1. Partial fractions and the Mittag-Leffler theorem

Isolated singularities, Laurent expansion, integral formula for the Laurent coefficients, calculation of Laurent coefficients at poles; poles and zeros, the field of meromorphic functions on a domain; principal part distributions, compact convergence of series of meromorphic functions, Mittag-Leffler theorem, partial fraction decomposition of a given meromorphic function, example: prescribing simple poles

2. Weierstrass product theorem

Entire functions without zeros, holomorphic logarithms of a function, infinite products and their convergence, absolute convergence, locally uniform convergence, Weierstrass product theorem, Weierstrass factors, corollaries: meromorphic functions are quotients of entire functions, surjective group homomorphism $\operatorname{div}\colon \mathcal{M}(U)^* \to \operatorname{Div}(U)$, prescribing values $f(a_n) = w_n$, $n \in \mathbb{N}$ of entire functions

3. Examples

Mittag-Leffler decompositions of cotangent, tangent and $1/\sin$, Bernoulli numbers and the values $\zeta(2n)$, the sine product, logarithmic derivative

4. Gamma function

How to discover Γ: Weierstrass product ansatz for $G = 1/\Gamma$, Euler-Mascheroni constant, definition of Γ according to Weierstrass, $\Gamma(z)\Gamma(1-z)$, Euler's definition of Γ as an infinite product; Wielandt's uniqueness theorem, proof using Liouville; Legendre duplication formula and Gauss' multiplication formula, Euler's integral formula for Γ; Stirling's formula, Binet's formula for Γ and its improvement

5. Elliptic functions, Liouville theorems

Periodic functions and their periods, classification of discrete subgroups $L \leq \mathbb{R}^2$, fundamental parallelogram of a lattice, Liouville's theorem on elliptic functions (4 parts), proof using $\operatorname{Res}_a(g\frac{f'}{f})$

6. Weierstrass $\wp$-functions

Definition of $\wp$ for a given lattice L, Laurent expansion of $\wp$ at 0, Eisenstein series, the differential equation $\wp(z)^2 = 4\,\wp(z)^3 - g_2\,\wp(z) - g_3$, rewritten using the half-period values e_1, e_2, e_3, these values are distinct, relation between half-period values and g_2, g_3, the discriminant Δ; how to write an arbitrary elliptic f as a rational function of $\wp$ and $\wp'$; Abel's theorem for tori $L\backslash\mathbb{C}$ and the Weierstrass sigma-function σ, construction of σ using the Weierstrass ζ-function, Weierstrass product representation of σ; remarks on the origin of elliptic functions; link to algebraic curves: embedding $L\backslash\mathbb{C}$ as a cubic curve $\phi\colon L\backslash\mathbb{C} \to \tilde{C} \subseteq \mathbb{C}P^2$

7. Modular forms

Automorphic functions, Möbius transformations and $\operatorname{Aut}(\mathbb{H}) \cong \operatorname{PSL}(2,\mathbb{R})$, the modular group $\operatorname{PSL}(2,\mathbb{Z})$, modular forms of weight k, modular functions, behaviour at

$i\infty$: $f(z) = g(q)$ with $q = e^{2\pi i z}$, Laurent expansion of g = Fourier expansion (or q-expansion) of the modular form f, entire modular forms; generators and fundamental domain $\mathbb{F}$ for $\mathrm{SL}(2,\mathbb{Z})$, connection with lattices: $L_\tau = \mathbb{Z} + \mathbb{Z}\tau$, Eisenstein series G_k, lattice invariants g_2, g_3 and discriminant Δ as functions of τ, the j-invariant; equivalence of lattices and $\mathrm{PSL}(2,\mathbb{Z})\backslash\mathbb{H}$; the $k/12$-formula, proof via residue theorem; applications of $k/12$: entire modular forms, zeros of G_4, G_6 and Δ, $j : \mathbb{F} \to \mathbb{C}$ is bijective; every modular function is a rational function of j; list of entire modular forms for $0 \le k \le 10$, cusp forms, $M_{k+12} = \Delta \cdot M_k + \mathbb{C}\, G_{k+12}$, $G_4^m G_6^n$ form a basis for M_k

8. Jensen's formula

Jensen's formula, proof starting from $f = e^g$ and the mean value property for g, lemma on counting functions, Jensen reformulated in terms of the Nevanlinna (pole) counting function $N(r,f)$, simple properties, comparison of $N(r,f)$ and $n(r,f)$, maximum modulus $M(r,f)$ of entire functions

9. Nevanlinna characteristic

Proximity function $m(r,f)$, the Nevanlinna characteristic $T(r,f) = N(r,f) + m(r,f)$, interpretation of $T(r, \frac{1}{f-a})$, Jensen reformulated in terms of Nevanlinna characteristic: $T(r,f) = T(r, \frac{1}{f}) + \log|c_l|$, Nevanlinna's first main (= fundamental) theorem, Landau's $\mathcal{O}$-notation, $T(r, \frac{1}{f-a}) = T(r,f) + \mathcal{O}(1)$ is asymptotically independent of a; example e^z (using Stirling's formula), example rational functions $T(r,f) = \deg(f)\log r + \mathcal{O}(1)$; properties of the Nevanlinna characteristic, $T(r, M \circ f)$, $T(r, P \circ f)$;

Cartan's formula for $T(r,f)$, proof: apply Jensen to $f(z) - e^{i\varphi}$ and integrate over φ; $N(r,f)$ and $T(r,f)$ are convex as functions of $\log r$; Nevanlinna characteristic of entire functions: comparison with $\log^+ M(r,f)$, Poisson kernel and the Poisson-Jensen formula, proof idea: reduce to Jensen formula via a suitable Möbius transformation; characterization of rational functions by $L := \liminf(T(r,f)/\log r)$, of polynomials by $\liminf(M(r,f)/\log r)$; proof trick: only finitely many poles, reduce to polynomial case; the (growth) order $\rho(f)$ of a meromorphic function, order of an entire function, examples

10. Nevanlinna's second main theorem

Statement of the theorem, $N_1(r)$ is non-negative, estimates $(q-2)T(r,f) \le \ldots$ and Picard's theorem; deficiency and ramification index of f at a value a, defect relations $\sum \Theta(a,f) \le 2$ and $\sum \delta(a,f) \le 2$, Nevanlinna's five value theorem

References and sources

Fischer, Lieb, Funktionentheorie
Freitag, Busam, Funktionentheorie I
Koecher, Krieg, Elliptische Funktionen und Modulformen
Serre, Cours d'arithméthique
Bergweiler, Entire and meromorphic functions (lecture notes 2018)