

Homotopy complex projective spaces with $Pin(2)$ -action

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Abstract

Let M be a manifold homotopy equivalent to the complex projective space $\mathbb{C}P^m$. Petrie conjectured that M has standard total Pontrjagin class if M admits a non-trivial action by S^1 . We prove the conjecture for $m < 12$ under the assumption that the action extends to a nice $Pin(2)$ -action with fixed point. The proof involves equivariant index theory for $Spin^c$ -manifolds and Jacobi functions as well as classical results from the theory of transformation groups. ¹

1 Introduction

Let M be a smooth closed manifold homotopy equivalent to the complex projective space $\mathbb{C}P^m$ and let G be a compact Lie group which acts smoothly and non-trivially on $\mathbb{C}P^m$. We consider the problem to determine how close M and $\mathbb{C}P^m$ are as differentiable manifolds if M also supports a non-trivial action by G . In this paper we give an answer to this problem if G is equal to $Pin(2)$, the normalizer of a maximal torus in S^3 , and the dimension of M is less than 24.

By simply-connected surgery theory one knows that for fixed $m \geq 3$ the set of diffeomorphism classes of homotopy $\mathbb{C}P^m$'s is infinite and partitioned into finite subsets by their total Pontrjagin class. In view of this classification one may think of a homotopy complex projective space M as being close to $\mathbb{C}P^m$ if the total Pontrjagin class of M is standard, i.e. if $p(M)$ takes the standard form $(1 + x^2)^{m+1}$, where x is a generator of $H^2(M; \mathbb{Z})$.

To be more specific on the problem above we ask the following strong (resp. weak) question: Is the total Pontrjagin class of a homotopy complex

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projective space M standard (resp. standard up to finite ambiguity), if M also supports a non-trivial action by G ?

The strong question has been answered in special cases by Petrie, Hattori, Masuda and many others (cf. [9] for a survey). Some of their work is stated below. Here we point out that by a result of Petrie (cf. [21]) the total Pontrjagin class of M is standard if M admits an effective action by the m -dimensional torus. The weak question has been considered for certain S^3 -actions in [8]. The questions above are motivated by the following conjecture of Petrie (cf. [20]).

Conjecture 1.1. *Let M be a homotopy $\mathbb{C}P^m$. If M supports a non-trivial smooth S^1 -action then its total Pontrjagin class is standard.*

Petrie's conjecture has been verified in several special cases (again cf. [9] for a survey). In particular, it holds for $m \leq 4$ (cf. [5], [16]) or if the number of fixed point components of the S^1 -action is ≤ 4 (cf. [23], [25], [22], [19]). In contrast the conclusion of the conjecture is known to fail if the S^1 -action is only locally linear or if the group acting on M is finite of arbitrary size (cf. [10], [9]).

One may argue that the positive results towards Petrie's conjecture described above merely express the general principle that the presence of a symmetry group imposes strong restrictions on the topology if the dimension of the symmetry group is 'large' compared to the dimension of the manifold or if the orbit structure is 'simple'. Note however, that by a result of Hattori (cf. [12]) the conclusion of Petrie's conjecture holds in *any* dimension if M admits an S^1 -equivariant stable almost complex structure with standard first Chern class. Hattori's argument is based on vanishing results for equivariant $Spin^c$ -Dirac operators derived from the Lefschetz fixed point formula in equivariant index theory (cf. [1], [2]).

In [7] generalizations of the rigidity theorems for elliptic genera (cf. [24], [3], [14], [18]) to $Spin^c$ -manifolds were given. These results led to a proof of Petrie's conjecture (cf. [6]) in the special case that the first Pontrjagin class of M is standard and M carries a smooth $Pin(2)$ -action with fixed point which is cohomologically trivial and almost effective (i.e. the action has finite kernel). We shall call such an action **nice with fixed point**. Note that such actions exist on any $\mathbb{C}P^m$ for $m > 1$. The main purpose of this paper is to show that in small dimensions the assumption on the first Pontrjagin class can be removed. This gives a partial answer to the strong question stated above.

Theorem 1.2. *Let M be a smooth closed homotopy $\mathbb{C}P^m$ which supports a nice $Pin(2)$ -action with fixed point. If $m < 12$ then the total Pontrjagin class of M is standard.*

To prove the theorem we first give a lower bound for the first Pontrjagin class of M (see Section 4) which we obtain by relating equivariant index theory to the theory of Jacobi functions (see Section 3). Then we combine results of [6] with the homotopy invariance of $p_1(M) \bmod 24$ to complete the proof (see Section 5).

Note that for m odd the $Pin(2)$ -action on M is trivial on cohomology if and only if $Pin(2)$ acts by orientation preserving diffeomorphisms. For m even the $Pin(2)$ -action has a fixed point if it acts trivially on cohomology (see Section 2). In particular, a non-trivial smooth S^3 -action on a homotopy $\mathbb{C}P^{2N}$ induces a nice $Pin(2)$ -action with fixed point by restricting the S^3 -action to the normalizer of a maximal torus. Hence, Theorem 1.2 implies

Corollary 1.3. *Let M be a smooth closed homotopy $\mathbb{C}P^{2N}$ which supports a non-trivial smooth S^3 -action. If $2N < 12$ then the total Pontrjagin class of M is standard. ■*

This paper is structured in the following way. In the next section we give some information on the weights of equivariant vector bundles with vanishing first Pontrjagin class. In Section 3 we define certain series of S^1 -equivariant twisted $Spin^c$ -Dirac operators and express their indices in terms of Jacobi functions. In Section 4 we establish a lower bound for the first Pontrjagin class of a cohomology $\mathbb{C}P^m$ which admits a nice $Pin(2)$ -action with fixed point. In the final section we give the proof for a slightly more general version of Theorem 1.2.

2 Weights and the first Pontrjagin class

In this section we give some information on the weights of $Pin(2)$ -equivariant vector bundles with vanishing first Pontrjagin class. Let M be a $2m$ -dimensional smooth oriented closed manifold with smooth $Pin(2)$ -action. We assume that the $Pin(2)$ -action is trivial on integral cohomology. This property guaranties that the action lifts to complex line bundles over M . For the induced action of $S^1 \subset Pin(2)$ let Y denote a connected component of the fixed point manifold M^{S^1} .

Let $\hat{\xi} \rightarrow M$ be a $Pin(2)$ -equivariant s -dimensional complex vector bundle. Since Y is a trivial S^1 -space the restriction of $\hat{\xi}$ to Y (viewed as an

S^1 -equivariant vector bundle) splits as a finite sum $\hat{\xi}|_Y = \sum_{k \in \mathbb{Z}} \hat{\xi}_k \otimes \lambda^k$. Here $\hat{\xi}_k$ is a complex vector bundle over Y which is trivial as an S^1 -space and λ denotes the standard complex one-dimensional representation of S^1 (to lighten the notation we suppress the dependence of $\hat{\xi}_k$ on Y). Let u_1, \dots, u_s denote the roots of $\sum_k \hat{\xi}_k$ defined using the splitting principle. Then the equivariant roots of $\hat{\xi}|_Y$ are defined as $u_1 + \omega_1 \cdot z, \dots, u_s + \omega_s \cdot z$, where ω_i is equal to k if u_i is a root of $\hat{\xi}_k$ and z is a formal variable. We call $\omega_1, \dots, \omega_s$ the weights of $\hat{\xi}$ at Y . Note that the character of the complex S^1 -representation $\hat{\xi}|_y$, $y \in Y$, is equal to $\sum_i \lambda^{\omega_i}$.

Next assume $\hat{\xi}$ is the complexification of an oriented $Pin(2)$ -equivariant real $2t$ -dimensional vector bundle ξ . Then $\hat{\xi}$ is invariant under conjugation and the equivariant roots of $\hat{\xi}|_Y$ occur in pairs $(u_i + \omega_i \cdot z, u_{t+i} + \omega_{t+i} \cdot z)$, $i = 1, \dots, t$, where $u_{t+i} + \omega_{t+i} \cdot z = -(u_i + \omega_i \cdot z)$. We call $\pm(u_i + \omega_i \cdot z)$, $i = 1, \dots, t$, the equivariant roots and $\pm\omega_i$ the weights of $\xi|_Y$. A spectral sequence argument for the Borel construction of ξ shows (cf. [6], Prop. 3.7)

Proposition 2.1. *If the first Pontrjagin class $p_1(\xi)$ is torsion then $\sum_{i=1}^t \omega_i^2$ is independent of $Y \subset M^{S^1}$. ■*

We apply the proposition in the case that M is a cohomology $\mathbb{C}P^m$, i.e. $H^*(M; \mathbb{Z}) \cong H^*(\mathbb{C}P^m; \mathbb{Z})$, and $p_1(M)$ is equal to $-n \cdot x^2$, where $x \in H^2(M; \mathbb{Z})$ is a generator and n is a non-negative integer. Let $\pm(x_i + m_{Y,i} \cdot z)$, $i = 1, \dots, m$, denote the equivariant roots of $TM|_Y$. Consider the complex line bundle γ over M with $c_1(\gamma) = x$. Since $Pin(2)$ acts trivially on integral cohomology there exists a unique lift of the $Pin(2)$ -action to γ (this follows from [13], cf. [6], Prop. 3.6). Let a_Y be the weight of the induced S^1 -action on γ at Y .

Note that by the assumption on $p_1(M)$ the first Pontrjagin class of the bundle $\xi = TM + n \cdot \gamma$ vanishes. Hence, Proposition 2.1 implies

Corollary 2.2. *There is a constant $C \in \mathbb{Z}$ such that $\sum_{i=1}^m m_{Y,i}^2 + n \cdot a_Y^2 = C$ for any connected component Y of M^{S^1} . ■*

Next we assume that the $Pin(2)$ -action is nice with fixed point, i.e. we assume in addition that the $Pin(2)$ -action has a fixed point $pt \in M$. A simple application of the Lefschetz fixed point formula for the Euler characteristic shows that such a fixed point exists if the Euler characteristic of M is odd (cf. [6], Lemma 3.8). Let Y_0 denote the connected component of M^{S^1} which

contains pt . Note that the representation $\gamma|_{pt}$ is trivial since it is a complex one-dimensional $Pin(2)$ -representation. Hence a_{Y_0} vanishes and we conclude from the corollary above that

$$\sum_{i=1}^m m_{Y,i}^2 + n \cdot a_Y^2 = \sum_{i=1}^m m_{Y_0,i}^2 \quad (*)$$

for any fixed point component Y . This formula is used in Section 4 to give a lower bound for the first Pontrjagin class.

3 Twisted $Spin^c$ -Dirac operators and Jacobi functions

In this section we consider certain series of equivariant twisted $Spin^c$ -Dirac operators closely related to elliptic genera and describe their indices in terms of Jacobi functions. Let M be a $2m$ -dimensional closed connected manifold with $Spin^c$ -structure given by a $Spin^c(2m)$ -principal bundle $P \rightarrow M$.² The $Spin^c$ -structure induces a complex line bundle over M and we denote its first Chern class by $c \in H^2(M; \mathbb{Z})$.

Let $V \rightarrow M$ be a complex vector bundle of dimension s . We fix connections on V and the $U(1)$ -part of P . Let $\partial_{\mathbf{c}} \otimes V$ denote the associated twisted $Spin^c$ -Dirac operator acting on sections of the tensor product of the complex spinor bundle and V . By the Atiyah-Singer index theorem (cf. [2]) its index $ind(\partial_{\mathbf{c}} \otimes V)$ is a topological invariant given by

$$ind(\partial_{\mathbf{c}} \otimes V) = \langle e^{c/2} \cdot \hat{\mathcal{A}}(M) \cdot ch(V), \mu_M \rangle.$$

Here $\hat{\mathcal{A}}(M)$ denotes the multiplicative series for M associated to the \hat{A} -genus, μ_M is the fundamental cycle of M and $\langle \ , \ \rangle$ denotes the pairing between cohomology and homology.

Next assume M carries an S^1 -action and the action lifts to the $Spin^c$ -structure P and to the complex vector bundle V . In this situation the twisted $Spin^c$ -Dirac operator $\partial_{\mathbf{c}} \otimes V$ refines to an S^1 -equivariant operator and its index refines to an element $ind_{S^1}(\partial_{\mathbf{c}} \otimes V)$ of the complex representation ring $R(S^1)$. For any topological generator $\lambda_0 \in S^1$ the equivariant index $ind_{S^1}(\partial_{\mathbf{c}} \otimes V)(\lambda_0)$ may be computed from the Lefschetz fixed point formula (cf. [1], [2]) in terms of local data at the fixed points

$$ind_{S^1}(\partial_{\mathbf{c}} \otimes V)(\lambda_0) = \sum_Y \tilde{\nu}_Y(\lambda_0).$$

²Together with a fixed isomorphism between the induced $SO(2m)$ -principal bundle and the frame bundle.

Here the sum runs over the connected components Y of the fixed point manifold M^{S^1} . To describe the local data $\tilde{\nu}_Y$ it is convenient to replace the S^1 -action by the two-fold action.³ Having done so it follows from the Lefschetz fixed point formula that each local contribution $\tilde{\nu}_Y(\lambda)$ is a rational function in $\lambda \in \mathbb{C}$ which only depends on the restriction of the $Spin^c$ -structure P and the bundle V to Y . Since $ind_{S^1}(\partial_{\mathbb{C}} \otimes V) \in R(S^1)$ is a finite Laurent polynomial in λ the sum $\sum_Y \tilde{\nu}_Y(\lambda)$ extends to a meromorphic function on \mathbb{C} without poles on \mathbb{C}^* .

Below we shall consider a certain series of S^1 -equivariant twisted $Spin^c$ -Dirac operators for which the equivariant index is related to a Jacobi function (see Prop. 3.1). In the next section we employ this relation to study the first Pontrjagin class of a homotopy complex projective space which admits a nice $Pin(2)$ -action with fixed point.

We digress and recall the definition of Jacobi functions. Let $SL_2(\mathbb{Z})$ act on \mathbb{Z}^2 by matrix multiplication from the right, i.e. $(\alpha, \beta) \mapsto (\alpha, \beta)A$ for $A \in SL_2(\mathbb{Z})$, and let \mathcal{H} denote the upper half-plane.

A meromorphic function $F(\tau, z)$ on $\mathcal{H} \times \mathbb{C}$ is called a Jacobi function for $SL_2(\mathbb{Z}) \times \mathbb{Z}^2$ of weight k and index I if

$$F(\tau, z + \alpha\tau + \beta) = F(\tau, z) \cdot e^{-2\pi i \cdot I \cdot (\alpha^2 \cdot \tau + 2\alpha z)}$$

for $(\alpha, \beta) \in \mathbb{Z}^2$ and

$$F\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = F(\tau, z) \cdot (c\tau + d)^k \cdot e^{2\pi i \cdot I \cdot \frac{c \cdot z^2}{c\tau + d}}$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. In view of these equations one may also define Jacobi functions of weight k and index I as fixed points under an action of $SL_2(\mathbb{Z}) \times \mathbb{Z}^2$ on the ring of meromorphic functions on $\mathcal{H} \times \mathbb{C}$ (cf. [11] where the definition also involves conditions for the cusps).

In topology Jacobi functions occur naturally as local contributions in the Lefschetz fixed point formula of elliptic genera for $Spin$, stable almost complex or $BO\langle 8 \rangle$ -manifolds (cf. [24], [3], [14], [18]).

We shall now consider some generalizations of elliptic genera to S^1 -equivariant $Spin^c$ -manifolds. As before let V be an S^1 -equivariant s -dimensional complex vector bundle over M . We define a q -power series $\mathcal{U}_V \in K_{S^1}(M)[[q]]$ of virtual S^1 -equivariant vector bundles by

$$\mathcal{U}_V := \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{TM} \otimes_{\mathbb{R}} \mathbb{C}) \otimes \Lambda_{-1}(V^*) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{-q^n}(\widetilde{V} \otimes_{\mathbb{R}} \mathbb{C}).$$

³This is not necessary but makes formulas easier. In particular, some functions which are only well defined on the covering $\mathbb{C}^* \rightarrow \mathbb{C}^*, \lambda \mapsto \lambda^2$, are defined on the base after passing to the two-fold action.

Here q is a formal variable, \tilde{E} denotes the reduced vector bundle $E - \dim(E)$ and $\Lambda_t := \sum \Lambda^i \cdot t^i$ (resp. $S_t := \sum S^i \cdot t^i$) denotes the exterior (resp. symmetric) power operation. The tensor product is, if not indicated otherwise, taken over the complex numbers.

The index of the equivariant $Spin^c$ -Dirac operator twisted with \mathcal{U}_V is a q -power series of representations $ind_{S^1}(\partial_c \otimes \mathcal{U}_V) \in R(S^1)[[q]]$. By the Lefschetz fixed point formula the equivariant index at a topological generator λ_0 of S^1 is a sum of local data

$$ind_{S^1}(\partial_c \otimes \mathcal{U}_V)(\lambda_0) = \sum_Y \tilde{\nu}_Y(q, \lambda_0).$$

Each local datum $\tilde{\nu}_Y(q, \lambda)$ is an element of $\mathbb{C}(\lambda)[[q]]$ which only depends on the restriction of the $Spin^c$ -structure P and \mathcal{U}_V to Y . In order to explain its relation to Jacobi functions we need to introduce some notation for the local data at the S^1 -fixed points.

For a connected component Y of M^{S^1} define $d(Y) := \dim(Y)/2$ (since M is of even dimension the same holds for Y). The tangent bundle TM restricted to Y splits equivariantly as the direct sum of TY and the normal bundle $\mathcal{N}(Y)$ which inherits a complex structure from the S^1 -action. Let $x_i + m_{Y,i} \cdot z$, $d(Y) < i \leq m$, denote the equivariant roots of $\mathcal{N}(Y)$ (to lighten the notation we suppress the dependence of x_i on Y).

On Y we choose the orientation which is compatible with the orientation of M and the complex normal bundle $\mathcal{N}(Y)$. Let $\pm x_1, \dots, \pm x_{d(Y)}$ denote a set of roots of TY such that $x_1 \cdot \dots \cdot x_{d(Y)}$ is equal to the Euler class of the oriented vector bundle TY and let $m_{Y,i} = 0$ for $i \leq d(Y)$. Note that $\pm(x_i + m_{Y,i} \cdot z)$ are the equivariant roots of $TM|_Y$ as introduced in Section 2.

Recall that the $Spin^c$ -structure induces a complex line bundle over M . Let l_Y be its weight at Y . The equivariant roots of V at Y shall be denoted by $v_1 + s_{Y,1} \cdot z, \dots, v_s + s_{Y,s} \cdot z$. Next let

$$I_Y := \frac{1}{2} \left(\sum_j s_{Y,j}^2 - \sum_i m_{Y,i}^2 \right),$$

which is an integer since we are looking at the two-fold action. Finally, define $n(V|_Y) := \dim_{\mathbb{C}}(V_0)$, where $V_0 := (V|_Y)^{S^1}$ denotes the subbundle of $V|_Y$ which is fixed under the S^1 -action.

We are now in the position to state the main result of this section which we use in the following section to derive a lower bound for the first Pontrjagin class of a cohomology $\mathbb{C}P^m$ with $Pin(2)$ -action.

Proposition 3.1. For $q = e^{2\pi i \cdot \tau}$, $\tau \in \mathcal{H}$, and $\lambda_0 = e^{2\pi i \cdot z_0}$ a topological generator of $S^1 \subset \mathbb{C}$ the series $\nu_Y(\tau, z_0) := \tilde{\nu}_Y(q, \lambda_0)$ converges to a meromorphic function on $\mathcal{H} \times \mathbb{C}$ also denoted by $\nu_Y(\tau, z)$. The sum $\sum_Y \nu_Y(\tau, z)$ has no poles on $z \in \mathbb{R}$.

1. If $n(V|_Y) > d(Y)$ then $\nu_Y(\tau, z)$ and $\tilde{\nu}_Y(q, \lambda)$ vanish identically.
2. If $n(V|_Y) = d(Y)$ then $\nu_Y(\tau, z)$ is the product of a holomorphic function $e(z)$ and a Jacobi function F_Y for $SL_2(\mathbb{Z}) \times \mathbb{Z}^2$ of index I_Y . For any fixed $\tau \in \mathcal{H}$ the set of poles of F_Y is contained in $\mathbb{Q} \cdot \tau + \mathbb{Q}$.

Proof: We describe the local datum $\tilde{\nu}_Y(q, \lambda_0)$ in terms of the Weierstraß' Φ -function and the equivariant roots. Recall that $\Phi(\tau, z)$ is a holomorphic function on $\mathcal{H} \times \mathbb{C}$ defined by the normally convergent infinite product

$$\phi(q, \lambda) := (\lambda^{\frac{1}{2}} - \lambda^{-\frac{1}{2}}) \cdot \prod_{n \geq 1} \frac{(1 - q^n \cdot \lambda) \cdot (1 - q^n \cdot \lambda^{-1})}{(1 - q^n)^2} \in \mathbb{C}[\lambda^{\frac{1}{2}}, \lambda^{-\frac{1}{2}}][[q]],$$

where $q = e^{2\pi i \cdot \tau}$ and $\lambda = e^{2\pi i \cdot z}$. For a topological generator λ_0 of S^1 the power series $\tilde{\nu}_Y(q, \lambda_0) \in \mathbb{C}[[q]]$ is given by

$$\tilde{\nu}_Y(q, \lambda_0) = \langle A(q, \lambda_0), \mu_Y \rangle \quad (1)$$

where $A(q, \lambda) \in H^*(Y; \mathbb{C}(\lambda)[[q]])$ is defined as⁴

$$e^{\frac{1}{2} \cdot (c - \sum_j v_j)} \cdot \lambda^{\frac{1}{2} \cdot (l_Y - \sum_j s_{Y,j})} \times \\ \times \prod_{m_{Y,i}=0} \frac{x_i}{\phi(q, e^{x_i})} \cdot \prod_{m_{Y,i} \neq 0} \frac{1}{\phi(q, e^{x_i} \cdot \lambda^{m_{Y,i}})} \cdot \prod_{j=1}^s \phi(q, e^{v_j} \cdot \lambda^{s_{Y,j}}).$$

Here μ_Y is the fundamental cycle of Y , the characteristic classes are expressed in terms of their formal roots and $\langle \ , \ \rangle$ denotes the pairing between cohomology and homology. Recall that l_Y is the weight at Y of the complex line bundle associated to the equivariant $Spin^c$ -structure. To prove formula (1) one computes the Chern character of \mathcal{U}_V and applies the Lefschetz fixed point formula to $ind_{S^1}(\partial_{\mathbf{c}} \otimes \mathcal{U}_V)(\lambda_0)$ (for details cf. [7]).

Recall that $\phi(q, \lambda)$ converges normally to $\Phi(\tau, z)$. This implies that for fixed $\lambda_0 = e^{2\pi i \cdot z_0}$ and any $q = e^{2\pi i \cdot \tau}$, $\tau \in \mathcal{H}$, the series $A(q, \lambda_0) \in H^*(Y; \mathbb{C})[[q]]$ converges to a well defined element $\mathcal{A}_{z_0}(\tau)$ in the cohomology of Y with values in the ring of holomorphic functions on \mathcal{H} . Moreover, there exists an element $\mathcal{A}(\tau, z)$ in the cohomology of Y with values in the ring $\mathcal{M}(\mathcal{H} \times \mathbb{C})$

⁴Since we have passed to the two-fold action this is an expression in λ rather than $\lambda^{\frac{1}{2}}$.

of meromorphic functions on $\mathcal{H} \times \mathbb{C}$ such that $\mathcal{A}(\tau, z_0) = \mathcal{A}_{z_0}(\tau)$ for any irrational $z_0 \in \mathbb{R}$ (for details cf. [7]). Changing variables we conclude from formula (1) that the series $\nu_Y(\tau, z)$ converges for any irrational real number z to the meromorphic function $\langle \mathcal{A}(\tau, z), \mu_Y \rangle$ (in the following also denoted by $\nu_Y(\tau, z)$).

Note that each coefficient of the q -power series $ind_{S^1}(\partial_{\mathbb{C}} \otimes \mathcal{U}_Y)$, being a finite Laurent polynomial in λ , is holomorphic on $S^1 \subset \mathbb{C}$. This implies that the sum $\sum_Y \nu_Y(\tau, z)$ has no poles on $z \in \mathbb{R}$ (again cf. [7] for details). Next we consider the statements involving the dimension $n(V_Y)$ of V_0 .

Ad (1): Assume $n(V_Y) > d(Y)$. Recall that the Weierstraß' Φ -function $\Phi(\tau, z)$ has a simple zero in $z = 0$ for any $\tau \in \mathcal{H}$. This implies that $\mathcal{A}(\tau, z)$ contains the Euler class $e(V_0) = \prod_{s_{Y,j}=0} v_j$ of V_0 as a factor. Since $n(V_Y) > d(Y)$ the function $\nu_Y(\tau, z) = \langle \mathcal{A}(\tau, z), \mu_Y \rangle$ vanishes for any irrational $z \in \mathbb{R}$. Being meromorphic this forces $\nu_Y(\tau, z)$ (and also $\tilde{\nu}_Y(q, \lambda)$) to vanish identically.

Ad (2): Assume $n(V_Y) = d(Y)$. Then $e(V_0)$ is in the top degree of the cohomology of Y . The local datum $\nu_Y(\tau, z_0)$ is equal to the product of the Euler number of V_0 and $\mathcal{A}_0(\tau, z_0)$, where $\mathcal{A}_0(\tau, z) = e^{\pi i(l_Y - \sum_j s_{Y,j} \cdot z)} \cdot F_Y(\tau, z)$ and

$$F_Y(\tau, z) = \prod_{m_{Y,i} \neq 0} \frac{1}{\Phi(\tau, m_{Y,i} \cdot z)} \cdot \prod_{s_{Y,j} \neq 0} \Phi(\tau, s_{Y,j} \cdot z).$$

To see this consider $A(q, \lambda)$, recall that $\phi(q, \lambda)$ converges to $\Phi(\tau, z)$ and note that $\phi(q, e^x)$ has the form $x + O(x^3)$. Whereas $\mathcal{A}(\tau, z)$ depends on the equivariant roots $\mathcal{A}_0(\tau, z)$ only depends on the weights of TM and V at Y .

We proceed to identify $F_Y(\tau, z)$ with a Jacobi function. The Weierstraß' Φ -function is a holomorphic Jacobi function for $SL_2(\mathbb{Z}) \times \mathbb{Z}^2$ of weight -1 and 'index $\frac{1}{2}$ with character' (cf. [11]). More precisely, Φ is holomorphic and satisfies

$$\Phi(\tau, z + \alpha \cdot \tau + \beta) = \Phi(\tau, z) \cdot e^{-\pi i(\alpha^2 \tau + 2\alpha \cdot z)} \cdot (-1)^{\alpha + \beta}$$

for $(\alpha, \beta) \in \mathbb{Z}^2$ and

$$\Phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = \Phi(\tau, z) \cdot (c\tau + d)^{-1} \cdot e^{\pi i \cdot \frac{c \cdot z^2}{c\tau + d}}$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. In particular, if s is even, then $\Phi(\tau, s \cdot z)$ is a holomorphic Jacobi function of weight -1 and index $\frac{s^2}{2}$ (as defined in the beginning of this section). For fixed τ the divisor of $\Phi_\tau(z) := \Phi(\tau, z)$ is equal to $\mathbb{Z}\tau + \mathbb{Z}$, i.e. Φ_τ has a simple zero in each lattice point. From these properties it follows that $F_Y(\tau, z)$ is a Jacobi function for $SL_2(\mathbb{Z}) \times \mathbb{Z}^2$ of index I_Y with poles in $\mathbb{Q} \cdot \tau + \mathbb{Q}$. This completes the proof of the proposition. \blacksquare

4 A lower bound for the first Pontrjagin class

In this section we give a lower bound for the first Pontrjagin class of a cohomology complex projective space which supports a nice $Pin(2)$ -action with fixed point. Let M be a $2m$ -dimensional manifold with $H^*(M; \mathbb{Z}) \cong H^*(\mathbb{C}P^m; \mathbb{Z})$. We assume that $Pin(2)$ acts almost effectively on M and trivially on cohomology. Also we assume that the action has a fixed point. As mentioned before the latter assumption is automatically satisfied if the Euler characteristic of M is odd, i.e. if m is even.

Let γ denote the complex line bundle with $c_1(\gamma) = x$, where x is a fixed generator of $H^2(M; \mathbb{Z})$. Since the $Pin(2)$ -action is trivial on integral cohomology, we can lift the action uniquely to γ (this follows from [13], cf. [6], Prop. 3.6). Let a_Y denote the weight of γ at a connected component Y of M^{S^1} for the induced S^1 -action.

Next we recall some well known facts about M and γ which may be induced from the localization theorem in K -theory applied to the S^1 -action and induced \mathbb{Z}_p -actions or by cohomological means (cf. for example [20], Th. 2.8, [4] Ch. VII, [15]):

- (i) The fixed point manifold M^{S^1} is a disjoint union $Y_0 \cup \dots \cup Y_k$, where the integral cohomology ring of Y_i is isomorphic to $\mathbb{C}P^{m_i}$ for some m_i .
- (ii) $\sum_{i=0}^k (m_i + 1) = m + 1$.
- (iii) The weights of γ are distinct.

Assume $a_{Y_0} = 0$, i.e. assume that the $Pin(2)$ -fixed point pt is in Y_0 . Let V be the sum of S^1 -equivariant complex line bundles over M given by

$$V := d(Y_0) \cdot \gamma + \sum_{i=1}^k (d(Y_i) + 1) \cdot \gamma \otimes \lambda^{-a_{Y_i}},$$

where λ denotes the standard complex one-dimensional representation of S^1 . We apply Proposition 3.1 to $ind_{S^1}(\partial_{\mathbf{c}} \otimes \mathcal{U}_V)$ to derive a lower bound for the first Pontrjagin class.

Proposition 4.1. *Let M be a cohomology $\mathbb{C}P^m$ as above. If $p_1(M) = -n \cdot x^2$ then $n < m$.*

Proof: We may assume that n is non-negative. Using the Atiyah-Singer index theorem one computes that the non-equivariant index $ind(\partial_{\mathbf{c}} \otimes \mathcal{U}_V)$ does not vanish. We proceed to describe the equivariant index in terms of

local data. For technical reasons we replace the S^1 -action by its two-fold action. Recall that $d(Y_i)$ denotes half of the dimension of Y_i , i.e. $d(Y_i) = m_i$, and $n(V_{|Y_i})$ denotes the complex dimension of the subbundle V_0 of $V_{|Y_i}$ which is fixed under the S^1 -action. Since the weights a_{Y_0}, \dots, a_{Y_k} of γ are distinct and $a_{Y_0} = 0$ we have $n(V_{|Y_i}) = d(Y_i) + 1$ if $i > 0$ and $n(V_{|Y_0}) = d(Y_0)$.

Next consider the local datum $\nu_{Y_i}(\tau, z)$ in the Lefschetz fixed point formula for $\text{ind}_{S^1}(\partial_{\mathbf{c}} \otimes \mathcal{U}_V)$. By Proposition 3.1 $\nu_{Y_i}(\tau, z)$ vanishes for $i > 0$, $\nu_{Y_0}(\tau, z)$ is the product of a holomorphic function $e(z)$ and a meromorphic function $F_{Y_0}(\tau, z)$ and $\text{ind}_{S^1}(\partial_{\mathbf{c}} \otimes \mathcal{U}_V)(\lambda)$ converges to

$$\sum_i \nu_{Y_i}(\tau, z) = \nu_{Y_0}(\tau, z) = e(z) \cdot F_{Y_0}(\tau, z)$$

for $q = e^{2\pi i \tau}$ and any topological generator $\lambda = e^{2\pi i z}$. Moreover $F_{Y_0}(\tau, z)$ is a Jacobi function for $SL_2(\mathbb{Z}) \times \mathbb{Z}^2$ of index

$$I_{Y_0} := \frac{1}{2} \left(\sum_{i=1}^k (d(Y_i) + 1) \cdot a_{Y_i}^2 - \sum_{j=1}^m m_{Y_0, j}^2 \right)$$

with poles in $\mathbb{Q}\tau + \mathbb{Q}$. Note that $F_{Y_0}(\tau, z)$ and $e(z)$ cannot vanish identically, since the non-equivariant index $\text{ind}(\partial_{\mathbf{c}} \otimes \mathcal{U}_V)$ does not vanish.

It follows from the proof of Proposition 3.1 that $e(z)$ has no zeros on $z \in \mathbb{R}$. Since $\sum_i \nu_{Y_i}(\tau, z)$ has no poles on $z \in \mathbb{R}$ the same holds for $F_{Y_0}(\tau, z)$.

Next consider the action of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ on $\mathcal{H} \times \mathbb{C}$ given by $A(\tau, z) = \left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d} \right)$. Note that the $SL_2(\mathbb{Z})$ -orbit of any element of $\{\tau\} \times (\mathbb{Q}\tau + \mathbb{Q})$ intersects with $\{\tau_0\} \times \mathbb{Q}$ for some $\tau_0 \in \mathcal{H}$. Since $F_{Y_0}(\tau, z)$ has poles in $\mathbb{Q}\tau + \mathbb{Q}$ but no poles on \mathbb{R} we conclude that F_{Y_0} has no poles at all. Hence, F_{Y_0} is a holomorphic Jacobi function of index I_{Y_0} which does not vanish identically. Since a holomorphic Jacobi function of negative index must vanish identically (cf. [11]) the index I_{Y_0} is non-negative, i.e.

$$\sum_{j=1}^m m_{Y_0, j}^2 \leq \sum_{i=1}^k (d(Y_i) + 1) \cdot a_{Y_i}^2.$$

Let Z be a connected fixed point component of M^{S^1} such that $a_Z^2 = \max_i \{a_{Y_i}^2\}$. By equation (*)

$$\sum_{j=1}^m m_{Y_0, j}^2 = \sum_{j=1}^m m_{Z, j}^2 + n \cdot a_Z^2.$$

Hence,

$$\sum_{j=1}^m m_{Z, j}^2 + n \cdot a_Z^2 \leq \sum_{i=1}^k (d(Y_i) + 1) \cdot a_{Y_i}^2.$$

This implies $n < m$ since $a_Z^2 \geq a_{Y_i}^2$ and $\sum_{i>0} (d(Y_i) + 1) \leq m$. ■

5 Rigidity of Pontrjagin classes

In this section we prove Theorem 1.2. In fact we show the following slightly more general result.

Theorem 5.1. *Let M be a smooth cohomology $\mathbb{C}P^m$, $m < 12$, which supports an almost effective smooth $Pin(2)$ -action which is trivial on cohomology. If m is odd assume in addition that the action has a fixed point. Then the total Pontrjagin class of M is standard, i.e. $p(M) = (1 + x^2)^{m+1}$.*

Remarks 5.2. 1. *The theorem above is slightly more general than Theorem 1.2 since a cohomology $\mathbb{C}P^m$ may have non-trivial fundamental group.*

2. *For m odd there are S^3 -actions on $\mathbb{C}P^m$ with fixed point free $Pin(2)$ -action. The homogeneous action on $S^3/S^1 \cong \mathbb{C}P^1$ is an example.*

Proof of Theorem 5.1: First note that the $Pin(2)$ -action on M always has a fixed point (for m even this is true since the Euler characteristic of M is odd). We order $H^4(M; \mathbb{Z}) = \mathbb{Z} \cdot x^2$ by identifying $H^4(M; \mathbb{Z})$ with the integers using $x^2 \mapsto 1$. By Proposition 4.1 the first Pontrjagin class satisfies $p_1(M) > -m \cdot x^2$. In [6], Th. 4.2, it was shown that $p_1(M) \leq (m + 1) \cdot x^2$ using methods similar to those of Section 3. For the convenience of the reader we sketch the argument below (see Th. 5.3). Hence,

$$-m \cdot x^2 < p_1(M) \leq (m + 1) \cdot x^2. \quad (2)$$

Next note that $p_1(M)$ is a cohomology invariant modulo 24, i.e. $p_1(M) \equiv (m + 1) \cdot x^2 \pmod{24}$. To see this choose a $Spin^c$ -structure on M with first Chern class $c = (m+1) \cdot x$ and let $V := (\gamma - 1)^{m-2}$. By the Atiyah-Singer index theorem the index of the (non-equivariant) $Spin^c$ -Dirac operator twisted with V is equal to

$$\langle e^{\frac{c}{2}} \cdot \hat{\mathcal{A}}(M) \cdot (e^x - 1)^{m-2}, \mu_M \rangle.$$

Note that $\hat{\mathcal{A}}(M) = 1 - \frac{p_1(M)}{24} +$ terms of higher order. Let b be the integer defined by $p_1(M) = b \cdot x^2$. Then the index takes the form $\frac{b}{24} - Q$, where Q is a rational number which only depends on the cohomology ring of M .

Since the index is an integer it follows that $\frac{b}{24} \equiv Q$ modulo the integers. The same computation shows for the standard complex projective space that $\frac{m+1}{24} \equiv Q \pmod{\mathbb{Z}}$. Hence, $b \equiv m+1 \pmod{24}$, i.e. $p_1(M) \equiv (m+1) \cdot x^2$ modulo 24. Since $m < 12$ it follows from equation (2) that the first Pontrjagin class is standard, i.e.

$$p_1(M) = (m+1) \cdot x^2.$$

As mentioned in the introduction it was shown in [6] that the total Pontrjagin class of M is standard if $p_1(M)$ is standard (see Th. 5.3 below). This completes the proof. \blacksquare

Theorem 5.3. *Let M be a cohomology $\mathbb{C}P^m$ with nice $Pin(2)$ -action. If m is odd assume in addition that the action has a fixed point. Then the first Pontrjagin class satisfies $p_1(M) \leq (m+1) \cdot x^2$. Moreover $p(M)$ is standard if $p_1(M) = (m+1) \cdot x^2$.*

Proof: For a detailed proof we refer to [6], Th. 4.2. Here is a sketch of the argument based on the following general vanishing result. Let V be a $Pin(2)$ -equivariant complex vector bundle and let W be a $Pin(2)$ -equivariant $2t$ -dimensional $Spin$ -vector bundle over M . Let $\pm(w_1 + t_{Y,1} \cdot z), \dots, \pm(w_t + t_{Y,t} \cdot z)$ denote the equivariant roots of W restricted to a connected component Y of M^{S^1} . The equivariant roots of V and TM shall be denoted as in Section 3. Assume $p_1(V+W) = p_1(M)$. Using a spectral-sequence argument one shows that

$$I := \frac{1}{2} \left(\sum_{j=1}^s s_{Y,j}^2 + \sum_{k=1}^t t_{Y,k}^2 - \sum_{i=1}^m m_{Y,i}^2 \right) \quad (3)$$

is independent of Y . Next consider the q -power series $\mathcal{U}_{V,W} \in K_{S^1}(M)[[q]]$ of virtual S^1 -equivariant vector bundles defined by

$$\mathcal{U}_{V,W} := \mathcal{U}_V \otimes \Delta(\widetilde{W}) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{q^n}(\widetilde{W} \otimes_{\mathbb{R}} \mathbb{C}).$$

Here $\Delta(W)$ denotes the full complex spinor bundle associated to the $Spin$ -vector bundle W . We fix a $Spin^c$ -structure on M and lift the induced S^1 -action. Next one shows (by arguments similar to the ones of the previous sections) that the equivariant index $ind_{S^1}(\partial_{\mathbf{c}} \otimes \mathcal{U}_{V,W})$ is equal to the product of a holomorphic function and a Jacobi function (for $\Gamma_0(2) \subset SL_2(\mathbb{Z})$) of index I if $p_1(V+W) = p_1(M)$ and $c_1(V)$ is equal to the first Chern class of the $Spin^c$ -manifold M . As in the proof of the rigidity of elliptic genera one

can show that the Jacobi function is in fact holomorphic. This implies that $ind_{S^1}(\partial_c \otimes \mathcal{U}_{V,W})$ vanishes identically if I is negative.

Now assume $p_1(M) > (m+1) \cdot x^2$. We want to show a contradiction. To this end fix a $Spin^c$ -structure on M with first Chern class equal to $(m+1) \cdot x$. Let $V := (m-1) \cdot \gamma + \gamma^2$ and let $W := (b-m-3) \cdot \gamma$, where $b > m+1$ is defined by $p_1(M) = b \cdot x^2$ (note that $b \geq m+3$ since $p_1(M) \equiv (m+1) \cdot x^2 \pmod{2}$). We lift the $Pin(2)$ -action to each line bundle occurring in V and W . Note that $p_1(V+W) = p_1(M)$ and $c_1(V)$ is equal to the first Chern class of M . Since the weights of V and W at the $Pin(2)$ -fixed point vanish it follows from equation (3) that I is negative. Hence, by the result above $ind_{S^1}(\partial_c \otimes \mathcal{U}_{V,W})$ vanishes identically. In particular, the non-equivariant index vanishes. However, one computes with the help of the Atiyah-Singer index theorem that the series $ind(\partial_c \otimes \mathcal{U}_{V,W})$ does not vanish. This contradicts the assumption on $p_1(M)$. Thus $p_1(M) \leq (m+1) \cdot x^2$.

Next assume $p_1(M) = (m+1) \cdot x^2$. We want to show that $p(M)$ is standard. To this end let $V_k := \gamma^2 + (m-3-2k) \cdot \gamma$, $W_k := (2k) \cdot \gamma$, $k \in \{0, \dots, \lfloor \frac{m-3}{2} \rfloor\}$, and choose a $Spin^c$ -structure on M with first Chern class equal to $c_1(V_k)$. Note that $p_1(V_k + W_k - TM) = 0$. Again we lift the $Pin(2)$ -action to each line bundle occurring in V_k and W_k and conclude from equation (3) that I is negative. By the result above $ind_{S^1}(\partial_c \otimes \mathcal{U}_{V,W})$ vanishes identically. In particular, the constant term in the q -power series is zero, i.e.

$$\left\langle \hat{\mathcal{A}}(M) \cdot (e^x - e^{-x}) \cdot (e^{\frac{x}{2}} - e^{-\frac{x}{2}})^{m-3-2k} \cdot (e^{\frac{x}{2}} + e^{-\frac{x}{2}})^{2k}, \mu_M \right\rangle = 0$$

for $k \in \{0, \dots, \lfloor \frac{m-3}{2} \rfloor\}$. These relations together with the signature theorem completely determine $\hat{\mathcal{A}}(M)$ and therefore determine the total Pontrjagin class $p(M)$. Since all these relations also hold true for $\mathbb{C}P^m$ we conclude that $p(M) = (1+x^2)^{m+1}$. Hence $p(M)$ is standard if $p_1(M)$ is standard. ■

References

- [1] M.F. Atiyah and G.B. Segal: The index of elliptic operators: II, *Ann. of Math.* **87** (1968), 531-545
- [2] M.F. Atiyah and I.M. Singer: The index of elliptic operators: III, *Ann. of Math.* **87** (1968), 546-604
- [3] R. Bott and C.H. Taubes: On the rigidity theorems of Witten, *J. of Amer. Math. Soc.* **2** (1989), 137-186
- [4] G. Bredon: *Introduction to compact transformation groups*, Pure and applied math., Academic Press, Vol. 46 (1972)

- [5] I.J. Dejter: Smooth S^1 -manifolds in the homotopy type of $\mathbb{C}P^3$, *Michigan Math. J.* **23** (1976), 83-95
- [6] A. Dessai: $Spin^c$ -manifolds with $Pin(2)$ -action, *Math. Ann.* **315** (1999), 511-528
- [7] A. Dessai: Rigidity theorems for $Spin^c$ -manifolds, *Topology* **39** (2000), 239-258
- [8] A. Dessai: Bordism-finiteness and semi-simple group actions, *preprint*, available at the xxx-archive: <http://arxiv.org/abs/math.DG/0003176> (2000)
- [9] K.H. Dovermann: *Cyclic group actions on homotopy complex projective spaces*, Algebra and Topology (edited by Koo and Suh), Conference Proceedings Korea Institute of Technology (1986), 164-271
- [10] K.H. Dovermann and M. Masuda: Exotic cyclic actions on homotopy complex projective spaces, *J. Fac. Sci. Univ. Tokyo* **37** (1990), 335-376
- [11] M. Eichler and D. Zagier: *The Theory of Jacobi Forms*, Progress in Mathematics, Vol 55, Birkhäuser (1985)
- [12] A. Hattori: $Spin^c$ -Structures and S^1 -Actions, *Invent. Math.* **48** (1978), 7-31
- [13] A. Hattori and T. Yoshida: Lifting compact group actions in fiber bundles, *Japan. J. Math.* **2** (1976), 13-25
- [14] F. Hirzebruch: Elliptic genera of level N for complex manifolds, in: K. Bleuler and M. Werner (Eds.): *Differential Geometrical Methods in Theoretical Physics (Como 1987)*, NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci., 250, Kluwer (1988)
- [15] W.Y. Hsiang: *Cohomology Theory of Topological Transformation Groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol 85, Springer (1975)
- [16] D.M. James: Smooth S^1 -actions on homotopy $\mathbb{C}P^4$'s, *Michigan Math. J.* **32** (1985), 259-266
- [17] P.S. Landweber (Ed.): *Elliptic Curves and Modular Forms in Algebraic Topology, Proceedings Princeton 1986*, Lecture Notes in Mathematics, 1326, Springer (1988)
- [18] K. Liu: On modular invariance and rigidity theorems, *J. Diff. Geo.* **41** (1995), 343-396

- [19] M. Masuda: On smooth S^1 -actions on cohomology complex projective spaces. The case where the fixed point set consists of four connected components, *J. Fac. Sci. Univ. Tokyo* **28** (1981), 127-167
- [20] T. Petrie: Smooth S^1 -actions on homotopy complex projective spaces and related topics, *Bull. Math. Soc.* **78** (1972), 105-153
- [21] T. Petrie: Torus actions on homotopy complex projective spaces, *Invent. Math.* **20** (1973), 139-146
- [22] E. Tsukada and R. Washiyama: Smooth S^1 -actions on cohomology complex projective spaces with three components of the fixed point set, *Hiroshima Math. J.* **9** (1979), 41-46
- [23] K. Wang: Differentiable Circle Group Actions on Homotopy Complex Projective Spaces, *Math. Ann.* **214** (1975), 73-80
- [24] E. Witten: The Index of the Dirac Operator in Loop Space, in: [17], 161-181
- [25] T. Yoshida: On smooth semi-free S^1 -actions on cohomology complex projective spaces, *Publ. Res. Inst. Math. Sci.* **11** (1976), 483-496

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