complex curves in $\mathbb{C}^2$, Riemann sphere and extended plane $\widehat{\mathbb{C}}$, stereographic projection; holomorphic maps between Riemann surfaces, meromorphic functions are holomorphic maps $f : X \to \widehat{\mathbb{C}}$; complex projective space and projective transformations, $\mathbb{C}P^1 \cong \widehat{\mathbb{C}}$, Hopf fibration, Möbius transformations correspond to projective transformations; holomorphic automorphisms of a Riemann surface, automorphism groups of $\mathbb{D}$, $\mathbb{C}$, $\widehat{\mathbb{C}}$ and $\mathbb{H}$; $\mathrm{Aut}(\widehat{\mathbb{C}}) \cong \mathrm{PSL}(2,\mathbb{C})$ and $\mathrm{Aut}(\mathbb{H}) \cong \mathrm{PSL}(2,\mathbb{R})$

7. Discrete groups and covering spaces (99-115)

Covering maps, universal covering; fundamental groups and the lifting theorem for covering spaces, uniqueness of universal coverings, deck transformations and normal coverings; isomorphism between the deck transformation group Γ of a universal covering and $\pi_1(X, x_0)$; group actions and orbit spaces, free actions, properly discontinuous actions, smooth (or holomorphic) covering map $M \to \Gamma\backslash M$; general Riemann surfaces as quotients $X \cong \Gamma\backslash\widetilde{X}$ where $\Gamma \leq \mathrm{Aut}(\widetilde{X})$ is isomorphic to $\pi_1(X)$; for $\Gamma \leq \mathrm{Aut}(\widetilde{X})$, the natural action on $\widetilde{X}$ is properly discontinuous iff Γ is a discrete subset of $\mathrm{Aut}(\widetilde{X}) \leq \mathrm{PSL}(2,\mathbb{C})$; X_1 and X_2 biholomorphic iff Γ_1 and Γ_2 are conjugate in $\mathrm{Aut}(\widetilde{X})$; classification of Riemann surfaces for which $\widetilde{X} = \mathbb{C}$ or $\widehat{\mathbb{C}}$, hyperbolic Riemann surfaces and Fuchsian groups; examples: X with non-abelian fundamental groups are hyberbolic, $\mathbb{C}$ minus 2 points is hyberbolic, open subsets of hyperbolic Riemann surfaces are hyperbolic; proof of Picard's little theorem via the lifting theorem for covering spaces; example of a discrete $\Gamma \leq \mathrm{Aut}(\mathbb{D})$ starting with a hyperbolic octagon, quotient space is a genus 2 surface

8. Riemannian metrics, the Poincaré disk (116-128)

Riemannian metrics and Riemannian manifolds, description in terms of local coordinates, euclidean metric, hyperbolic metric on the unit ball in $\mathbb{R}^n$, the Poincaré disk $(\mathbb{D}, g_{\mathbb{D}})$; conformally equivalent Riemannian metrics, arc length and distances in a Riemannian manifold, completeness; pullback metrics and isometries, $f^*|dw| = |f'(z)|\,|dz|$ when f is holomorphic; Poincaré: $\mathrm{Isom}^+(\mathbb{D}, g_{\mathbb{D}}) = \mathrm{Aut}(\mathbb{D})$; the Schwarz-Pick-Lemma for holomorphic maps $\mathbb{D} \to \mathbb{D}$; geodesics and distances in the Poincaré disk, $d_{\mathbb{D}}(z_1, z_2) = |\ln \mathrm{DV}(z_1, a, z_2, b)|$

9. Hyperbolic Riemann surfaces and Picard's theorem (129-138)

Normal coverings $\pi : \widehat{M} \to M$ and quotient Riemannian metrics; the Poincaré metric on a hyperbolic Riemann surface: $\pi^* g = g_{\mathbb{D}}$; the Schwarz-Pick-Lemma for holomorphic maps between hyperbolic Riemann surfaces; example: the Poincaré metric on the upper half plane $\mathbb{H}$; the Poincaré metric on $\mathbb{D}^* = \mathbb{D}\backslash\{0\}$, $g_{\mathbb{D}^*} = 1/(r \ln r)^2 |dz|^2$, $\mathrm{length}(c_\rho)$, proof of Picard's great theorem via Schwarz-Pick and the lifting theorem for covering spaces; Huber's removable singularities theorem; general Montel theorem for $f_n : X \to Y$ with Y hyperbolic, application: second proof of Picard's great theorem

10. Bloch's theorem (139-144)

Statement of the theorem, example $f(z) = (e^{nz} - 1)/n$, Landau constant and Bloch constant; lemma on open maps: $f(U) \supseteq B_\delta(f(a))$; lemma: $|f'(z)| \leq 2|f'(a)|$ implies $f(B_r(a)) \supseteq B_R(f(a))$ via estimate on $|f(z) - f'(0)z|$, Bloch: $f(B_t(a)) \supseteq B_R(f(a))$

11. Ahlfors' extension of the Schwarz Lemma (145-161)

Conformal metrics on $\Omega \subseteq \mathbb{C}$, regular conformal metrics and their Gauss curvature, Ahlfors lemma: $K_g \leq -1$ on $\mathbb{D}$ implies $g \leq g_{\mathbb{D}}$, extension to ultrahyperbolic metrics, pullbacks, Ahlfors' extension of the Schwarz Lemma for $f : \mathbb{D} \to \Omega$, the general Ahlfors-Schwarz-Pick theorem for maps $f : X_1 \to X_2$ between Riemann surfaces; two characterizations of the Poincaré metric on a hyperbolic Riemann surfaces: maximal ultrahyperbolic, complete regular conformal with $K = -1$; Ahlfors' improvement of Bloch's theorem, proof: the function σ and its properties, the ultrahyperbolic metric $g = \lambda^2 |dz|^2$, application of Ahlfors' lemma

12. Univalent functions (162-176)

The set S of univalent functions, Bieberbach's theorem and the Bieberbach conjecture, the Koebe function and its rotations; getting new elements of S out of old: square root transform, disk automorphism transform and omitted value transform; Gronwall's area theorem, $\sum_{n=1}^{\infty} n|b_n|^2 \leq 1$ for functions $g \in \Sigma$; proof of Bieberbach's theorem $|a_2| \leq 2$, Koebe's 1/4-theorem, counterexample if f is not injective; Koebe's distortion theorem, Koebe's growth theorem; consequence: sequential compactness for S with respect to locally uniform convergence, existence of extremals in S

Sample questions

What was the course about? Can you give a summary in ten (or twenty, or fifty) sentences? Explain Montel's theorem. What is its role in the proof of the Riemann mapping theorem? How is the length-area inequality used in the proof of Caratheodory's theorem? Why is the resulting map injective? What is a prime end? How is the map $\widehat{f} : \overline{\mathbb{D}} \to \widehat{\Omega}$ defined? Examples of inaccessible prime ends? What is a Riemann surface? What does the complex projective line $\mathbb{C}P^1$ have to do with the Riemann sphere? How can Riemann surfaces be classified? Prove that $\mathrm{Aut}(\widehat{\mathbb{C}}) \cong \mathrm{PSL}(2, \mathbb{C})$. Is $\mathbb{C}$ minus three points a hyperbolic Riemann surface? What does $\mathrm{Isom}^+(\mathbb{D}, g_{\mathbb{D}}) = \mathrm{Aut}(\mathbb{D})$ mean, and why is it useful? How does one compute distances in the Poincaré disk? What is the Schwarz-Pick Lemma? Describe a proof of Picard's theorem. Formulate Bloch's theorem. Where and how is the fact that f is holomorphic used in the proof? What is an ultrahyperbolic metric? How does the Gauss curvature enter into the Ahlfors Lemma? What is the Koebe function? What is its role? Gronwall's area theorem - statement and idea of proof? How can one compute the coefficients of the square root transform g from those of f? How is that used in Bieberbach's theorem? And how does Koebe's 1/4-theorem follow from Bieberbach's theorem? Is there an element of S with the following extremal property ... ?