# Quaternions and some global properties of hyperbolic 5–manifolds

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**Abstract.** We provide an explicit thick and thin decomposition for oriented hyperbolic manifolds $M$ of dimension 5. The result implies improved universal lower bounds for the volume $\mathrm{vol}_5(M)$ and, for $M$ compact, new estimates relating the injectivity radius and the diameter of $M$ with $\mathrm{vol}_5(M)$. The quantification of the thin part is based upon the identification of the isometry group of the universal space by the matrix group $PS_\Delta L(2,\mathbb{H})$ of quaternionic $2\times 2-$matrices with Dieudonné determinant $\Delta$ equal to 1 and isolation properties of $PS_\Delta L(2,\mathbb{H})$.

## 0. Introduction

The Margulis lemma for discrete groups of hyperbolic isometries has important consequences for the geometry and topology of hyperbolic manifolds of dimensions $n \geq 2$. There is a universal constant $\varepsilon = \varepsilon_n$ such that for each oriented hyperbolic $n-$manifold $M$ of finite volume there is a thick and thin decomposition

$$M = M_{\leq\varepsilon} \cup M_{>\varepsilon} \tag{0.1}$$

of $M$ as follows. The thick part $M_{>\varepsilon}$ having at each point an injectivity radius bigger than $\varepsilon/2$ is compact. The thin part $M_{\leq\varepsilon}$ of all points $p \in M$ with injectivity radius smaller than or equal to $\varepsilon/2$ consists of connected components of the following types. The bounded components are neighborhoods of simple closed geodesics in $M$ of length $\leq \varepsilon$ homeomorphic to ball bundles over the circle. The unbounded components are cusp neighborhoods homeomorphic to products of compact flat manifolds with a real half line.

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Estimates for the constant $\varepsilon_n$ induce universal bounds for various characteristic invariants of $M$ such as volume. Explicit values for $\varepsilon_n$ are known for $n = 2$ by work of P. Buser [Bu2, Chapter 4] and for $n = 3$ by work of R. Meyerhoff [M]. For $n = 4$, partial results are contained in [K3].

The aim of this work is to estimate the constant $\varepsilon_5$ and to derive some global properties such as new lower volume bounds for hyperbolic $5-$manifolds $M$ (cf. §2 and §3). We show that for $\varepsilon \leq \sqrt{3}/9\pi$ there is a decomposition of $M$ according to (0.1). Moreover, we prove the universal bound $\mathrm{vol}_5(M) > 0.000083$.

To this end, we analyse the thin part of $M$ and construct embedded tubes around simple closed geodesics of length $l \leq \sqrt{3}/8\pi$ of radius given by (cf. §2.1)

$$\cosh(2r) = \frac{1-3k}{k} \quad , \quad \text{where} \quad k = \frac{2\pi l}{\sqrt{3}} \quad . \tag{0.2}$$

The tubes around distinct closed geodesics of lengths $\leq \sqrt{3}/9\pi \simeq 0.0612$ are pairwise disjoint. In the non-compact case, they are also distinct from the canonical cusps associated to parabolic elements in the fundamental group of $M$.

Our considerations are based upon the identification of hyperbolic space $H^5$ and its boundary through quaternions such that $Iso^+(H^5)$ equals the group $PS_\Delta L(2;\mathbb{H})$ of quaternionic $2\times 2-$matrices with Dieudonné determinant $\Delta = 1$ as described by [H] and [Wil] (cf. §1.2). In this context, we characterise the isolation of the identity in $PS_\Delta L(2;\mathbb{H})$ (cf. §1.3). The strategies involved are standard and go back to [J], [Be] and [Wat].

The explicit tube construction (0.2) implies comparison results between injectivity radius, diameter and volume of compact hyperbolic $5-$manifolds $M$ (cf. §3.2). For example, we prove that the injectivity radius $i(M)$ of $M$ satisfies $i(M) \geq \mathrm{const}\cdot \mathrm{vol}_5(M)^{-1}$ improving results of P. Buser [Bu1] and A. Reznikov [Re].

In [CW, §9], C. Cao and P. Waterman constructed tubes around closed geodesics in hyperbolic $n-$manifolds $M$ for $n \geq 2$ and give a lower bound for the in-radius of $M$ by viewing isometries of hyperbolic $n-$space as Clifford matrices of pseudo-determinant 1. By different methods, Buser [Bu1, §4] obtained analogous results for compact hyperbolic manifolds of dimensions $> 2$. Both contributions provide clearly weaker bounds than ours when specialized to $n = 5$. As an illustration, the in-radius $r(M)$ measuring the radius of a largest embeddable ball in $M$ is bounded from below by $1/65536$ according to [Bu1, Theorem 4.11] and by $1/544$ according to [CW, Theorem 9.8] while we obtained the bound $1/30$ (cf. Lemma 5).

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# 1. The quaternion formalism for isometries of $H^5$

## 1.1. Loxodromic isometries of hyperbolic $n$–space

Let $\widehat{E}^n := E^n \cup \{\infty\}$. A Möbius transformation of $\widehat{E}^n$ is a finite composition of reflections in spheres or hyperplanes of $\widehat{E}^n$ and preserves cross ratios

$$[x,y;u,v] = \frac{|x-u|\cdot|y-v|}{|x-y|\cdot|u-v|}$$

for distinct points $x,y,u,v \in \widehat{E}^n$. The group of all Möbius transformations of $\widehat{E}^n$ is denoted by $M(\widehat{E}^n)$, or by $M(n)$ for short.

Consider hyperbolic space $H^n$ in the upper half space $E^n_+$, that is,

$$H^n = \left( E^n_+ \, , \, ds^2 = \frac{1}{x_n^2}\,(dx_1^2 + \cdots + dx_n^2) \right) \tag{1.1}$$

with distance between two points $x,y \in H^n$ given by

$$\cosh\, d(x,y) = 1 + \frac{|\,x-y\,|^2}{2x_n y_n} \quad . \tag{1.2}$$

By Poincaré extension, every Möbius transformation $T \in M(n-1)$ gives rise to an element in $M(E^n_+)$ again denoted by $T$. In fact, $T \in Iso(H^n)$ since it leaves invariant the hyperbolic metric (1.2).

According to the fixed point behavior a Möbius transformation is either elliptic, parabolic, or loxodromic. For example, if $T \in M(E^n_+)$ has precisely one resp. two fixed points in $\widehat{E}^{n-1}$ and none in $E^n_+$, then $T$ is parabolic resp. loxodromic.

Let $T \in Iso(H^n)$ be a loxodromic element, and denote by $q_1, q_2 \in \partial H^n$ its two different fixed points. They determine a unique geodesic $a_T \subset H^n$, the axis of $T$, along which $T$ acts as a translation. For $p \in a_T$, $d(p,T(p)) =: \tau$ is constant and called the translational length of $T$. Besides, $T$ consists of a rotational part $R$ such that – after a suitable conjugation – we obtain the representation

$$T = rA \quad , \quad \text{where} \quad r = e^{\tau} \quad , \quad A \in O(E^{n-1}) \quad . \tag{1.3}$$

For later purpose, we prove the following very useful property of $T$ (cf. [K3, Lemma 1.3] for $n=4$).

PROPOSITION 1.

*Let $T \in Iso(H^n)$ be a loxodromic element with axis $a_T$, with rotational part $R$ and with translational length $\tau$. Let $p \in H^n$ be such that $p \notin a_T$, and assume that the foot of the perpendicular from $p$ to $a_T$ is $\hat{p}$. Denote by $\omega = \omega(p)$ the angle at $\hat{p}$ in the triangle $(p,\hat{p},R(p))$. Let $d = d(p,T(p))$ and $\delta = d(p,a_T)$. Then,*

$$\cosh d = \cosh\tau + \sinh^2\delta \cdot \left( \cosh\tau - \cos\omega \right) \quad . \tag{1.4}$$

*Proof :* Without loss of generality, we may assume that $a_T = (0, \infty)$ . Then, $\hat{p} = |p|\, e_n$ . Let $a := d(p, R(p))$ , $b := d(R(p), T(p))$ , and $c := d(\hat{p}, T(p))$ (cf. Figure).

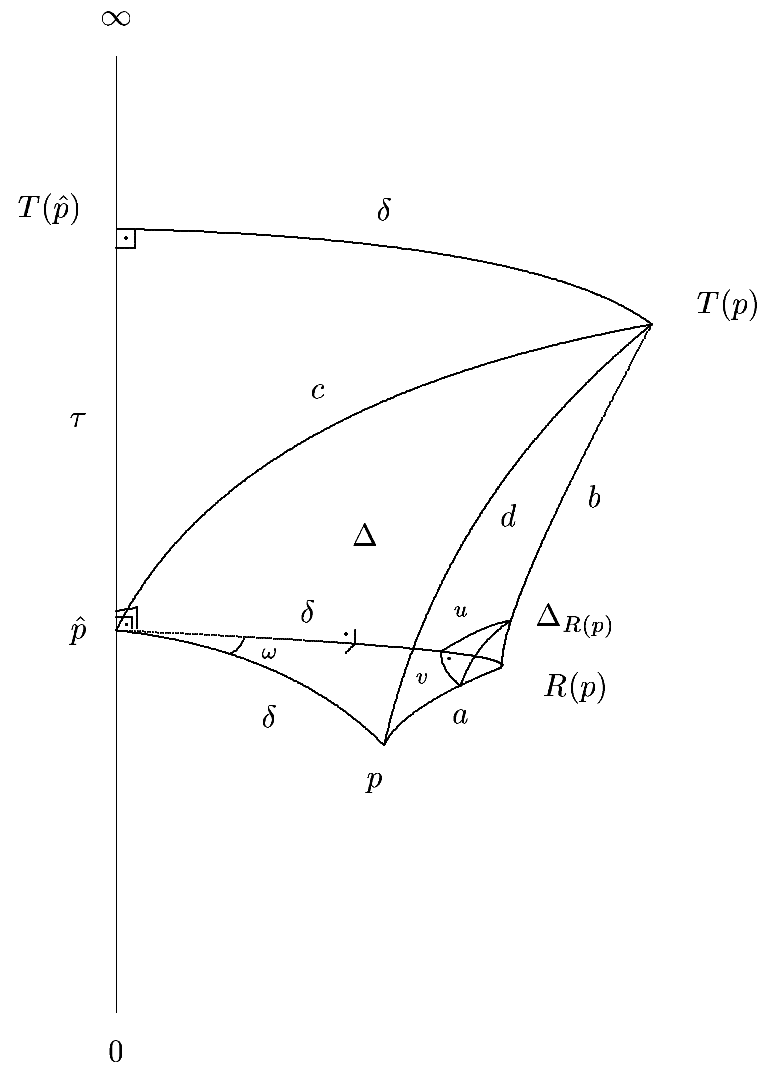


Figure.

Hyperbolic trigonometry yields with respect to the triangle $(p, \hat{p}, R(p))$

$$\cosh a = \cosh^2 \delta - \sinh^2 \delta \cos \omega = 1 + \sinh^2 \delta \,(1 - \cos \omega) \quad , \tag{1.5}$$

and with respect to the Saccheri quadrangle $(\hat{p}, T(\hat{p}), T(p), R(p))$

$$\cosh b = \cosh \tau \cosh^2 \delta - \sinh^2 \delta \quad , \tag{1.6}$$

and finally with respect to the right-angled triangle $(\hat{p}, T(\hat{p}), T(p))$

$$\cosh c = \cosh \tau \cosh \delta \quad . \tag{1.7}$$

Next, consider the hyperbolic tetrahedron $\Delta = \Delta(\hat{p}, p, R(p), T(p))$. The dihedral angle formed by the facets opposite to $p$ and $T(p)$, respectively, and attached at the edge $(\hat{p}, R(p))$ equals $\pi/2$. Denote by $\Delta_{R(p)}$ the spherical vertex figure of $\Delta$ at the vertex $R(p)$. $\Delta_{R(p)}$ is a right-angled triangle with hypotenuse $\beta$, say. Furthermore, let $u$ (resp. $v$) be the edge of $\Delta_{R(p)}$ in the facet opposite to $p$ (resp. $T(p)$) in $\Delta$. Then, $\cos\beta = \cos u \cos v$. By hyperbolic trigonometry, we deduce

$$\cosh d = \cosh a \cosh b - \sinh a \sinh b \cos\beta \quad , \tag{1.8}$$

as well as

$$\begin{aligned} \cosh c &= \cosh b \cosh \delta - \sinh b \sinh \delta \cos u \quad , \\ \cosh \delta &= \cosh a \cosh \delta - \sinh a \sinh \delta \cos v \quad . \end{aligned} \tag{1.9}$$

Hence, by (1.7) and (1.9),

$$\begin{aligned} \cos\beta = \cos u \cos v &= \frac{\cosh b \cosh \delta - \cosh \tau \cosh \delta}{\sinh b \sinh \delta} \cdot \frac{\cosh a \cosh \delta - \cosh \delta}{\sinh a \sinh \delta} \\ &= \coth^2 \delta \cdot \frac{\cosh b - \cosh \tau}{\sinh b} \cdot \frac{\cosh a - 1}{\sinh a} \quad . \end{aligned}$$

By using (1.5), (1.6) and (1.8), we obtain

$$\begin{aligned} \cosh d &= \cosh a \cosh b - \coth^2 \delta \,(\cosh b - \cosh \tau)\,(\cosh a - 1) \\ &= \cosh a \cosh b\,(1 - \coth^2 \delta) + \coth^2 \delta \cdot [\cosh b + (\cosh a - 1)\cosh \tau\,] \\ &= -\frac{1}{\sinh^2 \delta}\,[\cosh^2 \delta - \sinh^2 \delta \cos \omega\,] \cdot \cosh b + \\ &\qquad\qquad + \coth^2 \delta \cdot [\cosh b + \sinh^2 \delta\,(1 - \cos \omega) \cosh \tau\,] \\ &= \cosh b \cos \omega + \cosh^2 \delta \cosh \tau (1 - \cos \omega) \\ &= \cosh^2 \delta \cosh \tau \cos \omega - \sinh^2 \delta \cos \omega + \cosh^2 \delta \cosh \tau (1 - \cos \omega) \\ &= \cosh \tau + \sinh^2 \delta\,(\cosh \tau - \cos \omega) \quad . \end{aligned}$$

q.e.d.

**Remark.** Let $0 \le \alpha_0, \ldots, \alpha_r < 2\pi$, $0 \le r < [\frac{n}{2}]$, with $\cos\alpha_0 \ge \cdots \ge \cos\alpha_r$ denote the rotation angles of the loxodromic element $T \in Iso(H^n)$. Then,

$$\cos\alpha_0 \ge \cos\omega \ge \cos\alpha_r \quad .$$

To see this, pass to the normal form of the orthogonal part $R \in O(n-1)$ of $T$ and express $p = (p_0, \ldots, p_{n-2}, t) \in H^n$ with respect to the new basis in $E^{n-1} = \{\, t = 0 \,\}$. Then, project the triangle $(p, \hat{p}, R(p))$ orthogonally down to $\{\, t = 0 \,\}$ in order to compute

$$\begin{aligned} \cos\omega &= \frac{(p_0^2 + p_1^2)\cos\alpha_0 + \cdots + (p_{2r}^2 + p_{2r+1}^2)\cos\alpha_r + p_{2r+2}^2 + \cdots + p_{n-2}^2}{p_0^2 + \cdots + p_{n-2}^2} \\ &\ge \frac{(p_0^2 + \cdots + p_{n-2}^2)\ \cos\alpha_r}{p_0^2 + \cdots + p_{n-2}^2} = \cos\alpha_r \quad . \end{aligned}$$

### 1.2. Quaternions and $Iso^+(H^5)$

Consider the quaternion algebra $\mathbb{H} = \{\, q = q_0 + q_1 i + q_2 j + q_3 k \mid q_l \in \mathbb{R} \}$ with generators $i, j$, where $k = ij$ as usually. $\mathbb{H}$ is a Euclidean vector space with basis $1, i, j, k$. Decompose a quaternion $q = q_0 + q_1 i + q_2 j + q_3 k$ into scalar part $Sq := q_0$ and vector part $Vq := q_1 i + q_2 j + q_3 k$ so that $q = Sq + Vq$. The (quaternionic) conjugate of $q$ is given by $\overline{q} = Sq - Vq$ and satisfies $|q|^2 = q\overline{q} = \overline{q}q$. For a unit quaternion $a$, we can write

$$a = \exp(I\alpha) := \cos\alpha + I\,\sin\alpha \qquad \text{for some } \alpha \in [0, 2\pi) \quad , \tag{1.10}$$

where $I$ is a pure unit quaternion, i.e., the scalar part of $I$ vanishes and therefore $I = -\overline{I}$, or equivalently $I^2 = -1$. Furthermore, write $q =: u + vj$ with $u = q_0 + q_1 i$, $v = q_2 + q_3 i \in \mathbb{C}$. Then, there is the correspondence

$$q = (q_0 + q_1 i) + (q_2 + q_3 i)j = u + vj \quad \sim \quad Q := \begin{pmatrix} u & v \\ -\overline{v} & \overline{u} \end{pmatrix} \in \mathrm{Mat}(2; \mathbb{C}) \quad . \tag{1.11}$$

Consider a matrix $M \in \mathrm{Mat}(2; \mathbb{H})$ and associate to $M$ the complex block matrix

$$\mathcal{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Mat}(4; \mathbb{C})$$

according to (1.11). The trace $\mathrm{Tr}\, M$ of $M$ is defined by

$$\mathrm{Tr}\, M := \frac{1}{2}\,\mathrm{tr}\,\mathcal{M} = S(a + d) \quad \text{for} \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and is obviously conjugacy invariant. In order to establish a determinant of $M$ we adopt the point of view of J. Dieudonné (cf. [D], [As]) and consider again $\mathcal{M}$. By exploiting the correspondence (1.11), one calculates (cf. [Wil, §3])

$$\det \mathcal{M} = |l_{ij}|^2 = |r_{ij}|^2 \quad , \quad 1 \le i, j \le 2 \quad , \quad \text{where} \tag{1.12}$$

$$\begin{aligned} l_{11} &= da - dbd^{-1}c \quad , & l_{12} &= bdb^{-1}a - bc \quad , \\ l_{21} &= cac^{-1}d - cb \quad , & l_{22} &= ad - aca^{-1}b \quad ; \\ r_{11} &= ad - bd^{-1}cd \quad , & r_{12} &= db^{-1}ab - cb \quad , \\ r_{21} &= ac^{-1}dc - bc \quad , & r_{22} &= da - ca^{-1}ba \quad . \end{aligned} \tag{1.13}$$

In particular, $\det \mathcal{M} \ge 0$, and

$$\det \mathcal{M} = |ad - aca^{-1}b|^2 = |ad|^2 + |bc|^2 - 2\,S(a\overline{c}d\overline{b}) \quad . \tag{1.14}$$

The quantity

$$\Delta = \Delta(M) :=_+ \sqrt{\det \mathcal{M}} \tag{1.15}$$

is called the *Dieudonné determinant* of $M$.

PROPOSITION 2 [Wil, Theorem 1].

*Let* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Mat(2;\mathbb{H})$ *be such that* $\Delta(M) \neq 0$. *Then, $M$ is invertible, and*

$$M^{-1} = \begin{pmatrix} l_{11}^{-1} d & -l_{12}^{-1} b \\ -l_{21}^{-1} c & l_{22}^{-1} a \end{pmatrix} = \begin{pmatrix} d r_{11}^{-1} & -b r_{12}^{-1} \\ -c r_{21}^{-1} & a r_{22}^{-1} \end{pmatrix} \quad .$$

In order to abbreviate, we write

$$\begin{pmatrix} {}^{\sim}d & {}^{\sim}b \\ {}^{\sim}c & {}^{\sim}a \end{pmatrix} := \begin{pmatrix} l_{11}^{-1} d & l_{12}^{-1} b \\ l_{21}^{-1} c & l_{22}^{-1} a \end{pmatrix} \quad , \quad \begin{pmatrix} d_{\sim} & b_{\sim} \\ c_{\sim} & a_{\sim} \end{pmatrix} := \begin{pmatrix} d r_{11}^{-1} & b r_{12}^{-1} \\ c r_{21}^{-1} & a r_{22}^{-1} \end{pmatrix} \quad . \tag{1.16}$$

By coefficient comparison in $M M^{-1} = I = M^{-1} M$, one obtains the following useful identities.

**Lemma 1.**

*Let* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Mat(2;\mathbb{H})$ *be invertible. Then,*

(i) $ad_{\sim} - bc_{\sim} = da_{\sim} - cb_{\sim} = 1 \quad ; \quad {}^{\sim}da - {}^{\sim}bc = {}^{\sim}ad - {}^{\sim}cb = 1 \quad .$

(ii) $a\,{}^{\sim}d - b\,{}^{\sim}c = d\,{}^{\sim}a - c\,{}^{\sim}b = 1 \quad ; \quad d_{\sim}a - b_{\sim}c = a_{\sim}d - c_{\sim}b = 1 \quad .$

(iii) $ab_{\sim} = ba_{\sim} \quad , \quad cd_{\sim} = dc_{\sim} \quad ; \quad {}^{\sim}ac = {}^{\sim}ca \quad , \quad {}^{\sim}bd = {}^{\sim}db \quad .$

(iv) $a\,{}^{\sim}b = b\,{}^{\sim}a \quad , \quad c\,{}^{\sim}d = d\,{}^{\sim}c \quad ; \quad a_{\sim}c = c_{\sim}a \quad , \quad b_{\sim}d = d_{\sim}b \quad .$

By Lemma 1, the group $S_{\Delta}L(2;\mathbb{H})$ of all quaternionic $2 \times 2-$matrices with Dieudonné determinant $\Delta = 1$ can be identified according to *

$$S_{\Delta}L(2;\mathbb{H}) = \{ T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}(2;\mathbb{H}) \mid ad_{\sim} - bc_{\sim} = 1 \} \quad .$$

There is a close relationship to the group $Iso^{+}(H^5)$ of orientation preserving isometries of $H^5$ in the following way (cf. [H], [Wil]). Take the hyperbolic 5−space $H^5$ with its canonical orientation and parametrize the space with the aid of $\mathbb{H}$ by writing $E_{+}^{5} = \mathbb{H} \times \mathbb{R}_{+}$ so that $\partial H^n = \widehat{\mathbb{H}}$ (cf. (1.1)). The group $S_{\Delta}L(2;\mathbb{H})$ acts on $\widehat{\mathbb{H}}$ by linear fractional transformations

$$T(x) = (ax + b)(cx + d)^{-1}$$

* Following L. Ahlfors [Al], $SL(2;\mathbb{H})$ is used to denote the group of quaternionic Clifford matrices of pseudo-determinant equal to 1.

with $T(\infty)=\infty$ for $c=0$, and with $T(\infty)=ac^{-1}$ and $T(-c^{-1}d)=\infty$ for $c\neq 0$. By passing to the projectivized group

$$PS_{\Delta}L(2;\mathbb{H}) := S_{\Delta}L(2;\mathbb{H})\,/\,\{\pm E\} \quad ,$$

one gets the isomorphism

$$PS_{\Delta}L(2;\mathbb{H}) \simeq Iso^{+}(H^5) \quad .$$

In the following, we do not distinguish in the notation between elements of these groups. Let $T\in Iso^{+}(H^5)$ be a loxodromic element with rotational part $R$ (cf. (1.3)). Since $T$ is orientation preserving, $R$ is the Poincaré extension of the composition of either one or two rotations in planes of $\mathbb{H}$. In fact, $R\in SO(4)$ is given by (cf. [C2, (6.78)], [C1], [Po])

$$R(x)=axb \quad \text{with} \quad a,b\in\mathbb{H},\ |a|=|b|=1 \quad .$$

In particular, the rotation through the angles $\pm\alpha+\beta\in[0,2\pi)$, $0\leq\alpha\leq\beta<\pi$, about two completely orthogonal planes is given by

$$\begin{pmatrix} \exp(\alpha I) & 0 \\ 0 & \exp(-\beta J) \end{pmatrix} \tag{1.19}$$

for some unit pure elements $I,J\in\mathbb{H}$. Finally, consider a parabolic element $X\in Iso^{+}(H^5)$ which acts as a translation. Modulo conjugation in $PS_{\Delta}L(2;\mathbb{H})$, $X$ can be written in the form

$$X=\begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \quad \text{with} \quad \mu\in\mathbb{H}\cong E^4 \quad .$$

### 1.3. Isolation of the identity in $PS_{\Delta}L(2,\mathbb{H})$

Consider a non-elementary discrete two generator subgroup $<S,T>$ of $PSL(2,\mathbb{C})$. By Jørgensen's trace inequality [J],

$$|\,\mathrm{tr}^2T-4\,|+|\,\mathrm{tr}[S,T]-2\,|\geq 1 \quad , \tag{1.20}$$

where $[S,T]=STS^{-1}T^{-1}$. By specializing, for example to an element

$$T=\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \quad \text{with} \quad |\lambda|\neq 1 \quad ,$$

the inequality (1.20) takes the form

$$|\lambda-\lambda^{-1}|^2\cdot(\,1+|bc|\,)\geq 1 \quad . \tag{1.21}$$

By writing $\lambda=:e^{\frac{1}{2}(\tau+i\alpha)}$, (1.21) turns into

$$2\,(\cosh\tau-\cos\alpha)\cdot(\,1+|bc|\,)\geq 1 \quad . \tag{1.22}$$

Formulas avoiding trace such as (1.21) and (1.22) allow generalizations for $Iso^+(H^n)$ of geometrical relevance. In [Wat], P. Waterman presents various versions of (1.21) for the group $PSL(2; C_{n-2})$ of Clifford matrices associated to the Clifford algebra $C_{n-2}$ with $n-2$ generators.

Here, we derive a formula analoguous to (1.22) for $PS_\Delta L(2;\mathbb{H}) \simeq Iso^+(H^5)$ and for an element

$$T = \begin{pmatrix} e^{\tau/2}\exp(I\alpha) & 0 \\ 0 & e^{-\tau/2}\exp(-J\beta) \end{pmatrix}$$

with rotational part according to (1.19) by adapting suitably standard methods (cf. [Be], [Wat] and [K3]).

PROPOSITION 3.

*Let* $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $T = \begin{pmatrix} e^{\tau/2}\exp(I\alpha) & 0 \\ 0 & e^{-\tau/2}\exp(-J\beta) \end{pmatrix} \in PS_\Delta L(2;\mathbb{H})$ *be loxodromic elements generating a non-elementary discrete subgroup. Then,*

$$2\,(\cosh\tau - \cos(\alpha+\beta)) \cdot (\,1 + |bc|\,) \geq 1 \quad . \tag{1.23}$$

*Proof:* We follow the strategy of [Wat, Theorem I]. Suppose that

$$\mu := 2\,(\cosh\tau - \cos(\alpha+\beta)) \cdot (\,1 + |bc|\,) < 1 \quad , \tag{1.24}$$

and write $\rho := e^{\tau/2}$ for short, as well as

$$T =: \begin{pmatrix} A & 0 \\ 0 & B^{-1} \end{pmatrix} \quad .$$

Consider the Shimizu-Leutbecher sequence defined inductively by

$$S_0 = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} := S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad ;$$

$$S_{n+1} = \begin{pmatrix} a_{n+1} & b_{n+1} \\ c_{n+1} & d_{n+1} \end{pmatrix} := S_n T S_n^{-1} \quad \text{for} \quad n \geq 0 \quad .$$

By §1.2, Proposition 2 and (1.16), one computes

$$\begin{aligned} S_{n+1} &= \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & B^{-1} \end{pmatrix} \begin{pmatrix} {}^{\sim}d_n & -{}^{\sim}b_n \\ -{}^{\sim}c_n & {}^{\sim}a_n \end{pmatrix} \\ &= \begin{pmatrix} a_n A\,{}^{\sim}d_n - b_n B^{-1}\,{}^{\sim}c_n & -a_n A\,{}^{\sim}b_n + b_n B^{-1}\,{}^{\sim}a_n \\ c_n A\,{}^{\sim}d_n - d_n B^{-1}\,{}^{\sim}c_n & -c_n A\,{}^{\sim}b_n + d_n B^{-1}\,{}^{\sim}a_n \end{pmatrix} \quad . \end{aligned}$$

Since $\Delta(S_n) = 1$, we deduce that $|a_n| = |a_{n\sim}| = |{}^{\sim}a_n|$ and so forth. Therefore,

$$
\begin{aligned}
|b_{n+1}c_{n+1}| &= |\,(-a_nA\tilde{~}b_n + b_nB^{-1}\tilde{~}a_n\,)\cdot(\,c_nA\tilde{~}d_n - d_nB^{-1}\tilde{~}c_n\,)\,| \\
&= |a_nb_nc_nd_n|\cdot|A - a_n^{-1}b_nB^{-1}\tilde{~}a_n\tilde{~}b_n^{-1}|\cdot|A - c_n^{-1}d_nB^{-1}\tilde{~}c_n\tilde{~}d_n^{-1}| \quad .
\end{aligned}
\tag{1.25}
$$

For the middle factor in (2.6), for example, one gets the estimate (cf. §1.2)

$$
\begin{aligned}
|\,A - a_n^{-1}b_nB^{-1}\tilde{~}a_n\tilde{~}b_n^{-1}| &= |\,SA + VA - (SB^{-1})\cdot a_n^{-1}b_n\tilde{~}a_n\tilde{~}b_n^{-1} - a_n^{-1}b_n(VB^{-1})\tilde{~}a_n\tilde{~}b_n^{-1}\,| \\
&= |\,S(A-B^{-1}) + VA - a_n^{-1}b_n(VB^{-1})\tilde{~}a_n\tilde{~}b_n^{-1}\,| \\
&= \left\{\,S(A-B^{-1})^2 + |\,VA - a_n^{-1}b_n(VB^{-1})\tilde{~}a_n\tilde{~}b_n^{-1}\,|^2\,\right\}^{1/2} \\
&\le \left\{\,(\rho\cos\alpha - \rho^{-1}\cos\beta)^2 + \big(|VA| + |VB^{-1}|\big)^2\,\right\}^{1/2} \\
&= \left\{\,(\rho\cos\alpha - \rho^{-1}\cos\beta)^2 + (\rho|\sin\alpha| + \rho^{-1}|\sin\beta|)^2\,\right\}^{1/2} \\
&= \left\{\,\rho^2 + \rho^{-2} - 2\,c(\alpha,\beta)\,\right\}^{1/2} = \left\{\,2\,(\cosh\tau - c(\alpha,\beta))\,\right\}^{1/2} \quad ,
\end{aligned}
$$

where we used the notation

$$
c(\alpha,\beta) := \begin{cases} \cos(\alpha+\beta) & \text{if} \quad \alpha,\beta\in[0,\pi] \quad \text{or} \quad \alpha,\beta\in[\pi,2\pi)\,, \\ \cos(\alpha-\beta) & \text{else}\,. \end{cases}
$$

Hence, $c(0,\beta) = \cos\beta$, and by (1.19), $c(\alpha,\beta) \ge \cos(\alpha+\beta)$. The same estimate results for the third factor in (1.25). Therefore,

$$
|b_{n+1}c_{n+1}| \le |a_nb_nc_nd_n|\cdot\left\{\,2\,(\cosh\tau - \cos(\alpha+\beta))\,\right\} \quad .
$$

Since $|a_nd_n| \le 1 + |b_nc_n|$ by Lemma 1 (i), we obtain by induction

$$
|b_{n+1}c_{n+1}| \le \mu^n\,|bc| \quad ,
$$

and therefore, by (1.24), $b_nc_{n_\sim} \to 0$ and $a_nd_{n_\sim} \to 1$. Since

$$
|a_{n+1}| = |a_nA\tilde{~}d_n - b_nB^{-1}\tilde{~}c_n| \quad , |d_{n+1}| = |-c_nA\tilde{~}b_n - d_nB^{-1}\tilde{~}a_n| \quad ,
$$

we deduce that $|a_n| \to \rho$ and $|d_n| \to \rho^{-1}$. Moreover, we get the estimate

$$
|b_{n+1}| \le |a_nb_n|\cdot\left\{\,2\,(\cosh\tau - \cos(\alpha+\beta))\,\right\} \quad ,
$$

and by induction

$$
\frac{|b_n|}{\rho^n} \quad , \quad |c_n|\cdot\rho^n \quad \to \quad 0.
$$

Next, consider the elements

$$
\begin{aligned}
T_n := T^{-n}S_{2n}T^n &= \begin{pmatrix} A^{-n} & 0 \\ 0 & B^n \end{pmatrix}\begin{pmatrix} a_{2n} & b_{2n} \\ c_{2n} & d_{2n} \end{pmatrix}\begin{pmatrix} A^n & 0 \\ 0 & B^{-n} \end{pmatrix} \\
&= \begin{pmatrix} A^{-n}a_{2n}A^n & A^{-n}b_{2n}B^{-n} \\ B^nc_{2n}A^n & B^nd_{2n}B^{-n} \end{pmatrix} \\
&=: \begin{pmatrix} \alpha_n & \beta_n \\ \gamma_n & \delta_n \end{pmatrix} \quad \text{for} \quad n\ge 0 \quad .
\end{aligned}
$$

The sequence $\{T_n\}_{n\geq 0}$ has a convergent subsequence since

$$\begin{array}{rcl} |\alpha_n| = |a_{2n}| & \to & \rho \\ |\delta_n| = |d_{2n}| & \to & \rho^{-1} \\ |\beta_n| = \dfrac{|b_{2n}|}{\rho^{2n}} & \to & 0 \\ |\gamma_n| = |c_{2n}|\cdot\rho^{2n} & \to & 0 \quad . \end{array}$$

If we can show that the elements $T_n$ are all distinct, then the group $< S,T >$ is not discrete which yields the desired contradiction.

Suppose on the contrary that the sequence $\{T_n\}_{n\geq 0}$ stabilises, that is, $\beta_n = \gamma_n = 0$. Then, $b_{2n} = c_{2n} = 0$. Let $T_{n+1}$ be the first element such that $b_{n+1} = c_{n+1} = 0$. Since $\rho \neq 1$, (1.25) yields $a_n b_n = 0$ and $c_n d_n = 0$. But $\det S_n = |a_n d_n - a_n c_n a_n^{-1} b_n| = 1$, which leaves only two possibilities. In the first case, $b_n = c_n = 0$ which is impossible. In the second case, $a_n = d_n = 0$. For $n > 0$, $0 = \mathrm{Tr}\, S_n = S(a_n + d_n) = S(A + B^{-1}) = \rho\cos\alpha + \rho^{-1}\cos\beta$. It is easy to see that this contradicts $2\,(\cosh\tau - \cos(\alpha+\beta)) < 1$ given by the assumption (1.24). Therefore, $n = 0$ and $a = d = 0$. This is impossible since the group $< S,T >$ is supposed to be non-elementary.

q.e.d.

PROPOSITION 4.

*Let* $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $T = \begin{pmatrix} A & 0 \\ 0 & B^{-1} \end{pmatrix} \in PS_\Delta L(2;\mathbb{H})$ *be loxodromic elements such that* $2r := \mathrm{dist}\,(a_T, a_{STS^{-1}}) > 0$. *Then,*

$$\cosh r \geq |\,bc\,|^{1/2} \quad . \tag{1.26}$$

*Proof:* Denote by $p$ the common perpendicular of the axes $a_T, a_{STS^{-1}}$ whose end points equal $0, \infty, S(0), S(\infty)$ in $\partial H^5$. Choose a Möbius transformation

$$V = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in PS_\Delta L(2,\mathbb{H})$$

such that $0, \infty, S(0), S(\infty)$ are mapped to $-w, w, -1, 1$ with $|w| > 1$, say. That is, $p$ is mapped to the positive $t$–axis, and $2r = \mathrm{dist}\,(a_T, a_{STS^{-1}}) = \log|w|$. For the cross ratios, we obtain

$$\frac{|\,1-w\,|^2}{4|w|} = [-1,1,-w,w] = [bd^{-1}, ac^{-1}, 0, \infty] = \frac{|\,bd^{-1}\,|}{|\,bd^{-1} - ac^{-1}\,|} \quad .$$

By (1.12) and (1.13), this means that

$$\frac{|\,1-w\,|^2}{4|w|} = |\,bc\,| \quad .$$

By (1.10), we can write $w = \rho\,\exp(I\omega)$ in $E^4$ for some $\omega \in [0, 2\pi)$ and a unit pure element $I \in \mathbb{H}$. Hence, $2r = \log\rho$. Putting $z := (2r + I\omega)/2$, we deduce

$$w = e^{2r}\exp(I\omega) =: \exp(2r + I\omega) = \exp(2z) \quad .$$

Next, define

$$\sinh z := \frac{1}{2}\{\exp(z) - \exp(-z)\} \quad .$$

It follows that

$$|\sinh z|^2 = \frac{1}{4}|(1-w)^2 w^{-1}| = \frac{1}{2}(\cosh(2r) - \cos\omega) \le \frac{1}{2}(\cosh(2r) + 1) \quad .$$

Thus,

$$\cosh^2 r = \frac{1}{2}(\cosh(2r) + 1) \ge |\sinh z|^2 = |bc| \quad .$$

q.e.d.

PROPOSITION 5.

*Let* $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *and* $T = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \in PS_{\Delta}L(2;\mathbb{H})$ *with* $\mu \in E^4$ *generate a non-elementary discrete subgroup. Then,*

$$|c| \cdot |\mu| \ge 1 \quad . \tag{1.27}$$

The proof is a slight modification of the proof of [K3, Theorem 1.2] by using Lemma 1.

## 2. A thick and thin decomposition for hyperbolic 5–manifolds

Let $M$ denote an oriented complete hyperbolic 5−manifold of finite volume which consequently will be called hyperbolic 5−manifold for short. That is, $M$ is a Clifford-Klein space form $H^5/\Gamma$ where $\Gamma < PS_{\Delta}L(2,\mathbb{H})$ is discrete, torsion-free and cofinite. In particular, $\Gamma$ is non-elementary. Denote by $i_p(M)$ the injectivity radius of $M$ at $p$. By the Margulis Lemma for discrete groups of hyperbolic isometries (cf. [BGS, §9–10], [T], [R1]), there is a universal positive constant $\varepsilon$ such that there is a thick and thin decomposition

$$M = M_{\le\varepsilon} \cup M_{>\varepsilon} \tag{2.1}$$

of $M$ as follows. The thick part $M_{>\varepsilon} = \{p \in M \mid i_p(M) > \frac{\varepsilon}{2}\}$ of $M$ is compact.
The thin part $M_{\le\varepsilon} = \{p \in M \mid i_p(M) \le \frac{\varepsilon}{2}\}$ in (2.1) consists of connected components of the following types. The bounded components are neighborhoods $N$ of simple (i.e. with no self-intersection) closed geodesics $g$ through $p \in M_{\le\varepsilon}$ in $M$ of length $l(g) \le \varepsilon$

homeomorphic to ball bundles over the circle. In fact, $N$ is a quotient $U/\Gamma_U$ by an infinite cyclic group $\Gamma_U < \Gamma$ of loxodromic type with common axis projecting to $g$ and leaving precisely invariant some component $U \subset H^5$ lying above $N$. The unbounded components are cusp neighborhoods homeomorphic to products of compact flat manifolds with a real half line. Each cusp neighborhood can be written in the form $C = C_q = V_q/\Gamma_q$ with $\Gamma_q < \Gamma$ of parabolic type fixing some point $q \in \partial H^5$ and leaving precisely invariant some horoball $V_q \subset H^5$ based at $q$.

In fact, to each subgroup $\Gamma_q < \Gamma$ of parabolic type corresponds a particular extremal horoball $B_q$ such that $B_q/\Gamma_q$ embeds in $M$. We describe it for the case $q = \infty$, only. Denote by $\mu \neq 0$ a shortest vector in the translational lattice $\Lambda < \Gamma_\infty$ here identified with $E^4$. Then,

$$B(\mu) = B_\infty(\mu) := \{\, x \in H^5 \mid x_5 > |\mu| \,\}$$

is called the canonical horoball of $\Gamma_\infty$. $B(\mu)$ is precisely invariant with respect to $\Gamma_\infty$ and gives rise to a cusp neighborhood in $M$. Moreover, canonical horoballs associated to inequivalent parabolic transformations in $\Gamma$ are disjoint. The proofs are slight variations of those of [K3, Lemma 2.7] and [K3, Lemma 2.8].

## 2.1. The thin part of a hyperbolic 5–manifold

In the following, we construct neighborhoods of sufficiently small simple closed geodesics in $M$ such that they are disjoint from canonical cusp neighborhoods. If $g$ is a simple closed geodesic in $M$, denote by $r_g$ the injectivity radius for the exponential map of the normal bundle of $g$ into $M$. For $r \le r_g$, the set $T_g(r) = \{\, p \in M \mid \mathrm{dist}(p,g) < r \,\}$ is called a tube around $g$ of radius $r$. By making use of the description $Iso^+(H^5) \simeq PS_\Delta L(2,\mathbb{H})$, we construct tubes as follows.

PROPOSITION 6.

*Let* $l_0 = \frac{\sqrt{3}}{8\pi} \simeq 0.068916$. *Then, each simple closed geodesic* $g$ *in* $M$ *of length* $l(g) \le l_0$ *has a tube* $T_g(r)$ *of radius* $r$ *satisfying*

$$\cosh(2r) = \frac{1-3k}{k} \quad , \quad \text{where} \quad k = \frac{2\pi\, l(g)}{\sqrt{3}} \quad . \tag{2.2}$$

*Proof:* Consider two different lifts $\widetilde{g}_1, \widetilde{g}_2$ of $g$ in $H^5$. They give rise to $\Gamma-$conjugate loxodromic elements $T_1, T_2$ with disjoint axes $a_{T_1}, a_{T_2}$ but equal translational length $\tau$ and rotational angles $\pm\alpha+\beta$ with $0 \le \alpha \le \beta < \pi$. Denote by $p$ the common perpendicular of $a_{T_1}$ and $a_{T_2}$. We have to study the length $2r$ of $p$ in terms of $\tau = l(g)$. Without loss of generality assume that (cf. (1.19))

$$T_1 = \begin{pmatrix} e^{\tau/2}\exp(I\alpha) & 0 \\ 0 & e^{-\tau/2}\exp(-J\beta) \end{pmatrix} \quad ,$$

$$T_2 = S T_1 S^{-1} \quad \text{with} \quad S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad ,$$

for some unit pure quaternions $I, J$. Since $< T_1, T_2 >$ is non-elementary, $< T_1, S >$ is non-elementary as well. By Proposition 3, (1.23), applied to $< T_1, S >$, we obtain

$$2k \cdot \bigl(1 + |\, bc\,|\bigr) \geq 1 \quad , \quad \text{where} \quad k = \cosh\tau - \cos(\alpha+\beta) \quad . \tag{2.3}$$

Now, (1.26) of Proposition 4 yields $\cosh^2 r \geq |\, bc\,|$, that is,

$$\cosh(2r) \geq \frac{1-3k}{k} \quad , \tag{2.4}$$

which is nontrivial if

$$k = k(\tau; \alpha, \beta) = \cosh\tau - \cos(\alpha+\beta) \leq \frac{1}{4} \quad . \tag{2.5}$$

Next, observe that (2.4) remains valid for $k(n\tau; n\alpha, n\beta)$ by considering $n-$th iterates of $T_1, T_2$ for arbitrary $n \in \mathbb{N}$. In this situation, we make use of the modified Zagier inequality [CGM, Lemma 3.4] which says that for arbitrary $0 < \rho \leq \pi\sqrt{3}$ and $\nu \in [0, 2\pi)$, there exists a number $n_0 \in \mathbb{N}$ such that

$$\cosh(n_0\rho) - \cos(n_0\nu) \leq \frac{2\pi\,\rho}{\sqrt{3}} \quad . \tag{2.6}$$

By choosing $\tau = \rho \leq \frac{\sqrt{3}}{8\pi}$ and $\nu = \alpha + \beta$ according to (2.3), (2.5) and (2.6) imply that

$$k(n_0\tau; n_0\alpha, n_0\beta) \leq \frac{1}{4} \quad .$$

q.e.d.

**Lemma 2.**

*Let $g$ denote a simple closed geodesic in $M$ of length $l(g) \leq l_0$ with tube $T_g(r)$ of radius $r$ satisfying (2.2). Then,*

(a) *$r = r(l)$ is strictly decreasing.*

(b) *The volume $vol_5(T_g(r))$ is strictly decreasing with respect to $l$.*

*Proof:* Part (a) is obvious. As to part (b), observe that the volume of $T_g(r)$ equals the volume of a cylinder $\mathrm{Cyl}(r, l)$ of radius $r$ with axis of length $l$ which in general is given by (cf. [K3, Lemma 2.4])

$$\mathrm{vol}_n(\mathrm{Cyl}(r,l)) = \frac{2\pi}{n-1} \cdot l \cdot \sinh^{n-1} r \quad .$$

Hence,

$$\mathrm{vol}_5(T_g(r)) = \frac{\pi}{2} \cdot l \cdot \sinh^4 r = \frac{\sinh^2 r}{2} \cdot \mathrm{vol}_3(\mathrm{Cyl}(r,l)) \quad . \tag{2.7}$$

By (2.2),

$$\mathrm{vol}_3(\mathrm{Cyl}(r,l)) = \pi \cdot l \cdot \sinh^2 r = \frac{\sqrt{3}}{4} - 2\pi l \quad ,$$

which is a strictly decreasing function of $l$.

q.e.d.

**Remark.** Cao and Waterman [CW] obtained tubes around short closed geodesics of lengths $\le l_n$ in hyperbolic manifolds $M$ of arbitrary dimensions $n \ge 2$. They made use of certain extremal values associated to the rotational part of loxodromic elements loosing much accuracy when estimating the tube radius. For example, for $n = 5$, a closed geodesic $g$ of length $l_5 \simeq 0.0045$ in $M$ has a tube of radius $\simeq 0.9885$ and volume $\simeq 0.01269$ according to [CW, Corollary 9.5] while $g$ has a tube of radius $\simeq 2.3786$ and volume $\simeq 5.7846$ according to (2.2).

**Lemma 3.**

*Let $g, g'$ denote two simple closed geodesics in $M$ of lengths $l, l' \le l_1 := \frac{\sqrt{3}}{9\pi} \simeq 0.061258$ which do not intersect. Then, the tubes $T_g, T_{g'}$ of radii $r, r'$ subject to (2.2) are disjoint.*

*Proof:* Write $M = H^5/\Gamma$, and let $\widetilde{g}, \widetilde{g}'$ be lifts to $H^5$ of $g, g'$ which are the axes of loxodromic elements $T, T' \in \Gamma$ with translational lengths $\tau$ and $\tau'$ and angles of rotation $\pm\alpha + \beta$ and $\pm\alpha' + \beta'$ as usually. Let $\delta = \mathrm{dist}\,(\widetilde{g}, \widetilde{g}')$. We must prove that $\delta \ge r + r'$. For this, conjugate $T, T'$ in $PS_\Delta L(2, \mathbb{H})$ in order to obtain the elements

$$X = \begin{pmatrix} e^{\tau/2}\exp(I\alpha) & 0 \\ 0 & e^{-\tau/2}\exp(-J\beta) \end{pmatrix} \quad , \quad Y = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad .$$

The axis $a_{YXY^{-1}} = Y(a_X)$ of the element $YXY^{-1}$ is disjoint from $a_X$ and $a_Y$. Let $p \in a_X$ denote the point such that $\delta = \mathrm{dist}\,(a_X, a_Y) = \mathrm{dist}\,(p, a_Y)$, that is, $p$ is the foot point on $a_X$ of the common perpendicular of $a_X, a_Y$. By construction, $d := \mathrm{dist}\,(p, Y(p)) \ge 2r$. Denote by $k' := k(Y) = \cosh\tau' - \cos(\alpha' + \beta')$. Then, Proposition 1 implies that

$$\cosh(2r) \le \cosh d = \cosh\tau' + \sinh^2\delta\,(\cosh\tau' - \cos\omega) \quad .$$

Remark §1.1 yields $\cos\omega \ge \cos(\alpha' + \beta')$. Therefore,

$$\begin{aligned} \cosh(2r) &\le \cosh\tau' + \sinh^2\delta\,(\cosh\tau' - \cos(\alpha' + \beta')) \\ &\le k' + 1 + \sinh^2\delta \cdot k' = \cosh^2\delta \cdot k' + 1 \quad . \end{aligned}$$

By Proposition 6, we deduce that

$$\begin{aligned} \cosh(2\delta) &= 2\cosh^2\delta - 1 \ge 2 \cdot \frac{\cosh(2r) - 1}{k'} - 1 = 2 \cdot \frac{1 - 4k}{kk'} - 1 \\ &= \frac{1 - 4k}{kk'} + \frac{1 - 4k - kk'}{kk'} \quad . \end{aligned}$$

Suppose that $k' \geq k$ (otherwise, exchange the role of $X$ and $Y$). Then, we obtain

$$\cosh(2\delta) \geq \frac{\sqrt{1-4k}}{k} \cdot \frac{\sqrt{1-4k'}}{k'} + \frac{1-4k-kk'}{kk'} \quad .$$

By assumption, $l \leq l_1 = \frac{\sqrt{3}}{9\pi}$ so that, by Proposition 6,

$$k = \frac{2\pi\, l}{\sqrt{3}} < 2/9 \quad .$$

Hence,

$$\cosh(2r) = \frac{1-3k}{k} < \frac{\sqrt{1-4k}}{k} \quad ,$$

and similarly for $\cosh(2r')$. In order to conclude that $\cosh(2\delta) \geq \cosh(2r+2r')$, it suffices to show that

$$\frac{1-4k-kk'}{kk'} \geq \frac{\sqrt{1-4k-k^2}}{k} \cdot \frac{\sqrt{1-4k'-k'^2}}{k'} \geq \sinh(2r) \cdot \sinh(2r') \quad .$$

The verification is left to the reader (for details, cf. [K3, p. 64]). q.e.d.

**Lemma 4.**

*Let $M$ denote a non-compact hyperbolic $5-$manifold. Then, the canonical cusps and the tubes around closed geodesics according to* (2.2) *do not intersect in $M$.*

The proof of Lemma 4 is basically a consequence of Proposition 5. For details we refer to the analogous proof of [K3, Theorem 2.9].

## 2.2. A thick and thin decomposition

Let $M$ be a hyperbolic 5–manifold, and consider the thin and thick parts

$$M_{\leq\varepsilon} = \{\, p \in M \mid i_p(M) \leq \varepsilon/2 \,\} \quad \text{and} \quad M_{>\varepsilon} = \{\, p \in M \mid i_p(M) > \varepsilon/2 \,\}$$

of $M$ as in (2.1).

THEOREM I.

*For $\varepsilon \leq \frac{\sqrt{3}}{9\pi} \simeq 0.0612$, the thin part $M_{\leq\varepsilon}$ is a finite disjoint union of canonical cusps and tubes $T_g(r)$ around simple closed geodesics $g$ of length $\leq \varepsilon$ according to* (2.2).

*Proof:* We take up an idea of Meyerhoff [M]. Write $M = H^5/\Gamma$, where $\Gamma < Iso^+(H^5)$ is discrete, torsion-free and cofinite. The canonical cusps $C$ and the tubes $T$ around simple closed geodesics of lengths $\leq \frac{\sqrt{3}}{9\pi} \simeq 0.0612$ in $M$ as constructed in §2 are mutually disjoint. Hence, we must show that any cusp resp. any bounded component in $M_{\leq \varepsilon}$, $\varepsilon \leq 0.0612$, is contained in a canonical cusp $C$ resp. in a tube $T$. It is easy to verify the assertion for the canonical cusps (cf. §2).

Let $p \in M_{\leq \varepsilon}$ providing a loxodromic element $X \in \Gamma$ with distance $d := d(p, X(p)) \leq 0.0612$. Assume without loss of generality that $X$ has axis $a_X$ with end points $0, \infty$, and denote by $\tau > 0$ the translational length and by $\pm\alpha + \beta \in [0, 2\pi)$ the angles of rotation of $X$. Let $R$ be the rotational part of $X$. We show that $p \in T_{a_X}(r)$, where the tube radius is given by (2.3) and (2.4), that is,

$$\cosh(2r) = \frac{1-3k}{k} \quad \text{with} \quad k = k(X) = \cosh\tau - \cos(\alpha+\beta) \quad . \tag{2.8}$$

Let $\delta = d(p, a_X)$, and suppose that $\delta > 0$. By Proposition 1,

$$\cosh d = \cosh\tau + (\cosh\tau - \cos\omega) \cdot \sinh^2\delta \quad , \tag{2.9}$$

where $\omega = \omega(p)$ denotes the angle at the foot point $\hat{p}$ of the perpendicular from $p$ to $a_X$ in the triangle $(p, \hat{p}, R(p))$. Observe that $\cos(\alpha+\beta) \leq \cos\omega \leq \cos(\alpha-\beta)$.

By (2.9), we must show that for $d(p, X(p)) \leq d_0 := 0.0612$

$$\frac{\cosh d - \cosh\tau}{\cosh\tau - \cos\omega} = \sinh^2\delta \leq \sinh^2 r = \frac{1-4k}{2k} \quad , \tag{2.10}$$

where we may work with $k = k(X^n) < 1/4$ for any integer $n \geq 1$ (cf. proof of Proposition 6) and especially with

$$k = k(X^{n_0}) \leq \frac{2\pi\,\tau}{\sqrt{3}} \tag{2.11}$$

for $n_0 \in \mathbb{N}$ as given by (2.6). Now, write $p = (p_1, \ldots, p_5) \in H^5$ and consider the circular locus of all points $q \in H^5$ with $q_5 = p_5$ and $d(q, a_X) = \delta$. Varying over all such $q$, we find $d^-, d^+$ such that $0 < \tau < d^- \leq d \leq d^+ \leq d_0$ and (cf. (2.9) and Remark, §1.1)

$$\frac{\cosh d^+ - \cosh\tau}{\cosh\tau - \cos(\alpha+\beta)} = \sinh^2\delta = \frac{\cosh d^- - \cosh\tau}{\cosh\tau - \cos(\alpha-\beta)} \quad . \tag{2.12}$$

Therefore, it suffices to check that

$$\frac{\cosh d_0 - \cosh\tau}{\cosh\tau - \cos(\alpha+\beta)} \leq \frac{1-4k}{2k} \quad . \tag{2.13}$$

In order to verify (2.13), we distinguish between two cases.

Consider first the case $\cos(\alpha+\beta) > 1-\tau$. Choose $k$ according to (2.8). Then, (2.13) simplifies to

$$\cosh d_0 \leq \cosh\tau + \frac{1-4k}{2} \quad .$$

Since $k < \cosh\tau + \tau - 1 < \cosh d_0 + d_0 - 1 =: k_0$ with $\cosh d_0 \simeq 1.00187$, we see that the inequality

$$\cosh d_0 \leq 1 + \frac{1 - 4k_0}{2}$$

implying (2.13) is verified.

Next, suppose that $\cos(\alpha + \beta) \leq 1 - \tau$. Choose $k$ according to (2.11). Then, (2.13) turns into

$$\cosh d_0 \leq \cosh\tau + (\cosh\tau + \tau - 1) \cdot \frac{\sqrt{3} - 8\pi\tau}{4\pi\tau} \quad .$$

Since $\cosh\tau + \tau - 1 > \tau$, it suffices to verify

$$1.0019 < 1 + \tau \cdot \frac{\sqrt{3} - 8\pi\tau}{4\pi\tau} \quad . \tag{2.14}$$

The last term in (2.14) is strictly decreasing. Since $\tau < d_0$, we obtain the bound

$$\frac{\sqrt{3} - 8\pi\tau}{4\pi} > \frac{\sqrt{3} - 8\pi d_0}{4\pi} \simeq 0.0154 \quad ,$$

which proves (2.14).

q.e.d.

## 3. Consequences

### 3.1. Volume bounds

As first application, we derive some volume bounds.

PROPOSITION 7.

*Let $M$ be a hyperbolic $5-$manifold $M$ with $m$ cusps and $n$ distinct simple closed geodesics of lengths $\leq 0.059$. Then,*

$$\mathrm{vol}_5(M) > \frac{m+n}{96} \quad . \tag{2.15}$$

*Proof:* Replace each of the $m$ cusps by the canonical cusp neighborhood $C_i$, $1 \leq i \leq m$, as described above. $C_1, \ldots, C_m$ are pairwise disjoint. By methods based on results of Bieberbach and a sphere packing argument including the lattice constant computation $\delta_4 = \pi^2/16$ of Korkine-Zolotareff (cf. [K2, Remark (a), p. 726]), one has

$$\mathrm{vol}_5(C_i) > \frac{1}{96} \quad \text{for} \quad i = 1, \ldots, m \quad ,$$

whence

$$\mathrm{vol}_5(\cup_{i=1}^m C_i) = \sum_{i=1}^m \mathrm{vol}_5(C_i) > \frac{m}{96} \quad .$$

Suppose that $M$ carries $n \geq 1$ distinct simple closed geodesics of lengths $\leq 0.059 (< l_1 < l_0)$. By Proposition 6, Lemma 2, Lemma 3 and (2.7), $M$ contains $n$ mutually disjoint tubes $T_j$ , $1 \leq j \leq n$ , of total volume

$$\mathrm{vol}_5(\cup_{j=1}^n T_j) = \sum_{j=1}^n \mathrm{vol}_5(T_j) > n \cdot 0.01042 > \frac{n}{96} \quad .$$

Finally, by Lemma 4, the canonical cusps and the tubes are pairwise disjoint. This finishes the proof.

q.e.d.

**Remark.** Let $M$ be a (possibly non-orientable) hyperbolic 5−manifold $M$ with $m \geq 1$ cusps. In [K2] and by methods based on the theory of (horo-)sphere packings, we deduced the much better bound

$$\mathrm{vol}_5(M) > m \cdot 0.3922 \quad . \tag{2.16}$$

Adjusting suitably the estimate (2.15) requires to lower the upper length bound 0.059.

**Lemma 5.**

*Let $M$ be a hyperbolic 5−manifold. Then, there is a point $p \in M$ such that the injectivity radius $i_p(M)$ of $M$ at $p$ satisfies*

$$i_p(M) > 0.0343 > 1/30 \quad . \tag{2.17}$$

*Proof:* Suppose that a shortest closed geodesic of $M$ has length $l \leq l_2 := 0.0687526 < l_0$. Then, by Proposition 6, there is a tube $T$ embedded in $M$ of radius $r = r(l)$ according to (2.2). By a result of A. Przeworski (cf. [Pr, Proposition 4.1]), there is an embedded ball $B_p(\rho)$ centered at some point $p \in M$ which is of radius $\rho = \mathrm{arsinh}\,(\tanh(r)/2)$. Since $r(l)$ is strictly monotonely decreasing, it follows that $\rho \geq \rho(l_2) \simeq 0.03439$ and hence $i_p(M) \geq 0.03439$. If a shortest closed geodesics on $M$ is of length $> l_2$, then $i_p(M) > l_2/2 \simeq 0.03437$ for all $p \in M$. By comparison, the result (2.17) follows.

q.e.d.

THEOREM II.

*For a hyperbolic 5−manifold $M$,*

$$\mathrm{vol}_5(M) > 0.000083 \quad . \tag{2.18}$$

*Proof:* If $M$ is non-compact, the estimate follows from (2.16). Suppose that $M$ is compact. By Lemma 5, $M$ contains a ball $B$ of radius at least $0.0343$. This yields the estimate

$$\mathrm{vol}_5(M) \geq \mathrm{vol}_5(B) > 0.00000025 \quad , \tag{2.19}$$

which we improve as follows. Consider the in-radius $r(M) = \max_{p \in M} i_p(M)$ of $M$. Let $S_{reg} \subset H^5$ denote a regular hyperbolic simplex of edge length $2r(M)$ with spherical vertex simplex $s_{reg}$ of dimension 4. By [K1, Theorem], there is the volume bound

$$\mathrm{vol}_5(M) \geq \frac{4\pi^2}{9} \cdot \frac{\mathrm{vol}_5(S_{reg})}{\mathrm{vol}_4(s_{reg})} \quad . \tag{2.20}$$

By means of [K1, Lemma 4] and [K1, Lemma 5], the quotient $\mathrm{vol}_5(S_{reg})/\mathrm{vol}_4(s_{reg})$ in (2.20) can be estimated in terms of the dihedral angle $2\alpha$ as given by the edge length $2r(M)$ (cf. [K1, (3)]). Since $r(M) > 0.0343$, this leads to the asserted volume bound $\mathrm{vol}_5(M) > 0.000083$.

q.e.d.

**Remarks.** (a) Cao and Waterman derived the bound $r(M) \geq 1/544$ for the in-radius of a hyperbolic $5-$manifold $M$ (cf. [CW, Theorem 9.8]). By exploiting (2.20) as above, this yields the volume bound $\mathrm{vol}_5(M) > 0.00000023$.

(b) Ratcliffe and Tschantz (cf. [R2]) announced a geometrical construction of a non-orientable hyperbolic $5-$manifold with 10 cusps which is of volume $28\,\zeta(3) \simeq 33.6576$. By passing to its oriented double cover one obtains a hyperbolic $5-$manifold of volume $56\,\zeta(3)$ which to our knowledge represents the smallest known volume hyperbolic $5-$manifold. Therefore, a smallest volume hyperbolic $5-$manifold $M_0$ satisfies $0.000083 < \mathrm{vol}_5(M_0) \leq 67.3152$. Moreover, by Proposition 6 and Lemma 2, a shortest closed geodesic in $M_0$ has length $> 0.00043$.

### 3.2. Injectivity radius versus volume and diameter

Let $M$ be a compact hyperbolic $5-$manifold. Denote by $i(M) = \min_{p \in M} i_p(M)$ the injectivity radius of $M$ and by $\mathrm{diam}(M) = \max_{p,q \in M} \mathrm{dist}(p,q)$ the diameter of $M$. The injectivity radius $i(M)$ equals one half of the length of a shortest simple closed geodesic in $M$. By results of P. Buser [Bu1, Corollary 4.15] and A. Reznikov [R, Theorem],

$$i(M) \geq \mathrm{const} \cdot \mathrm{vol}_5(M)^{-3} \quad .$$

We improve this estimate as follows.

PROPOSITION 8.

*For a compact hyperbolic* $5-$*manifold* $M$,

$$i(M) \geq \mathrm{const} \cdot \mathrm{vol}_5(M)^{-1} \quad . \tag{2.21}$$

*Proof:* Assume that there is a short simple closed geodesic $g$ of length $l$ in $M$. Then, there is a tube $T_g(r)$ around $g$ of radius $r$ given by (cf. Proposition 6, (2.2))

$$\sinh^2 r = \frac{1}{2k} - 2 \quad , \quad \text{where} \quad k = \frac{2\pi l}{\sqrt{3}} \quad .$$

This implies

$$\mathrm{vol}_5(M) \geq \mathrm{vol}_5(T_g(r)) = \frac{\pi}{2} \cdot l \cdot \sinh^4 r \quad .$$

Since $\sinh^4 r \sim \mathrm{const} \cdot l^{-2}$ for small $l$, we deduce $l \geq \mathrm{const} \cdot \mathrm{vol}_5(M)^{-1}$ as desired.

q.e.d.

PROPOSITION 9.

*For a compact hyperbolic* $5-$*manifold* $M$,

$$i(M) \geq \mathrm{const} \cdot (\sinh(\mathrm{diam}(M))^{-2} \quad . \tag{2.22}$$

*Proof:* Let $g$ denote a simple closed geodesic in $M$. By a result due to E. Heintze and H. Karcher [HK, Corollary 2.3.2], the length $l$ of $g$ is bounded from below as follows.

$$l \geq \frac{2}{\pi^2} \cdot \frac{\mathrm{vol}_5(M)}{\sinh^4(\mathrm{diam}(M))} \quad .$$

This together with Proposition 8, (2.21), yields

$$l \geq \mathrm{const} \cdot \frac{1}{i(M) \cdot \sinh^4(\mathrm{diam}(M))} \quad ,$$

which implies the desired result.

q.e.d.

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