Cyclic actions and elliptic genera

Anand Dessai

Abstract

Let M be a *Spin*-manifold with S^1-action and let $\sigma \in S^1$ be of finite order. We show that the indices of certain twisted Dirac operators vanish if the action of σ has sufficiently large fixed point codimension. These indices occur in the Fourier expansion of the elliptic genus of M in one of its cusps. As a by-product we obtain a new proof of a theorem of Hirzebruch and Slodowy on involutions.

1 Introduction

Let M be a smooth closed connected *Spin*-manifold with smooth S^1-action and let $\sigma \in S^1$ be the element of order two. Hirzebruch and Slodowy [12] showed that the elliptic genus of M can be computed in terms of the transversal self-intersection of the fixed point manifold M^σ and used this property to deduce a vanishing theorem for certain characteristic numbers which occur in the Fourier expansion of the elliptic genus of M in one of its cusps.

In this note we extend this vanishing theorem from involutions to cyclic actions of arbitrary order. Our main result (see Theorem 2.1) is used in [5] to exhibit obstructions against the existence of positively curved metrics with symmetry on *Spin*-manifolds. The proof of Theorem 2.1 relies on the rigidity theorem for the elliptic genus which we shall recall first. As a general reference for the theory of elliptic genera we recommend [11, 14].

The elliptic genus Φ, in the normalization considered in [12, 17], is a ring homomorphism from the oriented bordism ring to the ring of modular functions (with $\mathbb{Z}/2\mathbb{Z}$-character) for $\Gamma_0(2) := \{A \in SL_2(\mathbb{Z}) \mid A \equiv \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \bmod 2\}$. In one of the cusps of $\Gamma_0(2)$ (the signature cusp) the Fourier expansion of $\Phi(M)$ has an interpretation as a series of twisted signatures

$$sign(M, \bigotimes_{n=1}^{\infty} S_{q^n} TM \otimes \bigotimes_{n=1}^{\infty} \Lambda_{q^n} TM) = sign(M) + 2 \cdot sign(M, TM) \cdot q + \dots .$$

Here $sign(M, E)$ denotes the index of the signature operator twisted with the complexified vector bundle $E_{\mathbb{C}}$, TM denotes the tangent bundle and $\Lambda_t = \sum \Lambda^i \cdot t^i$ (resp. $S_t = \sum S^i \cdot t^i$) denotes the exterior (resp. symmetric) power operation.

Following Witten [17] the series above is best thought of as the "signature" of the free loop space $\mathcal{L}M$ of M formally localized at the manifold M of constant loops. We denote the series of twisted signatures by $sign(q, \mathcal{L}M)$.

The main feature of the elliptic genus is its rigidity under S^1-actions. This phenomenon was first explained by Witten [17] using standard conjectures from quantum field theory and then shown rigorously by Taubes and Bott-Taubes in [3, 16] (cf. also [10, 15]).

If S^1 acts by isometries[1] on M and if E is a vector bundle associated to TM then the signature operator twisted with the complexified vector bundle $E_{\mathbb{C}}$ refines to an S^1-equivariant operator. Its index is a virtual S^1-representation which we denote by $sign_{S^1}(M, E) \in R(S^1)$. In particular, the expansion of the elliptic genus in the signature cusp refines to a series of equivariant twisted signatures $sign_{S^1}(q, \mathcal{L}M) \in R(S^1)[[q]]$.

Theorem 1.1 (Rigidity theorem [3, 16]). *Let M be a closed manifold with S^1-action. If M is Spin then each equivariant twisted signature occurring as coefficient in the series $sign_{S^1}(q, \mathcal{L}M)$ is constant as a character of S^1.* ■

We use the rigidity theorem to study the action of cyclic subgroups of S^1. Our investigation is inspired by work of Hirzebruch and Slodowy [12] on elliptic genera and involutions. As a motivation we shall briefly recall relevant aspects of their work.

Let M be a *Spin*-manifold with S^1-action and let $\sigma \in S^1$ be of order two. By the rigidity theorem the expansion of the elliptic genus in the signature cusp is equal to the S^1-equivariant expansion evaluated at $\sigma \in S^1$, i.e. $sign(q, \mathcal{L}M) = sign_{S^1}(q, \mathcal{L}M)(\sigma)$. The latter can be computed via the Lefschetz fixed point formula [2] as a sum of local contributions a_F at the connected components F of the fixed point manifold M^σ. Hirzebruch and Slodowy showed that a_F is equal to the expansion of the elliptic genus (in the signature cusp) of the transversal self-intersection $F \circ F$ (cf. [12] for details):

$$sign(q, \mathcal{L}M) = sign_{S^1}(q, \mathcal{L}M)(\sigma) \tag{1}$$

$$= \sum_{F \subset M^\sigma} sign(q, \mathcal{L}(F \circ F)) = sign(q, \mathcal{L}(M^\sigma \circ M^\sigma))$$

Note that, by taking constant terms, one obtains the classical formula $sign(M) = sign(M^\sigma \circ M^\sigma)$ for the ordinary signature which holds for the larger class of oriented manifolds (cf. [9, 13]).

Formula (1) has two immediate consequences. If the codimension of M^σ, codim $M^\sigma := \min_{F \subset M^\sigma}$ codim F, is greater than half of the dimension of M then the series $sign(q, \mathcal{L}M)$ vanishes identically. If the codimension of M^σ is equal to half of the dimension of M then all the twisted signatures occurring as coefficients of q^n, $n > 0$, in the series $sign(q, \mathcal{L}M)$ vanish, i.e. $sign(q, \mathcal{L}M) = sign(M)$.

If the codimension of M^σ is less than half of the dimension of M then formula (1) still gives some information on the action of the involution σ. Namely it implies that certain twisted Dirac operators have vanishing index provided that the codimension of M^σ is sufficiently large. These indices are related to the elliptic genus in the following way. Recall that the q-series $sign(q, \mathcal{L}M)$ is the expansion of the elliptic genus $\Phi(M)$ in one of the cusps of $\Gamma_0(2)$. In a different cusp (the $\hat{A}$-cusp) the expansion of $\Phi(M)$ may be described (using a suitable change of cusps) by

$$\Phi_0(M) := q^{-\dim M/8} \cdot \hat{A}(M, \bigotimes_{n=2m+1>0} \Lambda_{-q^n} TM \otimes \bigotimes_{n=2m>0} S_{q^n} TM)$$

[1] This is the case after averaging a given Riemannian metric over the S^1-action.

$$= q^{-\dim M/8} \cdot (\hat{A}(M) - \hat{A}(M,TM) \cdot q + \hat{A}(M, \Lambda^2 TM + TM) \cdot q^2 + \dots).$$

Here $\hat{A}(M,E)$ is a characteristic number of the pair (M,E) which, in the presence of a *Spin*-structure, is equal to the index of the Dirac operator twisted with the complexified vector bundle $E_{\mathbb{C}}$. We call the series above the expansion of $\Phi(M)$ in the $\hat{A}$-cusp.

Note that $\Phi_0(M)$ and $sign(q, \mathcal{L}(M))$ are different expansions of the same modular function $\Phi(M)$ and determine each other. By formula (1) $\Phi_0(M) = \Phi_0(M^\sigma \circ M^\sigma)$ which implies the following generalization of the Atiyah-Hirzebruch vanishing theorem for the $\hat{A}$-genus [1].

Theorem 1.2 ([12]). *Let M be a Spin-manifold with S^1-action and let $\sigma \in S^1$ be of order two. If* codim $M^\sigma > 4r$ *then the expansion of the elliptic genus of M in the $\hat{A}$-cusp has a pole of order less than $\frac{\dim M}{8} - r$.* ■

The reasoning indicated above also leads to obstructions against the existence of S^1-actions on highly connected manifolds which might be of independent interest.

Theorem 1.3. *Let M be a k-connected Spin-manifold. Assume $k \geq 4r$. If M admits a non-trivial S^1-action then the expansion of the elliptic genus of M in the $\hat{A}$-cusp has a pole of order less than $\frac{\dim M}{8} - r$.*

Note that for $r > 0$ the *Spin*-condition follows from the connectivity assumption. We remark that the conclusion of Theorem 1.3 also holds if M is a connected *Spin*-manifold with non-trivial S^1-action and $H^{4*}(M;\mathbb{Q}) = 0$ for $0 < * \leq r$ (see Section 4 for a proof).

The next result extends Theorem 1.2 to finite cyclic actions of arbitrary order .

Theorem 1.4. *Let M be a Spin-manifold with S^1-action and let $\sigma \in S^1$ be of order $o \geq 2$. If* codim $M^\sigma > 2o \cdot r$ *then the expansion of the elliptic genus of M in the $\hat{A}$-cusp has a pole of order less than $\frac{\dim M}{8} - r$.*

The theorem follows from a more general result (see Theorem 2.1 and the proof in Section 3). As indicated above the proof of Theorem 1.2 given in [12] is specific to actions of order two. To deal with the general situation we consider the expansion of the equivariant elliptic genus in the $\hat{A}$-cusp and study the local contributions of the S^1-fixed point components using the rigidity theorem. We close this section with some consequences of Theorem 1.4.

Corollary 1.5. *Let M be a Spin-manifold with S^1-action.*

1. *Let $\sigma \in S^1$ be of order* 3. *If* codim $M^\sigma > 0$ *then $\hat{A}(M)$ vanishes. If* codim $M^\sigma > 6$ *then $\hat{A}(M)$ and $\hat{A}(M,TM)$ vanish. If σ acts with isolated fixed points then $\Phi(M)$ vanishes identically.*

2. *Let $\sigma \in S^1$ be of order* 4. *If* codim $M^\sigma > 0$ *then $\hat{A}(M)$ vanishes. If* codim $M^\sigma > 8$ *then $\hat{A}(M)$ and $\hat{A}(M,TM)$ vanish. If σ acts with isolated fixed points then $\Phi(M)$ is equal to the signature of M.*

3. *Let $\sigma \in S^1$ be of order $o < \frac{dim M}{2}$. If σ acts with isolated fixed points then $\hat{A}(M)$ and $\hat{A}(M,TM)$ vanish.* ■

2 Cyclic actions

In this section we state the main result of this note. Let M be a connected S^1-manifold and let $o \geq 2$ be a natural number. At a connected component Y of the fixed point manifold M^{S^1} the tangent bundle TM splits equivariantly as the direct sum of TY and the normal bundle ν. The latter splits (non-canonically) as a direct sum $\nu = \bigoplus_{k\neq 0} \nu_k$ corresponding to the irreducible real 2-dimensional S^1-representations $e^{i\cdot\theta} \mapsto \left(\begin{smallmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{smallmatrix}\right)$, $k \neq 0$. We fix such a decomposition of ν. For each $k \neq 0$ choose $\alpha_k \in \{\pm 1\}$ such that $\alpha_k k \equiv \tilde{k} \bmod o$, $\tilde{k} \in \{0, \dots, [\frac{o}{2}]\}$. On each vector bundle ν_k introduce a complex structure such that $\lambda \in S^1$ acts on ν_k by scalar multiplication with $\lambda^{\alpha_k k}$. The $\alpha_k k's$ (taken with multiplicities) are called the rotation numbers of the S^1-action at Y. Finally define

$$m_o(Y) := (\sum_k d_k \cdot \tilde{k})/o \quad \text{and} \quad m_o := \min_Y m_o(Y),$$

where d_k denotes the complex dimension of ν_k and Y runs over the connected components of M^{S^1} (to keep notation light we have suppressed the dependence of ν, ν_k, d_k on Y). We are now in the position to state

Theorem 2.1. *Let M be a Spin-manifold with S^1-action. If $m_o > r$ then the expansion of the elliptic genus of M in the $\hat{A}$-cusp has a pole of order less than $\frac{\dim M}{8} - r$.*

If $\sigma \in S^1$ has order $o = 2$ then $\tilde{k} \in \{0, 1\}$ and $4 \cdot m_2(Y)$ is the codimension of the connected component of M^σ which contains Y. Thus codim $M^\sigma \leq 4 \cdot m_2$ and one recovers Theorem 1.2. In general if $\sigma \in S^1$ has order $o \geq 2$ then codim $M^\sigma \leq 2o \cdot m_o$ and one obtains Theorem 1.4. Note that without the *Spin* condition the conclusion of the theorem fails in general, e.g. for complex projective spaces of even complex dimension (see however Remark 3.1).

3 Proof of Theorem 2.1

We may assume that the dimension of M is divisible by 4 and that the fixed point manifold M^{S^1} is not empty since otherwise M is rationally zero bordant by the Lefschetz fixed point formula [2] and $\Phi(M)$ vanishes. We may also assume that the S^1-action lifts to the *Spin*-structure (otherwise the action is odd which forces the elliptic genus to vanish, see for example [12]). We fix an S^1-equivariant Riemannian metric on M. The proof is divided into three steps.

Step 1: We describe the equivariant elliptic genus at M^{S^1}. Consider the expansion of $\Phi(M)$ in the $\hat{A}$-cusp. Recall that the coefficients are indices of twisted Dirac operators associated to the *Spin*-structure. Since the S^1-action lifts to the *Spin*-structure each index refines to a virtual S^1-representation and the series refines to an element of $R(S^1)[q^{-\frac{1}{2}}][[q]]$ which we denote by $\Phi_{0,S^1}(M)$. Note that $sign_{S^1}(q, \mathcal{L}M)$ and $\Phi_{0,S^1}(M)$ are different expansions of the same function. Hence the rigidity of $sign_{S^1}(q, \mathcal{L}M)$ (see Theorem 1.1) is equivalent to the rigidity of $\Phi_{0,S^1}(M)$, i.e. each coefficient of the series $\Phi_{0,S^1}(M)$ is constant as a character of S^1.

Let $\lambda_0 \in S^1$ be a fixed topological generator. By the Lefschetz fixed point formula [2] the series $\Phi_{0,S^1}(M)(\lambda_0) \in \mathbb{C}[q^{-\frac{1}{2}}][[q]]$ is equal to a sum of local data

$$\Phi_{0,S^1}(M)(\lambda_0) = \sum_Y \mu_Y(q, \lambda_0),$$

where Y runs over the connected components of M^{S^1}.

Recall from Section 2 that we have decomposed the normal bundle ν of Y as a direct sum $\bigoplus_{k \neq 0} \nu_k$ of complex vector bundles. Fix the orientation for Y which is compatible with the orientation of M and the complex structure of ν. Let $\{\pm x_i\}$ denote the set of roots of Y and let $\{x_{k,j}\}_{j=1,\dots,d_k}$ denote the set of roots of the complex vector bundle ν_k. The local datum $\mu_Y(q, \lambda_0)$ may be described in cohomological terms as (cf. [2], Section 3):

$$\mu_Y(q, \lambda_0) = \left\langle \prod_i \frac{x_i}{f(q, x_i)} \cdot \prod_{\substack{k \neq 0 \\ j=1,\dots,d_k}} \frac{1}{f(q, x_{k,j} + \alpha_k k \cdot z_0)}, [Y] \right\rangle \tag{2}$$

Here $f(q, x) \in \mathbb{C}[[q^{\frac{1}{4}}]][[x]]$ is equal to

$$(e^{x/2} - e^{-x/2}) \cdot q^{1/4} \cdot \frac{\prod_{n=2m>0}(1 - q^n \cdot e^x) \cdot (1 - q^n \cdot e^{-x})}{\prod_{n=2m+1>0}(1 - q^n \cdot e^x) \cdot (1 - q^n \cdot e^{-x})},$$

$\lambda_0 = e^{z_0}$, $[Y]$ denotes the fundamental cycle of Y and $\langle \quad , \quad \rangle$ is the Kronecker pairing. In general each local datum $\mu_Y(q, \lambda_0)$ depends on λ_0. However, the sum $\sum_Y \mu_Y(q, \lambda_0)$ is equal to $\Phi_{0,S^1}(M)(\lambda_0)$ and therefore independent of λ_0 by the rigidity theorem.

Step 2: Each local datum is the expansion of a meromorphic function on $\mathcal{H} \times \mathbb{C}$ where $\mathcal{H}$ denotes the upper half plane. As in the proof of the rigidity theorem given in [3] (cf. also [6, 10, 15]) modularity properties of these functions will be central for the argument. In this step we examine some of their properties.

We begin to recall relevant properties of the series f (see for example [6, 11]). For $0 < |q| < 1$ and $z \in \mathbb{C}$ satisfying $|q| < |e^z| < |q|^{-1}$ the series $f(q, z)$ converges normally to a holomorphic function. This function extends to a meromorphic function $\widetilde{f}(\tau, z)$ on $\mathcal{H} \times \mathbb{C}$ after the change of variables $q = e^{2\pi i \cdot \tau}$ where τ is in $\mathcal{H}$. The function $\widetilde{f}(\tau, z)$ is elliptic in z for the lattice $L := 4\pi i \cdot \mathbb{Z}\langle 1, \tau \rangle$ and satisfies

$$\widetilde{f}(\tau, z + 2\pi i) = -\widetilde{f}(\tau, z), \widetilde{f}(\tau, z + 2\pi i \cdot \tau) = \widetilde{f}(\tau, z)^{-1}, \widetilde{f}(\tau + 2, z) = -\widetilde{f}(\tau, z).$$

The zeros of $\widetilde{f}(\tau, z)$ are simple and located at L and $L + 2\pi i$.

Let $q = e^{2\pi i \cdot \tau}$ and let $\lambda_0 = e^{z_0}$ be a topological generator of S^1. In view of formula (2) and the properties of f the local datum $\mu_Y(q, \lambda_0)$ converges to a meromorphic function $\widetilde{\mu}_Y$ on $\mathcal{H} \times \mathbb{C}$ evaluated at (τ, z_0). We proceed to explain how this function is related to $\widetilde{f}$. For a function F in the variables $x_i, x_{k,j}$ which is smooth in the origin let $\mathcal{T}(F)$ denote the Taylor expansion of F with respect to $x_i, x_{k,j} = 0$. It follows from formula (2) that $\widetilde{\mu}_Y$ is related to $\widetilde{f}$ by (see for example [6]):

$$\widetilde{\mu}_Y(\tau, z_0) = \left\langle \mathcal{T}\left(\prod_i \frac{x_i}{\widetilde{f}(\tau, x_i)} \cdot \prod_{\substack{k \neq 0 \\ j=1,\dots,d_k}} \frac{1}{\widetilde{f}(\tau, x_{k,j} + \alpha_k k \cdot z_0)} \right), [Y] \right\rangle$$

The properties of $\widetilde{f}$ stated above imply corresponding properties for $\widetilde{\mu}_Y$. In particular, $\widetilde{\mu}_Y$ is elliptic for the lattice L and satisfies

$$\widetilde{\mu}_Y(\tau+1,z)=(-1)^{\dim M/4}\cdot\widetilde{\mu}_Y(\tau,z),\quad \widetilde{\mu}_Y(\tau,z+2\pi i)=\pm\widetilde{\mu}_Y(\tau,z).$$

For fixed $\tau\in\mathcal{H}$ the poles of $\widetilde{\mu}_Y$ are contained in $\frac{1}{n}\cdot L$ for some $n\in\mathbb{N}$ depending on the rotation numbers of the S^1-action at Y (see for example [6, 11]).

In general $\widetilde{\mu}_Y(\tau,z)$ depends on z. If $\lambda=e^z$ is a topological generator of S^1, i.e. if $z/(2\pi i)$ is irrational, then $\Phi_{0,S^1}(M)(\lambda)$ converges to the sum $\sum_Y\widetilde{\mu}_Y(\tau,z)$ by the Lefschetz fixed point formula and the latter is independent of z by the rigidity theorem. Note that the original data may be recovered from $\widetilde{\mu}_Y(\tau,z)$ by taking the expansion of $\widetilde{\mu}_Y(\tau,z)$ with respect to $\tau\mapsto\tau+2$.

Step 3: In the final step we study the series $\sum_Y\mu_Y$ in terms of the sum $\sum_Y\widetilde{\mu}_Y(\tau,s(\tau))$ where $s:\mathcal{H}\to\mathbb{C}$ approximates $\tau\mapsto\frac{2}{o}\cdot 2\pi i\cdot\tau$. We choose $s(\tau)$ in such a way that $\widetilde{\mu}_Y(\tau,s(\tau))$ is periodic with respect to $\tau\mapsto\tau+N$ for some $N\in\mathbb{N}$ (see below).

Note that in general the series $\mu_Y(q,\lambda)$ does not converge if λ is close to $e^{\frac{2}{o}\cdot 2\pi i\cdot\tau}$ and the $q^{\frac{1}{N}}$-expansion of $\widetilde{\mu}_Y(\tau,s(\tau))$, denoted by a_Y, is different from the corresponding contribution $\mu_Y(q,\lambda_0)$ in the Lefschetz fixed point formula for $\Phi_{0,S^1}(M)(\lambda_0)$. In particular, we cannot compare $\mu_Y(q,e^{s(\tau)})$ and $\widetilde{\mu}_Y(\tau,s(\tau))$ directly. However, since the sum $\sum_Y\widetilde{\mu}_Y(\tau,z)$ is independent of z the sum $\sum_Y a_Y$ is equal to the elliptic genus in the $\hat{A}$-cusp (see last step). Using the properties of $\widetilde{\mu}_Y$ described above and the assumption on m_o we will show that $\sum_Y a_Y$ has a pole of order less than $\frac{\dim M}{8}-r$. This will complete the proof.

Here are the details. The discussion in the last step implies that the poles of $\widetilde{\mu}_Y$, $Y\subset M^{S^1}$, are contained in $\frac{1}{n}\cdot L$ for some $n\in\mathbb{N}$. Choose $s(\tau):=(1-\beta)\cdot\frac{2}{o}\cdot 2\pi i\cdot\tau$, where β is a fixed rational positive number $\ll\frac{1}{n}$. Hence, $s(\tau)$ is close to $\frac{2}{o}\cdot 2\pi i\cdot\tau$ and $\tau\mapsto\widetilde{\mu}_Y(\tau,s(\tau))$ is holomorphic on $\mathcal{H}$ for every Y. Using $\alpha_k k\equiv\tilde{k}\bmod o$, $\tilde{k}\in\{0,\dots,[\frac{o}{2}]\}$, and the transformation property $\widetilde{f}(\tau,z+4\pi i\cdot\tau)=\widetilde{f}(\tau,z)$ one computes that $\widetilde{\mu}_Y(\tau,s(\tau))$ is (up to sign) equal to $\langle\mathcal{T}(A_Y),[Y]\rangle$, where

$$A_Y:=\prod_i\frac{x_i}{\widetilde{f}(\tau,x_i)}\cdot\prod_{\substack{k\neq 0\\ j=1,\dots,d_k}}\frac{1}{\widetilde{f}(\tau,x_{k,j}+2\cdot(\frac{\tilde{k}}{o}\cdot(1-\beta)-\beta_k)\cdot(2\pi i\cdot\tau))}$$

and $\beta_k:=\beta\cdot\frac{\alpha_k k-\tilde{k}}{o}$.

Note that for some $N\in\mathbb{N}$ (depending on β and the rotation numbers) every summand $\widetilde{\mu}_Y(\tau,s(\tau))$ is periodic with respect to $\tau\mapsto\tau+N$. We claim that its expansion $a_Y\in\mathbb{C}[q^{-\frac{1}{N}}][[q^{\frac{1}{N}}]]$ has a pole of order less than $\frac{\dim M}{8}-r$.

Since the expansion of $\mathcal{T}\left(x_i/\widetilde{f}(\tau,x_i)\right)$ (with respect to $\tau\mapsto\tau+4$) is equal to $x_i/f(q,x_i)$ the expansion of

$$\mathcal{T}\left(\frac{1}{\widetilde{f}(\tau,x_{k,j}+2\cdot(\frac{\tilde{k}}{o}\cdot(1-\beta)-\beta_k)\cdot(2\pi i\cdot\tau))}\right)\qquad(*)$$

can be easily computed in terms of f. The computation shows that the expansion of $(*)$ has a pole of order $\leq\frac{1}{4}-\frac{\tilde{k}}{o}\cdot(1-\beta)+\beta_k$. Since $m_o(Y)\geq m_o>r$ and β, β_k are arbitrarily small it follows that $a_Y\in\mathbb{C}[q^{-\frac{1}{N}}][[q^{\frac{1}{N}}]]$ has a pole of order less than $\frac{\dim M}{8} - r$. As explained in the beginning of this step the sum $\sum_Y a_Y$ is equal to the expansion of the elliptic genus in the $\hat{A}$-cusp. Hence, $\Phi_0(M) \in \mathbb{C}[q^{-\frac{1}{2}}][[q]]$ has a pole of order less than $\frac{\dim M}{8} - r$. This completes the proof. ∎

Remark 3.1. *Essentially the same reasoning applies to orientable S^1-manifolds (not necessarily Spin) for which the equivariant elliptic genus is rigid. The rigidity theorem is known to hold for oriented manifolds with finite second homotopy group [8] and for Spin^c-manifolds with first Chern class a torsion class [4]. Theorem 2.1 is also true for these manifolds.*

4 Highly connected $\mathbf{S^1}$-manifolds

In this section we adapt the arguments of [12] to study the elliptic genus of certain S^1-manifolds including highly connected manifolds. To begin with we recall the Lefschetz fixed point formula for twisted signatures. Let M be an oriented closed S^1-manifold, E an S^1-equivariant vector bundle over M and $\sigma \in S^1$ the element of order 2 . In the following we shall always assume that the fixed point manifold M^σ is orientable (this is the case if M is *Spin* [3]). By the Lefschetz fixed point formula the equivariant twisted signature $sign_{S^1}(M,E) \in R(S^1)$ evaluated at σ is equal to a sum of local data $a_{F,E}$ at the connected components F of the fixed point manifold M^σ

$$sign_{S^1}(M,E)(\sigma) = \sum_F a_{F,E}.$$

The local contributions are given by (cf. [12])

$$a_{F,E} = \langle A_{F,E}, [F] \rangle$$

where

$$A_{F,E} = \prod_i \left(x_i \cdot \frac{1+e^{-x_i}}{1-e^{-x_i}} \right) \cdot \prod_j \left(y_j \cdot \frac{1+e^{-y_j}}{1-e^{-y_j}} \right)^{-1} \cdot ch(E_{|F})(\sigma) \cdot e(\nu_F).$$

Here $\pm x_i$ (resp. $\pm y_j$) denote the formal roots of F (resp. the normal bundle ν_F of F) for compatible orientations of F and ν_F, $e(\nu_F)$ is the Euler class of ν_F and $ch(E_{|F})$ denotes the equivariant Chern character of $E_{|F}$. The local datum $a_{F,E}$ is obtained by evaluating the cohomology class $A_{F,E}$ on the fundamental cycle $[F]$ via the Kronecker pairing $\langle \quad , \quad \rangle$. Note that $a_{F,E}$ vanishes if $e(\nu_F)$ is a torsion class. Hence, the following lemma is immediate.

Lemma 4.1. *Let M and E be as above and let $F \subset M^\sigma$ be of codimension k. If $H^k(F;\mathbb{Q}) = 0$ then the local datum $a_{F,E}$ vanishes.* ∎

For the proof of the next lemma recall that the Euler class of the normal bundle of $i : F \hookrightarrow M$ is equal to $i^*(i_!(1))$, where $i_! : H^*(F;\mathbb{Z}) \to H^{*+k}(M;\mathbb{Z})$ denotes the push forward in cohomology for the oriented normal bundle ν_F.

Lemma 4.2. *Let M and E be as above. If $H^k(M;\mathbb{Q}) = 0$ then $a_{F,E}$ vanishes for any connected component $F \subset M^\sigma$ of codimension k.* ■

We shall now apply these observations to the elliptic genus.

Theorem 4.3. *Let M be a Spin-manifold. Assume that $H^{4*}(M;\mathbb{Q}) = 0$ for $0 < * \leq r$. If M admits a non-trivial S^1-action then the expansion of $\Phi(M)$ in the $\hat{A}$-cusp has a pole of order less than $\frac{\dim M}{8} - r$.*

Proof: Let $\sigma \in S^1$ denote the element of order two. Arguing as in the proof of Theorem 2.1 we may assume that the dimension of M and the dimension of each connected component $F \subset M^\sigma$ is divisible by 4. Consider the expansion $sign_{S^1}(q, \mathcal{L}M)$ of the S^1-equivariant elliptic genus in the signature cusp. By the rigidity theorem $sign_{S^1}(q, \mathcal{L}M)(\sigma)$ is equal to the non-equivariant expansion $sign(q, \mathcal{L}M)$. By the Lefschetz fixed point formula $sign_{S^1}(q, \mathcal{L}M)(\sigma)$ is a sum of local contributions a_F at the connected components F of M^σ:

$$sign(q, \mathcal{L}M) = sign_{S^1}(q, \mathcal{L}M)(\sigma) = \sum_F a_F.$$

Note that each coefficient of the q-power series a_F is the local contribution in the Lefschetz fixed point formula of an equivariant twisted signature evaluated at $\sigma \in S^1$. Since $H^{4*}(M;\mathbb{Q}) = 0$ for $0 < * \leq r$ the contribution a_F vanishes if codim $F \leq 4r$ (see Lemma 4.2). If codim $F > 4r$ then a_F is equal to $sign(q, \mathcal{L}(F \circ F))$ (see formula (1)). Hence,

$$sign(q, \mathcal{L}M) = \sum_{\text{codim } F > 4r} a_F = \sum_{\text{codim } F \circ F > 8r} sign(q, \mathcal{L}(F \circ F)).$$

This implies that the expansion of $\Phi(M)$ in the $\hat{A}$-cusp has a pole of order less than $\frac{\dim M}{8} - r$. ■

Finally note that Theorem 1.3 is a direct consequence of the theorem above.

References

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Anand Dessai
e-mail: dessai@math.uni-augsburg.de
http://www.math.uni-augsburg.de/geo/dessai/homepage.html
Department of Mathematics, University of Augsburg, D-86159 Augsburg