1. Introduction and examples

Control systems, state variables and control variables, payoff or cost functionals, examples: production and consumption, bee population, harmonic oscillator with external force, moon landing, rocket car

2. Controllability for linear systems

Linear control systems, variation of constants and solution formula, properties of the controllable set: monotone, convex, symmetric; openness of the controllable set and rank condition on the controllability matrix, the supporting hyperplane theorem; eigenvalue criterion for complete controllability, example rocket car; observability and controllability

3. Functional analysis and the bang-bang-principle

The weak-*-topology on dual spaces; theorem of Banach-Alaoglu, proof using Tychonoff's theorem; locally convex topological vector spaces, extremal points and the Krein-Milman theorem; review of L^p-spaces, L^∞ as the dual of L^1, Banach-Alaoglu for L^∞; bang-bang-principle for linear control systems, extremal points of the set of successful controls exist and are bang-bang

4. Time optimal control for linear systems

Existence theorem for minimizers, properties of the reachable set: compact, convex, Hausdorff-continuous; Pontryagin's maximum principle (PMP) for linear time-optimal control, examples: rocket car, stopping a harmonic oscillator, switch curve; normality and uniqueness of the optimal control; criteria for normality; the number of switches; a converse of the PMP; general linear control systems

5. Pontryagin's maximum principle

PMP for the fixed time free endpoint problem; PMP for the free time fixed end point problem, the abnormal case; version for non-autonomous system; examples: PMP for linear systems (autonomous, non-autonomous), production and consumption, a linear system with quadratic cost; a general form of the PMP, transversality condition, special cases of the transversality condition, connecting submanifolds by optimal trajectories; more examples: shortest curve between submanifolds, commodity trading

6. Proof of the maximum principle

Needle perturbations of the control function, perturbation of initial conditions for ODE and Gronwall's lemma, effect of needle perturbations on the trajectories; proof of the PMP for free endpoint Mayer problems; problems with running payoff (Bolza problems) reduced to Mayer problems; multiple needle perturbations and their effect on trajectories, the terminal cone (tangent perturbation cone); proof of the PMP for the fixed endpoint problem, reduction to Mayer form, main lemma, geometric lemma and Brouwer's fixed point theorem

7. Dynamic programming

The value function, Hamilton-Jacobi-Bellman-equation (HJB), principle of optimality, dynamic programming method for optimization: find value function, then feedback control $\alpha(x,t)$, then trajectories; relation to the PMP: costate equals gradient of the value function, HJB implies PMP if value function is C^2; the method of characteristics for Hamilton-Jacobi equations, Hamiltonian systems of ODE, simple examples for value functions, linear systems with quadratic cost (linear quadratic regulator), HJB solved via Riccati equation

8. Existence of optimal controls

Examples for non-existence, admissible pairs (x, α), Filippov's existence theorem for Mayer problems, corollary for time-optimal control, proof of existence theorem: Arzela-Ascoli and Filippov's lemma; proof of Filippov's lemma (McShane-Warfield)

Sample questions

What was the course about? Can you give a summary in ten (or twenty) sentences?

Why does one admit bounded measurable controls, not just piecewise continuous ones for example? In what sense are the trajectories solutions of the ODE if they are not even assumed differentiable?

How would you solve the following optimal control problem: (insert problem here)? Do solutions for this problem actually exist? Do you expect all optimal controls to be bang-bang in this case? Can there be more than (insert number here) switches? Why or why not?

Explain the Banach-Alaoglu theorem. It was used twice for linear control systems in the course - where and how?

Formulate the Pontryagin maximum principle (PMP, several versions). Compute the Hamiltonian for the following example: (insert example here). Find out what the PMP gives in this case.

How does the PMP for time optimal control of linear systems follow from the general version? How can problems with running payoff be reduced to Mayer problems? What are multiple needle variations? What is the terminal cone? What is its role in the proof of the PMP? What does Brouwer's fixed point theorem have to do with it?

Do you know any sufficient conditions for optimality? Or how else could you be sure that some control is optimal? For the linear time optimal problem? In general?

What is the value function? What is it good for? How does one find it for the rocket car example? What is the Hamilton-Jacobi-Bellman equation (HJB)? Idea of proof? How does the HJB equation imply the PMP? Or does it, really?

What is the linear quadratic regulator? How does one find the HJB equation for this? And what is the ansatz for solving it?

Formulate Filippov's existence theorem. How is the Arzela-Ascoli theorem used in this? What is a minimizing sequence? What is the convexity assumption in Filippov's existence theorem? How does it enter the proof? Explain Filippov's lemma. What does it have to do with Cantor's middle third set?