# Obstructions to positive curvature and symmetry

Anand Dessai

*Mathematisches Institut, Universität Münster, 48149 Münster, Germany*
*dessai@math.uni-muenster.de*

**Abstract**

We discuss new obstructions to positive sectional curvature and symmetry. The main result asserts that the index of the Dirac operator twisted with the tangent bundle vanishes on a 2-connected manifold of dimension $\neq 8$ if the manifold admits a metric of positive sectional curvature and isometric effective $S^1$-action. The proof relies on the rigidity theorem for elliptic genera and properties of totally geodesic submanifolds.

## 1 Introduction

An important application of index theory in Riemannian geometry is in the study of manifolds of positive scalar curvature. Soon after Atiyah and Singer proved the index theorem Lichnerowicz used a Bochner type formula to show that for a closed Riemannian *Spin*-manifold $M$ the index of the Dirac operator $\hat{A}(M)$ vanishes if $M$ has positive scalar curvature.

Whereas the relation between index theory and positive scalar curvature (at least for simply connected manifolds of dimension $\geq 5$) is well understood [17,27,32,35,38,41] possible relations to stronger curvature conditions such as positive Ricci curvature or positive sectional curvature remain obscure (see however Stolz' conjecture in [42]).

In this paper we give index theoretical obstructions to the existence of positive sectional curvature (positive curvature for short) on *Spin*-manifolds under mild symmetry assumptions.

For a *Spin*-manifold $M$ let $\hat{A}(M, TM)$ denote the index of the Dirac operator twisted with the complexified tangent bundle (this operator is also known as the Rarita-Schwinger operator [50]). Our main result is the following

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**Theorem 1** *Let $M$ be a closed 2-connected manifold of dimension $\neq 8$. If $M$ admits a metric of positive curvature with effective isometric $S^1$-action then $\hat{A}(M, TM) = \hat{A}(M) = 0$.*

We note that the restriction on the dimension is necessary since $\hat{A}(M, TM)$ is non-zero for $M$ the quaternionic projective plane $\mathbb{H}P^2$. The curvature assumption cannot be weakened to positive scalar curvature. A simple example is given by the product of a 2-connected 8-dimensional manifold with $\hat{A} \neq 1$ and $\mathbb{H}P^2$ (see the discussion in Section 4). We expect that the theorem remains true if one drops the assumption on the connectivity. Partial results in this direction are discussed in Section 4 and Section 6.

For a 12-dimensional manifold $M$ Theorem 1 can be formulated in terms of the signature. In this dimension the signature satisfies

$$sign(M) = 8\,\hat{A}(M, TM) - 32\,\hat{A}(M).$$

This follows from a direct computation or by comparing the elliptic genus in different cusps.

**Corollary 2** *Let $M$ be a closed 2-connected 12-dimensional manifold. If $M$ admits a metric of positive curvature with effective isometric $S^1$-action then $sign(M) = 0$.* $\square$

Theorem 1 and its corollary hold under slightly weaker assumptions on the symmetry. It suffices to assume that the cyclic subgroup of $S^1$ of order 4 acts isometrically on $M$ (see Theorem 12). It would be interesting to understand how restrictive this symmetry assumption is. Looking at the literature one sees that all presently known examples of positively curved simply connected manifolds carry a metric of positive curvature with isometric $S^1$-action. Besides the biquotients found by Eschenburg [11,12] and Bazaikin [4], which are in dimension 6, 7 and 13, the other examples carry a homogeneous metric of positive curvature (the homogeneous examples were classified by Berger [6], Aloff, Wallach [1,46] and Bérard Bergery [5]).

To put Theorem 1 in perspective we briefly summarize the main obstructions to positive curvature presently known. Manifolds of positive curvature are classified in dimension $< 4$ [21]. In higher dimensions the only known obstructions are given by restrictions on the fundamental group [16,34,43], Gromov's Betti number theorem [16] and the obstructions to positive scalar curvature (coming from index theory and *Spin*-geometry, the minimal hypersurface method of Schoen-Yau or Seiberg-Witten theory). Further progress concerning obstructions and classification has been obtained for positively curved manifolds with large dimensional isometry group, large (discrete) symmetry rank or small cohomogeneity [13,18–20,28,37,39,40,45,47,48,51]. These results require that the dimension of the manifold is bounded from above by a constant depending on the symmetry. We like to stress that Theorem 1 has a quite different flavor since the dimension of the manifold can be arbitrarily large. Other results of this kind will be discussed in Section 6.

The main topological ingredient in the proof of Theorem 1 is the Bott-Taubes-Witten rigidity theorem [7,44,49] for the elliptic genus and its consequences for cyclic group actions (see Section 3). We recall that the (weight zero) elliptic genus $\Phi$ is a ring homomorphism from the oriented bordism ring to the ring of modular functions for $\Gamma_0(2)$ of weight zero. On the complex projective spaces $\mathbb{C}P^{2k}$ it is given by

$$\sum_{k\geq 0} \Phi(\mathbb{C}P^{2k})t^{2k} = (1 - 2\frac{\delta}{\sqrt{\epsilon}}t^2 + t^4)^{-1/2},$$

where $\delta$ and $\epsilon$ are certain modular forms of weight 2 and 4, respectively.

Following Witten [49] the elliptic genus $\Phi(M)$ of a *Spin*-manifold $M$ expands in one of the cusps of $\Gamma_0(2)$ as a series of twisted signatures which is best thought of as the index of a hypothetical signature operator on the free loop space of $M$. In a different cusp of $\Gamma_0(2)$ the elliptic genus expands as a series $\Phi_0(M)$ of twisted Dirac operators

$$\Phi_0(M) = q^{-\dim M/8} \cdot \hat{A}(M, \bigotimes_{n=2m+1>0} \Lambda_{-q^n}TM \otimes \bigotimes_{n=2m>0} S_{q^n}TM)$$

$$= q^{-\dim M/8} \cdot (\hat{A}(M) - \hat{A}(M,TM)\cdot q + \hat{A}(M, \Lambda^2 TM + TM)\cdot q^2 + \ldots).$$

Here $\hat{A}(M,E)$ denotes the index of the Dirac operator twisted with the complexification of the vector bundle $E$ (we refer to [25,31] and Section 3 for more information on the elliptic genus).

Theorem 1 says that for a 2-connected manifold $M$ of dimension $\neq 8$ the first two coefficients of $\Phi_0(M)$ vanish if $M$ admits a metric of positive curvature and isometric effective $S^1$-action. An important step in the proof is to relate $\Phi_0(M)$ to the action of finite cyclic subgroups of $S^1$. Using the rigidity theorem we derive the following result which might be of independent interest.

**Theorem 3** *Let $M$ be a closed Spin-manifold with smooth $S^1$-action and let $\sigma \in S^1$ be of order $o \geq 2$. If* codim $M^\sigma > 2o\cdot r$ *then the first $(r+1)$ coefficients of $\Phi_0(M)$ vanish.*

Here codim $M^\sigma$ denotes the minimal codimension of the connected components of $M^\sigma$ in $M$. If $\sigma$ is of order 2 the result is due to Hirzebruch and Slodowy [26] who showed that the elliptic genus of $M$ is equal to the elliptic genus of a transversal self-intersection of the fixed point manifold $M^\sigma$ (see Section 3). In Section 5 we give a different argument which generalizes to cyclic actions of arbitrary finite order.

Theorem 3 gives a purely topological relation between the codimension of $M^\sigma$ and the coefficients of $\Phi_0(M)$. Additional restrictions on $M^\sigma$ arise from the curvature assumption in Theorem 1. If $\sigma$ acts isometrically then any component of $M^\sigma$ is a totally geodesic submanifold of $M$. By a result of Frankel [14] totally geodesic submanifolds of a positively curved manifold intersect if they have sufficiently large dimension. This leads to restrictions on the dimension of the connected components of the fixed point manifold. Further constraints arise from the recent work of Wilking [47] on the connectivity of the inclusion map of totally geodesic submanifolds. These geometric properties together with the rigidity theorem and its consequences are the main ingredients in the proof of Theorem 1.

We don't know how to prove Theorem 1 by more direct methods such as a Bochner type argument for twisted Dirac operators. To prove the vanishing of $\hat{A}(M,TM)$ we need to use the entire elliptic genus. Note that any Bochner type argument for the vanishing of $\hat{A}(M,TM)$ must be sensitive to the dimension as no such argument can apply in dimension eight!

The paper is structured as follows. In the next section we review properties of totally geodesic submanifolds. In Section 3 and Section 5 we recall the rigidity theorem for elliptic genera and discuss relations between the fixed point codimension of cyclic actions and the elliptic genus as exemplified in Theorem 3. These sections do not rely on any curvature assumptions and might be of independent interest. Theorem 1 is proved in Section 4. In the final section we discuss related results for positive $k$th Ricci curvature and higher symmetry rank.

## 2 Totally geodesic submanifolds

Throughout this and the following sections all manifolds are assumed to be smooth closed manifolds and all actions are smooth. Let $M$ be a Riemannian manifold with positive sectional curvature (positive curvature for short). In this section we briefly review properties of totally geodesic submanifolds of $M$. Recall that a submanifold $N \subset M$ (equipped with the induced Riemannian metric) is *totally geodesic* if any geodesic of $N$ is also a geodesic of $M$. We begin with an old result of Frankel.

**Theorem 4 ([14])** *Let $N_1$ and $N_2$ be connected totally geodesic submanifolds of a positively curved connected manifold $M$. If* $\dim N_1 + \dim N_2 \geq \dim M$ *then $N_1$ and $N_2$ have non-empty intersection.* $\square$

The proof uses a Synge type argument for the parallel transport along a geodesic from $N_1$ to $N_2$ which minimizes the distance.

Whereas it is difficult to find totally geodesic submanifolds for generic metrics they do occur naturally as fixed point components in the presence of symmetry. Theorem 4 clearly imposes restrictions on the fixed point manifold of isometric actions. The following consequence is immediate.

**Corollary 5** *Let $\sigma$ be an isometry of a positively curved connected manifold $M$ and let $F$ be a connected component of the fixed point manifold $M^\sigma$ of minimal codimension. Then the dimension of every other component is less than the codimension of $F$.* $\square$

In [15] Frankel applied Theorem 4 to show that the inclusion $N \hookrightarrow M$ of a totally geodesic submanifold $N$ of codimension $\kappa$ is 1-connected provided $\kappa \leq \frac{\dim M}{2}$. Recently, Wilking generalized this result significantly.

**Theorem 6 ([47])** *Let $M$ be an $n$-dimensional positively curved connected manifold. Let $N \subset M$ be a connected submanifold of codimension $\kappa$. If $N$ is totally geodesic then the inclusion $N \hookrightarrow M$ is $(n - 2\kappa + 1)$-connected.* $\square$

The proof uses a Morse type argument on the space of path on $M$ which start and end in $N$. Following [47] we note that the existence of a highly connected inclusion map $N \hookrightarrow M$ is reflected in cohomology via Poincaré duality. More precisely, if $N$ and $M$ are oriented, $N$ has codimension $\kappa$ and $N \hookrightarrow M$ is $(n - \kappa - l)$-connected then the homomorphism

$$\cup \mathrm{PD}([N]) : H^i(M;\mathbb{Z}) \longrightarrow H^{i+\kappa}(M;\mathbb{Z})$$

given by taking the cup product with the Poincaré dual of $N$ is surjective for $l \leq i < n - \kappa - l$ and injective for $l < i \leq n - \kappa - l$.

Wilking applied Theorem 6 and variants of it to the study of positively curved manifolds with large symmetry [47,48]. Among other results he obtained structure theorems for positively curved $n$-dimensional manifolds with symmetry rank $\geq \frac{n}{4} + 1$.

## 3 Rigidity and cyclic actions

In this section we recall the Bott-Taubes-Witten rigidity theorem for elliptic genera and discuss applications to cyclic actions. For more information on elliptic genera we refer to [25,31].

A genus in the sense of Hirzebruch [23] is a ring homomorphism $\Psi$ from the oriented bordism ring $\Omega_*^{SO}$ to a $\mathbb{Q}$-algebra $R$ [23]. The genus is called elliptic (of level 2) if its logarithm $g(t) = \sum_{k\geq 0} \frac{\Psi(\mathbb{C}P^{2k})}{2k+1} t^{2k+1}$ is given by a formal elliptic integral

$$g(t) = \int_0^t \frac{du}{\sqrt{1 - 2\delta \cdot u^2 + \epsilon \cdot u^4}} \tag{1}$$

where $\delta, \epsilon \in R$ [36]. Classical examples of elliptic genera are the signature ($\delta = \epsilon = 1$) and the $\hat{A}$-genus ($\delta = -\frac{1}{8}$, $\epsilon = 0$).

The theory of elliptic genera (of level 2) is related to the theory of modular forms for

$$\Gamma_0(2) := \{A \in SL_2(\mathbb{Z}) \mid A \equiv \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \bmod 2\}.$$

Recall that the ring of modular forms $M_*(\Gamma_0(2))$ is a polynomial ring with generators $\delta$ and $\epsilon$ of weight 2 and 4, respectively (see for example [25]). If one chooses these generators in (1) one obtains the elliptic genus

$$\varphi : \Omega_*^{SO} \to M_*(\Gamma_0(2))$$

which is universal since $\delta$ and $\epsilon$ are algebraically independent.

As in [26,49] we shall consider the (weight zero) elliptic genus $\Phi$ which assigns to a $4k$-dimensional oriented manifold $M$ the modular function $\varphi(M)/\epsilon^{k/2}$. In one of the cusps (the signature-cusp) $\Phi(M)$ expands as a series of twisted signatures

$$sign(q, \mathcal{L}M) := sign(M, \bigotimes_{n=1}^{\infty} S_{q^n}TM \otimes \bigotimes_{n=1}^{\infty} \Lambda_{q^n}TM) \in \mathbb{Z}[[q]].$$

Here $\Lambda_t := \sum_i \Lambda^i \cdot t^i$ (resp. $S_t := \sum_i S^i \cdot t^i$) denotes the exterior (resp. symmetric) power operation and $sign(M, E)$ denotes the index of the signature operator twisted with the complexified vector bundle $E \otimes \mathbb{C}$. If $M$ is a *Spin*-manifold then - as explained by Witten [49] - the series $sign(q, \mathcal{L}M)$ describes the "signature" of the free loop space $\mathcal{L}M$ localized at the manifold $M$ of constant loops.

In a different cusp (the $\hat{A}$-cusp) $\Phi(M)$ expands as a series of characteristic numbers

$$\Phi_0(M) := q^{-k/2} \cdot \hat{A}(M, \bigotimes_{n=2m+1>0} \Lambda_{-q^n}TM \otimes \bigotimes_{n=2m>0} S_{q^n}TM)$$

$$= q^{-k/2} \cdot (\hat{A}(M) - \hat{A}(M, TM) \cdot q + \hat{A}(M, \Lambda^2 TM + TM) \cdot q^2 + \ldots).$$

Here $\hat{A}(M, E) := \langle \hat{\mathcal{A}}(M) \cdot ch(E \otimes \mathbb{C}), [M] \rangle$ where $\hat{\mathcal{A}}(M)$ denotes the multiplicative sequence for the $\hat{A}$-genus, $ch(E \otimes \mathbb{C})$ is the Chern character of the complexification of the vector bundle $E$, $[M]$ denotes the fundamental cycle and $\langle \ , \ \rangle$ is the Kronecker pairing. If $M$ is a *Spin*-manifold then $\hat{A}(M, E)$ is equal to the index of the Dirac operator twisted with $E \otimes \mathbb{C}$ by the Atiyah-Singer index theorem [2]. In this case $\Phi_0(M)$ has an interpretation as a series of indices of twisted Dirac operators (twisted Dirac-indices for short).

Now assume a compact Lie group $G$ acts smoothly on $M$ and preserves the *Spin*-structure (note that a smooth $G$-action always lifts to the *Spin*-structure after passing to a two-fold covering action). Then each twisted signature and each twisted Dirac-index occurring in $sign(q, \mathcal{L}M)$ or $\Phi_0(M)$ refines to a virtual $G$-representation which we identify with its character.

The main feature of the elliptic genus is its rigidity under actions of compact connected Lie groups. The rigidity was explained by Witten in [49] using heuristic arguments from quantum field theory and proved rigorously by Taubes and Bott-Taubes [7,44] (cf. also [24,33]).

**Rigidity Theorem 7** *Let $M$ be a $G$-equivariant Spin-manifold. If $G$ is connected then each equivariant twisted signature (resp. each equivariant twisted Dirac-index) occurring as coefficient in the expansion of $\Phi(M)$ in the signature-cusp (resp. in the $\hat{A}$-cusp) is constant as a character of $G$.* $\square$

The rigidity theorem imposes strong constraints on the $G$-action. In the following we shall discuss some consequences for cyclic actions which are relevant to the proof of Theorem 1.

We begin by recalling results of Hirzebruch and Slodowy on involutions. Assume $S^1$ acts on the *Spin*-manifold $M$ (not necessarily preserving the *Spin*-structure). Let $\sigma \in S^1$ be the element of order 2. In [26] Hirzebruch and Slodowy used the rigidity theorem to show that

$$sign(q, \mathcal{L}M) = sign(q, \mathcal{L}(M^\sigma \circ M^\sigma))$$

where $M^\sigma \circ M^\sigma$ denotes a transversal self-intersection of the fixed point manifold $M^\sigma$. Changing cusps one obtains

$$\Phi_0(M) = \Phi_0(M^\sigma \circ M^\sigma). \tag{2}$$

If the $S^1$-action does not lift to the *Spin*-structure (i.e. the $S^1$-action is odd) then the codimension of the connected components of $M^\sigma$ is always $\equiv 2 \bmod 4$ (cf. [3], Lemma 2.4). In this case $q^{\dim M/8} \cdot \Phi_0(M^\sigma \circ M^\sigma) \in q^{1/2} \cdot \mathbb{C}[[q]]$. Since $q^{\dim M/8} \cdot \Phi_0(M) \in \mathbb{C}[[q]]$ formula (2) implies the well known

**Lemma 8** *Let $M$ be a Spin-manifold with $S^1$-action. If the action is odd then $\Phi_0(M)$ vanishes identically.* $\square$

If the $S^1$-action lifts to the *Spin*-structure (i.e. the $S^1$-action is even) then the codimension of all connected components of $M^\sigma$ is divisible by 4. In this case formula (2) yields the following generalization of the Atiyah-Hirzebruch $\hat{A}$-vanishing theorem [3].

**Theorem 9 ([26])** *Let $M$ be a Spin-manifold with $S^1$-action and let $\sigma \in S^1$ be the element of order two. If* $\operatorname{codim} M^\sigma > 4r$ *then the first* $(r+1)$ *coefficients* *of* $\Phi_0(M)$ *vanish.* $\square$

Here codim $M^\sigma$ denotes the minimal codimension of the connected components of $M^\sigma$ in $M$.

In the remaining part of this section we present a generalization of this theorem to cyclic actions of arbitrary finite order. Let $M$ be a *Spin*-manifold with $S^1$-action and let $\sigma \in S^1$ be of order $o \geq 2$. To a connected component $Y$ of $M^{S^1}$ we attach a rational number $m_o(Y)$ as follows:

The tangent bundle $TM$ restricted to $Y$ splits equivariantly as the direct sum of $TY$ and the normal bundle $\nu$. The latter splits as a direct sum $\nu = \bigoplus_{l>0} \nu_l$ corresponding to the irreducible real 2-dimensional $S^1$-representations $e^{i\cdot\theta} \mapsto \left(\begin{smallmatrix} \cos l\theta & \sin l\theta \\ -\sin l\theta & \cos l\theta \end{smallmatrix}\right)$, $l > 0$. For each $l > 0$ we choose a complex structure on $\nu_l$ as follows. If $l \equiv \tilde{l} \bmod o$ for some $\tilde{l} \in \{0, \ldots, [\frac{o}{2}]\}$ we choose the complex structure such that $\lambda \in S^1$ acts by multiplication with $\lambda^l$. Otherwise we choose the complex structure such that $\lambda$ acts by multiplication with $\lambda^{-l}$. In this way we have fixed a complex structure on $\nu$ such that the rotation numbers of the $S^1$-action are all in $\{0, \ldots, [\frac{o}{2}]\} \bmod o$ . We denote by $\tilde{l} \in \{0, \ldots, [\frac{o}{2}]\}$ the mod $o$ reduction of the rotation number for $\nu_l$ (in other words $\tilde{l}$ is a number in $\{0, \ldots, [\frac{o}{2}]\}$ which satisfies $\tilde{l} \equiv l \bmod o$ or satisfies $\tilde{l} \equiv -l \bmod o$). Let $\dim_{\mathbb{C}} \nu_l$ denote the complex dimension of $\nu_l$.

With these conventions the number $m_o(Y)$ is defined by

$$m_o(Y) := (\sum_{l>0} \dim_{\mathbb{C}} \nu_l \cdot \tilde{l})/o.$$

Finally define

$$m_o := \min_Y m_o(Y)$$

where $Y$ runs over the connected components of $M^{S^1}$. We are now ready to state

**Theorem 10** *Let $M$ be a Spin-manifold with $S^1$-action. If $m_o > r$ then the first $(r+1)$ coefficients of $\Phi_0(M)$ vanish.*

To prove this theorem we analyze the expansion of the equivariant elliptic genus in the $\hat{A}$-cusp using the Lefschetz fixed point formula [2] and Theorem 7. The proof is carried out in Section 5.

Note that codim $M^\sigma \leq 2o \cdot m_o$. Hence, Theorem 10 implies the following

**Corollary 11** *(Theorem 3) Let $M$ be a closed Spin-manifold with $S^1$-action and let $\sigma \in S^1$ be of order $o \geq 2$. If* codim $M^\sigma > 2o \cdot r$ *then the first $(r+1)$ coefficients of $\Phi_0(M)$ vanish.* $\square$

## 4 Positive curvature and elliptic genera

In this section we prove the following generalization of Theorem 1.

**Theorem 12** *Let $M$ be a closed connected Spin-manifold of dimension $\neq 8$ with $b_2(M) = 0$. Assume $M$ admits a metric of positive curvature and an effective $S^1$-action such that the cyclic subgroup of order 4 acts by isometries. Then $\hat{A}(M)$ and $\hat{A}(M, TM)$ vanish, i.e. the first two coefficients in the expansion $\Phi_0(M)$ vanish.*

We remark that the vanishing of the second Betti number could be replaced by the condition that the second homotopy group $\pi_2(M)$ is finite since the theorem is only interesting for even-dimensional manifolds and an oriented even-dimensional manifold of positive curvature is simply connected [43]. Before we give the proof we will illustrate our result with a specific example.

**Example 13** Let $M$ be the product of the quaternionic projective plane $\mathbb{H}P^2$ and an 8-dimensional *Spin*-manifold $B$ with $b_2(B) = 0$ and $\hat{A}(B) = 1$ (the latter can be constructed for example via plumbing). Note that $M$ is *Spin* and $b_2(M) = 0$. Computing the elliptic genus of $M$ one finds $\hat{A}(M) = 0$ and $\hat{A}(M, TM) \neq 0$. On $\mathbb{H}P^2 = Sp(3)/(Sp(2) \times Sp(1))$ we choose a positively curved metric induced from a bi-invariant metric on $Sp(3)$. If we equip $B$ with some metric then after shrinking $\mathbb{H}P^2$, if necessary, the Riemannian product $M$ has positive scalar curvature and carries an isometric action of $Sp(3)$. By the work of Joyce (see [29], Table 1 on page 129) we can choose for $B$ a Ricci-flat Riemannian manifold with holonomy $Spin(7)$. For this choice $M$ has non-negative Ricci curvature as well as positive scaler curvature and carries an isometric action of $Sp(3)$. According to our theorem $M$ cannot carry a metric of positive curvature at least if one restricts to metrics which satisfy the symmetry assumptions given in Theorem 12. We don't know whether $M$ admits a metric of positive Ricci curvature.

**Proof of Theorem 12.** The statement is trivially true if the dimension of $M$ is 4 or if the dimension of $M$ is not divisible by 4. So assume $\dim M = 4k \geq 12$. Since $M$ is an orientable even-dimensional manifold of positive curvature $M$ is simply connected [43]. Note that $\hat{A}(M)$ vanishes by [3,32]. So we are left to show that the twisted Dirac-index $\hat{A}(M, TM)$ vanishes.

The proof is by contradiction. So assume $\hat{A}(M, TM) \neq 0$. Let $\rho \in S^1$ be an element of order 4 and let $\sigma := \rho^2$ denote the involution. Since $\sigma$ acts isometrically $M^\sigma$ is a totally geodesic submanifold of $M$. By Lemma 8 each connected component of $M^\sigma$ has codimension $\equiv 0 \bmod 4$. Theorem 9 implies that the codimension of $M^\sigma$ is 4. Let $X$ denote a connected component of codimension 4. It follows from Corollary 5 that any other connected component of $M^\sigma$ is an isolated $\sigma$-fixed point.

Using Theorem 10 (for $o = 4$) we see that either $X = X^\rho$ or the codimension of $X^\rho$ in $X$ is two. We claim that $X = X^\rho$. To prove this claim it suffices to show that $\hat{A}(M, TM)$ vanishes if $X \neq X^\rho$. So assume a connected component $F$ of $X^\rho$ has codimension two in $X$. Note that $F$ is totally geodesic. We apply Theorem 6 to $F \hookrightarrow X \hookrightarrow M$ and use Poincaré duality to obtain the following relations for the Betti numbers

$$b_2(M) = b_2(X) = b_2(F) = b_{4k-8}(F) = b_{4k-8}(X) = b_4(X) = b_4(M)$$

and

$$b_j(M) = b_j(X) = b_{4k-4-j}(X) = b_{4k-4-j}(M) = b_{j+4}(M)$$

for $3 < j < 4k - 7$. Since $b_2(M) = 0$ it follows that the Betti numbers $b_{4j}(M)$ vanish for $0 < 4j < 4k$. Combining this information with $\hat{A}(M) = 0$ we see that all Pontrjagin numbers of $M$ vanish. In particular, $\hat{A}(M, TM) = 0$ which proves the claim.

Since $X = X^\rho$ the action of $\rho$ on the normal bundle $\nu_X$ of $X \subset M$ induces a complex structure such that $\rho$ acts by multiplication with $i = \sqrt{-1}$. We fix the orientation of $X$ which is compatible with the orientation of $\nu_X$ (induced by the complex structure) and the given orientation of $M$. In the same way the action of $\rho$ induces a complex structure on the normal bundle of an isolated $\sigma$-fixed point and an orientation for this point.

We will now evaluate the expansion of the $S^1$-equivariant elliptic genus in the signature-cusp at $\rho \in S^1$. Recall that each coefficient of $sign(q, \mathcal{L}M)$ is a twisted signature which refines to a virtual $S^1$-representation. We identify the virtual representation with its character and denote the series of equivariant signatures by $sign_{S^1}(q, \mathcal{L}M)$. To compute $sign_{S^1}(q, \mathcal{L}M)(\rho)$ we apply the Lefschetz fixed point formula to each coefficient. Recall that the Lefschetz fixed point formula [2] computes an equivariant index evaluated at $\rho$ as a sum of local contributions at the $\rho$-fixed point components. In our situation $M^\rho$ is the union of $X$ and a possible empty set of isolated $\rho$-fixed points $p_l$. For the entire series one gets

$$sign_{S^1}(q, \mathcal{L}M)(\rho) = \mu_X + \sum_l \mu_{p_l},$$

where $\mu_X$ (resp. $\mu_{p_l}$) denotes the local contribution at $X$ (resp. at an isolated $\rho$-fixed point $p_l$). In cohomological terms $\mu_X$ is given by (cf. [2], Section 3)

$$\mu_X = \langle \mathcal{T}_X \cdot \mathcal{N}_X, [X] \rangle$$

where $\mathcal{T}_X$ is equal to

$$\prod_j \left( x_j \cdot \frac{1+e^{-x_j}}{1-e^{-x_j}} \cdot \prod_{n=1}^{\infty} \frac{(1+q^n \cdot e^{x_j}) \cdot (1+q^n \cdot e^{-x_j})}{(1-q^n \cdot e^{x_j}) \cdot (1-q^n \cdot e^{-x_j})} \right) \in H^*(X;\mathbb{Q})[[q]]$$

and $\mathcal{N}_X$ is equal to

$$\prod_{j=1,2} \left( \frac{1+e^{-y_j} \cdot (-i)}{1-e^{-y_j} \cdot (-i)} \cdot \prod_{n=1}^{\infty} \frac{(1+q^n \cdot e^{y_j} \cdot i) \cdot (1+q^n \cdot e^{-y_j} \cdot (-i))}{(1-q^n \cdot e^{y_j} \cdot i) \cdot (1-q^n \cdot e^{-y_j} \cdot (-i))} \right) \in H^*(X;\mathbb{Q})[[q]].$$

Here $\pm x_j$ (resp. $y_1, y_2$) denote the formal roots of $X$ (resp. $\nu_X$). Since $b_2(X) = b_2(M) = 0$ the first Chern class $y_1 + y_2$ of $\nu_X$ vanishes rationally. This implies

$$\mathcal{N}_X = \frac{1+e^{-y_1} \cdot (-i)}{1-e^{-y_1} \cdot (-i)} \cdot \frac{1+e^{y_1} \cdot (-i)}{1-e^{y_1} \cdot (-i)} = -1.$$

Hence, the expression for $\mu_X$ simplifies to

$$\mu_X = -\langle \mathcal{T}_X, [X] \rangle = -sign(q, \mathcal{L}X).$$

For an isolated $\rho$-fixed point $p_l$ the term $\mu_{p_l}$ is given by (cf. [2], Section 3)

$$\mu_{p_l} = \pm \left( \frac{1-i}{1+i} \prod_{n=1}^{\infty} \frac{(1+i \cdot q^n) \cdot (1-i \cdot q^n)}{(1-i \cdot q^n) \cdot (1+i \cdot q^n)} \right)^{2k} = \pm \left( \frac{1-i}{1+i} \right)^{2k} = \pm 1.$$

Using Theorem 7 we get $sign(q, \mathcal{L}M) = sign_{S^1}(q, \mathcal{L}M)(\rho) = -sign(q, \mathcal{L}X) + c$ where $c$ is the integer obtained by summing up $\mu_{p_l}$. Equivalently,

$$\Phi(M) = -\Phi(X) + c. \tag{3}$$

Note that $\Phi_0(M) \in q^{-k/2}\mathbb{C}[[q]]$ whereas $\Phi_0(X) \in q^{1/2} \cdot q^{-k/2}\mathbb{C}[[q]]$. Comparing the expansions in the $\hat{A}$-cusp of both sides of (3) we see that $\Phi_0(M) \in \mathbb{Z}$. Since $\dim M \geq 12$ this implies $\hat{A}(M, TM) = 0$ which gives the desired contradiction. $\square$

**Remark 14** H. and R. Herrera have shown that the rigidity theorem also holds for *oriented* manifolds with finite second homotopy group [22] (see also Remark 15). In can be shown (along the lines of the proof above) that Theorem 12 remains true for these manifolds.

## 5 Proof of Theorem 10

We may assume that the dimension of $M$ is divisible by 4 and that the fixed point manifold $M^{S^1}$ is not empty since otherwise $M$ is rationally zero bordant and $\Phi_0(M)$ vanishes. By Lemma 8 we may also assume that the $S^1$-action lifts to the *Spin*-structure. We fix a lift. The proof of Theorem 10 is divided into three steps.

*Step 1.* In this step we describe the equivariant elliptic genus in terms of fixed point data. Consider the expansion of $\Phi(M)$ in the $\hat{A}$-cusp. Recall from Section 3 that the coefficients are indices of twisted Dirac operators associated to the *Spin*-structure. Since the $S^1$-action has been lifted to the *Spin*-structure each index refines to a virtual $S^1$-representation which we identify with its character (an element of $\mathbb{Z}[\lambda,\lambda^{-1}]$) and the series $\Phi_0(M)$ refines to an element of $\mathbb{Z}[\lambda,\lambda^{-1}][q^{-\frac{1}{2}}][[q]]$ which we denote by $\Phi_{0,S^1}(M)$. By Theorem 7 each coefficient of the $q^{1/2}$-series $\Phi_{0,S^1}(M)$ is constant as a function on $S^1$.

Let $\lambda_0 \in S^1$ be a fixed topological generator. By the Lefschetz fixed point formula [2] the series $\Phi_{0,S^1}(M)(\lambda_0) \in \mathbb{C}[q^{-\frac{1}{2}}][[q]]$ is equal to a sum of local data

$$\Phi_{0,S^1}(M)(\lambda_0) = \sum_Y \mu_Y(q,\lambda_0)$$

where $Y$ runs over the connected components of $M^{S^1}$.

Recall from Section 3 that we have decomposed the normal bundle $\nu$ of $Y$ as a direct sum $\bigoplus_{l>0}\nu_l$ of complex vector bundles. The complex structure on $\nu_l$ was chosen such that $\lambda \in S^1$ acts on $\nu_l$ by multiplication with $\lambda^{\alpha_l l}$. Here $\alpha_l \in \{-1,+1\}$ was chosen such that $\alpha_l l$ is equivalent to some $\tilde{l} \in \{0,\ldots,[\frac{o}{2}]\}$ modulo $o$. Let $d_l := \dim_{\mathbb{C}} \nu_l$.

We fix the orientation for $Y$ which is compatible with the given orientation of $M$ and the orientation of $\nu$ induced from the complex structure of $\nu$. Let $\{\pm x_j\}$ denote the set of roots of $Y$ and let $\{x_{l,j}\}_{j=1,\ldots,d_l}$ denote the set of roots of $\nu_l$. The local datum $\mu_Y(q,\lambda_0)$ may be described in cohomological terms as (cf. [2], Section 3):

$$\mu_Y(q,\lambda_0) = \left\langle \prod_j \frac{x_j}{f(q,x_j)} \cdot \prod_{\substack{l>0 \\ j=1,\ldots,d_l}} \frac{1}{f(q,x_{l,j}+\alpha_l l\cdot z_0)}, [Y] \right\rangle \tag{4}$$

Here $f(q,x) \in \mathbb{C}[[q^{\frac{1}{4}}]][[x]]$ is equal to

$$(e^{x/2}-e^{-x/2})\cdot q^{1/4}\cdot \frac{\prod_{n=2m>0}(1-q^n\cdot e^x)\cdot(1-q^n\cdot e^{-x})}{\prod_{n=2m+1>0}(1-q^n\cdot e^x)\cdot(1-q^n\cdot e^{-x})},$$

$\lambda_0 = e^{z_0}$, $[Y]$ denotes the fundamental cycle of $Y$ and $\langle\ ,\ \rangle$ is the Kronecker pairing. In general each local datum $\mu_Y(q,\lambda_0)$ depends on $\lambda_0$. However, the sum $\sum_Y \mu_Y(q,\lambda_0)$ is equal to $\Phi_{0,S^1}(M)(\lambda_0)$ and therefore independent of $\lambda_0$ by Theorem 7.

*Step 2.* Each local datum is the expansion of a meromorphic function on $\mathcal{H}\times\mathbb{C}$ where $\mathcal{H}$ denotes the upper half plane. As in the proof of the rigidity theorem [7] (cf. also [24,33]) modularity properties of these functions will be central for the argument. In this step we examine some of their properties.

We begin to recall relevant properties of the series $f$ (see for example [8,25]). For $0 < |q| < 1$ and $z \in \mathbb{C}$ satisfying $|q| < |e^z| < |q|^{-1}$ the series $f(q,z)$ converges normally to a holomorphic function. This function extends to a meromorphic function $\widetilde{f}(\tau,z)$ on $\mathcal{H}\times\mathbb{C}$ after the change of variables $q = e^{2\pi i\cdot\tau}$ where $\tau$ is in $\mathcal{H}$. The function $\widetilde{f}(\tau,z)$ is elliptic in $z$ for the lattice $L := 4\pi i\cdot\mathbb{Z}\langle 1,\tau\rangle$ and satisfies $\widetilde{f}(\tau+1,z) = i\cdot\widetilde{f}(\tau,z)$, $\widetilde{f}(\tau,z+2\pi i) = -\widetilde{f}(\tau,z)$ and $\widetilde{f}(\tau,z+2\pi i\cdot\tau) = -\widetilde{f}(\tau,z)^{-1}$. The zeros of $\widetilde{f}(\tau,z)$ are simple and located at $L$ and $L+2\pi i$.

Let $q = e^{2\pi i\cdot\tau}$ and let $\lambda_0 = e^{z_0}$ be a topological generator of $S^1$. In view of formula (4) and the properties of $f$ the local datum $\mu_Y(q,\lambda_0)$ converges to a meromorphic function $\widetilde{\mu}_Y$ on $\mathcal{H}\times\mathbb{C}$ evaluated at $(\tau,z_0)$. We proceed to explain how this function is related to $\widetilde{f}$. For a function $F$ in the variables $x_j, x_{l,j}$ which is smooth in the origin let $\mathcal{T}(F)$ denote the Taylor expansion of $F$ with respect to $x_j, x_{l,j} = 0$. It follows from formula (4) that $\widetilde{\mu}_Y$ is related to $\widetilde{f}$ by (see for example [8]):

$$\widetilde{\mu}_Y(\tau,z_0) = \left\langle \mathcal{T}\left(\prod_j \frac{x_j}{\widetilde{f}(\tau,x_j)}\cdot \prod_{\substack{l>0\\ j=1,\dots,d_l}} \frac{1}{\widetilde{f}(\tau,x_{l,j}+\alpha_l l\cdot z_0)}\right), [Y]\right\rangle$$

The properties of $\widetilde{f}$ stated above imply corresponding properties for $\widetilde{\mu}_Y$. In particular, $\widetilde{\mu}_Y$ is elliptic for the lattice $L$ and satisfies

$$\widetilde{\mu}_Y(\tau+1,z) = (-1)^{\dim M/4}\cdot\widetilde{\mu}_Y(\tau,z), \quad \widetilde{\mu}_Y(\tau,z+2\pi i) = \pm\widetilde{\mu}_Y(\tau,z).$$

For fixed $\tau\in\mathcal{H}$ the poles of the functions $\widetilde{\mu}_Y$ are contained in $\frac{1}{n}\cdot L$ for some $n\in\mathbb{N}$ depending on the rotation numbers of the $S^1$-action at the connected components $Y\subset M^{S^1}$ (see for example [8]).

It follows that for any topological generator $\lambda_0 = e^{z_0}$ the series $\Phi_{0,S^1}(M)(\lambda_0)$ converges to the sum $\sum_Y \widetilde{\mu}_Y(\tau,z_0)$. In general $\widetilde{\mu}_Y(\tau,z)$ depends on $z$. By Theorem 7 the sum $\sum_Y \widetilde{\mu}_Y(\tau,z)$ is independent of $z$.

*Step 3.* In the final step we study the series $\sum_Y \mu_Y$ in terms of the sum $\sum_Y \widetilde{\mu}_Y(\tau,s(\tau))$ where $s:\mathcal{H}\to\mathbb{C}$ approximates $\tau\mapsto\frac{2}{o}\cdot 2\pi i\cdot\tau$. We choose $s(\tau)$ in such a way that $\widetilde{\mu}_Y(\tau,s(\tau))$ is periodic with respect to $\tau\mapsto\tau+N$ for some $N\in\mathbb{N}$ (see below).

Note that in general the series $\mu_Y(q,\lambda)$ does not converge if $\lambda$ is close to $e^{\frac{2}{o}\cdot 2\pi i\cdot\tau}$

and the $q^{\frac{1}{N}}$-expansion of $\tilde{\mu}_Y(\tau, s(\tau))$, denoted by $a_Y$, is different from the corresponding contribution $\mu_Y$ in the Lefschetz fixed point formula for $\Phi_{0,S^1}(M)$. In particular, we cannot compare $\mu_Y(q, e^{s(\tau)})$ and $\tilde{\mu}_Y(\tau, s(\tau))$ directly. However, since the sum $\sum_Y \tilde{\mu}_Y(\tau, z)$ is independent of $z$ the sum $\sum_Y a_Y$ is equal to the expansion of the elliptic genus in the $\hat{A}$-cusp (see last step). Using the properties of $\tilde{\mu}_Y$ described above and the assumption on $m_o$ we will show that $\sum_Y a_Y$ has a pole of order less than $\frac{\dim M}{8} - r$.

Here are the details. Recall from the last step that the poles of $\tilde{\mu}_Y$ are contained in $\frac{1}{n} \cdot L$ for some $n \in \mathbb{N}$. Choose $s(\tau) := (1-\beta) \cdot \frac{2}{o} \cdot 2\pi i \cdot \tau$ where $\beta$ is a fixed positive rational number $\ll \frac{1}{n}$. Hence, $s(\tau)$ is close to $\frac{2}{o} \cdot 2\pi i \cdot \tau$ and $\tau \mapsto \tilde{\mu}_Y(\tau, s(\tau))$ is holomorphic on $\mathcal{H}$ for every connected component $Y \subset M^{S^1}$. Using $\alpha_l l \equiv \tilde{l} \bmod o$, $\tilde{l} \in \{0, \ldots, [\frac{o}{2}]\}$, and the transformation property $\tilde{f}(\tau, z + 4\pi i \cdot \tau) = \tilde{f}(\tau, z)$ one computes that $\tilde{\mu}_Y(\tau, s(\tau))$ is equal to $\langle \mathcal{T}(A_Y), [Y] \rangle$ where

$$A_Y := \prod_j \frac{x_j}{\tilde{f}(\tau, x_j)} \cdot \prod_{\substack{l>0 \\ j=1,\ldots,d_l}} \frac{1}{\tilde{f}(\tau, x_{l,j} + 2 \cdot (\frac{\tilde{l}}{o} \cdot (1-\beta) - \beta_l) \cdot (2\pi i \cdot \tau))}$$

and $\beta_l := \beta \cdot \frac{\alpha_l l - \tilde{l}}{o}$. Note that for some $N \in \mathbb{N}$ (depending on $\beta$ and the rotation numbers) every summand $\tilde{\mu}_Y(\tau, s(\tau))$ is periodic with respect to $\tau \mapsto \tau + N$. We claim that its expansion $a_Y \in \mathbb{C}[q^{-\frac{1}{N}}][[q^{\frac{1}{N}}]]$ has a pole of order less than $\frac{\dim M}{8} - r$.

Since the expansion of $\mathcal{T}\left(x_j / \tilde{f}(\tau, x_j)\right)$ (with respect to $\tau \mapsto \tau + 4$) is equal to $x_j / f(q, x_j)$ the expansion of

$$\mathcal{T}\left(\frac{1}{\tilde{f}(\tau, x_{l,j} + 2 \cdot (\frac{\tilde{l}}{o} \cdot (1-\beta) - \beta_l) \cdot (2\pi i \cdot \tau))}\right) \tag{5}$$

can be easily computed in terms of $f$. The computation shows that the expansion of (5) has a pole of order $\leq \frac{1}{4} - \frac{\tilde{l}}{o} \cdot (1-\beta) + \beta_l$. Since $m_o(Y) \geq m_o > r$ and $\beta$, $\beta_l$ are arbitrarily small it follows that $a_Y \in \mathbb{C}[q^{-\frac{1}{N}}][[q^{\frac{1}{N}}]]$ has a pole of order less than $\frac{\dim M}{8} - r$. As explained in the beginning of this step the sum $\sum_Y a_Y$ is equal to the expansion of the elliptic genus in the $\hat{A}$-cusp. Hence, $\Phi_0(M) \in \mathbb{C}[q^{-\frac{1}{2}}][[q]]$ has a pole of order less than $\frac{\dim M}{8} - r$, i.e. the first $(r+1)$ coefficients of $\Phi_0(M)$ vanish. $\square$

**Remark 15** A similar argument applies to orientable $S^1$-manifolds for which the equivariant elliptic genus is rigid, e.g. oriented manifolds with finite second homotopy group [22] or *Spin*$^c$-manifolds with first Chern class a torsion class [9].

## 6 Higher symmetry rank and positive $k$th Ricci curvature

Theorem 12 indicates that the existence of a metric of positive sectional curvature on a *Spin*-manifold is reflected in the expansion of the elliptic genus in the $\hat{A}$-cusp (at least under mild assumptions on the symmetry). The purpose of this section is to discuss other results which illustrate this connection between the elliptic genus and positive curvature.

We first like to point out that under slightly different symmetry assumptions the conclusion of Theorem 12 also holds for *Spin*-manifolds with non-trivial second Betti number. More precisely one has

**Theorem 16 ([10])** *Let $M$ be a closed connected Spin-manifold of dimension $\neq 8$ and let $G$ be a compact connected Lie group which acts on $M$. Suppose $M$ admits a metric of positive curvature such that some subgroup $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ of $G$ acts effectively and isometrically. Then the first two coefficient of $\Phi_0(M)$ vanish, i.e. $\hat{A}(M) = \hat{A}(M, TM) = 0$.* $\square$

Looking at $\mathbb{H}P^2$ one sees that the restriction on the dimension is necessary. In dimension 12 the result can be reformulated in terms of the signature (see the paragraph before Corollary 2). As an illustration consider the product $M = \mathbb{H}P^2 \times K_3$ of the quaternionic projective plane and a $K_3$-surface. If we equip $K_3$ with a Ricci-flat metric then $M$ has non-negative Ricci curvature, positive scalar curvature and $Sp(3)$ acts isometrically on $M$. By the theorem above $M$ does not admit a positively curved metric with, say, isometric $Sp(3)$-action.

Higher vanishing results for the elliptic genus $\Phi_0(M)$ can be obtained under stronger symmetry assumptions as illustrated by the following

**Theorem 17** *Let $M$ be a closed connected Spin-manifold of dimension $> 12r - 4$. Suppose $M$ admits a metric of positive curvature and an action by a torus $T$ of rank $2r$ such that the 2-torus $T_2 \subset T$ acts isometrically and effectively. Then the first $(r+1)$ coefficients in the expansion $\Phi_0(M)$ vanish.*

**Proof.** The proof is by contradiction. So assume the first $(r+1)$ coefficients of $\Phi_0(M)$ do not all vanish. Note that $\dim M = n = 4k \geq 12r$ since $\Phi_0(M)$ vanishes if the dimension of $M$ is not divisible by 4. By [43] $M$ is simply connected.

In the lemma below we show that for some involution $\sigma \in T_2$ the fixed point manifold $M^\sigma$ is connected of codimension 4 and contains a totally geodesic connected submanifold $F \subset M^\sigma$ of codimension 2. It then follows from Theorem 6 that the inclusion maps $F \hookrightarrow M^\sigma$ and $M^\sigma \hookrightarrow M$ are both $(n-7)$-connected.

We apply Poincaré duality to conclude

$$b_i(M^\sigma) = b_i(F) = b_{n-6-i}(F) \geq b_{n-6-i}(M^\sigma) = b_{i+2}(M^\sigma) \text{ for } 1 \leq i < n-7$$

and

$$b_i(M) = b_i(M^\sigma) = b_{n-4-i}(M^\sigma) = b_{n-4-i}(M) = b_{i+4}(M) \text{ for } 3 < i < n-7.$$

In particular, since $b_1(M^\sigma) = b_1(M) = 0$, all odd Betti numbers of $M^\sigma$ and $M$ vanish.

The classical Lefschetz fixed point formula for the Euler characteristic gives $\chi(M) = \chi(M^T) = \chi((M^\sigma)^T) = \chi(M^\sigma)$. Hence,

$$\sum_{i\geq 0} b_i(M) = \chi(M) = \chi(M^\sigma) = \sum_{i\geq 0} b_i(M^\sigma). \tag{6}$$

On the other hand since $M^\sigma \hookrightarrow M$ is $(n-7)$-connected Poincaré duality gives

$$\sum_{i\geq 0} b_i(M) = \sum_{i\geq 0} b_i(M^\sigma) + b_4(M) + b_6(M). \tag{7}$$

Comparing equations (6) and (7) we find $b_4(M) = 0$. Since $b_i(M) = b_{i+4}(M)$ for $3 < i < n-7$ and $\hat{A}(M) = 0$ all Pontrjagin numbers of $M$ vanish. In particular, $\Phi_0(M) = 0$. This completes the proof based on the lemma below. $\square$

**Lemma 18** *Let $M$ be as in Theorem 17. Assume the first $(r+1)$ coefficients of $\Phi_0(M)$ do not all vanish. Then for some involution $\sigma \in T_2$ the fixed point manifold $M^\sigma$ is connected of codimension 4 and contains a totally geodesic connected submanifold $F \subset M^\sigma$ of codimension 2.*

**Proof.** As before we may assume that $\dim M = 4k \geq 12r$. We examine the action of the 2-torus $T_2 \cong (\mathbb{Z}/2\mathbb{Z})^{2r}$ at a $T$-fixed point $pt$ (which exists since we assume $\Phi_0(M) \neq 0$). For an involution $\sigma \in T_2$ let $N_\sigma$ denote the connected component of $M^\sigma$ which contains $pt$.

We split the tangent space $T_{pt}M$ into $2k$ complex one-dimensional $T$-representations $T_{pt}M = R_1 \oplus \ldots \oplus R_{2k}$. With respect to such a decomposition the $T$-action on $T_{pt}M$ is given by a homomorphism $T \to U(1)^{2k}$. Let $h : I_T \otimes (\mathbb{Z}/2\mathbb{Z}) \to (\mathbb{Z}/2\mathbb{Z})^{2k}$ denote the mod 2 reduction of the induced homomorphism of integral lattices $I_T \to I_{U(1)^{2k}} = \mathbb{Z}^{2k}$. Since the $T$-action is effective $h$ is injective. Let $\mathcal{C} := im(h) \subset (\mathbb{Z}/2\mathbb{Z})^{2k}$ be the induced binary code. We denote by $wt(\tilde{\sigma})$ the weight of $\tilde{\sigma} \in \mathcal{C}$ (i.e. the number of entries of $\tilde{\sigma}$ equal to 1) and by $cowt(\tilde{\sigma})$ the co-weight $2k - wt(\tilde{\sigma})$.

Note that the action of an involution $\sigma = \exp(c/2) \in T$, $c \in I_T$, on $T_{pt}M$ is described by the element $\tilde{\sigma} = h(\tilde{c})$ where $\tilde{c}$ denotes the mod 2 reduction of $c$. In particular, $\operatorname{codim} N_\sigma = 2wt(\tilde{\sigma})$ and $\dim N_\sigma = 2cowt(\tilde{\sigma})$.

By Theorem 9 and Lemma 8 the fixed point manifold $M^\sigma$ has codimension $\leq 4r$ and the dimension of each connected component of $M^\sigma$ is divisible by 4. Note that the connected components of $M^\sigma$ are totally geodesic. Since the dimension of $M$ is $\geq 8r$ it follows from Corollary 5 that either $\operatorname{codim} N_\sigma \leq 4r$ or $\dim N_\sigma \leq 4r-4$. Hence, for any $\tilde{\sigma} \in \mathcal{C}$ either $wt(\tilde{\sigma}) \leq 2r$ or $cowt(\tilde{\sigma}) \leq 2r-2$. It is an elementary fact from linear algebra that $\mathcal{C}$ has a basis $\tilde{\sigma}_1, \ldots, \tilde{\sigma}_{2r}$ such that

$$A := \begin{pmatrix} \tilde{\sigma}_1 \\ \vdots \\ \tilde{\sigma}_{2r} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots & 0 & ? & ? & ? & \ldots & ? \\ 0 & 1 & 0 & 0 & \ldots & 0 & ? & ? & ? & \ldots & ? \\ 0 & 0 & 1 & 0 & \ldots & 0 & ? & ? & ? & \ldots & ? \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 & ? & ? & ? & \ldots & ? \end{pmatrix}$$

after permuting the representations $R_i$ (i.e. the columns), if necessary. From the above inequalities for the weight and coweight we conclude that the elements $\tilde{\sigma}_i$ of the basis have weight $wt(\tilde{\sigma}_i) \leq 2r$. Since the weight function is sublinear, i.e. $wt(\tilde{\sigma} + \tilde{\sigma}') \leq wt(\tilde{\sigma}) + wt(\tilde{\sigma}')$, and $2k \geq 6r$ it follows that the set of elements in $\mathcal{C}$ with weight $\leq 2r$ is closed under addition. Hence, $wt(\tilde{\sigma}) \leq 2r$ for every $\tilde{\sigma} \in \mathcal{C}$. In particular, this inequality holds for $\sum_j \tilde{\sigma}_j$ and $\sum_{j \neq i} \tilde{\sigma}_j$ which implies $wt(\tilde{\sigma}_i) = 2$ and implies that each of the last $(2k - 2r)$ columns of $A$ has an even number of non-zero entries. In particular, we can order the elements $\tilde{\sigma}_i$ of the basis such that $wt(\tilde{\sigma}_1 + \tilde{\sigma}_2) = 2$.

Next we shall rephrase these observations in terms of the fixed point components $N_\sigma$. Let $\sigma_i$ denote the involution which corresponds to $\tilde{\sigma}_i$. Then by the above the codimension of $N_{\sigma_i}$ is 4 and the intersection $N_{\sigma_1} \cap N_{\sigma_2}$ has codimension 2 in $N_{\sigma_1}$. For an arbitrary involution $\sigma$ we know that the codimension of $N_\sigma$ is at most $4r$ since $wt(\tilde{\sigma}) \leq 2r$ for any $\tilde{\sigma} \in \mathcal{C}$.

We will now show that the conclusion of the lemma holds for $\sigma := \sigma_1$. Note that by Theorem 4 and Lemma 8 the fixed point manifold $M^\sigma$ is the union of $N_\sigma$ and a possible empty set of isolated $\sigma$-fixed points. Since an isolated $\sigma$-fixed point $q$ is also a $T$-fixed point we can argue as above to conclude that the connected component of $M^\sigma$ which contains $q$ has codimension $\leq 4r$. Since $\dim M > 4r$ we get a contradiction to $q$ being an isolated $\sigma$-fixed point. Hence, there are no isolated $\sigma$-fixed points and $M^\sigma = N_\sigma$ is connected. This proves the first part of the statement. The second part follows by taking $F$ to be the connected component of $M^{\sigma_1} \cap M^{\sigma_2}$ which contains the $T$-fixed point $pt$. $\square$

Next we will discuss relations between the elliptic genus and positive $k$th Ricci curvature. Recall that a Riemannian manifold has positive $k$th Ricci curvature (or $k$-positive Ricci curvature) if for any $(k+1)$ mutually orthogonal unit tangent vectors $e, e_1, \ldots, e_k$ (at any point of $M$) the sum of curvatures

$\sum_{i=1}^{k} sec(e \wedge e_i)$ is positive [30]. Thus, 1-positive Ricci curvature is equivalent to positive curvature and $(\dim M - 1)$-positive Ricci curvature is equivalent to positive Ricci curvature. Totally geodesic submanifolds in a manifold of positive $k$th Ricci curvature enjoy properties similar to the ones described in Section 2. This allows to extend our method to this situation. Theorem 4 generalizes to manifolds of positive $k$th Ricci curvature as follows [30]:

*Let $N_1$ and $N_2$ be connected totally geodesic submanifolds of a connected manifold $M$ of positive $k$th Ricci curvature. If $\dim N_1 + \dim N_2 \geq \dim M + (k-1)$ then $N_1$ and $N_2$ have non-empty intersection.*

Similarly, Theorem 6 generalizes to positive $k$th Ricci-curvature (see [47], Remark 2.1). Using these substitutes for Theorem 4 and Theorem 6 one can derive vanishing theorems for the elliptic genus of *Spin*-manifolds with positive $k$th Ricci curvature and symmetry. As an illustration we shall prove the following sample result.

Let $M$ be a closed connected *Spin*-manifold of positive $k$th Ricci curvature. Assume a torus $T$ of rank $R$ acts smoothly on $M$ such that for some prime $p$ the induced action of the $p$-torus $T_p \cong (\mathbb{Z}/p\mathbb{Z})^R$ is isometric and effective. To keep the exposition simple we shall assume the generous bounds $R \geq p \cdot r + \frac{k+1}{2}$ and $\dim M \geq 6p \cdot r + (k-1)$.

**Proposition 19** *Let $M$ be a Spin-manifold of positive $k$th Ricci curvature satisfying the assumptions above. Then the first $(r+1)$ coefficients in the expansion $\Phi_0(M)$ vanish.*

**Proof.** First note that if $F_1$ and $F_2$ are two different fixed point components of an isometry $\sigma$ then $\dim F_1 + \dim F_2 < \dim M + (k-1)$ by [30]. Now assume the first $(r+1)$ coefficients in the expansion $\Phi_0(M)$ do not vanish and consider the action of $\sigma \in T_p$ on $M$. By Theorem 3 the codimension of $M^\sigma$ is $\leq 2p \cdot r$. Hence, a connected component $F$ of $M^\sigma$ has either "small codimension", i.e. codim $F \leq 2p \cdot r$, or "small dimension", i.e. $\dim F < 2p \cdot r + (k-1)$.

Consider a $T$-fixed point $pt \in M$ (which exists since we assume $\Phi_0(M) \neq 0$) and let $F_\sigma \subset M^\sigma$ denote the component which contains $pt$. It is an elementary exercise to show that the $p$-torus $T_p$ has a basis $\sigma_1, \ldots, \sigma_R$ such that $\dim F_{\sigma_i} \geq 2R - 2 \geq 2p \cdot r + (k-1)$. This implies that the codimension of $F_{\sigma_i}$ is small (in the above sense). Next consider two elements $\sigma, \sigma' \in T_p$. Since the dimension of $M$ is $\geq 6p \cdot r + (k-1)$ the codimension of $F_{\sigma+\sigma'}$ is small provided this holds for $F_\sigma$ and $F_{\sigma'}$. Hence, the codimension of $F_\sigma$ is small for every $\sigma \in T_p$, i.e. codim $F_\sigma \leq 2p \cdot r$ for every $\sigma \in T_p$. However, it follows from elementary linear algebra that for some $\sigma \in T_p$ the codimension of $F_\sigma$ is at least $2R \geq 2p \cdot r + (k+1)$. This gives the desired contradiction. $\square$

We close this section with a speculation on the elliptic genus and positive curvature. As illustrated by the results in this paper the coefficients of the $\hat{A}$-expansion $\Phi_0(M)$ of the elliptic genus obstruct positive curvature on a *Spin*-manifold $M$ under mild symmetry assumptions. For our approach it is essential that a compact connected Lie group acts on a positively curved *Spin*-manifold and that the action contains sufficiently many isometries. Without assumptions on the symmetry our method breaks down and does not lead to any new obstructions. On the other hand all known examples of positively curved simply connected manifolds admit metrics of positive curvature with lots of symmetry. In fact in dimension divisible by 4 all known examples are homogeneous. For a *Spin*-homogeneous space $M$ Hirzebruch and Slodowy [26] have shown that the elliptic genus is strongly rigid in the sense that $\Phi(M)$ is constant as a $q$-power series (it reduces to the ordinary signature). This feature motivates the

**Question 20** *Let $M$ be a Spin-manifold. Is the elliptic genus $\Phi(M)$ strongly rigid if $M$ admits a metric of positive curvature?*

Less ambitious one might ask whether the twisted Dirac-index $\hat{A}(M, TM)$ vanishes if $M$ admits a metric of positive curvature and $\dim M \geq 12$.

A positive answer to any of these questions would give a new way to distinguish between positive curvature and positive Ricci curvature and could be applied in situation which are not obstructed by Gromov's Betti number theorem.

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