Bordism-finiteness and semi-simple group actions

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Abstract

We give bordism-finiteness results for smooth S^3-manifolds. Consider the class of oriented manifolds which admit an S^1-action with isolated fixed points such that the action extends to an S^3-action with fixed point. We exhibit various subclasses, characterized by an upper bound for the Euler characteristic and properties of the first Pontrjagin class p_1, for example $p_1 = 0$, which contain only finitely many oriented bordism types in any given dimension. Also we show finiteness results for homotopy complex projective spaces and complete intersections with S^3-action as above.

1 Introduction

A direct consequence of the Atiyah-Bott-Segal-Singer fixed point theorem in equivariant index theory is the fact that all the Pontrjagin numbers of an oriented closed manifold with smooth fixed point free S^1-action vanish. Since the oriented bordism ring is determined by Pontrjagin and Stiefel-Whitney numbers this result may be interpreted as a bordism-finiteness theorem: In a given dimension the class of oriented closed manifolds which support a fixed point free S^1-action contains only finitely many oriented bordism types.

In this paper we give bordism-finiteness results for smooth actions on connected closed oriented manifolds with possible non-empty fixed point set. We also look at the related question of diffeomorphism-finiteness in a given homotopy type.

In Riemannian geometry finiteness theorems (involving bounds for curvature, diameter, volume etc.) go back to the work of Cheeger (cf. [Ch]) and are the topic of active research since then. It is natural to look what happens if the geometrical bounds arise from symmetries, i.e. group actions.

Concerning the questions on bordism- and diffeomorphism-finiteness it is desirable to obtain positive results for groups as "small" as possible. Our

results involve smooth actions by the semi-simple group S^3. Before we state these we take a brief look at actions by "smaller" groups.

For finite groups one cannot expect any general finiteness results, even if one restricts to free actions or to a fixed homotopy type. Dovermann and Masuda used equivariant surgery to construct infinitely many homotopy $\mathbb{C}P^3$'s with effective smooth $\mathbb{Z}_p$-action for every prime number p (cf. [DoMa] and the references therein). Löffler and Raußen (cf. [LoRa]) used Sullivan's theory of minimal models to construct free $\mathbb{Z}_p$-actions on certain high connected manifolds for any large enough prime number p and conjectured that non-trivial $\mathbb{Z}_p$-actions exist on any simply-connected manifold (cf. also [We]; for the question of bordism-finiteness cf. [CoFl] and [Di]). On the other hand, by a result of Schultz, one knows that in dimension ≥ 4 any oriented bordism class contains infinitely many manifolds with infinite fundamental group which admit no effective finite group action (cf. [Sc]).

Next we discuss the question of bordism-finiteness for circle actions. By taking products or equivariant connected sum along invariant submanifolds one can construct new S^1-manifolds from given pieces. Using these methods it is easy to show that for any given oriented manifold a suitable multiple (disjoint union) is bordant to a connected oriented manifold with non-trivial S^1-action. On the other hand the Lefschetz fixed point formula in equivariant index theory (cf. [AtSe], [AtSiIII], [AtBo], [Bo]) implies that the bordism class of a manifold with S^1-action is modulo torsion completely determined by the local geometry of the action near the fixed point set (see Cor. 3.2).

In order to get interesting bordism-finiteness results one needs additional conditions which limit the constructions mentioned above but are not too restrictive on the local geometry of the action. This leads us to consider S^1-actions with *isolated* fixed points which satisfy a prescribed upper bound for the number of fixed points (by the classical Lefschetz fixed point formula the number of S^1-fixed points is just the Euler characteristic). In this way we exclude products with trivial S^1-manifolds and limit the use of the connected sum construction but still allow various local geometries although the non-equivariant part of these is trivial.

For these S^1-actions bordism-finiteness still fails (see the examples in Section 4). Taking products one concludes the same for torus actions as long as the dimension of the torus is small compared to the dimension of the manifold. We note that the absence of strong implications on the bordism type for torus actions is underlined by the following result of Buchstaber and Ray (cf. [BuRa]): The complex bordism ring is generated by toric manifolds. In particular, any $2n$-dimensional stable almost complex manifold is bordant to a manifold which admits an action by the n-dimensional torus.

Since bordism-finiteness fails for abelian actions it is natural to look at

semi-simple group actions next. Our results involve actions of S^3 with the following property:

S^3 acts with fixed point and the S^1-action has isolated fixed points   (∗)

for some (and hence every) fixed subgroup $S^1 \hookrightarrow S^3$. In turns out that for such actions the question whether bordism-finiteness holds depends on properties of the first Pontrjagin class. A special case is the following

Theorem 1.1. *Let C and m be natural numbers. The class of connected m-dimensional oriented manifolds with vanishing first Pontrjagin class and Euler characteristic $\leq C$ which admit an S^3-action satisfying (∗) contains only finitely many oriented bordism types.*

The conclusion holds in the more general situation where a negative multiple of the first Pontrjagin class p_1 is a sum of squares (see Th. 3.4). Moreover bordism-finiteness holds if one refines the bordism ring taking into account the condition on p_1 (see Th. 3.6). To prove Theorem 1.1 and its refinements we combine the Lefschetz fixed point formula in equivariant index theory with the following well known properties of S^3 (which hold for any semi-simple group): In any fixed dimension the group S^3 admits only finitely many non-equivalent representations. Any one-dimensional S^3-representation is trivial.

These properties imply that the local geometry of the induced S^1-action at the S^3-fixed point is determined up to finite ambiguity. The condition on the first Pontrjagin class guarantees that this is also true at the other S^1-fixed points. Finally one applies the Lefschetz fixed point formula in equivariant index theory to conclude that the oriented bordism type (resp. its refinement) is determined up to finite ambiguity.

Regarding the problem of diffeomorphism-finiteness in a given homotopy type we use similar arguments to show

Theorem 1.2. *In a fixed dimension $2n$ there are only finitely many homotopy complex projective spaces with an S^3-action satisfying (∗).*

By simply-connected surgery theory one knows that in any dimension $2n \geq 6$ there are infinitely many homotopy complex projective spaces distinguished by their Pontrjagin classes (cf. [Hs]). According to Theorem 1.2 almost all of them do not admit an S^3-action satisfying (∗). It is interesting to compare the result above with a conjecture of Petrie (cf. [Pe]) which states that the total Pontrjagin class of a homotopy $\mathbb{C}P^n$ with non-trivial S^1-action is standard. The conjecture is known to be true if $n \leq 4$ (cf. [Ja]; for related results cf. for example the survey [Do] as well as [De2] and the references therein).

It implies diffeomorphism-finiteness for homotopy complex projective spaces with non-trivial S^1-action.

As indicated before our methods are rather classical, involving index theory and properties of semi-simple groups, and work best if a negative multiple of the first Pontrjagin class is a sum of squares. On the other hand the theory of elliptic genera led to applications for $Spin^c$-manifolds with S^3-action if the first Pontrjagin class is a sum of squares (cf. [De2], [De3], [Li]). Combining both methods one obtains further information on manifolds with S^3-action.

This paper is structured in the following way. Section 2 deals with tangential weights and weights of equivariant complex line bundles for manifolds with admit an S^3-action with fixed point. We give conditions in terms of the first Pontrjagin class which guarantee that these weights are determined up to finite ambiguity. In Section 3 we extend Theorem 1.1 to the case that a negative multiple of the first Pontrjagin class is a sum of squares (see Theorem 3.4). To prove this theorem and related results we use the Lefschetz fixed point formula in equivariant index theory. In Section 4 we show, by example, that the bordism-finiteness result given in Theorem 3.4 fails if one relaxes the conditions on the S^3-action or the condition on the first Pontrjagin class. In Section 5 we prove the result on homotopy complex projective spaces (see Theorem 1.2 above). We use similar methods to show that in a given dimension there are only finitely many complete intersections which admit an S^3-action satisfying $(*)$.

2   Weights

In this section we give some information on the tangential weights and the weights of complex line bundles over a $2n$-dimensional manifold M which supports a non-trivial S^3-action with fixed point.

By a result of Hattori and Yoshida (cf. [HaYo]) the S^3-action lifts uniquely to any complex line bundle $L \to M$ and we shall always do so. Fix $S^1 \hookrightarrow S^3$. At an S^1-fixed point L reduces to a complex one-dimensional S^1-representation with character $\lambda \mapsto \lambda^a$. We call a the **weight** of L at this point. Note that at the S^3-fixed point the weight has to vanish since any complex one-dimensional S^3-representation is trivial (one way to characterize semi-simple compact Lie groups).

At an S^1-fixed point the tangent bundle TM reduces to a real S^1-representation which we identify (non-canonically) with a complex representation with character $\lambda \mapsto \sum \lambda^{m_i}$. We call $m_1, \ldots, m_n$ the **tangential weights** of M at the fixed point. Note that the tangential weights are only well defined up to sign. At an S^3-fixed point pt the S^1-representation $TM_{|pt}$

is induced from an S^3-representation. Since there are only finitely many equivalence classes of such representations in a given dimension (another way to characterize semi-simple compact Lie groups) the tangential weights at pt are determined by the dimension of M up to finite ambiguity, i.e. the set of tangential weights at pt belongs to a finite set which only depends on $2n$. The next lemma is used in the following section to derive the bordism-finiteness results mentioned in the introduction.

Lemma 2.1. *Let* $L_1, \ldots, L_k$ *be* S^3*-equivariant complex line bundles over* M *and* N *a positive integer. Assume the first Pontrjagin class of the bundle* $E := (L_1 + \ldots + L_k) + N \cdot TM$ *vanishes. Then the tangential weights at any* S^1*-fixed point are determined by the dimension* $2n$ *of* M *up to finite ambiguity. The weights of* L_i *are determined by* N *and* $2n$ *up to finite ambiguity.*

Proof: Note that the weights of E at an S^1-fixed point are just the weights of the line bundles L_j and the tangential weights taken with multiplicity N. Let $Y_1, \ldots, Y_{C'}$ denote the connected components of the fixed point manifold M^{S^1}. We assume that the S^3-fixed point pt belongs to Y_1. Let $m_{s,1}, \ldots, m_{s,n}$ denote the tangential weights and let $a_{s,j}$ denote the weight of L_j at Y_s.

Since the S^3-equivariant vector bundle E has vanishing first Pontrjagin class it follows from a spectral sequence argument (see the lemma below) that the weights of E at Y_s satisfy

$$\sum_{j=1}^{k} a_{s,j}^2 + N \cdot \sum_{i=1}^{n} m_{s,i}^2 = C'', \tag{1}$$

where C'' is a constant which does not depend on Y_s. At the S^3-fixed point $pt \in Y_1$ the weights of L_j vanish and the tangential weights $m_{1,i}$ are determined by the dimension of M up to finite ambiguity. Hence $C'' = N \cdot \sum_{i=1}^{n} m_{1,i}^2$ is bounded from above by N times a constant which only depends on $2n$. By formula (1) the tangential weights (resp. the weights of L_j) at any S^1-fixed point are determined by $2n$ (resp. $2n$ and N) up to finite ambiguity. ∎

Lemma 2.2. *Let* $E \to M$ *be a* $2r$*-dimensional* S^3*-equivariant vector bundle with weights* $e_{s,1}, \ldots, e_{s,r}$ *at the* S^1*-fixed point component* Y_s. *If* $p_1(E) = 0$ *then* $\sum_{i=1}^{r} e_{s,i}^2$ *is independent of* Y_s.

Proof: We use the Leray-Serre spectral sequence for the Borel construction of $E \to M$ (for details cf. [De1], Lemma A.6). Let $M_{S^3} := ES^3 \times_{S^3} M$

and let $\pi : M_{S^3} \to BS^3$ be the fibration with fibre $i : M \hookrightarrow M_{S^3}$ associated to the universal S^3-principal bundle $ES^3 \to BS^3$ and the S^3-space M. Since $H^*(BS^3; \mathbb{Z})$ is concentrated in degree $4*$ it follows from the Leray-Serre spectral sequence that the sequence

$$H^4(BS^3; \mathbb{Z}) \xrightarrow{\pi^*} H^4(M_{S^3}; \mathbb{Z}) \xrightarrow{i^*} H^4(M; \mathbb{Z}) \tag{2}$$

is exact. Next consider the bundle $E_{S^3} \to M_{S^3}$ associated to the S^3-equivariant vector bundle $E \to M$ via the Borel construction. We denote its first Pontrjagin class by $p_1(E)_{S^3} \in H^4(M_{S^3}; \mathbb{Z})$. Since $i^*(p_1(E)_{S^3}) = p_1(E) = 0$ it follows from (2) that $p_1(E)_{S^3} = \pi^*(\alpha)$ for some $\alpha \in H^4(BS^3; \mathbb{Z})$. By naturality, $p_1(E)_{S^1} = \pi^*(C'' \cdot x^2)$, where π also denotes $M_{S^1} \to BS^1$, $x \in H^2(BS^1; \mathbb{Z})$ denotes a generator and $C'' \cdot x^2$ is the image of α under the map induced by the inclusion $S^1 \hookrightarrow S^3$. Note that $p_1(E)_{S^1}$ reduces at $q \in Y_s$ to $(\sum_{i=1}^r e_{s,i}^2) \cdot x^2 \in H^4(BS^1; \mathbb{Z}) \cong H^4(q_{S^1}; \mathbb{Z})$. Hence $\sum_{i=1}^r e_{s,i}^2 = C''$ for any Y_s. ∎

3 Bordism-finiteness

In this section we apply the Lefschetz fixed point formula to the result of the previous section to derive bordism-finiteness theorems. Let ∂ be an S^1-equivariant elliptic operator over M with symbol $\sigma \in K_{S^1}(TM)$. By the fundamental work of Atiyah and Singer (cf. [AtSiI]) the equivariant index of ∂ is equal to the topological index $t\text{-}ind(\sigma)$, where $t\text{-}ind : K_{S^1}(TM) \to R(S^1)$ is defined as the push-forward in complex K-theory for $M \to pt$. Let N denote the normal bundle of $i : M^{S^1} \hookrightarrow M$ and $\Lambda_t := \sum_{i=0}^{\infty} \Lambda^i \cdot t^i$.

Theorem 3.1 (Lefschetz fixed point formula [AtSe]). *For an element $\sigma \in K_{S^1}(TM)$ and any topological generator $\lambda \in S^1$*

$$t\text{-}ind(\sigma)(\lambda) = t\text{-}ind\left(\frac{i^*(\sigma)(\lambda)}{\Lambda_{-1}(N \otimes_{\mathbb{R}} \mathbb{C})(\lambda)}\right). \tag{3}$$

∎

Note that the right hand side of formula (3) consists of a finite sum of local contributions at the fixed point components of the S^1-action. Since the set of topological generators is dense in S^1 the theorem above implies that the index function $t\text{-}ind : K_{S^1}(TM) \to R(S^1)$ vanishes identically if M^{S^1} is empty.

If the elliptic operator is geometrically defined using an H-structure (cf. [AtSiIII]), for example if ∂ is a twisted signature operator, then the local data in formula (3) may be expressed in terms of the tangent bundle and the vector bundles associated to the H-structure restricted to M^{S^1}.

Next recall that the Pontrjagin numbers of M are determined by twisted signatures, where the twist bundles are associated to the tangent bundle via some orthogonal representation. Since the S^1-action lifts canonically to these bundles one can consider the S^1-equivariant twisted signatures. If the S^1-action on M has no fixed point these indices have to vanish since the index function t-ind vanishes identically. In this case all Pontrjagin numbers of M vanish (for an elementary proof see [Bo]) which implies that M represents an element of order two in the oriented bordism ring.

The following reformulation is convenient for our purposes. Given two S^1-manifolds M_1 and M_2 we say that they have the same **local S^1-geometry** if there exists an S^1-equivariant orientation preserving diffeomorphism from the normal bundle of the fixed point manifold $M_1^{S^1}$ to the normal bundle of $M_2^{S^1}$. In this case one can glue M_1 and $-M_2$ along the fixed point manifold together to obtain a manifold with fixed point free S^1-action which is bordant to $M_1 - M_2$ (cf. for example [CoFl], Th. (22.1)). We summarize the discussion in

Corollary 3.2. *Let M_1 and M_2 be S^1-manifolds with the same local S^1-geometry. Then $M_1 - M_2$ represents a torsion element in the oriented bordism ring.* ∎

We shall combine the corollary with Lemma 2.1 to obtain bordism-finiteness for the class of manifolds for which a negative multiple of the first Pontrjagin class is a sum of squares. Let us call such manifolds **semi-negative**, i.e. a manifold M is semi-negative if there exists a finite number of classes $y_j \in H^2(M; \mathbb{Z})$ and a positive integer N such that $N \cdot p_1(M) + \sum y_j^2 = 0$. Note that M is semi-negative if and only if M admits complex line bundles L_j for which the first Pontrjagin class of $E := \sum L_j + N \cdot TM$ vanishes. Of course many manifolds such as 3-connected manifolds with non-vanishing first rational Pontrjagin class (e.g. quaternionic projective space $\mathbb{H}P^k$, $k > 1$) cannot be semi-negative. However the class of semi-negative manifolds is quite rich and contains some interesting families.

Remarks 3.3. *The class of semi-negative manifolds contains*

1. *manifolds with p_1 torsion, e.g. $BO\langle 8 \rangle$-manifolds or 4-connected manifolds.*

2. *manifolds with cohomology ring in degree ≤ 4 like a 4-manifold with indefinite intersection form.*

3. *in a given dimension all but a finite number of complete intersections.*

4. *infinitely many homotopy complex projective spaces in any given dimension $2n \geq 6$.*

Theorem 3.4. *Let C and m be natural numbers. The class of semi-negative connected m-dimensional oriented manifolds with Euler characteristic $\leq C$ and an S^3-action satisfying $(*)$ contains only finitely many oriented bordism types.*

Proof: The theorem follows from Corollary 3.2 once we know that the local S^1-geometry is determined by (m, C) up to finite ambiguity. Since the oriented bordism group Ω_m^{SO} is finite in dimension $m \not\equiv 0 \bmod 4$ we may assume $m = 4l$. Let M be a $4l$-dimensional manifold with S^3-action satisfying $(*)$ and Euler characteristic $\chi(M) = C' \leq C$. By the classical Lefschetz fixed point formula the induced S^1-action has C' isolated fixed points. Next assume that M is semi-negative, i.e. $p_1(L_1 + \ldots + L_k + N \cdot TM) = 0$ for a positive integer N and complex line bundles L_j. By Lemma 2.1 the tangential weights of M are determined by the dimension of M up to finite ambiguity. So all manifolds in the theorem with Euler characteristic equal to C' represent only a finite set of local S^1-geometries. Since $0 \leq C' \leq C$ the theorem follows from Corollary 3.2. ∎

The theorem immediately implies Theorem 1.1. It extends to the complex bordism ring if one assumes in addition that the induced S^1-action preserves the stable almost complex structure. A corresponding result for the $Spin^c$-bordism ring does not hold since an S^3-action always lifts to a given $Spin^c$-structure and there are just too many of them. However Theorem 3.4 admits the following refinement.

Let $X(k)$ be the Cartesian product of BSO and k copies of $\mathbb{C}P^\infty$. For a fixed positive integer N let $f_N : X(k) \to K(\mathbb{Z}, 4)$ be the map which classifies $-N \cdot p_1 + \sum_{j=1}^{k} x_j^2 \in H^4(X(k); \mathbb{Z})$. Here $p_1 \in H^4(BSO; \mathbb{Z})$ denotes the universal first Pontrjagin class and x_j denotes a generator for the j-th copy of $H^2(\mathbb{C}P^\infty; \mathbb{Z})$. Let $B(k, N) \to X(k)$ be the pullback of the path fibration $E(\mathbb{Z}, 4) \to K(\mathbb{Z}, 4)$ via f_N and let $\pi : B(k, N) \to BSO$ denote the projection. Fix a classifying map $\nu : M \to BSO$ for the stable normal bundle of M.

A $B(k, N)$-structure for M is the isotopy class h of a lift $M \to B(k, N)$ of ν in the fibration $\pi : B(k, N) \to BSO$. For fixed k and N we call (M, h)

a **B-manifold**. Note that (M, h) determines k complex line bundles L_j over M such that $N \cdot p_1(M) + \sum_{j=1}^k p_1(L_j) = 0$. In particular, M is semi-negative.

Any polynomial in $p_i(M)$ and $c_1(L_j)$ is called a **characteristic class** of the B-manifold (M, h). The **characteristic numbers** are defined as the values of the characteristic classes on the fundamental cycle of M. By the Pontrjagin lemma the characteristic numbers only depend on the bordism class of (M, h). The bordism group Ω_*^B of B-manifolds may be studied in terms of stable homotopy theory using the Pontrjagin-Thom construction.

Proposition 3.5. *For fixed k and N the bordism group Ω_n^B, $n \in \mathbb{Z}$, is finitely generated. The group $\Omega_n^B \otimes \mathbb{Q}$ is completely determined by the characteristic numbers.*

Proof: Since the argument is standard (cf. [La], [St]) we only sketch it. First apply the construction above to $BSO(r)$ for $r \in \mathbb{N}$. So X_r is the product of $BSO(r)$ and k copies of $\mathbb{C}P^\infty$, $B_r \to X_r$ is obtained as pullback of the path fibration $E(\mathbb{Z}, 4) \to K(\mathbb{Z}, 4)$ and $\pi_r : B_r \to BSO(r)$ is the projection. Note that $B(k, N) = \lim_{r \to \infty} B_r$. Let $\gamma_r \to B_r$ be the pullback of the universal vector bundle over $BSO(r)$ via π_r and let $M(\gamma_r)$ be its Thom space. The Pontrjagin-Thom construction gives an isomorphism $\Phi : \Omega_n^B \to \pi_{n+r}(M(\gamma_r))$ for $r \gg n$, which defines an isomorphism

$$\Phi : \Omega_n^B \to \lim_{r \to \infty} \pi_{n+r}(M(\gamma_r)) \cong \pi_n(M(\gamma))$$

between the bordism group of n-dimensional B-manifolds and the n-th homotopy of the associated spectrum $M(\gamma)$. Since $M(\gamma_r)$ is a CW-complex with finite skeletons Ω_n^B is finitely generated.

Next assume the n-dimensional B-manifold (M, h) has vanishing characteristic numbers. For the second statement it suffices to show that the bordism class of (M, h) vanishes in $\Omega_n^B \otimes \mathbb{Q}$. Note that the space of characteristic classes of (M, h) is equal to $\hat{h}^*(H^*(B_r; \mathbb{Q}))$, $r \gg n$, where $\hat{h} : M \to B_r$ represents h. Since (M, h) has vanishing characteristic numbers $\hat{h}_*(\mu_M)$ vanishes in $H_n(B_r; \mathbb{Q})$. Now apply the Thom isomorphism for the normal bundle of M and the bundle $\gamma_r \to B_r$ to conclude that the composition of the Pontrjagin-Thom map Φ and the rational Hurewicz homomorphism Ψ

$$\Omega_n^B \otimes \mathbb{Q} \xrightarrow{\Phi} \pi_{n+r}(M(\gamma_r)) \otimes \mathbb{Q} \xrightarrow{\Psi} H_{n+r}(M(\gamma_r); \mathbb{Q}),$$

$r \gg n$, maps (M, h) to zero. Since Φ and Ψ are isomorphisms (M, h) vanishes in $\Omega_n^B \otimes \mathbb{Q}$. $\blacksquare$

Theorem 3.6. *Let C and m be natural numbers. For fixed k and N the class of connected m-dimensional B-manifolds with Euler characteristic $\leq C$ and an S^3-action satisfying $(*)$ contains only finitely many B-bordism types.*

Proof: By Proposition 3.5 we may assume that $m = 2n$. Let (M, h) be a $2n$-dimensional B-manifold with Euler characteristic $\leq C$ and an S^3-action satisfying $(*)$. The map h induces via the projection of B to the k-fold product of $\mathbb{C}P^\infty$ a classifying map for k complex line bundles L_j which satisfy $N \cdot p_1(M) + \sum_{j=1}^{k} p_1(L_j) = 0$.

We want to show that the characteristic numbers of (M, h) are determined by (m, C, k, N) up to finite ambiguity. To this end let J denote the subring of $K(M)$ generated by the complex line bundles L_j and vector bundles associated to the tangent bundle. Let $ch : K(M) \to H^*(M; \mathbb{Q})$ be the Chern character. We note that $ch(J) \otimes \mathbb{Q}$ is the subspace V of $H^*(M; \mathbb{Q})$ which is spanned by the characteristic classes of the B-manifold (M, h).

Next we identify the characteristic numbers of (M, h) with certain twisted signatures. By the cohomological version of the index theorem (cf. [AtSiIII]) the index of the signature operator of M twisted with $F \in J$ is given by

$$sign(M; F) = \left\langle \prod_{i=1}^{n} \left(u_i \cdot \frac{1 + e^{-u_i}}{1 - e^{-u_i}} \right) \cdot ch(F), \mu_M \right\rangle,$$

where $\pm u_i$ are the formal roots of M, μ_M is the fundamental cycle of M and $\langle \ , \ \rangle$ denotes the pairing between cohomology and homology. This implies that $\{ sign(M; F) \mid F \in J \}$ spans the $\mathbb{Q}$-vector space $\langle V, \mu_M \rangle$. Hence the characteristic numbers of (M, h) are determined by twisted signatures, where the twist bundle is an element of J.

We are now in the position to prove the theorem from the Lefschetz fixed point formula. Choose a finite set $\{E_1, \dots, E_r\} \subset J$ (r depends on k and the dimension of M) of vector bundles which span $J \otimes \mathbb{Q}$. Here each E_i is given by a universal polynomial (which only depends on (m, k)) in the complex line bundles L_j and vector bundles associated to the tangent bundle. We view E_i as an S^3-equivariant vector bundle by lifting the S^3-action (uniquely) to each L_j. By Lemma 2.1 the tangential weights and the weights of L_j are determined by (m, N) up to finite ambiguity. This implies the same for the weights of E_i.

Next consider the signature operator twisted by E_i. By the discussion of the Lefschetz fixed point formula after Theorem 3.1 its S^1-equivariant index is equal to a sum of $\leq C$ local contributions which only depend on the weights of E_i and the tangential weights. Since these belong to a finite

set which only depends on (m, k, N) we conclude that the ordinary twisted signatures $sign(M; E_i)$, $i = 1, \ldots, r$, are determined by (m, C, k, N) up to finite ambiguity. This implies the same for the characteristic numbers of the B-manifold (M, h). Now the theorem follows from Proposition 3.5. $\blacksquare$

4   Examples of semi-simple group actions

In this section we show that Theorem 3.4 is sharp in the sense that bordism-finiteness fails if one weakens the assumptions on the S^3-action or the first Pontrjagin class. As explained in the introduction one cannot expect bordism-finiteness for S^1-actions if one allows arbitrary fixed point sets. We restrict to S^1-actions with isolated fixed points which satisfy a prescribed upper bound for the number of fixed points. Also we assume that the action extends to an action of S^3. In this situation, by Theorem 3.4, bordism-finiteness holds if the S^3-action has a fixed point and the manifolds are semi-negative. The following two propositions show that both assumptions are necessary.

Proposition 4.1. *There exist connected* 20*-dimensional semi-negative manifolds* M_l, $l \in \mathbb{N}$, *with Euler characteristic equal to* 12 *which represent distinct oriented bordism classes such that each* M_l *supports a fixed point free* S^3*-action with isolated* S^1*-fixed points.*

Proposition 4.2. *There exist connected* 20*-dimensional manifolds* N_l, $l \in \mathbb{N}$, *with Euler characteristic equal to* 23 *which represent distinct oriented bordism classes such that each* N_l *supports an* S^3*-action with fixed point and isolated* S^1*-fixed points.*

Note that by Theorem 3.4 almost all of the N_l are not semi-negative. We remark that examples as in the propositions are necessarily of dimension $4k \geq 8$ since in dimension 4 the upper bound on the number of isolated fixed points gives a bound on the absolute value of the signature. The remaining part of this section is devoted to the proof of the propositions above. We use a kind of induction to extend actions on the base of a fibre bundle to the total space if the fibration is associated to a principal torus bundle.

Let G be a compact connected Lie group which acts from the left on a connected manifold Z. Let F be a connected manifold with left $U(1)$-action. Assume the first Betti number $b_1(Z)$ vanishes or G is simply-connected.

For $y \in H^2(Z; \mathbb{Z})$ let $S \to Z$ denote the $U(1)$-principal bundle with first Chern class equal to y and let $M := S \times_{U(1)} F$ be the associated fibre bundle.

In [HaYo] it was shown that the G-action lifts to S (in fact uniquely if G is simply-connected). For a fixed lift G acts on M by $g(s, f)_\sim := (gs, f)_\sim$. Note that the projection $M \to Z$ is G-equivariant.

We now restrict to the case $G = S^3$ and fix $S^1 \hookrightarrow S^3$. Let a_Y denote the weight of the S^1-action on S restricted to a connected component Y of Z^{S^1}, i.e. at a point of Y (and hence at any point of Y) the S^1-action on the fibre of S has character $\lambda \mapsto \lambda^{a_Y}$. For further reference we note the elementary

Lemma 4.3. *Assume none of the weights a_Y vanish. Then the S^1-action on M has isolated fixed points if this holds for the S^1-action on Z and the $U(1)$-action on F.* ∎

We now begin with the construction of the examples mentioned before. The first series consists of Cayley plane bundles over $Z := S^2 \times S^2$ which support an S^3-action with isolated S^1-fixed points but no S^3-fixed point. We take the left S^3-action on Z induced from the homogeneous action on each copy of S^2. This action has isolated S^1-fixed points but no S^3-fixed points. Its principal stabilizer is equal to $\mathbb{Z}_2$, the center of S^3.

As fibre F we take the Cayley plane $Cl_2 = F_4/Spin(9)$. We fix an orientation of Cl_2. Next we choose an embedding $j : U(1) \hookrightarrow T$, where T is a maximal torus of $Spin(9)$, such that the induced $U(1)$-action on Cl_2 is effective and has isolated fixed points.

Let $\gamma_i \to Z$ denote the pullback of the Hopf bundle over S^2 under the projection of Z on the i-th factor and let $z_i := c_1(\gamma_i) \in H^2(Z; \mathbb{Z})$. Equip Z with the orientation dual to $z_1 \cdot z_2$. Note that the weights of the S^3-equivariant line bundle $\gamma_1^a \otimes \gamma_2^b$ at the S^1-fixed points of Z have the form $\pm a \pm b$.

To construct M_l fix positive integers $a \not\equiv b \bmod 2$, let S_l, $l \in \mathbb{N}$, be the $U(1)$-principal bundle associated to $(\gamma_1^a \otimes \gamma_2^b)^{2l+1}$ and set $M_l := S_l \times_{U(1)} Cl_2$. Note that M_l comes with an orientation induced from the orientations of Z and Cl_2. As explained above the S^3-action on Z lifts to M_l. For the induced S^1-action some of the tangential weights of M_l are odd since the weights of S_l have the form $(2l+1) \cdot (\pm a \pm b)$ and the action of $U(1)$ on Cl_2 is effective. In particular, the principal isotropy group of the S^3-manifold M_l is trivial.

Proof of Proposition 4.1: The Euler characteristic $\chi(M_l)$ of the oriented 20-dimensional manifold M_l is just the product of the Euler characteristic of the base and the fibre, thus $\chi(M_l) = 12$. The S^3-action on M_l has no fixed points since $p : M_l \to Z$ is equivariant. By Lemma 4.3 the induced S^1-action on M_l has isolated fixed points. Since $H^*(M_l; \mathbb{Z})$ is isomorphic to $H^*(Z; \mathbb{Z})$ in degree ≤ 4 (the fibre is 7-connected) each M_l is semi-negative (see Remarks 3.3).

We compute the Milnor number $\langle s_{10}(TM_l), \mu_{M_l} \rangle$ to show that the manifolds M_l represent distinct bordism classes (for a real vector bundle with formal roots $\pm u_i$ the class s_{2t} is defined as $\sum u_i^{2t}$). Let $\pi : E \to BU(1)$ denote the pullback of $BSpin(9) \to BF_4$ to $BU(1)$ via Bj and let $E^\triangle \to E$ denote the tangent bundle along the fibres. One computes using [BoHi] that $\pi_!(s_{10}(E^\triangle)) = \alpha \cdot x^2$, where $\alpha \neq 0$ (cf. for example [De1], Section 4.2). Here x is a generator of $H^2(BU(1); \mathbb{Z})$ and $\pi_!$ is the push-forward in cohomology. Next note that the map $f : Z \to BU(1)$ which classifies $(2l+1) \cdot (a \cdot z_1 + b \cdot z_2)$ is covered by $\tilde{f} : M_l \to E$ and the tangent bundle along the fibres of $p : M_l \to Z$ is isomorphic to $\tilde{f}^*(E^\triangle)$. Now compute

$$\langle s_{10}(TM_l), \mu_{M_l} \rangle = \langle s_{10}(\tilde{f}^*(E^\triangle) \oplus p^*(TZ)), \mu_{M_l} \rangle = \langle p_!(s_{10}(\tilde{f}^*(E^\triangle))), \mu_Z \rangle =$$

$$\langle f^*(\pi_!(s_{10}(E^\triangle))), \mu_Z \rangle = \langle \alpha \cdot (2l+1)^2 \cdot (a \cdot z_1 + b \cdot z_2)^2, \mu_Z \rangle = 2 \cdot \alpha \cdot (2l+1)^2 \cdot a \cdot b.$$

The computation shows that the M_l represent distinct bordism classes. ∎

The next series is constructed from the series above. Equip $\mathbb{C}P^{10}$ with the S^3-action induced by the direct sum of the trivial one-dimensional complex representation and the irreducible complex S^3-representation of dimension 10. Note that S^3 acts on $\mathbb{C}P^{10}$ with fixed point, the principal isotropy group is trivial and the induced S^1-action has isolated fixed points. The principal isotropy group of M_l is also trivial (see the discussion before the proof of Prop. 4.1). Now define N_l by taking the equivariant connected sum of M_l and $\mathbb{C}P^{10}$ along a principal orbit.

Proof of Proposition 4.2: First note that the disjoint union of M_l and $\mathbb{C}P^{10}$ is bordant to N_l. Also $\chi(N_l) = \chi(M_l) + \chi(\mathbb{C}P^{10})$. Hence, by Proposition 4.1 the manifolds N_l, $l \in \mathbb{N}$, represent distinct bordism classes and have Euler characteristic $\chi(N_l) = 23$. By construction S^3 acts on N_l with fixed point and the induced S^1-action has isolated fixed points. ∎

5 Homotopy complex projective spaces and complete intersections

In this section we prove the theorem on homotopy complex projective spaces given in the introduction and a related result for complete intersections (see Theorem 5.1 below).

Let M be a $2n$-dimensional closed manifold with $H^*(M; \mathbb{Z}) \cong H^*(\mathbb{C}P^n; \mathbb{Z})$ and non-trivial S^1-action. Let γ denote the complex line bundle with $c_1(\gamma) =$

x, where x is a fixed generator of $H^2(M;\mathbb{Z})$. By [HaYo] the S^1-action lifts to γ. We denote the weight of γ and the tangential weights at a connected component Y_s of M^{S^1} by a_s and $m_{s,1}, \ldots, m_{s,n}$, respectively. In the proof we use the following information on the weights: For fixed s the weights of γ are related to the tangential weights by

$$|\prod_{t \neq s}(a_t - a_s)^{n_t+1}| = |\prod_{m_{s,i} \neq 0} m_{s,i}|, \tag{4}$$

where n_t denotes the complex dimension of Y_t. To prove this identity one either applies the localization theorem in K-theory to the S^1-action and induced $\mathbb{Z}_p$-actions for p a large prime number (cf. [Pe], Th. 2.8) or uses cohomological means (cf. [Br], Ch. VII, Th. 5.5).

Proof of Theorem 1.2: For $n = 1$ the theorem is trivial. For $n = 2$ it follows for example from the classification of 4-dimensional S^3-manifolds (cf. [MePa]). So assume $n \geq 3$.

Let M be an oriented homotopy $\mathbb{C}P^n$ with S^3-action satisfying $(*)$. Let $Y_0, \ldots, Y_n$ denote the isolated fixed points under the induced S^1-action on M. We assume that Y_0 is also fixed by S^3. Hence, a_0 vanishes and the tangential weights $m_{0,1}, \ldots, m_{0,n}$ are determined by the dimension $m = 2n$ up to finite ambiguity. Therefore $|\prod_{i=1}^{n} m_{0,i}|$ is bounded from above by a constant which only depends on m. Now apply formula (4) twice to conclude that the same holds for the absolute value of all the weights a_s and $m_{s,i}$.

Next we argue as in the proof of Theorem 3.6 to show that the values of polynomials in the Pontrjagin classes and the generator $x \in H^2(M;\mathbb{Z})$ on the fundamental cycle belong to a finite set which only depends on m. Since the cohomology ring of M is a truncated polynomial ring in x the Pontrjagin classes are also determined up to finite ambiguity. By simply-connected surgery theory it follows that the diffeomorphism type of M belongs to a finite set which only depends on the dimension m. ∎

Next we consider complete intersections. A complete intersection $V_n^{(d_1, \ldots, d_r)}$ of complex dimension n and multidegree $(d_1, \ldots, d_r)$, $d_i \geq 2$, is defined as the transversal intersection of hypersurfaces of degree d_i, $i = 1, \ldots, r$, in $\mathbb{C}P^{n+r}$. Thom showed that the diffeomorphism type of $V_n^{(d_1, \ldots, d_r)}$ is completely determined by n and $(d_1, \ldots, d_r)$.

Theorem 5.1. *In a fixed complex dimension there are only finitely many complete intersections with an S^3-action satisfying $(*)$.*

We remark that the $\hat{A}$-vanishing theorem of Atiyah-Hirzebruch (cf. [AtHi]) may be used to show that there are only finitely many $Spin$-complete intersections with non-trivial S^1-action in a fixed even complex dimension. Also results of Hattori (cf. [Ha]) on the equivariant $Spin^c$-Dirac operator imply finiteness of the number of complete intersections with non-trivial S^1-action if one assumes that the action preserves the induced stable almost complex structure.

Proof of Theorem 5.1: Assume $M := V_n^{(d_1,\ldots,d_r)}$ admits an S^3-action satisfying ($*$). Let γ denote the pullback of the dual Hopf bundle over $\mathbb{C}P^{n+r}$ via $i : M \hookrightarrow \mathbb{C}P^{n+r}$ and let $x := c_1(\gamma) \in H^2(M;\mathbb{Z})$. Recall from [Hi] that $p(M) = (1 + x^2)^{n+r+1} \cdot \prod_{i=1}^r (1 + d_i^2 \cdot x^2)^{-1}$ and $\langle x^n, \mu_M \rangle = \prod_{i=1}^r d_i$. Hence, for all but a finite number of multidegrees M is semi-negative with $N = 1$ and $L_j = \gamma$. For semi-negative M the tangential weights and the weights of γ are determined by the complex dimension n up to finite ambiguity by Lemma 2.1. We argue as before (see the proof of Th. 3.6) to conclude that the values of polynomials in the Pontrjagin classes and x on the fundamental cycle μ_M belong to a finite set which only depends on n. In particular this holds for $\langle x^n, \mu_M \rangle = \prod d_i$. Since $d_i \geq 2$ the theorem follows. ∎

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