

RIGIDITY THEOREMS FOR $Spin^c$ -MANIFOLDS

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Abstract

We prove a rigidity and vanishing theorem for $Spin^c$ -manifolds. Special cases include the rigidity of the elliptic genus and the elliptic genera of higher level. We state some applications to $Spin^c$ -manifolds with nice $Pin(2)$ -action.

1 Introduction

The main aim of this paper is the proof of a rigidity and vanishing theorem for $Spin^c$ -manifolds. For a $Spin^c$ -manifold M , a complex vector bundle V over M and a $Spin$ -vector bundle W over M we define q -power series $\varphi_0^c(M; V, W)$ and $\varphi_N^c(M; V, W)$, where each coefficient is the index of the $Spin^c$ -Dirac operator of M twisted with a virtual vector bundle associated to M , V and W (see Definition 3.1). We assume S^1 acts on M and the action lifts to the $Spin^c$ -structure of M , to V and to W . In this situation the q -power series above refine to power series of S^1 -equivariant indices which are denoted by $\varphi_0^c(M; V, W)_{S^1}$ and $\varphi_N^c(M; V, W)_{S^1}$.

Our main result states that $\varphi_0^c(M; V, W)_{S^1}$ and $\varphi_N^c(M; V, W)_{S^1}$ are rigid or vanish identically provided the first equivariant Chern classes of M and V and the first equivariant Pontrjagin classes of M , V and W satisfy certain conditions (see Theorem 3.2). As special cases we recover the rigidity theorem of Bott-Taubes (cf. [3], [27]) for the elliptic genus and the rigidity theorem of Hirzebruch (cf. [10]) for the elliptic genera of higher level (see Corollary 3.3). The proof of Theorem 3.2 which uses the Lefschetz fixed point formula and the theory of Jacobi forms is in the same spirit as Liu's treatment of the rigidity of elliptic genera. For $Spin$ -manifolds vanishing type results as given in Theorem 3.2 were proven by Liu in [21].

The vanishing theorem implies the vanishing of certain mixed characteristic numbers, i.e. certain polynomials in the characteristic classes of the $Spin^c$ -structure, V and W vanish after evaluation on the fundamental cycle of M . It leads to many new applications to $Spin^c$ -manifolds which admit an almost effective $Pin(2)$ -action which is trivial on integral cohomology (such actions will be called nice). We state two of them which are proven in [6] (see also [5]).

The first application asserts the vanishing of $\varphi_0^c(M; 0, 0)$ if $c_1(M)$ and $p_1(M)$ are torsion classes. This result implies the vanishing of the Witten genus (see Theorem 3.4). For $Spin$ -manifolds this can also be derived from [21] (cf. [4]). The second application gives a partial answer to a conjecture of Petrie concerning the total Pontrjagin class of a homotopy $\mathbb{C}P^m$ with S^1 -symmetry (see Theorem 3.5).

The paper is structured in the following way. In Section 2 we briefly recall the rigidity theorems for classical operators and elliptic genera. In Section 3 we state our main theorem, a rigidity and vanishing theorem for $Spin^c$ -manifolds and derive the rigidity of elliptic genera as a corollary. The section ends with two applications to $Spin^c$ -manifolds with nice $Pin(2)$ -action. In Section 4 we give the proof of the rigidity and vanishing theorem. We show that the q -power series $\varphi_0^c(M; V, W)_{S^1}$ and

$\varphi_N^c(M; V, W)_{S^1}$ converge for $q \in \mathbb{C}$, $|q| < 1$, to holomorphic Jacobi functions of non-positive index (see Proposition 4.9). Now the theorem follows from standard properties of Jacobi functions.

2 Classical Operators and Elliptic Genera

In this section we recall the rigidity of classical operators and elliptic genera. Let M be a closed smooth connected manifold and let G be a compact Lie group which acts smoothly on M . For a G -equivariant elliptic differential operator D on M the equivariant index $ind_G(D)$ is defined as the (formal) difference of kernel and cokernel of D :

$$ind_G(D) = \ker(D) - \operatorname{coker}(D).$$

Since D is elliptic both spaces are finite dimensional G -representations and $ind_G(D)$ is an element of the representation ring $R(G)$. If $ind_G(D)$ is trivial, i.e. any $g \in G$ acts as the identity on $ind_G(D)$, we call the operator D and also its index $ind_G(D)$ **rigid**. Obviously the index is rigid if the G -action is trivial. If G is connected $ind_G(D)$ is rigid if and only if the restriction to any S^1 -subgroup is rigid.

In the following we restrict to the case that $G = S^1$ acts non-trivially on M , identify $R(S^1)$ with $\mathbb{Z}[\lambda, \lambda^{-1}]$ and view $ind_{S^1}(D)$ as a finite Laurent polynomial in λ . For any topological generator λ_0 of S^1 the Lefschetz fixed point formula (L-F-F) of Atiyah-Segal-Singer (cf. [1], Theorem 2.9) expresses the equivariant index $ind_{S^1}(D)(\lambda_0)$ in terms of local data ν_Y at the connected fixed point components Y of the S^1 -action:

$$ind_{S^1}(D)(\lambda_0) = \sum \nu_Y(\lambda_0).$$

It follows from the L-F-F that $\nu_Y(\lambda)$ is a meromorphic function with possible poles only on the unit circle S^1 or in 0 and ∞ . Since $ind_{S^1}(D)(\lambda) \in \mathbb{Z}[\lambda, \lambda^{-1}]$ has no poles on S^1 and the identity above holds on the dense subset of topological generators of S^1 the Laurent polynomial $ind_{S^1}(D)(\lambda)$ extends the sum $\sum \nu_Y(\lambda)$ holomorphically to S^1 . Thus $\sum \nu_Y(\lambda)$ is a meromorphic function with possible poles only in 0 and ∞ .

It is well-known that the classical operators are rigid. Here the classical operators we are referring to are the signature operator for oriented manifolds, the Dirac operator for $Spin$ -manifolds and the Dolbeault operator for complex manifolds.[†] In all these cases each summand $\nu_Y(\lambda)$ is holomorphic in 0 and bounded at ∞ . Hence by the theorem of Liouville the equivariant index of a classical operator has to be constant as a function of $\lambda \in S^1$.

For the Dirac operator ∂ this has a striking consequence: From the L-F-F follows directly that each local contribution vanishes in 0 and ∞ . Thus the rigidity of $ind_{S^1}(\partial)$ implies the vanishing of $ind_{S^1}(\partial)$. This is the famous vanishing theorem of Atiyah-Hirzebruch (cf. [2]). As a consequence the (non-equivariant) index of the Dirac operator, the \hat{A} -genus, is an obstruction to S^1 -actions on $Spin$ -manifolds.

In [28] Witten considered analogues of the classical operators on the free loop space \mathcal{LM} of a manifold M . Until now no mathematically rigorous definition for these “operators” is known. However, Witten derived a formula for their indices by formally applying the L-F-F to the natural S^1 -action.

The group S^1 acts on \mathcal{LM} by changing the parametrization of the loops. The fixed point set of this action consists of the constant loops, which can be identified with the underlying manifold M .

[†]The symbol of the Dolbeault operator is also defined for stable almost complex manifolds and its index is also rigid.

Applying formally the L-F-F for this S^1 -action Witten obtained invariants of the underlying manifold M .

The invariants which correspond to the signature and the Dirac operator are the elliptic genus φ_{ell} and the Witten genus φ_W (cf. [28]). The elliptic genus had been studied before by Landweber-Stong (cf. [18]) and Ochanine (cf. [23] and [24]). For a stable almost complex manifold with first Chern class divisible by an integer $N \geq 2$ the invariant which corresponds to the Dolbeault operator is the elliptic genus φ_N of level N (cf. [28]). For $N = 2$ it coincides with the elliptic genus. In all cases these genera are given by q -power series where each coefficient is the index of a twisted classical operator.

Witten conjectured that the elliptic genus is rigid for an S^1 -equivariant $Spin$ -manifold and that the elliptic genus of level N is rigid for an S^1 -equivariant stable almost complex manifold with first Chern class divisible by N . He also gave a heuristic proof using arguments from quantum field theory. In the semi-free case a proof of the first conjecture was given by Ochanine in [24] (cf. also [18]). The general case was proven by Taubes (cf. [27]) using a Dirac operator on the normal bundle to the embedding of the manifold in its free loop space and by Bott-Taubes (cf. [3]) using elliptic function theory. The second conjecture was proven by Hirzebruch (cf. [10] or [11], Appendix III). For some related results we refer to [12], [13], [14] and [17].

Later Liu observed that the rigidity theorems for the elliptic genera are consequences of the holomorphicity of associated Jacobi functions (cf. [20] and [21], cf. also [7]). Again the proof is based on the L-F-F. Each local datum ν_Y is now a function in λ and q and extends to a Jacobi functions $\mu_Y(\tau, z)$ of index zero after the substitution $\lambda = e^{2\pi i \cdot z}$ and $q = e^{2\pi i \cdot \tau}$. The equivariant genus converges normally to the meromorphic function $F(\tau, z) = \sum \mu_Y(\tau, z)$ which is elliptic in z for the lattice $\mathbb{Z}\langle 1, \tau \rangle$, where τ is any fixed element in the upper half plane. Under the action of the full modular group the function F transforms again to the limit of power series of equivariant indices which converge normally. Each coefficient is a finite Laurent polynomial in λ since it is an S^1 -equivariant index. A detailed analysis now gives the holomorphicity of F . Since for fixed τ the function F is elliptic in the variable z the theorem of Liouville shows that F is constant in z . This implies the rigidity of the equivariant genus.

Liu also proved that under certain assumptions on the first equivariant Pontrjagin class the equivariant Witten genus is constant zero. Based on Liu's work we showed in [4] (cf. also [6]) that the equivariant Witten genus vanishes for $BO\langle 8 \rangle$ -manifolds with non-trivial semi-simple group action. Independently, this was proven by Höhn, also using Liu's results. A generalization to nice $Pin(2)$ -actions will be given in the next section (see Theorem 3.4).

Finally we remark that elliptic genera are used to define elliptic homology (cf. [15], [16], [19] and the work of Hopkins and his collaborators). One hopes that a geometric definition of elliptic (co)-homology will play a similar role in topology as K -theory and ordinary cohomology does.

3 A Rigidity Theorem for $Spin^c$ -manifolds

In this section we state our main theorem, a rigidity and vanishing theorem. We derive the rigidity of elliptic genera as a corollary and state some applications to $Spin^c$ -manifolds with nice $Pin(2)$ -action.

Let M be a $2m$ -dimensional S^1 -equivariant $Spin^c$ -manifold and let ∂_c denote the $Spin^c$ -Dirac operator.[†] In contrast to the classical operators considered in the previous section the equivariant

[†]We speak of the $Spin^c$ -Dirac operator since we are only interested in its index which is independent of the choice of connections.

index of the $Spin^c$ -Dirac operator is in general not rigid. For example $\mathbb{C}P^2$ admits a $Spin^c$ -structure and a linear S^1 -action such that the action lifts to the $Spin^c$ -structure and for any lift the equivariant index of the $Spin^c$ -Dirac operator is not rigid (for details cf. [5], p. 22). For rigidity results under additional assumptions we refer to [9] and [22].

We now come to the construction of certain q -power series of equivariant indices of twisted $Spin^c$ -Dirac operators which are used in the rigidity und vanishing theorem (see Theorem 3.2). For a $2m$ -dimensional S^1 -equivariant $Spin^c$ -manifold M with $Spin^c$ -Dirac operator ∂_c the construction involves an S^1 -equivariant s -dimensional complex vector bundle V over M and an S^1 -equivariant $2t$ -dimensional $Spin$ -vector bundle W over M . From these data we build the q -power series $R_0 \in K_{S^1}(M)[[q]]$ of virtual S^1 -equivariant vector bundles defined by

$$R_0 := \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{TM} \otimes_{\mathbb{R}} \mathbb{C}) \otimes \Lambda_{-1}(V^*) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{-q^n}(\widetilde{V} \otimes_{\mathbb{R}} \mathbb{C}) \otimes \Delta(\widetilde{W}) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{q^n}(\widetilde{W} \otimes_{\mathbb{R}} \mathbb{C}).$$

Here q is a formal variable, \widetilde{E} denotes the reduced vector bundle $E - \dim(E)$, $\Delta(W)$ is the full complex spinor bundle associated to the $Spin$ -vector bundle W and $\Lambda_t := \sum \Lambda^i \cdot t^i$ (resp. $S_t := \sum S^i \cdot t^i$) denotes the exterior (resp. symmetric) power operation. The tensor product is, if not indicated otherwise, taken over the complex numbers. If $c_1(V)$ is divisible by an integer $N \geq 2$ and $y := -e^{2\pi i/N}$ we also consider $R_N \in K_{S^1}(M) \otimes_{\mathbb{Z}} \mathbb{C}[[q]]$ defined by

$$R_N := \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{TM} \otimes_{\mathbb{R}} \mathbb{C}) \otimes \Lambda_y(\widetilde{V}^*) \otimes \bigotimes_{n=1}^{\infty} (\Lambda_{y^{-1}q^n}(\widetilde{V}) \otimes \Lambda_{yq^n}(\widetilde{V}^*)) \otimes \Delta(\widetilde{W}) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{q^n}(\widetilde{W} \otimes_{\mathbb{R}} \mathbb{C}).$$

We extend the index function ind_{S^1} to power series.

Definition 3.1. Let $\varphi_0^c(M; V, W)_{S^1}$ be the S^1 -equivariant index of the $Spin^c$ -Dirac operator twisted with R_0 , i.e.

$$\varphi_0^c(M; V, W)_{S^1} := ind_{S^1}(\partial_c \otimes R_0) \in R(S^1)[[q]].$$

Let $\varphi_N^c(M; V, W)_{S^1}$ be the S^1 -equivariant index of the $Spin^c$ -Dirac operator twisted with R_N , i.e.

$$\varphi_N^c(M; V, W)_{S^1} := ind_{S^1}(\partial_c \otimes R_N) \in R(S^1) \otimes_{\mathbb{Z}} \mathbb{C}[[q]].$$

In the non-equivariant situation we write $\varphi_0^c(M; V, W)$ and $\varphi_N^c(M; V, W)$, respectively.

We remark that these power series specialize to the Witten genus and to the elliptic genera. In fact, if M is an S^1 -equivariant $Spin$ -manifold with tangent bundle TM and if ∂_c is the induced $Spin^c$ -Dirac operator then

$$\varphi_W(M)_{S^1} = \varphi_0^c(M; 0, 0)_{S^1} \quad \text{and} \quad \varphi_{ell}(M)_{S^1} = \varphi_0^c(M; 0, TM)_{S^1}.$$

If M admits an S^1 -equivariant stable almost complex structure $\tau(M)$ and if ∂_c is the induced $Spin^c$ -Dirac operator then

$$\varphi_N(M)_{S^1} = \varphi_N^c(M; \tau(M), 0)_{S^1}.$$

Here $\varphi_{ell}(M)_{S^1}$ and $\varphi_N(M)_{S^1}$ denote the equivariant elliptic genera and $\varphi_W(M)_{S^1}$ denotes the equivariant Witten genus.

In order to state the rigidity theorem for $\varphi_0^c(M; V, W)_{S^1}$ and $\varphi_N^c(M; V, W)_{S^1}$ we need to introduce some notation. For an S^1 -equivariant virtual vector bundle E over M and a characteristic class $u(E)$

the equivariant characteristic class will be denoted by $u(E)_{S^1}$. We recall that the $Spin^c$ -structure induces an S^1 -equivariant complex line bundle L_c over M . Let P be the $Spin^c$ -principal bundle over M . Then L_c is defined by the $U(1)$ -principal bundle $P/Spin(2m) \rightarrow P/Spin^c(2m) \cong M$ using the standard embedding $Spin(2m) \hookrightarrow Spin^c(2m)$. Its equivariant first Chern class $c_1(L_c)_{S^1}$ will also be denoted by $c_1(M)_{S^1}$. Finally, let π denote the projection of the fixed point manifold M^{S^1} to a point pt and let x be a fixed generator of $H^2(BS^1; \mathbb{Z})$. Our main result is

Theorem 3.2. *Assume that the equivariant class $p_1(V + W - TM)_{S^1}$ restricted to M^{S^1} is equal to $\pi^*(\mathcal{I} \cdot x^2)$ modulo torsion for some integer \mathcal{I} and assume that $c_1(M)_{S^1}$ and $c_1(V)_{S^1}$ restricted to M^{S^1} are equal modulo torsion.*

1. *If $\mathcal{I} = 0$ then $\varphi_0^c(M; V, W)_{S^1}$ is rigid, i.e. each coefficient of the q -power series does not depend on $\lambda \in S^1$. If, in addition, $c_1(V)$ is divisible by an integer $N \geq 2$ then $\varphi_N^c(M; V, W)_{S^1}$ is rigid.*
2. *If $\mathcal{I} < 0$ then $\varphi_0^c(M; V, W)_{S^1}$ vanishes identically. If, in addition, $c_1(V)$ is divisible by an integer $N \geq 2$ then $\varphi_N^c(M; V, W)_{S^1}$ vanishes identically.*

For $Spin$ -manifolds vanishing type results as given in Part 2 were proven by Liu in [21]. The proof of Theorem 3.2 will occupy the next section. We remark that in Part 2 the condition on the first equivariant Chern class can be relaxed to $c_1(M)|_{M^{S^1}} \equiv c_1(V)|_{M^{S^1}}$ modulo torsion (cf. [5] for details). As a special case we obtain the rigidity theorems for elliptic genera.

Corollary 3.3. *1. The elliptic genus is rigid for a $Spin$ -manifold with S^1 -action.*

2. *The elliptic genus of level N is rigid for an S^1 -equivariant stable almost complex manifold M with first Chern class divisible by N .*

Proof: Let M be a $Spin$ -manifold with S^1 -action. After doubling the action we may lift the S^1 -action to the $Spin$ -structure. Let $V = 0$ and $W = TM$. For the induced $Spin^c$ -structure one has $c_1(M)_{S^1} = c_1(V)_{S^1} = 0$, $p_1(V + W - TM)_{S^1} = 0$ and $\varphi_{ell}(M)_{S^1} = \varphi_0^c(M; 0, TM)_{S^1}$. Thus Theorem 3.2, Part 1, implies the rigidity of the elliptic genus.

If M admits an S^1 -equivariant stable almost complex structure $\tau(M)$ let $V = \tau(M)$ and $W = 0$. For the induced $Spin^c$ -structure one has $c_1(M)_{S^1} = c_1(V)_{S^1}$, $p_1(V + W - TM)_{S^1} = 0$ and $\varphi_N(M)_{S^1} = \varphi_N^c(M; \tau(M), 0)_{S^1}$. Thus Theorem 3.2, Part 1, implies the rigidity of the elliptic genus of level N . ■

The rigidity of elliptic genera implies their vanishing for actions of non-zero type (cf. for example [11], p. 181). An analogous result holds true for $\varphi_0^c(M; V, W)_{S^1}$ and $\varphi_N^c(M; V, W)_{S^1}$ (cf. [5], Theorem 3.9).

If M is a $Spin^c$ -manifold with $c_1(M)$ torsion then $\varphi_0^c(M; 0, 0)$ is equal to the Witten genus. If M is a $Spin$ -manifold then $\varphi_0^c(M; 0, 0)_{S^1}$ is equal to the equivariant Witten genus. In this situation the statement for $\varphi_0^c(M; 0, 0)_{S^1}$ in Theorem 3.2, Part 2, is a Theorem of Liu (cf. [21], Theorem 6).

One could also envisage more complicated rigidity theorems for $Spin^c$ -manifolds involving additional vector bundle data and corresponding conditions on the equivariant first Chern classes and the equivariant first Pontrjagin classes. Since we don't know of any interesting applications for these more complicated versions we have restricted to the situation described above.

Theorem 3.2 leads to many new results for $Spin^c$ -manifolds with nice $Pin(2)$ -action. The action is called nice if $Pin(2)$ acts almost effectively, i.e. with finite kernel, on the manifold and acts trivially on the integral cohomology ring. In particular any non-trivial semi-simple group action leads to a nice $Pin(2)$ -action. We state two applications. The proofs are given in [6] (for the special case of semi-simple group actions proofs are given in [5]).

Theorem 3.4. ([6], Th. 4.1) *Let M be a $2m$ -dimensional $Spin^c$ -manifold with nice $Pin(2)$ -action. Assume the first Chern class of the $Spin^c$ -structure and the first Pontrjagin class of M are torsion. Then the S^1 -action induced by $S^1 \hookrightarrow Pin(2)$ lifts to the $Spin^c$ -structure. For any lift the q -power series of S^1 -equivariant twisted $Spin^c$ -indices $\varphi_0^c(M; 0, 0)_{S^1}$ vanishes identically. In particular the Witten genus $\varphi_W(M)$ is zero. ■*

Theorem 3.4 generalizes previous results. The vanishing of the Witten genus was proven by the author for $BO(8)$ -manifolds with non-trivial S^3 -action in [4] and independently by Höhn in unpublished work. We remark that the vanishing of the Witten genus stated in Theorem 3.4 leads to some evidence for the following conjecture of Stolz and Höhn: Let M be a $Spin$ -manifold with $\frac{p_1}{2}(M) = 0$. If M admits a metric with positive Ricci curvature then $\varphi_W(M) = 0$. The conjecture implies the existence of a *simply connected* manifold which admits a positive scalar, but no positive Ricci curvature metric (cf. [26]).

The second application deals with cohomology $\mathbb{C}P^m$'s, i.e. manifolds having the same integral cohomology ring as $\mathbb{C}P^m$. Note that any homotopy $\mathbb{C}P^m$ is a cohomology $\mathbb{C}P^m$. The motivation is a conjecture of Petrie (cf. [25], Strong conjecture, p. 105) which we state in the following equivalent form: If M is a homotopy $\mathbb{C}P^m$ with non-trivial S^1 -action then the total Pontrjagin class has standard form, i.e. $p(M) = (1 + x^2)^{m+1}$ for a generator x of $H^2(M; \mathbb{Z})$. Using Theorem 3.2 we can show the

Theorem 3.5. ([6], Th. 4.2) *Let M be a cohomology $\mathbb{C}P^m$ with nice $Pin(2)$ -action. If m is odd assume in addition that the $Pin(2)$ -action has a fixed point. Let x be a generator of $H^2(M; \mathbb{Z})$ and let b be the integer defined by $p_1(M) = b \cdot x^2$. Then $b \leq m + 1$ and*

$$b = m + 1 \implies p(M) = (1 + x^2)^{m+1}. \quad \blacksquare$$

This theorem is based on Part 2 of Theorem 3.2. The proof involves the following steps (details are given in [6]):

1. The hypothesis implies that the S^1 -action induced by $S^1 \hookrightarrow Pin(2)$ has a fixed point (if m is even this follows from the Lefschetz fixed point formula for the Euler characteristic).
2. For $y \in H^2(M; \mathbb{Z})$ let L_y denote the complex line bundle with $c_1(L_y) = y$. For $k \in \{0, 1, \dots, \lfloor \frac{b-4}{2} \rfloor\}$ let $V_k := L_{2x} + (b - 4 - 2k) \cdot L_x$, $W_k := (2k) \cdot L_x$ and choose a $Spin^c$ -structure on M with $c_1(M) = c_1(V_k)$. Note that $p_1(V_k + W_k - TM) = 0$. One can show that the $Pin(2)$ -action lifts to each line bundle occurring in V_k and W_k . Also the S^1 -action lifts to the chosen $Spin^c$ -structure. At the $Pin(2)$ -fixed point the equivariant vector bundles V_k and W_k reduce to sums of trivial complex $Pin(2)$ -representation. This in turn implies that $p_1(V_k + W_k - TM)_{S^1} = \pi^*(\mathcal{I} \cdot x^2)$ for some *negative* number \mathcal{I} .

3. Part 2 of Theorem 3.2 implies the vanishing of $\varphi_0^c(M, V_k, W_k)$. In particular, the constant term in the q -power series $\varphi_0^c(M, V_k, W_k)$ is zero, i.e.

$$\left\langle \hat{\mathcal{A}}(M) \cdot (e^x - e^{-x}) \cdot (e^{\frac{x}{2}} - e^{-\frac{x}{2}})^{b-4-2k} \cdot (e^{\frac{x}{2}} + e^{-\frac{x}{2}})^{2k}, [M] \right\rangle = 0,$$

where $\hat{\mathcal{A}}$ denotes the multiplicative sequence for the \hat{A} -genus.

4. If $b > m + 1$ one of the above relations ($k = \frac{b-(m+3)}{2}$) gives the contradiction $\langle x^m, [M] \rangle = 0$. If $b = m + 1$ the relations above together with the signature theorem completely determine $\hat{\mathcal{A}}(M)$ and therefore determine the total Pontrjagin class $p(M)$. Since all these relations also hold true for $\mathbb{C}P^m$ we conclude that $p(M) = (1 + x^2)^{m+1}$.

4 Proof of Theorem 3.2

In this section we give the proof of the rigidity and vanishing theorem. In the following the (equivariant) index of a twisted $Spin^c$ -Dirac operator will be called (equivariant) twisted $Spin^c$ -index. For technical reasons we replace the action by the $2N$ -fold action. Note that replacing the action leads to a statement equivalent to Theorem 3.2.

Here is an outline of the proof where we restrict to the statement for $\varphi_N^c(M; V, W)$ (the reasoning for $\varphi_0^c(M; V, W)$ is similar): For any topological generator $\lambda_0 = e^{2\pi i \cdot z_0}$ of S^1 the coefficients of the q -power series $\varphi_N^c(M; V, W)_{S^1}(\lambda_0)$ of equivariant twisted $Spin^c$ -indices can be calculated via the Lefschetz fixed point formula. This formula gives a local datum for each connected component Y of M^{S^1} . We introduce a function $F(Y)(\tau, z, \xi) := F_{\frac{1}{N}, 0}^{(\frac{1}{2})}(Y)(\tau, z, \xi)$, where τ is in the upper half plane \mathcal{H} , $z \in \mathbb{C}$ and $\xi = (x_1, \dots, x_m, v_1, \dots, v_s, w_1, \dots, w_t) \in \mathbb{C}^{m+s+t}$. This function is meromorphic on $(\tau, z, \xi) \in \mathcal{H} \times \mathbb{C} \times \mathbb{C}^{m+s+t}$, has as building block the Weierstrass' Φ -function and has as input the local geometry of the S^1 -action on V, W and the $Spin^c$ -structure at Y (see Definition 4.2, Part 1). For a topological generator $\lambda_0 = e^{2\pi i \cdot z_0}$, z_0 close to zero, we show that the q -power series

$$\varphi_N^c(M; V, W)_{S^1}(\lambda_0)$$

converges normally on $B := \{q \in \mathbb{C} \mid |q| < 1\}$ for $q = e^{2\pi i \cdot \tau}$ and $\lambda_0 = e^{2\pi i \cdot z_0}$ to the meromorphic function

$$\sum_{Y \in \mathcal{Y}} \mathcal{P}_Y \left(F(Y)(\tau, z_0, \frac{\xi}{2\pi i}) \right) (\xi = 0).$$

Here \mathcal{Y} denotes the set of connected components of M^{S^1} and \mathcal{P}_Y is a polynomial in $\frac{\partial}{\partial x_i}$, $\frac{\partial}{\partial v_j}$ and $\frac{\partial}{\partial w_k}$ over \mathbb{Q} which corresponds to evaluation on the fundamental cycle of Y (see Corollary 4.4). At this point we use the condition on the equivariant first Chern classes of V and the $Spin^c$ -structure stated in Theorem 3.2.

Since each coefficient of the q -power series $\varphi_N^c(M; V, W)_{S^1}$ is an equivariant twisted $Spin^c$ -index, each coefficient is a finite Laurent polynomial in λ . In particular as a function in λ each coefficient has no poles on S^1 . Since $\varphi_N^c(M; V, W)_{S^1}$ converges normally to a meromorphic function it follows from a function theoretical lemma (see Lemma 4.5) that

$$F_N(\tau, z) := \sum_{Y \in \mathcal{Y}} \mathcal{P}_Y \left(F(Y)(\tau, z, \frac{\xi}{2\pi i}) \right) (\xi = 0)$$

is holomorphic on $(\tau, z) \in \mathcal{H} \times \mathbb{R}$ (see Proposition 4.6).

The condition on the equivariant first Pontrjagin class implies that $F_N(\tau, z)$ is a Jacobi function (i.e. transforms like a Jacobi form) of index \mathcal{I} for a subgroup of $SL_2(\mathbb{Z})$ of finite index (see Proposition 4.9). We show that under the action of the full group $SL_2(\mathbb{Z})$ the function $F_N(\tau, z)$ transforms to functions which are again the limit of normally convergent power series of twisted $Spin^c$ -indices. Here we need all conditions on the equivariant characteristic classes. By the same reasoning as above we conclude that all these transformed functions are holomorphic on $\mathcal{H} \times \mathbb{R}$. Now the transformation properties for Jacobi functions imply that $F_N(\tau, z)$ is holomorphic on $\mathcal{H} \times \mathbb{C}$. If $\mathcal{I} = 0$ this implies that $F_N(\tau, z)$ is constant in z and $\varphi_N^c(M; V, W)_{S^1}$ is rigid. If $\mathcal{I} < 0$ this implies that $F_N(\tau, z)$ and $\varphi_N^c(M; V, W)_{S^1}$ vanish identically. This finishes the outline.

We now begin with the proof of Theorem 3.2. The conditions on the first equivariant Pontrjagin classes and Chern classes will be assumed throughout this section. Since we have replaced the S^1 -action by the $2N$ -fold action \mathcal{I} is divisible by $(2N)^2$.

Let $P \rightarrow M$ denote the $Spin^c(2m)$ -principal bundle defining the $Spin^c$ -structure on M . For later purposes we fix a $U(s)$ -reduction P_V of V , i.e. P_V is an S^1 -equivariant $U(s)$ -principal bundle over M and there is a fixed S^1 -equivariant isomorphism $V \cong P_V \times_{U(s)} \mathbb{C}^s$ of complex vector bundles. Similarly, let P_W be an S^1 -equivariant $Spin(2t)$ -principal bundle over M with a fixed S^1 -equivariant isomorphism $W \cong P_W \times_{Spin(2t)} \mathbb{R}^{2t}$ of oriented vector bundles.

If $c_1(V)$ is divisible by an integer $N \geq 2$ the determinant line bundle $det(V)$ admits an equivariant N^{th} root L . In particular, $N \cdot c_1(L)_{S^1} = c_1(V)_{S^1}$.[†] At each point of a given fixed point component $Y \subset M^{S^1}$ the line bundle L reduces to a complex one-dimensional S^1 -representation $S^1 \rightarrow U(1)$, $\lambda \mapsto \lambda^{l(Y)}$, where $l(Y)$ only depends on Y .

During the proof we will need to study a family of series of twisted $Spin^c$ -indices $ind_{\alpha, k}^{(\epsilon)}(M)_{S^1}$. If not stated otherwise the index set will be always of the following form: $\epsilon \in \{\frac{1}{2}, \frac{\tau}{2}, \frac{\tau+1}{2}\}$, $k \in \mathbb{Z}$, and either $(\alpha, k) = 0$ or $\alpha = \frac{\alpha_1 \tau + \alpha_2}{N} \neq 0$, $\alpha_i \in \{0, \dots, N-1\}$, where N is an integer ≥ 2 . Note that for $(\alpha, k) \neq 0$ the integer N is part of the structure but is suppressed in the notation $ind_{\alpha, k}^{(\epsilon)}(M)_{S^1}$. If $c_1(V) \equiv 0 \pmod{N}$ and $(\alpha, k) \neq 0$ let

$$S_{\alpha, k}(V) := \Lambda_y(\widetilde{V}^*) \otimes \bigotimes_{n=1}^{\infty} (\Lambda_{y^{-1}q^n}(\widetilde{V}) \otimes \Lambda_{yq^n}(\widetilde{V}^*)) \otimes L^k,$$

where $y := -q^{\frac{\alpha_1}{N}} \cdot e^{2\pi i \frac{\alpha_2}{N}}$. Let $S_{0,0}(V) := \Lambda_{-1}(V^*) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{-q^n}(\widetilde{V} \otimes_{\mathbb{R}} \mathbb{C})$ and let

$$R^{(\frac{1}{2})}(W) := \Delta(\widetilde{W}) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{q^n}(\widetilde{W} \otimes_{\mathbb{R}} \mathbb{C}), \quad R^{(\frac{\tau}{2})}(W) := \bigotimes_{n=1}^{\infty} \Lambda_{q^{\frac{2n-1}{2}}}(\widetilde{W} \otimes_{\mathbb{R}} \mathbb{C}),$$

$$R^{(\frac{\tau+1}{2})}(W) := \bigotimes_{n=1}^{\infty} \Lambda_{-q^{\frac{2n-1}{2}}}(\widetilde{W} \otimes_{\mathbb{R}} \mathbb{C}).$$

Recall that ∂_c denotes the $Spin^c$ -Dirac operator on the S^1 -equivariant $Spin^c$ -manifold M . We extend the index function ind_{S^1} to power series and make the

[†]Note that the N -fold action always lifts to L with this property (cf. [11], p. 181).

Definition 4.1. The power series of equivariant $Spin^c$ -indices $ind_{\alpha,k}^{(\epsilon)}(M)_{S^1} \in R(S^1) \otimes_{\mathbb{Z}} \mathbb{C}[[q^{\frac{1}{2N}}]]$ is defined by

$$ind_{\alpha,k}^{(\epsilon)}(M)_{S^1} := ind_{S^1}(\partial_c \otimes \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{TM} \otimes_{\mathbb{R}} \mathbb{C}) \otimes S_{\alpha,k}(V) \otimes R^{(\epsilon)}(W)).$$

Note that $\varphi_N^c(M; V, W)_{S^1} = ind_{\frac{1}{N},0}^{(\frac{1}{2})}(M)_{S^1}$ and $\varphi_0^c(M; V, W)_{S^1} = ind_{0,0}^{(\frac{1}{2})}(M)_{S^1}$. The other indices will enter the proof when we consider the modularity properties of $\varphi_N^c(M; V, W)_{S^1}$ and $\varphi_0^c(M; V, W)_{S^1}$ (see Proposition 4.9 below).

We proceed to give the Lefschetz fixed point formula for $ind_{\alpha,k}^{(\epsilon)}(M)_{S^1}$ in terms of certain power series associated to the Weierstrass' Φ -function (see Proposition 4.3 below). In order to state this formula we describe the local geometry of the $Spin^c(2m)$ -principal bundle P and the principal bundles P_V and P_W at a connected component Y of M^{S^1} in terms of classifying spaces and universal equivariant roots.

We digress for a moment and consider an S^1 -equivariant H -principal bundle Q over M , where H is a connected compact Lie group. Let T_H be a fixed maximal torus of H . Since Y is a trivial S^1 -space the action of S^1 on the restriction $Q|_Y$ preserves the fibres and commutes with the principal action of H . It is well-known that $Q|_Y$ admits an S^1 -equivariant reduction $Q' \subset Q|_Y$ with respect to $H' \subset H$, where H' is the centralizer in H of the image $f(S^1)$ of a homomorphism $f : S^1 \rightarrow T_H$. The homomorphism f is unique up to the action of the Weyl group $W(H)$ on T_H . The S^1 -action on Q' is given by the principal action via f from the right (for details we refer to [5], Appendix A.3).

Let $\{\hat{u}_1, \dots, \hat{u}_l\}$ be a fixed basis of the Lie algebra \mathfrak{t}_H of $T_H = T_{H'}$ and let \hat{u} be the standard generator of the integral lattice of \mathfrak{t}_{S^1} . Then the differential df maps \hat{u} to $\sum n_i(Y) \cdot \hat{u}_i$. We identify the dual basis $\{u_i\}$ of $\{\hat{u}_i\}$ with elements in the second cohomology of the classifying space of T_H via transgression. So for any \mathbb{C} -algebra \mathcal{A}

$$H^{**}(BT_H; \mathcal{A}) = \mathcal{A}[[u_1, \dots, u_l]].$$

We call $\{u_i + n_i(Y) \cdot z\} \in H^*(BT_H; \mathbb{C}[z])$ the **universal equivariant roots** of Q at Y . Of course, the universal equivariant roots depend on the choices made above.

Notation: To simplify further calculations we will use the following notation, where u_i, z may be formal variables, cohomology classes or complex numbers, depending on the context.

- $\mathbf{u}_i(Y) := \mathbf{u}_i(Y)(z) := u_i + n_i(Y) \cdot z$.
- $\tilde{\mathbf{u}}_i(Y) := \tilde{\mathbf{u}}_i(Y)(z) := u_i + 2\pi i \cdot n_i(Y) \cdot z$.
- $|\mathbf{u}(Y)| := |\mathbf{u}(Y)|(z) := \sum_i u_i + \sum_i n_i(Y) \cdot z$.
- $|\tilde{\mathbf{u}}(Y)| := |\tilde{\mathbf{u}}(Y)|(z) := \sum_i u_i + 2\pi i \cdot \sum_i n_i(Y) \cdot z$.

Note that in the case of the principal bundles P, P_V and P_W above the $n_i(Y)$'s have a familiar interpretation as rotation numbers of associated vector bundles. For example, let $\hat{x}_1, \dots, \hat{x}_m, \hat{x}_c$ denote the standard basis of the integral lattice of $\mathfrak{t}_{SO(2m) \times U(1)}$. We identify $\mathfrak{t}_{Spin^c(2m)}$ with $\mathfrak{t}_{SO(2m) \times U(1)}$, using the two-fold covering. Let

$$\mathbf{x}_i(Y)(z) = x_i + m_i(Y) \cdot z, \quad i = 1, \dots, m, \quad \text{and} \quad \mathbf{x}_c(Y)(z) = x_c + m_c(Y) \cdot z$$

denote the universal equivariant roots of P at Y . Then $\{\pm m_i(Y)\}$ are the rotation numbers of TM at Y and $m_c(Y)$ is the rotation number of the complex line bundle L_c associated to the $Spin^c$ -structure. Similarly, for the universal equivariant roots

$$\mathbf{v}_j(Y)(z) = v_j + s_j(Y) \cdot z, \quad j = 1, \dots, s, \quad \text{and} \quad \mathbf{w}_k(Y)(z) = w_k + t_k(Y) \cdot z, \quad k = 1, \dots, t,$$

of V and W^\dagger the s_j (resp. the $\pm t_k$) are the rotation numbers of V and W at Y . Since we replaced the action by its $2N$ -fold action all rotation numbers are divisible by $2N$.

The reductions of the groups $Spin^c(2m)$, $U(s)$ and $Spin(2t)$ at Y will be denoted by H' , H'_V and H'_W , respectively. The reductions of the principal bundles at Y will be denoted by P' , P'_V and P'_W , respectively. Let

$$f_Y : Y \rightarrow B(H' \times H'_V \times H'_W)$$

denote a classifying map of the fibrewise product $P' \oplus P'_V \oplus P'_W$ (this is the pullback of $P' \times P'_V \times P'_W$ under the diagonal). Note that Y is even-dimensional and inherits an orientation from the reduction P' .

We will now introduce certain power series associated to the Weierstrass' Φ -function which will be used in the statement of the Lefschetz fixed point formula (see Proposition 4.3 below). Recall that the Weierstrass' Φ -function $\Phi(\tau, x)$ may be defined by the normally convergent infinite product

$$\phi_0(q, x) := (e^{\frac{2\pi i \cdot x}{2}} - e^{-\frac{2\pi i \cdot x}{2}}) \cdot \prod_{n=1}^{\infty} \frac{(1 - q^n e^{2\pi i \cdot x})(1 - q^n e^{-2\pi i \cdot x})}{(1 - q^n)^2} \in 2\pi i \cdot x \cdot (1 + x \cdot \mathbb{C}[[q, x]]),$$

where $q = e^{2\pi i \cdot \tau}$, τ is in the upper half plane \mathcal{H} and $x \in \mathbb{C}$. The function $\Phi(\tau, x)$ is holomorphic on $\mathcal{H} \times \mathbb{C}$. As a function in z it has simple zeros in each point of the lattice $\mathbb{Z} \cdot \tau + \mathbb{Z}$. The function $\Phi(\tau, x)$ is periodic with respect to $\tau \mapsto \tau + 1$ and $x \mapsto x + 2$. Its Fourier-Taylor expansion[‡] with respect to $\tau \mapsto \tau + 1$ and $x = 0$ is $\phi_0(q, x)$. For $\alpha = \frac{\alpha_1 \cdot \tau + \alpha_2}{N}$, $\alpha_i \in \mathbb{Z}$, $N \geq 2$, let $\phi_\alpha(q, x)$ denote the Fourier-Taylor expansion of $\Phi(\tau, x + \alpha)$ with respect to $\tau \mapsto \tau + 2N$ and $x = 0$.

We give the Lefschetz fixed point formula for $ind_{\alpha, k}^{(\epsilon)}(M)_{S^1}$ in terms of Fourier-Taylor expansions of a meromorphic function $F_{\alpha, k}^{(\epsilon)}(Y)(\tau, z, \xi)$ defined below. Let $\xi = (x_1, \dots, x_m, v_1, \dots, v_s, w_1, \dots, w_t) \in \mathbb{C}^{m+s+t}$ and let $D_Y^{(\frac{1}{2})}(z) := 1$, $D_Y^{(\frac{\tau}{2})}(z) := D_Y^{(\frac{\tau+1}{2})}(z) := e^{|\tilde{\mathbf{w}}(Y)|/2} = e^{(\sum w_k + 2\pi i \sum t_k(Y) \cdot z)/2}$.

Definition 4.2. 1. Let $F_{\alpha, k}^{(\epsilon)}(Y)(\tau, z, \xi)$, $(\alpha, k) \neq 0$, and $F_{0,0}^{(\epsilon)}(Y)(\tau, z, \xi)$ be the meromorphic functions on $\mathcal{H} \times \mathbb{C} \times \mathbb{C}^{m+s+t}$ defined by

$$\begin{aligned} F_{\alpha, k}^{(\epsilon)}(Y)(\tau, z, \xi) &:= \left(\prod_{m_i(Y)=0} \frac{2\pi i \cdot x_i}{\Phi(\tau, x_i)} \right) \cdot \left(\prod_{m_i(Y) \neq 0} \frac{1}{\Phi(\tau, \mathbf{x}_i(Y))} \right) \cdot \left(\prod_{j=1}^s \frac{\Phi(\tau, \mathbf{v}_j(Y) - \alpha)}{\Phi(\tau, -\alpha)} \right) \times \\ &\quad \left(\prod_{k=1}^t \frac{\Phi(\tau, \mathbf{w}_k(Y) + \epsilon)}{\Phi(\tau, \epsilon)} \right) \cdot D_Y^{(\epsilon)}(z) \cdot e^{\frac{k}{N} \cdot |\tilde{\mathbf{v}}(Y)|} \quad \text{and} \\ F_{0,0}^{(\epsilon)}(Y)(\tau, z, \xi) &:= \left(\prod_{m_i(Y)=0} \frac{2\pi i \cdot x_i}{\Phi(\tau, x_i)} \right) \cdot \left(\prod_{m_i(Y) \neq 0} \frac{1}{\Phi(\tau, \mathbf{x}_i(Y))} \right) \cdot \left(\prod_{j=1}^s \Phi(\tau, \mathbf{v}_j(Y)) \right) \times \end{aligned}$$

[†]Here $\hat{v}_1, \dots, \hat{v}_s$ is a standard basis for $\mathfrak{t}_{U(s)}$, $\hat{w}_1, \dots, \hat{w}_t$ one for $\mathfrak{t}_{SO(2t)}$ and we identify $\mathfrak{t}_{Spin(2t)}$ with $\mathfrak{t}_{SO(2t)}$, using the two-fold covering.

[‡]By the Fourier-Taylor expansion with respect to $\tau \mapsto \tau + 1$ and $x = 0$ we mean the series obtained by replacing each coefficient of the Fourier series associated to Φ and the period $\tau \mapsto \tau + 1$ by its Taylor series at $x = 0$.

$$\left(\prod_{k=1}^t \frac{\Phi(\tau, \mathbf{w}_k(Y) + \epsilon)}{\Phi(\tau, \epsilon)} \right) \cdot D_Y^{(\epsilon)}(z).$$

2. Let $m_{max} := \max\{|m_i(Y)| \mid Y \in \mathcal{Y}, i = 1, \dots, m\}$ and let $A_{\alpha,k}^{(\epsilon)}(Y)(z)$ and $A_{0,0}^{(\epsilon)}(Y)(z)$ denote the Fourier-Taylor expansions of $F_{\alpha,k}^{(\epsilon)}(Y)(\tau, z, \frac{\xi}{2\pi i})$ and $F_{0,0}^{(\epsilon)}(Y)(\tau, z, \frac{\xi}{2\pi i})$ with respect to $\tau \mapsto \tau + 2N$ and $\xi = 0$ for any fixed irrational number $z \in (-\frac{1}{m_{max}}, \frac{1}{m_{max}})$.

Note that $A_{\alpha,k}^{(\epsilon)}(Y)(z)$ and $A_{0,0}^{(\epsilon)}(Y)(z)$ converge normally[†] to $F_{\alpha,k}^{(\epsilon)}(Y)(\tau, z, \frac{\xi}{2\pi i})$ and $F_{0,0}^{(\epsilon)}(Y)(\tau, z, \frac{\xi}{2\pi i})$ for any fixed irrational number $z \in (-\frac{1}{m_{max}}, \frac{1}{m_{max}})$. For example, $A_{0,0}^{(\frac{1}{2})}(Y)(z)$ is equal to

$$\left(\prod_{m_i(Y)=0} \frac{x_i}{\phi_0(q, x_i/(2\pi i))} \right) \cdot \left(\prod_{m_i(Y) \neq 0} \frac{1}{\phi_0(q, \tilde{\mathbf{x}}_i(Y)/(2\pi i))} \right) \cdot \left(\prod_{j=1}^s \phi_0(q, \tilde{\mathbf{v}}_j(Y)/(2\pi i)) \right) \times \\ \left(\prod_{k=1}^t \frac{\phi_{\frac{1}{2}}(q, \tilde{\mathbf{w}}_k(Y)/(2\pi i))}{\phi_{\frac{1}{2}}(q, 0)} \right)$$

and converges normally on $B \times \{0\} \subset B \times \mathbb{C}^{m+s+t}$ to $F_{0,0}^{(\frac{1}{2})}(Y)(\tau, z, \xi)$ for any fixed irrational number $z \in (-\frac{1}{m_{max}}, \frac{1}{m_{max}})$, where $B = \{q \in \mathbb{C} \mid |q| < 1\}$. In the following we will identify the variables x_i , v_j and w_k with cohomology classes of the classifying spaces of the maximal tori of H' , H'_V and H'_W . Let $[Y]$ denote the fundamental cycle of Y and let $\langle \ , \ \rangle$ denote the Kronecker pairing. We are now ready to state the

Proposition 4.3. (Lefschetz Fixed Point Formula) *For any topological generator $\lambda_0 = e^{2\pi i \cdot z_0} \in S^1$ with $z_0 \in (-\frac{1}{m_{max}}, \frac{1}{m_{max}})$*

$$ind_{\alpha,k}^{(\epsilon)}(M)_{S^1}(\lambda_0) = \sum_{Y \in \mathcal{Y}} \left\langle f_Y^*(A_{\alpha,k}^{(\epsilon)}(Y)(z_0)), [Y] \right\rangle.$$

Proof: The proposition follows from the Lefschetz fixed point formula (cf. [1], Theorem 2.9) using the condition on the first equivariant Chern classes. We restrict to the statement for the coefficient of q^0 in $ind_{\alpha,k}^{(\epsilon)}(M)_{S^1}$, $(\alpha, k) \neq 0$, and leave the general case to the reader. This coefficient is equal to

$$ind_{S^1}(\partial_{\mathbb{C}} \otimes \Lambda_y(\tilde{V}^*) \otimes R_0^{(\epsilon)}(W) \otimes L^k),$$

where $R_0^{(\epsilon)}(W) = \Delta(\tilde{W})$ if $\epsilon = \frac{1}{2}$ and $R_0^{(\epsilon)}(W) = 1$ otherwise. The Lefschetz fixed point formula implies that for any topological generator $\lambda_0 = e^{2\pi i \cdot z_0}$ of S^1 with $z_0 \in (-\frac{1}{m_{max}}, \frac{1}{m_{max}})$

$$ind_{S^1}(\partial_{\mathbb{C}} \otimes \Lambda_y(\tilde{V}^*) \otimes R_0^{(\epsilon)}(W) \otimes L^k)(\lambda_0) = \sum_{Y \in \mathcal{Y}} a(Y)(\lambda_0),$$

where the local datum $a(Y)(\lambda_0)$ at the connected component Y of M^{S^1} is given by

$$\left\langle f_Y^* \left(e^{\tilde{\mathbf{x}}_{\mathbb{C}}(Y)(z_0)/2} \cdot \left(\prod_{m_i(Y)=0} \frac{x_i}{e^{x_i/2} - e^{-x_i/2}} \right) \cdot \left(\prod_{m_i(Y) \neq 0} \frac{1}{e^{\tilde{\mathbf{x}}_i(Y)(z_0)/2} - e^{-\tilde{\mathbf{x}}_i(Y)(z_0)/2}} \right) \right) \times \right.$$

[†]A series converges normally on a subset V of \mathbb{C}^r if it converges normally on an open neighborhood of V .

$$\left\langle \left(\prod_{j=1}^s \frac{1+y \cdot e^{-\tilde{v}_j(Y)(z_0)}}{1+y} \right) \cdot r_0^{(\epsilon)} \cdot e^{k \cdot (c_1(L) + 2\pi i \cdot l(Y) \cdot z_0)}, [Y] \right\rangle.$$

Here $r_0^{(\epsilon)} = \prod_{k=1}^t (e^{\tilde{w}_k(Y)(z_0)/2} + e^{-\tilde{w}_k(Y)(z_0)/2})/2$ if $\epsilon = \frac{1}{2}$ and $r_0^{(\epsilon)} = 1$ otherwise. Recall that $N \cdot c_1(L)_{S^1} = c_1(V)_{S^1}$ which implies

$$e^{k \cdot (c_1(L) + 2\pi i \cdot l(Y) \cdot z_0)} = f_Y^*(e^{\frac{k}{N} \cdot |\tilde{v}(Y)|(z_0)}).$$

Recall also that the restriction of $c_1(M)_{S^1}$ and $c_1(V)_{S^1}$ to Y are equal modulo torsion which implies

$$f_Y^*(e^{\tilde{x}_c(Y)(z_0)}) = f_Y^*(e^{|\tilde{v}(Y)|(z_0)}).$$

Thus $a(Y)(\lambda_0)$ is equal to

$$\left\langle f_Y^* \left(\prod_{m_i(Y)=0} \frac{x_i}{e^{x_i/2} - e^{-x_i/2}} \cdot \prod_{m_i(Y) \neq 0} \frac{1}{e^{\tilde{x}_i(Y)(z_0)/2} - e^{-\tilde{x}_i(Y)(z_0)/2}} \times \prod_{j=1}^s \frac{y^{-\frac{1}{2}} \cdot e^{\tilde{v}_j(Y)(z_0)/2} + y^{\frac{1}{2}} \cdot e^{-\tilde{v}_j(Y)(z_0)/2}}{y^{-\frac{1}{2}} + y^{\frac{1}{2}}} \cdot r_0^{(\epsilon)} \cdot e^{\frac{k}{N} \cdot |\tilde{v}(Y)|(z_0)} \right), [Y] \right\rangle.$$

A calculation shows that $r_0^{(\epsilon)}$ is the coefficient of q^0 in the Fourier-Taylor expansion of

$$\left(\prod_{k=1}^t \frac{\Phi(\tau, \tilde{w}_k(Y)(z_0)/(2\pi i) + \epsilon)}{\Phi(\tau, \epsilon)} \right) \cdot D_Y^{(\epsilon)}(z_0).$$

Comparing the factors of the last identity with $A_{\alpha,k}^{(\epsilon)}(Y)$ of Definition 4.2, Part 2, we conclude

$$a(Y)(\lambda_0) = \left\langle f_Y^*(A_{\alpha,k}^{(\epsilon)}(Y)_0(z_0)), [Y] \right\rangle,$$

where $A_{\alpha,k}^{(\epsilon)}(Y)_0$ is the coefficient of q^0 in $A_{\alpha,k}^{(\epsilon)}(Y)$. This completes the proof for the restricted statement. \blacksquare

Next we describe the evaluation on the fundamental cycle $[Y]$ of Y in terms of differential operators. Let \mathcal{A} be any \mathbb{C} -algebra, let

$$x_P \in H^{**}(BH'_P; \mathcal{A}), x_V \in H^{**}(BH'_V; \mathcal{A}), x_W \in H^{**}(BH'_W; \mathcal{A})$$

and regard $x := x_P \otimes x_V \otimes x_W$ as an element of $\mathcal{A}[[x_1, \dots, x_m, x_c, v_1, \dots, v_s, w_1, \dots, w_t]]$ using the cohomology of the classifying spaces of the maximal tori and the Künneth formula. Then

$$\langle f_Y^*(x), [Y] \rangle = \mathcal{P}_Y(x)(\xi = 0, x_c = 0),$$

where \mathcal{P}_Y is a homogeneous polynomial in $\mathbb{Q}[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_c}, \frac{\partial}{\partial v_j}, \frac{\partial}{\partial w_k}]$ of degree $\dim(Y)$ (we assign to $\frac{\partial}{\partial x_i}$, $\frac{\partial}{\partial x_c}$, $\frac{\partial}{\partial v_j}$, $\frac{\partial}{\partial w_k}$ the degree 2). This polynomial only depends on the mixed characteristic numbers of P' , P'_V and P'_W on Y but does not depend on the element x and the \mathbb{C} -algebra \mathcal{A} (for details we refer to [5], Proposition 7.7).

In particular, $\text{ind}_{\alpha,k}^{(\epsilon)}(M)_{S^1}(\lambda_0) = \sum_Y \mathcal{P}_Y \left(A_{\alpha,k}^{(\epsilon)}(Y)(z_0) \right) (\xi = 0)$. Since $A_{\alpha,k}^{(\epsilon)}(Y)(z)$ converges normally to $F_{\alpha,k}^{(\epsilon)}(Y)(\tau, z, \frac{\xi}{2\pi i})$ Proposition 4.3 gives the

Corollary 4.4. For a fixed topological generator $\lambda_0 = e^{2\pi i \cdot z_0} \in S^1$, $z_0 \in (-\frac{1}{m_{max}}, \frac{1}{m_{max}})$, the power series

$$ind_{\alpha,k}^{(\epsilon)}(M)_{S^1}(\lambda_0)$$

converges normally on $B = \{q \in \mathbb{C} \mid |q| < 1\}$ to

$$\sum_{Y \in \mathcal{Y}} \mathcal{P}_Y \left(F_{\alpha,k}^{(\epsilon)}(Y)(\tau, z_0, \frac{\xi}{2\pi i}) \right) (\xi = 0).$$

■

The next step will be to show that the power series of equivariant twisted $Spin^c$ -indices $ind_{\alpha,k}^{(\epsilon)}(M)_{S^1}(\lambda)$ converges normally on $(q, \lambda) \in B \times S^1 \subset B \times \mathbb{C}$. We need the following lemma which I learned from Thomas Berger and which is proven for example in [5], Lemma 7.13 or [7], Proposition 3.10.

Lemma 4.5. Let U be an open subset of $\mathbb{C} \times \mathbb{C}$. Consider a series $c = \sum_{n=0}^{\infty} c_n$ of holomorphic functions c_n on U such that c converges normally on $U' = \{(q, \lambda) \in U \mid \lambda \neq \mu\}$ for some $\mu \in \mathbb{C}$. Then c converges normally on all of U . ■

Proposition 4.6. (No Poles on S^1) The power series of equivariant twisted $Spin^c$ -indices

$$ind_{\alpha,k}^{(\epsilon)}(M)_{S^1}(\lambda)$$

converges normally on $(q^{\frac{1}{2N}}, \lambda) \in B \times S^1 \subset B \times \mathbb{C}$ (if $(\alpha, k) = 0$ we fix any natural number N). For $q^{\frac{1}{2N}} = e^{2\pi i \frac{\tau}{2N}}$ and $\lambda = e^{2\pi i \cdot z}$ the limit function is a holomorphic extension of

$$\sum_{Y \in \mathcal{Y}} \mathcal{P}_Y \left(F_{\alpha,k}^{(\epsilon)}(Y)(\tau, z, \frac{\xi}{2\pi i}) \right) (\xi = 0)$$

to $(\tau, z) \in \mathcal{H} \times \mathbb{R} \subset \mathcal{H} \times \mathbb{C}$.

Proof: Since $ind_{\alpha,k}^{(\epsilon)}(M)_{S^1}$ is a $q^{\frac{1}{2N}}$ -power series of S^1 -equivariant twisted $Spin^c$ -indices over \mathbb{C} each coefficient is an element of $R(S^1) \otimes_{\mathbb{Z}} \mathbb{C}$ which we have identified with its character in $\mathbb{C}[\lambda, \lambda^{-1}]$. For the following argument it is convenient to regard $\mathbb{C}[\lambda, \lambda^{-1}]$ as a subring of $\mathbb{C}[\lambda^{\frac{1}{2}}, \lambda^{-\frac{1}{2}}]$. Then

$$ind_{\alpha,k}^{(\epsilon)}(M)_{S^1}(\lambda) = \sum_{n=0}^{\infty} a_n(\lambda^{\frac{1}{2}}) q^{\frac{n}{2N}},$$

where $a_n(\lambda^{\frac{1}{2}}) \in \mathbb{C}[\lambda^{\frac{1}{2}}, \lambda^{-\frac{1}{2}}]$ is even. Now $\sum_Y \mathcal{P}_Y \left(F_{\alpha,k}^{(\epsilon)}(Y)(\tau, z, \frac{\xi}{2\pi i}) \right) (\xi = 0)$ is equal to a finite \mathbb{C} -linear combination of coefficients of the Taylor expansion of $F_{\alpha,k}^{(\epsilon)}(Y)(\tau, z, \frac{\xi}{2\pi i})$ with respect to $\xi = 0$. It follows from Definition 4.2 that for some natural number r the product of

$$C := \prod_{Y \in \mathcal{Y}} \left(\prod_{m_i(Y) \neq 0} (e^{2\pi i \cdot m_i(Y) \frac{\xi}{2}} - e^{-2\pi i \cdot m_i(Y) \frac{\xi}{2}})^r \right)$$

and any such Taylor coefficient is given by the limit value of a power series in $\mathbb{C}[\lambda^{\frac{1}{2}}, \lambda^{-\frac{1}{2}}][[q^{\frac{1}{2N}}]]$, which converges normally on $(q^{\frac{1}{2N}}, \lambda^{\frac{1}{2}}) \in B \times S^1 \subset B \times \mathbb{C}$ for $q^{\frac{1}{2N}} = e^{2\pi i \frac{\tau}{2N}}$ and $\lambda^{\frac{1}{2}} = e^{2\pi i \frac{z}{2}}$. Thus, there is a

power series $\sum_{n=0}^{\infty} b_n(\lambda^{\frac{1}{2}})q^{\frac{n}{2N}}$, such that $C \cdot b_n(\lambda^{\frac{1}{2}})$ is holomorphic on S^1 and $\sum_{n=0}^{\infty} C \cdot b_n(\lambda^{\frac{1}{2}})q^{\frac{n}{2N}}$ converges normally on $B \times S^1$ to

$$C \cdot \sum_{Y \in \mathcal{Y}} \mathcal{P}_Y \left(F_{\alpha,k}^{(\epsilon)}(Y)(\tau, z, \frac{\xi}{2\pi i}) \right) (\xi = 0).$$

By Corollary 4.4 the series $\sum_{n=0}^{\infty} a_n(\lambda_0)q^{\frac{n}{2N}}$ converges normally on B to

$$\sum_{Y \in \mathcal{Y}} \mathcal{P}_Y \left(F_{\alpha,k}^{(\epsilon)}(Y)(\tau, z, \frac{\xi}{2\pi i}) \right) (\xi = 0)$$

for any topological generator $\lambda_0 = e^{2\pi i \cdot z_0} \in S^1$, $z_0 \in (-\frac{1}{2m_{max}}, \frac{1}{2m_{max}})$. This implies that the functions $a_n(\lambda^{\frac{1}{2}})$ and $b_n(\lambda^{\frac{1}{2}})$ agree on a set with cluster point for any $n \in \mathbb{N}_0$. Thus $a_n(\lambda^{\frac{1}{2}})$ is a holomorphic extension of $b_n(\lambda^{\frac{1}{2}})$ to \mathbb{C}^* . Now Lemma 4.5 can be applied to $c_n := b_n \cdot q^{\frac{n}{2N}}$ and any $\mu \in S^1$. It follows that $ind_{\alpha,k}^{(\epsilon)}(M)_{S^1}(\lambda) = \sum_{n=0}^{\infty} a_n(\lambda)q^{\frac{n}{2N}}$ and $\sum_{n=0}^{\infty} b_n(\lambda)q^{\frac{n}{2N}}$ converge normally on $B \times S^1$ to $\sum_Y \mathcal{P}_Y \left(F_{\alpha,k}^{(\epsilon)}(Y)(\tau, z, \frac{\xi}{2\pi i}) \right) (\xi = 0)$. This completes the proof. \blacksquare

Recall from Definition 4.1 that $\varphi_N^c(M; V, W)_{S^1} = ind_{\frac{1}{N},0}^{(\frac{1}{2})}(M)_{S^1}$ and $\varphi_0^c(M; V, W)_{S^1} = ind_{0,0}^{(\frac{1}{2})}(M)_{S^1}$. By the last proposition these series converge to functions which are holomorphic on $\mathcal{H} \times \mathbb{R}$. Next we will study the modularity properties of $F_{\frac{1}{N},0}^{(\frac{1}{2})}(Y)(\tau, z, \xi)$ and $F_{0,0}^{(\frac{1}{2})}(Y)(\tau, z, \xi)$ under the assumption on $p_1(V + W - TM)_{S^1}$ given in Theorem 3.2. This will lead to modularity properties of $\varphi_N^c(M; V, W)_{S^1}$ and $\varphi_0^c(M; V, W)_{S^1}$ (see Proposition 4.9). Let

$$p_{(2,2)}(Y) = \sum_{j=1}^s s_j(Y) \cdot v_j + \sum_{k=1}^t t_k(Y) \cdot w_k - \sum_{i=1}^m m_i(Y) \cdot x_i \text{ and } p_{(0,4)} = \sum_{j=1}^s v_j^2 + \sum_{k=1}^t w_k^2 - \sum_{i=1}^m x_i^2.$$

Note that the condition on the equivariant first Pontrjagin class implies $f_Y^*(p_{(2,2)}(Y)) = f_Y^*(p_{(0,4)}) = 0$ and implies that $\sum_{j=1}^s s_j(Y)^2 + \sum_{k=1}^t t_k(Y)^2 - \sum_{i=1}^m m_i(Y)^2$ is equal to \mathcal{I} .

In the next Proposition we give transformation properties for $F_{\frac{1}{N},0}^{(\frac{1}{2})}(Y)$ and $F_{0,0}^{(\frac{1}{2})}(Y)$ which we derive from the Weierstrass' Φ -function. Recall that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ acts on the upper half plane \mathcal{H} by $A(\tau) = (a\tau + b)/(c\tau + d)$. Let $\Gamma_1(N) := \{A \in SL_2(\mathbb{Z}) \mid A \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{N}\}$.

We need the following notation: For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and $N \geq 2$ an integer define $\epsilon_c, \epsilon_d \in \{0, 1\}$ and $\alpha_c, \alpha_d \in \{0, \dots, N-1\}$ by $(\epsilon_c, \epsilon_d) \equiv (c, d) \pmod{2}$ and $(\alpha_c, \alpha_d) \equiv (c, d) \pmod{N}$. Let $\epsilon := (\epsilon_c \tau + \epsilon_d)/2$ and $\alpha := (\alpha_c \tau + \alpha_d)/N$. We are now in the position to give the modularity properties of $F_{\frac{1}{N},0}^{(\frac{1}{2})}(Y)$ and $F_{0,0}^{(\frac{1}{2})}(Y)$.

Proposition 4.7. 1. Let either $(\alpha, k) = 0$ or $(\alpha, k) = (\frac{1}{N}, 0)$, where N is an integer, $N \geq 2$. For any $(a, b) \in \mathbb{Z}^2$

$$F_{\alpha,k}^{(\frac{1}{2})}(Y)(\tau, z + a\tau + b, \xi) = F_{\alpha,k}^{(\frac{1}{2})}(Y)(\tau, z, \xi) \cdot e^{-2\pi i(a \cdot p_{(2,2)}(Y) + \mathcal{I}(\frac{a^2\tau}{2} + az))}.$$

2. Let $N \geq 2$ be an integer. For any $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$

$$F_{\frac{1}{N},0}^{(\frac{1}{2})}(Y)(A\tau, \frac{z}{c\tau + d}, \xi) = F_{\alpha,-\alpha_c}^{(\epsilon)}(Y)(\tau, z, \xi \cdot (c\tau + d)) \cdot (c\tau + d)^{m - \dim(Y)/2} \times$$

$$e^{2\pi i \cdot c \cdot \frac{\mathcal{I} \cdot z^2}{2(c\tau+d)}} \cdot e^{2\pi i \cdot c \left(\frac{p(0,A) \cdot (c\tau+d)}{2} + p(2,2)(Y) \cdot z \right)}.$$

3. For any $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$

$$F_{0,0}^{(\frac{1}{2})}(Y)(A\tau, \frac{z}{c\tau+d}, \xi) = F_{0,0}^{(\epsilon)}(Y)(\tau, z, \xi \cdot (c\tau+d)) \cdot (c\tau+d)^{m-s-\dim(Y)/2} \times \\ e^{2\pi i \cdot c \cdot \frac{\mathcal{I} \cdot z^2}{2(c\tau+d)}} \cdot e^{2\pi i \cdot c \left(\frac{p(0,A) \cdot (c\tau+d)}{2} + p(2,2)(Y) \cdot z \right)}.$$

Proof: Ad 1: The Weierstrass' Φ -function satisfies the following transformation property:

$$\Phi(\tau, z + a\tau + b) = \Phi(\tau, z) \cdot e^{-2\pi i \cdot \frac{a^2\tau + 2a \cdot z}{2}} \cdot (-1)^{a+b} \quad (1)$$

We apply this identity to each factor appearing in the definition of $F_{\frac{1}{N},0}^{(\frac{1}{2})}(Y)$ and $F_{0,0}^{(\frac{1}{2})}(Y)$. Since all rotation numbers are even the statement follows by direct calculation.

Ad 2: For any variable u let $u' := u \cdot (c\tau + d)$. The Weierstrass' Φ -function satisfies the following transformation property:

$$\Phi(A\tau, \frac{z}{c\tau+d}) = \Phi(\tau, z) \cdot (c\tau+d)^{-1} \cdot e^{2\pi i \cdot \frac{c}{2} \cdot \frac{z^2}{c\tau+d}} \quad (2)$$

We apply this identity to each factor appearing in the definition of $F_{\frac{1}{N},0}^{(\frac{1}{2})}(Y)$ and obtain

$$F_{\frac{1}{N},0}^{(\frac{1}{2})}(Y)(A\tau, \frac{z}{c\tau+d}, \xi) = \left(\prod_{m_i(Y)=0} \frac{2\pi i \cdot x'_i}{\Phi(\tau, x'_i)} \right) \cdot \left(\prod_{m_i(Y) \neq 0} \frac{1}{\Phi(\tau, x'_i + m_i(Y) \cdot z)} \right) \times \\ \left(\prod_{j=1}^s \frac{\Phi(\tau, v'_j + s_j(Y) \cdot z - \frac{c\tau+d}{N})}{\Phi(\tau, -\frac{c\tau+d}{N})} \right) \cdot \left(\prod_{k=1}^t \frac{\Phi(\tau, w'_k + t_k(Y) \cdot z + \frac{c\tau+d}{2})}{\Phi(\tau, \frac{c\tau+d}{2})} \right) \times \\ (c\tau+d)^{m-\dim Y/2} \cdot e^{2\pi i \cdot c \cdot \frac{\mathcal{I} \cdot z^2}{2(c\tau+d)}} \cdot e^{2\pi i \cdot c \left(\frac{p(0,A)}{2} + p(2,2)(Y) \cdot z \right)} \times \\ e^{2\pi i \cdot c \left(-\frac{1}{N} \sum (v'_j + s_j(Y) \cdot z) + \frac{1}{2} \sum (w'_k + t_k(Y) \cdot z) \right)}.$$

From the definition of α and ϵ follows directly that $\frac{c\tau+d}{N} = \alpha + a\tau + b$ and $\frac{c\tau+d}{2} = \epsilon + \hat{a}\tau + \hat{b}$, where a, b, \hat{a}, \hat{b} are integers. Using equation (1) one obtains the identity given in the second statement. The proof of the third statement is analogous. \blacksquare

Next we recall the transformation properties of Jacobi forms. For an introduction to the theory we refer to [8]. Any subgroup $\Gamma \leq SL_2(\mathbb{Z})$ of finite index acts on the lattice $L := \mathbb{Z} \times \mathbb{Z}$ by automorphisms, where the action of $A \in \Gamma$ is defined by matrix multiplication, $(\alpha, \beta) \mapsto (\alpha, \beta)A$. Let $\Gamma \ltimes L$ be the corresponding semi-direct product, i.e. the multiplication in $\Gamma \ltimes L$ is given by $(A, (\alpha, \beta)) \cdot (B, (\gamma, \delta)) := (A \cdot B, (\alpha, \beta)B + (\gamma, \delta))$.

Let f be a meromorphic function on $\mathcal{H} \times \mathbb{C}$. Let $(A, X) \in \Gamma \ltimes L$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $X = (\alpha, \beta)$. For fixed $k, I \in \mathbb{Z}$ one verifies that the assignments

$$f|[A]_{k,I}(\tau, z) := f(A\tau, \frac{z}{c\tau+d}) \cdot (c\tau+d)^{-k} \cdot e^{-2\pi i \cdot I \cdot \frac{c \cdot z^2}{c\tau+d}}, \\ f|[X]_I(\tau, z) := f(\tau, z + \alpha\tau + \beta) \cdot e^{2\pi i \cdot I(\alpha^2 \cdot \tau + 2\alpha \cdot z)}$$

define an action of $\Gamma \times L$ on the field of meromorphic functions on $\mathcal{H} \times \mathbb{C}$. Jacobi forms of weight k and index I for Γ are holomorphic functions on $\mathcal{H} \times \mathbb{C}$ which are fixed under the above action and satisfy certain conditions in the cusps (cf. [8]).

We say a meromorphic function f on $\mathcal{H} \times \mathbb{C}$ **transforms like a Jacobi form of weight k and index I for Γ** if and only if f is a fixed point under the action of $\Gamma \times L$ for k and I , i.e. if and only if

$$f|[A]_{k,I}(\tau, z) = f(\tau, z) \quad \text{and} \quad f|[X]_I(\tau, z) = f(\tau, z)$$

for any $(A, X) \in \Gamma \times L$. This is equivalent to

$$\begin{aligned} f\left(A\tau, \frac{z}{c\tau+d}\right) &= f(\tau, z) \cdot (c\tau + d)^k \cdot e^{2\pi i \cdot I \cdot \frac{c \cdot z^2}{c\tau+d}}, \\ f(\tau, z + \alpha\tau + \beta) &= f(\tau, z) \cdot e^{-2\pi i \cdot I(\alpha^2 \cdot \tau + 2\alpha \cdot z)}. \end{aligned}$$

Definition 4.8. *Let*

$$F_N(\tau, z) := \sum_{Y \in \mathcal{Y}} \mathcal{P}_Y \left(F_{\frac{1}{N}, 0}^{(\frac{1}{2})}(Y)(\tau, z, \frac{\xi}{2\pi i}) \right) (\xi = 0)$$

and

$$F_0(\tau, z) := \sum_{Y \in \mathcal{Y}} \mathcal{P}_Y \left(F_{0,0}^{(\frac{1}{2})}(Y)(\tau, z, \frac{\xi}{2\pi i}) \right) (\xi = 0).$$

By Corollary 4.4 $\varphi_N^c(M; V, W)$ and $\varphi_0^c(M; V, W)$ converge to F_N and F_0 , respectively, for any topological generator $\lambda_0 = e^{2\pi i \cdot z_0} \in S^1$, $z_0 \in (-\frac{1}{m_{max}}, \frac{1}{m_{max}})$. By Proposition 4.6 these functions are holomorphic on $\mathcal{H} \times \mathbb{R}$. The first two parts of the next proposition show that $F_N(\tau, z)$ (resp. $F_0(\tau, z)$) transforms like a Jacobi form of weight m (resp. $m - s$) and index $\mathcal{I}/2$ for $\Gamma_1(N)$ (resp. $\Gamma_1(2)$).

Proposition 4.9. *Let $N \geq 2$ be an integer.*

1. *For any $X = (a, b) \in \mathbb{Z}^2$*

$$F_N(\tau, z)|[X]_{\mathcal{I}/2} = F_N(\tau, z) \quad \text{and} \quad F_0(\tau, z)|[X]_{\mathcal{I}/2} = F_0(\tau, z).$$

2. *For any $A \in \Gamma_1(N)$ (resp. $A \in \Gamma_1(2)$)*

$$F_N(\tau, z)|[A]_{m, \mathcal{I}/2} = F_N(\tau, z) \quad (\text{resp. } F_0(\tau, z)|[A]_{m-s, \mathcal{I}/2} = F_0(\tau, z)).$$

3. *For any $A \in SL_2(\mathbb{Z})$ the function $F_N(\tau, z)|[A]_{m, \mathcal{I}/2}$ (resp. $F_0(\tau, z)|[A]_{m-s, \mathcal{I}/2}$) has a holomorphic extension to $\mathcal{H} \times \mathbb{R}$ which is given by the normally convergent series of twisted $Spin^c$ -indices $ind_{\alpha, -\alpha_c}^{(\epsilon)}(M)_{S^1}(\lambda)$ (resp. $ind_{0,0}^{(\epsilon)}(M)_{S^1}(\lambda)$), where $\lambda = e^{2\pi i \cdot z}$ and $q = e^{2\pi i \cdot \tau}$.*

Proof: To show the first statement we use Proposition 4.7, Part 1, which specializes to

$$F_{\frac{1}{N}, 0}^{(\frac{1}{2})}(Y)(\tau, z + a\tau + b, \frac{\xi}{2\pi i}) = F_{\frac{1}{N}, 0}^{(\frac{1}{2})}(Y)(\tau, z, \frac{\xi}{2\pi i}) \cdot e^{-2\pi i \cdot (a \cdot \frac{p(2,2)(Y)}{2\pi i} + \mathcal{I}(\frac{a^2\tau}{2} + az))}.$$

Since \mathcal{P}_Y commutes with $z \mapsto z + a\tau + b$ and $f_Y^*(p(2,2)(Y)) = 0$ we get

$$F_N(\tau, z + a\tau + b) = \sum_{Y \in \mathcal{Y}} \mathcal{P}_Y \left(F_{\frac{1}{N}, 0}^{(\frac{1}{2})}(Y)(\tau, z + a\tau + b, \frac{\xi}{2\pi i}) \right) (\xi = 0) =$$

$$\begin{aligned} & \sum_{Y \in \mathcal{Y}} \mathcal{P}_Y \left(F_{\frac{1}{N}, 0}^{(\frac{1}{2})}(Y)(\tau, z, \frac{\xi}{2\pi i}) \cdot e^{-2\pi i \cdot (a \cdot \frac{p(2,2)(Y)}{2\pi i} + \mathcal{I}(\frac{a^2\tau}{2} + az))} \right) (\xi = 0) = \\ & \sum_{Y \in \mathcal{Y}} \mathcal{P}_Y \left(F_{\frac{1}{N}, 0}^{(\frac{1}{2})}(Y)(\tau, z, \frac{\xi}{2\pi i}) \right) (\xi = 0) \cdot e^{-2\pi i \cdot \mathcal{I}(\frac{a^2\tau}{2} + az)} = \\ & F_N(\tau, z) \cdot e^{-2\pi i \cdot \mathcal{I}(\frac{a^2\tau}{2} + az)}. \end{aligned}$$

Thus $F_N(\tau, z)|[X]_{\mathcal{I}/2} = F_N(\tau, z)$ which proves the first statement for $F_N(\tau, z)$. The calculation for $F_0(\tau, z)$ is analogous.

Next we show the other two statements. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and let $\epsilon = (\epsilon_c\tau + \epsilon_d)/2$ and $\alpha = (\alpha_c\tau + \alpha_d)/N$ be as given before Proposition 4.7. For any variable u let $u' := (c\tau + d)u$.

Recall that \mathcal{P}_Y is a homogeneous polynomial in $\mathbb{Q}[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_c}, \frac{\partial}{\partial v_j}, \frac{\partial}{\partial w_k}]$ of degree $\dim(Y)$, where $\frac{\partial}{\partial x_i}$, $\frac{\partial}{\partial x_c}$, $\frac{\partial}{\partial v_j}$ and $\frac{\partial}{\partial w_k}$ have degree 2. Also recall that the number of $m_i(Y)$ which are zero is equal to $\dim(Y)/2$. We have the following identities:

$$\begin{aligned} F_N(A\tau, \frac{z}{c\tau + d}) &= \sum_{Y \in \mathcal{Y}} \mathcal{P}_Y \left(F_{\frac{1}{N}, 0}^{(\frac{1}{2})}(Y)(A\tau, \frac{z}{c\tau + d}, \frac{\xi}{2\pi i}) \right) (\xi = 0) = \\ & \sum_{Y \in \mathcal{Y}} \mathcal{P}_Y \left(F_{\alpha, -\alpha_c}^{(\epsilon)}(Y)(\tau, z, \frac{\xi'}{2\pi i}) \right) (\xi = 0) \cdot e^{2\pi i \cdot c \cdot \frac{\mathcal{I} \cdot z^2}{2(c\tau + d)}} \cdot (c\tau + d)^{m - \dim(Y)/2} = \\ & \sum_{Y \in \mathcal{Y}} \mathcal{P}_Y \left(F_{\alpha, -\alpha_c}^{(\epsilon)}(Y)(\tau, z, \frac{\xi}{2\pi i}) \right) (\xi = 0) \cdot e^{2\pi i \cdot c \cdot \frac{\mathcal{I} \cdot z^2}{2(c\tau + d)}} \cdot (c\tau + d)^m. \end{aligned}$$

Here the first identity holds since \mathcal{P}_Y commutes with $(\tau, z) \mapsto (A\tau, \frac{z}{c\tau + d})$, the second identity follows from Proposition 4.7, Part 2, using $f_Y^*(p_{(0,4)}) = f_Y^*(p_{(2,2)}) = 0$ and the last equation uses the fact that \mathcal{P}_Y has degree $\dim(Y)$. Using Proposition 4.7, Part 3, a similar calculation gives

$$F_0(A\tau, \frac{z}{c\tau + d}) = \sum_{Y \in \mathcal{Y}} \mathcal{P}_Y \left(F_{0,0}^{(\epsilon)}(Y)(\tau, z, \frac{\xi}{2\pi i}) \right) (\xi = 0) \cdot e^{2\pi i \cdot c \cdot \frac{\mathcal{I} \cdot z^2}{2(c\tau + d)}} \cdot (c\tau + d)^{m-s}.$$

Thus $F_N(\tau, z)|[A]_{m, \mathcal{I}/2}$ and $F_0(\tau, z)|[A]_{m-s, \mathcal{I}/2}$ are equal to $\sum_Y \mathcal{P}_Y \left(F_{\alpha, -\alpha_c}^{(\epsilon)}(Y)(\tau, z, \frac{\xi}{2\pi i}) \right) (\xi = 0)$ and $\sum_Y \mathcal{P}_Y \left(F_{0,0}^{(\epsilon)}(Y)(\tau, z, \frac{\xi}{2\pi i}) \right) (\xi = 0)$, respectively. Note that Part 2 of the statement is a special case of these calculations. By Proposition 4.6 the q -power series of twisted $Spin^c$ -indices $ind_{\alpha, -\alpha_c}^{(\epsilon)}(M)_{S^1}$ and $ind_{0,0}^{(\epsilon)}(M)_{S^1}$ converge normally on $(q^{\frac{1}{2N}}, \lambda) \in B \times S^1 \subset B \times \mathbb{C}$ for $q^{\frac{1}{2N}} = e^{2\pi i \frac{\tau}{2N}}$ and $\lambda = e^{2\pi i \cdot z}$ to $F_N(\tau, z)|[A]_{m, \mathcal{I}/2}$ and $F_0(\tau, z)|[A]_{m-s, \mathcal{I}/2}$. Thus $F_N(\tau, z)|[A]_{m, \mathcal{I}/2}$ and $F_0(\tau, z)|[A]_{m-s, \mathcal{I}/2}$ admit holomorphic extensions to $\mathcal{H} \times \mathbb{R}$. \blacksquare

Proof of Theorem 3.2: We first show that $F_N(\tau, z)$ and $F_0(\tau, z)$ are holomorphic on $\mathcal{H} \times \mathbb{C}$. Recall that the Weierstrass' Φ -function $\Phi(\tau, z)$ is holomorphic on $\mathcal{H} \times \mathbb{C}$. Let $\tau \in \mathcal{H}$ be fixed. The set of zeros of $\Phi(\tau, z)$ is equal to $\mathbb{Z} \cdot \tau + \mathbb{Z}$. It follows from Definition 4.2 that the poles of any Taylor coefficient of $F_{Y, \frac{1}{N}}^{(\frac{1}{2})}(\tau, z, \xi)$ or $F_{Y, 0}^{(\frac{1}{2})}(\tau, z, \xi)$ with respect to $\xi = 0$ are in $\mathbb{Q} \cdot \tau + \mathbb{Q}$. From Definition 4.8 follows that $F_N(\tau, z)$ and $F_0(\tau, z)$ have only poles in $z \in \mathbb{Q} \cdot \tau + \mathbb{Q}$. Let $f(\tau, z)$ denote one of these functions and let $f_\tau(z) := f(\tau, z)$. For rational numbers α and β we want to show that $f_\tau(z)$ has no pole in $z = \alpha\tau + \beta$. We may choose $A \in SL_2(\mathbb{Z})$ such that $\frac{z}{c\tau + d} \in \mathbb{R}$. Since $f|[A]_{k, \mathcal{I}/2}$ ($k = m$ if $f = F_N$ and $k = m - s$ if $f = F_0$) has no poles on \mathbb{R} by Proposition 4.9, Part 3, the function f has no pole in $z = \alpha\tau + \beta$.

So $f(\tau, z)$ is meromorphic on $\mathcal{H} \times \mathbb{C}$ and for any $\tau \in \mathcal{H}$ the function $f_\tau(z)$ is holomorphic on \mathbb{C} . Thus $f(\tau, z)$ is holomorphic on $\mathcal{H} \times \mathbb{C}$.

Next recall from Proposition 4.6 and Definition 4.8 that the power series of equivariant twisted $Spin^c$ -indices $\varphi_N^c(M; V, W)_{S^1}(\lambda)$ converges normally on $B \times S^1$ to the function $F_N(\tau, z)$. By Proposition 4.9, Part 2, and the last argument $F_N(\tau, z)$ is a holomorphic Jacobi function of index $\mathcal{I}/2$. Now the statement for $\varphi_N^c(M; V, W)_{S^1}(\lambda)$ follows from general properties of Jacobi functions. In fact, for any given Jacobi function of index I which does not vanish identically the difference of the number of zeros and the number of poles (counted with multiplicities) in a fundamental domain of the action of the lattice $\mathbb{Z} + \mathbb{Z} \cdot \tau$ on \mathbb{C} is equal to $2 \cdot I$ (cf. [8], Theorem 1.2). Thus, if \mathcal{I} is negative $\varphi_N^c(M; V, W)_{S^1}(\lambda)$ and $F_N(\tau, z)$ have to vanish identically. If \mathcal{I} is zero it follows from the transformation properties given in Proposition 4.9, Part 1, that for fixed $\tau \in \mathcal{H}$ the function $F_N(\tau, z)$ is elliptic in the variable z with respect to the lattice $\mathbb{Z} + \mathbb{Z} \cdot \tau$. Since $F_N(\tau, z)$ is holomorphic $F_N(\tau, z)$ is constant in z by the theorem of Liouville. Since $\varphi_N^c(M; V, W)_{S^1}(\lambda)$ converges normally to $F_N(\tau, z)$ each coefficient of this power series is constant in λ , i.e. $\varphi_N^c(M; V, W)_{S^1}$ is rigid. The argument for $\varphi_0^c(M; V, W)_{S^1}(\lambda)$ is analogous.

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References

- [1] M.F. Atiyah and I.M. Singer: The index of elliptic operators: III, *Ann. of Math.* **87** (1968), 546-604
- [2] M.F. Atiyah and F. Hirzebruch: Spin-Manifolds and Group Actions, in: *Essays on Topology and Related Topics. Memoires dédiés à Georges de Rham*, Springer (1970), 18-28
- [3] R. Bott and C.H. Taubes: On the rigidity theorems of Witten, *J. of Amer. Math. Soc.* **2** (1989), 137-186
- [4] A. Dessai: *The Witten genus and S^3 -actions on manifolds*, preprint 94/6, University of Mainz (1994)
- [5] A. Dessai: *Rigidity Theorems for $Spin^c$ -Manifolds and Applications*, doctoral thesis, University of Mainz (1996), published by Shaker (1997)
- [6] A. Dessai: *$Spin^c$ -Manifolds with $Pin(2)$ -Action*, submitted to Math. Ann.
- [7] A. Dessai and R. Jung: Some remarks on the rigidity theorem for elliptic genera, *Trans. Amer. Math. Soc.* **350** (1998), 4195-4220
- [8] M. Eichler and D. Zagier: *The Theory of Jacobi Forms*, Progress in Mathematics, Vol 55, Birkhäuser (1985)
- [9] A. Hattori: $Spin^c$ -Structures and S^1 -Actions, *Invent. Math.* **48** (1978), 7-31
- [10] F. Hirzebruch: Elliptic genera of level N for complex manifolds, in: K. Bleuler and M. Werner (Eds.): *Differential Geometrical Methods in Theoretical Physics (Como 1987)*, NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci., 250, Kluwer (1988)
- [11] F. Hirzebruch, Th. Berger and R. Jung: *Manifolds and Modular Forms*, Aspects of Mathematics, Vol. E20, Vieweg (1992)
- [12] F. Hirzebruch and P. Slodowy: Elliptic Genera, Involutions and Homogeneous Spin Manifolds, *Geom. Dedicata* **35** (1990), 309-343

[†]EU-project CHRX-CT94-0441.

- [13] G. Höhn: *Komplexe elliptische Geschlechter und S^1 -äquivariante Kobordismustheorie*, Diplomarbeit, Bonn 1991
- [14] I.M. Kričever: Generalized Elliptic Genera and Baker-Akhiezer Functions, *Math. Notes* **47** (1990), 132-142
- [15] M. Kreck and S. Stolz: $\mathbb{H}P^2$ -bundles and elliptic homology, *Acta Math.* **171** (1993), 231-261
- [16] P.S. Landweber, Elliptic Cohomology and Modular Forms, in: [17], 55-68
- [17] P.S. Landweber (Ed.): *Elliptic Curves and Modular Forms in Algebraic Topology, Proceedings Princeton 1986*, Lecture Notes in Mathematics, 1326, Springer (1988)
- [18] P.S. Landweber, R.E. Stong: Circle actions on spin manifolds and characteristic numbers, *Topology* **27** (1988), 145-161
- [19] P.S. Landweber, D.C. Ravenel and R.E. Stong: Periodic Cohomology Theories defined by Elliptic Curves, *Contemporary Mathematics* **181** (1995), 317-337
- [20] K. Liu: *On Elliptic Genera, Theta-Functions and Elliptic Modular Surfaces*, preprint (1992)
- [21] K. Liu: On modular invariance and rigidity theorems, *J. Diff. Geo.* **41** (1995), 343-396
- [22] K.H. Mayer and R.L.E. Schwarzenberger: Lefschetz formulae for modest vector bundles, *Proc. Camb. Phil. Soc.* **73** (1973), 439-453
- [23] S. Ochanine: *Genres Elliptiques Équivariants*, in [17], 107-122
- [24] S. Ochanine: Sur les genres multiplicatifs définis par des intégrales elliptiques, *Topology* **26** (1987), 143-151
- [25] T. Petrie: Smooth S^1 -actions on homotopy complex projective spaces and related topics, *Bull. Math. Soc.* **78** (1972), 105-153
- [26] S. Stolz: A conjecture concerning positive Ricci curvature and the Witten genus, *Math. Ann.* **304** (1996), 785-800
- [27] C.H. Taubes: S^1 Actions and Elliptic Genera, *Comm. Math. Phys.* **122** (1989), 455-526
- [28] E. Witten: The Index of the Dirac Operator in Loop Space, in: [17], 161-181

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