1. Introduction, examples of PDE

Examples: Laplace equation, Poisson equation, heat and wave equation, nonlinear examples, systems; multi-index notation; well posed problems, example: Dirichlet problem for the Laplace equation.

2. Diffusion and heat flow

Conservation of mass, continuity equation, Fick's law and the diffusion equation, Fourier's law and the heat equation, stationary solutions and Laplace equation.

3. Maximum principles for elliptic PDE

Linear elliptic equations of 2nd order, weak maximum principle, uniqueness for the Dirichlet problem; the boundary point lemma of Hopf, strong maximum principle, uniqueness for the Neumann problem.

4. A priori estimates based on maximum principles

$Lu \geq f$ implies estimates for $\sup u$, $Lu = f$ for $\sup |u|$; eigenvalue estimate; quasilinear elliptic equations, example H-equation and mimimal surfaces, comparison theorem, application to minimal surfaces; general nonlinear elliptic PDE of 2nd order, example Monge-Ampère equations, comparison theorem, uniqueness for the Dirichlet problem for Monge-Ampère; a priori estimates for quasilinear elliptic equations via comparison theorem; for fully nonlinear equations by reduction to the quasilinear case.

5. Distributions, Sobolev spaces

Test functions and distributions, examples of distributions: L^1_{loc}, Dirac's delta, measures; derivatives of distributions, examples, fundamental solutions, a fundamental solution for Δ, Newton potentials; Sobolev spaces $W^{k,p}(\Omega)$, Sobolev norms, $W^{k,p}(\Omega)$ is a Banach space, separable for $p < \infty$.

6. Approximation by smooth functions

Mollification and its properties: C^∞, convergence $u^h \to u$: almost everywhere, locally uniformly, in L^p_{loc}, in $W^{k,p}_{\text{loc}}$ depending on u; partitions of unity, global smooth approximation of Sobolev functions, segment property.

7. Extensions and traces

Behaviour of Sobolev functions under diffeomeorphisms, extension operators, trace theorem.

8. Sobolev inequalities

Theorem of Gagliardo and Nirenberg, why p^*; Sobolev embedding and estimate for $W^{1,p}(\Omega)$, $1 \leq p < n$; Poincaré-inequality for $W^{1,p}_0(\Omega)$, $1 \leq p < \infty$; Hölder norms, Morrey's inequality and Morrey embedding for $n < p \leq \infty$; general Sobolev inequalities for $W^{k,p}(\Omega)$.

9. Compactness

Compact linear maps, compactness theorem of Rellich-Kondrachov, Arzela-Ascoli, compactness of the inclusion maps $W_0^{1,p}(\Omega) \subseteq L^p(\Omega)$ and $W^{1,p}(\Omega) \subseteq L^p(\Omega)$.

10. Weak solutions of elliptic equations.

The bilinear form associated with L, weak solution of the Dirichlet problem, description of $H^{-1}(\Omega)$, reduction to boundary value 0; Lax-Milgram theorem, short proof for symmetric B, Poincaré-inequality, boundedness of B "Gardings inequality", properties of $L : H_0^1(\Omega) \to H^{-1}(\Omega)$, existence theorem in case $\gamma = 0$.

11. Fredholm alternative, eigenvalues

The map $J : H_0^1(\Omega) \to H^{-1}(\Omega)$, the operator L_σ, Fredholm altenative, $\hat{L} : H^1(\Omega) \to H^{-1}(\Omega) \oplus B(\Omega)$, open mapping theorem and a priori estimate; eigenfunctions, eigenvalues and their properties; general properties of compact self adjoint operators, application to L, eigenbasis; convergence of the generalized Fourier series in $H_0^1(\Omega)$; the Rayleigh quotient, variational characterization of the eigenvalues, Courant's max-min and min-max theorem; Fourier's method for the generalized heat equation.

12. Regularity of weak solutions

Interior regularity, regularity up to the boundary and the corresponding estimates; difference quotients and membership in $W^{1,p}$.

Sample questions

What is a "well posed" problem? Explain the continuity equation. Why does the Laplace operator appear in the diffusion equation? Formulate and explain the strong maximum principle. Is it stronger than the weak one? Can you explain the main idea for the proof of the boundary point lemma? What is the role of that auxiliary function v in the proof? When is a general nonlinear 2nd order equation called elliptic? Give an example. Formulate a comparison theorem for such equations and show how it can be obtained from the strong maximum principle for *linear* equations. In what sense are elements of is $L^1_{\text{loc}}(\Omega)$ distributions? Are there other kinds of distributions? Do you know a fundamental solution for the Laplace operator? What is the purpose of Sobolev spaces in the context of this course? When and how can Sobolev functions be approximated by smooth functions? How can they be extended? What is role of the inequality $||Tu||_{L^p(\Omega)} \leq c||u||_{W^{1,p}\Omega)}$ in the construction of the trace operator? Why does the derivative of u enter in this inequality? What does the theorem of Gagliardo-Nirenberg say? Why the exponent p^* and not some other exponent? Why does the derivative of u enter in this inequality? Formulate Morrey's inequalty. Explain how one can obtain pointwise bounds from integral bounds in the proof of Morrey's inequality. Explain the weak formulation of the Dirichlet problem. Why does $H^{-1}(\Omega)$ come into play? How do estimates imply an existence theorem? What is the role of Poincaré's inequality in this context? What is the σ in L_σ good for? Why and how do compact operators play a role? What is an "eigenbasis" for L? In what sense is it orthonormal? Why could it be useful? What is the relation between the eigenvalues of L and the Rayleigh quotient? Prove Courant's min-max theorem. How could it be used to estimate eigenvalues? Formulate a regularity theorem. What does that theorem imply for eigenfunctions? ...