Chapter 2

Laplace's Equation

Let Ω be a domain in $\mathbb{R}^n$ and u a $C^2(\Omega)$ function. The Laplacian of u, denoted Δu, is defined by

$$(2.1) \qquad \Delta u = \sum_{i=1}^{n} D_{ii} u = \operatorname{div} Du.$$

The function u is called *harmonic* (*subharmonic*, *superharmonic*) in Ω if it satisfies there

$$(2.2) \qquad \Delta u = 0 \quad (\geqslant 0, \ \leqslant 0).$$

In this chapter we develop some basic properties of harmonic, subharmonic and superharmonic functions which we use to study the solvability of the classical Dirichlet problem for *Laplace's equation*, $\Delta u = 0$. As mentioned in Chapter 1, Laplace's equation and its inhomogeneous form, Poisson's equation, are basic models of linear elliptic equations.

Our starting point here will be the well known divergence theorem in $\mathbb{R}^n$. Let Ω_0 be a bounded domain with C^1 boundary $\partial \Omega_0$ and let v denote the unit outward normal to $\partial \Omega_0$. For any vector field $\mathbf{w}$ in $C^1(\bar{\Omega}_0)$, we then have

$$(2.3) \qquad \int_{\Omega_0} \operatorname{div} \mathbf{w} \, dx = \int_{\partial \Omega_0} \mathbf{w} \cdot v \, ds$$

where ds indicates the $(n-1)$-dimensional area element in $\partial \Omega_0$. In particular if u is a $C^2(\bar{\Omega}_0)$ function we have, by taking $\mathbf{w} = Du$ in (2.3),

$$(2.4) \qquad \int_{\Omega_0} \Delta u \, dx = \int_{\partial \Omega_0} Du \cdot v \, ds = \int_{\partial \Omega_0} \frac{\partial u}{\partial v} \, ds.$$

(For a more general formulation of the divergence theorem, see [KE 2].)

2.1. The Mean Value Inequalities

Our first theorem, which is a consequence of the identity (2.4), comprises the well known mean value properties of harmonic, subharmonic and superharmonic functions.

Theorem 2.1.   *Let* $u \in C^2(\Omega)$ *satisfy* $\Delta u = 0$ $(\geqslant 0, \leqslant 0)$ *in* Ω. *Then for any ball* $B = B_R(y) \subset\subset \Omega$, *we have*

$$(2.5) \qquad u(y) = (\leqslant, \geqslant) \frac{1}{n\omega_n R^{n-1}} \int\limits_{\partial B} u \, ds,$$

$$(2.6) \qquad u(y) = (\leqslant, \geqslant) \frac{1}{\omega_n R^n} \int\limits_{B} u \, dx.$$

For harmonic functions, Theorem 2.1 thus asserts that the function value at the center of the ball B is equal to the integral mean values over both the surface ∂B and B itself. These results, known as the *mean value theorems*, in fact also characterize harmonic functions; (see Theorem 2.7).

Proof of Theorem 2.1.   Let $\rho \in (0, R)$ and apply the identity (2.4) to the ball $B_\rho = B_\rho(y)$. We obtain

$$\int\limits_{\partial B_\rho} \frac{\partial u}{\partial \nu} \, ds = \int\limits_{B_\rho} \Delta u \, dx = (\geqslant, \leqslant) \, 0.$$

Introducing radial and angular coordinates $r = |x - y|$, $\omega = \dfrac{x - y}{r}$, and writing $u(x) = u(y + r\omega)$, we have

$$\int\limits_{\partial B_\rho} \frac{\partial u}{\partial \nu} \, ds = \int\limits_{\partial B_\rho} \frac{\partial u}{\partial r}(y + \rho\omega) \, ds = \rho^{n-1} \int\limits_{|\omega| = 1} \frac{\partial u}{\partial r}(y + \rho\omega) \, d\omega$$

$$= \rho^{n-1} \frac{\partial}{\partial \rho} \int\limits_{|\omega| = 1} u(y + \rho\omega) \, d\omega = \rho^{n-1} \frac{\partial}{\partial \rho} \left[\rho^{1-n} \int\limits_{\partial B_\rho} u \, ds \right]$$

$$= (\geqslant, \leqslant) \, 0.$$

Consequently for any $\rho \in (0, R)$,

$$\rho^{1-n} \int\limits_{\partial B_\rho} u \, ds = (\leqslant, \geqslant) R^{1-n} \int\limits_{\partial B_R} u \, ds$$

and since

$$\lim_{\rho \to 0} \rho^{1-n} \int\limits_{\partial B_\rho} u \, ds = n\omega_n u(y)$$

relations (2.5) follow. To get the solid mean value inequalities, that is, relations (2.6), we write (2.5) in the form

$$n\omega_n \rho^{n-1} u(y) = (\leqslant, \geqslant) \int\limits_{\partial B_\rho} u \, ds, \quad \rho \leqslant R$$

and integrate with respect to ρ from 0 to R. The relations (2.6) follow immediately. $\square$

2.2. Maximum and Minimum Principle

With the aid of Theorem 2.1 the *strong maximum principle* for subharmonic functions and the *strong minimum principle* for superharmonic functions may be derived.

Theorem 2.2. *Let* $\Delta u \geqslant 0$ ($\leqslant 0$) *in* Ω *and suppose there exists a point* $y \in \Omega$ *for which* $u(y) = \sup_\Omega u$ (inf$_\Omega u$). *Then* u *is constant. Consequently a harmonic function cannot assume an interior maximum or minimum value unless it is constant.*

Proof. Let $\Delta u \geqslant 0$ in Ω, $M = \sup_\Omega u$ and define $\Omega_M = \{x \in \Omega \mid u(x) = M\}$. By assumption Ω_M is not empty. Furthermore since u is continuous, Ω_M is closed relative to Ω. Let z be any point in Ω_M and apply the mean value inequality (2.6) to the subharmonic function $u - M$ in a ball $B = B_R(z) \subset \subset \Omega$. We therefore obtain

$$0 = u(z) - M \leqslant \frac{1}{\omega_n R^n} \int_B (u - M) \, dx \leqslant 0,$$

so that $u = M$ in $B_R(z)$. Consequently Ω_M is also open relative to Ω. Hence $\Omega_M = \Omega$. The result for superharmonic functions follows by replacement of u by $-u$. $\square$

The strong maximum and minimum principles immediately imply global estimates, namely the following *weak maximum* and *minimum principles*.

Theorem 2.3. *Let* $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ *with* $\Delta u \geqslant 0$ ($\leqslant 0$) *in* Ω. *Then, provided* Ω *is bounded,*

$$(2.7) \qquad \sup_\Omega u = \sup_{\partial\Omega} u \quad (\inf_\Omega u = \inf_{\partial\Omega} u).$$

Consequently, for harmonic u

$$\inf_{\partial\Omega} u \leqslant u(x) \leqslant \sup_{\partial\Omega} u, \quad x \in \Omega.$$

A uniqueness theorem for the classical Dirichlet problem for Laplace's and Poisson's equation in bounded domains now follows from Theorem 2.3.

Theorem 2.4. *Let* $u, v \in C^2(\Omega) \cap C^0(\bar{\Omega})$ *satisfy* $\Delta u = \Delta v$ *in* Ω, $u = v$ *on* $\partial\Omega$. *Then* $u = v$ *in* Ω.

Proof. Let $w = u - v$. Then $\Delta w = 0$ in Ω and $w = 0$ on $\partial\Omega$. It follows from Theorem 2.3 that $w = 0$ in Ω. $\square$

Note that also by Theorem 2.3, we have that if u and v are harmonic and subharmonic functions respectively, agreeing on the boundary $\partial\Omega$, then $v \leqslant u$ in Ω.

Hence the term subharmonic. A corresponding remark is true for superharmonic functions. Later in this chapter, we employ this property of $C^2(\Omega)$ subharmonic and superharmonic functions to expand their definition to larger classes of functions. In the next chapter, an alternate method of proof for Theorems 2.2, 2.3 and 2.4 will be supplied when we treat maximum principles for general elliptic equations; (see also Problem 2.1).

2.3. The Harnack Inequality

A further consequence of Theorem 2.1 is the following Harnack inequality for harmonic functions.

Theorem 2.5. *Let u be a non-negative harmonic function in Ω. Then for any bounded subdomain $\Omega' \subset\subset \Omega$, there exists a constant C depending only on n, Ω' and Ω such that*

$$(2.8) \qquad \sup_{\Omega'} u \leqslant C \inf_{\Omega'} u.$$

Proof. Let $y \in \Omega$, $B_{4R}(y) \subset \Omega$. Then for any two points $x_1, x_2 \in B_R(y)$, we have by the inequalities (2.6)

$$u(x_1) = \frac{1}{\omega_n R^n} \int\limits_{B_R(x_1)} u \, dx \leqslant \frac{1}{\omega_n R^n} \int\limits_{B_{2R}(y)} u \, dx.$$

$$u(x_2) = \frac{1}{\omega_n (3R)^n} \int\limits_{B_{3R}(x_2)} u \, dx \geqslant \frac{1}{\omega_n (3R)^n} \int\limits_{B_{2R}(y)} u \, dx.$$

Consequently we obtain

$$(2.9) \qquad \sup_{B_R(y)} u \leqslant 3^n \inf_{B_R(y)} u.$$

Now let $\Omega' \subset\subset \Omega$ and choose $x_1, x_2 \in \overline{\Omega}'$ so that $u(x_1) = \sup_{\Omega'} u$, $u(x_2) = \inf_{\Omega'} u$. Let $\Gamma \subset \overline{\Omega}'$ be a closed arc joining x_1 and x_2 and choose R so that $4R < \text{dist}\,(\Gamma, \partial\Omega)$. By virtue of the Heine-Borel theorem, Γ can be covered by a finite number N (depending only on Ω' and Ω) of balls of radius R. Applying the estimate (2.9) in each ball and combining the resulting inequalities, we obtain.

$$u(x_1) \leqslant 3^{nN} u(x_2).$$

Hence the estimate (2.8) holds with $C = 3^{nN}$. $\square$

Note that the constant in (2.8) is invariant under similarity and orthogonal transformations. A Harnack inequality for weak solutions of homogeneous elliptic equations will be established in Chapter 8.

2.4. Green's Representation

As a prelude to *existence* considerations we derive now some further consequences of the divergence theorem, namely, the Green identities. Let Ω be a domain for which the divergence theorem holds and let u and v be $C^2(\bar{\Omega})$ functions. We select $w = vDu$ in the identity (2.3) to obtain *Green's first identity*:

$$(2.10) \qquad \int_\Omega v \, \Delta u \, dx + \int_\Omega Du \cdot Dv \, dx = \int_{\partial\Omega} v \frac{\partial u}{\partial v} \, ds.$$

Interchanging u and v in (2.10) and subtracting, we obtain *Green's second identity*:

$$(2.11) \qquad \int_\Omega (v \, \Delta u - u \, \Delta v) \, dx = \int_{\partial\Omega} \left(v \frac{\partial u}{\partial v} - u \frac{\partial v}{\partial v} \right) ds.$$

Laplace's equation has the radially symmetric solution r^{2-n} for $n > 2$ and $\log r$ for $n = 2$, r being radial distance from a fixed point. To proceed further from (2.11), we fix a point y in Ω and introduce the normalized *fundamental solution* of Laplace's equation:

$$(2.12) \qquad \Gamma(x-y) = \Gamma(|x-y|) = \begin{cases} \dfrac{1}{n(2-n)\omega_n} |x-y|^{2-n}, & n > 2 \\[2ex] \dfrac{1}{2\pi} \log |x-y|, & n = 2. \end{cases}$$

By simple computation we have

$$D_i \Gamma(x-y) = \frac{1}{n\omega_n} (x_i - y_i)|x-y|^{-n};$$

$$(2.13)$$

$$D_{ij} \Gamma(x-y) = \frac{1}{n\omega_n} \{|x-y|^2 \delta_{ij} - n(x_i - y_i)(x_j - y_j)\} |x-y|^{-n-2}.$$

Clearly Γ is harmonic for $x \neq y$. For later purposes we note the following derivative estimates:

$$|D_i \Gamma(x-y)| \leqslant \frac{1}{n\omega_n} |x-y|^{1-n};$$

$$(2.14)$$

$$|D_{ij} \Gamma(x-y)| \leqslant \frac{1}{\omega_n} |x-y|^{-n}.$$

$$|D^\beta \Gamma(x-y)| \leqslant C |x-y|^{2-n-|\beta|}, \qquad C = C(n, |\beta|).$$

The singularity at $x = y$ prevents us from using Γ in place of v in Green's second identity (2.11). One way of overcoming this difficulty is to replace Ω by $\Omega - \bar{B}_\rho$

where $B_\rho = B_\rho(y)$ for sufficiently small ρ. We can then conclude from (2.11) that

$$(2.15) \qquad \int_{\Omega - B_\rho} \Gamma \, \Delta u \, dx = \int_{\partial\Omega} \left(\Gamma \frac{\partial u}{\partial v} - u \frac{\partial \Gamma}{\partial v} \right) ds + \int_{\partial B_\rho} \left(\Gamma \frac{\partial u}{\partial v} - u \frac{\partial \Gamma}{\partial v} \right) ds.$$

Now

$$\int_{\partial B_\rho} \Gamma \frac{\partial u}{\partial v} \, ds = \Gamma(\rho) \int_{\partial B_\rho} \frac{\partial u}{\partial v} \, ds$$

$$\leq n\omega_n \rho^{n-1} \Gamma(\rho) \sup_{B_\rho} |Du| \to 0 \quad \text{as } \rho \to 0$$

and

$$\int_{\partial B_\rho} u \frac{\partial \Gamma}{\partial v} \, ds = -\Gamma'(\rho) \int_{\partial B_\rho} u \, ds \qquad \text{(recall that } v \text{ is } outer \text{ normal to } \Omega - B_\rho)$$

$$= \frac{-1}{n\omega_n \rho^{n-1}} \int_{\partial B_\rho} u \, ds \to -u(y) \quad \text{as } \rho \to 0.$$

Hence letting ρ tend to zero in (2.15) we arrive at *Green's representation formula*:

$$(2.16) \qquad u(y) = \int_{\partial\Omega} \left(u \frac{\partial \Gamma}{\partial v} (x-y) - \Gamma(x-y) \frac{\partial u}{\partial v} \right) ds + \int_{\Omega} \Gamma(x-y) \, \Delta u \, dx, \qquad (y \in \Omega).$$

For an integrable function f, the integral $\int_\Omega \Gamma(x - y)f(x) \, dx$ is called the *Newtonian potential* with density f. If u has compact support in $\mathbb{R}^n$, then (2.16) yields the frequently useful representation formula,

$$(2.17) \qquad u(y) = \int \Gamma(x-y) \, \Delta u(x) \, dx.$$

For harmonic u, we also obtain the representation

$$(2.18) \qquad u(y) = \int_{\partial\Omega} \left(u \frac{\partial \Gamma}{\partial v} (x-y) - \Gamma(x-y) \frac{\partial u}{\partial v} \right) ds, \qquad (y \in \Omega).$$

Since the integrand above is infinitely differentiable and, in fact, also analytic with respect to y, it follows that u is also analytic in Ω. Thus harmonic functions are analytic throughout their domain of definition and therefore uniquely determined by their values in any open subset.

Now suppose that $h \in C^1(\bar\Omega) \cap C^2(\Omega)$ satisfies $\Delta h = 0$ in Ω. Then again by Green's second identity (2.11) we obtain

$$(2.19) \qquad - \int_{\partial\Omega} \left(u \frac{\partial h}{\partial v} - h \frac{\partial u}{\partial v} \right) ds = \int_\Omega h \, \Delta u \, dx.$$

Writing $G = \Gamma + h$ and adding (2.16) and (2.19) we then obtain a more general version of Green's representation formula:

$$(2.20) \qquad u(y) = \int_{\partial\Omega} \left(u\, \frac{\partial G}{\partial v} - G\, \frac{\partial u}{\partial v} \right) ds + \int_{\Omega} G\, \Delta u\, dx.$$

If in addition $G = 0$ on $\partial\Omega$ we have

$$(2.21) \qquad u(y) = \int_{\partial\Omega} u\, \frac{\partial G}{\partial v}\, ds + \int_{\Omega} G\, \Delta u\, dx$$

and the function $G = G(x, y)$ is called the (Dirichlet) *Green's function* for the domain Ω, sometimes also called the *Green's function of the first kind* for Ω. By Theorem 2.4, the Green's function is unique and from the formula (2.21) its existence implies a representation for a $C^1(\bar{\Omega}) \cap C^2(\Omega)$ harmonic function in terms of its boundary values.

2.5. The Poisson Integral

When the domain Ω is a ball the Green's function can be explicitly determined by the method of images and leads to the well known Poisson integral representation for harmonic functions in a ball. Namely, let $B_R = B_R(0)$ and for $x \in B_R$, $x \neq 0$ let

$$(2.22) \qquad \bar{x} = \frac{R^2}{|x|^2}\, x$$

denote its inverse point with respect to B_R; if $x = 0$, take $\bar{x} = \infty$. It is then easily verified that the Green's function for B_R is given by

$$(2.23) \qquad G(x, y) = \begin{cases} \Gamma(|x - y|) - \Gamma\left(\dfrac{|y|}{R}\, |x - \bar{y}| \right), & y \neq 0 \\[2ex] \Gamma(|x|) - \Gamma(R), & y = 0. \end{cases}$$

$$= \Gamma(\sqrt{|x|^2 + |y|^2 - 2x \cdot y}) - \Gamma\left(\sqrt{\left(\frac{|x||y|}{R} \right)^2 + R^2 - 2x \cdot y} \right)$$

$$\text{for all } x, y \in B_R, \ x \neq y.$$

The function G defined by (2.23) has the properties

$$(2.24) \qquad G(x, y) = G(y, x), \quad G(x, y) \leq 0 \quad \text{for } x, y \in \bar{B}_R.$$

Furthermore, direct calculation shows that at $x \in \partial B_R$ the normal derivative of G is given by

$$(2.25) \qquad \frac{\partial G}{\partial v} = \frac{\partial G}{\partial |x|} = \frac{R^2 - |y|^2}{n\omega_n R} |x - y|^{-n} \geqslant 0.$$

Hence if $u \in C^2(B_R) \cap C^1(\bar{B}_R)$ is harmonic, we have by (2.21) the *Poisson integral formula*:

$$(2.26) \qquad u(y) = \frac{R^2 - |y|^2}{n\omega_n R} \int_{\partial B_R} \frac{u \, ds_x}{|x - y|^n}.$$

The right hand side of formula (2.26) is called the Poisson integral of u. A simple approximation argument shows that the Poisson integral formula continues to hold for $u \in C^2(B_R) \cap C^0(\bar{B}_R)$. Note that by taking $y = 0$, we recover the mean value theorem for harmonic functions. In fact all the previous theorems of this chapter could have been derived as consequences of the representation (2.21) with $\Omega = B_R(0)$.

To establish the existence of solutions of the classical Dirichlet problem for balls we need the converse result to the representation (2.26), and we prove this now.

Theorem 2.6. *Let* $B = B_R(0)$ *and* φ *be a continuous function on* ∂B. *Then the function* u *defined by*

$$(2.27) \qquad u(x) = \begin{cases} \dfrac{R^2 - |x|^2}{n\omega_n R} \displaystyle\int_{\partial B} \dfrac{\varphi(y) \, ds_y}{|x - y|^n} & \text{for } x \in B \\[4mm] \varphi(x) & \text{for } x \in \partial B \end{cases}$$

belongs to $C^2(B) \cap C^0(\bar{B})$ *and satisfies* $\Delta u = 0$ *in* B.

Proof. That u defined by (2.27) is harmonic in B is evident from the fact that G, and hence $\partial G/\partial v$, is harmonic in x, or it may be verified by direct calculation. To establish the continuity of u on ∂B, we use the Poisson formula (2.26) for the special case $u = 1$ to obtain the identity.

$$(2.28) \qquad \int_{\partial B} K(x, y) \, ds_y = 1 \quad \text{for all } x \in B$$

where K is the *Poisson kernel*

$$(2.29) \qquad K(x, y) = \frac{R^2 - |x|^2}{n\omega_n R |x - y|^n}; \quad x \in B, \, y \in \partial B.$$

Of course the integral in (2.28) may be evaluated directly but this is a complicated calculation. Now let $x_0 \in \partial B$ and ε be an arbitrary positive number. Choose $\delta > 0$

so that $|\varphi(x) - \varphi(x_0)| < \varepsilon$ if $|x - x_0| < \delta$ and let $|\varphi| \leqslant M$ on ∂B. Then if $|x - x_0| < \delta/2$, we have by (2.27) and (2.28)

$$|u(x) - u(x_0)| = \left| \int_{\partial B} K(x, y)(\varphi(y) - \varphi(x_0)) \, ds_y \right|$$

$$\leqslant \int_{|y - x_0| \leqslant \delta} K(x, y)|\varphi(y) - \varphi(x_0)| \, ds_y$$

$$+ \int_{|y - x_0| > \delta} K(x, y)|\varphi(y) - \varphi(x_0)| \, ds_y$$

$$\leqslant \varepsilon + \frac{2M(R^2 - |x|^2)R^{n-2}}{(\delta/2)^n}.$$

If now $|x - x_0|$ is sufficiently small it is clear that $|u(x) - u(x_0)| < 2\varepsilon$ and hence u is continuous at x_0. Consequently $u \in C^0(\bar{B})$ as required. $\square$

We note that the preceding argument is local; that is, if φ is only bounded and integrable on ∂B, and continuous at x_0, then $u(x) \rightarrow \varphi(x_0)$ as $x \rightarrow x_0$.

2.6. Convergence Theorems

We consider now some immediate consequences of the Poisson integral formula. The following three theorems will not however be required for the later development. We show first that harmonic functions can in fact be characterized by their mean value property.

Theorem 2.7. *A $C^0(\Omega)$ function u is harmonic if and only if for every ball $B = B_R(y) \subset\subset \Omega$ it satisfies the mean value property,*

$$(2.30) \qquad u(y) = \frac{1}{n\omega_n R^{n-1}} \int_{\partial B} u \, ds.$$

Proof. By Theorem 2.6, there exists for any ball $B \subset\subset \Omega$ a harmonic function h such that $h = u$ on ∂B. The difference $w = u - h$ will then be a function satisfying the mean value property on any ball in B. Consequently the maximum principle and uniqueness results of Theorems 2.2, 2.3 and 2.4 apply to w since the mean value inequalities were the only properties of harmonic functions used in their derivation. Hence $w = 0$ in B and consequently u must be harmonic in Ω. $\square$

As an immediate consequence of the preceding theorem we have:

Theorem 2.8. *The limit of a uniformly convergent sequence of harmonic functions is harmonic.*

It follows from Theorem 2.8, that if $\{u_n\}$ is a sequence of harmonic functions in a bounded domain Ω, with continuous boundary values $\{\varphi_n\}$ which converge uniformly on $\partial\Omega$ to a function φ, then the sequence $\{u_n\}$ converges uniformly (by the maximum principle) to a harmonic function having the boundary values φ on $\partial\Omega$. By means of Harnack's inequality, Theorem 2.5, we can also derive, from Theorem 2.8, *Harnack's convergence theorem*.

Theorem 2.9.   *Let* $\{u_n\}$ *be a monotone increasing sequence of harmonic functions in a domain Ω and suppose that for some point $y \in \Omega$, the sequence $\{u_n(y)\}$ is bounded. Then the sequence converges uniformly on any bounded subdomain $\Omega' \subset\subset \Omega$ to a harmonic function.*

Proof.   The sequence $\{u_n(y)\}$ will converge, so that for arbitrary $\varepsilon > 0$ there is a number N such that $0 \leqslant u_m(y) - u_n(y) < \varepsilon$ for all $m \geqslant n > N$. But then by Theorem 2.5, we must have

$$\sup_{\Omega'} |u_m(x) - u_n(x)| < C\varepsilon$$

for some constant C depending on Ω' and Ω. Consequently $\{u_n\}$ converges uniformly and by virtue of Theorem 2.8, the limit function is harmonic. $\square$

2.7. Interior Estimates of Derivatives

By direct differentiation of the Poisson integral it is possible to obtain interior derivative estimates for harmonic functions. Alternatively, such estimates also follow from the mean value theorem. For let u be harmonic in Ω and $B = B_R(y) \subset\subset \Omega$. Since the gradient Du is also harmonic in Ω it follows by the mean value and divergence theorems that

$$Du(y) = \frac{1}{\omega_n R^n} \int_B Du \, dx = \frac{1}{\omega_n R^n} \int_{\partial B} uv \, ds,$$

$$|Du(y)| \leqslant \frac{n}{R} \sup_{\partial B} |u|$$

and hence

$$(2.31) \qquad |Du(y)| \leqslant \frac{n}{d_y} \sup_{\Omega} |u|,$$

where $d_y = \text{dist}(y, \partial\Omega)$. By successive application of the estimate (2.31) in equally spaced nested balls we obtain an estimate for higher order derivatives:

Theorem 2.10. *Let u be harmonic in Ω and let Ω' be any compact subset of Ω. Then for any multi-index α we have*

$$(2.32) \qquad \sup_{\Omega'} |D^\alpha u| \leqslant \left(\frac{n|\alpha|}{d}\right)^{|\alpha|} \sup_{\Omega} |u|$$

where $d = \text{dist}\,(\Omega', \partial\Omega)$.

An immediate consequence of the bound (2.32) is the equicontinuity on compact subdomains of the derivatives of any bounded set of harmonic functions. Consequently by Arzela's theorem, we see that any bounded set of harmonic functions forms a *normal family*; that is, we have:

Theorem 2.11. *Any bounded sequence of harmonic functions on a domain Ω contains a subsequence converging uniformly on compact subdomains of Ω to a harmonic function.*

The previous convergence theorem, Theorem 2.8, would also follow immediately from Theorem 2.11.

2.8. The Dirichlet Problem; the Method of Subharmonic Functions

We are in a position now to approach the question of existence of solutions of the classical Dirichlet problem in arbitrary bounded domains. The treatment here will be accomplished by *Perron's method of subharmonic functions* [PE] which relies heavily on the maximum principle and the solvability of the Dirichlet problem in balls. The method has a number of attractive features in that it is elementary, it separates the interior existence problem from that of the boundary behaviour of solutions, and it is easily extended to more general classes of second order elliptic equations. There are other well known approaches to existence theorems such as the method of integral equations, treated for example in the books [KE 2] [GU], and the variational or Hilbert space approach which we describe in a more general context in Chapter 8.

The definition of $C^2(\Omega)$ subharmonic and superharmonic function is generalized as follows. A $C^0(\Omega)$ function u will be called *subharmonic* (*superharmonic*) in Ω if for every ball $B \subset\subset \Omega$ and every function h harmonic in B satisfying $u \leqslant (\geqslant)h$ on ∂B, we also have $u \leqslant (\geqslant)h$ in B. The following properties of $C^0(\Omega)$ subharmonic functions are readily established:

(i) If u is subharmonic in a domain Ω, it satisfies the strong maximum principle in Ω; and if v is superharmonic in a bounded domain Ω with $v \geqslant u$ on $\partial\Omega$, then either $v > u$ throughout Ω or $v \equiv u$. To prove the latter assertion, suppose the contrary. Then at some point $x_0 \in \Omega$ we have

$$(u - v)(x_0) = \sup_{\Omega} (u - v) = M \geqslant 0,$$

and we may assume there is a ball $B = B(x_0)$ such that $u - v \neq M$ on ∂B. Letting $\bar{u}, \bar{v}$ denote the harmonic functions respectively equal to u, v on ∂B (Theorem 2.6), one sees that

$$M \geqslant \sup_{\partial B} (\bar{u} - \bar{v}) \geqslant (\bar{u} - \bar{v})(x_0) \geqslant (u - v)(x_0) = M,$$

and hence the equality holds throughout. By the strong maximum principle for harmonic functions (Theorem 2.2) it follows that $\bar{u} - \bar{v} \equiv M$ in B and hence $u - v \equiv M$ on ∂B, which contradicts the choice of B.

(ii) Let u be subharmonic in Ω and B be a ball strictly contained in Ω. Denote by $\bar{u}$ the harmonic function in B (given by the Poisson integral of u on ∂B) satisfying $\bar{u} = u$ on ∂B. We define in Ω the *harmonic lifting* of u (in B) by

$$(2.33) \qquad U(x) = \begin{cases} \bar{u}(x), & x \in B \\ u(x), & x \in \Omega - B. \end{cases}$$

Then the function U is also subharmonic in Ω. For consider an arbitrary ball $B' \subset\subset \Omega$ and let h be a harmonic function in B' satisfying $h \geqslant U$ on $\partial B'$. Since $u \leqslant U$ in B' we have $u \leqslant h$ in B' and hence $U \leqslant h$ in $B' - B$. Also since U is harmonic in B, we have by the maximum principle $U \leqslant h$ in $B \cap B'$. Consequently $U \leqslant h$ in B' and U is subharmonic in Ω.

(iii) Let $u_1, u_2, \ldots, u_N$ be subharmonic in Ω. Then the function $u(x) = \max \{u_1(x), \ldots, u_N(x)\}$ is also subharmonic in Ω. This is a trivial consequence of the definition of subharmonicity. Corresponding results for superharmonic functions are obtained by replacing u by $-u$ in properties (i), (ii) and (iii).

Now let Ω be bounded and φ be a bounded function on $\partial \Omega$. A $C^0(\bar{\Omega})$ subharmonic function u is called a *subfunction* relative to φ if it satisfies $u \leqslant \varphi$ on $\partial \Omega$. Similarly a $C^0(\bar{\Omega})$ superharmonic function is called a *superfunction* relative to φ if it satisfies $u \geqslant \varphi$ on $\partial \Omega$. By the maximum principle every subfunction is less than or equal to every superfunction. In particular, constant functions $\leqslant \inf_{\partial\Omega} \varphi$ ($\geqslant \sup_{\partial\Omega} \varphi$) are subfunctions (superfunctions). Let S_φ denote the set of subfunctions relative to φ. The basic result of the Perron method is contained in the following theorem.

Theorem 2.12. *The function* $u(x) = \sup_{v \in S_\varphi} v(x)$ *is harmonic in* Ω.

Proof. By the maximum principle any function $v \in S_\varphi$ satisfies $v \leqslant \sup \varphi$, so that u is well defined. Let y be an arbitrary fixed point of Ω. By the definition of u, there exists a sequence $\{v_n\} \subset S_\varphi$ such that $v_n(y) \to u(y)$. By replacing v_n with max $(v_n, \inf \varphi)$, we may assume that the sequence $\{v_n\}$ is bounded. Now choose R so that the ball $B = B_R(y) \subset\subset \Omega$ and define V_n to be the harmonic lifting of v_n in B according to (2.33). Then $V_n \in S_\varphi$, $V_n(y) \to u(y)$ and by Theorem 2.11 the sequence $\{V_n\}$ contains a subsequence $\{V_{n_k}\}$ converging uniformly in any ball $B_\rho(y)$ with $\rho < R$ to a function v that is harmonic in B. Clearly $v \leqslant u$ in B and $v(y) = u(y)$. We claim now that in fact $v = u$ in B. For suppose $v(z) < u(z)$ at some $z \in B$. Then there exists

a function $\bar{u} \in S_\varphi$ such that $v(z) < \bar{u}(z)$. Defining $w_k = \max(\bar{u}, V_{n_k})$ and also the harmonic liftings W_k as in (2.33), we obtain as before a subsequence of the sequence $\{W_k\}$ converging to a harmonic function w satisfying $v \leqslant w \leqslant u$ in B and $v(y) = w(y) = u(y)$. But then by the maximum principle we must have $v = w$ in B. This contradicts the definition of $\bar{u}$ and hence u is harmonic in Ω. $\square$

The preceding result exhibits a harmonic function which is a prospective solution (called the *Perron solution*) of the classical Dirichlet problem: $\Delta u = 0$, $u = \varphi$ on $\partial\Omega$. Indeed, if the Dirichlet problem is solvable, its solution is identical with the Perron solution. For let w be the presumed solution. Then clearly $w \in S_\varphi$ and by the maximum principle $w \geqslant u$ for all $u \in S_\varphi$. We note here also that the proof of Theorem 2.12 could have been based on the Harnack convergence theorem, Theorem 2.9, instead of the compactness theorem, Theorem 2.11; (see Problem 2.10).

In the Perron method the study of boundary behaviour of the solution is essentially separate from the existence problem. The continuous assumption of boundary values is connected to the geometric properties of the boundary through the concept of *barrier* function. Let ξ be a point of $\partial\Omega$. Then a $C^0(\bar\Omega)$ function $w = w_\xi$ is called a *barrier* at ξ relative to Ω if:

(i) w is superharmonic in Ω;
(ii) $w > 0$ in $\bar\Omega - \xi$; $w(\xi) = 0$.

A more general definition of barrier requires only that the superharmonic function w be continuous and positive in Ω, and that $w(x) \to 0$ as $x \to \xi$. The results of this section are valid for these *weak* barriers as well (see [HL, p. 168], for example). An important feature of the barrier concept is that it is a local property of the boundary $\partial\Omega$. Namely, let us define w to be a *local barrier* at $\xi \in \partial\Omega$ if there is a neighborhood N of ξ such that w satisfies the above definition in $\Omega \cap N$. Then a barrier at ξ relative to Ω can be defined as follows. Let B be a ball satisfying $\xi \in B \subset\subset N$ and $m = \inf\limits_{N-B} w > 0$. The function

$$\bar{w}(x) = \begin{cases} \min(m, w(x)), & x \in \bar\Omega \cap B \\ m, & x \in \bar\Omega - B \end{cases}$$

is then a barrier at ξ relative to Ω, as one sees by confirming properties (i) and (ii). Indeed, $\bar{w}$ is continuous in $\bar\Omega$ and is superharmonic in Ω by property (iii) of subharmonic functions; property (ii) is immediate.

A boundary point will be called *regular* (with respect to the Laplacian) if there exists a barrier at that point.

The connection between the barrier and boundary behavior of solutions is contained in the following.

Lemma 2.13. *Let u be the harmonic function defined in Ω by the Perron method (Theorem 2.12). If ξ is a regular boundary point of Ω and φ is continuous at ξ, then $u(x) \to \varphi(\xi)$ as $x \to \xi$.*

Proof.   Choose $\varepsilon > 0$, and let $M = \sup |\varphi|$. Since ξ is a regular boundary point, there is a barrier w at ξ and, by virtue of the continuity of φ, there are constants δ and k such that $|\varphi(x) - \varphi(\xi)| < \varepsilon$ if $|x - \xi| < \delta$, and $kw(x) \geqslant 2M$ if $|x - \xi| \geqslant \delta$. The functions $\varphi(\xi) + \varepsilon + kw$, $\varphi(\xi) - \varepsilon - kw$ are respectively superfunction and subfunction relative to φ. Hence from the definition of u and the fact that every superfunction dominates every subfunction, we have in Ω.

$$\varphi(\xi) - \varepsilon - kw(x) \leqslant u(x) \leqslant \varphi(\xi) + \varepsilon + kw(x)$$

or

$$|u(x) - \varphi(\xi)| \leqslant \varepsilon + kw(x).$$

Since $w(x) \to 0$ as $x \to \xi$, we obtain $u(x) \to \varphi(\xi)$ as $x \to \xi$. $\square$
        This leads immediately to

Theorem 2.14.   *The classical Dirichlet problem in a bounded domain is solvable for arbitrary continuous boundary values if and only if the boundary points are all regular.*

Proof.   If the boundary values φ are continuous and the boundary $\partial\Omega$ consists of regular points, the preceding lemma states that the harmonic function provided by the Perron method solves the Dirichlet problem. Conversely, suppose that the Dirichlet problem is solvable for all continuous boundary values. Let $\xi \in \partial\Omega$. Then the function $\varphi(x) = |x - \xi|$ is continuous on $\partial\Omega$ and the harmonic function solving the Dirichlet problem in Ω with boundary values φ is obviously a barrier at ξ. Hence ξ is regular, as are all points of $\partial\Omega$. $\square$

The important question remains: For what domains are the boundary points regular? It turns out that general sufficient conditions can be stated in terms of local geometric properties of the boundary. We mention some of these conditions below.

If $n = 2$, consider a boundary point z_0 of a bounded domain Ω and take the origin at z_0 with polar coordinates r, θ. Suppose there is a neighborhood N of z_0 such that a single valued branch of θ is defined in $\Omega \cap N$, or in a component of $\Omega \cap N$ having z_0 on its boundary. One sees that

$$w = -\operatorname{Re} \frac{1}{\log z} = -\frac{\log r}{\log^2 r + \theta^2}$$

is a (weak) local barrier at z_0 and hence z_0 is a regular point. In particular, z_0 is a regular boundary point if it is the endpoint of a simple arc lying in the exterior of Ω. Thus the Dirichlet problem in the plane is always solvable for continuous boundary values in a (bounded) domain whose boundary points are each accessible from the exterior by a simple arc. More generally, the same barrier shows that the boundary value problem is solvable if every component of the complement of the domain consists of more than a single point. Examples of such domains are domains bounded by a finite number of simple closed curves. Another is the unit disc slit along an arc; in this case the boundary values can be assigned on opposite sides of the slit.

For higher dimensions the situation is substantially different and the Dirichlet problem cannot be solved in corresponding generality. Thus, an example due to Lebesgue shows that a closed surface in three dimensions with a sufficiently sharp inward directed cusp has a non-regular point at the tip of the cusp; (see for example [CH]).

A simple sufficient condition for solvability in a bounded domain $\Omega \subset \mathbb{R}^n$ is that Ω satisfy the *exterior sphere condition*; that is, for every point $\xi \in \partial\Omega$, there exists a ball $B = B_R(y)$ satisfying $\bar{B} \cap \bar{\Omega} = \xi$. If such a condition is fulfilled, then the function w given by

$$(2.34) \qquad w(x) = \begin{cases} R^{2-n} - |x-y|^{2-n} & \text{for } n \geqslant 3 \\ \log \dfrac{|x-y|}{R} & \text{for } n = 2 \end{cases}$$

will be a barrier at ξ. Consequently the boundary points of a domain with C^2 boundary are all regular points; (see Problem 2.11).

2.9. Capacity

The physical concept of *capacity* provides another means of characterizing regular and exceptional boundary points. Let Ω be a bounded domain in $\mathbb{R}^n (n \geqslant 3)$ with smooth boundary $\partial\Omega$, and let u be the harmonic function (often called the *conductor potential*) defined in the complement of $\bar{\Omega}$ and satisfying the boundary conditions $u = 1$ on $\partial\Omega$ and $u = 0$ at infinity. The existence of u is easily established as the (unique) limit of harmonic functions u' in an expanding sequence of bounded domains having $\partial\Omega$ as an inner boundary (on which $u' = 1$) and with outer boundaries (on which $u' = 0$) tending to infinity. If Σ denotes $\partial\Omega$ or any smooth closed surface enclosing Ω, then the quantity

$$(2.35) \qquad \text{cap } \Omega = -\int_{\Sigma} \frac{\partial u}{\partial v} \, ds = \int_{\mathbb{R}^n - \Omega} |Du|^2 \, dx \qquad v = \text{outer normal}$$

is defined to be the capacity of Ω. In electrostatics, cap Ω is within a constant factor the total electric charge on the conductor $\partial\Omega$ held at unit potential (relative to infinity).

Capacity can also be defined for domains with nonsmooth boundaries and for any compact set as the (unique) limit of the capacities of a nested sequence of approximating smoothly bounded domains. Equivalent definitions of capacity can be given directly without use of approximating domains (e.g., see [LK]). In particular, we have the variational characterization

$$(2.36) \qquad \text{cap } \Omega = \inf_{v \in K} \int |Dv|^2.$$

where

$$K = \{v \in C_0^1(\mathbb{R}^n) | v = 1 \text{ on } \Omega\}.$$

To investigate the regularity of a point $x_0 \in \partial\Omega$, consider for any fixed $\lambda \in (0, 1)$ the capacity

$$C_j = \text{cap } \{x \notin \Omega | | x - x_0 | \leqslant \lambda^j\}.$$

The *Wiener criterion* states that x_0 is a regular boundary point of Ω if and only if the series

$$(2.37) \qquad \sum_{j=0}^{\infty} C_j / \lambda^{j(n-2)}$$

diverges.

For a discussion of capacity and proof of the Wiener criterion we refer to the literature, e.g., [KE 2, LK]. In Chapter 8 this condition for regularity will be proved for a general class of elliptic operators in divergence form.

Problems

2.1. Derive the weak maximum principle for $C^2(\Omega)$ subharmonic functions from a consideration of necessary conditions for a relative maximum.

2.2. Prove that if $\Delta u = 0$ in Ω and $u = \partial u / \partial v = 0$ on an open, smooth portion of $\partial\Omega$, then u is identically zero.

2.3. Let G be the Green's function for a bounded domain Ω. Prove

   a) $G(x, y) = G(y, x)$ for all $x, y \in \Omega$, $x \neq y$;

   b) $G(x, y) < 0$ for all $x, y \in \Omega$, $x \neq y$;

   c) $\int_\Omega G(x, y) f(y) \, dy \rightarrow 0$ as $x \rightarrow \partial\Omega$, if f is bounded and integrable on Ω.

2.4. (*Schwarz reflection principle*.) Let Ω^+ be a subdomain of the half-space $x_n > 0$ having as part of its boundary an open section T of the hyperplane $x_n = 0$. Suppose that u is harmonic in Ω^+, continuous in $\Omega^+ \cup T$, and that $u = 0$ on T. Show that the function U defined by

$$U(x_1, \ldots, x_n) = \begin{cases} u(x_1, \ldots, x_n), & x_n \geqslant 0 \\ -u(x_1, \ldots, -x_n), & x_n < 0 \end{cases}$$

is harmonic in the domain $\Omega^+ \cup T \cup \Omega^-$, where Ω^- is the reflection of Ω^+ in $x_n = 0$ (*i.e.*, $\Omega^- = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid (x_1, \ldots, -x_n) \in \Omega^+\}$).

2.5. Determine the Green's function for the annular region bounded by two concentric spheres in $\mathbb{R}^n$.

2.6. Let u be a non-negative harmonic function in a ball $B_R(0)$. Deduce from the Poisson integral formula, the following version of *Harnack's inequality*

$$\frac{R^{n-2}(R-|x|)}{(R+|x|)^{n-1}} u(0) \leqslant u(x) \leqslant \frac{R^{n-2}(R+|x|)}{(R-|x|)^{n-1}} u(0).$$

2.7. Show that a $C^0(\Omega)$ function u is subharmonic in Ω if and only if it satisfies the mean value inequality locally; that is, for every $y \in \Omega$ there exists $\delta = \delta(y) > 0$ such that

$$u(y) \leqslant \frac{1}{n\omega_n R^{n-1}} \int\limits_{\partial B_R(y)} u \, ds \quad \text{for all } R \leqslant \delta.$$

2.8. An integrable function u in a domain Ω is called *weakly harmonic* (*subharmonic, superharmonic*) in Ω if

$$\int\limits_{\Omega} u \, \Delta\varphi \, dx = (\geqslant, \leqslant) \, 0$$

for all functions $\varphi \geqslant 0$ in $C^2(\Omega)$ having compact support in Ω. Show that a $C^0(\Omega)$ weakly harmonic (subharmonic, superharmonic) function is harmonic (subharmonic, superharmonic).

2.9. Show that for $C^2(\Omega)$ functions u, the conditions: (i) $\Delta u \geqslant 0$ in Ω; (ii) u is subharmonic in Ω; (iii) u is weakly subharmonic in Ω, are equivalent.

2.10. Prove Theorem 2.12 using Theorem 2.9 instead of Theorem 2.11.

2.11. Show that a domain Ω with C^2 boundary $\partial\Omega$ satisfies an exterior sphere condition.

2.12. Show that the Dirichlet problem is solvable for any domain Ω satisfying an *exterior cone condition*; that is, for every point $\xi \in \partial\Omega$ there exists a finite right circular cone K, with vertex ξ, satisfying $\overline{K} \cap \overline{\Omega} = \xi$. At each point $\xi \in \partial\Omega$ taken as origin, show that a suitable local barrier can be chosen in the form $w = r^\lambda f(\theta)$ where θ is the polar angle.

2.13. Let u be harmonic in $\Omega \subset \mathbb{R}^n$. Use the argument leading to (2.31) to prove the interior gradient bound,

$$|Du(x_0)| \leqslant \frac{n}{d_0} [\sup_\Omega u - u(x_0)], \quad d_0 = \text{dist}(x_0, \partial\Omega).$$

If $u \geqslant 0$ in Ω infer that

$$|Du(x_0)| \leqslant \frac{n}{d_0} u(x_0).$$

2.14. (a) Prove *Liouville's theorem*: A harmonic function defined over $\mathbb{R}^n$ and bounded above is constant.

(b) If $n = 2$ prove that the Liouville theorem in part (a) is valid for *subharmonic* functions.

(c) If $n > 2$ show that a bounded subharmonic function defined over $\mathbb{R}^n$ need not be constant.

2.15. Let $u \in C^2(\bar{\Omega})$, $u = 0$ on $\partial\Omega \in C^1$. Prove the interpolation inequality: For every $\varepsilon > 0$,

$$\int_\Omega |Du|^2 \, dx \leqslant \varepsilon \int_\Omega (\Delta u)^2 \, dx + \frac{1}{4\varepsilon} \int_\Omega u^2 \, dx.$$

2.16. Prove Theorem 2.12 by finding in every ball $B \subset \subset \Omega$ a monotone increasing sequence of harmonic functions that are restrictions of subfunctions on B and that converge uniformly to u on a dense set of points in B. Hence show that Theorems 2.12 and 2.14 can be proved without use of the strong maximum principle.

2.17. Show that the volume integral in (2.35) is defined, and prove the equivalence of the capacity definitions (2.35) and (2.36).

2.18. Let u be harmonic in (open, connected) $\Omega \subset \mathbb{R}^n$, and suppose $B_c(x_0) \subset \subset \Omega$. If $a \leqslant b \leqslant c$, where $b^2 = ac$, show that

$$\int_{|\omega| = 1} u(x_0 + a\omega)u(x_0 + c\omega) \, d\omega = \int_{|\omega| = 1} u^2(x_0 + b\omega) \, d\omega.$$

Hence, conclude that if u is constant in a neighbourhood it is identically constant. (Cf. [GN].)