{\sqrt{d}}$ if and only if $\left(\frac{d}{p}\right) = 1$, where $\left(\frac{d}{p}\right)$ denotes the usual Legendre symbol. In this case, the primes π_1 and π_2 are not associate but π_1 and the Galois conjugate of π_2 are.*

2. *If d is odd, then:*

   - *2 is the associate of the square of a prime if $d \equiv 3$ (4).*
   - *2 remains prime if $d \equiv 5$ (8).*
   - *2 is the product of two distinct primes if (i.e. non-associates) primes if $d \equiv 1$ (8).*

3. *Any prime $p \in \mathbb{P}$ which divides d is the associate of the square of a prime.*

If $p \in \mathbb{P}$ is a rational prime which is not a prime in $\mathcal{O}_K$, we want to find its prime factor(s). If d is 2 or 3 mod 4, we have to find one solution to the generalized Pell equation

$$x^2 - dy^2 = \pm p.$$

In [Rob04], Robertson provides different algorithms to find solutions to this equation as well as information to derive a solution from the *minimal* solution of $x^2 - dy^2 = 1$.

If d is 1 mod 4, then we look for a solution of the equation

$$x^2 - dy^2 = \pm 4p.$$

Such a solution has $x \equiv y$ (2) and we can write $\pi = \frac{x-y}{2} + y \cdot \Theta$.

Remark 6.5.4

In the program, we store, for each value of d mentioned in the beginning of the section, a minimal solution of the equation $x^2 - dy^2 = 1$. From that, we find the factors.

Integer square roots of a quotient Let $\lambda, \mu \in \mathcal{O}_K$ be two quadratic integers of the same sign. We want to compute the integer square root s, and the supremal integer square root s', of the quotient $\frac{\lambda}{\mu}$. These are rational integers such that:

$$s^2 \leq \frac{\lambda}{\mu} < (s+1)^2, \quad (s'-1)^2 < \frac{\lambda}{\mu} \leq s'^2.$$

First, we write $\frac{\lambda}{\mu} = \frac{x}{z} + \frac{y}{z}\sqrt{d}$. If $\lambda = a + b \cdot \Theta$, $\mu = \alpha + \beta \cdot \Theta$, with $a, b, \alpha, \beta \in \mathbb{Z}$, we have

$$x = a\alpha - b\beta d, \quad y = b\alpha - a\beta, \quad z = N(\mu), \quad \text{if } d \equiv 2, 3 \ (4),$$
$$x = \operatorname{Tr}(\lambda) \cdot \operatorname{Tr}(\mu) - b\beta d, \quad y = b \operatorname{Tr}(\lambda) - \beta \operatorname{Tr}(\mu), \quad z = 4 \cdot N(\mu), \quad \text{if } d \equiv 1 \ (4).$$

Up to a sign change, we can suppose that $z \geq 0$. We consider now separately the integer square root and the supremal integer square root.

For the integer square root s, we distinguish two cases:

$y \geq 0$ Let $t := \text{SQRTinf}\,\frac{x+\text{SQRTinf}(y^2 d)}{z}$. Note that t may be smaller than the required value s. While $(t+1)^2 \leq \frac{\lambda}{\mu}$ (or equivalently $z(t+1)^2 - x)^2 \leq y^2 d$), we add 1 to t.

$y < 0$ We consider $t := \text{SQRTsup}\,\frac{x-\text{SQRTinf}(y^2 d)}{z}$ which may be bigger than the required value s. While $\frac{\lambda}{\mu} < t^2$, we subtract one to t. Note that, again, we can perform this test without risking any approximation error as follows: while $zt^2 - x > 0$ or $zt^2 - x \leq 0$ and $(zt^2 - x)^2 < y^2 d$, subtract 1 from t.

For the supremal integer square root s', we distinguish two cases:

$y \geq 0$ Let $t := \text{SQRTsup}\,\frac{x+\text{SQRTsup}(y^2 d)}{z}$. Note that t may be greater than the required value s'. While $y^2 d \leq \left((t-1)^2 z - x\right)^2$, subtract 1 from t.

$y < 0$ We consider $t := \text{SQRTsup}\,\frac{x-\text{SQRTinf}(y^2 d)}{z}$. Note that t may be greater than the required value s'. While $x - (t-1)^2 z \leq \sqrt{y^2 d}$, subtract one from t.

6.5.2.2 Possible values for (e, e)

It is known (see the Dirichlet's unit theorem, Theorem 2.2.1) that units of $\mathcal{O}_K$ are given by $\pm\eta^n$, $n \in \mathbb{N}_0$, for some $\eta \in \mathcal{O}_K^*$. The element η is called the *fundamental unit* of $\mathcal{O}_K$ and can be deduced from the minimal solution of the Pell equation; alternatively, a list of all fundamental units can be found here: `http://mathworld.wolfram.com/FundamentalUnit.html`. Using this fact, we can see that the possible values for (e, e) are of the following form:

$$(e, e) = \eta^n \cdot \rho_1^{0,1} \cdot \ldots \cdot \rho_k^{0,1} \cdot \prod_{i=1}^{r} \pi_i, \quad n \in \mathbb{N}_0, \quad k = 1, 2, \quad \pi_i \in \mathcal{P},$$

where $\rho_1, \ldots, \rho_k \in \mathcal{O}_K$, $k = 1, 2$, are the prime factors of 2 (see also Section 6.2.2). Moreover, up to a rescaling, we can suppose that

$$(e, e) = \eta^{0,1} \cdot \rho_1^{0,1} \cdot \ldots \cdot \rho_k^{0,1} \cdot \prod_{i=1}^{r} \pi_i, \quad k = 1, 2, \quad \pi_i \in \mathcal{P}.$$

As before, we must also have $\overline{(e, e)} > 0$, where $\overline{(e, e)}$ denotes the non-trivial Galois conjugate of (e, e).

6.5.2.3 Couples $(k_0, (e, e))$

As explained in Section 6.3.2, in order to determine all the fractions satisfying $m < \frac{k_0^2}{\varepsilon} \leq M$ for some integers $m < M$, we have to solve the following system:

$$\begin{cases} m \cdot \varepsilon < k_0^2 \leq M \cdot \varepsilon \\ \left|\overline{k}_0\right| \leq \sqrt{\frac{\overline{\varepsilon}}{-\overline{\alpha}_0}}. \end{cases}$$

We write $k_0 = x_0 + y_0 \cdot \Theta$ and distinguish the following two possible cases, depending on d.

$d \equiv 2, 3$ (4) We have the system

$$\begin{cases} \sqrt{m\varepsilon} < x_0 + y_0 \cdot \Theta \leq \sqrt{M\varepsilon} \\ \left|x_0 - y_0 \cdot \Theta\right| \leq \sqrt{\frac{\overline{\varepsilon}}{-\overline{\alpha}_0}}, \end{cases}$$

and a parametrizations of the solutions is given by

$$\begin{cases} \frac{\sqrt{m\varepsilon} - \sqrt{\frac{\overline{\varepsilon}}{-\overline{\alpha}_0}}}{2} < x_0 \leq \frac{\sqrt{M\varepsilon} + \sqrt{\frac{\overline{\varepsilon}}{-\overline{\alpha}_0}}}{2} \\ \sqrt{\frac{m\varepsilon}{d}} - \frac{x_0}{\sqrt{d}} < y_0 \leq \frac{\sqrt{M\varepsilon} - x_0}{\sqrt{d}}. \end{cases}$$

We look for solutions in the following domain

$$\begin{cases} \text{ceil}\, \frac{\text{SQRTinf}\, m\varepsilon - \text{SQRTsup}\, \frac{\overline{\varepsilon}}{-\overline{\alpha}_0}}{2} \leq x_0 \leq \text{floor}\, \frac{\text{SQRTsup}\, M\varepsilon + \text{SQRTsup}\, \frac{\overline{\varepsilon}}{-\overline{\alpha}_0}}{2} \\ \text{SQRTinf}\, \frac{m\varepsilon}{d} - \text{SQRTinf}\, \frac{x_0^2}{d} \leq y_0 \leq \text{SQRTinf}\, \frac{M\varepsilon}{d} - \text{SQRTinf}\, \frac{x_0^2}{d}. \end{cases}$$

$d \equiv 1 \ (4)$ We have the system

$$\begin{cases} \sqrt{m\varepsilon} < x_0 + y_0 \cdot \Theta \leq \sqrt{M\varepsilon} \\ \left|(x_0 + y_0) - y_0 \cdot \Theta\right| \leq \sqrt{\frac{\overline{\varepsilon}}{-\overline{\alpha}_0}}, \end{cases}$$

and a parametrizations of the solutions is given by

$$\begin{cases} \sqrt{\frac{m\varepsilon}{d}} - \sqrt{\frac{\overline{\varepsilon}}{-d\cdot\overline{\alpha}_0}} < y_0 \leq \sqrt{\frac{M\varepsilon}{d}} + \sqrt{\frac{\overline{\varepsilon}}{-d\cdot\overline{\alpha}_0}} \\ \sqrt{m\varepsilon} - y_0 \cdot \Theta < x_0 \leq \sqrt{M\varepsilon} - y_0 \cdot \Theta. \end{cases}$$

We look for solutions in the following domain

$$\begin{cases} \text{SQRTinf}\left(\frac{m\varepsilon}{d}\right) - \text{SQRTinf}\left(\frac{\overline{\varepsilon}}{-d\cdot\overline{\alpha}_0}\right) \leq y_0 \\ \text{SQRTinf}\left(\frac{M\varepsilon}{d}\right) + \text{SQRTinf}\left(\frac{\overline{\varepsilon}}{-d\cdot\overline{\alpha}_0}\right) + 1 \geq y_0 \\ \text{ceil}\left(\frac{\text{SQRTinf}(4m\varepsilon) - y_0 - \text{SQRTinf}(y_0^2\, d) - 1}{2}\right) \leq x_0 \\ \text{SQRTinf}(M\varepsilon) + 1 - \text{floor}\,\frac{y_0 + \text{SQRTinf}(y_0^2\, d)}{2} \geq x_0. \end{cases}$$

6.5.2.4 Solving the norm equation

We solve the system (6.3) of Section 6.2.4 depending on d.

$d \equiv 2, 3 \ (4)$ We find

$$\begin{cases} -\sqrt{\frac{\overline{S}_j}{2\cdot\overline{\alpha}_j}} \leq x \leq \sqrt{\frac{S_j}{2\cdot\alpha_j}} + \sqrt{\frac{\overline{S}_j}{2\cdot\overline{\alpha}_j}} \\ -\text{SQRTinf}\,\frac{x^2}{d} - 1 \leq y \leq \text{SQRTinf}\,\frac{S_j}{d\cdot\alpha_j} - \text{SQRTinf}\,\frac{x^2}{d}. \end{cases}$$

$d \equiv 1 \ (4)$ We find

$$\begin{cases} -\sqrt{\frac{\overline{S}_j}{d\cdot\overline{\alpha}_j}} \leq y \leq \sqrt{\frac{S_j}{d\cdot\alpha_j}} + \sqrt{\frac{\overline{S}_j}{d\cdot\overline{\alpha}_j}} \\ -y \cdot \Theta \leq x \leq \sqrt{\frac{S_j}{\alpha_j}} - y \cdot \Theta, \end{cases}$$

and

$$\begin{cases} -\,\text{floor}\,\frac{y + \text{SQRTinf}(y^2 d) + 1}{2} \leq x \\ \text{SQRTinf}\,\frac{S_j}{\alpha_j} + 1 - \text{ceil}\,\frac{y + \text{SQRTinf}(y^2 d) + 1}{2} \geq x. \end{cases}$$

6.5.3 Maximal real subfield of the 7-cyclotomic field

Although we presented in Section 2.2.6 information about the maximal real subfield $K = \mathbb{Q}\big[\cos\frac{2\pi}{q}\big]$ of the cyclotomic extension $\mathbb{Q}[\mu_q]$ with $q = 7, 11, 13, 17$ and 19, we only focus here on the case $q = 7$. The methods presented in this section could be generalized to the other cases but the resolution of the different equations becomes overly complicated.

In our setting, the ring of integers of K has $\mathbb{Z}$-basis $\lambda_i = 2\cos\frac{2i\pi}{7}$, for $i = 1, 2, 3$, and we sometimes write $[a_1, a_2, a_3]$ instead of $a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_3$.

6.5.3.1 Operations

The two main operations we need to perform are comparisons of elements and decomposition of rational primes. The latter is the starting point for gcd computations and finding the list of possible values for the norm (e, e) of a root e.

Comparisons of elements During the process of the algorithm, we often have to compare different elements of $\mathcal{O}_K$. This reduces to be able to decide whether a given $x = [a_1, a_2, a_3] \in \mathcal{O}_K$ is smaller than zero or not. Of course, we want to conclude without worrying about approximation errors which occur during the computation of $a_1 \cdot \cos\frac{2\pi}{7} + a_2 \cdot \cos\frac{4\pi}{7} + a_3 \cdot \cos\frac{6\pi}{7}$. One solution is to find rational numbers q_i, q_i', $1 \leq i \leq 3$, such that $q_i < \cos\frac{2\pi i}{7} < q'$. Hence, we want to find some rationals q, q' such that $q < x < q'$. If both q and q' are of the same sign, then we can decide whether x is smaller than zero or not. If they are of different sign, then we have to improve the precision of our rational approximations of the numbers $\cos\frac{2\pi i}{7}$ and do the test again.

Luckily for us, there exist libraries which implement *interval arithmetic*: every number is represented as an interval and all the operations take intervals as parameters and return an interval (hence, the final result is completely reliable). We chose to use the library `Gaol`[9] in our project. This library, written in C++ by Frederic Goualard, allows us to do all the operations we need: comparisons, arithmetic operations, trigonometric functions and square roots.

Decomposition of rational primes: first remarks Let $p \in \mathbb{P}$ be a rational prime and let $\pi \in \mathcal{O}_K$ be a prime above p. In order to find π, we could solve the equation $N(\pi) = p$, as we did with the Pell equation for a quadratic field (see Section 6.5.2.1). Although feasible (see [HS04], for example), this approach requires more advanced tools and is not suitable for an integration inside our computer program. Thus, it may be better to use Theorem 2.2.7 instead. To find the prime factors π_i of p, we proceed as follows:

1. We first factor the minimal polynomial T of $\lambda = 2\cos\frac{2\pi}{7}$ over $\mathbb{F}_p[x]$:

$$T(x) \equiv \prod_{i=1}^{g} T_i(x)^{e_i}, \quad (\text{mod } p).$$

[9] *Gaol: NOT Just Another Interval Library*, `http://gaol.sourceforge.net/`. Licence: GNU Library or Lesser General Public License version 2.0 (LGPLv2)

2. We choose π_i as a generator of the ideal $\langle p, T_i(\lambda)\rangle = p\mathcal{O}_K + T_i(\lambda)\mathcal{O}_K$ (recall that $\mathcal{O}_K$ is a PID).

Remark 6.5.5
We can use the `C++` library `PARI` to do the factorization of the minimal polynomial T over $\mathbb{F}_p[x]$.

Unfortunately, π_i is the gcd of p and $T_i(\lambda)$ in $\mathcal{O}_K$ but in order to compute the gcd we need the decomposition in primes (see Section 2.2.2). If the norm $N(T_i(\lambda))$ is a power of p, then $\texttt{gcd}(p, T_i(\lambda)) = T_i(\lambda)$ and we are done. If it is not the case, we can try to remove the others factors of the norm, as shown in the next example.

Example 6.5.6
Suppose we want to find the decomposition of 13. We have $T(x) = x^3+x^2-2x-1$ and we find the factorization

$$T(x) \equiv (x+5)(x+3)(x+6) \in \mathbb{F}_{13}[x],$$

and we see that $N(\lambda+5) = 13\cdot 7$. Therefore, if we know the prime decomposition of 7, we will be able to compute the gcd. Hence, we can construct a table of the decomposition of small rational prime numbers to help to find the decomposition of bigger primes.

Another possibility is to choose other lifts of $T_i(x)$ in $\mathbb{Z}[x]$, as shown in the next example.

Example 6.5.7
For $p = 19$, we find the factorization

$$T(x) \equiv (x+11)(x+22)(x+26) \in \mathbb{F}_{29}[x],$$

and we find the following values

$$N(\lambda+11) = 29\cdot 41, \quad N(\lambda+22) = 29\cdot 349, \quad N(\lambda+26) = 29\cdot 581.$$

Since the "residual factor" is always bigger than 29, we cannot use the same trick as in Example 6.5.6. However, if we consider the lift $x-7 \in \mathbb{Z}[x]$ of $x+22 \in \mathbb{F}_{29}[x]$, we see that $N(\lambda-7) = -29$, which means that $\lambda-7$ is a prime above 29, as desired. We will formalize this below.

Decomposition of rational primes: formalization To find the decomposition of a rational prime p, we first compute the factorization of the polynomial $T(x) = x^3+x^2-2x-1$ in $\mathbb{F}_p[x]$:

$$T(x) \equiv \prod_{i=1}^{g} T_i(x)^e \pmod{p},$$

where each T_i has degree f. We then have the following three possibilities:

$g = e = 1$, $f = 3$: p is prime in $\mathcal{O}_K$ (p is the inert).

$e = f = 1$, $g = 3$: p decomposes as a product of three different primes (p splits completely).

$g = f = 1$**,** $e = 3$**:** p is associate to π^3 for some prime π. Note that this ramified case happens if and only if $p = 7$, since the discriminant of the field is 49.

As explained above, we have to find a generator π_i of the ideal $\langle p, T_i(\lambda)\rangle$ if p splits completely or is ramified. If $N(T_i(\lambda)) = p$, then $\pi_i = T_i(\lambda)$. Otherwise, we have to cancel the other prime factors which appear in $T_i(\lambda)$. This can be made easier if we manage to get the norm as small as possible, which can be achieved by choosing a suitable lift of $T_i(x)$ in $\mathbb{Z}[x]$. Since $T_i(x)$ is of degree 1, we write $T_i(x) = x + a_i \in \mathbb{F}_p[x]$ with $0 \leq a_i < p$. For a lift $T_i(x) = x + a_i + k \cdot p \in \mathbb{Z}[x]$, we have

$$\begin{aligned} f(k) &:= N\big(\lambda + a_i + k \cdot p\big) \\ &= k^3p^3 + k^2 \cdot p^2(3a_i - 1) + k \cdot (3a_i^2p - 2p - 2a_ip) + 1 - 2a_i - a_i^2 + a_i^3. \end{aligned}$$

The zeroes of this polynomial are

$$\begin{aligned} k_1 &= -\frac{1}{3p}\left(3a_i - 1 + 2\sqrt{7}\cdot\cos\frac{\theta}{3}\right) \in \left[-1 - \frac{0.24698}{p}, -\frac{1.24698}{p}\right], \\ k_2 &= -\frac{1}{3q}\left(3a_i - 1 - \sqrt{7}\left(cos\frac{\theta}{3} - \sqrt{3}\cdot\sin\frac{\theta}{3}\right)\right) \in \left[-1 + \frac{1.44504}{p}, \frac{0.445042}{p}\right], \\ k_3 &= -\frac{1}{3q}\left(3a_i - 1 - \sqrt{7}\left(cos\frac{\theta}{3} + \sqrt{3}\cdot\sin\frac{\theta}{3}\right)\right) \in \left[-1 + \frac{2.80194}{p}, \frac{1.80194}{q}\right], \end{aligned}$$

where $\theta = \arctan(3\sqrt{3})$. In particular, the zeroes are between -2 and 1, so we can consider the elements $\lambda + a_i + k \cdot p$ with $k \in \{-2, -1, 0, 1\}$. With this technique, we can find the factorization of all rational primes below 10000. These factorizations, computed once and for all, are used by `AlVin` for the computations.

6.5.3.2 Possible values for (e, e)

In $K = \mathbb{Q}\big[\cos\frac{2\pi}{7}\big]$, the rational prime 2 is a prime element. Moreover, the group of positive units is generated by $-\lambda_3$ and $-\lambda_2 - \lambda_3$ (see Section 2.2.6). Therefore, if e is a root, the its norm (e, e) can be written as follows:

$$(e, e) = 2^{0,1} \cdot (-\lambda_3)^{0,1} \cdot (-\lambda_2 - \lambda_3)^{0,1} \cdot \prod_{i=1}^{r} \pi_i, \quad \pi_i \in \mathcal{P},$$

where $\mathcal{P}$ denotes the set of prime elements of $\mathcal{O}_K$ which divide at least one coefficient of the quadratic form. As before, in order for (e, e) to be admissible, we must have $\sigma(e, e) > 0$ for the two non-trivial Galois embeddings.

6.5.3.3 Couples $(k_0, (e, e))$

Let $m < M$ be two positive integers. In order to parametrize the fractions $m < \frac{k_0^2}{\varepsilon} \leq M$, where $e = (k_0, \ldots, k_n)$ is a root such that $\varepsilon = (e, e)$ is admissible, we have to consider the following system (see Section 6.3.2)

$$\begin{cases} m < \frac{k_0^2}{\varepsilon} \leq M \\ |\sigma_i(k_0)| \leq \sqrt{\sigma_i\left(\frac{\varepsilon}{-\alpha_0}\right)}, \quad i = 2, 3. \end{cases}$$

We write $k_0 = a\lambda_1 + b\lambda_2 + c\lambda_3$ for some $a, b, c \in \mathbb{Z}$, $R_i = \sqrt{\sigma_i\left(\frac{\varepsilon}{-\alpha_0}\right)}$ and $m' = \sqrt{m\varepsilon}$, $M' = \sqrt{M\varepsilon}$ which gives the system

$$\begin{cases} m' < a\lambda_1 + b\lambda_2 + c\lambda_3 \leq M' & (6.7) \\ -R_2 \leq c\lambda_1 + a\lambda_2 + b\lambda_3 \leq R_2 & (6.8) \\ -R_3 \leq b\lambda_1 + c\lambda_2 + a\lambda_3 \leq R_3. & (6.9) \end{cases}$$

Taking the sum of the three equations and multiplying (6.8) by $\lambda_2\lambda_3$ and (6.9) by $\lambda_1\lambda_3$ leads to

$$\begin{cases} -R_2 - R_3 + m' < -a - b - c \leq R_2 + R_3 + M' \\ -R_2\lambda_2\lambda_3 \leq a \cdot \lambda_2^2\lambda_3 + b \cdot \lambda_2\lambda_3^2 + c \leq R_2\lambda_2\lambda_3 \\ R_3\lambda_1\lambda_3 \leq a \cdot \lambda_1\lambda_3^2 + b \cdot \lambda_1^2\lambda_3 + c \leq -R_3\lambda_1\lambda_3. \end{cases}$$

And thus

$$\begin{cases} -R_2 - R_3 + m' < -a - b - c \leq R_2 + R_3 + M' & (6.10) \\ \lambda_3 R_2 - R_3 + m' < a \cdot (\lambda_2^2\lambda_3 - 1) + b \cdot (\lambda_2\lambda_3^2 - 1) & \\ \qquad \leq -\lambda_3 R_2 + R_3 + M' & (6.11) \\ -R_2 + (\lambda_1\lambda_3 - 1)R_3 + m' < a \cdot (\lambda_1\lambda_3^2 - 1) + b \cdot (\lambda_1^2\lambda_3 - 1) & \\ \qquad \leq R_2 - (\lambda_1\lambda_3 - 1)R_3 + M'. & (6.12) \end{cases}$$

Multiplying equation (6.12) by $\frac{1-\lambda_2^2\lambda_3}{\lambda_1\lambda_3^2-1}$ and adding it to (6.11) yields the two inequalities

$$\begin{aligned} R_2(\lambda_2 + \lambda_3) + R_3(\lambda_2 - 2) + m'(1 - \lambda_2) &< b \cdot (3\lambda_2 + \lambda_3 - 1) \\ -R_2(\lambda_2 + \lambda_3) - R_3(\lambda_2 - 2) + M'(1 - \lambda_2) &\geq b \cdot (3\lambda_2 + \lambda_3 - 1). \end{aligned}$$

And finally

$$\begin{aligned} 7b &\geq R_2(2\lambda_1 + 2\lambda_2 + 3\lambda_3) + R_3(\lambda_1 + 4\lambda_2 + 2\lambda_3) + M'(2\lambda_1 + 3\lambda_2 + 2\lambda_3) \\ 7b &< -R_2(2\lambda_1 + 2\lambda_2 + 3\lambda_3) - R_3(\lambda_1 + 4\lambda_2 + 2\lambda_3) + m'(2\lambda_1 + 3\lambda_2 + 2\lambda_3). \end{aligned}$$

Using equation (6.11), we get

$$\begin{aligned} a &\geq (1 - \lambda_2^2\lambda_3)^{-1} \cdot \big(b \cdot (\lambda_2\lambda_3^2 - 1) + \lambda_3 R_2 - R_3 - M'\big) \\ a &< (1 - \lambda_2^2\lambda_3)^{-1} \cdot \big(b \cdot (\lambda_2\lambda_3^2 - 1) - \lambda_3 R_2 + R_3 - m'\big). \end{aligned}$$

Finally, we have

$$\frac{M' - a\lambda_1 - b\lambda_2}{\lambda_3} \leq c < \frac{m' - a\lambda_1 - b\lambda_2}{\lambda_3}.$$

6.5.3.4 Solving the norm equation

Suppose the first j components of a root $e = (k_0, \ldots, k_n)$ have been chosen. We write $k_j = a\lambda_1 + b\lambda_2 + c\lambda_3$ and $S_j = (e, e) + \alpha_0 k_0^2 - \sum_{i=1}^{j-1} \alpha_i k_i^2$. Bounds for k_j are obtained by solving the system (6.3) of Section 6.2.4. We write

$$R_i := \sqrt{\sigma_i\left(\frac{S_j}{\alpha_j}\right)}, \quad i = 1, 2, 3,$$

and we proceed as in Section 6.5.3.3 to get:

$$\begin{aligned}
7b &\geq R_2(2\lambda_1 + 2\lambda_2 + 3\lambda_3) + R_3(\lambda_1 + 4\lambda_2 + 2\lambda_3) + R_1(2\lambda_1 + 3\lambda_2 + 2\lambda_3)\\
7b &\leq -R_2(2\lambda_1 + 2\lambda_2 + 3\lambda_3) - R_3(\lambda_1 + 4\lambda_2 + 2\lambda_3)\\
a &\geq (1-\lambda_2^2\lambda_3)^{-1} \cdot \big(b \cdot (\lambda_2\lambda_3^2 - 1) + \lambda_3 R_2 - R_3 - R_1\big)\\
a &< (1-\lambda_2^2\lambda_3)^{-1} \cdot \big(b \cdot (\lambda_2\lambda_3^2 - 1) - \lambda_3 R_2 + R_3\big).
\end{aligned}$$

Finally, we have

$$\frac{R_1 - a\lambda_1 - b\lambda_2}{\lambda_3} \leq c < \frac{-a\lambda_1 - b\lambda_2}{\lambda_3} = \big[2a - 2b, -b, a - b\big].$$

Note that when R_1, R_2 and R_3 are integers, then the solutions reduce to the following:

$$\begin{aligned}
7b &\geq \big[2R_2 + R_3 + 2R_1, 2R_2 + 4R_3 + 3R_1, 3R_2 + 2R_3 + 2R_1\big]\\
7b &\leq -\big[2R_2 + R_3, 2R_2 + 4R_3, 3R_2 + 2R_3\big]\\
a &\geq \frac{1}{7} \cdot \big[2R_1 + 3R_2 + 2R_3, R_1 + 5R_2 + R_3, 7b + 4R_1 + 6R_2 + 4R_3\big]\\
a &\leq -\frac{1}{7} \cdot \big[3R_2 + 2R_3, 5R_2 + R_3, -7b + 6R_2 + 4R_3\big].
\end{aligned}$$

6.5.4 Non-reflectivity of a quadratic form

As explained in Section 6.2.6.2, one way to prove that a quadratic form is non-reflective is to exhibit integral symmetries of the fundamental polyhedron (i.e. symmetries of the polyhedron which preserve the quadratic form and the lattice spanned over $\mathcal{O}_K$ by the canonical basis) which generate a subgroup whose action on $\mathcal{H}^n$ has no fixed points. This method has been implemented into `AlVin` and can be used for a quadratic form of small dimension (see Section 6.8.1.2 and Appendix A.3 for some examples).

Suppose that the algorithm `AlVin` finds the roots $e_1, \dots, e_r$, and let $T \subset \{1, \dots, r\}$ and $\mathcal{G}_T$ be the Coxeter graph corresponding to the vectors $\{e_t\}_{t \in T}$. In order for an involution of the graph $\mathcal{G}_T$ to define an integral symmetry of the fundamental polyhedron, we need the following:

1. Since the set $\{e_t\}_{t\in T}$ has to contain a basis of the space K^{n+1}, and since Γ_T has to contain a maximal spherical or Euclidean subgraph, we must have $|T| \geq n$. Moreover, tests suggest that good sizes of T are $n+1 \leq |T| \leq n+3$. For this step, we use the library `Eigen` to extract $n+1$ linearly independent vectors from the list $\{e_t\}_{t\in T}$. The second condition is checked with `CoxIter`: the f-vector of the polyhedron associated to the group Γ_T is computed.
2. Using the library `igraph`, we can then find involutions of the graph $\mathcal{G}_T$. Since the isometry η of the space should preserve the quadratic form, two vertices of $\mathcal{G}_T$ can be swapped only if the corresponding vectors have the same norm. We use a colouring of the vertices to limit the number of false candidates[10].

[10] In fact, we assign to each vertex a weight equal to $\lfloor (e,e) \rfloor$, where e is its corresponding vector.

3. For each candidate η found in step 2., we check whether the corresponding map indeed preserves the products of the vectors as well as the lattice generated by the canonical basis or not. This last condition is satisfied if and only if the matrix of η with respect to the canonical basis has coefficients in $\mathcal{O}_K$. Again, computations are done using `Eigen`.

Once different transformations η are found (corresponding to different subsets T) we compute the space F of fixed points. If the dimension of F is zero, or if F is spanned by a single vector of positive norm or by two perpendicular vectors of positive norm, then the form is non-reflective. Otherwise, we continue looking for other involutions.

6.6 Using `AlVin`

The goal of this section is to provide examples illustrating the use of `AlVin`. We also explain how we can use it to help deciding whether a quadratic form is reflective or not. If the algorithm terminates for a given quadratic form, then the program displays the complete list of normal vectors, saves the Coxeter graph in a file and, if possible, creates an image to help visualize the graph.

The parameters which can be given to the program `AlVin` are the following:

qf quadratic form (mandatory)
The coefficients of the diagonal quadratic form, separated by commas.

k field of definition (optional)
It can take the following values: "Q[sqrt d]", where d is a positive square-free integer or "RC7", for the maximal real subfield of the cyclotomic field. If omitted, `AlVin` assumes that the quadratic form is defined over the rationals.

ip invariants of the polyhedron (optional)
If the algorithm terminates, the invariants of the polyhedron are computed (Euler characteristic, f-vector, number of vertices at infinity, growth series and growth rate).

iqf commensurability invariant of the quadratic form
If the quadratic form is defined over $\mathbb{Q}$, the program will compute the commensurability invariants of the group (see Section 4.3.1.3).

oformat output format
Can be used to write the normal vectors in another format (for example: "-oformat mathematica"). Possible values: mathematica, latex, pari.

maxv maximal number of vectors (optional)
If specified, the program will stop after finding a specific number of vector, regardless whether the polyhedron is of finite volume or not.

minv minimal number of vectors (optional)
If specified, the program will not test whether the polyhedron has finite volume as long as the number of computed vectors is smaller than the given value.

nr non-reflectivity (optional)
If given, then AlVin will try to determine if the quadratic form is non-reflective (see Section 6.6.5 for an example).

nrequations non-reflectivity (optional)
If given and if $K = \mathbb{Q}$, AlVin will try to print the system of equations which can be used to prove that the form is not reflective (see Section 6.2.6.1).

6.6.1 A first example over the rationals

As a first example, we apply the algorithm to the quadratic form $\langle -2, 3, 1, 1, 1 \rangle$. We call the program as follows:

```
./alvin -qf -2,3,1,1,1
```

and the output is then the following:

```
Quadratic form (4,1): -2, 1, 1, 1, 3
Field of definition: Q

Vectors:
    e1 = (0, -1, 1, 0, 0)
    e2 = (0, 0, -1, 1, 0)
    e3 = (0, 0, 0, -1, 0)
    e4 = (0, 0, 0, 0, -1)
    e5 = (1, 1, 0, 0, 1)
    e6 = (1, 2, 0, 0, 0)
    e7 = (1, 1, 1, 1, 0)
    e8 = (3, 3, 3, 0, 1)
Algorithm ended

Graph written in the file:
    ../output/4-2,1,1,1,3.coxiter

Computation time: 0.00305064s
```

The normal vectors of the 8 facets of the polyhedron are displayed. The name of the file containing the Coxeter graph (which can be given to CoxIter) is written. If the number of found vectors is smaller than 25, then an image containing the Coxeter graph is created.

The group for this quadratic form is the same as the one associated to the quadratic form $\langle -6, 1, 1, 1, 1 \rangle$ (for the Coxeter graph, see Figure 6.4 page 144).

6.6.2 A cocompact group defined over $\mathbb{Q}[\sqrt{2}]$

We consider now the quadratic form $\langle -5-4\sqrt{2}, 1, 1, 1, 1, 1 \rangle$ defined over $\mathbb{Q}[\sqrt{2}]$. Recall that an integer basis for the ring $\mathcal{O}_{\mathbb{Q}[\sqrt{d}]}$ is given by $\{1, T\}$, with $T = \sqrt{d}$ if $d \equiv 2, 3 \bmod 4$, and with $T = \frac{1+\sqrt{d}}{2}$ if $d \equiv 1 \bmod 4$. We call AlVin as follows (the last parameter makes the invariants of the fundamental polyhedron to be displayed):

```
./alvin -k=Q[sqrt 2] -qf -5-4T,1,1,1,1,1 -ip
```

The algorithm ends after finding the normal vectors of the 20 bounding hyperplanes. The end of the output is then

```
--------------------------------
Information about the polyhedron:
--------------------------------
Euler characteristic: 0
f-vector: (96, 240, 228, 102, 20, 1)
Number of vertices at infinity: 0

Growth series:
f(x) = C(2,2,2,2,2,3,3,4,4,5,6,6,8,10,12)/(1 - 15 * x + 12 *
     x^2 + 39 * x^4 + 3 * x^5 + 67 * x^6 + 27 * x^7 + 126 * x
    ^8 - 2 * x^9 + 152 * x^10 + 12 * x^11 + 185 * x^12 - 47 *
     x^13 + 159 * x^14 - 77 * x^15 + 152 * x^16 - 152 * x^17
    + 77 * x^18 - 159 * x^19 + 47 * x^20 - 185 * x^21 - 12 *
    x^22 - 152 * x^23 + 2 * x^24 - 126 * x^25 - 27 * x^26 -
    67 * x^27 - 3 * x^28 - 39 * x^29 - 12 * x^31 + 15 * x^32
    - x^33)

Growth rate: 14.137172610056932590629682183699363504
    Perron number: yes
    Pisot number: no
    Salem number: no

Computation time: 1.10686s
```

6.6.3 Quadratic forms over $\mathbb{Q}[\cos\frac{2\pi}{7}]$

An integral basis of $\mathcal{O}_{\mathbb{Q}[\cos\frac{2\pi}{7}]}$ is $\lambda_i = 2\cos\frac{2\pi\, i}{7}$, $i = 1, 2, 3$. An element x of this ring is given by specifying its three components into the basis λ_1, λ_2 and λ_3 into brackets, that is

$$x = [x_1, x_2, x_3] := x_1\lambda_1 + x_2\lambda_2 + x_3\lambda_3, \quad x_1, x_2, x_3 \in \mathbb{Z}.$$

For example, to execute the program with the quadratic form $\langle -2\cos\frac{2\pi}{7},\ 1, -4\cos\frac{2\pi}{7} - 10\cos\frac{4\pi}{7} - 14\cos\frac{6\pi}{7}\rangle$, we use

```
./alvin -k=RC7 -qf [-1,0,0],1,[-2,-5,-7]
```

and the output is given by

```
Quadratic form (2,1): [-1,0,0], 1, [-2,-5,-7]
Field of definition: RC7

Vectors:
        e1 = (0, -1, 0)
        e2 = (0, 0, -1)
        e3 = (1, [0,0,-1], 0)
        e4 = ([0,-1,-2], [0,0,-1], [1,0,0])
        e5 = ([-1,-2,-4], 0, [0,-1,-1])
Algorithm ended

Graph written in the file:
        output/2-[1,0,0],1,[-2,-5,-7].coxiter
```

```
Computation time: 1.0383s
```

The corresponding Coxeter graph is shown in Figure 6.3.

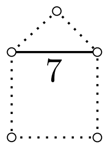


Figure 6.3 – Maximal reflection groups associated to the quadratic form $\langle -2\cos\frac{2\pi}{7}, 1, -4\cos\frac{2\pi}{7} - 10\cos\frac{4\pi}{7} - 14\cos\frac{6\pi}{7}\rangle$

6.6.4 Non-reflectivity of the form $\langle -1, 1, \ldots, 1, 3\rangle$ for $n = 11$

Using AlVin, we see that the quadratic form $\langle -1, 1, \ldots, 1, 3\rangle$ is reflective for $n \leq 10$. The corresponding polyhedra are different from the ones found by McLeod for the form $\langle -3, 1 \ldots, 1\rangle$ but they are commensurable when $n = 3, 5, 7$ and 9 (see Section 4.4.2). The ranks of the groups Γ_f^n are the following:

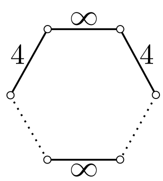
n	2	3	4	5	6	7	8	9	10
rank	4	6	7	8	9	10	12	13	16

For $n = 11$, the associated polyhedron is not of finite volume when the first 25 vectors are found, which indicates that the form may not be reflective. We launch again the program with the following command:

```
./alvin -qf -1,3,1,1,1,1,1,1,1,1,1,1 -maxv 14 -nrequations
```

The parameter "-maxv 14" makes the program stop after the first 14 vectors while the "-nrequations" option specifies that AlVin will try to find a Euclidean subgraph which cannot be extended and derive the system of equations from it (see Proposition 6.3.1 and Section 6.2.6.1). At the end of the computations, the program outputs two systems of equations. Unfortunately, both of them have solutions, which means that we cannot decide about the non-reflectivity of the quadratic form. We start again but with the first 15 vectors and this time, among the three systems of equations, one of them has no solution. Therefore, the form is not reflective. It follows that it is not reflective for every $n \geq 11$ (see Proposition 3.9.5).

Remark 6.6.1
For $n = 3$, the group Γ corresponding to the quadratic form $\langle -1, 1, 1, 3\rangle$ is the following:



Using [Kel91, Theorem 14.6], we can compute

$$\operatorname{covol}\Gamma = Л\left(\frac{\pi}{6}\right) + \frac{1}{2}Л\left(\frac{\pi}{12}\right) + \frac{1}{2}Л\left(\frac{7\pi}{12}\right) \simeq 0.634338504006034,$$

where Л is the *Lobachevsky function*, defined by

$$Л(x) = -\int_0^x \log|2\sin t|\mathrm{d}t, \quad x \in \mathbb{R}.$$

According to Scharlau's classification, our group Γ is a subgroup of one of the following groups in his list: Γ_1, Γ_6, Γ_{27}, Γ_{38}, Γ_{44}, Γ_{47}, Γ_{48} (see [Sch89, p. 21]). The group Γ_1, corresponds to the group of units of the form $\langle -3, 1, 1, 1\rangle$ as shown by McLeod. It preserves a lattice of the same discriminant as the lattice preserved by Γ, which suggest that $\Gamma \leq \Gamma_1$. Moreover, using [RT13], we see that

$$\operatorname{covol}\Gamma_1 = \frac{5\sqrt{3}}{64} \cdot L(2, -3) \simeq 0.105723084010027,$$

where $L(s, D)$ is the Dirichlet L-series given by

$$L(s, d) = \sum_{k=1}^{\infty} \left(\frac{D}{k}\right) k^{-s}, \quad s > 1,$$

where $\left(\frac{D}{k}\right)$ is a Kronecker symbol. Numerical computations show that the quotient $\operatorname{covol}\Gamma / \operatorname{covol}\Gamma_1$ is equal to 6, up to machine precision. This strongly indicates that Γ is a subgroup of index 6 in Γ_1.

6.6.5 Non-reflectivity of the form $\langle -1, 1, 1, 13\rangle$

The quadratic form $\langle -1, 1, 13\rangle$ is reflective and the rank of the associated reflection group is 8. When running the algorithm with $\langle -1, 1, 1, 13\rangle$ we see that the components of the vectors grow quickly and that more of 20 vectors are computed without the polyhedron $\bigcap_i H_{e_i}^-$ becoming of finite volume. The method used for the example of Section 6.6.4 does not seem to work here: for a few numbers of vectors, we see that among the Euclidean subgraphs, there is none which cannot we extended to a Euclidean graph of rank 2. We will thus see that the form is not reflective by showing that the fundamental polyhedron P has an infinite number of symmetries.

We again call the program as follows:

```
./alvin -qf -1,1,1,13 -maxv 15 -nr[5,6]
```

The option "-maxv 15" is to limit the search to the first 15 vectors. The last option indicates that `AlVin` will try to find integral symmetries of the fundamental polyhedron by looking at involutions of subgraphs of sizes 5 and 6 (see Section 6.2.6.2). The output is then the following:

```
Quadratic form (3,1): -1, 1, 1, 13
Field of definition: Q

Vectors:
    e1 = (0, -1, 1, 0)
```

```
    e2 = (0, 0, -1, 0)
    e3 = (0, 0, 0, -1)
    e4 = (1, 1, 1, 0)
    e5 = (4, 2, 1, 1)
    e6 = (13, 13, 0, 1)
    e7 = (6, 5, 0, 1)
    e8 = (11, 2, 1, 3)
    e9 = (18, 1, 0, 5)
    e10 = (52, 13, 0, 14)
    e11 = (91, 13, 0, 25)
    e12 = (34, 25, 8, 6)
    e13 = (143, 104, 39, 25)
    e14 = (72, 55, 17, 12)
    e15 = (312, 234, 78, 53)

The algorithm did not terminate; the polyhedron may be of
    infinite volume

Checking if the form is non-reflective...
    The form is non-reflective
```

If we add the parameter "-debug" to get more information, we get at the end the following additional data:

```
Checking if the form is non-reflective...
    The form is non-reflective
    List of used involutions:
        e2 <-> e4, e3 <-> e6, e5
        e1 <-> e5, e2, e3 <-> e10
```

It means that the two given permutations of vectors extend to two integral symmetries of the polyhedron which don't have common fixed points inside $\mathcal{H}^3$. Hence, the form is non-reflective.

Remark 6.6.2
When trying to exploit this method to show that a quadratic form is not reflective, there are two parameters:

1. the bounds for the size of the subgraphs;
2. the number "-maxv" of vectors to compute.

It is advised to start with small parameters to avoid long computations and then gradually increase them. Usually, the bounds for the size of the subgraphs can be taken to be $[n+1, n+2]$, $[n+1, n+3]$ or $[n+1, n+4]$. If `AlVin` cannot conclude (this often happens when the roots can have many possible norms) but the quadratic form is thought to be non-reflective, one can increase the number of "maxv" vectors by small steps.

6.6.6 Running time for some well-known examples

The following table gives some typical running time of `AlVin`. In some cases, a large part of the computation time is allocated to check that the final polyhedron of the sequence has indeed finite volume.

n	K	form	time (s)
18	$\mathbb{Q}$	$\langle -1, 1, \dots, 1\rangle$	170
14	$\mathbb{Q}$	$\langle -2, 1, \dots, 1\rangle$	0.3
13	$\mathbb{Q}$	$\langle -3, 1, \dots, 1\rangle$	0.2
6	$\mathbb{Q}[\sqrt{2}]$	$\langle -1-\Theta_2, 1, \dots, 1\rangle$	16.8
7	$\mathbb{Q}[\sqrt{5}]$	$-\Theta_5, 1, \dots, 1$	0.1
4	$\mathbb{Q}\left[\cos\frac{2\pi}{7}\right]$	$\langle -2\cos\frac{2\pi}{7}, 1, 1, 1, 1\rangle$	9.5

6.7 Program testing

Consider the quadratic form $f_\alpha^n = \langle -\alpha, 1, \dots, 1\rangle$ of signature $(n, 1)$. We tested our program for the following quadratic form:

α	n	Reference
1	$2 \to 19$	[Vin72],[KV78]
2	$2 \to 14$	[Vin72]
3	$2 \to 13$	[Mcl11]
5	$2 \to 8$	[Mar12]
$\frac{1+\sqrt{5}}{2}$	$2 \to 7$	[Bug84]
$1+\sqrt{2}$	$2 \to 6$	[Bug90]
$2\cos\frac{2\pi}{7}$	$2 \to 4$	[Bug92]

The two-dimensional forms without square factors of [Gro08] were also compared. This corresponds to the following quadratic forms:

- $\langle -1, 1, \alpha\rangle$ for $\alpha \in \{1, 2, 3, 5, 6, 7, 10, 11, 13, 14\}$;
- $\langle -1, \alpha, \alpha\rangle$ for $\alpha \in \{1, 2, 3, 5, 6, 7, 10, 11\}$.

6.8 Applications of the algorithm

In this section, we present numerous applications of the Vinberg algorithm, including new polyhedra and a classification of diagonal quadratic forms of signature $(3, 1)$ with small coefficients. The applications are distributed in three parts, according to the defining field of their quadratic form.

6.8.1 Rational integers

6.8.1.1 Scharlau's quadratic forms

In [Sch89], Scharlau classified the 49 reflective quadratic forms f of signature $(3, 1)$ such that the group Γ_f are non-cocompact. For some of these quadratic forms, we were able to find a diagonal quadratic form with the same group of units. We list them in the following table, together with some properties of the associated Coxeter polyhedra:

- "No." is the number in Scharlau's list
- "v, e, f" corresponds to the number of vertices, edges and facets.
- "cu." (cusp) denotes the number of vertices at infinity.
- The last columns gives the numbers of faces with 3 vertices, 4 vertices, etc. For example, "4, 1" indicate that the associated polyhedron has 4 triangular facets and 1 facet which is a 4-gon.

No.	Quad. form	δ	v, e, f	cu.	facets with x vertices
1	$\langle -3, 1, 1, 1\rangle$	-3	$4, 6, 4$	1	4
2	$\langle -1, 1, 1, 1\rangle$	-1	$4, 6, 4$	1	4
4	$\langle -1, 1, 1, 2\rangle$	-2	$5, 8, 5$	1	$4, 1$
11	$\langle -1, 1, 1, 5\rangle$	-5	$7, 11, 6$	1	$2, 4$
12	$\langle -1, 1, 2, 3\rangle$	-6	$7, 11, 6$	1	$2, 4$
13	$\langle -1, 1, 1, 6\rangle$	-6	$7, 11, 6$	1	$3, 2, 1$
14	$\langle -1, 1, 1, 7\rangle$	-7	$10, 16, 8$	2	$4, 0, 4$
16	$\langle -1, 1, 3, 3\rangle$	-1	$6, 9, 5$	1	$2, 3$
18	$\langle -1, 1, 2, 5\rangle$	-10	$12, 19, 9$	2	$4, 2, 0, 3$
19	$\langle -1, 1, 1, 10\rangle$	-10	$9, 14, 7$	1	$1, 5, 1$
20	$\langle -11, 1, 1, 1\rangle$	-11	$9, 14, 7$	1	$2, 3, 2$
22	$\langle -1, 1, 1, 14\rangle$	-14	$12, 19, 9$	2	$3, 2, 3, 1$
23	$\langle -1, 1, 1, 15\rangle$	-15	$14, 22, 10$	2	$2, 4, 2, 2$
25	$\langle -1, 1, 1, 17\rangle$	-17	$19, 30, 13$	3	$6, 2, 0, 2, 2, 1$
26	$\langle -1, 1, 3, 6\rangle$	-2	$10, 16, 8$	2	$2, 4, 2$
29	$\langle -21, 1, 1, 1\rangle$	-21	$11, 17, 8$	1	$2, 3, 2, 1$
30	$\langle -1, 1, 3, 7\rangle$	-21	$17, 26, 11$	1	$0, 4, 6, 1$
31	$\langle -1, 1, 5, 5\rangle$	-1	$8, 12, 6$	2	$0, 6$
32	$\langle -1, 1, 2, 15\rangle$	-30	$17, 26, 11$	1	$3, 2, 3, 2, 0, 1$

Continued on next page

No.	Quad. form	δ	v, e, f	cu.	facets with x vertices
33	$\langle -1, 1, 1, 30\rangle$	-30	$22, 34, 14$	2	$2, 4, 2, 6$
34	$\langle -1, 1, 5, 6\rangle$	-30	$20, 31, 13$	2	$2, 6, 2, 0, 2, 1$
35	$\langle -1, 1, 3, 10\rangle$	-30	$22, 34, 14$	2	$0, 8, 2, 2, 2$
37	$\langle -35, 1, 1, 1\rangle$	-35	$17, 26, 11$	1	$0, 4, 6, 1$
39	$\langle -1, 1, 6, 7\rangle$	-42	$30, 46, 18$	2	$4, 2, 6, 0, 6$
40	$\langle -1, 1, 3, 14\rangle$	-42	$30, 46, 18$	2	$0, 6, 8, 1, 2, 1$
41	$\langle -1, 1, 3, 15\rangle$	-5	$23, 36, 15$	3	$2, 6, 0, 7$
42	$\langle -1, 1, 7, 7\rangle$	-1	$14, 21, 9$	2	$0, 3, 6$
43	$\langle -1, 1, 5, 10\rangle$	-2	$20, 31, 13$	2	$0, 8, 0, 5$
45	$\langle -1, 1, 6, 15\rangle$	-10	$34, 52, 20$	2	$4, 2, 6, 6, 0, 0, 2$
46	$\langle -1, 1, 3, 30\rangle$	-10	$34, 52, 20$	2	$0, 10, 4, 3, 2, 0, 0, 0, 0, 1$
49	$\langle -1, 1, 15, 15\rangle$	-1	$32, 48, 18$	4	$0, 4, 4, 10$

6.8.1.2 Classification of diagonal quadratic forms of signature $(3, 1)$ with small coefficients

We consider diagonal quadratic forms $\langle -\alpha_0, \alpha_1, \alpha_2, \alpha_3\rangle$ where $\alpha_i \in \{1,\ 2,\ 3,\ 5,\ 6,\ 7,\ 10\}$. Up to the ordering of the coefficients α_1, α_2 and α_3, there are 543 of them. For each such quadratic form, we test whether it is reflective or not and we classify the associated maximal subgroup generated by reflections up to commensurability and isomorphism.

Among the 543 forms, 248 of them are non-reflective and 295 are reflective. The latter ones give rise to groups which fall into 20 commensurability classes and 49 isomorphism classes. In the tables below, we omit the form brackets $\langle\ \rangle$ for brevity.

Non-reflective forms The 248 non-reflective are the following:

$-10, 1, 1, 7$	$-10, 1, 2, 5$	$-10, 1, 2, 7$	$-10, 1, 3, 6$	$-10, 1, 3, 7$
$-10, 1, 5, 7$	$-10, 1, 6, 7$	$-10, 1, 6, 10$	$-10, 1, 7, 7$	$-10, 1, 7, 10$
$-10, 2, 2, 7$	$-10, 2, 3, 7$	$-10, 2, 3, 10$	$-10, 2, 5, 7$	$-10, 2, 6, 7$
$-10, 2, 7, 7$	$-10, 2, 7, 10$	$-10, 3, 3, 7$	$-10, 3, 3, 10$	$-10, 3, 5, 6$
$-10, 3, 5, 7$	$-10, 3, 6, 7$	$-10, 3, 6, 10$	$-10, 3, 7, 7$	$-10, 3, 7, 10$
$-10, 5, 5, 7$	$-10, 5, 6, 7$	$-10, 5, 7, 7$	$-10, 5, 7, 10$	$-10, 6, 6, 7$
$-10, 6, 7, 7$	$-10, 6, 7, 10$	$-10, 7, 7, 7$	$-10, 7, 7, 10$	$-10, 7, 10, 10$
$-7, 1, 1, 2$	$-7, 1, 1, 3$	$-7, 1, 1, 5$	$-7, 1, 1, 6$	$-7, 1, 1, 10$

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$-7, 1, 2, 2$	$-7, 1, 2, 3$	$-7, 1, 2, 5$	$-7, 1, 2, 6$	$-7, 1, 2, 7$
$-7, 1, 2, 10$	$-7, 1, 3, 3$	$-7, 1, 3, 5$	$-7, 1, 3, 6$	$-7, 1, 3, 7$
$-7, 1, 3, 10$	$-7, 1, 5, 5$	$-7, 1, 5, 6$	$-7, 1, 5, 7$	$-7, 1, 5, 10$
$-7, 1, 6, 6$	$-7, 1, 6, 7$	$-7, 1, 6, 10$	$-7, 1, 7, 10$	$-7, 1, 10, 10$
$-7, 2, 2, 2$	$-7, 2, 2, 3$	$-7, 2, 2, 5$	$-7, 2, 2, 6$	$-7, 2, 2, 7$
$-7, 2, 2, 10$	$-7, 2, 3, 5$	$-7, 2, 3, 6$	$-7, 2, 3, 7$	$-7, 2, 3, 10$
$-7, 2, 5, 5$	$-7, 2, 5, 6$	$-7, 2, 5, 7$	$-7, 2, 5, 10$	$-7, 2, 6, 7$
$-7, 2, 6, 10$	$-7, 2, 7, 10$	$-7, 2, 10, 10$	$-7, 3, 3, 5$	$-7, 3, 3, 7$
$-7, 3, 3, 10$	$-7, 3, 5, 5$	$-7, 3, 5, 6$	$-7, 3, 5, 7$	$-7, 3, 5, 10$
$-7, 3, 6, 6$	$-7, 3, 6, 7$	$-7, 3, 6, 10$	$-7, 3, 7, 7$	$-7, 3, 7, 10$
$-7, 3, 10, 10$	$-7, 5, 5, 6$	$-7, 5, 5, 7$	$-7, 5, 5, 10$	$-7, 5, 6, 6$
$-7, 5, 6, 7$	$-7, 5, 6, 10$	$-7, 5, 7, 7$	$-7, 5, 7, 10$	$-7, 5, 10, 10$
$-7, 6, 6, 7$	$-7, 6, 6, 10$	$-7, 6, 7, 7$	$-7, 6, 7, 10$	$-7, 6, 10, 10$
$-7, 7, 7, 10$	$-7, 7, 10, 10$	$-7, 10, 10, 10$	$-6, 1, 2, 7$	$-6, 1, 5, 5$
$-6, 1, 5, 6$	$-6, 1, 5, 7$	$-6, 1, 6, 7$	$-6, 1, 7, 7$	$-6, 1, 7, 10$
$-6, 1, 10, 10$	$-6, 2, 3, 7$	$-6, 2, 5, 7$	$-6, 2, 6, 7$	$-6, 2, 7, 7$
$-6, 2, 7, 10$	$-6, 3, 3, 7$	$-6, 3, 5, 7$	$-6, 3, 5, 10$	$-6, 3, 6, 7$
$-6, 3, 7, 7$	$-6, 3, 7, 10$	$-6, 5, 5, 6$	$-6, 5, 5, 7$	$-6, 5, 6, 7$
$-6, 5, 6, 10$	$-6, 5, 7, 7$	$-6, 5, 7, 10$	$-6, 6, 6, 7$	$-6, 6, 7, 7$
$-6, 6, 7, 10$	$-6, 7, 7, 10$	$-6, 7, 10, 10$	$-5, 1, 1, 7$	$-5, 1, 2, 7$
$-5, 1, 2, 10$	$-5, 1, 3, 7$	$-5, 1, 5, 6$	$-5, 1, 5, 7$	$-5, 1, 6, 7$
$-5, 1, 7, 7$	$-5, 1, 7, 10$	$-5, 2, 2, 7$	$-5, 2, 3, 5$	$-5, 2, 3, 6$
$-5, 2, 3, 7$	$-5, 2, 5, 7$	$-5, 2, 6, 7$	$-5, 2, 7, 7$	$-5, 2, 7, 10$
$-5, 3, 5, 6$	$-5, 3, 5, 7$	$-5, 3, 6, 7$	$-5, 3, 6, 10$	$-5, 3, 7, 7$
$-5, 3, 7, 10$	$-5, 5, 5, 7$	$-5, 5, 6, 6$	$-5, 5, 6, 7$	$-5, 5, 7, 7$
$-5, 5, 7, 10$	$-5, 6, 6, 7$	$-5, 6, 7, 7$	$-5, 6, 7, 10$	$-5, 7, 7, 10$
$-5, 7, 10, 10$	$-3, 1, 3, 7$	$-3, 1, 5, 7$	$-3, 1, 7, 7$	$-3, 1, 7, 10$
$-3, 2, 2, 7$	$-3, 2, 3, 7$	$-3, 2, 3, 10$	$-3, 2, 5, 5$	$-3, 2, 5, 7$
$-3, 2, 6, 7$	$-3, 2, 7, 7$	$-3, 2, 7, 10$	$-3, 2, 10, 10$	$-3, 3, 3, 7$
$-3, 3, 5, 7$	$-3, 3, 5, 10$	$-3, 3, 6, 7$	$-3, 3, 7, 7$	$-3, 3, 7, 10$
$-3, 3, 10, 10$	$-3, 5, 5, 7$	$-3, 5, 6, 7$	$-3, 5, 6, 10$	$-3, 5, 7, 7$
$-3, 5, 7, 10$	$-3, 6, 6, 7$	$-3, 6, 7, 7$	$-3, 6, 7, 10$	$-3, 7, 7, 10$
$-3, 7, 10, 10$	$-2, 1, 5, 7$	$-2, 1, 5, 10$	$-2, 1, 6, 7$	$-2, 1, 7, 7$
$-2, 1, 7, 10$	$-2, 2, 3, 7$	$-2, 2, 5, 7$	$-2, 2, 7, 7$	$-2, 2, 7, 10$
$-2, 3, 5, 5$	$-2, 3, 5, 6$	$-2, 3, 5, 7$	$-2, 3, 6, 7$	$-2, 3, 7, 7$

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$-2,3,7,10$	$-2,3,10,10$	$-2,5,5,7$	$-2,5,6,7$	$-2,5,7,7$
$-2,5,7,10$	$-2,6,7,7$	$-2,6,7,10$	$-2,7,7,7$	$-2,7,7,10$
$-2,7,10,10$	$-1,1,5,7$	$-1,1,7,10$	$-1,2,5,7$	$-1,2,5,10$
$-1,2,6,7$	$-1,2,7,7$	$-1,2,7,10$	$-1,3,5,7$	$-1,3,6,7$
$-1,3,6,10$	$-1,3,7,10$	$-1,5,5,6$	$-1,5,5,7$	$-1,5,6,7$
$-1,5,7,7$	$-1,5,7,10$	$-1,6,6,7$	$-1,6,7,7$	$-1,6,7,10$
$-1,6,10,10$	$-1,7,7,10$	$-1,7,10,10$		

Reflective forms The reflective forms are presented by their commensurability invariants and isomorphism classes. The rank of the groups ranges between 4 and 22. The following table presents the number of groups with a given rank.

4	5	6	7	8	9	10	11	13	14	15	18	19	20	22
5	8	58	12	15	30	19	14	22	48	1	19	8	20	16

The classification of the reflective forms is presented in the following tables.

$\{\mathbb{Q},-1,\{\}\}$	$-1,1,1,1$			
	$-2,1,1,2$	$-1,1,2,2$		
	$-3,1,1,3$	$-1,1,3,3$		
	$-5,1,1,5$	$-1,1,5,5$		
	$-7,1,1,7$	$-1,1,7,7$		
	$-5,3,3,5$	$-3,3,5,5$		
	$-10,1,1,10$	$-5,2,2,5$	$-2,2,5,5$	$-1,1,10,10$
	$-6,1,1,6$	$-6,1,2,3$	$-3,1,2,6$	$-3,2,2,3$
	$-2,1,3,6$	$-2,2,3,3$	$-1,1,6,6$	$-1,2,3,6$
$\{\mathbb{Q},-10,\{\}\}$	$-10,1,1,1$	$-10,1,10,10$	$-5,2,2,2$	$-5,2,5,5$
	$-2,2,2,5$	$-2,5,5,5$	$-1,1,1,10$	$-1,10,10,10$
	$-10,1,3,3$	$-10,2,3,6$	$-6,1,6,10$	$-5,1,3,6$
	$-5,2,6,6$	$-3,2,3,5$	$-2,3,6,10$	$-2,5,6,6$
	$-1,3,3,10$	$-1,3,5,6$		
	$-10,1,6,6$	$-6,1,3,5$	$-6,2,3,10$	$-6,2,5,6$
	$-5,2,3,3$	$-3,1,3,10$	$-3,1,5,6$	$-3,2,6,10$
	$-2,3,3,5$	$-1,6,6,10$		
	$-10,1,2,2$	$-10,1,5,5$	$-10,2,5,10$	$-5,1,1,2$
	$-5,1,5,10$	$-5,2,10,10$	$-2,1,1,5$	$-2,1,2,10$

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	$-2,5,10,10$	$-1,1,2,5$	$-1,2,2,10$	$-1,5,5,10$
$\{\mathbb{Q},-14,\{3\}\}$	$-7,2,3,3$	$-7,2,6,6$	$-6,1,3,7$	$-3,1,6,7$
	$-2,3,3,7$	$-2,6,6,7$		
$\{\mathbb{Q},-14,\{\}\}$	$-7,2,7,7$	$-2,1,1,7$	$-2,2,2,7$	$-1,1,2,7$
$\{\mathbb{Q},-15,\{2\}\}$	$-3,1,1,5$			
	$-5,3,3,3$	$-3,5,5,5$		
	$-10,1,2,3$	$-10,3,3,6$	$-6,1,1,10$	$-6,5,5,10$
	$-5,1,2,6$	$-5,3,6,6$	$-3,2,2,5$	$-3,5,10,10$
	$-2,1,3,10$	$-2,1,5,6$	$-1,2,3,10$	$-1,2,5,6$
$\{\mathbb{Q},-15,\{\}\}$	$-10,1,1,6$	$-10,3,5,10$	$-10,5,5,6$	$-6,1,2,5$
	$-6,3,3,10$	$-6,3,5,6$	$-5,1,1,3$	$-5,2,2,3$
	$-5,3,5,5$	$-5,3,10,10$	$-5,5,6,10$	$-3,1,2,10$
	$-3,3,3,5$	$-3,3,6,10$	$-3,5,6,6$	$-2,2,3,5$
	$-1,1,3,5$	$-1,1,6,10$		
$\{\mathbb{Q},-2,\{3\}\}$	$-6,1,1,3$	$-6,2,2,3$	$-3,1,1,6$	$-3,2,2,6$
	$-2,1,3,3$	$-2,1,6,6$	$-1,2,3,3$	$-1,2,6,6$
	$-10,3,3,5$	$-10,5,6,6$	$-6,3,5,5$	$-6,3,10,10$
	$-5,3,3,10$	$-5,6,6,10$	$-3,5,5,6$	$-3,6,10,10$
$\{\mathbb{Q},-2,\{\}\}$	$-2,1,1,1$	$-2,1,2,2$	$-1,1,1,2$	$-1,2,2,2$
	$-6,1,2,6$	$-3,1,2,3$	$-2,2,3,6$	$-1,1,3,6$
	$-10,1,1,5$	$-10,1,2,10$	$-10,2,2,5$	$-5,1,1,10$
	$-5,1,2,5$	$-5,2,2,10$	$-2,1,5,5$	$-2,1,10,10$
	$-2,2,5,10$	$-1,1,5,10$	$-1,2,5,5$	$-1,2,10,10$
$\{\mathbb{Q},-21,\{\}\}$	$-1,1,3,7$			
	$-7,3,3,3$	$-3,1,1,7$	$-3,7,7,7$	
$\{\mathbb{Q},-3,\{7\}\}$	$-1,3,7,7$			
$\{\mathbb{Q},-3,\{\}\}$	$-3,1,5,5$			
	$-3,1,1,1$	$-1,3,3,3$		
	$-6,1,1,2$	$-6,1,3,6$	$-6,2,3,3$	$-3,1,2,2$
	$-3,1,3,3$	$-3,1,6,6$	$-3,2,3,6$	$-2,1,1,6$
	$-2,1,2,3$	$-2,3,3,6$	$-1,1,1,3$	$-1,1,2,6$
	$-1,2,2,3$	$-1,3,6,6$		
	$-10,1,3,10$	$-10,1,5,6$	$-10,2,3,5$	$-6,1,5,10$
	$-6,2,5,5$	$-5,1,3,5$	$-5,1,6,10$	$-5,2,3,10$
	$-5,2,5,6$	$-3,1,10,10$	$-3,2,5,10$	$-2,3,5,10$

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	$-2,5,5,6$	$-1,3,5,5$	$-1,3,10,10$	$-1,5,6,10$
$\{\mathbb{Q},-30,\{\}\}$	$-10,1,1,3$	$-10,3,6,6$	$-6,5,10,10$	$-5,2,2,6$
	$-5,3,3,6$	$-3,5,5,10$	$-2,2,5,6$	$-1,1,3,10$
	$-10,2,2,3$	$-10,3,3,3$	$-6,1,2,10$	$-6,5,5,5$
	$-5,1,1,6$	$-5,6,6,6$	$-3,1,2,5$	$-3,10,10,10$
	$-2,2,3,10$	$-1,1,5,6$		
	$-10,3,5,5$	$-10,5,6,10$	$-6,2,2,5$	$-6,3,3,5$
	$-6,3,6,10$	$-5,3,5,10$	$-5,6,10,10$	$-3,1,1,10$
	$-3,3,5,6$	$-3,6,6,10$		
	$-10,1,2,6$	$-10,3,10,10$	$-6,1,1,5$	$-6,5,6,6$
	$-5,1,2,3$	$-5,5,5,6$	$-3,2,2,10$	$-3,3,3,10$
	$-2,1,3,5$	$-2,1,6,10$	$-1,2,3,5$	$-1,2,6,10$
$\{\mathbb{Q},-35,\{3\}\}$	$-5,3,3,7$			
$\{\mathbb{Q},-35,\{\}\}$	$-7,5,5,5$	$-5,7,7,7$		
$\{\mathbb{Q},-42,\{\}\}$	$-7,3,3,6$	$-6,2,2,7$	$-3,1,2,7$	$-1,1,6,7$
	$-7,6,6,6$	$-6,1,1,7$	$-6,7,7,7$	$-2,1,3,7$
	$-2,2,6,7$	$-1,2,3,7$		
$\{\mathbb{Q},-5,\{3\}\}$	$-5,1,3,3$	$-1,3,3,5$		
	$-10,2,3,3$	$-6,1,3,10$	$-6,2,3,5$	$-5,1,6,6$
	$-3,1,6,10$	$-3,2,5,6$	$-2,3,3,10$	$-1,5,6,6$
$\{\mathbb{Q},-5,\{\}\}$	$-3,1,3,5$			
	$-5,1,1,1$	$-5,1,5,5$	$-1,1,1,5$	$-1,5,5,5$
	$-10,1,1,2$	$-10,1,5,10$	$-10,2,5,5$	$-5,1,2,2$
	$-5,1,10,10$	$-5,2,5,10$	$-2,1,1,10$	$-2,1,2,5$
	$-2,5,5,10$	$-1,1,2,10$	$-1,2,2,5$	$-1,5,10,10$
$\{\mathbb{Q},-6,\{\}\}$	$-6,1,2,2$	$-6,1,6,6$	$-3,1,1,2$	$-3,2,3,3$
	$-2,2,2,3$	$-2,3,6,6$	$-1,1,1,6$	$-1,3,3,6$
	$-10,1,3,5$	$-10,2,5,6$	$-6,2,5,10$	$-5,1,3,10$
	$-5,2,6,10$	$-3,1,5,10$	$-2,5,6,10$	$-1,3,5,10$
	$-6,1,1,1$	$-6,1,3,3$	$-6,2,3,6$	$-3,1,3,6$
	$-3,2,2,2$	$-3,2,6,6$	$-2,1,1,3$	$-2,1,2,6$
	$-2,3,3,3$	$-1,1,2,3$	$-1,2,2,6$	$-1,6,6,6$
$\{\mathbb{Q},-7,\{2\}\}$	$-1,3,3,7$			
	$-7,1,1,1$	$-1,7,7,7$		

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$\{\mathbb{Q}, -7, \{\}\}$	$-7, 1, 7, 7$	$-2, 1, 2, 7$	$-1, 1, 1, 7$	$-1, 2, 2, 7$

6.8.1.3 Reflectivity of the forms $\langle -d, 1, \ldots, 1\rangle$ and $\langle -1, d, 1, \ldots, 1\rangle$

Using the program `AlVin`, we investigate the reflectivity of the two quadratic forms $\langle -d, 1, \ldots, 1\rangle$ and $\langle -1, d, 1, \ldots, 1\rangle$, where $1 \leq d \leq 30$ is a square-free integer. The first collection of forms was already studied by McLeod in [Mcl11] but a systematic study of the quadratic forms $\langle -1, d, 1, \ldots, 1\rangle$ is new.

The table below provides the maximal dimension $n_{\max}$ (see Proposition 3.9.5) for which each of the quadratic form is reflective. The non-reflective forms marked with a $*$ have been tested with the method presented in Section 6.2.6.1. For the other forms, we used the method explained in Section 6.2.6.2 and the involutions used are given in Appendix A.3. A sign "?" in the column concerning the non-reflectivity indicates that we were not able to prove the non-reflectivity. The columns labelled "new" indicate which quadratic forms lead to previously unknown polyhedra.

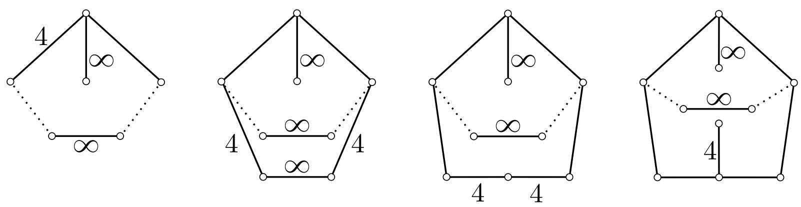
d	$\langle -d, 1, \ldots, 1\rangle$			$\langle -1, d, 1, \ldots, 1\rangle$		
	$n_{\max}$	non-ref	new	$n_{\max}$	non-ref	new
1	19	20$*$		19	20$*$	
2	14	15$*$		14	15$*$	
3	13	14$*$		10	11$*$	$3-10$
5	8	9$*$		8	9$*$	
6	9	10$*$	$4-9$	10	11$*$	$4-10$
7	3	4$*$		6	7$*$	$3-6$
10	6	7$*$	$3-6$	6	7$*$	$3-6$
11	4	5$*$		2	3	
13	2	3		2	3	
14	2	3		5	?	$3-5$
15	5	6$*$		5	?	$3-5$
17	3	4		3	4	
19	2	3		–	2	
21	4	5		2	3	
22	2	3		2	3	
23	2	3		–	2	
26	2	3		2	3	
29	–	2		–	2	
30	3	?	3	3	4	3

6.8.1.4 Some new Coxeter polyhedra

Some new Coxeter polyhedra were found by studying the quadratic forms $\langle -d, 1, \ldots, 1\rangle$ and $\langle -1, d, 1, \ldots, 1\rangle$ and when doing the classification of the quadratic forms of signature $(n, 1)$ with small coefficients. We present now some other of these polyhedra.

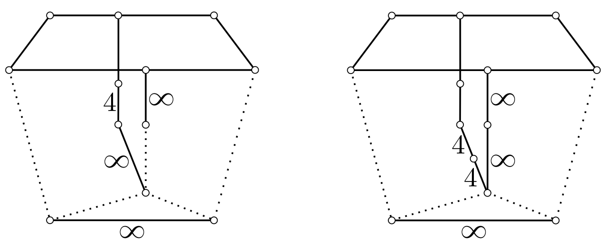
The form $\langle -6, 1, \ldots, 1\rangle$ The form $\langle -6, 1, \ldots, 1\rangle$ is reflective for $3 \leq n \leq 9$ and not reflective for $n = 10$. These new polyhedra, which have a symmetry group of order 2 when $n \geq 4$, are presented in figures 6.4, 6.5 and 6.6. The f-vectors of the polyhedra are the following:

n	f-vector
2	$(4, 4, 1)$
3	$(7, 11, 6, 1)$
4	$(13, 28, 23, 8, 1)$
5	$(19, 51, 57, 32, 9, 1)$
6	$(25, 80, 114, 90, 41, 10, 1)$
7	$(36, 135, 235, 240, 152, 58, 12, 1)$
8	$(46, 200, 411, 511, 413, 217, 71, 13, 1)$



$n = 3$ $n = 4$ $n = 5$ $n = 6$

Figure 6.4 – Maximal reflection groups associated to the quadratic form $\langle -6, 1, \ldots, 1\rangle$ for $n = 3, 4, 5, 6$



$n = 7$ $n = 8$

Figure 6.5 – Maximal reflection groups associated to the quadratic form $\langle -6, 1, \ldots, 1\rangle$ for $n = 7, 8$

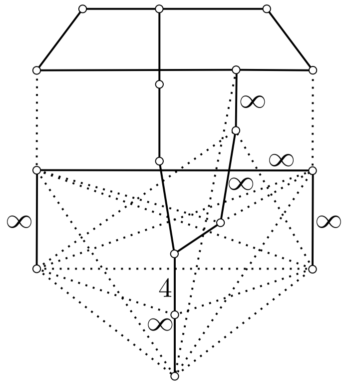


Figure 6.6 – Maximal reflection group associated to the quadratic form $\langle -6, 1, 1, \dots, 1\rangle$ of signature $(9, 1)$

6.8.1.5 Cocompact groups with $k = \mathbb{Q}$

The following quadratic forms are reflective and anisotropic, which implies that the associated polyhedra are compact (see Remark 3.9.2). For each form, we give the p-adic field $\mathbb{Q}_p$ over which the form is anisotropic, together with the f-vector of the associated Coxeter polyhedron in $\mathcal{H}^3$.

Quad. form	anis. over	$f-$vector
$\langle -1, 2, 3, 3\rangle$	$\mathbb{Q}_3$	$(8, 12, 6, 1)$
$\langle -1, 2, 6, 6\rangle$	$\mathbb{Q}_3$	$(8, 12, 6, 1)$
$\langle -7, 1, 1, 1\rangle$	$\mathbb{Q}_2$	$(6, 9, 5, 1)$
$\langle -1, 3, 3, 5\rangle$	$\mathbb{Q}_3$	$(10, 15, 7, 1)$
$\langle -15, 1, 1, 1\rangle$	$\mathbb{Q}_2$	$(8, 12, 6, 1)$
$\langle -6, 2, 2, 3\rangle$	$\mathbb{Q}_3$	$(8, 12, 6, 1)$

6.8.1.6 About the combinatorics

For a reflective quadratic form, it seems impossible to predict the f-vector $f(P)$ of the associated Coxeter polyhedron $P \subset \mathcal{H}^n$, or even the rank of the group Γ_f, which can be big when compared with the dimension n, as shown in Table 6.7.

6.8.1.7 Errors in literature

As mentioned above, Lemma 3.1.5 of [Mcl13] is true only if d is prime. This leads in particular to incorrect groups when $d = 6, 15, 30, 33, 39$. Moreover, the form $\langle -d, 1, \dots, 1\rangle$ is reflective at least until $n = 6$ when $d = 6$ and is reflective exactly when $n \leq 6$ for $d = 10$ (compare to [Mcl13, Table 3.1]).

n	Quad. form	cusps	invariant	$f(P)$
3	$\langle -1, 2, 2, 3\rangle$	1	$\{\mathbb{Q}, -3, \{\}\}$	$(7, 11, 6, 1)$
4	$\langle -1, 2, 2, 3, 3\rangle$	2	$\{\mathbb{Q}, \{3, \infty\}\}$	$(18, 38, 29, 9, 1)$
5	$\langle -1, 2, 2, 3, 3, 3\rangle$	10	$\{\mathbb{Q}, 3, \{\}\}$	$(66, 188, 193, 85, 16, 1)$
6	$\langle -1, 2, 2, 3, 3, 3, 3\rangle$	57	$\{\mathbb{Q}, \{2, \infty\}\}$	(653, 2286, 2896, 1660, 444, $47, 1)$
7	$\langle -1, 2, 2, 3, 3, 3, 3, 3\rangle$	328	$\{\mathbb{Q}, -3, \{\}\}$	(4536, 19136, 29816, 22364, $8684, 1654, 120, 1)$

Table 6.7 – The forms $\langle -1, 2, 2, 3, \ldots, 3\rangle$

6.8.2 Maximal real subfield of the 7-cyclotomic field

Unfortunately, it seems that our implementation of the algorithm for the field $K = \mathbb{Q}[\cos \frac{2\pi}{7}]$ does not lead to new polyhedra in dimensions above 3. One of the reason is that our method for solving the norm equation is not suitable when the degree of the number field increases.

We give here two quadratic forms of signature $(2, 1)$ which are reflective (recall that the integers between the brackets give the decomposition of the element in the standard $\mathbb{Z}$-basis, as indicated in Section 6.5.3):

- $\langle [-1, 0, 0], 1, [-2, -4, -4]\rangle$
- $\langle [-1, 0, 0], 1, [-2, -5, -7]\rangle$

6.8.3 Quadratic integers

6.8.3.1 Some reflective forms

We present some reflective quadratic forms over the first quadratic fields. It follows from equation (6.2) page 112 that the number of possibilities for the norm (e, e) of a root grows exponentially with the number of prime factors which divide one of the factor of the quadratic form. Therefore, we consider only quadratic forms with a few prime factors. As above, we denote by f_α the quadratic form $\langle -\alpha, 1, \ldots, 1\rangle$.

In what follows, we denote by Θ_d the generator of the ring of integers of $\mathbb{Q}[\sqrt{d}]$, that is, $\Theta_d = \sqrt{d}$ if $d \equiv 2, 3 \bmod 4$ and $\Theta_d = \frac{1+\sqrt{d}}{2}$ if $d \equiv 1 \bmod 4$.

Over $\mathbb{Q}[\sqrt{2}]$ The following quadratic forms are reflective:

- $\langle -\Theta_2, 3 + \Theta_2, 1, 1\rangle$
- $\langle -\Theta_2, 2 + \Theta_2, 1, 1, 1, 1\rangle$
- $\langle 1 - 2\Theta_2, 1, 1, 1, 1\rangle$
- $\langle 1 - 2\Theta_2, 3 + \Theta_2, 1\rangle$

Over $\mathbb{Q}[\sqrt{5}]$ The quadratic form $\langle -\Theta_5, 2+\Theta_5, 1, \dots, 1\rangle$ is reflective if and only if $n \leq 5$. In the following table, we present the commensurability invariants of this quadratic form together with the ones of the form $\langle -\Theta_5, 1, \dots, 1\rangle$. Note that since $1+\Theta_5$ is a square in $\mathbb{Q}[\sqrt{5}]$, then the invariants are the same and the groups are commensurable.

n	$\langle -\Theta_5, 2+\Theta_5, 1, \dots, 1\rangle$	$\langle -\Theta_5, 1, \dots, 1\rangle$
2	$\mathbb{Q}[\sqrt{5}], \{2\}$	$\mathbb{Q}[\sqrt{5}], \{\sqrt{5}\}$
3	$\mathbb{Q}[\sqrt{5}], -\Theta_5, \emptyset$	$\mathbb{Q}[\sqrt{5}], -\Theta_5(1+\Theta_5), \emptyset$
4	$\mathbb{Q}[\sqrt{5}], \emptyset$	$\mathbb{Q}[\sqrt{5}], \{2, \sqrt{5}\}$
5	$\mathbb{Q}[\sqrt{5}], \Theta_5, \emptyset$	$\mathbb{Q}[\sqrt{5}], \Theta_5(1+\Theta_5), \emptyset$

6.8.3.2 Quadratic Salem numbers

In [ERT15], Emery, Ratcliffe and Tschantz show that for every Salem number λ and every positive integer n, there exists an arithmetic group of the simplest type $\Gamma < \operatorname{Isom}\mathbb{H}^n$ containing an hyperbolic element whose translation length is equal to $\ln\lambda$. In this setting, the group Γ is defined over $\mathbb{Q}[\lambda+\lambda^{-1}]$. Moreover, they provide an interesting way to create admissible quadratic forms over a totally real number field K:

- Since K is a totally real number field, there exists a Pisot number α such that $K = \mathbb{Q}[\alpha]$.
- The solution of the equation $x + x^{-1} = 2\alpha$ which is bigger than 1 is a Salem number λ.
- The quadratic form $f_\alpha = \langle -(\lambda-\lambda^{-1})^2, 1 \dots, 1\rangle$ defined over K is admissible.

In the next table, we present the reflectivity of the form f_α for small quadratic Pisot numbers α. It turns out that among the groups of units of these quadratic forms, we find the groups of Bugaenko (see [Bug84] and [Bug90]) as well as some new compact polyhedra in dimensions 3, 4 and 5 (the last columns indicates the dimension of the new groups).

α	λ	quad. form	ref	reflective	new
$\frac{1+\sqrt{5}}{2}$	$2(1+\sqrt{5})$	$\langle -\Theta_5, 1 \dots, 1\rangle$	[Bug84]	$n \leq 7$	
$1+\sqrt{2}$	$8(1+\sqrt{2})$	$\langle -1-\Theta_2, 1 \dots, 1\rangle$	[Bug90]	$n \leq 6$	
$\frac{3+\sqrt{5}}{2}$	$10+6\sqrt{5}$	$\langle -1-3\Theta_5, 1, \dots, 1\rangle$		$n \leq 5$	5
$1+\sqrt{3}$	$12+8\sqrt{3}$	$\langle -3-2\Theta_3, 1 \dots, 1\rangle$		$n \leq 4$	4
$\frac{3+\sqrt{13}}{2}$	$6(3+\sqrt{13})$	$\langle -3-3\Theta_{13}, 1 \dots, 1\rangle$		$n \leq 2$	
$2+\sqrt{2}$	$4(5+4\sqrt{2})$	$\langle -5-4\Theta_2, 1 \dots, 1\rangle$		$n \leq 5$	$4, 5$

Continued on next page

α	λ	quad. form	ref	reflective	new
$\frac{3+\sqrt{17}}{2}$	$22+6\sqrt{17}$	$\langle 2-\Theta_{17}, 1, \dots, 1\rangle$		$n \leq 3$	3
$2+\sqrt{3}$	$8(3+2\sqrt{3})$	$\langle -\Theta_3, 1, \dots, 1\rangle$		$n \leq 3$	3
$\frac{3+\sqrt{21}}{2}$	$26+6\sqrt{21}$	$\langle -5-3\Theta_{21}, 1, \dots, 1\rangle$		?	
$2+\sqrt{5}$	$16(2+\sqrt{5})$	$\langle -1-2\Theta_5, 1 \dots, 1\rangle$	[Bug84]	$n \leq 7$	

Remarks:

- When the considered quadratic forms are reflective, the associated group Γ often has a small rank $n+k$, $k \geq 1$, compared to the dimension n. However, this is not always the case. For example, for $\alpha = 2+\sqrt{2}$ and $n = 5$ we find a polyhedron whose f-vector is $(96, 240, 228, 102, 20, 1)$, so that the rank of Γ equals 20.
- The values $\alpha = 1+\sqrt{3}, \frac{3+\sqrt{5}}{2}, 2+\sqrt{2}, \frac{3+\sqrt{17}}{2}$ and $2+\sqrt{3}$ lead to new polyhedra. The value $\alpha = 2+\sqrt{3}$ gives rise to Coxeter diagrams with weight 12.
- For $\alpha = \frac{3+\sqrt{21}}{2}$ and $n = 2$, we were not able to decide whether the quadratic form is reflective or not.

6.8.3.3 Errors in literature

In [Mcl13], the vector e_4 for the quadratic form $\langle -1-4\cdot\Theta_5, 1, 1\rangle$ is $(2+4\cdot\Theta_5, 7+10\cdot\Theta_5, 1)$. However, the vector $(1+\Theta_5, 1+4\cdot\Theta_5, 0)$ seems to be admissible and have a smaller fraction (1.5 compared to 35.9). It seems that non-integer possible values for (e, e) were forgotten. The same issue seems to apply to the form $\langle 5-4\cdot\Theta_5, 1, 1\rangle$. This leads to incorrect polytopes (i.e. some of the normal vectors he found contain errors).

CHAPTER 7

Index two subgroups and an infinite sequence

In [Gal05], Gal explains when a Coxeter group arises as a semi-direct product of two Coxeter subgroups, with one of them being a standard parabolic subgroup, and gives the presentation of the two subgroups. We will present his result and then study the situation when one of the two subgroups is of index two. We will see that under certain conditions, this construction gives rise to an infinite sequence of index two subgroups. Our sequence of abstract Coxeter groups generalizes the geometric approach described by Allcock : in [All06], he gave a condition under which a hyperbolic Coxeter polyhedron gives rise to an infinite sequence of polyhedra called "doublings".

In the first section, we will describe a way to construct an index two Coxeter subgroup Γ_1 of a given abstract Coxeter group Γ and give the explicit presentation of the subgroup. In the second section, we will see that under a simple assumption this process may be repeated to give rise to an infinite sequence of index two subgroups

$$\Gamma > \Gamma_1 > \Gamma_2 > \dots$$

The explicit presentation of the subgroups allows us to give a formula for the rank of any member of this sequence. Then, we restrict ourselves to geometric Coxeter groups (see Definition 3.4.6) and compute the f-vector of the associated polyhedron P_n of Γ_n. Finally, we will see how the growth series of the group Γ_n is related to the growth series of the first group of the sequence and address the question of the evolution of the growth rate τ_n.

7.1 Description of the construction

In this section, we will consider an abstract Coxeter group Γ with generating set $S = \{s_1, \dots, s_n\}$.

Definition 7.1.1 (Standard parabolic subgroup)
We say that $\Gamma' < \Gamma$ is a *standard parabolic subgroup* of Γ if there exists $I \subset S$ such that $\Gamma' = \Gamma_I$, where Γ_I denotes the subgroup generated by the elements in I.

Definition 7.1.2 (Admissible subset, admissible vertex)
A subset $I \subset S$ is called *admissible* if $m(s_i, s_j)$ is even[1] for every $s_i \in I$ and

[1] We will consider ∞ as an even number.

every $s_j \in S \setminus I$. If $I = \{s\} \subset S$ is an admissible subset, then we call s an *admissible element* or *admissible vertex*[2].

Remark 7.1.3
A subset I of S is admissible if and only if no element of I is conjugate to an element of $S \setminus I$. Indeed, if $m(s_i, s_j) = 2k+1$, then we have

$$(s_i \cdot s_j)^k \cdot s_i \cdot (s_i \cdot s_j)^{-k} = s_j,$$

and thus s_i and s_j are conjugate. For the converse, see [Bou68b, Chapter IV, §1, Proposition 3].

Proposition 7.1.4 ([Gal05, Proposition 2.1])
Let $I \subset S$ be an admissible set and let $J = S \setminus I$. Then

1. *We have the decomposition $\Gamma = \overline{\Gamma_J} \rtimes \Gamma_I$, where $\overline{\Gamma_J}$ is the normal closure of Γ_J in Γ.*
2. *The group $\overline{\Gamma_J}$ is a Coxeter group. Moreover, we have $\overline{\Gamma_J} = (\Gamma_{\tilde{J}}, \tilde{J})$, where $\tilde{J}$ consists of all the Γ_I-conjugates of J.*

Proposition 7.1.5 ([Bou68b, Chapter IV, §1, Exercise 3])
Let (Γ, S) be a Coxeter group and let $S_1, S_2 \subset S$. Then, for every $g \in \Gamma$ there exists a unique element of minimal length in the double coset $\Gamma_{S_1} \cdot g \cdot \Gamma_{S_2}$, where Γ_{S_1} and Γ_{S_2} are the two standard parabolic subgroups associated to S_1 and S_2 respectively.

Notation 7.1.6
For $T \subset S$, we will write

$$T^{\perp} = \{s \in S : m(s,t) = 2, \forall t \in T\}.$$

Moreover, for $t \in S$, we write $t^{\perp}$ instead of $\{t\}^{\perp}$. Let $I \subset S$ and $t_1, t_2 \in S$. For every $g \in \Gamma$, we will denote by $\mathrm{short}_{t_1,t_2,I}(g)$ the unique (exists by Proposition 7.1.5) shortest element in the set $\Gamma_{t_1^{\perp} \cap I} \cdot g \cdot \Gamma_{t_2^{\perp} \cap I}$.

We are now able to describe the presentation of the normal closure $\Gamma_{\tilde{J}}$ of Γ_J.

Proposition 7.1.7 ([Gal05, Corollary 3.3])
Let $\tau_i = w_i\, t_i\, w_i^{-1}$, $i = 1, 2$, be two elements of $\tilde{J}$, that is $w_i \in \Gamma_I$ and $t_i \in J$. Then:

1. *We have $\tau_1 = \tau_2$ if and only if $t_1 = t_2$ and $\mathrm{short}_{t_1,t_2,I}(w_1^{-1}\, w_2) = 1$.*
2. *It $\tau_1 \neq \tau_2$, we have:*

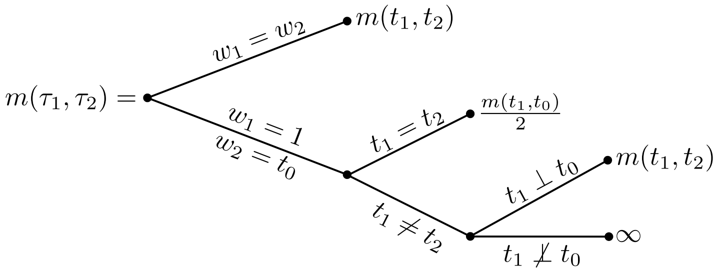
$$m(\tau_1, \tau_2) = \begin{cases} m(t_1, t_2) & \textit{if } t_1 \neq t_2 \textit{ and } \mathrm{short}_{t_1,t_2,I}(w_1^{-1}\, w_2) = 1. \\ \frac{m(t_1,s)}{2} & \textit{if } t_1 = t_2 \textit{ and } \mathrm{short}_{t_1,t_2,I}(w_1^{-1}\, w_2) = s \in I, \\ \infty & \textit{otherwise.} \end{cases}$$

Remark 7.1.8
In the case where $\Gamma < \mathrm{Isom}\,\mathbb{H}^n$ and $m(\tau_1, \tau_2) = \infty$, the Proposition 7.1.7 does not tell us whether the hyperplanes corresponding to τ_1 and τ_2 are parallel or ultra-parallel. We will discuss that in the next proposition.

[2]Recall that we make no distinction between the presentation of the Coxeter group and the Coxeter graph.

We consider now the case where the induced subgroup is of index two.

Proposition 7.1.9
Let $I = \{t_0\} \subset S$ be an admissible subset, $J = S \setminus I$ and let $\tau_i = w_i\, t_i\, w_i^{-1}$, $i = 1, 2$, be two elements of $\tilde{J}$, that is $w_i \in \{1, t_0\}$ and $t_i \in J$. Then, the weight $m(\tau_1, \tau_2)$ is given as follows:



Moreover, if Γ is a hyperbolic Coxeter group, then the "nature" of the ∞ is preserved in the following sense:

- *In the first and in the third case, the hyperplanes H_{τ_1} and H_{τ_2} corresponding to τ_1 and τ_2 respectively are ultra-parallel if and only if the hyperplanes H_{t_1} and H_{t_2} are ultra-parallel.*
- *In the second case, the hyperplanes H_{τ_1} and H_{τ_2} are ultra-parallel if and only if the hyperplanes H_{t_1} and H_{t_0} are ultra-parallel.*

Finally, in the last case, H_{τ_1} is ultra-parallel to H_{τ_2} except when $t_1 \perp t_2$ and $m(t_0, t_1) = m(t_0, t_2) = 4$.

Proof. We prove only the second part since the first is a direct consequence of Proposition 7.1.7. For an element $\tau \in \tilde{J} \cup \{t_0\}$, we denote by v_τ the outward unit normal vector to the corresponding facet of the associated polyhedron.
The first case is trivial. Suppose now that $w_1 = 1$, $w_2 = t_0$ and $t_1 = t_2$. Then, we have

$$\langle v_{\tau_1}, v_{\tau_2} \rangle = \langle v_{t_1}, r_{t_0}(v_{t_2}) \rangle = 1 - 2 \cdot \langle v_{t_0}, v_{t_1} \rangle^2.$$

The last term is less than -1 if and only if so is $\langle v_{t_0}, v_{t_1} \rangle$, as required. In the second case, we have $\langle v_{\tau_1}, v_{\tau_2} \rangle = \langle v_{t_1}, v_{t_2} \rangle$. Finally, in the last case, we get

$$\langle v_{\tau_1}, v_{\tau_2} \rangle = \langle r_{t_0}(v_{t_1}), v_{t_2} \rangle = \langle v_{t_1}, v_{t_2} \rangle - 2 \cdot \langle v_{t_0}, v_{t_1} \rangle \cdot \langle v_{t_0}, v_{t_2} \rangle.$$

Since $v_{t_0} \not\perp v_{t_i}$ and since t_0 is admissible, then $m(t_0, t_i) \geq 4$ and the result follows. □

Implementation in `CoxIter` The computation of the presentation of $\Gamma_{\tilde{J}}$, when the admissible subset is of the form $I = \{t_0\}$, as in Proposition 7.1.9, have been implemented in `CoxIter`. For example, if we want to read the graph "input.coxiter", then do the computations for the admissible vertex "5" and write the result in the file "subgroup.output", then we would call `CoxIter` as follows (the "wg", write graph, option is for `CoxIter` to write the resulting presentation in a file):

```
./coxiter -i input.coxiter -o subgroup -wg -index2 5
```

7.2 An infinite sequence

The aim of this section is to present the construction of an infinite sequence of index two subgroups in a Coxeter group (Γ, S). In what follows, when considering an element $\tau = w\,t\,w^{-1} \in \tilde{J}$, with $t \in J$, it will be assumed that t does not commute with w.

Proposition 7.2.1
Let (W_0, S_0) be a Coxeter system. If s_0 and t_0 are two admissible elements of S with $m(s_0, t_0) = \infty$, then there exists an infinite sequence $\mathcal{W}$ of groups $W_0 \geq W_1 \geq W_2 \geq \ldots$ where each W_{i+1} is of index two in W_i.

Proof. For $n \geq 1$ define $t_n = t_{n-1}s_0t_{n-1} = \big(t_0s_0\big)^{2^n-1}t_0$. We construct a subgroup W_1 of W_0 as above:

$$I_1 := \{t_0\}, \quad J_1 := S \setminus I_1, \quad \tilde{J}_1 := J_1 \cup t_0 \cdot J_1 \cdot t_0, \quad W_1 = \langle \tilde{J}_1 \rangle.$$

We want to show that t_1 and s_0 are admissible in $\tilde{J}_1$. Let $\tilde{t} = w \cdot t \cdot w$, with with $w \in \{1, t_0\}$ and $t \in J_1$. If $w = t_0$, then $m(t_1, \tilde{t}) = m(s_0, t)$ which is even by hypothesis. If $w = 1$, then we get $m(t_1, t) = m(s_0, t)$ or $m(t_1, t) = \infty$, depending on the fact that t_0 commutes with t or not. Hence, t_1 is admissible. Similarly, s_0 is admissible. Thus, we can apply again the process with

$$I_2 := \{t_1\}, \quad J_2 := S \setminus I_2, \quad \tilde{J}_2 := J_2 \cup t_1 \cdot J_2 \cdot t_1, \quad W_2 = \langle \tilde{J}_2 \rangle,$$

to get a subgroup W_2 of W_1 of index 2.
Suppose now we are given a sequence $(W_0, S_0) \geq (W_1, \tilde{J}_1) \geq \ldots \geq (W_n, \tilde{J}_n)$ which satisfies:

- The involutions s_0 and t_i are admissible in $\tilde{J}_i$ for every $0 \leq i \leq n-1$.
- The group W_{i+1} is obtained from W_i by removing t_{i-1} from I_i and applying the process.

To see that this sequence can be extended with a subgroup $(W_{n+1}, \tilde{J}_{n+1})$ of index 2 in W_0, we show that s_0, t_n are admissible in $\tilde{J}_n = J_n \cup t_{n-1} \cdot J_n \cdot t_{n-1}$. We start with s_0. By induction hypothesis, we only need to consider the weights $m(s_0, \tilde{t})$ where $\tilde{t} \in t_{n-1} \cdot J_n \cdot t_{n-1}$. Moreover, if $\tilde{t} = t_{n-1} \cdot t \cdot t_{n-1}$, we can suppose that t does not commute with t_{n-1}, because otherwise $\tilde{t} = t \in J_n$. Since $m(s_0, t_0) = \infty$, then s_0 does not commute with t_{n-1} and thus $m(s_0, t_{n-1} \cdot t \cdot t_{n-1}) = \infty$, which implies that s_0 is admissible. It remains to compute the weights $m(t_n, w \cdot t \cdot w)$ to show that t_n is admissible. By induction hypothesis, the only case to consider is when $w = 1$. In this, setting we get $m(t_n, t) = \infty$, since t does not commute with t_{n-1}. Hence, t_n is admissible, as required. Thus, the sequence can be extended. $\square$

Remark 7.2.2
The construction of the sequence $\mathcal{W}$ is purely algebraic and can be done for Coxeter groups which are not hyperbolic Coxeter groups.

7.2.1 Tests of `CoxIter` and the conjecture about Perron numbers

Using the different graphs encoded for `CoxIter` (see Section 5.5), we created more than three thousands groups. The goal was to test the output of `CoxIter` (cocompactness, cofiniteness, f-vector) and that the growth rate of all these groups was a Perron number (see Section 5.3.5).

7.2.2 Rank of the groups of the sequence $\mathcal{W}$

As before, we consider two admissible vertices t_0 and s_0 such that $m(t_0, s_0) = \infty$ and the corresponding sequence $\mathcal{W}$ given by

$$(W_0, S_0) \geq (W_1, \tilde{J}_1) \geq (W_2, \tilde{J}_2) \geq \dots$$

For a vertex $s \in \tilde{J}_i$, we consider the neighbours of s:

$$N^i(s) = \big\{t \in \tilde{J}_i : t \not\perp s\big\}.$$

We also consider

$$N^i_\infty(s) = \big\{t \in \tilde{J}_i : m(s,t) = \infty\big\}.$$

Using Proposition 7.1.9, we get the following result.

Proposition 7.2.3
Let $n \geq 1$. The neighbours $N^n(s_0)$ of s_0 in the Coxeter graph of W_n consist of the following:

- $N^{n-1}(s_0)$*;*
- $t_{n-1} \cdot N^{n-1}(t_{n-1}) \cdot t_{n-1} \subset N^n_\infty(s_0)$*.*

The neighbours $N^n(t_n)$ of t_n consist of the following:

- $N^{n-1}(t_{n-1}) \subset N^n_\infty(t_n)$*;*
- $N^{n-1}(s_0)$
- $t_{n-1} \cdot N^{n-1}(s_0) \cdot t_{n-1}$*.*

For $n \geq 0$, consider the counting functions $a_n = \big|N^n(s_0)\big|$, $b_n = \big|N^n(t_n)\big|$ and $c_n = \big|N^n(s_0) \cap N^n(t_0)\big|$. Using Proposition 7.2.3, we easily deduce the following recurrence relations:

$$\begin{aligned} a_n &= \big(a_{n-1} - 1\big) + b_{n-1} \\ b_n &= \big(a_{n-1} + b_{n-1} - c_{n-1} - 1\big) + c_{n-1} \\ &= a_{n-1} + b_{n-1} - 1 \\ c_n &= a_{n-1} + c_{n-1} - 1. \end{aligned}$$

Hence, we get the following proposition.

Proposition 7.2.4
For a_n, b_n and c_n as above, we have for $n \geq 1$

$$a_n = b_n = 2^{n-1}(a_0 + b_0 - 2) + 1, \quad c_n = 2^{n-1}(a_0 + b_0 - 2) - b_0 + c_0 + 1.$$

In particular, we have

$$\begin{aligned}|\tilde{J}_n| &= 2^{n-1}(a_0+b_0-2)+1+b_0-c_0+\iota \\ &= 2^{n-1}(a_0+b_0-2)+1+|J_0|-a_0,\end{aligned}$$

where ι is the number of vertices which commute with both s_0 and t_n; the number ι is constant. Finally, we get

$$|\tilde{J}_{n+1}|-|\tilde{J}_n| = 2^{n+1}(a_0+b_0-2).$$

7.2.3 f-vector of the polyhedra of the sequence

We suppose now that (Γ_0, S_0) is a geometric Coxeter group[3] (i.e. Γ_0 is a discrete group generated by finitely many reflections in hyperplanes of $\mathbb{S}^n, \mathbb{H}^n, \mathbb{E}^n$) and, as before, we fix two admissible vertices $t_0, s_0 \in S_0$ such that the corresponding hyperplanes are (ultra-)parallel. The goal is then to describe the f-vector of the polyhedron P_n which arises after n doublings (the n here is independent of the dimension of the space).

Let F be a face of P_{n-1} and denote by $T \subset \tilde{J}_{n-1}$ the corresponding spherical/Euclidean subgraph of the Coxeter graph $\mathcal{G}_{n-1}$ of Γ_{n-1}. We then have the following three possibilities when creating the doubling P_n of P_{n-1} (for illustration see Figure 7.1):

F **is deleted** This is the case when F is contained in $H_{t_{n-1}}$ and all facets of P_{n-1} going through F except $H_{t_{n-1}}$ are perpendicular to $H_{t_{n-1}}$ (edges AB and AC and vertex A of the figure). In terms of the graph, this is equivalent to $t_{n-1} \in T$ and $T \setminus \{t_{n-1}\} \subset t_{n-1}^{\perp}$.

F **is duplicated** This happens when F has no intersection with $H_{t_{n-1}}$ (for example all facets of the triangle $A'B'C'$ in Figure 7.1) or when F is contained in an hyperplane which is not perpendicular to $H_{t_{n-1}}$ (for example edges BB', C, C' and vertices B and C). The former is equivalent to $t_{n-1} \notin T$ while the later is equivalent to $T \setminus \{t_{n-1}\} \not\subset t_{n-1}^{\perp}$.

F **is kept** In the other cases (edge BC and vertices B and C of the Figure). Note that in this case, the corresponding figure around the face in P_n may be different.

Hence, a face F_0 of P_0 can be deleted, can give rise to many faces or can contribute only to one face in P_n. We summarize this in the next proposition.

Proposition 7.2.5
Let $F \subset P_0$ be a face of the first polyhedron and consider the associated subgraph $T \subset S_0$. For $n \geq 1$, the number of faces of the polyhedron P_n arising from F after n doublings is the following:

1. $t_0 \notin T, s_0 \notin T$:

[3] In this context, the bijection of Theorem 3.7.5 in Chapter 3 is still true (see for example [Vin67, Page 431] for the Euclidean case).

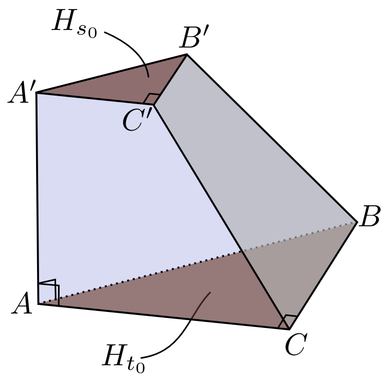


Figure 7.1 – First polyhedron of the sequence

Condition	Number of faces
$T \subset t_0^\perp, T \subset s_0^\perp$	1
$T \not\subset t_0^\perp, T \not\subset s_0^\perp$	2^n
$T \not\subset t_0^\perp, T \subset s_0^\perp$	$1 + 2^{n-1}$
$T \subset t_0^\perp, T \not\subset s_0^\perp$	2^{n-1}

2. $s_0 \in T, t_0 \notin T$*:*

Condition	Number of faces
$T \setminus \{s_0\} \subset s_0^\perp$	2
$T \setminus \{s_0\} \not\subset s_0^\perp$	$1 + 2^{n-1}$

3. $s_0 \notin T, t_0 \in T$*:*

Condition	Number of faces
$T \setminus \{t_0\} \subset t_0^\perp$	0
$T \setminus \{t_0\} \not\subset t_0^\perp$	2^{n-1}

4. $s_0 \in T, t_0 \in T$*: 2 faces.*

Proof. We consider all the possibilities:

1. $t_0 \notin T, s_0 \notin T$

   (a) $T \subset t_0^\perp, T \subset s_0^\perp$: F stays the same and $t_1 \notin T$, $T \subset t_1^\perp$. Inductively, we get $t_n \notin T$ and $T \subset t_n^\perp$.

   (b) $T \not\subset t_0^\perp, T \not\subset s_0^\perp$: The face F is duplicated and this new face $F^{(1)}$ will also satisfy $t_1 \notin \tilde{T}^{(1)}$, $s_0 \notin T^{(1)}$ and $T^{(1)} \not\subset t_1^\perp, T^{(1)} \not\subset s_0^\perp$. Inductively, we see that this will yield to 2^n faces.

(c) $T \not\subset t_0^\perp, T \subset s_0^\perp$: The face F is duplicated and this new face $F^{(1)}$ will be such that $T^{(1)} \subset t_1^\perp$ and $T^{(1)} \subset N_\infty^1(s_0)$. Hence, the new face $F^{(1)}$ won't be duplicated during the second doubling but will be duplicated during each doubling after (see $1.(d)$).

(d) $T \subset t_0^\perp, T \not\subset s_0^\perp$: We have $T \not\subset t_1^\perp$ and $T \not\subset s_0^\perp$. Hence, this face is not duplicated during the first doubling but is during all the next doublings.

2. $s_0 \in T, t_0 \notin T$: The face is duplicated into $F^{(1)} \subset H_{t_1}$.

   (a) $T \setminus \{s_0\} \subset s_0^\perp$: The new face $F^{(1)}$ is not duplicated nor removed after this first step.

   (b) $T \setminus \{s_0\} \not\subset s_0^\perp$: $F^{(1)}$ is not duplicated during the second doubling but is afterwards.

3. $s_0 \notin T, t_0 \in T$

   (a) $T \setminus \{t_0\} \subset t_0^\perp$: The face is removed.

   (b) $T \setminus \{t_0\} \not\subset t_0^\perp$: Is not duplicated during the first doubling but is afterwards.

4. We have $F \subset H_{t_n}$ for for every n.

□

Remark 7.2.6
In all cases except $3(b)$, the type of the graph corresponding to the face F is preserved. In the latter case, the corresponding graph $\mathcal{G}_T$ is modified according to Proposition 7.1.9 once and then remains unchanged.

Example 7.2.7
We consider again the family of cocompact Coxeter prisms $\Gamma^m := \Gamma^3_{4,m}$ (see Example 4.4.3) and we choose s_0, t_0 as in Figure 7.2. The spherical subgraphs

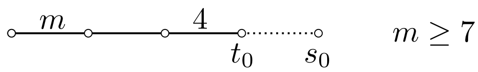


Figure 7.2 – The family of prisms $\Gamma_0^m := \Gamma^3_{4,m}$

of $\mathcal{G}_{\Gamma_0^m}$ together with their multiplicities are the following:

$5 * A_1$	$1 * G_2^{(m)}$	$1 * G_2^{(4)}$
$1 * A_2$	$6 * A_1 \times A_1$	$1 * B_3$
$2 * A_1 \times G_2^{(m)}$	$2 * A_1 \times G_2^{(4)}$	$1 * A_1 \times A_2$
$1 * A_1 \times A_1 \times A_1$.		

Using Proposition 7.2.5 and Remark 7.2.6, we compute the contribution of each of these graphs in the graph $\mathcal{G}_{\Gamma_n^m}$ obtained after n doublings:

- $G_2^{(m)}$: $1 * G_2^{(m)}$;
- $G_2^{(4)}$: $2^{n-1} * A_1 \times A_1$;

- A_2: $(1+2^{n-1}) * A_2$;
- $6 * A_1 \times A_1$: $(7+2^{n-1}) * A_1 \times A_1$;
- B_3: $2^{n-1} * A_3$;
- $2 * A_1 \times G_2^{(m)}$: $2 * A_1 \times G_2^{(m)}$;
- $A_1 \times G_2^{(4)}$: $2^{n-1} * A_1 \times A_1 \times A_1$;
- $A_1 \times A_2$: $2 * A_1 \times A_2$;
- $A_1 \times A_1 \times A_1$: $2 * A_1 \times A_1 \times A_1$.

In particular, the f-vector of Γ_n^m is given by $\big(2^n+6, 9+3\cdot 2^{n-1}, 5+2^{n-1}, 1\big)$.

Implementation in `CoxIter` The computation of the f-vector after n doublings has been implemented in `CoxIter`. To use this feature, one should use the option "is" (for **i**nfinite **s**equence), together with the name of the vertices t_0 and s_0. For example, for the group $\Gamma_{4,7}^3$ of the previous example, we call `CoxIter` as follows:

```
./coxiter -i ../graphs/3-Gamma-4,7.coxiter -is=[4,5]
```

And the output is then the following:

```
Reading graph:
    Number of vertices: 5
    Dimension: 3
    Vertices: 1, 2, 3, 4, 5
    Field generated by the entries of the Gram matrix: ?
File read

Infinite sequence:
    f-vector after n doubling:
    (6, 9, 5, 1) + 2^(n-1)*(2, 3, 1, 0)
    Number of new hyperplanes after first doubling: 1
```

7.2.4 Evolution of the growth rate

Example 7.2.8
We continue the investigation of the properties of the groups Γ_n^m (see Example 7.2.7). Using the list of spherical subgroups, we see that the growth series is given by

$$f_{\Gamma_n^m}(x) = \frac{2(x+1)^3\left(x^2+1\right)\left(x^2+x+1\right)(x^m-1)}{(x+1)\cdot q_m^n(x)},$$

with

$$\begin{aligned} q_n^m(x) = {} & -2+2^n x+2^{1+n}x^2+2x^3+3\cdot 2^n x^3+2x^4+2^{1+n}x^4+2x^5+2^n x^5 \\ & -2x^{1+m}-2^n x^{1+m}-2x^{2+m}-2^{1+n}x^{2+m}-2x^{3+m}-3\cdot 2^n x^{3+m} \\ & -2^{1+n}x^{4+m}-2^n x^{5+m}+2x^{6+m}, \end{aligned}$$

which factors as $x-1$ times a palindromic polynomial. Therefore, finding the growth rate τ_n^m of Γ_n^m is equivalent as finding the biggest positive real root of q_n^m. Luckily for us, by Descartes' rule of signs (see Theorem 2.6.2), each q_n^m has only 3 positive real roots (and thus only one root bigger than 1).

We see that the difference $q_n^{m+1}(x) - q_n^m(x)$ is equal to

$$(x-1)x^{m+1}\left(2x^5 - 2^n x^4 - 2^{n+1}x^3 - 3\cdot 2^n x^2 - 2^{n+1}x - 2^n - 2x^2 - 2x - 2\right).$$

Theorem 2.6.8 implies that all the real roots of the polynomial $h_n(x) = 2x^5 - 2^n x^4 - 2^{n+1}x^3 - 3\cdot 2^n x^2 - 2^{n+1}x - 2^n - 2x^2 - 2x - 2$ lie in the interval $\left(-2^{n+2}, 2^{n+2}\right)$. Hence, we can use Sturm's theorem (see Theorem 2.6.5) to determine the number of positive real roots bigger than 1 of h_n. The signs of the Sturm sequence are the following:

α_i	0	1	2	3	4	5	$\sigma(\alpha_i)$
1	$-$	$-$	$+$	$-$	$+$	$+$	3
2^{n+2}	$+$	$+$	$+$	$-$	$+$	$+$	2

If we denote by β_n the unique root of h_n bigger than 1, then we have $q_n^m(\beta_n) = q_n^7(\beta_n)$ for every $m \geq 7$. If $q_n^7(\beta_n) \geq 0$, then $\{\tau_n^m\}_m$ is an increasing sequence bounded above by β_n. On the other hand, if $q_n^7(\beta_n) < 0$, then $\{\tau_n^m\}_m$ is a decreasing sequence bounded below by β_n. In both cases, $\{\tau_n^m\}_{m\geq 7}$ converges to some value τ_∞^n depending on n. We will see that

$$\tau_n^m \in \left(2^{n-1} + 2 - \frac{1}{2^{n-1}}, 2^{n-1} + 2 - \frac{1}{2^{n+1}}\right), \quad \forall n \geq 1, m \geq 7.$$

In particular, for every $m \geq 7$, the ratio $\frac{\tau_{n+1}^m}{\tau_n^m}$ goes to 2 as n goes to infinity. We show separately that $q_n^m\left(2^{n-1} + 2 - \frac{1}{2^{n-1}}\right) < 0$ and $q_n^m\left(2^{n-1} + 2 - \frac{1}{2^{n+1}}\right) > 0$. First, remark that the last 5 monomials of $q_n^m(x)$ where m arises in the power factors as follows when evaluated in $2^{n-1} + 2 - \frac{1}{2^{n-1}}$:

$$\begin{aligned} &-2^{-6n-2}\left(2^{2n} + 2^{n+2} - 4\right)\left(-2^{1-n} + 2^{n-1} + 2\right)^m \\ &\quad \cdot \left(2^{7n} - 5\cdot 2^{n+7} + 9\cdot 2^{2n+7} - 49\cdot 2^{3n+4} + 33\cdot 2^{5n+2} + 3\cdot 2^{6n+3} + 128\right). \end{aligned}$$

Since this expression is negative and decreasing in m, we only have to show that $q_n^7\left(2^{n-1} + 2 - \frac{1}{2^{n-1}}\right) < 0$. Now, we have

$$q_n^7\left(2^{n-1} + 2 - \frac{1}{2^{n-1}}\right) = -2^{-13n-9}\left(2^{2n} + 2^{n+1} - 4\right)\cdot p_n(2),$$

where $p(x) \in \mathbb{Z}[x]$ is an integer polynomial whose exponents depend on n. Now, to prove that $p_n(2)$ is positive for every $n \geq 1$, we replace each positive coefficient a_i of p by the biggest power of 2 which is smaller than a_i and each negative coefficient $-a_i$ by -2^{m_i} where m_i is the smallest integer such that $2^{m_i} > a_i$. Hence, we find

$$\begin{aligned} p_n(2) &> -2097152 - 2^{2n+28} - 2^{4n+30} - 2^{6n+30} - 2^{9n+27} - 2^{10n+26} - 2^{13n+23} \\ &\quad - 2^{14n+22} + 2^{21n} + 2^{n+24} + 2^{3n+28} + 2^{5n+30} + 2^{7n+23} + 2^{8n+28} \\ &\quad + 2^{11n+25} + 2^{12n+23} + 2^{15n+17} + 2^{16n+18} + 2^{17n+16} + 2^{18n+13} \\ &\quad + 2^{19n+10} + 2^{20n+5} \\ &> -2097152 - 7\cdot 2^{14n+22} + 2^{20n+5}, \end{aligned}$$

which is bigger than 0 if $n > 2$. For $n = 1$ and $n = 2$, a direct computation shows that $p_n(2) > 0$, as required. With the same kind of comparisons, we can show that $q_n^m\left(2^{n-1} + 2 - \frac{1}{2^{n+1}}\right) > 0$, as required.

We saw in the previous example that for a fixed m, the quotient $\frac{\tau_{n+1}}{\tau_n}$ goes to 2 as n goes to infinity when the considered sequence is built from one family of Kaplinskaya prisms. Next proposition relates the growth series of two successive terms in the sequence.

Proposition 7.2.9
Let (Γ_0, S_0) be a hyperbolic Coxeter group and let $t_0, s_0 \in S_0$ be two admissible vertices with $m(t_0, s_0) = \infty$. Then, there exists a polynomial $q \in \mathbb{Z}[x]$ such that the rational expansion of the growth series f_{Γ_n} of the group after n doublings can be written

$$f_{\Gamma_n}(x) = \frac{q(x)}{g_n(x)}, \quad g_n(x) \in \mathbb{Z}[x].$$

Moreover, there exists $r \in \mathbb{N}$ such that the g_n are related as follows:

$$g_{n+1}(x) = (2^n - 1) \cdot x^r \cdot g(x) + g_1(x)$$

Proof. We know from Steinberg's formula that the growth series is given by

$$\frac{1}{f_{(\Gamma_n, S_n)}(x)} = \sum_{T \in \mathcal{F}_n} \frac{(-1)^{|T|} \cdot x^{m_T}}{f_T(x)},$$

where $\mathcal{F}_n$ is the collection of subsets T of S_n such that the subgroup generated by T is finite, and where m_T is some exponent depending on the type of T (see Figure 5.3 on page 83). Since the list of spherical subgroups of Γ_n remains unchanged if $n \geq 1$ (see Remark 7.2.6), we can regroup terms corresponding to the same subgroups and write

$$\frac{1}{f_{(\Gamma_n, S_n)}(x)} = \sum_{T \in \mathcal{F}} \frac{(-1)^{|T|} \cdot x^{m_T}}{f_T(x)} \cdot \alpha_{T,n},$$

where $\mathcal{F}$ is the list of representatives of distinct finite subgroups of Γ_1. Moreover, Proposition 7.2.5 implies that $\alpha_{T,n} \in \{1, 2, 1 + 2^{n-1}, 2^{n-1}, 2^n\}$. We now have

$$\frac{1}{f_{(\Gamma_n, S_n)}(x)} = \frac{g_n(x)}{q(x)},$$

where $g_n(x)$ has constant coefficient 1, a degree independent of n, and where $q(x)$ is independent of n. Now, each coefficient of $g_n(x)$ splits as a sum of a term which is independent of n and a term which is divisible by 2^{n-1}. Therefore, we have

$$g_{n+1}(x) - g_n(x) = 2^{n-1} \cdot x^d \cdot g(x),$$

for some positive integer r and some polynomial $g \in \mathbb{Z}[x]$ and thus

$$g_{n+1}(x) = (2^n - 1) \cdot x^r \cdot g(x) + g_1(x).$$

□

Corollary 7.2.10
Consider the g_n as in Proposition 7.2.9 and let $r \in \mathbb{N}$ and $g \in \mathbb{Z}[x]$ be such that

$$g_{n+1}(x) = (2^n - 1) \cdot x^r \cdot g(x) + g_1(x).$$

If there exists an index m such that the smallest positive root of g_m is strictly less than the smallest positive root of g, then

$$\lim_{n\to\infty} \frac{\tau_{n+1}}{\tau_n} = 2^r.$$

As indicated above, we constructed the first terms of all possible sequences of known cofinite hyperbolic Coxeter groups (see all groups mentioned in Section 5.5). For all these sequences (more than 500), we checked numerically that the situation of the Corollary 7.2.10 applies; more precisely:

- we have $\frac{\tau_{n+1}}{\tau} \longrightarrow 2$;
- often, the smallest positive root of g_1 is smaller than the smallest positive root of g.

These experimental observations lead us to state the following.

Conjecture
Let (Γ_0, S_0) be a hyperbolic Coxeter group and let $t_0, s_0 \in S_0$ be two admissible vertices with $m(t_0, s_0) = \infty$. Then, the growth rate τ_n of the group Γ_n arising after n doublings satisfies

$$\lim_{n\to\infty} \frac{\tau_{n+1}}{\tau_n} = 2.$$

CHAPTER 8

Clifford algebras and isometries of (infinite dimensional) hyperbolic spaces

Clifford algebras are unitary associative algebras which provide a way to generalize complex numbers and quaternions. Moreover, by considering a certain group of two-by-two matrices with coefficients in a subset $\Gamma_n \cup \{0\}$ of a Clifford algebra, one gets a way to describe the group of orientation preserving isometries of the hyperbolic n-space, i.e. we have an isomorphism

$$\mathrm{PSL}(2;\Gamma_n) \cong \mathrm{Isom}^+ \mathbb{H}^{n+2}.$$

In low dimensions, this result gives the well know isomorphisms

$$\mathrm{PSL}(2;\mathbb{R}) \cong \mathrm{Isom}^+ \mathbb{H}^2, \quad \mathrm{PSL}(2;\mathbb{C}) \cong \mathrm{Isom}^+ \mathbb{H}^3.$$

The construction of the group $\mathrm{PSL}(2;\Gamma_n)$, its basic properties and its action by isometries on the hyperbolic $(n+2)$-space via Poincaré extension have been extensively studied over the last decades (see [Ahl85] and [Wat93] for example).

The goal of this chapter is to explain how we can generalize the construction of the group $\mathrm{PSL}(2;\Gamma_n)$ to the infinite-dimensional setting. For a given infinite-dimensional Hilbert space $\mathscr{H}$, we will be able to give a description of the group $\mathrm{M\ddot{o}b}^*(\mathscr{H})$ (see Definition 2.7.17) in terms of Clifford matrices. Then, we will see that this group induces isometries of the associated upper half-space model $\mathcal{U}_{\mathscr{H}}$ (see Section 3.2.1). Other authors studied Clifford matrices induced Möbius transformations in the infinite-dimensional setting (see [Li11] and [Fru91], for example) but they considered only the case $\mathscr{H} = \ell^2$. Our approach, which is more canonical, allows to consider any Hilbert space and its induced hyperbolic space. In this way, we are able to use the work of [Das12].

In the first section, we will present the construction of the Clifford algebra associated to any quadratic space (V, q) and study its basic properties. In the second part, we will determine under which conditions a two-by-two matrix with coefficients in the Clifford group induces a bijection of the ambient space and see how the usual transformations (reflections, translations, sphere inversions, etc.) can be rewritten by means of Clifford matrices. Most results of this part are adaptations of Waterman's results to the infinite-dimensional setting. Finally, we will show that Clifford matrices induce isometries of the associated hyperbolic space and discuss the classification of the isometries in this setting.

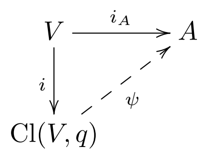
In this chapter, any field will be supposed to have characteristic different from 2. Moreover, we will denote invertible elements of a ring R by $R^\times$ instead of R^* in order to avoid confusion with the involution $*$ of the Clifford algebra.

8.1 Construction and basic properties

8.1.1 Clifford algebras

Let (V, q) be a quadratic space defined over some field K. A *Clifford algebra* associated to (V, q) is a unitary associative algebra over K denoted by $\mathrm{Cl}(V, q)$, together with a K-linear map $i : V \longrightarrow \mathrm{Cl}(V, q)$ which satisfies the following two properties:

- *Clifford identity*
  We have $i(v)^2 = -q(v) \cdot 1_{\mathrm{Cl}(V,q)}$ for every $v \in V$.
- *Universal property*
  If A is another unitary associative K-algebra together with a K-linear function $i_A : V \longrightarrow A$ such that $i_A(v)^2 = -q(v) \cdot 1_A$ for every $v \in V$, then there exists a unique morphism of K-algebras $\psi : \mathrm{Cl}(V, f) \longrightarrow A$ such that the following diagram commutes:



The universal property implies that if the algebra $\mathrm{Cl}(V, q)$ exists, then it is unique up to isomorphism. In fact, the Clifford algebra can be constructed explicitly as follows. We start with the tensor algebra $T(V) = \bigoplus_{i \in \mathbb{N}_0} V^{\otimes n}$, where $V^{\otimes n} = V \otimes \ldots \otimes V$, n times, and $V^{\otimes 0} = K$. Then, we consider the ideal I in $T(V)$ generated by the elements $v^2 + q(v)$. It follows that the quotient $T(V)/_I$ satisfies the universal property, as required. Thus, we have $\mathrm{Cl}(V, q) = T(V)/_I$. Since the composition

$$V \hookrightarrow T(V) \twoheadrightarrow T(V)/_I$$

is injective, we will write v instead of $i(v)$ for the image of a vector in $\mathrm{Cl}(V, q)$. Moreover, when working in $\mathrm{Cl}(V, f)$ we will write the product $a \cdot b$ instead of $a \otimes b$.

Let $\{v_i\}_{i \in \mathcal{I}}$ be an algebraic basis of V (see Remark 2.7.6) and consider some well-order on $\mathcal{I}$. A *multi-index* I will denote a finite ordered subset of $\mathcal{I}$. If $I = (i_1, \ldots, i_k)$, we will say that I has length k and write $|I| = k$. Any element $a \in \mathrm{Cl}(V, q)$ can be written as $a = \sum_I a_I \cdot v_I$, where the finite sum is taken over multi-indices $I = (i_1, \ldots, i_k)$, $a_I \in K$, and $v_I := v_{i_1} \cdot \ldots \cdot v_{i_k}$, with the additional convention that $v_\emptyset := 1 = 1_K$.

The natural $\mathbb{N}_0$-grading on the tensor algebra $T(V)$ induces a $\left(\mathbb{Z}/_{2\mathbb{Z}}\right)$-grading on $\mathrm{Cl}(V, f)$ which is compatible with the algebra structure. In other words, the vector space

$$\mathrm{Cl}_0(V, f) := \mathrm{span}_K \left\{ v_{i_1} \cdot \ldots \cdot v_{i_k} : 1 \leq i_1 < i_2 < \ldots < i_k \leq n, k \text{ even} \right\}$$

is a sub-algebra of $\mathrm{Cl}(V, f)$ called the *even* part of $\mathrm{Cl}(V, f)$. More canonically, $\mathrm{Cl}_0(V, f)$ is the image of $\bigoplus_{i \in \mathbb{N}_0} V^{\otimes 2i}$ by the corresponding quotient map.

Notice, that $\mathrm{Cl}(V, q)$, when viewed as a vector space, also has a $\mathbb{Z}$-grading. However, this grading is not compatible with the algebra structure.

8.1.1.1 Functoriality and three involutions

If (V', q') is another quadratic space over K and $\phi : V \longrightarrow V'$ is a morphism of quadratic spaces (i.e. a linear map preserving the quadratic forms), then the composition $i_{v'} \circ \phi$ satisfies the condition of the universal property, which means that there exists a morphism of K-algebras, again denoted ϕ, from $\mathrm{Cl}(V, q)$ to $\mathrm{Cl}(V', q')$. With the explicit construction of the Clifford algebra, the map ϕ can be described as follows:

$$\begin{aligned} \phi : \mathrm{Cl}(V, q) &\longrightarrow \mathrm{Cl}(V', q') \\ v_1 \cdot \ldots \cdot v_m &\longmapsto \phi(v_1) \cdot \ldots \cdot \phi(v_m). \end{aligned}$$

Therefore, Cl is a functor from the category of quadratic spaces over a field K to the category of unitary associative algebras over K.

The automorphism of (V, q) which sends a vector v to $-v$ then induces an automorphism $' : \mathrm{Cl}(V, q) \longrightarrow \mathrm{Cl}(V, q)$ which sends an element $v_1 \cdot \ldots \cdot v_m$ to $(-1)^m \cdot v_1 \cdot \ldots \cdot v_m$. The inclusion of $\mathrm{Cl}(V, q)$ in its opposite algebra $\mathrm{Cl}(V, q)^{\mathrm{op}}$ gives rise to an anti-automorphism $* : \mathrm{Cl}(V, q) \longrightarrow \mathrm{Cl}(V, q)$ which sends an element $v_1 \cdot \ldots \cdot v_m$ to $v_m \cdot \ldots \cdot v_1$. Since these two maps commute, we can define the following anti-automorphism

$$\begin{aligned} \overline{} : \mathrm{Cl}(V, q) &\longrightarrow \mathrm{Cl}(V, q) \\ a &\longmapsto \bar{a} = (a')^* = (a^*)'. \end{aligned}$$

Example 8.1.1 (Standard Clifford algebra)
For a field K of characteristic different from two, and for $n \in \mathbb{N}_0$, we let $V_n = K^n$, with the convention that K^0 is the null space $\{0\}$. Any orthonormal basis $\{e_1, \ldots, e_n\}$ of V_n gives rise to the diagonal quadratic form $q_n = \langle 1, \ldots, 1 \rangle$. In the Clifford algebra $\mathrm{Cl}(V_n, q_n)$, we have the identities $e_i^2 = -1$ and $e_i e_j = -e_j e_i$, when $i \neq j$. Moreover, it is easy to see that

$$\left\{ e_{i_1} \cdot \ldots \cdot e_{i_k} : 1 \leq i_1 < i_2 < \ldots < i_k \leq n \right\}$$

is a K-basis of $\mathrm{Cl}(V_n, q_n)$ and in particular $\dim_K \mathrm{Cl}(V_n, q_n) = 2^{\dim_K V_n} = 2^n$. We will denote this Clifford algebra by $\mathrm{Cl}_n(K)$, or Cl_n if $K = \mathbb{R}$. For $n = 0, 1$ and 2 we get successively $\mathrm{Cl}_n = \mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$, with the standard basis $\{1\}$, $\{1, i\}$ and $\{1, i, j, ij\}$. Moreover, the anti-automorphism $^-$ coincides with the usual conjugation in $\mathbb{C}$ and $\mathbb{H}$.

8.1.1.2 Center of the algebra

Proposition 8.1.2 (Center of Cl, finite-dimensional case)
Let (V, q) be an n-dimensional quadratic space over K with orthogonal basis $\{v_1, \ldots, v_n\}$. Then, the center of $\mathrm{Cl}(V, q)$ is K if n is even, and it equals $\mathrm{span}\{1, v_1 \cdot \ldots \cdot v_n\}$ if n is odd.

Proof. First, we remark that the following two equalities hold for every multi-index I:

$$\begin{aligned} v_j \cdot v_I &= (-1)^{|I|} \cdot v_I \cdot v_j, \quad j \notin I \\ v_j \cdot v_I &= (-1)^{|I|+1} \cdot v_I \cdot v_j, \quad j \in I. \end{aligned} \tag{8.1}$$

Let $a \in \mathrm{Cl}(V, q)$ and write $a = \sum a_I \cdot v_I$, where all the multi-indices I are distinct and totally ordered, that is $I = (i_1, \dots, i_k)$ for some $1 \leq i_1 < i_2 < \dots < i_k \leq n$. If J is any multi-index, then we have $v_I \cdot v_J \in \mathbb{R} \cdot v_{I\Delta J}$, where $I\Delta J$ denotes the symmetric difference between I and J, that is $I\Delta J = (I \setminus J) \cup (J \setminus I)$. Since $I\Delta J = I'\Delta J$ implies $I = I'$, it follows that a is in the center of the Clifford algebra if and only if every v_I such that $a_I \neq 0$ is also in the center. Moreover, this last condition is equivalent to $v_I \cdot v_j = v_j \cdot v_I$ for every $1 \leq j \leq n$. We conclude using equations (8.1). □

Proposition 8.1.3 (Center of Cl, infinite-dimensional case)
Let $(V, \|\cdot\|)$ be an infinite-dimensional Hilbert space, let $\lambda \in \mathbb{R} \setminus \{0\}$ and consider the quadratic form $q : V \longrightarrow \mathbb{R}$ defined by $q(v) = \lambda \cdot \|v\|^2$. Then, the center of $\mathrm{Cl}(V, q)$ is $\mathbb{R}$.

Proof. First, we pick a Hilbert orthonormal basis $\{e_i\}_{i \in \mathcal{I}}$ of V (see Definition 2.7.2). We thus have the following relations:

$$e_i^2 = -\lambda, \quad e_i \cdot e_j = -e_j \cdot e_i, \quad \forall i \neq j \in \mathcal{I}.$$

We know (see Proposition 2.7.14) that the set of products

$$e_{i_1} \otimes \dots \otimes e_{i_k}, \quad i_1, \dots, i_k \in \mathcal{I}$$

is an orthonormal Hilbert basis of $V^{\hat{\otimes} k}$ (see Definition 2.7.13). Hence, the closure of the linear span of the family of vectors

$$\mathcal{C} = \big\{ e_{i_1} \cdot \ldots \cdot e_{i_k} : k \in \mathbb{N},\ i_1, \dots, i_k \in \mathcal{I} \big\}$$

is $\mathrm{Cl}(V, q)$. Let $x \in Z\big(\mathrm{Cl}(V, q)\big)$ and write

$$x = \sum_{J \in \mathcal{J}} a_J e_J, \quad a_J \neq 0, \forall J \in \mathcal{J},$$

where $\{e_J\}_{J \in \mathcal{J}}$ is a finite or countable subset of $\mathcal{C}$ consisting of linearly independent vectors. If $x \notin \mathbb{R}$, then there exists a non-empty $J \in \mathcal{J}$ such that $a_J \neq 0$. Moreover, we must have $e_i e_J = e_J e_i$ for every $i \in \mathcal{I}$ (see proof of Proposition 8.1.2). If $|J|$ is odd this is possible only for $i \in J$, and only for $i \notin J$ if $|J|$ is even (see equations (8.1) of Proposition 8.1.2). Therefore, we must have $x \in \mathbb{R}$, as required. □

8.1.1.3 The Clifford algebra as a quadratic space

The vector space $\mathrm{Cl}(V, q)$ has a natural grading given by the one on the tensor algebra and the projection $T(V) \longrightarrow T(V)_0 = K$ gives rise to the linear map

$$\begin{aligned} \Re : \mathrm{Cl}(V, q) &\longrightarrow K \\ x &\longmapsto x_0. \end{aligned}$$

This map yields a quadratic form q on $\mathrm{Cl}(V,q)$ and its associated bilinear form

$$\begin{aligned} q : \mathrm{Cl}(V,q) &\longrightarrow K & B : \mathrm{Cl}(V,q)\times \mathrm{Cl}(V,q) &\longrightarrow K \\ x &\longmapsto \Re(\overline{x}x) & (x,y) &\longmapsto \frac{1}{2}\Re(\overline{x}y+\overline{y}x) = \Re(\overline{x}y). \end{aligned}$$

We note that the quadratic form q, when restricted to V, agrees with the initial one. Moreover, if $\phi : (V,q) \longrightarrow (V',q')$ is a morphism of quadratic spaces, then the induced map $\phi : \mathrm{Cl}(V,q) \longrightarrow \mathrm{Cl}(V',q')$ preserves the bilinear forms on $\mathrm{Cl}(V,q)$ and $\mathrm{Cl}(V',q')$.

Proposition 8.1.4
Let $x_1,\dots,x_k,x,y \in K \oplus V$, $v,\tilde{v} \in V$, $\alpha,\tilde{\alpha} \in K$ and $z \in \mathrm{Cl}(V,q)$. Then, we have the following properties:

(i) $B(\alpha+v,\tilde{\alpha}+\tilde{v}) = \alpha\tilde{\alpha} + B(v,\tilde{v})$;

(ii) $q(\alpha+v) = \alpha^2 + q(v)$;

(iii) $q(x_1\cdot\ldots\cdot x_k) = \prod_{i=1}^k q(x_i)$;

(iv) $q(xz) = q(x)\cdot q(z)$;

(v) $x\overline{x} = q(x)$;

(vi) $x\overline{y} + y\overline{x} = 2\cdot b(x,y)$.

Proof. For the sake of clarity, we will write q for the quadratic form defined on V (respectively b for its associated bilinear form) and Q for the one defined on $\mathrm{Cl}(V,q)$ (respectively B for its associated bilinear form).

(i) We will prove the equality $B(\alpha+v,\tilde{\alpha}+\tilde{v}) = \alpha\tilde{\alpha}+b(v,\tilde{v})$. First we remark that by definition of b, we have

$$\begin{aligned} b(v,\tilde{v}) &= \frac{1}{2}\big(q(v+\tilde{v}) - q(v) - q(\tilde{v})\big) = \frac{1}{2}\left(-(v+\tilde{v})^2 + v^2 + \tilde{v}^2\right) \\ &= -\frac{1}{2}(v\tilde{v}+\tilde{v}v) = -\Re(v\tilde{v}). \end{aligned}$$

Then, we get

$$\begin{aligned} B(\alpha+v,\tilde{\alpha}+\tilde{v}) &= \Re\left((\overline{\alpha+v})(\tilde{\alpha}+\tilde{v})\right) = \Re\left(\alpha\tilde{\alpha} - \tilde{\alpha}v + \alpha\tilde{v} - v\tilde{v}\right) \\ &= \alpha\tilde{\alpha} - \Re\left(v\tilde{v}\right) = \alpha\tilde{\alpha} + b(v,\tilde{v}), \end{aligned}$$

as required.

(ii) Follows directly from (i).

(iii) We write $x_1 = \alpha_1 + v_1$ and we get

$$\begin{aligned} Q(x_1x_2) &= \Re\big(\overline{x}_2\overline{x}_1x_1x_2\big) = \Re\big(\overline{x}_2(\alpha_1^2 - v_1^2)x_2\big) \\ &= Q(x_1)\cdot\Re\big(\overline{x}_2x_2\big) = Q(x_1)\cdot Q(x_2). \end{aligned}$$

(iv)-(vi) Follow easily from direct computations.

□

Remark 8.1.5
The points (v) and (vi) are not true for arbitrary elements of $\mathrm{Cl}(V,q)$. For example, $x\overline{x}$ may fail to be an element of K.

8.1.2 Vectors and the Clifford group

Definition 8.1.6 (Vectors)
The elements of the subspace $K \oplus V$ will be called *vectors*. We will denote the set $K \oplus V \cup \{\infty\}$ by $\hat{V}_{\text{ext}}$, and its elements will be called *extended vectors*.

As we saw before, the quadratic form is multiplicative when restricted to vectors. Moreover, vectors lead to the construction of a multiplicative subgroup of $\mathrm{Cl}(V, q)$ called the *Clifford group* as follows.

Proposition 8.1.7
Let $x \in \mathrm{Cl}(V, q)$ be a product of vectors. Then, x is invertible if and only if $q(x) \neq 0$, and its inverse is then given by $x^{-1} = \frac{1}{q(x)}\overline{x}$. In particular, the inverse of a product of vectors is again a product of vectors.

Definition 8.1.8 (Clifford group)
The group of all products of invertible vectors is called the *Clifford group* of $\mathrm{Cl}(V, q)$ and is denoted by $\Gamma_{\mathrm{Cl}(V,q)}$, or just by Γ. For the standard Clifford algebras (see Example 8.1.1), we will denote the Clifford group by $\Gamma_n(K)$ or just Γ_n if $K = \mathbb{R}$.

Remark 8.1.9
Later on we will restrict ourselves to an *anisotropic quadratic form*, so that the Clifford group consists of *all* products of non-zero vectors.

Example 8.1.10
We have $\Gamma_0 = \mathbb{R}^\times$, $\Gamma_1 = \mathbb{C}^\times$ and $\Gamma_2 = \mathbb{H}^\times$. In higher dimensions, it is not true that $\Gamma_n = \mathrm{Cl}_n \setminus \{0\}$.

Any invertible element $x \in \mathrm{Cl}(V, q)$ gives rise to a linear automorphism

$$\begin{aligned} \rho_x : \mathrm{Cl}(V, q) &\longrightarrow \mathrm{Cl}(V, q) \\ y &\longmapsto xy{x'}^{-1}. \end{aligned}$$

Lemma 8.1.11
For any $x \in \Gamma$, then $\rho_x \in O(K \oplus V, q)$.

Proof. First, we prove that $\rho_x(y) \in K \oplus V$ for any $y \in K \oplus V$. If x is a vector, then the equality $x\overline{y} + y\overline{x} = 2 \cdot b(x, y)$ implies $x\overline{y}x = 2 \cdot b(x, y)x - q(x)y$ and thus

$$x\overline{y}x^* = x\overline{y}x = 2 \cdot b(x, y)x - q(x)y \in K \oplus V.$$

Hence, we have $xy{x'}^{-1} \in K \oplus V$. Now, proceed by induction for finite products of vectors. Finally, we obtain:

$$\begin{aligned} q\big(\rho_x(y)\big) &= xy{x'}^{-1} \cdot \overline{xy{x'}^{-1}} = xy\,q\big(x^{-1}\big)\,\overline{y}\,\overline{x} \\ &= \frac{1}{q(x)} xy\overline{y}\,\overline{x} = q(y). \end{aligned}$$

□

Remark 8.1.12
The map ρ_x corresponds to $R_x \circ R_1$, where R_x, respectively R_1, denotes the

reflection with respect to the hyperplane perpendicular to x, respectively to 1. Indeed, we see that the two reflections are given by

$$R_x(y) = y - 2\frac{b(x,y)}{q(x)} \cdot x$$
$$R_1(y) = y - 2\frac{\Re(y)}{1}1 = -\overline{y}.$$

Then, an easy computation gives $R_x \circ R_1 = \rho_x$.

Lemma 8.1.13
If a and b are two elements of Γ, then

$$ab^{-1} \in K \oplus V \Leftrightarrow a^*b \in K \oplus V.$$

Proof. We have

$$\begin{aligned} ab^{-1} \in K \oplus V &\Leftrightarrow b^{*-1}a^* \in K \oplus V \\ &\Leftrightarrow \rho_{b^*}\left(b^{*-1}a^*\right) \in K \oplus V \\ &\Leftrightarrow a^*b\frac{1}{q(b)} \in K \oplus V. \end{aligned}$$

□

Lemma 8.1.14
For $x, y \in \Gamma$, then we have $\Re(x \cdot y \cdot x^{-1}) = \Re y$.

Proof. First, notice that if the claim is true when x is a vector, then the general case follows by induction. Hence, we suppose that we have $x = x_0 + x_v \in K \oplus V$ with $x_0 \in K$ and $x_v \in V$. We also write $y = \sum_I y_I \cdot v_I$, where $y_I \in K$ are the components of y with respect to the basis $\{v_I\}$ of $\mathrm{Cl}(V, q)$. Now, using the K-linearity of $\Re$, we get

$$\begin{aligned} \Re(x \cdot y \cdot x^{-1}) &= \sum_I y_I \cdot \Re\big(x \cdot v_I \cdot x^{-1}\big) \\ &= \sum_I y_I \cdot \Re\left((x_0 + x_v) \cdot v_I \cdot \frac{1}{q(x)}(x_0 - x_v)\right) \\ &= \Re(y) + \frac{1}{q(x)} \sum_{I \neq \emptyset} \Re\big(x_0^2 \cdot v_I - x_0 \cdot v_I\, x_v + x_0 \cdot x_v\, v_I - x_v\, v_I\, x_v\big) \\ &= \Re(y) + \frac{1}{q(x)} \sum_{I \neq \emptyset} \Re\left(-x_0 \cdot v_I\, x_v + x_0 \cdot x_v\, v_I - x_v\, v_I\, x_v\right). \end{aligned}$$

If $|I| = 1$, then $\Re(x_0 \cdot v_I\, x_v) = \Re(x_0 \cdot x_v\, v_I)$ and $\Re(x_v\, v_I\, x_v) = 0$. On the other hand, if $|I| \geq 2$, then all three terms are zero. Therefore, we have $\Re(x \cdot y \cdot x^{-1}) = \Re(y)$, as required. □

8.2 Clifford matrices and their action on the ambient space

Consider $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}(2; \Gamma_{\mathrm{Cl}(V,q)} \cup \{0\})$. We want to determine the conditions under which g induces a bijection $g : \hat{V}_{\mathrm{ext}} \longrightarrow \hat{V}_{\mathrm{ext}}$ (see Definition

8.1.6) via $x \longmapsto (ax+b)(cx+d)^{-1}$. First, we remark that we must have

$$g(0), g(\infty), g^{-1}(0), g^{-1}(\infty) \in \hat{V}_{\text{ext}},$$

which gives the following.

Proposition 8.2.1
If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Mat}(2; \Gamma_{\text{Cl}(V,q)} \cup \{0\})$ *induces a bijection* $\hat{V}_{\text{ext}} \longrightarrow \hat{V}_{\text{ext}}$*, then we must have*

$$bd^{-1}, ac^{-1}, -a^{-1}b, -c^{-1}d \in \hat{V}_{\text{ext}}$$

or equivalently (see Lemma 8.1.13)

$$b^*d, a^*c, ab^*, cd^* \in \hat{V}_{\text{ext}}.$$

Another obvious condition is that $cx+d$ has to be either zero or invertible (see Proposition 8.1.7). This is automatically fulfilled if V is a normed vector space and q is a non-zero multiple of the norm or, more generally, if q is anisotropic. Also, we don't want $ax+b$ and $cx+d$ to be zero at the same time. If this happens, then a and c must be different from zero which gives $a^{-1}b = c^{-1}d$ and thus

$$ad^* = a\left(b^* {a^{-1}}^* c^*\right) = a\left(a^{-1}b\right)^* c^* \overset{8.2.1}{=} aa^{-1}bc^* = bc^*.$$

Definition 8.2.2 (Determinant of a Clifford matrix)
Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Mat}(2; \Gamma_{\text{Cl}(V,q)} \cup \{0\})$. The *determinant* of g is $\Delta(g) = ad^* - bc^*$.

Proposition 8.2.3
If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Mat}(2; \Gamma_{\text{Cl}(V,q)} \cup \{0\})$ *induces a bijection* $\hat{V}_{\text{ext}} \longrightarrow \hat{V}_{\text{ext}}$*, then we must have* $\Delta(g) \neq 0$*.*

Proposition 8.2.4
The composition of applications is induced by the composition of matrices.

Proof. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\tilde{g} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{Mat}(2; \Gamma_{\text{Cl}(V,q)} \cup \{0\})$, we compute:

$$\begin{aligned} g\big(\tilde{g}(x)\big) &= \big(a(\alpha x+\beta)(\gamma x+\delta)^{-1} + b\big)\big(c(\alpha x+\beta)(\gamma x+\delta)^{-1} + d\big)^{-1} \\ &= \big(a(\alpha x+\beta) + b(\gamma x+\delta)\big)(\gamma x+\delta)^{-1}\big(c(\alpha x+\beta)(\gamma x+\delta)^{-1} + d\big)^{-1} \\ &= \big(a(\alpha x+\beta) + b(\gamma x+\delta)\big)\Big(\big(c(\alpha x+\beta)(\gamma x+\delta)^{-1} + d\big)(\gamma x+\delta)\Big)^{-1} \\ &= \big((a\alpha+b\gamma)x + a\beta + b\delta\big)\big(c(\alpha x+\beta) + d(\gamma x+\delta)\big)^{-1} \\ &= \big((a\alpha+b\gamma)x + (a\beta+b\delta)\big)\big((c\alpha+d\gamma)x + (c\beta+d\delta)\big)^{-1} \\ &= (g\cdot\tilde{g})(x). \end{aligned}$$

∎

In general, Δ is not a multiplicative function from the group of matrices inducing a bijection $\hat{V}_{\text{ext}} \longrightarrow \hat{V}_{\text{ext}}$ to $\mathrm{Cl}(V,q)$. However, we have the following result.

Proposition 8.2.5
Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tilde{g} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{Mat}(2; \Gamma_{\mathrm{Cl}(V,q)} \cup \{0\})$ *inducing bijections such that* $\Delta(\tilde{g}) = \Delta(\tilde{g})^*$. *Then,* $\Delta(g \cdot \tilde{g}) = \Delta(g) \cdot \Delta(\tilde{g})$.

Proof. We compute

$$\begin{aligned}\Delta(g\tilde{g}) &= a\alpha\beta^* c^* + a\alpha\delta^* d^* + b\gamma\beta^* c^* + b\gamma\delta^* d^* \\ &\quad - a\beta\gamma^* d^* - a\beta\alpha^* c^* - b\delta\gamma^* d^* - b\delta\alpha^* c^* \\ &\overset{8.2.1}{=} a\alpha\delta^* d^* + b\gamma\beta^* c^* - a\beta\gamma^* d^* - b\delta\alpha^* c^* \\ &= a\,\Delta(\tilde{g})\,d^* - b\,\Delta(\tilde{g})^*\,c^* \\ &= \Delta(g) \cdot \Delta(\tilde{g}).\end{aligned}$$

□

Proposition 8.2.6
Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}(2; \Gamma_{\mathrm{Cl}(V,q)} \cup \{0\})$. *Then, g induces the identity if and only if g is a diagonal matrix with* $a = d \in Z(\mathrm{Cl}(V,q))^\times$.

Proof. Suppose that g induces the identity. Substituting $x = 0, \infty$ and 1 gives $b = 0$, $c = 0$ and $a = d$. Now, we have $ax = xa$ for every $x \in V$ which means $ax = xa$ for every $x \in \mathrm{Cl}(V,q)$, as required. The converse is obvious. □

We want to determine the inverse of the matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If we write $y = (ax+b)(cx+d)^{-1}$, then we have $(yc-a)x = b-yd$. Except when $y = ac^{-1}$, which happens when $x = \infty$, the term $(yc-a)$ is invertible, which yields

$$x = (yc-a)^{-1}(b-yd) = (d^*y - b^*)\,(-c^*y + a^*)^{-1}.$$

This motivates to look at $g^* = \begin{pmatrix} d^* & -b^* \\ -c^* & a^* \end{pmatrix}$, and we find the products

$$\begin{aligned} g \cdot g^* &= \begin{pmatrix} \Delta(g) & 0 \\ 0 & \Delta(g)^* \end{pmatrix} \\ g^* \cdot g &= \begin{pmatrix} \Delta(g^*) & 0 \\ 0 & \Delta(g^*)^* \end{pmatrix}. \end{aligned} \tag{8.2}$$

A necessary condition for the invertibility of the map induced by g is that $\Delta(g) \in Z(\mathrm{Cl}(V,q))^\times$, $\Delta(g) = \Delta(g)^*$ and $\Delta(g^*) = \Delta(g^*)^*$.

Definition 8.2.7 (Clifford matrices)
We define $\mathrm{GL}(\Gamma_{\mathrm{Cl}(V,q)})$, or just $\mathrm{GL}(\Gamma)$, to be the set of matrices of the form $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}(2; \mathrm{Cl}(V,q))$ which satisfy the following conditions:

- $a, b, c, d \in \Gamma \cup \{0\}$;

- $\Delta(g) = ad^* - bc^* \in K^\times$;
- $ab^*, cd^*, c^*a, d^*b \in K \oplus V$.

We also define $\mathrm{SL}(\Gamma) = \{g \in \mathrm{GL}(\Gamma) : \Delta(g) = 1\}$ and $\mathrm{PSL}(\Gamma) = \mathrm{SL}(\Gamma)/_{\{\pm I\}}$.

Remarks 8.2.8
- Using Theorem 8.2.11 below, we will see that $\mathrm{GL}(\Gamma)$ is a group.
- If $\Delta(g), \Delta(g^*) \in K$, then equations (8.2) imply that $g = \Delta(g) \cdot g^{*-1}$ and $g^* = \Delta(g^*) \cdot g^{-1}$ which gives $\Delta(g) = \Delta(g^*)$.
- Scaling all the coefficients of a matrix by an element of $K^\times$ does not change the induced application. Therefore, if $K = \mathbb{R}$, we can suppose that $\Delta(g) = \pm 1$.

Definition 8.2.9 (Trace of a matrix)
Let $g \in \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}(2; \Gamma_{\mathrm{Cl}(V,q)} \cup \{0\})$. Then, the *trace* of g, denoted by $\mathrm{Tr}\, g$, is defined to be $\mathrm{Tr}(g) = a + d^*$.

Following [Li11], we also have the following definition.

Definition 8.2.10 (Vectorial element)
A non-trivial Clifford matrix $g \in \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(\Gamma)$ is called *vectorial* if $b^* = b$, $c^* = c$ and $\mathrm{Tr}\, g \in \mathbb{R}$.

8.2.1 Algebraic characterization

From now on, we suppose that the quadratic form q is anisotropic. In this setting, every non-zero vector $v \in V$ is invertible and in particular, Γ consists of all products of non-zero vectors. We also request that the only matrices inducing the identity map are multiples of the identity matrix by an element of $K^\times$. This is the case in the following two situations:

1. The center of the Clifford algebra is trivial.
   This happens when $\dim_K V$ is even (see Proposition 8.1.2) or when $K = \mathbb{R}$, V is an infinite-dimensional Hilbert space with norm $\|\cdot\|^2$ and $q(v) = \lambda \cdot \|v\|^2$ for some fixed $\lambda \in \mathbb{R}^\times$ (see Proposition 8.1.3).
2. The Clifford algebra is a standard Clifford algebra (see Example 8.1.1). Then, a matrix $g \in \mathrm{Mat}(2; \Gamma_n(K) \cup \{0\})$ induces a map on both $\hat{K}^n_{\mathrm{ext}}$ and $\hat{K}^{n+1}_{\mathrm{ext}}$. Hence, we can consider matrices $g \in \mathrm{Mat}(2, \mathrm{Cl}(V_n, q_n))$ which induce bijections on $\hat{K}^n_{\mathrm{ext}} \longrightarrow \hat{K}^n_{\mathrm{ext}}$ and $\hat{K}^{n+1}_{\mathrm{ext}} \longrightarrow \hat{K}^{n+1}_{\mathrm{ext}}$. If such a matrix induces the identity on both $\hat{K}^n_{\mathrm{ext}}$ and $\hat{K}^{n+1}_{\mathrm{ext}}$, then g is the multiple of the identity matrix by an element of $K^\times$ (see Proposition 8.1.3). This second instance is used by [Ahl85] and [Wat93].

Theorem 8.2.11
Let (V, q) be an anisotropic quadratic space over a field K such that the center of the Clifford algebra $\mathrm{Cl}(V, q)$ is K, and let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}(2; \Gamma_{\mathrm{Cl}(V,q)} \cup \{0\})$. Then, g induces a bijection on $\hat{V}_{\mathrm{ext}}$ if and only if $g \in \mathrm{GL}(\Gamma)$.

Theorem 8.2.12
Let $n \in \mathbb{N}_0$ *and* $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}(2; \Gamma_n(K) \cup \{0\})$. *Then,* g *induces a bijection on* $\hat{K}^n_{\mathrm{ext}}$ *and* $\hat{K}^{n+1}_{\mathrm{ext}}$ *if and only if* $g \in \mathrm{GL}(\Gamma)$.

Proof. We already saw that these algebraic conditions are necessary.
Suppose now that $g \in \mathrm{GL}(\Gamma)$. First, we want to check whether the expression $(cx+d)^{-1}$ actually makes sense. If $c = 0$, then d is non-zero and thus is invertible. On the other hand, if $c \neq 0$, then we have $cx+d = c(x+c^{-1}d)$, where the second term is either zero or an element of $K \oplus V$ (and thus invertible). We now prove that the image of g lies in $\hat{V}_{\mathrm{ext}}$. For $x, y \in K \oplus V$, a direct computation gives

$$\begin{aligned}(yc^* + d^*)(ax+b) - (ya^* + b^*)(cx+d) &= -y \cdot \Delta(g) + \Delta(g^{-1})x \\ &= \Delta(g)\,(x-y)\end{aligned}$$

which leads to

$$gx - (gy)^* = \Delta(g)\,(yc^* + d^*)^{-1}(x-y)(cx+d)^{-1}. \tag{8.3}$$

Letting $y = 0$ gives

$$(gx - g0)^{-1} = \Delta(g) \cdot \left(cd^* + dx^{-1}d^*\right),$$

which is a vector (or ∞) by Proposition 8.2.1 and Lemma 8.1.11. Hence, $gx - g0$ is a vector (or ∞) and so is gx, as required. Now, equation (8.3) can be rewritten

$$gx - gy = \Delta(g)\,(yc^* + d^*)^{-1}(x-y)(cx+d)^{-1},$$

which implies the injectivity of g. Now, the matrix $g^* = \begin{pmatrix} d^* & -b^* \\ -c^* & a^* \end{pmatrix}$ satisfies the algebraic conditions which insure that g^* induces a map from $\hat{V}_{\mathrm{ext}}$ to $\hat{V}_{\mathrm{ext}}$, and this map is the inverse to the one induced by g. □

8.2.2 Typical transformations of a Hilbert space

In this section, we suppose that V is a real Hilbert space and $q = \lambda \cdot \|\cdot\|^2$, for some $\lambda \in \mathbb{R}^\times$. We express the typical transformations of the space V by means of Clifford matrices as follows. First, we write the inversion ι with respect to the unit sphere $S(0,1)$:

$$\iota = \begin{pmatrix} 0 & -\lambda \\ 1 & 0 \end{pmatrix} \in \mathrm{GL}(\Gamma).$$

The inversion with respect to the sphere with center $a \in V$ and radius r, is written

$$\iota_{S(a,r)} = f \circ \iota \circ f^{-1}, \quad f = \begin{pmatrix} r & a \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}(\Gamma).$$

The reflection with respect to the hyperplane $P(a,\alpha) = \{v \in V : \langle v, a\rangle = \alpha\}$ is given by:

- For $\alpha \neq 0$: $f \circ \iota \circ f^{-1}$, with $f = \iota \circ \varphi$, where

$$\varphi(x) = \left\|\frac{a}{2\alpha}\right\| \cdot x + \frac{a}{2\alpha}.$$

- For $\alpha = 0$: $g \circ \iota \circ g$ with $g = \psi \circ \iota \circ \varphi$, where

$$\varphi(x) = \|a\| \cdot x + a, \quad \psi(x) = x - \frac{a}{2 \cdot \|a\|^2}.$$

We know present a lemma which will have many important consequences.

Lemma 8.2.13
Any Clifford matrix g induces an application $g : \hat{V}_{\mathrm{ext}} \longrightarrow \hat{V}_{\mathrm{ext}}$ which is a composition of the following transformations:

translation	$T_\mu = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}$	$x \longmapsto x + \mu$, $\mu \in K \oplus V$
inversion	$\iota = \begin{pmatrix} 0 & -\lambda \\ 1 & 0 \end{pmatrix}$	$x \longmapsto -\lambda x^{-1}$
dilatation	$D_r = \begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix}$	$x \longmapsto r^2\, x$, $r \in \mathbb{R}_+^\times$
orthogonal	$S_a = \begin{pmatrix} a & 0 \\ 0 & a' \end{pmatrix}$	$x \longmapsto axa^*$, $a \in \Gamma$, $Q(a) = 1$
reflection	$R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$x \longmapsto -x$.

In the sequel, we use the function

$$\varepsilon : \mathbb{R}^\times \longrightarrow \{0,1\},\ x \longmapsto \begin{cases} 0 & \text{if } x > 0 \\ 1 & \text{otherwise.} \end{cases}$$

Remarks 8.2.14 (i) If $Q(a) \neq 1$, then the transformation $x \longmapsto axa^*$ can be written as follows:

$$S_a = D_{\sqrt{|Q(a)|}} \cdot S_{a/Q(a)} \cdot R^{\varepsilon(Q(a))}.$$

Hence, we can assume without loss of generality that $Q(a)$ is equal to 1.

(ii) If a is a vector, then S_a corresponds to the map ρ_a, which consists of the reflection with respect to the hyperplane perpendicular to 1, followed by the reflection with respect to the hyperplane perpendicular to a (see Remark 8.1.12).

(iii) In particular, the lemma implies that every Clifford matrix induced map is in Möb*. The reverse inclusion follows from (ii).

(iv) If $x \in V$, then $\iota(x) = \frac{x}{\|x\|^2}$. More generally, if $x = \alpha + v$ with $v \in V$ and $\alpha \in \mathbb{R}$, then

$$\iota(x) = \frac{-\lambda}{Q(\alpha + v)}\alpha + \frac{\lambda}{Q(\alpha + v)}v.$$

Proof. We distinguish the cases where $c = 0$ and $c \neq 0$.

- Let $c = 0$. In this setting, we have

$$gx = \frac{1}{\Delta(g)}axa^* + bd^{-1},$$

that is,

$$g = T_{bd^{-1}} \circ \underbrace{\left(D_{\sqrt{|\Delta(g)|}^{-1}} \circ R^{\varepsilon(\Delta(g))}\right)}_{1/\Delta(g)} \circ \underbrace{\left(D_{\sqrt{|Q(a)|}} \circ S_{a/Q(a)} \circ R^{\varepsilon(Q(a))}\right)}_{S_a}.$$

- Let $c \neq 0$. First, we see that $b = ac^{-1}d - \Delta(g) \cdot c^{*-1}$. Then, we compute

$$\begin{aligned} gx &= \Big(ac^{-1}(cx+d) - \Delta(g) \cdot c^{*-1}\Big)(cx+d)^{-1} \\ &= ac^{-1} - \Delta(g) \cdot c^{*-1}(x + c^{-1}d)^{-1}c^{-1}. \end{aligned}$$

  Furthermore, we can write

$$x^{-1} = R^{\varepsilon(\lambda)} \circ D_{1/\sqrt{|\lambda|}} \circ \iota(x).$$

  Since $ac^{-1}, c^{-1}d \in V$, we can combine the different transformations above, and conclude.

□

Using the decomposition provided by the Lemma 8.2.13, we get the following result.

Proposition 8.2.15
Let W be a subspace of the vector space V and let $g : \hat{W}_{\text{ext}} \longrightarrow \hat{W}_{\text{ext}}$ be a map induced by a Clifford matrix. Then, the trivial extension $g : \hat{V}_{\text{ext}} \longrightarrow \hat{V}_{\text{ext}}$ corresponds to the Poincaré extension of g (see Definition 3.2.4).

Since the group of Möbius transformations of $\mathbb{R}^{n+1}$ is generated by reflections in spheres and hyperplanes, we have the following corollary.

Corollary 8.2.16
The map which sends a Clifford matrix $g \in \mathrm{GL}(\Gamma_n)$ to the induced map $g : \hat{\mathbb{R}}^{n+1}_{\text{ext}} \longrightarrow \hat{\mathbb{R}}^{n+1}_{\text{ext}}$ yields a group isomorphism $\mathrm{GL}(\Gamma_n)/_{\mathbb{R}^\times} \cong \mathrm{Möb}(\mathbb{R}^{n+1})$.

The Lemma 8.2.13, together with Remark 8.2.14, also give the analogue of the Corollary 8.2.16 in the infinite-dimensional setting.

Proposition 8.2.17
Let $\mathscr{H}$ be a Hilbert space, of finite or infinite dimension, and consider a codimension one subspace $\mathscr{H}'$ of $\mathscr{H}$. If we let Γ' be the Clifford group associated to $\mathrm{Cl}(\mathscr{H}', q')$, where q' is the restriction of $\|\cdot\|^2$ to $\mathscr{H}'$, then we get

$$\mathrm{GL}(\Gamma')/_{\mathbb{R}^*} \cong \mathrm{Möb}^*(\mathscr{H}).$$

The *motto* behind the result is the following: any map coming from a Clifford matrix can be written as the Poincaré extension of a map coming from a Clifford matrix $g \in \mathrm{GL}(\Gamma_{V_f})$, where V_f is a *finite-dimensional* subspace of $\mathscr{H}$. This will be made more precise by Proposition 8.2.18 below. Before that, let us recall the definition of a direct system and a direct limit of groups[1]. The starting point is a *directed*, or *filtered*, set $(I, \leq)$. It means that $\leq$ is a preorder on I such that any pair of elements $i, j \in I$ has an upper bound (i.e. there exists $k \in I$ such that $i \leq k$ and $j \leq k$). Then, we consider a collection of groups $\{G_i\}_{i \in I}$ and we suppose that for every $i \leq j$ we have a homomorphism $\varphi_{i,j} : G_i \longrightarrow G_j$. Finally, it is required that the morphisms $\varphi_{i,j}$ enjoy the following properties:

- For every $i \in I$, we have $\varphi_{i,i} = \mathrm{id}_{G_i}$.

[1] More generally, the notion of direct limit can be defined using a universal property in any category.

- For every $i \leq j \leq k$ in I, we have $\varphi_{j,k} \circ \varphi_{i,j} = \varphi_{i,k}$.

The collection $(G_i, \varphi_{i,j})$ is called a *direct system.*

Now, we consider on the disjoint union $\bigsqcup_i G_i$ the equivalence relation $\sim$ defined as follows: for $g \in G_i$ and $h \in G_j$, we have $g \sim h$ if and only if there exists $k \in I$ such that $\varphi_{i,k}(g) = \varphi_{j,k}(h)$ in G_k. We can endow the set $\varinjlim_{i\in I} G_i := \bigsqcup_{i\in I} G_i/_{\sim}$ with a group structure as follows: for $g \in G_i$ and $h \in G_j$, we consider an upper bound k of i and j and we define

$$[g] \cdot [h] := \big[\varphi_{i,k}(g) \cdot \varphi_{j,k}(h)\big],$$

where $[-]$ denotes the equivalence class of an element. It is easily shown that this definition is independent of the choice of the upper bound and indeed defines a group structure on $\varinjlim_{i\in I} G_i$. The group $\varinjlim_{i\in I} G_i$ is called the *direct limit* of the system $\big(G_i, \varphi_{i,j}\big)$. A typical direct system is given by the lattice of finite-dimensional subspaces of a given vector space and the morphisms are just the inclusions. We are now ready to present the proposition.

Proposition 8.2.18
For $(\mathscr{H}, \|\cdot\|)$ a Hilbert space, we denote by $\mathcal{V}$ the set of all finite-dimensional subspaces of $\mathscr{H}$. For every $V \in \mathcal{V}$, we denote by $\mathrm{GL}(\Gamma_V)$ the group of Clifford matrices corresponding to the quadratic space $(V, \|\cdot\|^2)$. Then, we have

$$\varinjlim_{V\in\mathcal{V}} \mathrm{GL}(\Gamma_V) \cong \mathrm{GL}(\Gamma),$$

where $\mathrm{GL}(\Gamma)$ is the group of Clifford matrices corresponding to $(\mathscr{H}, \|\cdot\|^2)$.

Proof. First, we notice that the direct limit is well-defined: the inclusion of finite subspaces of $\mathscr{H}$ induces an order on $\mathcal{V}$ and the upper bound of two elements in $\mathcal{V}$ is given by the direct sum. Moreover, if $V \subset V'$ are two elements of $\mathcal{V}$, then the map $\varphi_{V,V'} : \mathrm{GL}(\Gamma_V) \longrightarrow \mathrm{GL}(\Gamma_{V'})$ is given by the Poincaré extension. Therefore, $(\mathrm{GL}(\Gamma_V), \varphi_{V,V'})$ is a direct system.

Now, the collection of group homomorphisms $\mathrm{GL}(\Gamma_V) \longrightarrow \mathrm{GL}(\Gamma)$ given by Poincaré extensions is compatible with the direct system, which means that we get an injective homomorphism $\psi : \varinjlim_{V\in\mathcal{V}} \mathrm{GL}(\Gamma_V) \longrightarrow \mathrm{GL}(\Gamma)$. The surjectivity of ψ is given by Lemma 8.2.13. □

Lemma 8.2.13 is at the basis of the following result.

Proposition 8.2.19
Let $g \in \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}(2; \Gamma_{\mathrm{Cl}(V,q)} \cup \{0\})$. Then, $\Re(\mathrm{Tr}(g))$ is invariant under conjugation by an element of $\mathrm{GL}(\Gamma)$.

Proof. Let $h \in \mathrm{GL}(\Gamma)$. In order to show that $\Re\left(\mathrm{Tr}(h \cdot g \cdot h^{-1})\right) = \Re(\mathrm{Tr}(g))$, it is sufficient to prove the equality for $h = T_\mu, \iota, D_r, S_a$ and R of Lemma 8.2.13. We consider these cases separately.

- For $\mathbf{h} = \mathbf{T}_\mu$: We have $h = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}$, for some $\mu \in K \oplus V$, which gives

$$\mathrm{Tr}(h \cdot g \cdot h^{-1}) = a + \mu\, c - \mu^*\, c^* + d^* = \mathrm{Tr}(g) - \mu(c - c^*).$$

  Now, since μ is an element of $K \oplus V$ and since both the real and the 1-graded part of $c - c^*$ are zero, then we have $\Re\left(\mu(c - c^*)\right) = 0$.

- For $\mathbf{h} = \iota$: A direct computation shows that $\mathrm{Tr}(h \cdot g \cdot h^{-1}) = d + a^*$, which proves the claim.

- For $\mathbf{h} = \mathbf{S}_\alpha$: We first get

$$\mathrm{Tr}(h \cdot g \cdot h^{-1}) = \alpha \cdot (a + d^*) \cdot \bar{\alpha} = \alpha \cdot (a + d^*) \cdot \alpha^{-1}.$$

  Lemma 8.1.14 allows us to conclude.

The two remaining cases are very easy. □

8.3 Isometries of the hyperbolic space

In this section, we suppose that $(\mathscr{H}, \|\cdot\|)$ is a Hilbert space of Hilbert dimension $\mathfrak{n}$ and we let $q = \|\cdot\|^2$, i.e. $\lambda = 1$ in the above notations. We consider two orthogonal unit vectors $e_0, u \in \mathscr{H}$ which lead to the orthogonal decomposition

$$\mathscr{H} = \langle e_0 \rangle \oplus V \oplus \langle u \rangle. \tag{8.4}$$

As we saw in Section 3.2.1, the set

$$\mathcal{U}^{\mathfrak{n}} = \mathcal{U}_{\mathscr{H}} = \{x \in \mathscr{H} : l_u(x) > 0\},$$

together with the distance function given by

$$d = d_{\mathcal{U}^{\mathfrak{n}}} = \operatorname{arcosh}\left(1 + \frac{d_{\mathscr{H}}(x, y)}{2 \cdot l_u(x) \cdot l_u(y)}\right), \quad \forall x, y \in \mathcal{U}^{\mathfrak{n}},$$

is a model of the hyperbolic space of dimension $\mathfrak{n}$. If we identify $\langle e_0 \rangle \oplus V$ with $\mathbb{R} \oplus V$, then $\mathrm{GL}(\Gamma_{(V \oplus \langle u \rangle, q)})$ acts on $\hat{\mathscr{H}}$. Moreover, since $\mathrm{GL}(\Gamma_{(V,q)})$ is a subgroup of $\mathrm{GL}(\Gamma_{(V \oplus \langle u \rangle, q)})$, it also acts on $\hat{\mathscr{H}}$. In fact, we will show that matrices of positive determinant Δ in $\mathrm{GL}(\Gamma_{(V,q)})$ preserve $\mathcal{U}$.

Lemma 8.3.1
With the above decomposition, we have:

1. $\overline{x\,u} = -x\,u$, *for every* $x \in \mathbb{R} \oplus V$;

2. $u \cdot \overline{x} = x^* \cdot u$, *for every* $x \in \Gamma_{\mathbb{R} \oplus V}$.

Proof. The first property comes from a direct computation using (vi) of Proposition 8.1.4. The second property follows from the first. Indeed, if x is an element of $\mathbb{R} \oplus V$, then we have

$$u \cdot \overline{x} = -\overline{u} \cdot \overline{x} = -\overline{x \cdot u} = xu = x^*\, u.$$

The general case follows by induction. □

Proposition 8.3.2
Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(\Gamma_{(V,q)})$ *such that* $\Delta(g) > 0$. *Then,* g *preserves* $\mathcal{U}$.

Proof. We write $y = x + \mu u$ with $x \in \mathbb{R} \oplus V$ and $\mu > 0$. We then compute

$$\begin{aligned} gy &= (ay+b)(cy+d)^{-1} \\ &= \frac{1}{q(cy+d)} \cdot (ay+b) \cdot \left(\overline{y}\,\overline{c} + \overline{d}\right) \\ &= \frac{1}{q(cy+d)} \cdot \left(q(y)a\overline{c} + b\overline{d} + ax\overline{d} + b\overline{x}\,\overline{c} + \mu(au\overline{d} - bu\overline{c})\right) \\ &\overset{8.3.1}{=} \frac{1}{q(cy+d)} \cdot \left(q(y)a\overline{c} + b\overline{d} + ax\overline{d} + b\overline{x}\,\overline{c}\right) + \frac{\Delta(g)}{q(cy+d)}\mu \cdot u. \end{aligned}$$

Now, the first term is an element of $\hat{V}_{\text{ext}}$ and the coefficient of u in the second term is positive, as required. ☐

Theorem 8.3.3
Let $(\mathscr{H}, \|\cdot\|)$ be a Hilbert space of finite or infinite dimension. Let $\mathcal{U} = \mathcal{U}_{\mathscr{H}}$ be the upper half-space model of the hyperbolic space of Hilbert dimension $\dim \mathscr{H}$ *defined by $\mathscr{H}$. If V is a codimension 2 subspace of $\mathscr{H}$, with the decomposition of $\mathscr{H}$ as in (8.4), then $\mathrm{SL}(\Gamma_{\mathrm{Cl}(V,\|\cdot\|)})$ acts by isometries on $\mathcal{U}$ via Poincaré extension.*

Remark 8.3.4
Combined with Corollary 8.2.16, the theorem gives the following well-known result: for every $n \in \mathbb{N}$, we have $\mathrm{Isom}^+ \mathbb{H}^{n+2} \cong \mathrm{PSL}(\Gamma_n)$.

Proof. We use the orthogonal decomposition

$$\mathscr{H} = \langle e_0 \rangle \oplus V \oplus \langle u \rangle,$$

where e_0 and u are two unit vector and we identify $\langle e_0 \rangle \oplus V$ with $\mathbb{R} \oplus V$. Because of Proposition 8.3.2, we only need to show that $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(\Gamma)$ preserves the hyperbolic distance. For two elements $y, \tilde{y} \in \mathcal{U}$, we write $y = x + \mu u$ and $\tilde{y} = \tilde{x} + \tilde{\mu} u$, with $\mu, \tilde{\mu} > 0$. Recall from the proof of Theorem 8.2.11

$$gy - g\tilde{y} = (\tilde{y}c^* + d^*)^{-1}(y - \tilde{y})(cy+d)^{-1}.$$

Now, we compute:

$$\begin{aligned} \frac{d_{\mathscr{H}}(gy, g\tilde{y})}{2 \cdot l_u(gy) \cdot l_u(g\tilde{y})} &= \frac{q(\tilde{y}c^* + d^*)^{-1} \cdot q(y - \tilde{y}) \cdot q(cy+d)^{-1}}{2 \cdot q(cy+d)^{-1} \cdot \mu \cdot q(c\tilde{y}+d)^{-1} \cdot \tilde{\mu}} \\ &= \frac{q\left((c\tilde{y}+d)^*\right)^{-1} \cdot q(y - \tilde{y})}{2 \cdot \mu \cdot q(c\tilde{y}+d)^{-1} \cdot \tilde{\mu}} \\ &= \frac{d_{\mathcal{H}}(y, \tilde{y})}{2 \cdot l_u(y) \cdot l_u(\tilde{y})}. \end{aligned}$$

This implies that $d_{\mathcal{U}}(gy, g\tilde{y}) = d_{\mathcal{U}}(y, \tilde{y})$, as required. ☐

Now that we have proved that Clifford matrices act by isometries of hyperbolic spaces, we can use Proposition 3.2.11 and Lemma 8.2.13 to distinguish the type of an isometry. In particular, we have the two following propositions.

Proposition 8.3.5 (Loxodromic Clifford isometries)
Let $g \in \mathrm{SL}(\Gamma_{\mathrm{Cl}(V,\|\cdot\|)})$ be a Clifford matrix. Then, g is loxodromic if and only if g is conjugate to a Clifford matrix of the type $\begin{pmatrix} r \cdot a & 0 \\ 0 & \frac{1}{r} \cdot a' \end{pmatrix}$, with $r \in \mathbb{R}$, $r > 0$ and $r \neq 1$, and $a \in \Gamma$ with $q(a) = 1$.

Proof. Follows from Proposition 3.2.11 and Lemma 8.2.13. □

Proposition 8.3.6 (Parabolic Clifford isometries)
Let $g \in \mathrm{SL}(\Gamma_{\mathrm{Cl}(V,\|\cdot\|)})$ be a Clifford matrix. Then, g is parabolic if and only if g is conjugate to a Clifford matrix of the type $\begin{pmatrix} a & b \\ 0 & a' \end{pmatrix}$, where $a \in \Gamma$, $q(a) = 1$, and $b \in \langle e_0 \rangle \oplus V$ is such that $a \cdot b = b \cdot a'$.

Proof. Follows from Proposition 3.2.11 and Lemma 8.2.13. The fact that b can be chosen such that $a \cdot b = b \cdot a'$ is a consequence of from Proposition 3.2.12. □

8.4 Final remarks and further questions

The machinery we have built up allows one to generalize directly several results known in the finite-dimensional setting as well as some results known in the separable infinite-dimensional case due to [Li11] to the arbitrary infinite-dimensional setting. We will mention a few of these generalizations without providing details and indicate some further interesting questions.

For a Hilbert space $\mathscr{H}$ with norm $\|\cdot\|$, the quadratic form

$$q : \mathrm{Cl}(\mathscr{H}, \|\cdot\|^2) \longrightarrow \mathbb{R}, \quad q(x) = \Re(\overline{x}x)$$

as presented in Section 8.1.1.3 naturally defines a norm $\|\cdot\|$ on the Clifford algebra $\mathrm{Cl}(V, q)$ via $\|x\|^2 = q(x)$ which coincides with the "natural" norm on the tensor algebra in the following sense:

- It is multiplicative on (extended) vectors:

$$q(v \cdot v') = q(v) \cdot q(v') = \|v\|^2 \cdot \|v'\|^2, \quad \forall v, v' \in \mathscr{H}.$$

- If $\{v_i\}_{i \in \mathcal{I}}$ is an algebraic basis of $\mathscr{H}$ and if $I, J \subset \mathcal{I}$ are two different multi-indices (see Section 8.1.1), then one easily checks that

$$q(a_I\, v_I + a_J\, v_J) = q(a_I\, v_I) + q(a_J\, v_J), \quad \forall a_I, a_J \in \mathbb{R}.$$

This allows to define the norm of a Clifford matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(\Gamma)$ via

$$\|g\|^2 = \|a\|^2 + \|b\|^2 + \|c\|^2 + \|d\|^2.$$

In this setting, we have the following definition.

Definition 8.4.1 (Discrete subgroup of GL(Γ))
A subgroup G of $\mathrm{GL}(\Gamma)$ is *discrete* if for every convergent sequence $\{g_n\} \subset G$ such that $\lim_{n\to\infty} g_n = g \in G$, one has $g_n = g$ for all sufficiently large n.

We have now a direct generalization of Li's result (see [Li11, Theorem 3.1]) to the non-separable case.

Theorem 8.4.2 (A generalization of Jørgensen's trace inequality)
Let $f, g \in \mathrm{SL}(\Gamma_V)$ be such that f is loxodromic, the commutator $[f, g]$ is vectorial (see Definition 8.2.10) and such that the group $\langle f, g\rangle$ is discrete and non-elementary (i.e. every $\langle f, g\rangle$-orbit is infinite). Then, we have

$$\left|\mathrm{Tr}(f)^2 - 4\right| + |\mathrm{Tr}([f, g]) - 2| \geq 1.$$

Since any isometry induced from a Clifford matrix in $\mathrm{GL}(\Gamma_V)$ comes from an element in $\mathrm{GL}(\Gamma_{V_f})$ for a finite-dimensional subspace V_f of $\mathscr{H}$ by Proposition 8.2.18, we can also derive generalizations of classical results. For example, we obtain the following description of elliptic elements.

Lemma 8.4.3 ([Wat93, Lemma 13])
Let $g \in \mathrm{SL}(\Gamma_V)$ be a matrix inducing an elliptic isometry. Then, there exist an even positive integer $n = 2k$, real numbers $\theta_0, \dots, \theta_k \in [0, 2\pi)$ and an orthonormal family of vectors $\{v_0, \dots, v_{n-1}\} \subset V$ such that g is conjugate to the Clifford matrix $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda' \end{pmatrix}$, where $\lambda = \prod_{i=0}^{k-1} r_i$, with

$$r_i = \cos\theta_i + v_{2i}\, v_{2i+1} \sin\theta_i, \quad \forall 0 \leq i \leq k-1.$$

Moreover, the r_i commute, and so do $\lambda, \lambda', \lambda^*, \overline{\lambda}$.

In a different context, isometries of infinite-dimensional hyperbolic spaces can be constructed with tools from algebraic geometry and more specifically by means of the *Cremona group.* For a field k and a positive integer n, the Cremona group $\mathrm{Cr}_n(k)$ is the group of k-automorphisms of the k-algebra $k(x_1, \dots, x_n)$. This group can also be viewed as the group $\mathrm{Bir}(\mathbb{A}^n_k)$ of birational maps[2] of the affine space $\mathbb{A}^n_k$, which is in turn equal to $\mathrm{Bir}(\mathbb{P}^n_k)$. Now, it can be shown that the second Cremona group $\mathrm{Cr}_2(k)$ acts by isometries on a certain infinite-dimensional hyperbolic space $\mathbb{H}^\infty(\mathbb{P}^2_k)$ associated to the projective surface $\mathbb{P}^2_k$. The construction of the space $\mathbb{H}^\infty(\mathbb{P}^2_k)$ is not immediate at all and relies on a direct limit of groups associated to blow ups. References are [CL13] and [Can15], for example, wherein many typical results are shown to remain valid. In particular, there exists a way to detect if an isometry obtained in such an algebro-geometrical way is elliptic, parabolic or loxodromic (see [Can15, Theorem 4.6]). It would be interesting to study the relation between our group of isometries $\mathrm{PSL}(\Gamma_{\mathrm{Cl}(V, \|\cdot\|)})$ and the Cremona group $\mathrm{Cr}_2(k)$.

[2]For two algebraic varieties X and Y, a rational map $f : X \dashrightarrow Y$ is a morphism of algebraic varieties defined on a dense open subset of X. We identify two rational maps $f, f' : X \dashrightarrow Y$ if they coincide on a dense open subset of X. If a rational map $f : X \dashrightarrow Y$ is *dominant* (i.e. if its image is dense in Y) and if there exists another dominant rational map $f' : Y \dashrightarrow X$ such that $f \circ f'$ and $f' \circ f$ are equivalent to the identity, then f is called *birational.* For a given algebraic variety X, the set $\mathrm{Bir}(X)$ is a group.

APPENDIX A

Data

A.1 Kaplinskaya's prisms in dimension 3

A.1.1 The compact case

We present in this section the invariants of the compact Coxeter prisms given in Figure A.1 (see [Kap74]). Some details regarding the groups Γ^1_m, Γ^2_m and $\Gamma^3_{k,m}$ can be found in Section 4.4.1.

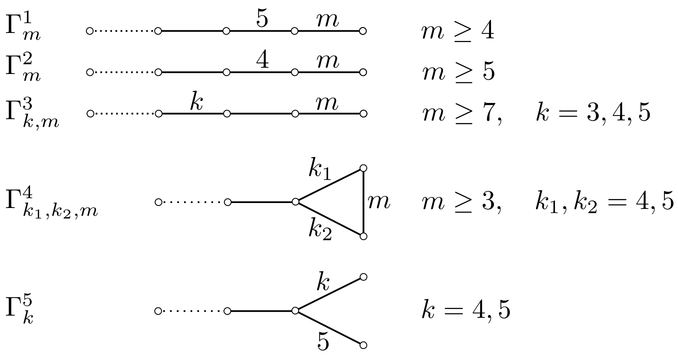


Figure A.1 – Families of 3-dimensional compact prisms

Arithmeticity The following table contains the values for which the group is (quasi-)arithmetic (arithmetic groups are indeed also quasi-arithmetic).

Group	Quasi-arithmetic	(Quasi-)Arithmetic
Γ^1_m	$m = 5, 10$	$m = 4, 6$
Γ^2_m	$m = 12$	$m = 5, 6, 8$
$\Gamma^3_{3,m}$	$m = 12, 18, 24, 30$	$m = 7, 8, 9, 10, 14$
$\Gamma^3_{4,m}$	$m = 18$	$m = 7, 8, 10, 12$

Continued on next page

Group	Quasi-arithmetic	(Quasi-)Arithmetic
$\Gamma^3_{5,m}$	$m = 30$	$m = 10$
$\Gamma^4_{4,4,m}$	$m = 6$	$m = 3, 4$
$\Gamma^4_{4,5,m}$	$m = 4$	$\emptyset$
$\Gamma^4_{5,5,m}$	$m = 3, 5$	$\emptyset$
Γ^5_k	$\emptyset$	$m = 4, 5$

Some constants We give in the next table the value α_m, which are used to simplify some computations.

Group	α_m
Γ^1_m	$\frac{3\cos\frac{2\pi}{m}+\sqrt{5}}{4\cos\frac{2\pi}{m}+\sqrt{5}-1}$
Γ^2_m	$\frac{3\cos\frac{2\pi}{m}+1}{\cos\frac{2\pi}{m}}$
$\Gamma^3_{3,m}$	$\frac{3\cos\left(\frac{2\pi}{m}\right)-1}{4\cos\left(\frac{2\pi}{m}\right)-2}$
$\Gamma^3_{4,m}$	$\frac{\cos\left(\frac{2\pi}{m}\right)}{2\cos\left(\frac{2\pi}{m}\right)-1}$
$\Gamma^3_{5,m}$	$\frac{-\left(\sqrt{5}-5\right)\cos^2\left(\frac{\pi}{m}\right)+\sqrt{5}-3}{4\cos\left(\frac{2\pi}{m}\right)-2}$

Coefficients of the diagonal quadratic form The following table contains the coefficients of a diagonal form $\langle a_0, a_1, a_2, a_3\rangle$ of the associated quadratic forms of signature $(3, 1)$, with the convention that $a_0 < 0$ and $a_1, a_2, a_3 > 0$. We omit the form brackets $\langle\ \rangle$ for brevity.

Group	Coefficients a_0, a_1, a_2, a_3
Γ^1_m	$-2\left(5+2\sqrt{5}\right)\cos^2\left(\frac{\pi}{m}\right)\cdot\left(3\cos\frac{2\pi}{m}+\sqrt{5}\right), 1, \alpha_m, 2\left(5+\sqrt{5}\right)\alpha_m$
Γ^2_m	$-\left(1+\sec\frac{2\pi}{m}\right)\alpha_m, 2\alpha_m, 2\alpha_m, \left(\sec\left(\frac{2\pi}{m}\right)+3\right)\alpha_m$
$\Gamma^3_{3,m}$	$-\cos\left(\frac{2\pi}{m}\right)-\cos\left(\frac{4\pi}{m}\right), \alpha_m, 3\alpha_m, 6\alpha_m$
$\Gamma^3_{4,m}$	$-\left(1+\sec\frac{2\pi}{m}\right)\alpha_m, 2\alpha_m, 3\alpha_m, 6\alpha_m$
$\Gamma^3_{5,m}$	$\frac{\left(7+3\sqrt{5}\right)\cos^2\left(\frac{\pi}{m}\right)}{2-4\cos\left(\frac{2\pi}{m}\right)}, \alpha_m, 3\alpha_m, 3\alpha_m$
$\Gamma^4_{4,4,m}$	$-6\cos\left(\frac{\pi}{m}\right)-2, 2, 3, 6$

Continued on next page

Group	Coefficients a_0, a_1, a_2, a_3
$\Gamma^4_{4,5,m}$	$2\left(\sqrt{5}-5\right)\left(8\sqrt{10}\cos\frac{\pi}{m}-3\left(\sqrt{5}-5\right)\cos\frac{2\pi}{m}+\sqrt{5}+15\right), 1, 3, 6$
$\Gamma^4_{5,5,m}$	$-\left(\cos\frac{\pi}{m}+1\right)\left(3\cos\frac{\pi}{m}+\sqrt{5}\right), 1, 3, 3$
Γ^5_4	$-2-\sqrt{5}, 2, 2, 2$
Γ^5_5	$-10-6\sqrt{5}, 1, 1, 1$

Determinant δ of the quadratic form

Group	δ
Γ^1_m	$-\left(5+2\sqrt{5}\right)\cos^2\left(\frac{\pi}{m}\right)\cdot\left(3\left(5+\sqrt{5}\right)\cos\left(\frac{2\pi}{m}\right)+5\left(1+\sqrt{5}\right)\right)$
Γ^2_m	$-2\cos^2\left(\frac{\pi}{m}\right)\cdot\left(3\cos\left(\frac{2\pi}{m}\right)+1\right)$
$\Gamma^3_{3,m}$	$\cos^2\left(\frac{\pi}{m}\right)\cdot\left(1-2\cos\left(\frac{2\pi}{m}\right)\right)\alpha_m$
$\Gamma^3_{4,m}$	$-\left(\cos\frac{2\pi}{m}+\cos\frac{4\pi}{m}\right)\alpha_m$
$\Gamma^3_{5,m}$	$-\cos^2\left(\frac{\pi}{m}\right)\cdot\left(\left(25+11\sqrt{5}\right)\cos\left(\frac{2\pi}{m}\right)+5\sqrt{5}+11\right)$
$\Gamma^4_{4,4,m}$	$-6\cos\left(\frac{\pi}{m}\right)-2$
$\Gamma^4_{4,5,m}$	$-2\cos^2\left(\frac{\pi}{m}\right)\left(4\sqrt{3+\sqrt{5}}\cos\frac{\pi}{m}+3\cos\frac{2\pi}{m}+\sqrt{5}+4\right)$
$\Gamma^4_{5,5,m}$	$-2\left(3\cos\left(\frac{\pi}{m}\right)+\sqrt{5}\right)$
Γ^5_4	$-2\left(2+\sqrt{5}\right)$
Γ^5_5	$-10-6\sqrt{5}$

Invariant trace field

Group	$K(\Gamma)$
Γ^1_m	$\mathbb{Q}\left[\cos\frac{2\pi}{m}, \sqrt{5}, \sqrt{\delta}\right]$
Γ^2_m	$\mathbb{Q}\left[\cos\frac{2\pi}{m}, \sqrt{\delta}\right]$
$\Gamma^3_{3,m}$	$\mathbb{Q}\left[\cos\frac{2\pi}{m}, \sqrt{\delta}\right]$
$\Gamma^3_{4,m}$	$\mathbb{Q}\left[\cos\frac{2\pi}{m}, \sqrt{\delta}\right]$
$\Gamma^3_{5,m}$	$\mathbb{Q}\left[\cos\frac{2\pi}{m}, \sqrt{5}, \sqrt{\delta}\right]$

Continued on next page

Group	$K(\Gamma)$
$\Gamma^4_{4,4,m}$	$\mathbb{Q}\left[\cos\frac{\pi}{m}, \sqrt{\delta}\right]$
$\Gamma^4_{4,5,m}$	$\mathbb{Q}\left[\sqrt{5}, \sqrt{3+\sqrt{5}}\cdot\cos\frac{\pi}{m}, \sqrt{2}\cdot\cos\frac{\pi}{m}\right]$
$\Gamma^4_{5,5,m}$	$\mathbb{Q}\left[\cos\frac{\pi}{m}, \sqrt{5}, \sqrt{\delta}\right]$
Γ^5_m	$\mathbb{Q}\left[\sqrt{5}, \sqrt{\delta}\right]$

Growth rate (limit value) For each family of groups, we give all the coefficients γ_k, $k \geq 0$, of a polynomial $\sum \gamma_k\, x^k$ whose biggest positive real root is the limit of the growth rate τ and an approximate value for τ.

Group	$\gamma_0, \gamma_1, \dots$	$\lim \tau_m \simeq$
Γ^1_m	$-1,-1,-2,-2,-3,-2,-3,-2,-3,-1,-2,0,-1,1$	1.982497369
Γ^2_m	$-1,-1,-2,-1,-2,0,-1,1$	1.906484762
$\Gamma^3_{3,m}$	$-1,-2,-2,-1,0,1$	1.734691346
$\Gamma^3_{4,m}$	$-1,-1,-2,-1,-2,0,-1,1$	1.906484762
$\Gamma^3_{5,m}$	$-1,-1,-2,-2,-3,-2,-3,-2,-3,-1,-2,0,-1,1$	1.982497369
$\Gamma^4_{4,4,m}$	$-1,-1,-1,0,-2,1$	2.305223929
$\Gamma^4_{4,5,m}$	$-1,-1,-2,-2,-3,-2,-3,-2,-3,-1,-3,1,-2,1$	2.349133213
$\Gamma^4_{5,5,m}$	$-1,-1,-1,-2,-1,-2,-1,-2,-1,0,-2,1$	2.389464732
Γ^5_4	$1,-2,2,-4,2,-5,2,-5,2,-5,2,-4,2,-2,1$	2.045214041
Γ^5_5	$1,-2,0,0,0,-2,0,0,0,-2,1$	2.105678915

A.1.2 The non-compact case

In this section, we present the invariants of the non-compact prisms given in Figure A.2 (see [Kap74]).

Arithmeticity The following table contains the values for which the group is (quasi-)arithmetic (arithmetic groups are indeed also quasi-arithmetic).

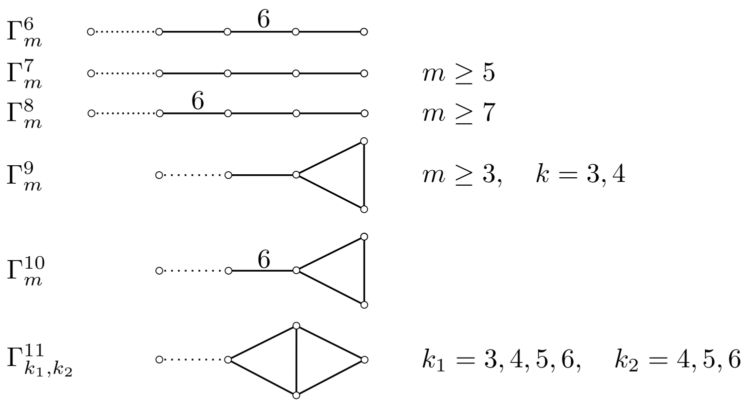


Figure A.2 – Families 3-dimensional non-compact prisms

Group	Quasi-arithmetic	(Quasi-)Arithmetic
Γ_m^6	$m = 6$	$m = 4$
Γ_m^7	$\emptyset$	$m = 6$
Γ_m^8	$\emptyset$	$\emptyset$
$\Gamma_{3,m}^9$	$m = 4$	$\emptyset$
$\Gamma_{4,m}^9$	$\emptyset$	$m = 3$
Γ_m^{10}	$\emptyset$	$\emptyset$
Γ_{k_1,k_2}^{11}	$\emptyset$	$k_1 = k_2 = 4, 6$

Determinant δ of the quadratic form

Group	δ	Group	δ
Γ_m^6	-3	$\Gamma_{3,m}^9, \Gamma_{4,m}^9$	-1
Γ_m^7	-1	Γ_m^{10}	-3
Γ_m^8	-3	Γ_{k_1,k_2}^{11}	-3

Invariant trace field

Group	$K(\Gamma)$	Group	$K(\Gamma)$
Γ_m^6	$\mathbb{Q}\left[i\sqrt{3}, \cos\frac{2\pi}{m}\right]$	$\Gamma_{3,6}^{11}$	$\mathbb{Q}\left[i, \sqrt{3}\right]$
Γ_m^7	$\mathbb{Q}\left[i, \cos\frac{2\pi}{m}\right]$	$\Gamma_{4,4}^{11}$	$\mathbb{Q}\left[i\sqrt{3}\right]$
Γ_m^8	$\mathbb{Q}\left[i\sqrt{3}, \cos\frac{2\pi}{m}\right]$	$\Gamma_{4,5}^{11}$	$\mathbb{Q}\left[i\sqrt{3}, \sqrt{2}, \sqrt{5}\right]$

Continued on next page

Group	$K(\Gamma)$	Group	$K(\Gamma)$
$\Gamma^9_{3,m}$	$\mathbb{Q}\left[i, \sqrt{2} \cdot \cos\frac{\pi}{m}\right]$	$\Gamma^{11}_{4,6}$	$\mathbb{Q}\left[i\sqrt{3}, \sqrt{6}\right]$
$\Gamma^9_{4,m}$	$\mathbb{Q}\left[i, \cos\frac{\pi}{m}\right]$	$\Gamma^{11}_{5,5}$	$\mathbb{Q}\left[i\sqrt{3}, \sqrt{5}\right]$
Γ^{10}_{m}	$\mathbb{Q}\left[i\sqrt{3}, \cos\frac{\pi}{m}\right]$	$\Gamma^{11}_{5,6}$	$\mathbb{Q}\left[i, \sqrt{3}, \sqrt{5}\right]$
$\Gamma^{11}_{3,4}$	$\mathbb{Q}\left[i\sqrt{3}, \sqrt{2}\right]$	$\Gamma^{11}_{6,6}$	$\mathbb{Q}\left[i\sqrt{3}\right]$
$\Gamma^{11}_{3,5}$	$\mathbb{Q}\left[i\sqrt{3}, \sqrt{5}\right]$		

Growth rate (limit value) For each infinite family of groups, we give all the coefficients γ_k, $k \geq 0$, of a polynomial $\sum \gamma_k\, x^k$ whose biggest positive real root is the limit of the growth rate τ and an approximate value for τ.

Group	$\gamma_0, \gamma_1, \dots$	$\lim \tau_m \simeq$
Γ^6_m	$-1, -2, -1, -1, -1, -1, 1$	2.015614858
Γ^7_m	$-1, -2, -1, -1, 1$	2.065994892
Γ^8_m	$-1, -2, -1, -1, -1, -1, 1$	2.015614858
$\Gamma^9_{3,m}$	$-1, -2, -3, -3, -3, -3, -1, -1, 1$	2.352041123
$\Gamma^9_{4,m}$	$-2, -3, -2, -1, 1$	2.451109537
Γ^{10}_{m}	$-2, -1, -1, 0, -2, 1$	2.330809311

We now give the growth rate of the remaining 9 prisms $\Gamma^{11}_{k1,k2}$.

Group	Coefficients $\gamma_0, \gamma_1, \dots$ and growth rate
$\Gamma^{11}_{3,4}$	$-2, -2, -3, -3, -2, -2, -1, 1$ 2.459101131
$\Gamma^{11}_{3,5}$	$-2, -2, -5, -6, -7, -9, -8, -9, -8, -8, -6, -5, -3, -1, -1, 1$ 2.496678489
$\Gamma^{11}_{3,6}$	$-1, -3, -3, -4, -4, -2, -2, -1, 1$ 2.510800457
$\Gamma^{11}_{4,4}$	$-2, -2, -4, -3, -3, -2, -1, 1$ 2.548573757
$\Gamma^{11}_{4,5}$	$-2, -2, -6, -6, -9, -9, -10, -9, -10, -8, -8, -5, -4, -1, -1, 1$ 2.581500889
$\Gamma^{11}_{4,6}$	$-1, -3, -3, -5, -4, -3, -2, -1, 1$ 2.593583812

Continued on next page

Group	Coefficients $\gamma_0, \gamma_1, \ldots$ and growth rate
$\Gamma^{11}_{5,5}$	$-2, -2, -4, -5, -5, -6, -5, -6, -5, -4, -3, -2, -1, 1$
	2.612522099
$\Gamma^{11}_{5,6}$	$-1, -3, -3, -5, -6, -6, -6, -6, -6, -6, -4, -3, -2, -1, 1$
	2.623816333
$\Gamma^{11}_{6,6}$	$-2, -4, -2, -2, -3, -1, 1$
	2.634795548

A.2 Polytopes with $n+3$ facets and one non-simple vertex

In [Rob15, v3], Roberts presents the classification of hyperbolic Coxeter polytopes in $\mathbb{H}^n$ with $n+3$ facets and one non-simple vertex. We present the invariants of the commensurability class for the arithmetic groups of his list (38 groups among 144). We omit the brackets in the notation of the quadratic form. The invariants were found using a computer and `AlVin`.

n	Group	Quadratic form	$c(V)$	δ	Invariant
4	8 (a)	$-6, 1, 3, 6, 6$	$(-1, -1)$	-2	$\{\mathbb{Q}, \{0, 2\}\}$
4	12(a)	$-6, 1, 2, 3, 6$	$(-2, -1)$	-6	$\{\mathbb{Q}, \{0, 2\}\}$
5	13(a)	$-6, 1, 3, 3, 3, 6$	$(-3, -1)$	3	$\{\mathbb{Q}, 3, \{\}\}$
5	13(b)	$-10, 1, 5, 5, 5, 10$	$(-5, -1)$	5	$\{\mathbb{Q}, 5, \{\}\}$
5	15(a)	$-5, 1, 5, 10, 15, 30$	$(-5, -1)$	5	$\{\mathbb{Q}, 5, \{\}\}$
5	15(b)	$-5, 1, 2, 3, 5, 6$	$(-2, -1)$	1	$\{\mathbb{Q}, \{0, 2\}\}$
5	15(d)	$-3, 1, 2, 3, 3, 6$	$(-2, -1)$	1	$\{\mathbb{Q}, \{0, 2\}\}$
5	15(f)	$-3, 1, 2, 3, 3, 6$	$(-2, -1)$	2	$\{\mathbb{Q}, \{0, 2\}\}$
5	16(a)	$-3, 1, 3, 3, 3, 3$	$(-3, -1)$	3	$\{\mathbb{Q}, 3, \{\}\}$
5	16(b)	$-15, 1, 5, 5, 5, 15$	$(-1, -1)$	5	$\{\mathbb{Q}, 5, \{\}\}$
5	18(a)	$-2, 1, 1, 2, 2, 2$	$(-2, -1)$	1	$\{\mathbb{Q}, \{0, 2\}\}$
5	18(d)	$-7, 1, 1, 2, 2, 7$	$(-2, -1)$	1	$\{\mathbb{Q}, \{0, 2\}\}$
5	18(f)	$-2, 1, 1, 2, 2, 2$	$(-2, -1)$	1	$\{\mathbb{Q}, \{0, 2\}\}$
5	20(a)	$-1, 1, 1, 1, 2, 2$	$(-2, -1)$	1	$\{\mathbb{Q}, \{0, 2\}\}$
5	20(d)	$-2, 1, 1, 2, 2, 2$	$(-2, -1)$	1	$\{\mathbb{Q}, \{0, 2\}\}$
5	20(f)	$-1, 1, 1, 1, 2, 2$	$(-2, -1)$	1	$\{\mathbb{Q}, \{0, 2\}\}$

Continued on next page

n	Group	Quadratic form	$c(V)$	δ	Invariant
5	21(c)	$-5, 1, 2, 3, 5, 6$	$(-2, -1)$	1	$\{\mathbb{Q}, \{0, 2\}\}$
5	21(e)	$-21, 1, 2, 3, 6, 21$	$(-2, -1)$	1	$\{\mathbb{Q}, \{0, 2\}\}$
5	22(b)	$-1, 1, 2, 3, 10, 15$	$(-5, -1)$	1	$\{\mathbb{Q}, \{0, 2\}\}$
5	22(c)	$-2, 1, 1, 1, 3, 6$	$(-1, -1)$	1	$\{\mathbb{Q}, \{0, 2\}\}$
5	22(d)	$-3, 1, 1, 1, 1, 3$	$(-1, -1)$	1	$\{\mathbb{Q}, \{0, 2\}\}$
5	22(e)	$-3, 1, 1, 1, 1, 3$	$(-1, -1)$	1	$\{\mathbb{Q}, \{0, 2\}\}$
5	22(f)	$-1, 1, 2, 3, 10, 15$	$(-5, -1)$	1	$\{\mathbb{Q}, \{0, 2\}\}$
5	22(g)	$-1, 1, 1, 2, 3, 6$	$(-2, -1)$	1	$\{\mathbb{Q}, \{0, 2\}\}$
6	23(a)	$-3, 1, 2, 2, 2, 2, 2$	$(-2, -1)$	6	$\{\mathbb{Q}, \{0, 2\}\}$
6	24(a)	$-1, 1, 1, 1, 2, 2, 2$	$(-2, -1)$	2	$\{\mathbb{Q}, \{0, 2\}\}$
6	24(d)	$-2, 1, 1, 1, 3, 6, 6$	$(-2, -1)$	6	$\{\mathbb{Q}, \{0, 2\}\}$
6	25(b)	$-1, 1, 1, 1, 2, 2, 2$	$(-2, -1)$	2	$\{\mathbb{Q}, \{0, 2\}\}$
6	25(e)	$-1, 1, 1, 2, 2, 3, 6$	$(-2, -1)$	2	$\{\mathbb{Q}, \{0, 2\}\}$
7	28(a)	$-3, 1, 2, 2, 2, 2, 2, 2$	$(-1, -6)$	-3	$\{\mathbb{Q}, -3, \{\}\}$
7	28(d)	$-5, 1, 1, 1, 3, 6, 10, 15$	$(-3, -15)$	-15	$\{\mathbb{Q}, -15, \{\}\}$
7	29(a)	$-3, 1, 2, 2, 2, 2, 2, 2$	$(-1, -3)$	-3	$\{\mathbb{Q}, -3, \{\}\}$
8	30(a)	$-5, 1, 1, 1, 3, 3, 6, 10, 15$	$(-1, 10)$	-5	$\{\mathbb{Q}, \{\}\}$
9	32(a)	$-5, 1, 1, 1, 3, 6, 7, 10, 15, 21$	1	5	$\{\mathbb{Q}, 5, \{\}\}$
9	32(e)	$-5, 1, 1, 3, 5, 6, 7, 10, 15, 21$	1	1	$\{\mathbb{Q}, \{\}\}$
9	32(f)	$-3, 1, 1, 3, 3, 6, 7, 10, 15, 21$	1	1	$\{\mathbb{Q}, \{\}\}$
9	32(g)	$-3, 1, 1, 3, 3, 6, 7, 10, 15, 21$	1	1	$\{\mathbb{Q}, \{\}\}$
10	33(a)	$-1, 1, 1, 1, 1, 2, 3, 3, 6, 10, 15$	1	2	$\{\mathbb{Q}, \{\}\}$

A.3 Non-reflectivity of quadratic forms

The method presented in Section 6.2.3 to prove the non-reflectivity of quadratic forms requires to produce involutions of subgraphs with some properties. Since finding these transformations can be tricky, we describe them explicitly below.

Quadratic form	Involution
$\langle -1, 1, 19 \rangle$	$e_4 \leftrightarrow e_8, e_6 \leftrightarrow e_7$ $e_1 \leftrightarrow e_6, e_5 \leftrightarrow e_8$

Continued on next page

Quadratic form	Involution
$\langle -1,1,23\rangle$	$e_5 \leftrightarrow e_9, e_6 \leftrightarrow e_8$ $e_5 \leftrightarrow e_{13}, e_6 \leftrightarrow e_7$
$\langle -1,1,1,11\rangle$	$e_1 \leftrightarrow e_8, e_4, e_7 \leftrightarrow e_9$ $e_1 \leftrightarrow e_{10}, e_2, e_5 \leftrightarrow e_7$
$\langle -1,1,1,13\rangle$	$e_1 \leftrightarrow e_7, e_2 \leftrightarrow e_4, e_3 \leftrightarrow e_6, e_5$ $e_1 \leftrightarrow e_5, e_2, e_3 \leftrightarrow e_{10}, e_4 \leftrightarrow e_8$
$\langle -1,1,1,21\rangle$	$e_2 \leftrightarrow e_4, e_3 \leftrightarrow e_7, e_6$ $e_1 \leftrightarrow e_6, e_4 \leftrightarrow e_8, e_5$
$\langle -1,1,1,22\rangle$	$e_1 \leftrightarrow e_6, e_2, e_7 \leftrightarrow e_9$ $e_1 \leftrightarrow e_8, e_6, e_7 \leftrightarrow e_{10}$
$\langle -1,1,1,26\rangle$	$e_1, e_2 \leftrightarrow e_{11}, e_6 \leftrightarrow e_7$ $e_1 \leftrightarrow e_{14}, e_2, e_6 \leftrightarrow e_8$
$\langle -13,1,1,1\rangle$	$e_3 \leftrightarrow e_4, e_5 \leftrightarrow e_7, e_8 \leftrightarrow e_{10}$ $e_1 \leftrightarrow e_2, e_3 \leftrightarrow e_8, e_5 \leftrightarrow e_{11}$
$\langle -14,1,1,1\rangle$	$e_1 \leftrightarrow e_2, e_3 \leftrightarrow e_6, e_4 \leftrightarrow e_{10}$ $e_1 \leftrightarrow e_5, e_2 \leftrightarrow e_9, e_3, e_4 \leftrightarrow e_8$
$\langle -19,1,1,1\rangle$	$e_1 \leftrightarrow e_4, e_2, e_3 \leftrightarrow e_6$ $e_2 \leftrightarrow e_4, e_3, e_5 \leftrightarrow e_8$
$\langle -22,1,1,1\rangle$	$e_1 \leftrightarrow e_7, e_2 \leftrightarrow e_6, e_3, e_4$ $e_1 \leftrightarrow e_6, e_3 \leftrightarrow e_8, e_4 \leftrightarrow e_{12}$
$\langle -23,1,1,1\rangle$	$e_2 \leftrightarrow e_{11}, e_3, e_4 \leftrightarrow e_5$ $e_1 \leftrightarrow e_8, e_2 \leftrightarrow e_4, e_3 \leftrightarrow e_6$
$\langle -26,1,1,1\rangle$	$e_1 \leftrightarrow e_7, e_2, e_3 \leftrightarrow e_{10}$ $e_1 \leftrightarrow e_8, e_3, e_9 \leftrightarrow e_{15}$
$\langle -1,1,1,1,17\rangle$	$e_1, e_2 \leftrightarrow e_6, e_3 \leftrightarrow e_{20}$ $e_2 \leftrightarrow e_5, e_3, e_4 \leftrightarrow e_8, e_7$
$\langle -1,1,1,1,30\rangle$	$e_2 \leftrightarrow e_{10}, e_8, e_{15} \leftrightarrow e_{20}$ $e_1 \leftrightarrow e_2, e_9 \leftrightarrow e_{24}, e_{14}, e_{16}$
$\langle -17,1,1,1,1\rangle$	$e_3 \leftrightarrow e_6, e_4, e_5, e_7 \leftrightarrow e_{13}$ $e_1 \leftrightarrow e_6, e_2 \leftrightarrow e_5, e_3, e_4 \leftrightarrow e_8$
$\langle -21,1,1,1,1,1\rangle$	$e_1 \leftrightarrow e_8, e_2, e_3, e_4, e_5 \leftrightarrow e_{10}$ $e_2 \leftrightarrow e_3, e_4 \leftrightarrow e_8, e_5, e_{12} \leftrightarrow e_{15}$

Quadratic form	Involution
$\langle -\Theta_{13}, 1, 1, 1\rangle$	$e_4, e_5 \leftrightarrow e_8, e_6 \leftrightarrow e_{10}$ $e_4 \leftrightarrow e_{13}, e_6 \leftrightarrow e_7, e_8$
$\langle -\Theta_3, 1, 1, 1, 1\rangle$	$e_1 \leftrightarrow e_3, e_2, e_4 \leftrightarrow e_{16}$ $e_1 \leftrightarrow e_2, e_4 \leftrightarrow e_{17}, e_{13} \leftrightarrow e_{14}$
$\langle 2-\Theta_{17}, 1, 1, 1, 1\rangle$	$e_1 \leftrightarrow e_{11}, e_3 \leftrightarrow e_{15}, e_6$ $e_1, e_2 \leftrightarrow e_{10}, e_3, e_{15} \leftrightarrow e_{20}$
$\langle -3-2\Theta_3, 1, 1, 1, 1, 1\rangle$	$e_1 \leftrightarrow e_3, e_5, e_6 \leftrightarrow e_9, e_7$ $e_2 \leftrightarrow e_4, e_6, e_7 \leftrightarrow e_{10}, e_8$ $e_9, e_{12} \leftrightarrow e_{15}, e_{18} \leftrightarrow e_{29}, e_{27}$
$\langle -1-3\Theta_5, 1, 1, 1, 1, 1, 1\rangle$	$e_2 \leftrightarrow e_{14}, e_3 \leftrightarrow e_5, e_7, e_8 \leftrightarrow e_9$ $e_2 \leftrightarrow e_{14}, e_3 \leftrightarrow e_5, e_4, e_7, e_8 \leftrightarrow e_9$
$\langle -\Theta_5, 2+\Theta_5, 1, 1, 1, 1, 1\rangle$	$e_1 \leftrightarrow e_3, e_2, e_4 \leftrightarrow e_{13}, e_5, e_8$ $e_1 \leftrightarrow e_{10}, e_2, e_3, e_4, e_7 \leftrightarrow e_9$ $e_2 \leftrightarrow e_8, e_3 \leftrightarrow e_7, e_4 \leftrightarrow e_{13}, e_5$

APPENDIX B

Codes

B.1 Mathematica® codes

B.1.1 Lemma 3.6.6

The following code shows how to compute the Sturm sequence and the signs of Lemma 3.6.6.

```
f[x_,m_]:=x^(m+8)-x^(m+7)-2x^(m+5)-x^(m+4)-2x^(m+3)-x^(m+2)-x^(m+1)
    +x^7+x^6+2x^5+x^4+2x^3+x-1;
Df[x_,m_]:=1+6*x^2+4*x^3+10*x^4+6*x^5+7*x^6-(1+m)*x^m-(2+m)*x^(1+m)
    -2*(3+m)*x^(2+m)-(4+m)*x^(3+m)-2(5+m)*x^(4+m)-(7+m)*x^(6+m)+(8+
    m) x^(7+m);
Df[x,2m+2]-Df[x,2m]//Factor

(* We have Df[x,2m+2]-Df[x,2m]=x^(2m)*hm[x] and we want to count
    the zeros of hm[x] using Sturm's theorem *)
hm[x_]:=1+2m+2(1+m)x+x^2(3+2m)-6*x^5-2m*x^5-7*x^6-2m*x^6-8*x^7-2m*x
    ^7-9*x^8-2m*x^8+10*x^9+2m*x^9;
S:={hm[x], D[hm[x],x]} (* First two elements of the Sturm sequence
    *)

(* We construct the Sturm sequence *)
For[i=1,i<=8,i++,
  S=Join[S,{-FullSimplify[PolynomialRemainder[S[[i]],S[[i+1]],x
      ]]}];
]

(* Some verifications *)
Length[S]
Exponent[S[[Length[S]]],x]
NSolve[{S[[10]]==0},Reals]

(* Computing the signs *)
For[val=-2,val<=2,val++, (* For val=-2,-1,0,1,2 *)
  Print["Val: ", val];
  For[i=1,i<=Length[S],i++,
    ExprM1=S[[i]]/.{x->val};
    SolPos:=Reduce[{ExprM1>0&&m>=2}];
    SolNeg:=Reduce[{ExprM1<0&&m>=2}];
    Print[i-1];
    Print["        -: ",SolNeg];
    Print["        +: ",SolPos];
  ]
```

```
30   Print[" "];
31 ]
```

The output of the lines 15-17 is the following:

```
10
0
{{m->-6.78853}, {m->-6.78853}, {m->-5.1517}, {m->-4.59411},
 {m->-2.35966}, {m->-2.35966}, {m->-1.10695},
 {m->-1.10695}, {m->0.790002}, {m->0.790002},
 {m->-0.709781}, {m->-0.639181}, {m->-0.425418}}
```

This means that the sequence contains 10 terms, the last term is a constant polynomial which is non-zero if m is an integer number. Thus, the sequence is indeed a Sturm sequence.

B.1.2 Example 4.4.1

For $m \in \{9, 13, 14, 16, 20, 21, 22, 28, 30, 36, 54\}$, we consider the Gram matrix

$$G_m = \begin{pmatrix} 1 & -\frac{\sqrt{3\cos\left(\frac{2\pi}{m}\right)+1}}{2\sqrt{\cos\left(\frac{2\pi}{m}\right)}} & 0 & 0 & 0 \\ -\frac{\sqrt{3\cos\left(\frac{2\pi}{m}\right)+1}}{2\sqrt{\cos\left(\frac{2\pi}{m}\right)}} & 1 & -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & 1 & -\cos\left(\frac{\pi}{m}\right) \\ 0 & 0 & 0 & -\cos\left(\frac{\pi}{m}\right) & 1 \end{pmatrix},$$

whose first non-trivial minor is $\frac{1}{8}\left(2 - 6\cos^2\left(\frac{\pi}{m}\right)\right)$. For each root α of the minimal polynomial of $\cos\frac{\pi}{m}$ such that the induced embedding is not the identity on $\cos\frac{2\pi}{m}$ (i.e. such that $2\alpha^2 - 1 \neq \cos\frac{2\pi}{m}$), we check whether $\frac{1}{8}\left(2 - 6\alpha^2\right)$ is negative. If it is the case, then the group is not quasi-arithmetic (see Theorem 3.9.7). Numerical computations show that we can always find such an α. The code to achieve this is the following.

```
S := {9, 13, 14, 16, 20, 21, 22, 28, 30, 36, 54}
For[i = 1, i ≤ 11, i++,
  m := S[[i]];
  Print["m = ", m];
  f[x] = MinimalPolynomial[Cos[π/m], x];

  Rf := x /. NSolve[f[x] == 0, 50]; (* Roots of the minimal
      polynomial with high precision *)

  For[r = 1, r ≤ Length[Rf], r++, (* For each root *)
    If[Abs[2*Rf[[r]]^2 - 1 - Cos[2π/m]] < 10^-20, Continue[]]; (*
        If the corresponding Galois embedding is the identity on K,
        we skip to the next root *)

  If[1/8*(2 - 6*Rf[[r]]^2) < 0, Print["    NOT quasi-arithmetic"];
      Break[];]
  ]
]
```

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Education

2012–2017 **PhD Studies in Mathematics**, *University of Fribourg*.
Advisor: Prof. Dr Ruth Kellerhals

2011–2012 **Master of Science MSc in Mathematics**, *École Polytechnique Fédérale de Lausanne*.
Master thesis: "The Brauer-Grothendieck group". Advisors: Eva Bayer (EPFL) and Alexei Skorobogatov (Imperial College, London).

2007–2011 **Bachelor of Science BSc in Mathematics**, *École Polytechnique Fédérale de Lausanne*.

Invited Talks

12.2016 **Clifford algebras and isometries of (infinite dimensional) hyperbolic spaces**.
Oberseminar Geometry, University of Fribourg.

04.2016 **(CoxIter -) Computing invariants of hyperbolic Coxeter groups**.
Geometry Seminar, Vanderbilt University.

10.2015 **Commensurability of arithmetic hyperbolic Coxeter groups**.
Journées de Géométrie Hyperbolique, University of Fribourg.

06.2015 **Hyperbolic Coxeter groups and commensurability - Algorithmic and arithmetic aspects**.
Workshop: Geometry and Analysis of Discrete Groups and Hyperbolic Spaces, RIMS, Kyoto University.

11.2014 **Groupes et polyèdres de Coxeter hyperboliques: invariants et commensurabilité**, *with M. Jacquemet*.
Paroles aux jeunes chercheurs en géométrie et dynamique, Université of Bordeaux.

11.2014 **Groupes et polyèdres de Coxeter hyperboliques: invariants et commensurabilité**, *with M. Jacquemet*.
Oberseminar Geometry, University of Fribourg.

12.2012 **The Brauer-Grothendieck group**.
Oberseminar Geometry, University of Fribourg.

Research Stays

04.2016 **Vanderbilt University**, *4 weeks*.
Host: Prof. J. Ratcliffe.

Publications

Commensurability of hyperbolic Coxeter groups: theory and computation, *with M. Jacquemet and R. Kellerhals*.
50 pp., accepted for publication in RIMS Kôkyûroku Bessatsu: Discrete Groups and Hyperbolic Geometry, ed. M. Fujii.

Polyhedra and commensurability, *with M. Jacquemet*.
Snapshots of Modern Mathematics, MFO Oberwolfach (2016).

On commensurable hyperbolic Coxeter groups, *with M. Jacquemet and R. Kellerhals*.
Geometriae Dedicata 183.1 (2016), pp. 143–167.

CoxIter - Computing invariant of hyperbolic Coxeter groups.
LMS Journal of Computation and Mathematics 18.1 (Dec. 2015), pp. 754–773.

Popularization of Mathematics

11.2016 **Cryptographie et codage: protéger vos secrets**, *Workshop*, with B. Reinhard.
TecDay@Porrentry, Lycée cantonal de Porrentry, Porrentry.

04.2015 **Parfois, la taille compte aussi!**, *Workshop*, with M. Jacquemet.
TecDay@Madame-de-Staël, Collège Madame-de-Staël, Carouge.

10.2014 **Parfois, la taille compte aussi!**, *Workshop*, with M. Jacquemet.
TecDay@Beaulieu, Gymnase de Beaulieu, Lausanne.

04.2015 **Le dilemme du prisonnier**, *Talk*.
Mathematikòn, University of Fribourg.

10.2014 **Des chapeaux, des groupes et l'axiome du choix**, *Talk*.
Mathematikòn, University of Fribourg.

04.2014 **Introduction à la théorie des codes**, *Talk*.
Mathematikòn, University of Fribourg.

12.2013 **A quel point un groupe non-abélien peut-il être abélien**, *Talk*.
Mathematikòn, University of Fribourg.

05.2012 **Le groupe de Brauer-Grothendieck**, *Talk*.
Séminaire Master, EPFL.

05.2012 **La théorie des catégories**, *Talk*, with D. Zaganidis.
Séminaire Bachelor, EPFL.

11.2011 **Groupes profinis**, *Talk*.
Séminaire Master, EPFL.

10.2011 **Le problème des chapeaux**, *Talk*.
Séminaire Bachelor, EPFL.

10.2011 **Jeux mathématiques**, *Talk*.
Séminaire Bachelor, EPFL.

10.2010 **De quelle couleur est votre chapeau?**, *Talk*.
Séminaire Bachelor, EPFL.

12.2009 **Introduction à la théorie algébrique des nombres**, *Talk*.
Séminaire Bachelor, EPFL.

10.2009 **Théorie des jeux et logique**, *Talk*, with D. Zaganidis.
Séminaire Bachelor, EPFL.

10.2008 **Introduction à la théorie algébrique des nombres**, *Talk*.
Séminaire Bachelor, EPFL.