1. Introduction, review of holomorphic functions

Real and complex linear algebra, five characterizations of holomorphic functions, angle preservation and conformal maps, holomorphic inverse function theorem, open mapping property, local normal form for holomorphic functions, examples z^2, e^z, Lipschitz bounds and Liouville's theorem

2. The Riemann mapping theorem

Riemann mapping theorem, elementary examples: Möbius transformations, mapping a slit half plane to the unit disk, logarithm; theorem of Arzela-Ascoli, Montel's theorem, uniform limits of holomorphic functions are holomorphic, Hurwitz theorem on the injectivity of limits, argument principle, Schwarz lemma and holomorphic automorphisms of the unit disk, holomorphic square roots, stretching lemma, proof of the Riemann mapping theorem

3. Boundary continuity of conformal maps

Continuous extension of $f : \Omega_1 \to \Omega_2$, counterexamples; Jordan domains, Caratheodory's theorem, application to the Dirichlet problems for harmonic functions; Schwarz reflection principle and its generalizations, application: biholomorphic extensions of $f : \Omega_1 \to \Omega_2$ for Jordan domains with analytic boundaries; proof of Caratheodory's theorem: length-area estimate, lemma on "small arcs", continuous extension $f : \overline{\mathbb{D}} \to \overline{\Omega}_2$, injectivity of the extended map, continuity of f^{-1}

4. Prime ends

Crosscuts and crosscut neighbourhoods, null chains, prime ends and their impressions on $\partial\Omega$, the Caratheodory compactification $\widehat{\Omega} = \Omega \cup P(\Omega)$, homeomorphism $\widehat{f} : \overline{\mathbb{D}} \to \widehat{\Omega}$, the impression of $\widehat{f}(\xi)$ is the cluster set of f at $\xi \in \partial\mathbb{D}$; homeomorphism $\mathrm{imp} : \widehat{\Omega} \to \overline{\Omega}$ for Jordan domains; endcuts and their landing points, accessible prime ends; principal points; accessible prime ends have only one principal point - the landing point of an endcut

5. Riemann surfaces and uniformization

Riemann surfaces and complex manifolds, examples: curves in $\mathbb{C}^2$, Riemann sphere and extended plane $\widehat{\mathbb{C}}$; complex projective space and projective transformations, $\mathbb{C}P^1 \cong \widehat{\mathbb{C}}$, Möbius transformations correspond to projective transformations; holomorphic maps between Riemann surfaces, meromorphic functions are holomorphic maps $f : X \to \widehat{\mathbb{C}}$; statement of the uniformisation theorem; holomorphic automorphisms of a Riemann surface, $\mathrm{Aut}(\widehat{\mathbb{C}}) \cong \mathrm{PSL}(2, \mathbb{C})$ and $\mathrm{Aut}(\mathbb{H}) \cong \mathrm{PSL}(2, \mathbb{R})$; universal covering space, group actions and quotient spaces; general Riemann surfaces as quotients $X \cong \widetilde{X}/\Gamma$ where $\Gamma \leq \mathrm{Aut}(\widetilde{X})$ is isomorphic to $\pi_1(X)$; Kleinian and Fuchsian groups, X_1 and X_2 biholomorphic iff Γ_1 and Γ_2 are conjugate; classification of Riemann surfaces for which $\widetilde{X} = \mathbb{C}$ or $\widehat{\mathbb{C}}$, proof of Picard's little theorem, lifting theorem for covering spaces

6. Hyperbolic metrics

Riemannian metrics and Riemannian manifolds, description in terms of local coordinates, hyperbolic metric on the unit ball in $\mathbb{R}^n$, the Poincaré disk $(\mathbb{D}, g_{\mathbb{D}})$; arc lengh and distances in a Riemannian manifold, completeness; pullback metrics and isometries, $f^*|dz| = |f'(z)|\,|dz|$; Poincaré: $\mathrm{Isom}^+(\mathbb{D}, g_{\mathbb{D}}) = \mathrm{Aut}(\mathbb{D})$, geodesics and distance formula for the Poincaré disk, the hyperbolic Riemannian metric on a hyperbolic Riemann surface; the Schwarz-Pick-Lemma for holomorphic maps $\mathbb{D} \to \mathbb{D}$ and for maps between hyperbolic Riemann surfaces; hyperbolic metric on the upper half plane $\mathbb{H}$ and on $\mathbb{D}^*$; proof of Picard's great theorem via Schwarz-Pick and the lifting theorem for covering spaces, Huber's removable singularities theorem; general Montel theorem for $f_n : X \to Y$ with Y hyperbolic, application: second proof of Picard's theorem

7. Univalent functions, mapping properties

The set S of univalent functions, Bieberbach's theorem and the Bieberbach conjecture, Koebe function and its rotations; getting new elements of S out of old: square root transform, disk automorphism transform and omitted value transform; Gronwall's area theorem for functions $g \in \Sigma$, proof of Bieberbach's theorem $|a_2| \leq 2$, Koebe's 1/4-theorem, counterexample if f is not injective; mapping properties for general holomorphic $f : \mathbb{D} \to \mathbb{C}$: Bloch's theorem, Landau's constant and Bloch's constant; back to univalent functions: Koebe distortion theorem, Koebe growth theorem, consequence: sequential compactness for S

Sample questions

What was the course about? Can you give a summary in ten (or twenty) sentences? Explain Montel's theorem. What is its role in the proof of the Riemann mapping theorem? How is the length-area inequality used in the proof of Caratheodory's theorem? Why is the resulting map injective? What is a prime end? How is the map $\widehat{f} : \overline{\mathbb{D}} \to \widehat{\Omega}$ defined? Examples of inaccessible prime ends? What is a Riemann surface? What does the complex projective line $\mathbb{C}P^1$ have to do with the Riemann sphere? How can Riemann surfaces be classified? Prove that $\mathrm{Aut}(\widehat{\mathbb{C}}) \cong \mathrm{PSL}(2, \mathbb{C})$. Is $\mathbb{C}$ minus three points a hyperbolic Riemann surface? What does $\mathrm{Isom}^+(\mathbb{D}, g_{\mathbb{D}}) = \mathrm{Aut}(\mathbb{D})$ mean, and why is it useful? How does one compute distances in the Poincaré disk? What is the Schwarz-Pick Lemma? Describe a proof of Picard's theorem. What is the Koebe function? What is its role? Gronwall's area theorem - statement and idea of proof? How can one compute the coefficients of the square root transform g from those of f? How is that used in Bieberbach's theorem? And how does Koebe's 1/4-theorem follow from Bieberbach's theorem? Formulate Bloch's theorem. Where is the fact that f is holomorphic used in the proof? ...