On the topology of scalar-flat manifolds

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Abstract

Let M be a simply-connected closed manifold of dimension ≥ 5 which does not admit a metric with positive scalar curvature. We give necessary conditions for M to admit a scalar-flat metric. These conditions involve the first Pontrjagin class and the cohomology ring of M. As a consequence any simply-connected scalar-flat manifold of dimension ≥ 5 with vanishing first Pontrjagin class admits a metric with positive scalar curvature. We also describe some relations between scalar-flat metrics, almost complex structures and the free loop space.[1]

1. *Introduction*

In this paper we give restrictions on the possible topological type of connected closed scalar-flat manifolds. A Riemannian manifold is called scalar-flat if its scalar curvature vanishes identically. One way to obtain such manifolds (in dimension ≥ 3) is to start with a metric with positive scalar curvature and decrease the scalar curvature by changing the metric. By the results of Kazdan-Warner (cf. [**14**]) one can deform the metric globally by a conformal change to obtain a scalar-flat metric. Recently Lohkamp (cf. [**16**]) has shown that scalar decreasing deformations can even be carried out locally to achieve changes of the scalar curvature arbitrarily close to a prescribed decrease.

However, not every scalar-flat metric arises from a metric with positive scalar curvature. The positive energy theorem and the Lichnerowicz formula for *Spin*-manifolds imply that it is in general impossible to increase scalar curvature locally or globally. In the following we shall call a scalar-flat manifold *strongly scalar-flat* if it does not admit a metric with positive scalar curvature. By a result of Bourguignon such manifolds are Ricci-flat. Otherwise one could deform the

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metric using the Ricci tensor to obtain a new metric which would be pointwise conformal to one with positive scalar curvature (cf. [**14**]).

Although strongly scalar-flat manifolds seem to be quite special very little is known about their topological type. In contrast the surgery theorem of Gromov-Lawson and Schoen-Yau (cf. [**7**], [**18**]) and Stolz' solution of the Gromow-Lawson conjecture (cf. [**19**]) give a complete topological classification for simply-connected manifolds of dimension ≥ 5 which admit a metric with positive scalar curvature.

In Theorem 2.1 we give partial information on the topological type of strongly scalar-flat manifolds. Our result shows that many manifolds are not strongly scalar-flat. In particular, it implies the following

Theorem 1.1. *Let M be a simply-connected scalar-flat manifold of dimension ≥ 5. If the first real Pontrjagin class of M vanishes then M is not strongly scalar-flat, i.e. M admits a metric with positive scalar curvature.*

As a consequence of Stolz' theorem (cf. [**19**]) a simply-connected strongly scalar-flat manifold of dimension ≥ 5 admits a non-vanishing harmonic spinor. Futaki proved in [**6**] that this implies that the manifold decomposes as a Riemannian product of irreducible strongly scalar-flat manifolds which are Ricci-flat Kähler or have holonomy $Spin(7)$ (cf. also [**9**]). Theorem 2.1 follows from properties of these factors. Futaki used this decomposition to give an upper bound for the absolute value of the $\hat{A}$-genus of strongly scalar-flat manifolds.

The paper is structured in the following way. In Section 2 we state our main result on the topology of strongly scalar-flat manifolds (see Theorem 2.1) and give a partial answer to the question whether such manifolds are almost complex (see Proposition 2.2). In Section 3 we prove these statements and observe that irreducible Ricci-flat manifolds with vanishing first Pontrjagin class have generic holonomy. In Section 4 we rephrase some of the results in terms of the free loop space of the manifold and point out more recent results related to this paper.

2. *Topology of strongly scalar-flat manifolds*

In this section we state some necessary conditions for a manifold to be strongly scalar-flat. We also give a partial answer to the question whether strongly scalar-flat manifolds are almost complex. In the following all manifolds shall be smooth, connected and closed.

Theorem 2.1. *Let M be a simply-connected manifold of dimension ≥ 5. If M is strongly scalar-flat then M carries a closed 2-form ω and* *a closed* 4*-form* Ω *such that*

$$\langle e^{[\omega]} \cdot e^{[\Omega]} \cdot p_1(M), \mu_M \rangle \neq 0.$$

Here $[\omega]$ and $[\Omega]$ denote the induced real cohomology classes, μ_M is the fundamental cycle of M and $\langle \quad , \quad \rangle$ denotes the Kronecker pairing. Theorem 1.1 follows directly from Theorem 2.1.

We deduce Theorem 2.1 in the next section from properties of Ricci-flat Kähler manifolds and manifolds with holonomy $Spin(7)$. First compact examples of $Spin(7)$-holonomy manifolds were constructed by Joyce (cf. [**12**]). Very little is known about their topological type. In all known examples the signature is divisible by 8 (cf. [**12**], [**13**]). In particular their Euler characteristic is even. If this were true for all $Spin(7)$-holonomy manifolds then any strongly scalar-flat manifold would be almost complex. A more precise statement is given in the following

Proposition 2.2. *Let* M *be a simply-connected strongly scalar-flat manifold of dimension* ≥ 5*. Assume for some scalar-flat metric on* M *that any* $Spin(7)$*-holonomy factor of* M *has even Euler characteristic. Then* M *admits an almost complex structure with vanishing first Chern class. If* $\dim M = 4k$ *then* $sign(M) \equiv (-1)^k \cdot \chi(M) \bmod 4$.

We close this section with a few remarks.

Remark 2.3.
1. *The method used to prove Theorem 2.1 also shows that the odd Stiefel-Whitney classes of a simply-connected strongly scalar-flat manifold vanish.*
2. *In contrast to the case of positive scalar curvature strongly scalar-flat structures are not preserved under surgery. In fact, performing a surgery (of any codimension) turns a strongly scalar-flat manifold with infinite fundamental group into a manifold for which the scalar curvature of any metric is negative somewhere. The situation for finite fundamental groups is similar.*

3. *Holonomy of strongly scalar-flat manifolds*

In this section we prove Theorem 2.1, Proposition 2.2 and the following observation concerning the question whether there exist irreducible Ricci-flat manifolds with generic holonomy (cf. [**1**], p. 19).

Observation 3.1. *A simply-connected irreducible Ricci-flat manifold with vanishing real first Pontrjagin class has generic holonomy.*

Theorem 2.1 relies on the following result of Futaki which he deduces from the surgery theorem (cf. [**7**]) and Stolz' solution of the Gromov-Lawson conjecture (cf. [**19**]) using the de Rham splitting theorem and Berger's classification of holonomy groups (cf. [**1**], Chapter 10).

Theorem 3.2. ([6], Th. 1) *Let M be a simply-connected strongly scalar-flat manifold of dimension ≥ 5. Then M is a Spin-manifold and splits as a Cartesian product of irreducible manifolds, where each factor admits a Ricci-flat Kähler metric or a Riemannian metric with holonomy $Spin(7)$.* ■

In the next proposition we collect those topological properties of Ricci-flat Kähler manifolds and $Spin(7)$-holonomy manifolds which are used in the proof of Theorem 2.1. Note that a holonomy reduction leads to a corresponding topological reduction of the frame bundle. We always fix the orientation of the manifold induced by this reduction.

Proposition 3.3. *Let (M, g) be a simply-connected irreducible Riemannian manifold of dimension $n > 0$. Let H be the holonomy group of (M, g).*

1. *If $H \subset SU(n/2)$ (i.e. (M, g) is Ricci-flat Kähler) and ω denotes the Kähler form then*

$$\langle e^{[\omega]} \cdot p_1(M), \mu_M \rangle < 0 \quad \textit{and} \quad \langle e^{[\omega]}, \mu_M \rangle > 0.$$

2. *If $H = Spin(7)$ (implying $n = 8$) then M admits a closed 4-form Ω such that*

$$\langle e^{[\Omega]} \cdot p_1(M), \mu_M \rangle < 0 \quad \textit{and} \quad \langle e^{[\Omega]}, \mu_M \rangle > 0.$$

Proof: Let $H \subset SU(m)$ where $m = n/2$. We note that $\langle e^{[\omega]}, \mu_M \rangle > 0$ since the m-fold wedge of the Kähler form ω is (up to a positive constant) the volume form. Also $c_1(M) = 0$ implies that $p_1(M) = -2 \cdot c_2(M)$. Thus $\langle e^{[\omega]} \cdot p_1(M), \mu_M \rangle$ is equal to $\langle [\omega]^{m-2} \cdot c_2(M), \mu_M \rangle$ times a negative constant. We now use the following formula of Apte (cf. [**1**], formula (2.80a))

$$\langle [\omega]^{m-2} \cdot c_2(M), \mu_M \rangle = C \cdot \int_M |R|^2 \, \mu_g.$$

Here C is a positive constant and R denotes the curvature tensor viewed as a symmetric endomorphism on 2-forms. Note that (M, g) as a simply-connected manifold of positive dimension cannot be flat. Hence

$$\langle e^{[\omega]} \cdot p_1(M), \mu_M \rangle < 0.$$

Next let $\dim M = 8$ and $H = Spin(7) \hookrightarrow SO(8)$. We adapt the argument for G_2-manifolds given in [**11**], p. 333. First we recall some standard facts about $Spin(7)$-holonomy manifolds (cf. [**3**], [**12**]). Associated to the holonomy reduction is a parallel closed self-dual 4-form $\tilde{\Omega}$ (for an explicit local description see for example [**12**], p. 510). Its stabilizer at each point is isomorphic to $Spin(7)$. Up to a positive constant $\tilde{\Omega} \wedge \tilde{\Omega}$ is the volume form of M.

The space of k-forms $\Gamma(\Lambda^k(M))$ splits orthogonally into components parametrized by the irreducible representations of the $Spin(7)$-module $\Lambda^k(\mathbb{R}^8)$. In particular $\Gamma(\Lambda^2(M))$ decomposes as $\Gamma(\Lambda^2_7) \oplus \Gamma(\Lambda^2_{21})$, where $\Gamma(\Lambda^2_7)$ corresponds to the $Spin(7)$-representation $\Lambda^1(\mathbb{R}^7)$ and $\Gamma(\Lambda^2_{21})$ corresponds to the adjoint representation of $Spin(7)$. Local computations (cf. [**3**], p. 546) show that $\int_M \tilde{\Omega} \wedge \xi \wedge \xi = -\int_M |\xi|^2 \, \mu_g$ for any $\xi \in \Gamma(\Lambda^2_{21})$. So for $\Omega := -\tilde{\Omega}$ we get

$$\int_M \Omega \wedge \Omega > 0 \quad \text{and} \quad \int_M \Omega \wedge \xi \wedge \xi = \int_M |\xi|^2 \, \mu_g.$$

By the Ambrose-Singer theorem (cf. [**1**], p. 291) the components R_{ij} of the curvature tensor R are in $\Gamma(\Lambda^2_{21})$. Since the first Pontrjagin form $p_1(M,g)$ is given by a negative multiple of the trace $tr(R \wedge R) = \sum R_{ij} \wedge R_{ij}$ we conclude

$$\int_M \Omega \wedge p_1(M,g) = C \cdot \int_M |R|^2 \, \mu_g,$$

where $C < 0$ is a constant. Again the integral is non-zero since (M,g) is not flat. Hence, $\langle e^{[\Omega]} \cdot p_1(M), \mu_M \rangle = \int_M \Omega \wedge p_1(M,g) < 0$. ∎

Proof of Theorem 2.1: Let $M = K_1 \times \ldots \times K_r \times J_1 \times \ldots \times J_s$ be the splitting of the strongly scalar-flat manifold M into irreducible factors, where the holonomy of K_i is contained in the special unitary group (i.e. K_i is Ricci-flat Kähler) and the holonomy of J_j is $Spin(7)$ (see Theorem 3.2). On each factor we choose the orientation induced by the holonomy reduction. This gives an orientation of M. Let ω_i denote the Kähler form of K_i and let Ω_j denote the 4-form given in Proposition 3.3. We put $\omega := \sum_i \omega_i$, $\Omega := \sum_j \Omega_j$, $A_i := \langle e^{[\omega_i]}, \mu_{K_i} \rangle$, $B_j := \langle e^{[\Omega_j]}, \mu_{J_j} \rangle$, $A := \prod_{i=1}^r A_i$ and $B := \prod_{j=1}^s B_j$. Next we note that $\langle e^{[\omega]} \cdot e^{[\Omega]} \cdot p_1(M), \mu_M \rangle$ may be written as a sum, where each summand has one of the following forms

$$\langle e^{[\omega_i]} \cdot p_1(K_i), \mu_{K_i} \rangle \cdot (A \cdot B)/A_i \quad \text{or} \quad \langle e^{[\Omega_j]} \cdot p_1(J_j), \mu_{J_j} \rangle \cdot (A \cdot B)/B_j.$$

By Proposition 3.3 each summand is negative. ∎

Proof of Observation 3.1: Assume M is an irreducible Ricci-flat manifold with reduced holonomy. We apply Berger's classification (cf. [**1**], Chapter 10) and conclude that the holonomy of M is G_2, $Spin(7)$ or contained in the special unitary group. In all these cases the first real Pontrjagin class of M is non-zero giving the desired contradiction (see [**11**] and Proposition 3.3). ■

For the proof of Proposition 2.2 we use the following

Lemma 3.4. *Let N be an 8-dimensional Spin-manifold which admits a topological $Spin(7)$-reduction. Then N admits an almost complex structure with vanishing first Chern class if and only if the Euler characteristic $\chi(N)$ is even.*

Proof: In [**8**] Heaps showed, using results of Massey, that an oriented 8-dimensional manifold X admits a stable almost complex structure (s.a.c.s.) if and only if $w_2(X)$ is integral and $w_8(X) \in Sq^2(H^6(X;\mathbb{Z}))$. Note that the Steenrod operation $Sq^2 : H^6(N;\mathbb{Z}/2\mathbb{Z}) \to H^8(N;\mathbb{Z}/2\mathbb{Z})$ vanishes identically since it is given by multiplication with $w_2(N) = 0$. Since $\chi(N)$ reduces to $w_8(N)$ modulo 2 we conclude that N admits a s.a.c.s. if and only if $\chi(N)$ is even. This gives one direction. Next assume $\chi(N)$ is even and let ξ denote a s.a.c.s.. We may assume $c_1(\xi) = 0$ (replace ξ by $\xi + L - \bar{L}$, where L is a complex line bundle with $2 \cdot c_1(L) = -c_1(\xi)$). It is well-known that ξ induces an almost complex structure if and only if $c_4(\xi)$ is equal to the Euler class $e(N)$. We now use that N admits a topological $Spin(7)$-reduction. In cohomological terms the existence of such a reduction is equivalent to $8 \cdot e(N) = 4 \cdot p_2(N) - p_1(N)^2$ (cf. for example [**15**], p. 349). Expressing the Pontrjagin classes of N in terms of the Chern classes of ξ we get $c_4(\xi) = e(N)$. Thus ξ induces an almost complex structure on N with vanishing first Chern class. ■

Proof of Proposition 2.2: By Theorem 3.2 the strongly scalar-flat manifold M splits as a Cartesian product of manifolds with holonomy contained in the special unitary group and manifolds with $Spin(7)$-holonomy. The former are complex manifolds with vanishing first Chern class. The latter support almost complex structures with vanishing first Chern class by Lemma 3.4 and the assumption. For the last statement we recall from [**10**], p. 777, that any $4k$-dimensional almost complex manifold M satisfies the congruence $sign(M) \equiv (-1)^k \cdot \chi(M) \bmod 4$. ■

4. *Relations to the free loop space*

Some of the previous results indicate that the first Pontrjagin class of a manifold M takes a special role in questions concerning strongly scalar-flat metrics or Ricci-flat metrics with generic holonomy. On the other hand the vanishing of $p_1(M)$ is closely related to the existence of a *Spin*-structure on the free loop space $\mathcal{L}M$ of M.

Motivated by a paper of Stolz (cf. [**20**]) we rephrase Theorem 1.1 and Observation 3.1 in terms of $\mathcal{L}M$ at the end of this section. In [**20**] Stolz gives a heuristical treatment of a 'Weitzenböck formula' for the free loop space of a *Spin*-manifold M with $\frac{p_1}{2}(M) = 0$ and conjectures that the Witten genus of M vanishes if M admits a metric with positive Ricci curvature. The conjecture implies the existence of simply-connected Riemannian manifolds with positive scalar curvature which do not admit metrics with positive Ricci curvature.

Let M be a closed oriented connected n-dimensional Riemannian manifold with free loop space $\mathcal{L}M = C^\infty(S^1, M)$, where we identify S^1 with $\mathbb{R}/\mathbb{Z}$. Following [**20**] we define the scalar curvature of $\mathcal{L}M$ to be the map $scal_{\mathcal{L}M} : \mathcal{L}M \to \mathbb{R}$, which assigns to a loop $\gamma \in \mathcal{L}M$ the average of the Ricci curvature in direction of γ, i.e.

$$scal_{\mathcal{L}M}(\gamma) := \int_0^1 Ric(\gamma'(z))dz.$$

Hence $scal_{\mathcal{L}M}$ is positive on non-constant loops if M has positive Ricci curvature. Also $\mathcal{L}M$ is scalar-flat, i.e. $scal_{\mathcal{L}M}$ vanishes identically, if M is Ricci-flat. Conversely, if $Ric(v) \neq 0$ for some unit tangent vector it is easy to find a loop γ with $\gamma'(0) = v$ such that $scal_{\mathcal{L}M}(\gamma) \neq 0$. Hence, if $\mathcal{L}M$ is scalar-flat then M must be Ricci-flat.

In the following let M be a simply-connected manifold which carries a *Spin*-structure given by a $Spin(n)$-principal bundle $P \to M$. The free loop space $\mathcal{L}M$ of M is called *Spin* if the $\mathcal{L}Spin(n)$-principal bundle $\mathcal{L}P \to \mathcal{L}M$ admits a reduction to the basic central extension of $\mathcal{L}Spin(n)$ by S^1 (for $n \neq 4$ this is the universal extension). As shown in [**17**] the free loop space is *Spin* if $\frac{p_1}{2}(M)$ vanishes. Here $\frac{p_1}{2}$ denotes the generator of $H^4(BSpin(n); \mathbb{Z})$ which is half of the universal first Pontrjagin class. The converse holds at least if M is two-connected.

Next assume $\mathcal{L}M$ is *Spin*. Guided by the finite dimensional situation one expects that $\mathcal{L}M$ carries a 'Dirac operator' acting on sections of the spinor bundle associated to the *Spin*-structure of $\mathcal{L}M$. In [**21**] Witten applied formally the Lefschetz fixed point formula to this hypothetical Dirac operator and derived a bordism invariant of the underlying manifold M. This invariant, known as the Witten genus, expresses the equivariant index of the 'Dirac operator' localized at the constant loops $M \subset \mathcal{L}M$.

In the finite dimensional setting the α-invariant, i.e. the index of the ordinary Dirac operator, is an obstruction to positive scalar curvature metrics on a *Spin*-manifold by the Weitzenböck formula (cf. [**9**]). In analogy one expects that there exists an obstruction (namely the Witten genus) to positive scalar curvature on the free loop space if $\mathcal{L}M$ is *Spin*. The first part of the following corollary shows that if one replaces positive scalar curvature by scalar-flatness the α-invariant is an obstruction.

Corollary 4.1. *Let M be a Riemannian manifold for which its free loop space $\mathcal{L}M$ is simply-connected and Spin.*

1. *Assume M has dimension ≥ 5. If $\mathcal{L}M$ is scalar-flat then $\alpha(M) = 0$, i.e. M admits a metric with positive scalar curvature.*
2. *Assume M is irreducible. If $\mathcal{L}M$ is scalar-flat then M is Ricci-flat with generic holonomy group.*

Proof: Note that the homotopy groups of $\mathcal{L}M$ may be computed from the homotopy groups of M using the path fibration $\Omega M \to \Gamma M \to M$ and the fibration $\Omega M \to \mathcal{L}M \to M$. In particular, M is two-connected since $\mathcal{L}M$ is simply-connected. As mentioned before the existence of a *Spin*-structure for $\mathcal{L}M$ implies that $p_1(M)$ vanishes. Hence the statements follow from Theorem 1.1 and Observation 3.1. ■

Final remark 4.2.

1. *Some of the results of this paper have counterparts for non-trivial fundamental groups. For example Theorem 1.1 is still true if $\pi_1(M)$ is a finite group for which the Gromov-Lawson-Rosenberg conjecture holds (cf. [**5**], cf. also the recent paper [**2**] of Botvinnik and McInnes for relations between holonomy groups and the Gromov-Lawson-Rosenberg conjecture).*
2. *The class of simply-connected strongly scalar-flat manifolds of dimension $n \geq 5$ which satisfy an upper diameter bound and a lower curvature bound contains only finitely many diffeomorphism classes (for details cf. [**5**]). This follows from the results of Cheeger-Fukaya-Gromov on collapsed manifolds in [**4**].*

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