Introduction to Symplectic Topology I/II

Fall 2017/Spring 2018

The following survey of the material covered in the course is rough, incomplete and may contain typos. Please use your course notes.

1. Symplectic Linear Algebra

Symplectic vector space (V, ω), $\dim V < \infty$, example: $(\mathbb{R}^{2n}, \omega_0)$, existence of a symplectic basis, $\dim V$ necessarily of even dimension $2n$ and (V, ω) isomorphic to $(\mathbb{R}^{2n}, \omega_0)$,

alternating k-form, $Alt^k(V)$, examples: linear forms and determinant, pullback of k-forms, exterior (wedge) product $\wedge$, properties of $\wedge$ and pullback, alternating 2-form ω on V is non-degenerate $\iff$ $\omega^n \neq 0$,

linear symplectomorphism, $Sp(V, \omega)$, $Sp(2n) := Sp(\mathbb{R}^{2n}, \omega_0)$,

characterization of $A \in Sp(2n)$ in terms of the complex structure J_0,

$$Sp(2n) \cap O(2n) = Sp(2n) \cap Gl_n(\mathbb{C}) = O(2n) \cap Gl_n(\mathbb{C}) = U(n),$$

properties of the eigenvalues of $\Psi \in Sp(2n)$,

$\Psi \in Sp(2n)$ symmetric, $\alpha > 0 \implies \Psi^\alpha \in Sp(2n)$ symmetric, $Sp(2n)/U(n)$ contractible,

affine symplectomorphism, affine non-squeezing theorem

2. Symplectic Manifolds

Topological manifold, change of chartes, smooth atlas, differentiable structure, smooth manifold M,

tangent space, equivalence class of curves, derivations, tangent bundle TM, vector fields, Lie bracket, cotangent bundle T^*M,

differential forms, exterior derivative, de Rham complex, exact and closed forms, de Rham cohomology ring, orientation,

symplectic form/structure on M, symplectic manifold (M, ω), examples: oriented surfaces or $(\mathbb{R}^{2n}, \omega_0)$,

λ_{can} canonical 1-form on $M := T^*L$ (cotangent bundle of L), canonical symplectic 2-form $\omega_{can} := -d(\lambda_{can})$ on M,

a symplectic manifold is oriented and, if compact, c-symplectic, examples of manifolds which are not symplectic,

symplectomorphisms, symplectic vector fields, Hamiltonian vector fields, symplectic gradient X_H of a function H, Hamiltonian flow, X_H tangent to the level sets, example: height function of S^2,

Theorem of Darboux, X symplectic vector field $\iff$ flow Ψ_t symplectic, Poissson bracket,

3. Symplectic Group Actions

S^1-action, symplectic S^1-action on (M, ω), Hamiltonian S^1-action, Hamiltonian function H = moment map, examples: rotation of $\mathbb{R}^2$, rotation of S^2 or S^1-action on the torus T^2, Hamiltonian action is symplectic action, when is a symplectic S^1-action Hamiltonian?

torus actions, T^n-action on $\mathbb{C}^n = \mathbb{R}^{2n}$, moment map μ,

f Poisson commutes with $H \implies f$ is constant on level sets of H/integral curves of X_H, Excursion Hamiltonian Mechanics: configuration space Q of particles, Lagrangian L, action integral, principle of least action, Euler-Lagrange equations, example: planet orbiting around the sun (Kepler problem), Legendre condition, Hamiltonian H, Hamiltonian differential equations, examples: Kepler problem or simple pendulum, $L : TQ \to T^*Q$, symplectic form ω on T^*Q, symplectic gradient X_H of H, evolution of the physical system can be described by the Hamiltonian flow.

4. Almost Complex Structures

Almost complex manifold (M, J), compatible structures, every symplectic manifolds admits a compatible almost complex structure,

space of compatible almost complex structures is contractible,

$N \subset (M, \omega, J)$ submanifold and $J(TN) \subset TN \implies N$ is a symplectic submanifold

5. Convexity Theorem

Statements of the convexity theorem of Atiyah and Guillemin-Sternberg and the theorem of Schur and Horn.

Elements of Morse Theory: critical points, Hessian, index, nullity, Lemma of Morse, non-degenerate critical points are isolated, Riemannian gradient, sublevel sets M^a, criterion for $M^a \cong M^b$, attaching a cell, relation to the index,

gradient like vector fields, stable/unstable manifolds, Morse-Bott functions,

index/coindex $\neq 1 \implies$ level sets are connected, Hamiltonian torus action on a symplectic manifold admits an invariant compatible almost complex structure, fixed point sets are symplectic submanifolds (uses facts from Riemannian geometry), Hamiltonian functions associated to the moment map μ are Morse-Bott function with even index and coindex, level sets are connected,

almost periodic vector fields, assertions (A_n) and (B_n), Atiyah's proof of the convexity theorem (modulo some minor details),

$U(n)$-action on the space of $(n \times n)$-hermitian matrices $\mathcal{H}$, characterization of an orbit $M := M_\lambda$ in terms of eigenvalues, $u(n)$ = skew-hermitian matrices, exponential map for matrices, T_hM, symplectic form ω_h, T^n-fixed points of M, Theorem of Schur and Horn as a corollary of the convexity theorem, example: hermitian (2×2)-matrices,

complex projective space $\mathbb{C}P^{n-1}$, desciption as a $U(n)$-orbit $M = U(n) \cdot h$, example: $M = \mathbb{C}P^1$,

$\mu(\mathbb{C}P^{n-1})$ = convex hull of standard basis vectors, polytopes, Delzant polytopes, toric manifolds, characterization of these in terms of Delzant polytopes (without proof)

Discussion of J-holomorphic curves and Gromov's proof of the non-squeezing theorem (very rough).