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CIRCLE ACTIONS ON COMPLETE INTERSECTIONS

THESIS

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On ne voit bien qu'avec le cœur.
L'essentiel est invisible pour les yeux.

— Antoine de Saint-Exupéry

Dedicated to my beloved wife Rosalie Chevalley.

ABSTRACT

We study complete intersections with S^1-symmetry after reviewing general properties of these manifolds. To do so, we use equivariant cohomology, equivariant characteristic classes, the Atiyah-Bott integration formula and the Lefschetz fixed point formula for the equivariant signature.

We prove that an odd dimensional complete intersection with an S^1-action admitting a fixed point component of real codimension 2 or 4 is a complex projective space or a complex quadric. In addition to this result, we study in greater details complex 5-dimensional complete intersections with S^1-symmetry. As a corollary, we establish the classification of 5-dimensional complete intersections with T^2-symmetry.

RÉSUMÉ

Nous étudions les intersections complètes munient d'une action de cercle lisse après un survol des propriétés générales de cette famille de variétés. Pour se faire, on utilise la cohomologie équivariante, les classes caractéristiques équivariantes, la formule d'intégration d'Atiyah-Bott ainsi que la formule des points fixes de Lefschetz pour la signature équivariante.

Nous démontrons notamment qu'une intersection complète de dimension impaire munie d'une action de cercle admettant une composante de point fixe de codimension 2 ou 4 est un espace complexe projectif ou une quadrique. De plus, nous étudions de manière plus approfondie les intersections complètes de dimension 5 munient d'une action de cercle lisse. Ceci nous permet notamment de classifier les intersections complètes de dimension 5 munient d'une action lisse de tore T^2.

If we knew what it was we were doing,
it would not be called research, would it?

— Albert Einstein

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CONTENTS

1 INTRODUCTION

Human beings have always been fascinated by the regularity, harmony and symmetry of objects. Mathematicians have tried to express these notions in more formal ways since the early days of our science. In the nineteenth century, the concept of a group has been invented. This notion allowed mathematicians to define in an insightful way the concept of symmetry. In 1872, Felix Klein proposed in his Erlangen program to use group theory to organize geometrical knowledge. Few years later, Sophus Lie developed the foundations of the theory of transformation groups. The context was then set for geometers to study manifolds and their symmetries.

In the beginning of the twentieth century, finite group actions on spheres were studied. It was a matter of interest to determine if a periodic transformation of an n-sphere resembles a rotation or more generally an orthogonal transformation. Brouwer and Kérékjártó proved independently that every periodic transformation of a 2-sphere is topologically equivalent to an orthogonal transformation. However, no such result was given for $n > 2$.

The early twentieth century has been also the rise of homology theory. These tools turned out to be very relevant in the study of group actions on manifolds. Smith suggested in [49] to focus on the homological properties of transformation groups. Regarding the above problem on spheres, he proved the following result.

THEOREM (SMITH, 1938). *Let M^n be a manifold equipped with a smooth $\mathbb{Z}/p$-action, where p is a prime number. If M is an $\mathbb{F}_p$ homology n-sphere, i.e. $H_*(M;\mathbb{F}_p) \cong H_*(S^n;\mathbb{F}_p)$, then the fixed point set of the $\mathbb{Z}/p$-action is again an $\mathbb{F}_p$ homology r-sphere, i.e $H_*(F;\mathbb{F}_p) \cong H_*(S^r;\mathbb{F}_p)$ where $F = M^{\mathbb{Z}/p}$, and $-1 \leqslant r \leqslant n$.*

As a general assumption, we assume that manifolds are closed smooth and oriented. Smith actually proved the above result in a much greater generality (cf. [12, Chapter III]), but for our expository purpose this is enough. Furthermore, Conner and Floyd proved that the above Theorem holds if one replaces the $\mathbb{Z}/p$-action by an S^1-action, and the coefficient field $\mathbb{F}_p$ by $\mathbb{Q}$ (cf. [23, Theorem 3.2] and [15, Section 3]).

In the late fifties, Borel introduced a theory to generalize Smith's results in order to prove this kind of results in a more systematic way. Namely, he developed his famous Borel construction and equivariant homology theory. In [11], Bredon used this techniques to prove a similar theorem for cohomology complex projective spaces.

THEOREM (BREDON, 1964). *Let M^n be a manifold equipped with a smooth $\mathbb{Z}/p$-action, where p is an odd prime number. If M is an $\mathbb{F}_p$ cohomology $\mathbb{C}P^n$, i.e. $H^*(M;\mathbb{F}_p) \cong H^*(\mathbb{C}P^n;\mathbb{F}_p)$, then any fixed point component F is again an $\mathbb{F}_p$ cohomology $\mathbb{C}P^r$ i.e. $H^*(F;\mathbb{F}_p) \cong H^*(\mathbb{C}P^r;\mathbb{F}_p)$, for $0 \leqslant r \leqslant n$.*

The result also holds if one replaces $\mathbb{Z}/p$ by S^1, and $\mathbb{F}_p$ by $\mathbb{Q}$.

Note that Bredon also carried out the case where $p = 2$. However, we do not include his result for such p here. The reader can read it in [12, Chapter VII, Section 3].

In this thesis, we follow the philosophy of Smith and Bredon in the case of S^1-actions on *complete intersections*. The later are complex manifolds given by transversal intersections of finitely many non-singular hypersurfaces in $\mathbb{C}P^N$. By a non-singular hypersurface we mean that there is a homogeneous polynomial with complex coefficients, say $f(z_0, \dots, z_N)$, such that the maps $f^j : \mathbb{C}^N \to \mathbb{C}$ defined by $f^j(z_1, \dots, z_N) := f(z_1, \dots, z_j, 1, z_{j+1}, \dots, z_N)$ have 0 as regular value for all $j = 0, \dots, N$. In this case, the hypersurface is defined as the zero set of f.

Thom showed that the diffeomorphism type of a complete intersection depends only on its dimension and the degrees of the homogeneous polynomials. Thus, we denote by $X_n(\mathbf{d})$ a complete intersection in $\mathbb{C}P^{n+r}$ of complex dimension n and multi-degree $\mathbf{d} = (d_1, \dots, d_r)$. Note that if $r > 1$, we can assume $d_j > 1$, since an intersection with a hypersurface of degree one cuts the dimension by one of the ambient complex projective space. Furthermore $X_n(1) = \mathbb{C}P^n$. A short review of complete intersections is given in Chapter 2.

One can think of complete intersections as a natural generalization of complex projective spaces. These manifolds are classical submanifolds of the complex projective spaces and share many properties with them. In particular, odd dimensional complete intersections have the same even cohomology ring with rational coefficients as complex projective spaces, i.e.

$$H^{ev}(X_n(\mathbf{d});\mathbb{Q}) \cong \mathbb{Q}[x]/(x^{n+1}).$$

Thus, using the same tools as Bredon we prove the following result, in Chapter 4.

PROPOSITION A. *Let $X_n(\mathbf{d})$ be a complete intersection with odd $n > 1$. If $X_n(\mathbf{d})$ admits a smooth circle action, then for any connected fixed point component $F_i \subset X_n(\mathbf{d})^{S^1}$, we have*

$$H^{ev}(F_i;\mathbb{Q}) \cong \mathbb{Q}[x|_{F_i}]/(x|_{F_i}^{m_i+1}),$$

where $x|_{F_i} \in H^2(F_i;\mathbb{Q})$ and $2m_i$ is the dimension of F_i. Moreover, we have $\sum_i (m_i + 1) = n + 1$.

Another approach to prove the above result would be to use Theorem 3.8.12 in [1]. The later theorem gives a rather complete picture of the cohomological structure of the fixed point set in the case of S^1 and $\mathbb{Z}/p$-actions in terms of generators and relations. Furthermore, this therorem follows the above philosophy of Smith and Bredon and generalizes their results in an insightful way.

We use Proposition A to classify complete intersections with S^1-symmetry. This classification has been started by Dessai and Wiemeler in [16]. They classified complete intersections with S^1-symmetry up to dimension 3. In complex dimension 1, they showed that only $X_1(1)$, $X_1(2)$, $X_1(3)$, and $X_1(2,2)$ admit a smooth non-trivial S^1-action. In dimension 2, they find out again that $X_2(1)$, $X_2(2)$, $X_2(3)$, and $X_2(2,2)$ admit a smooth non-trivial S^1-action. However in complex dimension 3, only $X_3(1)$ and $X_3(2)$ admit S^1-symmetry.

It seems that in higher dimensions, only the complex projective spaces $X_n(1)$ and the complex quadrics $X_n(2)$ admit a smooth non-trivial S^1-actions. Linear actions on complex projective spaces and complex quadrics are discussed in great details in Chapter 2. The low dimensional classification together with the lack of examples in high dimensions suggests the following result.

THEOREM B & C. *Let $X_n(\mathbf{d})$ be a complete intersection with $n \geqslant 5$ odd. If $X_n(\mathbf{d})$ admits a smooth circle action with a fixed point component of codimension 2 or 4, then*

$\mathbf{d} = (1)$ *and* $X_n(\mathbf{d}) = \mathbb{C}P^n$, *or*

$\mathbf{d} = (2)$ *and* $X_n(\mathbf{d})$ *is a complex quadric.*

The proof of this theorem is a case by case study of the possible fixed point configurations. Theorem B treats the codimension 2 case, and Theorem C the codimension 4 case. They both are proven in Chapter 5. The proof of Theorem B is rather concise and does not involve many technicalities. However, the proof of Theorem C is fairly computational and uses extensively the Atiyah-Bott integration formula, characteristic classes and the Lefschetz fixed point formula for the equivariant signature. Chapter 3 provides a turnkey review of these tools.

Restricting our attention to complete intersections of complex dimension 5, we prove a better result.

THEOREM D. *Let $X_5(\mathbf{d})$ be a complete intersection equipped with a smooth and effective circle action which admits a fixed point component of dimension strictly bigger than 2, then*

$\mathbf{d} = (1)$ *and* $X_5(\mathbf{d}) = \mathbb{C}P^5$, *or*

$\mathbf{d} = (2)$ *and* $X_5(\mathbf{d})$ *is a complex quadric.*

The proof of Theorem D is again a case by case study on the possible fixed point configurations and is very computational. However the tools remain the same, i.e. Atiyah-Bott integration formula, characteristic classes and the Lefschetz fixed point formula for the equivariant signature. The difficulty comes essentially from the structure of the normal bundle of the fixed point components.

In the appendix, we give the *Mathematica*® computations for this proof, cf. Chapter B. It helps to figure out all the missing computations.

An interesting corollary of the previous result is the classification of 5-dimensional complete intersections with T^2-symmetry. Together with Theorem D, the following result is discussed in Chapter 6.

THEOREM E. *Let $X_5(\mathbf{d})$ be a complete intersection equipped with a smooth and effective rank 2 torus action, then*

$\mathbf{d} = (1)$ *and* $X_5(\mathbf{d}) = \mathbb{C}P^5$, *or*

$\mathbf{d} = (2)$ *and* $X_5(\mathbf{d})$ *is a complex quadric.*

From a more philosophical viewpoint, these results satisfy the intuitive feeling that manifolds have little symmetry. An interesting survey on this regard was written by Puppe [48]. Although the author focus more on asymmetric manifolds (i.e. manifolds admitting no finite group action), the results above illustrate in our context what Raymond and Schultz said in [14, p. 260]: *"It is generally felt that a manifold 'chosen at random' will have very little symmetry"*.

Part I

GENERAL RESULTS ABOUT CIRCLE ACTIONS ON COMPLETE INTERSECTIONS

We start in Chapter 2 with a reminder about complete intersections, linear actions on complex projective spaces, and linear actions on complex quadrics. We then review, in Chapter 3, the equivariant cohomology, the Atiyah-Bott integration formula, equivariant characteristic classes and the Lefschetz fixed point formula for the equivariant signature. In Chapter 4, we study the cohomology structure of the fixed point components of complete intersections with S^1-symmetry.

2 COMPLETE INTERSECTIONS AND EXAMPLES OF CIRCLE ACTIONS

2.1 COMPLETE INTERSECTIONS

A *complete intersection* is a complex manifold given by a transversal intersection of finitely many non-singular hypersurfaces in $\mathbb{C}P^N$. By a non-singular hypersurface we mean that there is a homogeneous polynomial with $\mathbb{C}$ coefficients, say $f(z_0, \dots, z_N)$, such that the maps $f^j : \mathbb{C}^N \to \mathbb{C}$ defined by $f^j(z_1, \dots, z_N) := f(z_1, \dots, z_j, 1, z_{j+1}, \dots, z_N)$ have 0 as regular value for all $j = 0, \dots, N$. In this case, the hypersurface is defined as the zero set of f.

Thom showed that the diffeomorphism type of a complete intersection depends only on its dimension and the degrees of the homogeneous polynomials. One gets in $\mathbb{C}P^N$ an ambient isotopy from one complete intersection to another [34, Section 4.]. Thus, we denote by $X_n(\mathbf{d})$ a complete intersection in $\mathbb{C}P^{n+r}$ of complex dimension n and multi-degree $\mathbf{d} = (d_1, \dots, d_r)$. Moreover, if $r > 1$ we can assume $d_j > 1$, since an intersection with a hypersurface of degree one cuts the dimension by one of the ambient complex projective space.

The Lefschetz hyperplane theorem [9] gives us that the inclusion map $X_n(\mathbf{d}) \hookrightarrow \mathbb{C}P^{n+r}$ is n-connected. Together with Poincaré duality we have that the cohomology groups of $X_n(\mathbf{d})$ are torsion-free and are the same as the cohomology groups of $\mathbb{C}P^n$ except in middle dimension. If $n > 1$, then $X_n(\mathbf{d})$ is simply-connected. If $n > 2$, then $H^2(X_n(\mathbf{d}); \mathbb{Z})$ is generated by the first Chern class $x := c_1(\gamma)$ where γ is the restriction to $X_n(\mathbf{d})$ of the dual Hopf line bundle over $\mathbb{C}P^{n+r}$. Thus the cohomology ring $H^*(X_n(\mathbf{d}); \mathbb{Z})$ is torsion-free, and outside the middle dimension $X_n(\mathbf{d})$ is a homology $\mathbb{C}P^n$.

By [28, Chapter 3, Section 3.1], or [27] the total Pontrjagin class and Chern class are given respectively by

$$p(X_n(\mathbf{d})) = (1 + x^2)^{n+r+1} \cdot \prod_{j=1}^{r} (1 + d_j^2 \cdot x^2)^{-1}, \quad \text{and}$$

$$c(X_n(\mathbf{d})) = (1 + x)^{n+r+1} \cdot \prod_{j=1}^{r} (1 + d_j \cdot x)^{-1}.$$

In particular, we have that $p_1(X_n(\mathbf{d})) = (n + r + 1 - \sum_i d_i^2) \cdot x^2$. In addition, one can also show that the evaluation of x^n on the fundamental cycle is given by

$$\int_{X_n(\mathbf{d})} x^n = d_1 \cdots d_r,$$

(cf. [34, Section 5]). The product $d_1 \cdots d_r$ is denoted d and called the *total degree* of $X_n(\mathbf{d})$.

Let us give some examples of complete intersections. First, we have $X_n(1) = \mathbb{C}P^n$. Which is why, complete intersections are an interesting generalization of the complex projective space. One can also show that $X_n(2) = SO(n+2)/(SO(n) \times SO(2))$. If $n = 1$, the Euler characteristic can be computed, the later is given by

$$\chi(X_1(\mathbf{d})) = (d_1 \cdots d_r) \cdot \left(2 - \sum_{j=1}^{r} (d_j - 1)\right).$$

Thus, it is easy to see that $X_1(1) = X_1(2) = S^2$ and $X_1(3) = X_1(2,2) = T^2$. However, it is not easy to give an explicit formula for the Euler characteristic in any degree n. Hirzebruch gave in [27] a formula which can be used to compute the Euler characteristic, namely

$$\sum_{n \geqslant 0} \chi(X_n(\mathbf{d})) z^n = \frac{d_1 \cdots d_r}{(1-z)^2} \cdot \prod_{j=1}^{r} \frac{1}{1 + (d_j - 1)z}.$$

Using this formula, Ewing and Moolgavkar showed in [18] that if $n > 1$ is even then $\chi(X_n(\mathbf{d})) = n + 1$ is equivalent to $(\mathbf{d}) = (1)$, and if n is odd then $\chi(X_n(\mathbf{d})) = n + 1$ is equivalent to $(\mathbf{d}) = (1)$ or (2).

The classification of complete intersection up to homotopy, homeomorphism and diffeomorphism type has been a very active field of research in the last fifty years. Let us recall some major results.

In [39], Libgober and Wood gave a connected sum decomposition for complex odd dimensional complete intersections. The later involves d-twisted $\mathbb{C}P^n$, Kervaire manifolds and connected sums of $S^n \times S^n$. This allowed them to prove that for odd dimensional complete intersections $X_n(\mathbf{d})$ and $X_n(\mathbf{d}')$ with the same total degree d, if d has no prime factor less than $(n+3)/2$, then $X_n(\mathbf{d})$ and $X_n(\mathbf{d}')$ are homotopy equivalent if and only if their Euler characteristic agree. Fifteen years later, Fang [19] completed this homotopy classification started by Libgober and Wood for even dimensional complete intersection. Precisely, he showed that for even dimensional complete intersections $X_n(\mathbf{d})$ and $X_n(\mathbf{d}')$ with $n > 2$ and the same total degree d, if d has no prime factor less than $(n+3)/2$, then $X_n(\mathbf{d})$ and $X_n(\mathbf{d}')$ are homotopy equivalent if and only if their Euler characteristic and signature agree.

Note that the result given by Libgober, Wood and Fang applies in the case when d is larger than the dimension of the complete intersection. Thus, in [2] Astey, Gitler, Micha and Pastor discussed the cases when the complex dimension is larger than the total degree.

Furthermore, it is interesting to notice the polarization between even and odd dimensional complete intersection. Even dimensional complete intersection seems to have a more complicated topology compared to the odd dimensional ones. A reference of this fact can be found in [37, 40].

Let us say few words about the diffeomorphism type of complete intersections. Their classification in this context has been studied in [38, 39, 40]. In particular, Libgober and Wood showed uniqueness in low codimension. More precisely, if the codimension r of $X_n(\mathbf{d}) \subset \mathbb{C}P^{n+r}$ satisfies $2r \leqslant n$ and $n > 2$ then any complete intersection $X_n(\mathbf{d}')$ satisfying $2r' \leqslant n$ and diffeomorphic to $X_n(\mathbf{d})$ has the same multidegrees.

However, it is well known that uniqueness does not hold in general (cf. [40, Theorem 6.1]). The classification is organized around the Sullivan Conjecture.

CONJECTURE 2.1 (SULLIVAN). *The complete intersections $X_n(\mathbf{d_1})$ and $X_n(\mathbf{d_2})$ are diffeomorphic if and only if they share the same total degree, the same Euler characteristic and the same Pontrjagin classes.*

The previous conjecture has been shown for $n = 1$ using the classification of surfaces, and $n = 3$ using the classification of simply connected 6-manifolds (cf. [51, 30]). For $n = 2$, the conjecture is true in the topological category by [24], but fails in the smooth category. In [17], the author gave an example of homeomorphic but not diffeomorphic complete intersections. For $n = 4, 5, 6$ and 7, the conjecture has been proven in the topological category with the work of [20] and [22], but is still open in the smooth category.

During her Diploma Arbeit, Traving showed that the Sullvan Conjecture holds for $n > 2$ under the assumption that the total degree d is divisible by $p^{[(2n+1)/(2p-1)]+1}$ for all primes p with $p(p-1) \leqslant n+1$ [50, 33].

Other properties have been studied on complete intersections like positive scalar curvature [21], real homotopy properties [6], rational homotopy groups [46], the signature [36, 35], ... Furthermore, complete intersections play an important role in algebraic geometry. However, we will not discuss it, since we are interested in them from a topological and geometrical viewpoint.

2.2 CIRCLE ACTIONS ON COMPLETE INTERSECTIONS

In this thesis, we are interested to classify complete intersections with S^1-symmetry. These are complete intersections admitting a smooth non-trivial S^1-action.

Since complete intersections are Kähler manifolds, it is possible to consider S^1-actions preserving some additional structure, for example the complex structure. In this case, we know by [31, Chapter III, Theorem 2.1] that the group of holomorphic transformations is finite, if the first Chern class is negative. Thus, if $c_1(X_n(\mathbf{d})) = c_1 x$ where $c_1 = n + r + 1 - \sum_i d_i < 0$, then a non-trivial S^1-action preserving the complex structure cannot exists on $X_n(\mathbf{d})$. Another interesting research direction would be to study complete intersections admitting a symplectic or Hamiltonian S^1-action. However, this will not be discussed at all in this thesis.

2.2.1 *Complete intersections of complex dimension 1*

Complete intersections of dimension 1 admitting a smooth S^1-actions can be easily classified. One only needs to use that the Euler characteristic is equivariant, i.e.

$$\chi(X) = \chi(X^{S^1}).$$

A proof of this statement can be found in [28, p. 70]. Furthermore, the fixed point set of a smooth non-trivial S^1-action on an closed smooth oriented surface can only be composed with isolated fixed points. This implies that the Euler characteristic of the fixed point set is greater or equal to 0. It is now easy to see that except $X_1(1)$, $X_1(2)$, $X_1(3)$, and $X_1(2,2)$ all other complete intersections have a negative Euler characteristic. This shows that the only complete intersections admitting a smooth non-trivial S^1-action are those with a non-negative Euler characteristic. Namely $X_1(1)$ and $X_1(2)$ which are both isomorphic to the sphere S^2, and $X_1(3)$ and $X_1(2,2)$ which are isomorphic to the torus T^2.

2.2.2 *Complete intersections of complex dimension 2*

In [16, Theorem 2.2], Dessai and Wiemeler used Seiberg-Witten theory to prove the following theorem.

THEOREM 2.2 (DESSAI, WIEMELER, 2011). *A complete intersection $X_2(\mathbf{d})$ admits a smooth non-trivial S^1-action if and only if $X_2(\mathbf{d})$ is diffeomorphic to a complex projective plane $X_2(1)$, a quadric $X_2(2)$, a cubic $X_2(3)$ or an intersection of two quadrics $X_2(2,2)$.*

To show that $X_2(3)$ and $X_2(2,2)$ admit smooth non-trivial S^1-actions, they identified these complete intersections with a blowing up of $\mathbb{C}P^2$ at 6 and respectively 5 points in general position [32, Section 3.5], [37, p. 653], and [41]. Thus we have that $X_2(3) = \mathbb{C}P^2\#6\overline{\mathbb{C}P^2}$ and $X_2(2,2) = \mathbb{C}P^2\#5\overline{\mathbb{C}P^2}$, which allows us to define easily the smooth S^1-actions.

Note that in complex dimension 2, the main tools in use cannot be generalized in higher dimensions. In particular, the Seiberg-Witten invariants are special invariants of 4-dimensional manifolds, while the above identifications are special to this dimension.

2.2.3 *Complete intersections of complex dimension 3*

The classification of complete intersections of complex dimension 3 admitting a smooth non-trivial S^1-action follows from a more general result proven by Dessai and Wiemeler in [16].

THEOREM 2.3 (DESSAI, WIEMELER, 2011). *Let* M *be a closed smooth oriented* 6*-dimensional manifold with torsion-free homology,* $b_1(M) = 0$, $H^2(M;\mathbb{Z}) = \langle x \rangle$, $p_1(M) = \rho x^2$ *with* $\rho \leqslant 0$, $x^3 \neq 0$ *and* $\chi(M) < 4$. *Then M does not support a non-trivial smooth circle action.*

This implies in particular that only the complex projective space $X_3(1)$ and the complex quadric $X_3(2)$ admit a smooth non-trivial S^1-action.

The proof of this statement consists of a case by case study of the possible fixed point configurations. In particular, they used Bredon's techniques (cf. [12, Chapter VII]) to show that the sum of even Betti numbers

$$b_{\mathrm{ev}}(M) = b_{\mathrm{ev}}(M^{S^1}) = 4.$$

This limits the number of fixed point configurations to seven different cases, which they treated separately. To do so, they used equivariant characteristic classes, the Atiyah-Bott integration formula, the Lefschetz fixed point formula for the equivariant signature, all together with acute arithmetic relations in cohomology. Therefore, the techniques used in [16] serve as a pillar to this thesis.

2.3 LINEAR ACTIONS ON COMPLEX PROJECTIVE SPACES

In this section, we discuss linear circle actions on complex projective spaces. This very basic example of S^1-actions is interesting since it illustrates very well a set of parameters which appears recurrently in the thesis, namely the *Hopf weights* and the *normal weights*. Furthermore in Section 2.4, these linear actions also serve us to define linear S^1-actions on the complex quadrics.

Let $V = \mathbb{C}^{n+1}$ equipped with an Hermitian inner product. Let us consider a Lie group homomorphism

$$S^1 \to U(n+1) : \lambda \mapsto A_\lambda.$$

Define a smooth S^1-action on V via $\lambda \cdot \mathbf{z} := A_\lambda \mathbf{z}$, for any $\lambda \in S^1$. Since the irreducible complex representations of S^1 are given by the characters $\lambda \mapsto \lambda^a$ for a certain integer a, the vector space V decomposes into

$$V = \bigoplus_{a \in \mathbb{Z}} E_a$$

where E_a is the isotypical summand associated to the weight $\lambda \mapsto \lambda^a$. Note that there are only finitely many a's where E_a is not trivial. Thus let us denote them $a_0, \ldots, a_k$.

Consider now the complex projective space $P(V) = \mathbb{C}P^n$. The later is therefore equipped with an isometric S^1-action (for the Fubini-Study metric), and the fixed point set is given by

$$P(V)^{S^1} = \coprod_i P(E_{a_i}).$$

Note that $P(E_{a_i}) = \mathbb{C}P^{m_i}$ where $m_i + 1 = \dim(E_{a_i})$. In particular, we have that $n + 1 = \sum_i (m_i + 1)$.

Let γ denote the Hopf line bundle over $P(V)$. Recall that the total space $E(\gamma)$ consists of pairs $(\mathbf{w}, [\mathbf{z}]) \in V \times P(V)$ with $\mathbf{w} \in [\mathbf{z}]$, and where $[\mathbf{z}]$ denotes the complex line through $\mathbf{z} \in V$. It is easy to see that we can lift the S^1-action to γ. We define the action on $E(\gamma)$ simply by

$$\lambda \cdot (\mathbf{w}, [\mathbf{z}]) := (A_\lambda \mathbf{w}, [A_\lambda \mathbf{z}])$$

for any $\lambda \in S^1$. If we restrict γ to a fixed point component $P(E_{a_i})$, then $\gamma|_{P(E_{a_i})}$ is an S^1-equivariant complex line bundle over an S^1-trivial space. It is then easy to see that the S^1-representation on the fibers of $\gamma|_{P(E_{a_i})}$ is given by the complex multiplication with λ^{a_i} for any $\lambda \in S^1$. For this reason, we call these a_i's the *Hopf weights* of the S^1-action.

Let us now consider $\gamma^\perp$ the orthogonal complement of γ. The total space $E(\gamma^\perp)$ consists of pairs $(\mathbf{w}, [\mathbf{z}]) \in V \times P(V)$ with $\mathbf{w} \in [\mathbf{z}]^\perp$. Similarly, we can lift the S^1-action to $\gamma^\perp$. Thus, the complex bundle $\hom(\gamma, \gamma^\perp)$ over $P(V)$ admits an S^1-action. Recall that the total space of $\hom(\gamma, \gamma^\perp)$ consists of pairs $(\Phi, [\mathbf{z}])$ where $\Phi : \gamma_{[\mathbf{z}]} \to \gamma^\perp_{[\mathbf{z}]}$ is linear and $[\mathbf{z}] \in P(V)$. With the above notation, the S^1-action on $\hom(\gamma, \gamma^\perp)$ is defined by

$$\lambda \cdot (\Phi, [\mathbf{z}]) := (A_\lambda \Phi A_\lambda^{-1}, [A_\lambda \mathbf{z}])$$

for any $\lambda \in S^1$.

We use the bundle $\hom(\gamma, \gamma^\perp)$ to understand the tangent bundle of $P(V)$, in particular the restriction to a fixed point component $TP(V)|_{P(E_{a_i})}$. Let us briefly recall from [44, Theorem 14.10] that the tangent bundle $TP(V)$ is isomorphic to $\hom(\gamma, \gamma^\perp)$. Indeed, first let us identify $T_{[\mathbf{z}]}P(V)$ with the open neighborhood $U_{[\mathbf{z}]} := \{[\mathbf{z} + \mathbf{w}] : \mathbf{w} \in [\mathbf{z}]^\perp\}$ of $P(V)$. This can be done easily with the map

$$T_{[\mathbf{z}]}P(V) \longrightarrow U_{[\mathbf{z}]} : [t \mapsto [\mathbf{z} + t\mathbf{w}]] \longmapsto [\mathbf{z} + \mathbf{w}].$$

Moreover, the fiber $\hom(\gamma, \gamma^\perp)_{[\mathbf{z}]}$ of $\hom(\gamma, \gamma^\perp)$ at $[\mathbf{z}]$ is isomorphic to $U_{[\mathbf{z}]}$ using the map

$$\hom(\gamma, \gamma^\perp)_{[\mathbf{z}]} \longrightarrow U_{[\mathbf{z}]} : (\Phi, [\mathbf{z}]) \longmapsto [\mathbf{z} + \Phi\mathbf{z}].$$

These maps define an isomorphism between $TP(V)$ and $\hom(\gamma, \gamma^{\perp})$.

Note that the S^1-action on $P(V)$ induces an S^1-action on $TP(V)$, where $\lambda \in S^1$ acts by the differential $\lambda_* : TP(V) \to TP(V) : [\gamma] \mapsto [\lambda \cdot \gamma]$. Furthermore, we can see that the isomorphism above is S^1-equivariant because the following diagram

$$\begin{tikzcd}
T_{[\mathbf{z}]}P(V) \arrow[r] \arrow[d, "\lambda_*"'] & U_{[\mathbf{z}]} \arrow[d, "\lambda"] & \hom(\gamma,\gamma^{\perp})_{[\mathbf{z}]} \arrow[l] \arrow[d, "\lambda"] \\
T_{\lambda[\mathbf{z}]}P(V) \arrow[r] & U_{\lambda[\mathbf{z}]} & \hom(\gamma,\gamma^{\perp})_{\lambda[\mathbf{z}]} \arrow[l]
\end{tikzcd}$$

commutes for any $[\mathbf{z}] \in P(V)$ and $\lambda \in S^1$.

As we mentioned above, we want now to study the restriction of the tangent bundle $TP(V)|_{P(E_{a_i})}$ to a fixed point component $P(E_{a_i})$. First remark that we have an S^1-equivariant direct sum decomposition

$$TP(V)|_{P(E_{a_i})} = TP(E_{a_i}) \oplus \nu_{P(E_{a_i})},$$

where $TP(E_{a_i})$ is the tangent bundle of $P(E_{a_i})$, and $\nu_{P(E_{a_i})}$ is the normal bundle of $P(E_{a_i})$ in $P(V)$. Thus $TP(E_{a_i})$ is a trivial S^1-bundle, and $\nu_{P(E_{a_i})}$ is an S^1-equivariant complex bundle over a trivial S^1-space. The later implies that at any fixed point $[\mathbf{z}] \in P(E_{a_i})$, the fiber $\left(\nu_{P(E_{a_i})}\right)_{[\mathbf{z}]}$ of $\nu_{P(E_{a_i})}$ at $[\mathbf{z}]$ decomposes into the direct sum

$$\left(\nu_{P(E_{a_i})}\right)_{[\mathbf{z}]} = \bigoplus_{n_{i,j} \in \mathbb{Z}} \nu(n_{i,j})_{[\mathbf{z}]}$$

where $\nu(n_{i,j})_{[\mathbf{z}]}$ is the isotypical summand associated to the weight $\lambda \mapsto \lambda^{n_{i,j}}$ for any $\lambda \in S^1$, i.e. the S^1-representation on $\nu(n_{i,j})_{[\mathbf{z}]}$ is given by complex multiplication with $\lambda^{n_{i,j}}$.

Furthermore, the isomorphism type of the fibers at $[\mathbf{z}]$ is independent of the choice of $[\mathbf{z}] \in P(E_{a_i})$. This yields an S^1-equivariant direct sum decomposition of the normal bundle $\nu_{P(E_{a_i})}$,

$$\nu_{P(E_{a_i})} = \bigoplus_{n_{i,j} \in \mathbb{Z}} \nu(n_{i,j}).$$

Note that only finitely many $n_{i,j}$'s give us a non-trivial weight space $\nu(n_{i,j})_{[\mathbf{z}]}$. We call these weights the *normal weights* of the S^1-action at the fixed point component $P(E_{a_i})$.

In our case when the S^1-action on $P(V)$ is linear, the normal weights have a very particular form. Using the above identification of the tangent bundle $TP(V)$ with $\hom(\gamma, \gamma^{\perp})$, we show that the weights at the fibers of $TP(V)|_{P(E_{a_i})}$ are $\lambda \mapsto \lambda^{a_j - a_i}$ with multiplicities $m_j + 1$ for any $j \neq i$, and $\lambda \mapsto 1$ with multiplicity m_i. Thus the normal weights at $P(E_{a_i})$ are $n_{i,j} = a_j - a_i$ with multiplicities $m_j + 1$.

Indeed, the S^1-representation at the fibers of γ over $P(E_{a_i})$ is given by the complex multiplication with λ^{a_i}, for any $\lambda \in S^1$. Furthermore, we have that $\gamma \oplus \gamma^{\perp}$ is equivariantly isomorphic to the trivial bundle $V \times P(V)$. This implies that the weights of the S^1-representation at the fibers of $\gamma^{\perp}$ over $P(E_{a_i})$ are $\lambda \mapsto \lambda^{a_j}$ with multiplicities $m_j + 1$ except for $j = i$ where the multiplicity is m_i. The result now follows easily from the definition of the S^1-action on $\hom(\gamma, \gamma^{\perp})$ given above.

2.4 LINEAR ACTIONS ON THE COMPLEX QUADRICS

In this section, we explore linear actions on complex quadrics. This gives rise to an interesting family of circle actions with many different fixed point configurations.

Let $V = \mathbb{C}^{n+2}$ equipped with an Hermitian inner product that we denote by $\langle \cdot, \cdot \rangle$. Let us consider a Lie group homomorphism

$$S^1 \to O(n+2) \subset U(n+2) : \lambda \mapsto A_\lambda.$$

Define an S^1-action on V via $\lambda \cdot \mathbf{z} := A_\lambda \mathbf{z}$, for any $\lambda \in S^1$. Furthermore, let $\overline{(\cdot)}$ denote the conjugation map.

Using as above E_a to denote the isotypical summand associated to the weight $\lambda \mapsto \lambda^a$, we can see that if E_a is a non-trivial weight space, then $E_{-a} = \overline{E_a}$ is also a non-trivial weight space. Furthermore, note that $E_a \perp E_{-a}$ whenever $a \neq 0$. This gives us an S^1-equivariant direct sum decomposition

$$V = E_0 \bigoplus_{a_i} (E_{a_i} \oplus E_{-a_i}).$$

In other terms, whenever a_i is a Hopf weight, then $-a_i$ is also a Hopf weight. We fix the convention that $a_0 = 0$.

Consider now the complex quadric $X_n(2)$ defined by

$$Q(V) := \{[\mathbf{z}] \in P(V) \mid \langle \mathbf{z}, \overline{\mathbf{z}} \rangle = 0\}.$$

The above S^1-action restricts to $Q(V)$. Indeed, for any $\lambda \in S^1$, we have

$$\langle \lambda \cdot \mathbf{z}, \overline{\lambda \cdot \mathbf{z}} \rangle = \langle A_\lambda \mathbf{z}, \overline{A_\lambda \mathbf{z}} \rangle = \langle A_\lambda \mathbf{z}, A_\lambda \overline{\mathbf{z}} \rangle = \langle \mathbf{z}, \overline{\mathbf{z}} \rangle = 0.$$

This shows that the S^1-action is well defined on $Q(V)$. Since the action is isometric on the level of the complex projective space with the Fubini-Study metric, this yields an isometric S^1-action on the complex quadric for the induced metric.

Like in the previous section, we want now to understand the structure of the fixed point set of this action. We first remark that $Q(V)^{S^1} = P(V)^{S^1} \cap Q(V)$. Secondly, note that for any Hopf weight a_i where $i \neq 0$, we have that the fixed point set $P(E_{a_i}) \subset Q(V)$. This follows directly form the fact that $E_{a_i} \perp E_{-a_i}$. However, for $a_0 = 0$ the projectivization $P(E_0)$ of the weight space E_0 is not contained in $Q(E_0)$ if E_0 is non-trivial. Indeed, if $\mathbf{z} \neq 0$ is in E_0, then $\overline{\mathbf{z}}$ is also in E_0. Therefore,

$\mathbf{w} := \mathbf{z} + \overline{\mathbf{z}} = 2 \cdot \Re(\mathbf{z})$ is also contained in E_0 and $\langle \mathbf{w}, \overline{\mathbf{w}} \rangle$ is a positive real number, showing us that $P(E_0) \not\subset Q(E_0)$.

Note that when $i \neq 0$ (i.e. $a_i \neq 0$) the fixed point component $Q(E_{a_i})$ can be identified with $P(E_{a_i}) = \mathbb{C}P^{m_i}$ where $m_i + 1 = \dim(E_{a_i})$. This is exactly like in the previous section, i.e. for a linear S^1-action on the complex projective space. However, for $i = 0$ (i.e $a_i = 0$) the fixed point component $Q(E_0)$ is a quadric $X_{m_0}(2)$ where $m_0 + 2 = \dim(E_0)$. The previous arguments sum up to the following proposition.

PROPOSITION 2.4. *Let S^1 act on $X_n(2)$ linearly, then the fixed point set $X_n(2)^{S^1}$ is of the form*

$$X_{m_0}(2) \amalg (\mathbb{C}P^{m_1} \amalg \mathbb{C}P^{m_1}) \amalg \cdots \amalg (\mathbb{C}P^{m_l} \amalg \mathbb{C}P^{m_l}), \tag{1}$$

with $(m_0 + 1) + 2\sum_{i=1}^{l}(m_i + 1) = n + 1$. We assume in (1) that m_0 can be equal to -1, in which case $X_{-1}(2) = \emptyset$.

We now focus our interest on the structure of the normal bundle at a fixed point component. Let $Q(E_{a_i}) \subset Q(V)^{S^1}$ be a fixed point component of the S^1-action on $Q(V)$. Then we have an S^1-equivariant direct sum decomposition

$$TQ(V)|_{Q(E_{a_i})} = TQ(E_{a_i}) \oplus \nu_{Q(E_{a_i}) \subset Q(V)},$$

where $TQ(E_{a_i})$ is the tangent bundle of $Q(E_{a_i})$ and $\nu_{Q(E_{a_i}) \subset Q(V)}$ is the normal bundle of $Q(E_{a_i})$ in $Q(V)$. Thus $TQ(E_{a_i})$ is a trivial S^1-bundle, and $\nu_{Q(E_{a_i}) \subset Q(V)}$ is an S^1-equivariant bundle over a trivial S^1-space. Like in the previous section, one can define the *normal weights* at the fixed point component $Q(E_{a_i})$. These are given by the S^1-representations at the fibers of $\nu_{Q(E_{a_i}) \subset Q(V)}$.

PROPOSITION 2.5. *Let $X_n(2)$ be a complex quadric equipped with a linear S^1-action. Let $F \subset X_n(2)^{S^1}$ be a fixed point component.*

IF $F = X_{m_0}(2)$. *The normal weights at F are the a_j's of multiplicity $m_j + 1$, for $j \neq 0$.*

IF $F = \mathbb{C}P^{m_i}$. *The normal weights at F are:*

- *the differences $(a_j - a_i)$'s of multiplicity $m_j + 1$, for $j \neq 0, i$ and $a_j \neq -a_i$,*
- *the numbers $-a_i$'s of multiplicity $m_0 + 2$,*
- *the numbers $-2a_i$'s of multiplicity m_i.*

PROOF. Let us first consider the case $F = X_{m_0}(2)$. By Proposition 2.4, the fixed point component corresponds to $Q(E_0)$. Consider then the normal bundle $\nu_{Q(E_0) \subset Q(V)}$. We note that the inclusion $Q(V) \subset P(V)$ induces an S^1-equivariant isomorphism

$$\nu_{Q(E_0) \subset Q(V)} \cong \left(\nu_{P(E_0) \subset P(V)}\right)\big|_{Q(V)}.$$

This gives us that the normal weights at $Q(E_0)$ in $Q(V)$ are exactly the normal weights at $P(E_0)$ in $P(V)$ computed in the previous section. These are the differences $(a_j - a_0)$'s of multiplicity $m_j + 1$ for $j \neq 0$. Since $a_0 = 0$, this concludes the first part of the proof.

Let us now consider the case where $F = \mathbb{C}P^{m_i}$. By Proposition 2.4, this fixed point component corresponds to the space $Q(E_{a_i})$ where $a_i \neq 0$. Note that in this case, we have $Q(E_{a_i}) = P(E_{a_i})$. We can then consider the normal bundle $\nu_{Q(E_{a_i})\subset P(V)}$ of $Q(E_{a_i})$ in $P(V)$. The later gives us an S^1-equivariant direct sum decomposition

$$\nu_{Q(E_{a_i})\subset P(V)} = \nu_{Q(E_{a_i})\subset Q(V)} \oplus \left(\nu_{Q(V)\subset P(V)}\right)\big|_{Q(E_{a_i})}.$$

From the above splitting, we deduce that the normal weights at $Q(E_{a_i})$ in $Q(V)$ come all from $Q(E_{a_i}) = P(E_{a_i})$ in $P(V)$ which have been computed in the previous section. Thus, we need to show that the normal weight at fibers of $\left(\nu_{Q(V)\subset P(V)}\right)\big|_{Q(E_{a_i})}$ is $-2a_i$.

Let us consider the S^1-invariant subspace $V_{a_i} := E_{a_i} \oplus E_{-a_i}$ of V. The inclusion induces an equivariant isomorphism

$$\nu_{Q(V_{a_i})\subset P(V_{a_i})} \cong \left(\nu_{Q(V)\subset P(V)}\right)\big|_{Q(V_{a_i})}.$$

Therefore, it is sufficient to show that the normal weight at the fibers of $\nu_{Q(V_{a_i})\subset P(V_{a_i})}$ above $Q(E_{a_i}) = P(E_{a_i})$ is $-2a_i$. Consider then the following direct sum decomposition

$$\nu_{P(E_{a_i})\subset P(V_{a_i})} = \nu_{P(E_{a_i})\subset Q(V_{a_i})} \oplus \left(\nu_{Q(V_{a_i})\subset P(V_{a_i})}\right)\big|_{P(E_{a_i})}.$$

By the computation in the previous section, we know that the normal weights on the left hand side are all $-2a_i$. $\square$

EXAMPLE. To illustrate the previous proposition, let us consider the linear S^1-action on $X_n(2)$ given by the Lie group homomorphism

$$\begin{aligned} S^1 &\longrightarrow SO(n+2) \\ e^{i\theta} &\longmapsto \begin{pmatrix} \cos\theta & -\sin\theta & & & \\ \sin\theta & \cos\theta & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \end{aligned}$$

The fixed point components of this action are two isolated fixed points $P_1 = [1 : -i : 0 : \cdots : 0]$ and $P_{-1} = [1 : i : 0 : \cdots : 0]$, and a $n-2$ dimensional complex quadric $X_{n-2}(2)$ consisting of points $[0 : 0 : z_2 : \cdots : z_{n+1}] \in \mathbb{C}P^{n+1}$ such that $\langle \mathbf{z}, \bar{\mathbf{z}} \rangle = 0$. The point P_1 corresponds to the weight $\lambda \mapsto \lambda$, while the point P_{-1} corresponds to its conjugate weight $\lambda \mapsto \lambda^{-1}$. Therefore, the Hopf weights at P_1 and P_{-1} are 1 and -1 respectively. Furthermore, the Hopf weight at $X_{n-2}(2)$ is 0. By Proposition 2.5, we see that all the normal weights at any fixed point component are equal to ± 1, i.e. the action is semi-free around the fixed point components.

2.5 exotic circle action on complex projective space

An important example of non-linear smooth circle actions on the complex projective spaces was constructed by Petrie in [47, p. 148]. These are called *exotic actions* because the normal weights at a fixed points are not the differences of the Hopf weights.

Petrie constructed an example of such an action on $\mathbb{C}P^3$ admitting four isolated fixed point P_0, P_1, P_2, P_3. The Hopf weights at these points are respectively $a_0 = 0$, $a_1 = 7$, $a_2 = 1$ and $a_3 = 6$. Furthermore, the normal weights are:

at P_0: 7, 2 and 3,

at P_1: -7, 2 and 3,

at P_2: 5, 2 and 3,

at P_3: -5, 2 and 3.

Note that if the action were linear with these Hopf weights, the normal weights at P_0 should be instead 7, 1 and 6.

3 REVIEW OF EQUIVARIANT COHOMOLOGY AND RELATED TOPICS

In this chapter, we provide the reader with a short review of all the tools we use to study complete intersections with S^1-symmetry.

3.1 BOREL CONSTRUCTION AND EQUIVARIANT COHOMOLOGY

Let G be a topological group and let X be a (left) G-space. Consider then a *universal* G*-bundle* $EG \to BG$, where BG is a *classifying space*. Recall that G acts freely on EG, and $EG \to BG$ is a G-principal bundle. We assume the convention that EG is a right G-space.

We define the space X_G as

$$X_G := EG \times_G X,$$

and $EG \times_G X$ is the quotient $EG \times X/\sim$ for the equivalence relation defined by $(eg, x) \sim (e, gx)$ for any $(e, x) \in EG \times X$ and $g \in G$. Note that X_G can also be seen as the orbit space $(EG \times X)/G$ of the G-action on $EG \times X$ defined by $g(e, x) := (eg^{-1}, gx)$. The *equivariant cohomology* of X is defined by taking the ordinary cohomology of the space X_G, i.e.

$$H^*_G(X) := H^*(X_G).$$

The construction above gives us also the following commutative diagram

$$\begin{tikzcd} EG & EG \times X \arrow[l] \arrow[r] \arrow[d] & X \arrow[d] \\ BG \arrow[from=u] & X_G \arrow[l] \arrow[r] & X/G. \end{tikzcd}$$

In addition, this construction is functorial. If $f : X \to Y$ is a G-equivariant map, then we have an induced map $f_G : X_G \to Y_G$ which induces in cohomology

$$f^*_G : H^*_G(Y) \to H^*_G(X).$$

Another interesting consequence is that we have a fibration $X \to X_G \to BG$. This fibration plays an important role, because it allows us to understand (via the Leray-Serre spectral sequence) the equivariant cohomology of X via the ordinary cohomology of X and BG. Furthermore, the map $X_G \to BG$ equips the equivariant cohomology ring $H^*_G(X)$ with a structure of $H^*(BG)$-module via $H^*(BG) \to H^*_G(X)$.

Consequently, $H^*_G(X)$ has the structure of an $H^*(BG)$-algebra. This turns out to be very useful to understand $H^*_G(X)$.

Recall that for $G = S^1$, the infinite complex projective space $\mathbb{C}P^\infty$ can be identified with the classifying space BS^1. Therefore, we have the following isomorphism of rings

$$H^*(BS^1) \cong \mathbb{Z}[z],$$

where z is a generator of degree 2. For $G = \mathbb{Z}/2$, the infinite real projective space $\mathbb{R}P^\infty$ can now be identified with the classifying space $B\mathbb{Z}/2$, and the cohomology ring over $\mathbb{F}_2$ coefficients is

$$H^*(B\mathbb{Z}/2; \mathbb{F}_2) \cong \mathbb{F}_2[t],$$

where t is a generator of degree 1. Similarly for $G = \mathbb{Z}/p$, where p is an odd prime number, we have that $B\mathbb{Z}/p$ can be identified with a lens space, its cohomology ring is given by

$$H^*(B\mathbb{Z}/p; \mathbb{F}_p) \cong \mathbb{F}_p[s,t]/(s^2),$$

where s is of degree 1, t is of degree 2 and $\beta(s) = t$ where β is the Bockstein homomorphism.

3.2 ATIYAH-BOTT INTEGRATION FORMULA

From now on, we assume manifolds to be closed, smooth and oriented.

3.2.1 *Review of the Umkehrung homomorphism*

Let $f : N \to M$ be a smooth map between two manifolds. The classical *Umkehrung* homomorphism (also called the *pushforward*) is a map in cohomology

$$f_! : H^*(N) \longrightarrow H^{*+k}(M)$$

where $k = \dim M - \dim N$. This map is functorial in the sense that the Umkehrung homomorphism of a composition satisfies $(f \circ g)_! = f_! \circ g_!$.

Furthermore, this map satisfies some interesting properties. Let us define an $H^*(M)$-module structure on $H^*(N)$ via the map $f^* : H^*(M) \to H^*(N)$ and the cup product. Then $f_!$ is a map of $H^*(M)$-modules. More precisely, we have that

$$f_!(\mu \cdot f^*(\omega)) = f_!(\mu) \cdot \omega$$

for any $\mu \in H^*(N)$, and $\omega \in H^*(M)$.

Most importantly, when we have an inclusion $i : N^{n-k} \hookrightarrow M^n$ of a submanifold of codimension k, then the Umkehrung homomorphism *factors through the Thom isomorphism*. To be precise, the Umkehrung homomorphism is the composition

$$H^*(N) \xrightarrow{\phi} H^{*+k}(M, M-N) \xrightarrow{\text{res}} H^{*+k}(M).$$

We identify the cohomology groups $H^{*+k}(E(\nu_N), E(\nu_N) - N)$ and $H^{*+k}(M, M-N)$, where ν_N denotes the normal bundle of N in M, and $E(\nu_N)$ is its total space. This isomorphism follows easily using the excision isomorphism, and identifying the total space $E(\nu_N)$ as a tubular neighborhood of N in M. A precise proof can be found in [44, Corollary 11.2].

In particular, this factorization through the Thom isomorphism implies that $i_!(1)$ is the *Poincaré-dual cohomology class* of N in $H^k(M)$. Consequently, [44, Theorem 11.3] gives us the following corollary.

COROLLARY 3.1. *Let* $i : N \hookrightarrow M$ *be as above, then* $i^* \circ i_!(1) = e(\nu_N)$ *where* $e(\nu_N)$ *is the Euler class of the normal bundle of* N *in* M.

Another interesting property occurs when $p : E \to B$ is a fiber bundle of dimension k, then the Umkehrung homomorphism

$$p_! : H^*(E) \longrightarrow H^{*-k}(B)$$

is the *integration along the fibers* of p. In particular, considering the projection of a manifold M^n to a point pt, i.e. $M^n \to pt$, the Umkehrung homomorphism gives us the standard integration map

$$\int_M : H^n(M) \to H^0(pt), \text{ where } \int_M \omega = \langle \omega, [M] \rangle,$$

for any cohomology class $\omega \in H^n(M)$. Recall that $\langle \cdot, [M] \rangle$ denotes the evaluation with $[M] \in H_n(M)$, the later being the fundamental homology class of M.

Let $i : N^{n-k} \hookrightarrow M^n$ be an inclusion of a submanifold. Then by the functoriality of the Umkehrung homomorphism, we have the following commutative diagram

$$\begin{tikzcd}
H^n(M) \arrow[r, "\int_M"] & H^0(pt) \\
H^{n-k}(N). \arrow[u, "i_!"] \arrow[ur, "\int_N"'] &
\end{tikzcd}$$

Therefore, if one considers a cohomology class $\omega \in H^{n-k}(M)$, then with all we have seen above we can deduce that

$$\int_N \omega|_N = \int_M \eta_N \cdot \omega,$$

where η_N denotes the Poincaré-dual cohomology class of N.

3.2.2 *Equivariant Umkehrung homorphism and equivariant Euler class*

We now focus our interest on defining the *equivariant Umkehrung homomorphism*. In particular, let $f : N \to M$ be an equivariant map between G-manifolds, where G is a compact Lie group acting smoothly on N and M. We want to define a homomorphism

$$f_!^G : H_G^*(N) \longrightarrow H_G^{*+k}(M),$$

where $k = \dim(M) - \dim(N)$, having similar properties as the classical Umkehrung homomorphism. To do this, note first that any equivariant map $f : N \to M$ can be factored

$$\begin{tikzcd} N \arrow[r, "i"'] \arrow[rr, bend left, "f"] & N \times M \arrow[r, "p"'] & M \end{tikzcd}$$

where $i : N \to N \times M$ is the graph embedding and $p : N \times M \to M$ is the projection on M. Furthermore, i is an equivariant inclusion of a submanifold, and p is an equivariant fiber bundle. Using the above properties of the Umkehrung homomorphism, we only need to translate in the equivariant theory the *Thom isomorphism* and the *integration along the fibers*.

Let E^n and B^{n-k} be G-manifolds and $p : E \to B$ be a G-fibering. One can easily see that applying the Borel construction to p gives us again a fibering $p_G : E_G \to B_G$. If the fibers of p are oriented compact manifolds, so are those of p_G. Therefore, the *equivariant integration along the fibers* is well defined, the later being simply the classical integration along the fibers of $p_G : E_G \to B_G$. Thus this gives us an equivariant Umkehrung homomorphism

$$p_!^G : H_G^*(E) \longrightarrow H_G^{*-k}(B),$$

whenever $p : E \to B$ is an equivariant fibering.

Let $i : N^{n-k} \to M^n$ be an equivariant inclusion of G-manifolds. The *equivariant normal bundle* of N, denoted ν_N^G, is the bundle obtained by applying the Borel construction to ν_N, i.e. we have

$$\begin{tikzcd} EG \times_G E(\nu_N) \arrow[r] \arrow[d, "\pi_G"'] & EG \times_G TM \arrow[d] \\ N_G \arrow[r] & M_G. \end{tikzcd}$$

As above, we have that the fibers of ν_N^G are of the same dimension as the fibers of ν_N, i.e. they are k-dimensional, and the Thom isomorphism theorem gives us the existence of a Thom class $\mu_G \in H^k(E(\nu_N^G), E(\nu_N^G) - N_G)$ such that the composition

$$H^*(N_G) \xrightarrow{\pi_G^*} H^*(E(\nu_N^G)) \xrightarrow{\cdot \mu_G} H^{*+k}(E(\nu_N^G), E(\nu_N^G) - N_G)$$

is an isomorphism. Thus, we have an *equivariant Thom isomorphism*

$$\phi_G : H^*_G(N) \longrightarrow H^{*+k}_G(M, M - N).$$

Therefore, the following composition gives us an equivariant Umkehrung homomorphism

$$H^*_G(N) \xrightarrow{\phi_G} H^{*+k}_G(M, M - N) \xrightarrow{\text{res}} H^{*+k}_G(M),$$

whenever $i : N \to M$ is an equivariant inclusion of G-manifolds.

The reader can check that this equivariant Umkehrung homomorphism is functorial, and that it is an $H^*(BG)$-module homomorphism. Furthermore, the above construction of the equivariant Umkehrung homomorphism suggests us the following definition of the equivariant Euler class.

DEFINITION. Let $E \xrightarrow{\xi} B$ be an oriented k-dimensional equivariant vector bundle. The *equivariant Euler class* of ξ is defined as

$$e_G(\xi) := e(\xi_G) \in H^*_G(E),$$

where ξ_G is the k-dimensional vector bundle obtained by applying the Borel construction to ξ.

With this definition, we have that if $i : N \hookrightarrow M$ is an equivariant inclusion of G-manifolds, then

$$i^*_G \circ i^G_!(1) = e_G(\nu_N).$$

3.2.3 *Integration formula*

From now on, let us assume $G = S^1$. Let M^n be a S^1-manifold. In virtue of 3.2.2 we can now use the equivariant Umkehrung homomorphism associated to $M \to pt$ to talk about *equivariant integration*. Since $H^*_{S^1}(pt) = H^*(BS^1)$, the integration map

$$\int_M : H^*_{S^1}(M) \longrightarrow H^*(BS^1)$$

takes values in $H^*(BS^1)$ instead of $H^*(pt)$. Recall from before that $\int_M$ is an $H^*(BS^1)$-module homomorphism. Therefore, if z denotes the generator of $H^*(BS^1)$ such that $H^*(BS^1) \cong \mathbb{Z}[z]$ and $\omega \in H^*_{S^1}(M)$, then

$$\int_M (\omega \cdot z^k) = \left(\int_M \omega\right) \cdot z^k.$$

Furthermore, the equivariant integration of a cohomology class coming from $H^n(M)$ corresponds to the integration in the non equivariant setting. This fact will be very useful later.

Another interesting fact is given by integrating any cohomology class $\omega \in H^n_{S^1}(M)$ of degree n, then the equivariant integration of ω takes value in $H^0(BS^1) \cong \mathbb{Z}$. With an appropriate choice of ω, we can obtain very useful information on the equivariant cohomology of M.

In order to simplify the notations, let H^* denote the cohomology ring with coefficients in $\mathbb{Q}$. Let us now recall an important theorem about equivariant cohomology, namely the so called Localization Theorem. We consider here only a weaker statement, but the reader interested in the more general one can find it in [1, Theorem 3.1.6].

LOCALIZATION THEOREM. *Let M be a closed smooth oriented manifold equipped with a smooth S^1-action. Then the localized homomorphism*

$$S^{-1}H^*_{S^1}(M) \xrightarrow{S^{-1}i^*} S^{-1}H^*_{S^1}(M^{S^1})$$

is an isomorphism of $S^{-1}H^*(BS^1)$-algebras, where $S = H^*(BS^1) - \{0\}$. Furthermore, the localized equivariant cohomology of M^{S^1} can be identified via

$$S^{-1}H^*_{S^1}(M^{S^1}) \cong \bigoplus_{F \subset M^{S^1}} S^{-1}H^*(BS^1) \otimes_{\mathbb{Q}} H^*(F).$$

We use this theorem to explain the *Atiyah-Bott-Berline-Vergne integration formula* which has been discovered in [7] and [4]. However, we only refer to the paper of Atiyah and Bott.

Let F be a fixed point component of M^{S^1} and $i_F : F \hookrightarrow M$ denote the inclusion map. Recall that the composition $i_F^* \circ (i_F)_!(1)$ in equivariant cohomology is equal to the equivariant Euler class $e_{S^1}(\nu_F)$. On the other hand, we have an isomorphism

$$H^*_{S^1}(F) \cong H^*(BS^1) \otimes_{\mathbb{Q}} H^*(F)$$

of $H^*(BS^1)$-algebras. Therefore, we can deduce that the homomorphism

$$i_F^* \circ (i_F)_! : H^*_{S^1}(F) \longrightarrow H^{*+k}_{S^1}(F)$$

is the multiplication by $e_{S^1}(\nu_F)$. Since this homomorphism induces, in the localization $S^{-1}H^*_{S^1}(F)$, an isomorphism of $S^{-1}H^*(BS^1)$-algebras, we deduce that the kernel and the cokernel of $i_F^* \circ (i_F)_!$ are $H^*(BS^1)$-torsion modules. Thus the element $e_{S^1}(\nu_F)$ is invertible in $S^{-1}H^*_{S^1}(F)$.

ATIYAH-BOTT INTEGRATION FORMULA. *Let M be a closed smooth oriented manifold equipped with a smooth S^1-action. Let $F \subset M^{S^1}$ be a connected component of the fixed point set, and let $i_F^* : H^*_{S^1}(M;\mathbb{Q}) \to H^*_{S^1}(F;\mathbb{Q})$ be the restriction to F in equivariant cohomology. Furthermore, let ν_F denote the normal bundle of F and $e_{S^1}(\nu_F)$ the corresponding equivariant Euler class. Then we have*

$$\int_M \omega = \sum_{F \subset M^{S^1}} \int_F \frac{i_F^*(\omega)}{e_{S^1}(\nu_F)}$$

for any equivariant cohomology class $\omega \in H^*_{S^1}(M;\mathbb{Q})$.

The proof of this statement (also explained in [4]) is rather simple. As a matter of interest let us sketch it briefly here. Define the homomorphism

$$\theta : S^{-1}H^*_{S^1}(M) \longrightarrow S^{-1}H^*_{S^1}(M^{S^1}) \cong \bigoplus_{F\subset M^{S^1}} S^{-1}H^*(BS^1)\otimes_{\mathbb{Q}} H^*(F)$$

$$\theta(\omega) := \sum_{F\subset M^{S^1}} \frac{i_F^*(\omega)}{e_{S^1}(\nu_F)},$$

for any localized equivariant class $\omega \in S^{-1}H^*_{S^1}(M)$. We can see that θ is the inverse of $i_!$. Indeed,

$$\theta \circ i_! = \left(\sum_{F\subset M^{S^1}} \frac{i_F^*}{e_{S^1}(\nu_F)}\right) \circ i_! = \sum_{F\subset M^{S^1}} \frac{i_F^* \circ (i_F)_!}{e_{S^1}(\nu_F)}$$

and since $i_F^* \circ (i_F)_!$ is the multiplication by $e_{S^1}(\nu_F)$, the above composition is the identity. Therefore, if ω is an equivariant cohomology class in $H^*_{S^1}(M)$, then

$$\omega = i_! \circ \theta(\omega) = i_! \left(\sum_{F\subset M^{S^1}} \frac{i_F^*(\omega)}{e_{S^1}(\nu_F)}\right) = \sum_{F\subset M^{S^1}} \frac{(i_F)_! \circ i_F^*(\omega)}{e_{S^1}(\nu_F)}.$$

If we equivariantly integrate both side, we then have

$$\int_M \omega = \int_M \sum_{F\subset M^{S^1}} \frac{(i_F)_! \circ i_F^*(\omega)}{e_{S^1}(\nu_F)} = \sum_{F\subset M^{S^1}} \int_F \frac{i_F^*(\omega)}{e_{S^1}(\nu_F)}.$$

The very last equality follows simply by functoriality of the Umkehrung homomorphism.

3.3 LEFSCHETZ FIXED POINT FORMULA FOR THE EQUIVARIANT SIGNATURE

The formula we introduce in this section will also be of great importance for our computations. This formula appears especially in [3], and [5]. However, we use the approach of [28, Section 5.8] since it is already specific for an S^1-action.

Let M be a manifold equipped with a smooth S^1-action. Let M^{S^1} denote the fixed point set of the action. Then, at any fixed point component $F \subset X^{S^1}$, we have an S^1-equivariant direct sum decomposition

$$TM|_F = TF \oplus \nu_F$$

where ν_F is the normal bundle of F in TM. Note that TF is a trivial S^1-bundle, while ν_F is an S^1-bundle over a trivial S^1-space.

From representation theory, we know that the real non-trivial irreducible representations of S^1 are two dimensional and given by

$$e^{i\theta} \longmapsto \begin{pmatrix} \cos(n\theta) & -\sin(n\theta) \\ \sin(n\theta) & \cos(n\theta) \end{pmatrix}$$

with $n \in \mathbb{Z}$. Note that the representations given by n and $-n$ are equivalent, thus we assume n to be positive. Furthermore, such representations are of complex type, which defines a natural complex structure where the S^1-action is given by complex multiplication with λ^n for $\lambda \in S^1$.

Therefore, the above argument induces for a fixed point $x \in F$ a complex structure at the fibers of ν_F at x together with a direct sum decomposition

$$(\nu_F)_x = \bigoplus_{j=1}^{\frac{\operatorname{codim} F}{2}} \nu(n_{F,j})_x$$

where $\nu(n_{F,j})_x$ is the isotypical summand associated to the weight $\lambda \mapsto \lambda^{n_{F,j}}$, and $n_{F,j} > 0$. Furthermore, the isomorphism type of the fibers at x is independent of the choice of $x \in F$. This yields an S^1-equivariant direct sum decomposition of the normal bundle

$$\nu_F = \bigoplus_j \nu(n_{F,j}).$$

Note that there are only finitely many $n_{F,j}$'s giving us a non-trivial weight space $\nu(n_{F,j})_x$. These weights are called the *normal weights* at the fixed point component F.

We fix the orientation on the normal bundle ν_F by means of the complex structure mentioned above. Thus, we can now choose the orientation on F to make it compatible with the one of ν_F and M.

In addition, we use the splitting principle [10, Section 21]. Let the $x_{F,i}$'s denote the formal roots of TF and let the $y_{F,j}$'s denote those of ν_F. Let $\lambda \in S^1$ be a topological generator and let $\operatorname{sign}(\lambda, M) \in \mathbb{Z}[\lambda, \lambda^{-1}]$ denote the equivariant signature which we see in the ring of finite Laurent series in λ with integer coefficients. We can now apply the equivariant Atiyah-Singer index theorem to the equivariant signature. This yields the formula

$$\operatorname{sign}(\lambda, M) = \sum_{F \subset M^{S^1}} \mu(\lambda, F),$$

where the *local datum* at F is given by

$$\mu(\lambda, F) := \int_F \prod_{i=1}^{\frac{\dim F}{2}} \left(x_{F,i} \cdot \frac{1 + e^{-x_{F,i}}}{1 - e^{-x_{F,i}}} \right) \prod_{j=1}^{\frac{\operatorname{codim} F}{2}} \left(\frac{1 + \lambda^{-n_{F,j}} e^{-y_{F,j}}}{1 - \lambda^{-n_{F,j}} e^{-y_{F,j}}} \right).$$

Because S^1 is connected, $\mathrm{sign}(\lambda, M) = \mathrm{sign}(M)$ by homotopy invariance. Furthermore, if we let $\lambda \to \infty$, then the local datum $\mu(\lambda, F)$ on the right hand side becomes simply $\mathrm{sign}(F)$, thus

$$\mathrm{sign}(M) = \sum_{F \subset M^{S^1}} \mathrm{sign}(F).$$

EXAMPLE. Let P be an isolated fixed point of a smooth S^1-action on a manifold M^{2n}. Let us compute the local datum at P. We use the same notation as above for the normal weights. Note that since P is a point there is no formal tangential roots and the normal roots are all trivial. Let us denote $\varepsilon_P = \pm 1$ the orientation of P according to the convention above, then

$$\mu(\lambda, P) = \varepsilon_P \prod_{j=1}^{n} \frac{1 + \lambda^{-n_{P,j}}}{1 - \lambda^{-n_{P,j}}}.$$

Furthermore the term $\frac{1+\lambda^{-n_{P,j}}}{1-\lambda^{-n_{P,j}}}$ is expressed as the formal power series

$$-(1 + 2\lambda^{n_{P,j}} + 2\lambda^{2\,n_{P,j}} + 2\lambda^{3\,n_{P,j}} + 2\lambda^{4\,n_{P,j}} + 2\lambda^{5\,n_{P,j}} + \cdots).$$

The formula above shows us in particular that a manifold M^{2n} cannot admit P as the only fixed point of the S^1-action. Indeed, one would have that

$$\mathrm{sign}(M) = (-1)^n \varepsilon_P \prod_{j=1}^{n} (1 + 2\lambda^{n_{P,j}} + 2\lambda^{2\,n_{P,j}} + 2\lambda^{3\,n_{P,j}} + \cdots).$$

Since the left hand side is an integer and does not depend on λ, and the right hand side is a non-trivial power series in λ, we conclude this cannot happen.

4 THE COHOMOLOGY STRUCTURE OF THE FIXED POINT SET

Our goal in this chapter is to study which homological properties of complete intersections transfer to their fixed point set for an S^1-action. Having in mind Smith's and Bredon's theorems (cf. [49] and [11] respectively), we conjectured that the fixed point components of a complete intersection with an S^1-action would again be complete intersections, or at least cohomologically. It turned out that such a result could not be proven. However, if we consider odd dimensional complete intersections, we are able to show that their fixed point components under an S^1-action have a very understandable cohomology ring in even degrees, cf. Proposition A. This result suggests to consider the work of Hsiang [29] and Masuda [42], both focussing on rational cohomology complex projective spaces. Thus, we show in this chapter how to adapt their work into our context.

4.1 BETTI NUMBERS AND DEGENERACY OF SPECTRAL SEQUENCES

In this section we discuss the degeneracy of the Leray-Serre cohomology spectral sequences associated to the Borel construction. Moreover, we use the same techniques as Bredon in [12, Chapter VII] to obtain data on the cohomology ring of the fixed point set.

LEMMA 4.1. *Let $X_n(\mathbf{d})$ be a complete intersection with $n > 1$ and equipped with a smooth S^1-action. Then the Leray-Serre cohomology spectral sequence associated to $X_n(\mathbf{d}) \to X_n(\mathbf{d})_{S^1} \to BS^1$ degenerates at the E_2-term with $\mathbb{Q}$ coefficients.*

PROOF. This fact follows from [8, Théorème II.1.2] or [43] since complete intersections are Kähler manifolds. However, we can give a more direct proof by generalizing [16, Lemma 3.1].

First, note that if n is even, then the spectral sequence degenerates trivially since $H^*(X_n(\mathbf{d}))$ is concentrated in even degrees. Thus let us assume n to be odd.

Let x be the generator of $H^2(X_n(\mathbf{d});\mathbb{Q})$ introduced in Section 2.1, and let z be a generator of $H^2(BS^1;\mathbb{Q})$ such that $H^*(BS^1;\mathbb{Q}) \cong \mathbb{Q}[z]$. Thus, x and z are generators of $E_2^{*,ev} \cong H^{ev}(X_n(\mathbf{d});\mathbb{Q}) \otimes H^*(BS^1;\mathbb{Q})$ and we have the relation $x^{n+1} = 0$. Since all differentials d_r vanish on x and z, they also vanish on $E_r^{*,ev}$ for multiplicative reasons.

We now show that the differentials $d_r : E_r^{0,n} \to E_r^{r,n-r+1}$ are trivial. Let us first consider $d_2 : E_2^{0,n} \to E_2^{2,n-1}$. By above, we know that the class $x^{\frac{n+1}{2}} \cdot z \in E_2^{2,n+1}$ survives until E_∞. Thus, $x^{\frac{n+1}{2}} \cdot z$ is non-zero viewed as a class in E_∞. Hence, the same is true for $x^{\frac{n-1}{2}} \cdot z$ for multiplicative reasons. Since $x^{\frac{n-1}{2}} \cdot z$ generates $E_2^{2,n-1}$, this implies that the differential $d_2 : E_2^{0,n} \to E_2^{2,n-1}$ is trivial. By induction we can show that all differentials $d_r : E_r^{0,n} \to E_r^{r,n-r+1}$ are trivial. By multiplicativity, the spectral sequence degenerates entirely at the E_2-term. □

NOTATION. For any topological space X, we denote by

$$b_{ev}(X) := \sum_{k \text{ even}} \operatorname{rk} H^k(X;\mathbb{Q})$$
$$b_{odd}(X) := \sum_{k \text{ odd}} \operatorname{rk} H^k(X;\mathbb{Q})$$

the sum of *even* and *odd Betti numbers* of X.

It follows from [12, Chapter VII, Theorem 2.1] that

$$b_{ev}(X_n(\mathbf{d})^{S^1}) = b_{ev}(X_n(\mathbf{d})), \text{ and } \quad b_{odd}(X_n(\mathbf{d})^{S^1}) = b_{odd}(X_n(\mathbf{d})).$$

Later on, we will improve the above equalities. In particular, we obtain the same equalities for the fixed point sets of the induced $\mathbb{Z}/p^r$-actions, where p^r is the power of a prime number. Note that these equations allow us already to classify odd dimensional complete intersection with a smooth S^1-action admitting only isolated fixed points.

COROLLARY 4.2. *Let $X_n(\mathbf{d})$ be a complete intersection with $n > 1$ odd. If $X_n(\mathbf{d})$ admits a smooth circle action with only isolated fixed points, then*

$\mathbf{d} = (1)$ *and* $X_n(\mathbf{d}) = \mathbb{C}P^n$, *or*

$\mathbf{d} = (2)$ *and* $X_n(\mathbf{d})$ *is a complex quadric.*

PROOF. By Lemma 4.1, we have

$$b_{ev}(X_n(\mathbf{d})^{S^1}) = b_{ev}(X_n(\mathbf{d})) = n+1, \text{ and}$$
$$b_{odd}(X_n(\mathbf{d})^{S^1}) = b_{odd}(X_n(\mathbf{d})) = 0.$$

Therefore we can deduce that $\chi(X_n(\mathbf{d})) = n+1$. This implies by Section 2.1 that $\mathbf{d} = (1)$ or (2). □

It is interesting to note that if n is even, we cannot prove this result as easily. For even $n \geqslant 4$, the classification of complete intersections with smooth S^1-action admitting isolated fixed points is completely open. However, the above results shift the discussion of smooth circle actions on odd dimensional complete intersections to the cases where the fixed point components are not simply isolated fixed points. This is our main focus for the rest of the chapter.

For the rest of this section, let X denote a complete intersection $X_n(\mathbf{d})$ with $n > 1$, and equipped with a smooth S^1-action.

PROPOSITION 4.3. *Let p^r be the power of a prime number. We consider on X the induced cyclic group action $\mathbb{Z}/p^r \subset S^1$. Then the following equalities hold:*

$$b_{ev}(X^{\mathbb{Z}/p^r}) = \operatorname{rk} H^{ev}(X^{\mathbb{Z}/p^r}; \mathbb{F}_p) = \operatorname{rk} H^{ev}(X; \mathbb{F}_p) = b_{ev}(X), \text{ and}$$
$$b_{odd}(X^{\mathbb{Z}/p^r}) = \operatorname{rk} H^{odd}(X^{\mathbb{Z}/p^r}; \mathbb{F}_p) = \operatorname{rk} H^{odd}(X; \mathbb{F}_p) = b_{odd}(X).$$

Consequently, $H^*(X^{S^1}; \mathbb{Z})$ is torsion free.

The proof of this proposition is the same in essence as [16, Proposition 3.2] or [42, Lemma 3.1].

PROOF. Let us fix a prime q large enough such that

- $X^{\mathbb{Z}/q} = X^{S^1}$,
- $X^{\mathbb{Z}/p^r}$ has no q-torsion,
- X^{S^1} has no q-torsion.

It follows by [12, Chapter VII, Theorem 2.2] that for any manifold Z with a smooth S^1-action, and any prime number p

$$\operatorname{rk} H^{ev}(Z^{\mathbb{Z}/p}; \mathbb{F}_p) \leqslant \operatorname{rk} H^{ev}(Z; \mathbb{F}_p), \text{ and}$$
$$\operatorname{rk} H^{odd}(Z^{\mathbb{Z}/p}; \mathbb{F}_p) \leqslant \operatorname{rk} H^{odd}(Z; \mathbb{F}_p).$$

By induction, one can show that

$$\operatorname{rk} H^{ev}(Z^{\mathbb{Z}/p^r}; \mathbb{F}_p) \leqslant \operatorname{rk} H^{ev}(Z; \mathbb{F}_p), \text{ and}$$
$$\operatorname{rk} H^{odd}(Z^{\mathbb{Z}/p^r}; \mathbb{F}_p) \leqslant \operatorname{rk} H^{odd}(Z; \mathbb{F}_p).$$

Therefore, we have the following inequalities:

$$\begin{aligned} b_{ev}(X^{S^1}) &= \operatorname{rk} H^{ev}(X^{S^1}; \mathbb{F}_q) \\ &= \operatorname{rk} H^{ev}((X^{\mathbb{Z}/p^r})^{\mathbb{Z}/q}; \mathbb{F}_q) \\ &\leqslant \operatorname{rk} H^{ev}(X^{\mathbb{Z}/p^r}; \mathbb{F}_q) = b_{ev}(X^{\mathbb{Z}/p^r}) \\ &\leqslant \operatorname{rk} H^{ev}(X^{\mathbb{Z}/p^r}; \mathbb{F}_p) \\ &\leqslant \operatorname{rk} H^{ev}(X; \mathbb{F}_p) = b_{ev}(X). \end{aligned}$$

The same holds in odd dimensions. Since $b_{ev}(X^{S^1}) = b_{ev}(X)$ and $b_{odd}(X^{S^1}) = b_{odd}(X)$, all the above inequalities are in fact equalities, and this concludes the proof of the proposition. □

COROLLARY 4.4. *Let n be an integer and $F \subset X^{\mathbb{Z}/n}$ be any fixed point component. Then F is orientable.*

PROOF. Let F be the fixed point component of the induced $\mathbb{Z}/n \subset S^1$ action. We decompose the $\mathbb{Z}/n$-action as follow. Write $n = 2^r k$ where

k is an odd number. We first show that the connected components of $X^{\mathbb{Z}/2^r}$ are orientable. Indeed, by Proposition 4.3 we know that

$$b_{ev}(X^{\mathbb{Z}/2^r}) = \operatorname{rk} H^{ev}(X^{\mathbb{Z}/2^r}; \mathbb{F}_2), \text{ and}$$
$$b_{odd}(X^{\mathbb{Z}/2^r}) = \operatorname{rk} H^{odd}(X^{\mathbb{Z}/2^r}; \mathbb{F}_2).$$

This implies in particular that $H_*(X^{\mathbb{Z}/2^r}; \mathbb{Z})$ does not have 2-torsion. By [25, Chapter 3, Corollary 3.28] we can deduce that any component of $X^{\mathbb{Z}/2^r}$ is orientable.

Furthermore, $X^{\mathbb{Z}/2^r}$ is equipped with a $\mathbb{Z}/k$-action such that

$$(X^{\mathbb{Z}/2^r})^{\mathbb{Z}/k} = X^{\mathbb{Z}/n}.$$

This implies that F is a fixed point component of a $\mathbb{Z}/k$-action on a certain orientable connected component $Y \subseteq X^{\mathbb{Z}/2^r}$. Since Y is an orientable manifold and since the $\mathbb{Z}/k$-action induces a complex structure on the normal bundle of F in Y, we have that F is an orientable manifold. □

REMARK. From the proofs of Proposition 4.3 and Corollary 4.4, we extract an interesting criterion concerning the orientability of a fixed point component of a cyclic group action coming from an S^1-action (cf. Appendix A).

COROLLARY 4.5. *For any prime number p, the Leray-Serre spectral sequence associated to $X \to X_{\mathbb{Z}/p} \to B\mathbb{Z}/p$ degenerates at the E_2-term with $\mathbb{F}_p$ coefficients.*

PROOF. This follows directly by [12, Chapter VII, Theorem 1.6]. □

COROLLARY 4.6. *The Leray-Serre spectral sequence associated to $X \to X_{S^1} \to BS^1$ degenerates with $\mathbb{F}_p$ coefficients for any prime number p. Consequently the above spectral sequence degenerates at the E_2-term over $\mathbb{Z}$.*

PROOF. Recall that the inclusion map $i : \mathbb{Z}/p \hookrightarrow S^1$ induces in cohomology a monomorphism $i^* : H^*(BS^1; \mathbb{F}_p) \to H^*(B\mathbb{Z}/p; \mathbb{F}_p)$. Comparing the spectral sequences with the above map, we can easily deduce that all differentials are trivial on

$$H^*(BS^1; H^*(X; \mathbb{F}_p)) \Longrightarrow H^*(X_{S^1}; \mathbb{F}_p),$$

since this applies to

$$H^*(B\mathbb{Z}/p; H^*(X; \mathbb{F}_p)) \Longrightarrow H^*(X_{\mathbb{Z}/p}; \mathbb{F}_p).$$ □

COROLLARY 4.7. *Let p^r be the power of a prime number, and $F \subset X^{\mathbb{Z}/p^r}$ be a connected component. Then*

$$b_{ev}(F^{S^1}) = b_{ev}(F), \quad \textit{and} \quad b_{odd}(F^{S^1}) = b_{odd}(F).$$

Therefore, the Leray-Serre spectral sequence associated to $F \to F_{S^1} \to BS^1$ degenerates at the E_2-term over $\mathbb{Q}$.

PROOF. By [12, Chapter VII, Theorem 2.1], we know that the above spectral sequence degenerates over $\mathbb{Q}$ if and only if the inequalities

$$b_{ev}(F^{S^1}) \leqslant b_{ev}(F), \quad \text{and} \quad b_{odd}(F^{S^1}) \leqslant b_{odd}(F)$$

are equalities, which is straight forward by Proposition 4.3. □

4.2 COHOMOLOGY OF THE FIXED POINT COMPONENTS

We prove Proposition A with the same techniques as Bredon used in [12, Chapter VII, Theorem 3.1]. The main idea is based on the fact that for an odd dimensional complete intersection, the even cohomology looks like the one from a complex projective space.

From now on, we assume that the complex dimension of our complete intersections is odd and strictly greater than 1. Furthermore, we denote by x the generator of $H^2(X_n(\mathbf{d}))$ introduced in Section 2.1.

PROPOSITION A. *Let $X_n(\mathbf{d})$ be a complete intersection with odd $n > 1$. If $X_n(\mathbf{d})$ admits a smooth circle action, then for any connected fixed point component $F_i \subset X_n(\mathbf{d})^{S^1}$ we have*

$$H^{ev}(F_i;\mathbb{Q}) \cong \mathbb{Q}[x|_{F_i}]/(x|_{F_i}^{m_i+1}),$$

where $x|_{F_i} \in H^2(F_i;\mathbb{Q})$ and $2m_i$ is the dimension of F_i. Moreover, we have $\sum_i (m_i + 1) = n + 1$.

DEFINITION. A manifold F^{2m} satisfying

$$H^{ev}(F;\mathbb{Q}) \cong \mathbb{Q}[x_F]/(x_F^{m+1})$$

is called *rational cohomology $\mathbb{C}P^m$ in even degrees*.

PROOF. Let $*$ be a fixed point, and F_i a connected component of the fixed point set $X_n(\mathbf{d})^{S^1}$ containing $*$. Furthermore, let j_i denote the composition of inclusions

$$F_i \longrightarrow X_n(\mathbf{d})^{S^1} \xrightarrow{\ j\ } X_n(\mathbf{d}).$$

For this proof, we use H^* to denote the singular cohomology with rational coefficients. Similarly, let $H^*_{S^1}$ denote the S^1-equivariant cohomology with rational coefficients.

By Lemma 4.1, there exists a unique equivariant cohomology class $\xi \in H^2_{S^1}(X_n(\mathbf{d}), *)$ which restricts to our generator $x \in H^2(X_n(\mathbf{d}))$ and generates $H^2_{S^1}(X_n(\mathbf{d}), *)$ over the field of rational numbers. Let $\xi_i \in H^2_{S^1}(F_i, *)$ denote

$$(j_i)^*_{S^1}(\xi) = \xi_i \in H^2_{S^1}(F_i, *).$$

Since the diagram

$$\begin{tikzcd} H^*_{S^1}(X_n(\mathbf{d}),*) \arrow[r, "(j_i)^*_{S^1}"] \arrow[d, "\mathrm{res}"'] & H^*_{S^1}(F_i,*) \arrow[d, "\mathrm{res}"] \\ H^*(X_n(\mathbf{d}),*) \arrow[r, "(j_i)^*"] & H^*(F_i,*) \end{tikzcd}$$

commutes, we can see that $\mathrm{res}(\xi_i) = x|_{F_i}$.

We claim that the even degree cohomology $H^{ev}(F_i,*)$ is a $\mathbb{Q}$-algebra generated by $x|_{F_i}$. Indeed, by [12, Chapter VII, Theorem 1.5] the map $(j_i)^*_{S^1}$ induced in equivariant cohomology is onto in high degrees, and since $H^{ev}_{S^1}(X_n(\mathbf{d}),*)$ is a free $H^*(BS^1)$-module generated by the $\xi, \xi^2, \ldots, \xi^n$, the image of $(j_i)^*_{S^1}$ in $H^{ev}_{S^1}(F_i,*)$ is generated over $H^*(BS^1)$ by

$$\xi_i, \xi_i^2, \ldots, \xi_i^{m_i},$$

where m_i is the biggest power such that $\xi_i^{m_i} \neq 0$. Since $H^*_{S^1}(F_i,*)$ is a free $H^*(BS^1)$-module, it follows that $H^{ev}(F_i,*)$ is a $\mathbb{Q}$-algebra generated by $x|_{F_i}$. Therefore, the ring structure of $H^{ev}(F_i)$ is given by

$$H^{ev}(F_i;\mathbb{Q}) \cong \mathbb{Q}[x|_{F_i}]/(x|_{F_i}^{m_i+1}),$$

and the fact that $\sum_i m_i + 1 = n + 1$ follows directly from Proposition 4.3. $\square$

Note that the odd dimensional hypothesis on our complete intersection ensures us that $H^{ev}_{S^1}(X_n(\mathbf{d}),*)$ is a free $H^*(BS^1)$-module generated by $\xi, \xi^2, \ldots, \xi^n$, without all the other generators in the middle dimension $H^n_{S^1}(X_n(\mathbf{d}),*)$ messing up.

In view of all the results in Section 4.1, we might be tempted to say that any component $F \subset X^{\mathbb{Z}/p^r}$ is a rational cohomology $\mathbb{C}P^m$ in even degrees. This is unfortunately not true in view of Proposition 2.5. In particular, we can construct on the quadric $X_3(2)$ an S^1-action whose fixed point set $X_3(2)^{S^1}$ consists of $\mathbb{C}P^1 \amalg \mathbb{C}P^1$, but the fixed point set $X_3(2)^{\mathbb{Z}/2}$ is the complex quadric $X_2(2)$. The later quadric is not a rational cohomology $\mathbb{C}P^2$ in even degrees.

Note that we could have used [1, Theorem 3.8.12] to prove the above result. Furthermore, this theorem explains in some sense the situation above with the $\mathbb{Z}/2$ fixed point set. Indeed, if p divides the total degree $d = d_1 \cdots d_r$, then the even cohomology ring $H^{ev}(X_n(\mathbf{d});\mathbb{F}_p)$ has two generators.

4.3 EQUIVARIANT COHOMOLOGY OF COMPLETE INTERSECTIONS

We try now to understand the structure of the equivariant cohomology ring. More precisely, we will adapt to odd dimensional complete intersections results from Bredon and Hsiang (cf. [12, Chapter VII] and [29] respectively).

As above, let $X = X_n(\mathbf{d})$ be an odd dimensional complete intersection with $n > 1$ equipped with a smooth S^1-action. Let γ be the restriction to X of the dual Hopf line bundle over $\mathbb{C}P^{n+r}$. We know from [26] or more recently from [45] that the S^1-action can be lifted to γ if and only if $x = c_1(\gamma) \in i^*H^2_{S^1}(X;\mathbb{Z})$, where $c_1(\gamma)$ denotes the first Chern class of γ and i^* is the map induced in cohomology by $i : X \to X_{S^1}$.

We know by Corollary 4.6 that the Leray-Serre spectral sequence associated to $X \to X_{S^1} \to BS^1$ degenerates at the E_2-term with $\mathbb{Z}$ coefficients. This yields a direct sum decomposition

$$H^2_{S^1}(X;\mathbb{Z}) \cong H^2(X;\mathbb{Z}) \oplus H^2(BS^1;\mathbb{Z})$$

and i^* is given by the projection on $H^2(X;\mathbb{Z})$. Thus, the S^1-action lifts to γ. If we restrict γ to a fixed point component $F_j \subset X^{S^1}$, then we have on the fibers a complex one dimensional S^1-representation whose isomorphism type is constant on F_j. We denote by $a_j \in \mathbb{Z}$ the weight of the S^1-action at F_j, i.e. the S^1-action on the fibers is given by complex multiplication with λ^{a_j} for any $\lambda \in S^1$. We call the a_j's the *Hopf weights* of the S^1-action on X.

Let z be a generator of $H^*(BS^1;\mathbb{Z})$, such that $H^2(BS^1;\mathbb{Z}) \cong \mathbb{Z}[z]$. One can remark that these Hopf weights are not unique. Indeed, by [45] we know that such lifts of the S^1-action are classified by $(i^*)^{-1}(c_1(\gamma)) \subset H^2_{S^1}(X;\mathbb{Z})$. Thus, if one chooses $\xi \in H^2_{S^1}(X;\mathbb{Z})$ such that $i^*(\xi) = x$, then for any $l \in \mathbb{Z}$, the cohomology class $\xi' = \xi + l \cdot z$ is also mapped on x. However, we know that the inclusion of a point $p_j \in F_j$ into X induces in equivariant cohomology a restriction homomorphism

$$H^*_{S^1}(X;\mathbb{Z}) \to H^*_{S^1}(p_j;\mathbb{Z}) \cong \mathbb{Z}[z]$$

which maps ξ to $a_j \cdot z$. Therefore, ξ' is mapped on $(a_j + l) \cdot z$. This means that the Hopf weights are unique up to a translation by $l \in \mathbb{Z}$.

The geometric interpretation of this unicity up to translation is explained using the tensor product of S^1-bundles. Indeed, if we choose a fixed lift of the S^1-action to γ, then we can tensor γ with a trivial complex line bundle, where the S^1-action on the fibers is the multiplication by λ^l for any $\lambda \in S^1$. This defines another lift of the S^1-action on γ, where the new Hopf weights differ from the previous one by $l \in \mathbb{Z}$.

The following proposition is an adaptation of a theorem of Hsiang concerning smooth S^1-actions on $\mathbb{C}P^n$ (cf. [29, Theorem IV.1]). The proof we give is almost the same. We simply adapted the discussion to the even cohomology ring.

PROPOSITION 4.8. *Let $X = X_n(\mathbf{d})$ be an odd dimensional complete intersection with a smooth S^1-action. Then*

$$H^{ev}_{S^1}(X;\mathbb{Q}) \cong H^*(BS^1;\mathbb{Q})[\xi]/(f(\xi))$$

as $H^*(BS^1;\mathbb{Q})$-algebras, where ξ is a lift of $c_1(\gamma) = x \in H^2(X)$, and the polynomial $f(\xi) = (\xi - a_1 z)^{m_1+1} \cdots (\xi - a_k z)^{m_k+1}$ corresponds to $X^{S^1} = F_1 \cup \cdots \cup F_k$. Furthermore, the a_j's are the Hopf weights of the S^1-action on X, and $2m_j$ is the dimension of F_j.

PROOF. Like in [29, Theorem IV.1], let us denote by $R_0 = S^{-1}H^*(BS^1;\mathbb{Q})$ the quotient field of $H^*(BS^1;\mathbb{Q})$. Moreover, let

$$A = R_0[\xi] \xrightarrow{\rho} S^{-1}H^{ev}_{S^1}(X;\mathbb{Q})$$

be an epimorphism which sends ξ on a fixed lift of $c_1(\gamma) = x$ in $H^2_{S^1}(X;\mathbb{Q})$. Let I be the kernel of ρ. By the localization theorem there is an isomorphism

$$S^{-1}H^{ev}_{S^1}(X;\mathbb{Q}) \cong \bigoplus_j R_0 \otimes H^{ev}(F_j;\mathbb{Q}).$$

Let ρ_j be the composition of ρ with the projection on $R_0 \otimes H^{ev}(F_j;\mathbb{Q})$, and let $I_j := \ker \rho_j$. Therefore, the I_j's satisfies $I = I_1 \cap \cdots \cap I_k$ and $I_j + I_1 \cap \cdots \cap I_{j-1} \cap I_{j+1} \cap \cdots \cap I_k = 1$. This implies that $I = I_1 \cdots I_k$.

Let $p_j \in F_j$ be a fixed point. The equivariant restriction of ξ to p_j is given by $\xi|_{p_j} = a_j z \in H^2_{S^1}(p_j;\mathbb{Q}) \cong H^2(BS^1;\mathbb{Q})$. Therefore, we can see that $\rho_j(\xi - a_j z) \in \widetilde{H}^*(F_j;\mathbb{Q}) \otimes R_0$. Since any element of $\widetilde{H}^*(F_j;\mathbb{Q})$ is nilpotent, for a sufficiently large integer N we have that $(\xi - a_j z)^N \in I_j$. More precisely, it is enough to take $N \geqslant m_j + 1$ where $2m_j$ is the dimension of F_j.

We deduce that the radical ideal $\sqrt{I_j} = M_j$ is the maximal ideal generated by $(\xi - a_j z)$, and the M_j's are mutually distinct. Since $A - M_j$ contains no zero divisors of A, we can see A as a subset of the localization A_{M_j}. Furthermore, one can prove that $I_j = I_{M_j} \cap A$.

Finally, let us assume that I is generated by the polynomial $f(\xi)$, i.e. $I = (f(\xi))$, then we have that $f(\xi)$ factorizes into linear factors in $R_0[\xi]$. By the Gauss lemma, $f(\xi)$ factorizes with coefficients in $H^*(BS^1;\mathbb{Q})$, thus

$$f(\xi) = (\xi - a_1 z)^{m_1+1} \cdots (\xi - a_k z)^{m_k+1}. \qquad \square$$

We prove now some interesting divisibility properties between the normal weights and the Hopf weights.

PROPOSITION 4.9. *Let $Y, Z \subset X_n(\mathbf{d})^{S^1}$ be two connected components. Let $m \in \mathbb{Z}$ such that $Y \cup Z \subset F$ where F is a connected component of $X_n(\mathbf{d})^{\mathbb{Z}/m}$. Then m divides $a_Y - a_Z$, where a_Y and a_Z are the Hopf weights of Y and Z respectively.*

Proposition 4.8 allows us to give two different proofs of the same statement. The following proof is adapted from the proof of [12, Proposition 5.6]

ALGEBRAIC PROOF. Let $p \in Y$ and $q \in Z$. By hypothesis $p, q \in F$. Let $\varphi : I \to F$ be an arc from p to q. Therefore, φ induces an equivariant map $\varphi : I \times S^1 \to F$, where S^1 acts on $I \times S^1$ with $\lambda \cdot (t, z) = (t, \lambda^m z)$. By collapsing the ends, we get an equivariant map

$$\varphi : S^2 \to F.$$

Let u, v denote the isolated fixed points of this action. By constructions we also have that $\varphi(u) = p$ and $\varphi(v) = q$.

Since the spectral sequence associated to $X_n(\mathbf{d}) \to X_n(\mathbf{d})_{S^1} \to BS^1$ degenerates over $\mathbb{Z}$, we know that $H^2_{S^1}(X_n(\mathbf{d}); \mathbb{Z}) \cong H^2(X_n(\mathbf{d})) \otimes H^2(BS^1)$. Thus, let $\xi \in H^2_{S^1}(X_n(\mathbf{d}); \mathbb{Z})$ be a fixed lift of our generator $x \in H^2(X_n(\mathbf{d}))$. Without loss of generality, we can choose the lift ξ such that $a_Y = 0$. This implies in particular that $\xi \in H^2_{S^1}(X, p)$. Furthermore, the restriction of ξ to q is by definition $a_Z \cdot z$, where a_Z is the weight of the S^1-representation at the fixed point component Z.

Let $x' \in H^2(S^2, u)$ be a generator and let $k \in \mathbb{Z}$ such that $\varphi^*(x) = kx'$. We show that $a_Z = \pm km$. Consider the diagram

$$\begin{tikzcd}
H^2_{S^1}(X_n(\mathbf{d}), p) \arrow[rr, "\varphi^*_{S^1}"] \arrow[d, "i^*_Z"'] & & H^2_{S^1}(S^2, u) \arrow[d, "i^*_v"] \\
H^2_{S^1}(Z) \arrow[r, "(\varphi|_v)^*_{S^1}"] & H^2_{S^1}(v) \cong H^2(BS^1) \otimes H^0(v). &
\end{tikzcd}$$

We can see that $\varphi^*_{S^1}(\xi) = k\xi'$, where ξ' is the generator of $H^2_{S^1}(S^2, u)$ representing $x' \in H^2(S^2, u)$.

By [12, Chapter VII, Theorem 5.5], we have that $i^*_v(\xi') = \pm m \cdot z$, hence $i^*_v \circ \varphi^*_{S^1}(\xi) = \pm km \cdot z$. On the other hand,

$$i^*_Z(\xi) = a_Z \cdot z + (\text{terms in } H^2(Z) \otimes H^0(BS^1))$$

which is sent by $(\varphi|_v)^*_{S^1}$ on $a_Z \cdot z$. This shows that $a_Z = \pm km$ and concludes our proof. □

GEOMETRICAL PROOF. Consider the equivariant map

$$\varphi : S^2 \to F$$

constructed above. We denote u and v the isolated fixed points of the S^1-action on S^2. Furthermore, we also have that $\varphi(u) = p$ and $\varphi(v) = q$.

Let γ denote the restriction to $X_n(\mathbf{d})$ of the dual Hopf line bundle over $\mathbb{C}P^{n+r}$. By construction, $\mathbb{Z}/m$ acts trivially on S^2. Without loss of generality suppose that the S^1-action on $\gamma|_p$ is trivial. Therefore, the $\mathbb{Z}/m$-action on $\gamma|_p$ is also trivial and by connectedness, the $\mathbb{Z}/m$-action is trivial on all $\gamma|_{S^2}$. In particular the $\mathbb{Z}/m$-action is trivial on $\gamma|_q$, hence m divides a_Z. □

COROLLARY 4.10. *Let p be a prime number. Then for any connected component $Y \subset X_n(\mathbf{d})^{S^1}$ admitting a normal weight $n_{Y,i}$ divisible by p^r, there exists another connected component $Z \subset X_n(\mathbf{d})^{S^1}$ also admitting a normal weight $n_{Z,j}$ divisible by p^r and*

$$a_Z - a_Y \equiv 0 \mod p^r.$$

PROOF. Let Y^{2m} be as above. Since $n_{Y,i}$ is divisible by the power of a prime number p^r, there exists a fixed point component $F \subset X_n(\mathbf{d})^{\mathbb{Z}/p^r}$ containing Y. Note that Y is an S^1-fixed point component of F.

Since $x|_Y^m \neq 0$ in $H^{2m}(Y;\mathbb{Q})$, the cohomology class $x|_F^m \neq 0$ in $H^{2m}(F;\mathbb{Q})$. Thus, we can deduce that $b_{ev}(F) > b_{ev}(Y)$. By Corollary 4.7, there must be another fixed point component $Z \subset X_n(\mathbf{d})^{S^1}$ contained in F. In particular, Z is also an S^1-fixed point component of F. Thus there exists a normal weight $n_{Z,j}$ divisible by p^r. By Proposition 4.9, we conclude that p^r divides the difference $a_Z - a_Y$. □

COROLLARY 4.11. *Let $P \in X_n(\mathbf{d})$ be an isolated fixed point of the S^1-action, and $n_{P,i}$ be a normal weight at P. Then there exists a component $Y \subset X_n(\mathbf{d})^{S^1}$ not containing P such that $a_Y - a_P \equiv 0 \mod n_{P,i}$.*

PROOF. Let P be as above, and let F be the connected component of $X_n(\mathbf{d})^{\mathbb{Z}/n_{P,i}}$ containing P. Therefore, P is an isolated fixed point of the S^1-action on F. Since a closed smooth oriented manifold cannot admit a single isolated S^1-fixed point (cf. Section 3.3), F must contain another S^1-fixed point component, say Y. The result follows by Proposition 4.9. □

4.4 DEFECT OF A FIXED POINT COMPONENT AND FORMULAE

In this section, we adapt to odd dimensional complete intersections results from Masuda [42]. The author studied smooth S^1-action on twisted complex projective spaces. His methods can easily be carried out for odd dimensional complete intersections, since our manifolds are integral cohomology twisted complex projective spaces in even degrees.

DEFINITION. Let $F \subset X_n(\mathbf{d})^{S^1}$ be a fixed point component, we define the *defect* of F as

$$D(F) := \left| \int_F x|_F^{m_F} \right|$$

where $2m_F$ is the dimension of F.

The following result is an adaptation of [42, Lemma 3.6].

PROPOSITION 4.12. *Let* $X_n(\mathbf{d})^{S^1} = \bigcup_i F_i$ *be the fixed point components,* $j : F_i \hookrightarrow X_n(\mathbf{d})$ *the inclusion,* $n_{i,j}$ *the normal weights at* F_i*, and* a_k *the Hopf weight at* F_k*. Then the following equation holds*

$$d \cdot \prod_j n_{i,j} = \pm D(F_i) \prod_{k \neq i} (a_i - a_k)^{m_k+1},$$

for every F_i*, and where* $d = d_1 \cdots d_r$ *denotes the total degree of* $X_n(\mathbf{d})$*.*

PROOF. Let $j_! : H^*_{S^1}(F_i) \to H^*_{S^1}(X_n(\mathbf{d}))$ be the equivariant Umkehrung homomorphism which increases the degree by the codimension of F_i in $X_n(\mathbf{d})$. Furthermore, we know that the restriction of $j_!(1)$ to F_k vanishes for $k \neq i$, and $j^* j_!(1)$ is equal to the equivariant Euler class $e_{S^1}(\nu_{F_i})$. Using Proposition 4.8 we deduce that

$$j_!(1) = C \cdot \prod_{k \neq i} (\xi - a_k z)^{m_k+1}.$$

Since $(\xi - a_k z)$ restricts to x for any k in $H^*(X_n(\mathbf{d}))$, we can deduce that up to sign $C = D(F_i) / \prod_k d_k$. Hence,

$$j_!(1) \cdot \prod_{k=1}^{r} d_k = D(F_i) \cdot \prod_{k \neq i} (\xi - a_k z)^{m_k+1}.$$

The restriction to a point of F_i gives us the desired equation since ξ is sent on $a_i z$. □

We make no use of the above formula except to check our computations in Part II. Indeed, the Atiyah-Bott integration formula will give us the exact same equations with the correct sign. Moreover, this formula is a generalization to complete intersection of the Petrie's formula for complex projective spaces [47, Theorem 2.8].

Part II

APPLICATIONS

Using equivariant cohomology, characteristic classes, the Atiyah-Bott integration formula and the Lefschetz fixed point formula for the equivariant signature, we study odd dimensional complete intersections with S^1-symmetry.

In Chapter 5, we deal with general odd dimensional complete intersections admitting a high dimensional fixed point component. In Chapter 6, we study 5-dimensional complete intersections with S^1-symmetry, and give a classification of 5-dimensional complete intersections with T^2-symmetry.

5 CIRCLE ACTIONS ON ODD DIMENSIONAL COMPLETE INTERSECTIONS

In this chapter, we consider odd dimensional complete intersections with S^1-symmetry. Using Proposition A (cf. Section 4.2), we can see that whenever we have a fixed point component of high dimension, the number of fixed point components is low. Therefore, we give a classification of odd dimensional complete intersections admitting an S^1-action with a codimension 2 or a codimension 4 fixed point component.

5.1 THE CODIMENSION 2 CASE

In this short section, we prove the following theorem.

THEOREM B. *Let $X_n(\mathbf{d})$ be a complete intersection with $n \geqslant 5$ odd. If $X_n(\mathbf{d})$ admits a smooth circle action with a fixed point component of codimension 2, then $\mathbf{d} = (1)$ and $X_n(\mathbf{d}) = \mathbb{C}P^n$.*

The proof consists mainly in using that the signature of a closed smooth oriented manifold M with a smooth circle action can be computed as the signature of the fixed point set, i.e.

$$\operatorname{sign}(M) = \sum_{Z \subset M^{S^1}} \operatorname{sign}(Z)$$

where $Z \subset M^{S^1}$ runs over all the connected components of the fixed point set (cf. Section 3.3 or [28, Section 5.8]).

Let $X_n(\mathbf{d})$ be a complete intersection with $n > 1$ odd, and with multi-degree $\mathbf{d} = (d_1, \dots, d_r)$. We assume that $X_n(\mathbf{d})$ admits a smooth S^1-action with codimension 2 fixed point set. Using Proposition A (cf. Section 4.2), we can deduce that $X_n(\mathbf{d})^{S^1}$ is the disjoint union of an isolated point P and a submanifold F^{2n-2}, such that

$$H^{ev}(F; \mathbb{Q}) \cong \mathbb{Q}[x|_F]/(x|_F^n).$$

Since $\operatorname{sign}(X_n(\mathbf{d})) = 0$ we have that $\operatorname{sign}(F) = -\operatorname{sign}(P) = \pm 1$ depending on the orientation of the normal bundle of F.

On the other hand, let $\eta x \in H^2(X_n(\mathbf{d}); \mathbb{Q})$ be the Poincaré-dual class of F in our complete intersection $X_n(\mathbf{d})$, with $\eta \in \mathbb{Z}$. Furthermore, we have that

$$0 \neq \int_F x|_F^{n-1} = \int_{X_n(\mathbf{d})} x^{n-1} \cdot (\eta x) = \eta \cdot d,$$

this implies that $\eta \neq 0$.

LEMMA 5.1.

$$\operatorname{sign}(F) = \begin{cases} \operatorname{sign}(X_{n-1}(d_1, \dots, d_r, \eta)), & \text{if } \eta > 0 \\ -\operatorname{sign}(X_{n-1}(d_1, \dots, d_r, -\eta)), & \text{if } \eta < 0. \end{cases}$$

PROOF. Let us first assume $\eta > 0$. Then by [28, Chapter 3], we can think of the class $\eta x \in H^2(X_n(\mathbf{d}); \mathbb{Q})$ as a virtual submanifold of $X_n(\mathbf{d})$, where $x \in H^2(\mathbb{C}P^{n+r}; \mathbb{Q})$ is the first Chern class of the dual Hopf line bundle over $\mathbb{C}P^{n+r}$. In particular, $(d_1 x, \dots, d_r x, \eta x)$ is a virtual submanifold of codimension $2(r+1)$ of $\mathbb{C}P^{n+r}$ represented by F. Thus, we have that the total Pontrjagin class of F is given by

$$p(F) = p(X_n(\mathbf{d})) \cdot (1 + (\eta x|_F)^2)^{-1} = p(X_{n-1}(d_1, \dots, d_r, \eta)).$$

We know from the Hirzebruch Signature Theorem, that the signature can be computed with the L-polynomial in the Pontrjagin classes. Thus, the signature of F is given by the signature of the complete intersection $X_{n-1}(d_1, \dots, d_r, \eta)$, since their Pontrjagin classes coincide.

Note that if $\eta < 0$, we just have to reverse the orientation on the normal bundle of F. The Poincaré-dual class of F is then $-\eta x$ and the sign of the signature changes. □

To conclude the proof of Theorem B, we need the following theorem due to Libgober [35].

THEOREM 5.2 (LIBGOBER, 1980). *Let $X_{2k}(d_1, \dots, d_r)$ be an even dimensional complete intersection. Then the signature is monotone as a function on the d_i's; increasing when k is even and decreasing when k is odd. Moreover, we have that $|\operatorname{sign}(X_{2k}(d_1, \dots, d_r))| > 2$ except for:*

- *the complex projective space, for which the signature is 1,*
- *the quadric $X_{2k}(2)$ with k odd, for which the signature is zero.*
- *the quadric $X_{2k}(2)$ with k even, for which the signature is 2.*

PROOF OF THEOREM B. We have seen previously that the signature of F is equal to ± 1 depending on the orientation. Since $\operatorname{sign}(F) = \pm \operatorname{sign}(X_{n-1}(d_1, \dots, d_r, |\eta|))$ we deduce from the above theorem that all the d_i's are equal to 1. Thus $X_n(\mathbf{d})$ is the complex projective space. □

5.2 THE CODIMENSION 4 CASE

In this section, we prove the following theorem.

THEOREM C. *Let $X_n(\mathbf{d})$ be a complete intersection with $n \geqslant 5$ odd. If $X_n(\mathbf{d})$ admits a smooth and effective circle action with a fixed point component of codimension 4, then*

$\mathbf{d} = (1)$ *and* $X_n(\mathbf{d}) = \mathbb{C}P^n$, *or*

$\mathbf{d} = (2)$ *and* $X_n(\mathbf{d})$ *is a quadric.*

According to Proposition A (cf. Section 4.2) there are exactly two different fixed point configurations. Indeed, let $X_n(\mathbf{d})$ be a complete intersection with $n > 1$ odd, then we have either that

$$X_n(\mathbf{d})^{S^1} = Y^{2n-4} \cup Z^2 \quad \text{or} \quad X_n(\mathbf{d})^{S^1} = Y^{2n-4} \cup P_1 \cup P_2,$$

where Y is a rational cohomology $\mathbb{C}P^{n-2}$ in even degrees, Z is a rational cohomology $\mathbb{C}P^1$ in even degrees, and P_1 and P_2 are isolated fixed points.

5.2.1 *The case* $X_n(\mathbf{d})^{S^1} = Y^{2n-4} \cup Z^2$

We consider first the case where the fixed point set consists of exactly two connected components, namely a codimension 4 manifold and a surface. To prove Theorem C in this case, we use the Atiyah-Bott integration formula in equivariant cohomology and compute

$$\int_{X_n(\mathbf{d})} x^n \quad \text{and} \quad \int_{X_n(\mathbf{d})} p_1(X_n(\mathbf{d})) \cdot x^{n-2}.$$

We recall briefly the integration formula. The reader can find more information about this formula in Chapter 3 or in the original paper of Atiyah and Bott [4].

ATIYAH-BOTT INTEGRATION FORMULA. *Let* M *be a closed smooth oriented manifold equipped with a smooth* S^1*-action. Let* $F \subset M^{S^1}$ *be a connected component of the fixed point set, and let* $i_F^* : H^*_{S^1}(M;\mathbb{Q}) \to H^*_{S^1}(F;\mathbb{Q})$ *be the restriction to* F *in equivariant cohomology. Furthermore, let* ν_F *denote the normal bundle of* F *and* $e_{S^1}(\nu_F)$ *the corresponding equivariant Euler class. Then we have*

$$\int_M \omega = \sum_{F \subset M^{S^1}} \int_F \frac{i_F^*(\omega)}{e_{S^1}(\nu_F)}$$

for any equivariant cohomology class $\omega \in H^*_{S^1}(M;\mathbb{Q})$.

Let us define first the *Hopf weights* of the S^1-action, like in Section 4.3. Recall that if γ denotes the restriction to $X_n(\mathbf{d})$ of the dual Hopf line bundle over $\mathbb{C}P^{n+r}$, then the first Chern class $x := c_1(\gamma)$ is a generator of $H^2(X_n(\mathbf{d});\mathbb{Z})$. We know from [26] or [45] that the S^1-action can be lifted to γ if and only if $c_1(\gamma) \in i^* H^2_{S^1}(X_n(\mathbf{d}))$, where i^* is the map induced by $X_n(\mathbf{d}) \to X_n(\mathbf{d})_{S^1}$. By Corollary 4.6,

$$H^2_{S^1}(X_n(\mathbf{d});\mathbb{Z}) \cong H^2(X_n(\mathbf{d});\mathbb{Z}) \oplus H^2(BS^1;\mathbb{Z})$$

and i^* is the projection on $H^2(X_n(\mathbf{d}))$. Thus, the S^1-action can be lifted to γ. Furthermore, the restriction of γ to the fixed point set gives us on the fibers complex one dimensional S^1-representations, whose isomorphism types are constant on each fixed point components. Let $a_Y, a_Z \in \mathbb{Z}$ be the weights of these S^1-representations. We call these integers the *Hopf weights* of the S^1-action. From an algebraic point of view, the lift of γ gives us an equivariant cohomology class $\xi \in H^2_{S^1}(X_n(\mathbf{d}))$, whose restriction to any point in $Y \subset X_n(d)^{S^1}$ (respectively $Z \subset X_n(d)^{S^1}$) is the class $a_Y \cdot z$ (resp. $a_Z \cdot z$) where z is the generator of $H^*(BS^1; \mathbb{Z})$. Thus the equivariant first Chern classes of Y and Z are respectively $x|_Y + a_Y \cdot z$ and $x|_Z + a_Z \cdot z$.

Note that, the lift of the S^1-action to γ is not unique. For any $l \in \mathbb{Z}$ we can choose a lift such that the equivariant first Chern classes of γ at Y and Z respectively are of the form $x|_Y + (a_Y + l) \cdot z$ and $x|_Z + (a_Z + l) \cdot z$ (cf. Section 4.3 for more details). Without loss of generality we assume that $a_Y = 0$, i.e. S^1 acts trivially on the fibers of γ over Y.

We define the *normal weights* at Y and Z like in Section 3.3. Let ν_Y and ν_Z be the normal bundles of Y and Z respectively. Thus, ν_Y and ν_Z are S^1-bundles over trivial S^1-spaces.

From representation theory, we know that the real non-trivial irreducible representations of S^1 are two dimensional and given by

$$e^{i\theta} \longmapsto \begin{pmatrix} \cos(n\theta) & -\sin(n\theta) \\ \sin(n\theta) & \cos(n\theta) \end{pmatrix}$$

with $n \in \mathbb{Z}$. Note that the representations given by n and $-n$ are equivalent, thus we can always assume n to be positive. Furthermore, such representations are of complex type, which defines a natural complex structure where the S^1-action is given by complex multiplication with λ^n for $\lambda \in S^1$.

Therefore, the later argument induces for any fixed point x in $F \subset X_n(\mathbf{d})^{S^1}$ a complex structure at the fibers of ν_F at x together with a direct sum decomposition

$$(\nu_F)_x = \bigoplus_{n_{F,j}} \nu(n_{F,j})_x$$

where $\nu(n_{F,j})_x$ is the isotypical summand associated to the weight $\lambda \mapsto \lambda^{n_{F,j}}$ and $n_{F,j} > 0$. Furthermore, the isomorphism type of the fibers at x is independent of the choice of $x \in F$. This yields an S^1-equivariant direct sum decomposition of the normal bundle

$$\nu_F = \bigoplus_{n_{F,j}} \nu(n_{F,j}).$$

Note that there are only finitely many $n_{F,j}$'s giving us a non-trivial weight space $\nu(n_{F,j})_x$. We call these $n_{F,j}$'s the *normal weights* at the fixed point component F.

We fix the orientation of the normal bundle ν_F by means of the complex structure mentioned above. Thus, we can now choose the orientation on F to make it compatible with the one of ν_F and $X_n(\mathbf{d})$. In addition, we use the splitting principle [10, Section 21]. Let $y_{F,j} + n_{F,i} \cdot z$ denote the formal roots of ν_F. Thus, the equivariant normal roots at Y are given by $y_{Y,1} + n_{Y,1} \cdot z$ and $y_{Y,2} + n_{Y,2} \cdot z$, while those at Z are given by $y_{Z,1} + n_{Z,1} \cdot z, \dots, y_{Z,n-1} + n_{Z,n-1} \cdot z$.

LEMMA 5.3. *Let $X_n(\mathbf{d})$ be a complete intersection with $n \geqslant 5$ odd. Suppose that $X_n(\mathbf{d})$ is equipped with a smooth effective S^1-action, such that the fixed point set is $Y^{2n-4} \cup Z^2$. If the action is semi-free around the fixed points components, then $\mathbf{d} = (1)$ and $X_n(\mathbf{d}) = \mathbb{C}P^n$*

PROOF. By hypothesis, we assume that the action is semi-free around the fixed point components, i.e. the normal weights $n_{Y,i}$'s and $n_{Z,i}$'s are all equal to 1. By obstruction theory (cf. [44]) we can also assume that $y_{Z,2} = \cdots = y_{Z,n-1} = 0$ since Z is a surface. Therefore, the equivariant Euler classes of ν_Y and ν_Z are given by

$$e_{S^1}(\nu_Y) = (y_{Y,1} + z)(y_{Y,2} + z) \quad \text{and} \quad e_{S^1}(\nu_Z) = (y_{Z,1} + z)z^{n-2}.$$

Using the Atiyah-Bott integration formula, we have

$$\int_{X_n(\mathbf{d})} x^n = \int_{Y^{2n-4}} i_Y^*(x^n) \cdot e_{S^1}(\nu_Y)^{-1} + \int_{Z^2} i_Z^*(x^n) \cdot e_{S^1}(\nu_Z)^{-1}. \quad (2)$$

By Section 2.1 we know that the left hand side of the above equation is equal to the product $d = d_1 \cdots d_r$. Furthermore, we can see that the right hand side is a polynomial in the variable l. Indeed, let us make the local datum at Y more explicit.

$$\begin{aligned}
&\int_{Y^{2n-4}} i_Y^*(x^n) \cdot e_{S^1}(\nu_Y)^{-1} \\
&\qquad = \int_Y (x|_Y + l \cdot z)^n (y_{Y,1} + z)^{-1} (y_{Y,2} + z)^{-1} \\
&\qquad = \binom{n}{2} \cdot \int_Y x|_Y^{n-2} \cdot l^2 + (\text{terms of higher order in } l)
\end{aligned}$$

Next, let us compute the local datum at Z.

$$\begin{aligned}
&\int_{Z^2} i_Z^*(x^n) \cdot e_{S^1}(\nu_Z)^{-1} \\
&= \int_Z (x|_Z + (a_Z + l) \cdot z)^n (y_{Z,1} + z)^{-1} z^{-(n-2)} \\
&= n \cdot (a_Z + l)^{n-1} \int_Z x|_Z - (a_Z + l)^n \int_Z y_{Z,1} \\
&= \sum_{k=0}^{n-1} \left[\binom{n-1}{k} a_Z^{n-1-k} \cdot n \int_Z x|_Z - \binom{n}{k} a_Z^{n-k} \int_Z y_{Z,1} \right] l^k - \int_Z y_{Z,1} \cdot l^n
\end{aligned}$$

As above, we use the Atiyah-Bott integration formula to compute

$$\begin{aligned}\int_{X_n(\mathbf{d})} p_1(X_n(\mathbf{d}))\cdot x^{n-2} = &\int_{Y^{2n-4}} p_{1,S^1}(Y)\cdot i_Y^*(x^{n-2})\cdot e_{S^1}(\nu_Y)^{-1}\\ &+\int_{Z^2} p_{1,S^1}(Z)\cdot i_Z^*(x^{n-2})\cdot e_{S^1}(\nu_Z)^{-1}.\end{aligned} \tag{3}$$

The left hand side is equal to $p_1 \cdot d$, where $p_1 = (n+r+1-\sum_{i=1}^r d_i^2)$ according to Section 2.1. The right hand side is also a polynomial in l, but we are interested in its constant term only. Furthermore, $p_{1,S^1}(Y)$ (respectively $p_{1,S^1}(Z)$) denotes the equivariant first Pontrjagin classes of $X_n(\mathbf{d})$ restricted to Y (resp. Z). Let us first compute the local datum at Y.

$$\begin{aligned}&\int_{Y^{2n-4}} p_{1,S^1}(Y)\cdot i_Y^*(x^{n-2})\cdot e_{S^1}(\nu_Y)^{-1}\\ &\qquad=\int_Y \big(p_1(Y)+(y_{Y,1}+z)^2+(y_{Y,2}+z)^2\big)\\ &\qquad\qquad (x|_Y+l\cdot z)^{n-2}(y_{Y,1}+z)^{-1}(y_{Y,2}+z)^{-1}\end{aligned}$$

As above, one can find that

$$\begin{aligned}&\int_{Y^{2n-4}} p_{1,S^1}(Y)\cdot i_Y^*(x^{n-2})\cdot e_{S^1}(\nu_Y)^{-1}\\ &\qquad= 2\cdot\int_Y x^{n-2}|_Y + (\text{terms of higher order in } l)\end{aligned}$$

The local datum at Z is computed in a similar fashion.

$$\begin{aligned}&\int_{Z^2} p_{1,S^1}(Z)\cdot i_Z^*(x^{n-2})\cdot e_{S^1}(\nu_Z)^{-1}\\ &=\int_Z \big(p_1(Z)+(y_{Z,1}+z)^2+(n-2)z^2\big)\\ &\qquad\cdot (x|_Z+(a_Z+l)z)^{n-2}(y_{Z,1}+z)^{-1}z^{2-n}\\ &=(n-2)(n-1)\cdot a_Z^{n-3}\int_Z x|_Z-(n-3)\cdot a_Z^{n-2}\int_Z y_{Z,1}\\ &\qquad+(\text{terms of higher order in } l)\end{aligned}$$

If we wrap up our computations of the local data in (2) and (3), we have two polynomials in l which are constant equal to d and $p_1\cdot d$ respectively. Therefore, by comparing the coefficients we can extract the following system of equations

$$\begin{aligned}d &= n\cdot a_Z^{n-1}\int_Z x|_Z - a_Z^n\int_Z y_{Z,1},\\ 0 &= \binom{n-1}{1} n\cdot a_Z^{n-2}\int_Z x|_Z - \binom{n}{1} a_Z^{n-1}\int_Z y_{Z,1},\\ 0 &= \binom{n-1}{2} n\cdot a_Z^{n-3}\int_Z x|_Z - \binom{n}{2} a_Z^{n-2}\int_Z y_{Z,1} + \binom{n}{2}\int_Y x|_Y^{n-2},\end{aligned}$$

$$p_1 \cdot d = (n-1)(n-2) \cdot a_Z^{n-3} \int_Z x|_Z \\ - (n-3) \cdot a_Z^{n-2} \int_Z y_{Z,1} + 2 \int_Y x|_Y^{n-2}.$$

This system is linear in the variables p_1, $\int x|_Z$, $\int y_{Z,1}$, $\int x|_Y^{n-2}$ and has a unique solution, namely

$$p_1 = \frac{n+1}{a_Z^2}, \quad \int_Z x|_Z = \frac{d}{a_Z^{n-1}}, \quad \int_Z y_{Z,1} = \frac{d \cdot (n-1)}{a_Z^n}, \quad \int_Y x|_Y^{n-2} = \frac{d}{a_Z^2}.$$

By assumption we took $n \geqslant 5$, thus by Section 2.1 and Poincaré duality the class $x|_Y^{n-2}$ is the product of d with a cohomology class in $H^{2n-4}(Y;\mathbb{Z})$. Therefore, we have that $\int_Y x|_Y^{n-2} \in d\mathbb{Z}$. One can then directly obtain $a_Z = \pm 1$. Since $p_1 = n + r + 1 - \sum_{i=1}^r d_i^2$ we deduce that all the d_i's are equal to 1, thus $X_n(\mathbf{d}) = \mathbb{C}P^n$. □

Let us now treat the general case.

PROOF OF THEOREM C. PART 1. Let us assume by Lemma 5.3 that the action is not semi-free around the fixed point set. Then there exists a prime number p and a connected component $\widetilde{F} \subset X_n(\mathbf{d})^{\mathbb{Z}/p}$ which contains strictly a component of $F \subset X_n(\mathbf{d})^{S^1}$. Since $x|_F^{m_F} \in H^{ev}(F;\mathbb{Q})$ is non-zero, $x|_{\widetilde{F}}^{m_F} \in H^{ev}(\widetilde{F};\mathbb{Q})$ is also non-zero. Therefore, by counting the even Betti numbers, we have $b_{ev}(\widetilde{F}) > b_{ev}(F)$. In virtue of Corollary 4.7, we can conclude that $\widetilde{F}$ contains all of $X_n(\mathbf{d})^{S^1}$ and is of codimension 2 in $X_n(\mathbf{d})$. Using Poincaré duality, we can deduce that the even Betti numbers of $\widetilde{F}$ are given by $b_{2k}(\widetilde{F}) = 1$ except for $b_{n-1}(\widetilde{F}) = 2$.

Therefore, we can think of the Poincaré-dual class $\eta x \in H^2(X_n(\mathbf{d}))$ of $\widetilde{F}$ as a virtual complete intersection in $X_n(\mathbf{d})$ (cf. [28, Chapter 3]). Thus $\widetilde{F}$ represents a virtual complete intersection in $\mathbb{C}P^{n+r}$ with multidegree $(d_1, \dots, d_r, |\eta|)$. In addition, we have that the signature of $\widetilde{F}$ is 0 or ± 2. Thus by Theorem 5.2, we get that $\mathbf{d} = (1)$ or (2). □

5.2.2 *The case* $X_n(\mathbf{d})^{S^1} = Y^{2n-4} \cup P_1 \cup P_2$

Let us consider the case where the fixed point set consists of a codimension 4 manifold and two isolated points.

Let us use the same notations as in Section 5.2.1. As above, we lift the S^1-action to γ such that the equivariant first Chern class of the restriction to Y is equal to $x|_Y + l \cdot z$. Recall that γ is the restriction to $X_n(\mathbf{d})$ of the dual Hopf line bundle over $\mathbb{C}P^{n+r}$, and that such lift exists for any $l \in \mathbb{Z}$. Moreover, $a_1 + l$ and $a_2 + l$ denote the (Hopf) weights of the induced S^1-representations on P_1 and P_2 respectively.

We denote by ν_1 (respectively ν_2) the normal bundle of P_1 (resp. P_2). The S^1-action on ν_1 and ν_2 give us the normal weights $n_{1,1}, \dots, n_{1,n}$ and $n_{2,1}, \dots, n_{2,n}$ which we assume positive. This induces a complex structure on ν_1 and ν_2 together with an orientation. Furthermore, we choose the orientation of P_1 and P_2 in order to make it compatible with the orientation of their respective normal bundle and $X_n(\mathbf{d})$. Since the signature of $X_n(\mathbf{d})$ is the sum of the signature of the fixed points, then one of these isolated fixed point has to be positively oriented, say P_1. Hence P_2 is negatively oriented.

Before we dive into the computations, we need the following lemma to understand the normal weights at P_1 and P_2.

LEMMA 5.4. *Let N^{2k} be a closed smooth oriented manifold equipped with a smooth S^1-action such that the fixed point set consists of two isolated points P_1 and P_2. Then the normal weights at P_1 and P_2, are equal up to ordering.*

PROOF. As above, let us denote the normal weights at P_1 (respectively P_2) by $(n_{1,1}, \dots, n_{1,k})$ (resp. $(n_{2,1}, \dots, n_{2,k})$) and assume them all positive. Without loss of generality, one can order them in the following way,

$$1 \leqslant n_{1,1} = n_{1,2} = \cdots = n_{1,r} < n_{1,r+1} \leqslant \cdots \leqslant n_{1,k}$$
$$1 \leqslant n_{2,1} = n_{2,2} = \cdots = n_{2,s} < n_{2,s+1} \leqslant \cdots \leqslant n_{2,k}.$$

Let $\lambda \in S^1$ be a topological generator. By the equivariant signature formula we know that

$$\operatorname{sign}(N) = \mu(\lambda, P_0) + \mu(\lambda, P_1).$$

The local datum at P_i is given by the Laurent polynomial in $\mathbb{Z}[\lambda, \lambda^{-1}]$

$$\mu(\lambda, P_i) = \int_{P_i} \prod_{j=1}^{k} \frac{1 + \lambda^{-n_{i,j}}}{1 - \lambda^{-n_{i,j}}}.$$

Note that each term of the above product is given by the formal power serie in λ given by

$$\frac{1 + \lambda^{-n_{i,j}}}{1 - \lambda^{-n_{i,j}}} = -(1 + 2\lambda^{n_{i,j}} + 2\lambda^{2n_{i,j}} + 2\lambda^{3n_{i,j}} + 2\lambda^{4n_{i,j}} + \cdots).$$

Since the signature of N is a constant polynomial in λ, the isolated fixed points P_1 and P_2 need to be of different orientations in order for the power series to cancel themselves. By looking at the constant term, we see that the signature of N is equal to 0. Thus, we have the following equality,

$$\prod_{j=1}^{k} \frac{1 + \lambda^{-n_{1,j}}}{1 - \lambda^{-n_{1,j}}} = \prod_{j=1}^{k} \frac{1 + \lambda^{-n_{2,j}}}{1 - \lambda^{-n_{2,j}}}.$$

If we now look on both sides at the coefficients with the smallest power of λ, we get

$$1 + 2r\lambda^{n_{1,1}} + O(\lambda^{>n_{1,1}}) = 1 + 2s\lambda^{n_{2,1}} + O(\lambda^{>n_{2,1}}).$$

From this, we deduce that $r = s$ and $n_{1,1} = n_{2,1}$. By induction we get that the normal weights $(n_{1,1}, \ldots, n_{1,k})$ and $(n_{2,1}, \ldots, n_{2,k})$ are equal up to ordering. $\square$

We are now ready to dive into the computations. In particular, we prove the following lemma.

LEMMA 5.5. *Let $X_n(\mathbf{d})$ be a complete intersection with $n \geqslant 5$ odd. Suppose that $X_n(\mathbf{d})$ is equipped with a smooth and effective S^1-action, such that the fixed point set is $Y^{2n-4} \cup P_1 \cup P_2$. If the action is semi-free around Y^{2n-4}, then $\mathbf{d} = (1)$ and $X_n(\mathbf{d}) = \mathbb{C}P^n$, or $\mathbf{d} = (2)$ and $X_n(\mathbf{d})$ is a complex quadric.*

PROOF. By the Atiyah-Bott integration formula, we have

$$\begin{aligned}\int_{X_n(\mathbf{d})} x^n = &\int_{Y^{2n-4}} i_Y^*(x)^n \cdot e_{S^1}(\nu_Y)^{-1} \\ &+ \int_{P_1} i_1^*(x)^n \cdot e_{S^1}(\nu_1)^{-1} + \int_{P_2} i_2^*(x)^n \cdot e_{S^1}(\nu_2)^{-1}.\end{aligned} \tag{4}$$

The local datum at Y is computed exactly the same way as in the previous part. Therefore we have

$$\begin{aligned}\int_{Y^{2n-4}} i_Y^*(x^n) \cdot e_{S^1}(\nu_Y)^{-1} = &\binom{n}{2} \cdot \int_Y x|_Y^{n-2} \cdot l^2 \\ &+ (\text{terms of higher order in } l).\end{aligned}$$

We now compute the local datum at P_1 or P_2. According to the previous lemma, we denote by $(n_1, \ldots, n_n)$ their normal weights. Thus the local datum is given by

$$\int_{P_k} i_k^*(x)^n \cdot e_{S^1}(\nu_k)^{-1} = \int_{P_k} \frac{(a_k + l)^n}{n_1 \cdots n_n} = \varepsilon_k \frac{(a_k + l)^n}{n_1 \cdots n_n},$$

and where ε_k is the orientation of P_k, i.e positive for P_1 and negative for P_2.

Again by the Atiyah-Bott integration formula, we now compute

$$\begin{aligned}\int_{X_n(\mathbf{d})} p_1(X_n(\mathbf{d})) \cdot x^{n-2} = &\int_{Y^{2n-4}} p_{1,S^1}(Y) \cdot i_Y^*(x)^{n-2} \cdot e_{S^1}(\nu_Y)^{-1} \\ &+ \int_{P_1} p_{1,S^1}(P_1) \cdot i_1^*(x)^{n-2} \cdot e_{S^1}(\nu_1)^{-1} \\ &+ \int_{P_2} p_{1,S^1}(P_2) \cdot i_2^*(x)^{n-2} \cdot e_{S^1}(\nu_2)^{-1}.\end{aligned} \tag{5}$$

The local datum at Y is given as above by

$$\begin{aligned}\int_{Y^{2n-4}} p_{1,S^1}(Y) \cdot i_Y^*(x^{n-2}) \cdot e_{S^1}(\nu_Y)^{-1} = 2 \cdot &\int_Y x^{n-2}|_Y \\ &+ (\text{terms of higher order in } l),\end{aligned}$$

while the local datum at P_1 or P_2 is given by

$$\int_{P_k} p_{1,S^1}(P_k) \cdot i_k^*(x)^{n-2} \cdot e_{S^1}(\nu_k)^{-1} = \varepsilon_k (n_1^2 + \cdots n_n^2) \cdot \frac{(a_k + l)^{n-2}}{n_1 \cdots n_n}.$$

As in the first part, if we now put our computations of the local data in (4) and (5), we have two polynomials in l which are constant equal to d and $p_1 \cdot d$ respectively. Therefore, by comparing the coefficients we get the following system of equations

$$\begin{aligned} d &= \frac{a_1^n}{n_1 \cdots n_n} - \frac{a_2^n}{n_1 \cdots n_n}, \\ 0 &= \frac{a_1^{n-1}}{n_1 \cdots n_n} - \frac{a_2^{n-1}}{n_1 \cdots n_n}, \\ 0 &= \int_Y x|_Y^{n-2} + \frac{a_1^{n-2}}{n_1 \cdots n_n} - \frac{a_2^{n-2}}{n_1 \cdots n_n}, \\ p_1 \cdot d &= 2 \cdot \int_Y x|_Y^{n-2} + \left(\sum_{i=1}^n n_i^2\right) \cdot \left(\frac{a_1^{n-2}}{n_1 \cdots n_n} - \frac{a_2^{n-2}}{n_1 \cdots n_n}\right), \end{aligned}$$

where $d = d_1 \cdots d_r$ and $p_1 = n + r + 1 - \sum_{i=1}^r d_i^2$.

We can see that the first and second equalities imply that $a_1 = -a_2 > 0$. Moreover, this system has a unique solution, namely

$$d = \frac{2 \cdot a_1^n}{n_1 \cdots n_n}, \quad p_1 = \frac{\sum_{i=1}^n n_i^2 - 2}{a_1^2}, \quad \int_Y x|_Y^{n-2} = \frac{-2 \cdot a_1^{n-2}}{n_1 \cdots n_n}.$$

By assumption we took $n \geqslant 5$, thus by Section 2.1 and Poincaré duality we have $\int_Y x|_Y^{n-2} \in d\mathbb{Z}$. Therefore, we directly get that $a_1 = 1$. This leaves us with, $d = 2/n_1 \cdots n_n$, and we have two possibilities. Either all the d_i's are equal to 1 and $X_n(\mathbf{d}) = \mathbb{C}P^n$, or $\mathbf{d} = (2)$ and $X_n(\mathbf{d}) = X_n(2)$ is a quadric. □

PROOF OF THEOREM C. PART 2. In virtue of Lemma 5.5, suppose the action is not semi-free around Y. Then there exists a prime number p and a fixed point component $F^{2n-2} \subset X_n(\mathbf{d})^{\mathbb{Z}/p}$ containing Y. Therefore, by Proposition 4.3 the component F must contain at least one of the isolated fixed points. Since $\operatorname{sign}(F) = \operatorname{sign}(F^{S^1})$, the signature of F is either ± 1 in the case F contains only one of the isolated fixed point, or the signature of F is 0 or ± 2 if F contains both isolated fixed points.

Since F is of codimension 2, we can picture the Poincaré-dual class of F, namely $\eta x \in H^2(X_n(\mathbf{d}); \mathbb{Z})$, as a virtual complete interesection in $X_n(\mathbf{d})$ (cf. [28, Chapter 3]). Therefore, F represents a virtual complete intersection in $\mathbb{C}P^{n+r}$ with multidegree $(d_1, \ldots, d_r, |\eta|)$. Thus by Theorem 5.2, we have that $\mathbf{d} = (1)$ or (2). □

6 COMPLETE INTERSECTIONS OF COMPLEX DIMENSION 5

In this chapter, we consider 5-dimensional complete intersections with S^1-symmetry. The idea is to extend the classification of complete intersections with S^1-symmetry in low degrees given in Section 2.2.

THEOREM D. *Let $X_5(\mathbf{d})$ be a complete intersection equipped with a smooth and effective circle action which admits a fixed point component of dimension strictly bigger than 2, then*

$\mathbf{d} = (1)$ *and* $X_5(\mathbf{d}) = \mathbb{C}P^5$*, or*

$\mathbf{d} = (2)$ *and* $X_5(\mathbf{d})$ *is a complex quadric.*

Before we prove this theorem, let us give an interesting corollary.

THEOREM E. *Let $X_5(\mathbf{d})$ be a complete intersection equipped with a smooth and effective rank 2 torus action, then*

$\mathbf{d} = (1)$ *and* $X_5(\mathbf{d}) = \mathbb{C}P^5$*, or*

$\mathbf{d} = (2)$ *and* $X_5(\mathbf{d})$ *is a complex quadric.*

PROOF. First, remark that if there exists a fixed point component $Y \subset X_5(\mathbf{d})^{T^2}$ of dimension $4, 6$ or 8, then there exists $S^1 \subset T^2$ acting smoothly on $X_5(\mathbf{d})$ admitting Y as fixed point component. This fact follows easily from the classification of real T^2-representations [13, Proposition 8.5]. Thus, by Theorem D the multidegree $\mathbf{d}$ needs to be equal to (1) or (2). The same holds in the case where $X_5(\mathbf{d})^{T^2}$ happens to be a union of isolated fixed points. Indeed, the result follows by Corollary 4.2.

Therefore, let us assume there exists a fixed point component $Z^2 \subset X_5(\mathbf{d})^{T^2}$ of dimension 2. Let ν_Z denote its normal bundle. Since Z^2 is fixed by the torus action, we know that T^2 acts on ν_Z which we equip with a complex structure using the action. This allows us to decompose ν_Z into a sum of irreducible T^2-modules. Since any irreducible complex representation of T^2 is of dimension 1, the kernel of this action on any irreducible T^2-module contains a circle S^1. One can use this S^1 to equip $X_5(\mathbf{d})$ with a circle action admitting a fixed point component of dimension greater or equal to 4 containing Z. By Theorem D, it follows that $\mathbf{d}$ is equal to (1) or (2). □

Let us now prove Theorem D. The proof is nothing more sophisticated than a case by case study among all possible S^1-fixed point configurations. In virtue of Proposition A (cf. Section 4.2), any complete intersection of complex dimension 5 equipped with a non-trivial smooth S^1-actions admits one of these fixed point configuration:

1. $X_5(\mathbf{d})^{S^1} = Y^8 \cup P^0$,
2. $X_5(\mathbf{d})^{S^1} = Y^6 \cup Z^2$,
3. $X_5(\mathbf{d})^{S^1} = Y^6 \cup P^0_1 \cup P^0_2$,
4. $X_5(\mathbf{d})^{S^1} = Y^4_1 \cup Y^4_2$,
5. $X_5(\mathbf{d})^{S^1} = Y^4 \cup Z^2 \cup P^0$,
6. $X_5(\mathbf{d})^{S^1} = Y^4 \cup P^0_1 \cup P^0_2 \cup P^0_3$,
7. $X_5(\mathbf{d})^{S^1} = Z^2_1 \cup Z^2_2 \cup Z^2_3$,
8. $X_5(\mathbf{d})^{S^1} = Z^2_1 \cup Z^2_2 \cup P^0_1 \cup P^0_2$,
9. $X_5(\mathbf{d})^{S^1} = Z^2 \cup P^0_1 \cup P^0_2 \cup P^0_3 \cup P^0_4$,
10. $X_5(\mathbf{d})^{S^1} = P^0_1 \cup P^0_2 \cup P^0_3 \cup P^0_4 \cup P^0_5 \cup P^0_6$.

One can see that Chapter 5 deals in a greater generality with the cases 1, 2 and 3. Therefore, to prove Theorem D, one needs to deal with the cases 4, 5 and 6. Thus, the proof of Theorem D is three parts, the later corresponding to the cases 4, 5 and 6. Note also that Corollary 4.2 deals with case 10 for any odd dimensional complete intersection.

6.1 THE CASE WHERE $X_5(\mathbf{d})^{S^1} = Y^4_1 \cup Y^4_2$

Let γ denote the restriction to $X_5(\mathbf{d})$ of the dual Hopf line bundle over $\mathbb{C}P^{5+r}$, then the first Chern class $x := c_1(\gamma)$ is the generator of $H^2(X_5(\mathbf{d}); \mathbb{Z})$. Like in Chapter 5, we can lift the S^1-action to γ (cf. [26] or [45]). The restriction of γ to the fixed point set gives us on the fibers a complex one-dimensional S^1-representations whose isomorphism types are constant on each fixed point components. Let $a_1, a_2 \in \mathbb{Z}$ be the weights of these S^1-representations. As above, we call these integers the *Hopf weights* of the S^1-action. Thus, the equivariant first Chern classes of γ at Y_1 and Y_2 are respectively $x|_{Y_1} + a_1 \cdot z$ and $x|_{Y_2} + a_2 \cdot z$, where z is the generator of $H^*(BS^1; \mathbb{Z})$.

Furthermore, the lift of the S^1-action to γ is not unique. For any $l \in \mathbb{Z}$ we can choose a lift such that the equivariant first Chern classes of γ at Y_1 and Y_2 are respectively of the form $x|_{Y_1} + (a_1 + l) \cdot z$ and $x|_{Y_2} + (a_2 + l) \cdot z$. Without loss of generality we assume that the Hopf weight $a_1 = 0$, i.e. S^1 acts trivially on the fibers of γ over Y_1.

We denote by ν_1 and ν_2 the normal bundle of Y_1 and Y_2 respectively. Like in Section 3.3, we orient these bundles by mean of the S^1-action. The orientation on ν_1 and ν_2 is chosen such that their normal weights $n_{1,1}, n_{1,2}, n_{1,3}$ and $n_{2,1}, n_{2,2}, n_{2,3}$ are all positive. Furthermore, we choose the orientations of Y_1 and Y_2 in order to make them compatible with the orientations of their respective normal bundles and the orientation of $X_n(\mathbf{d})$. Since the signature of $X_5(\mathbf{d})$ is the sum of the signature of the fixed points, then the signature of one of the Y_i has to be 1, say Y_1. Hence the signature of Y_2 is -1.

We denote by x_1 a fixed generator of $H^2(Y_1;\mathbb{Z})$ such that $\int_{Y_1} x_1^2 = 1$. Similarly, x_2 denotes a generator of $H^2(Y_2;\mathbb{Z})$ and satisfies $\int_{Y_2} x_2^2 = -1$. Using this notation, we can write $x|_{Y_1} = r_1 x_1$ and $x|_{Y_2} = r_2 x_2$, where $r_1, r_2 \in \mathbb{Z}$.

Furthermore, we apply the splitting principle to ν_1 and ν_2. We denote by $\{y_{i,j} + n_{i,j} \cdot z\}$ the three equivariant normal roots at Y_i.

LEMMA 6.1. *Let $X_5(\mathbf{d})$ be a complete intersection with a smooth circle action such that the fixed point components are $X_5(\mathbf{d})^{S^1} = Y_1^4 \cup Y_2^4$. If the action is semi-free around the fixed point components, then $\mathbf{d} = (1)$ and $X_5(\mathbf{d})$ is the complex projective space.*

Since the computations will be more and more involved in this chapter, we invite the reader to be aware of the *Mathematica*® computations in the appendix, cf. Chapter B. An explanation of the *Mathematica*® code is given using the computations of Lemma 6.1 as an example.

PROOF. By hypothesis, we have that all the $n_{i,j}$ are equal to 1. Thus one can compute the Euler classes of ν_1 and ν_2 using elementary symmetric functions in the $y_{i,j}$'s,

$$\begin{aligned} e_{S^1}(\nu_i) &= (y_{i,1} + z)(y_{i,2} + z)(y_{i,3} + z) \\ &= c_2(\nu_i)z + c_1(\nu_i)z^2 + z^3. \end{aligned}$$

If we now write $c_2(\nu_i) = c_{i,2}x_i^2$ and $c_1(\nu_i) = c_{i,1}x_i$ with $c_{i,j} \in \mathbb{Z}$, then one can compute $e_{S^1}(\nu_i)^{-1}$ as a power series in x_i.

$$e_{S^1}(\nu_i)^{-1} = \frac{1}{z^3} - \frac{c_{i,1}x_i}{z^4} + \frac{(c_{i,1}^2 - c_{i,2})x_i^2}{z^5}$$

We now compute locally

$$\int_{X_5(\mathbf{d})} x^5, \qquad \text{and} \qquad \int_{X_5(\mathbf{d})} p_1(X_5(\mathbf{d})) \cdot x^3.$$

The local datum of x^5 at Y_i is given by

$$\int_{Y_i^4} (r_i x_i + (a_i + l) \cdot z)^5 \left(\frac{1}{z^3} - \frac{c_{i,1}x_i}{z^4} + \frac{(c_{i,1}^2 - c_{i,2})x_i^2}{z^5} \right).$$

If we now develop $\int_{X_5(\mathbf{d})} x^5$ as a polynomial in l which is constant equal to d, this gives us the system of equations

$$\begin{aligned}
d &= -a_2^3(a_2^2c_{2,1}^2 - a_2^2c_{2,2} - 5a_2c_{2,1}r_2 + 10r_2^2)\\
0 &= -5a_2^2(a_2^2c_{2,1}^2 - a_2^2c_{2,2} - 4a_2c_{2,1}r_2 + 6r_2^2)\\
0 &= -10a_2(a_2^2c_{2,1}^2 - a_2^2c_{2,2} - 3a_2c_{2,1}r_2 + 3r_2^2)\\
0 &= -10(a_2^2c_{2,1}^2 - a_2^2c_{2,2} - r_1^2 + 2a_2c_{2,1}r_2 + r_2^2)\\
0 &= -5(a_2c_{2,1}^2 - a_2c_{2,2} + c_{1,1}r_1 - c_{2,1}r_2)\\
0 &= c_{1,1}^2 - c_{1,2} - c_{2,1}^2 + c_{2,2}.
\end{aligned}$$

If one now solve this system of equations in the variables d, r_2, $c_{1,1}$, $c_{1,2}$, $c_{2,1}$ and $c_{2,2}$ we get two solutions, namely

$$\begin{aligned}
d &= -a_2^3r_1^2, & c_{1,1} &= -\frac{3r_1}{a_2}, & c_{1,2} &= \frac{3r_1^2}{a_2^2},\\
r_2 &= \pm r_1, & c_{2,1} &= \pm\frac{3r_1}{a_2}, & c_{2,2} &= \frac{3r_1^2}{a_2^2}.
\end{aligned}$$

To solve the above system of equations, note the similarities inside the parenthesis of the equations on line 2, 3 and 4. This can be used to find that $r_2^2 = r_1^2$, and $c_{2,1} = 3r_2/a_2$. The rest can then be computed easily.

We now compute the local datum of $p_1(X_5(\mathbf{d})) \cdot x^3$ at Y_i, i.e.

$$\int_{Y_i^4} p_{1,S^1}(Y_i)\,(r_ix_i + (a_i + l)\cdot z)^3\, e_{S^1}(\nu_i)^{-1}.$$

Note that we already computed above $e_{S^1}(\nu_i)^{-1}$, and the equivariant Pontrjagin class are given by

$$p_{1,S^1}(Y_i) = (p_1(Y_i) + (y_{i,1}+z)^2 + (y_{i,2}+z)^2 + (y_{i,3}+z)^2).$$

If we now identify the elementary symmetric functions in the $y_{i,j}$'s using the Chern classes of ν_i, we then get

$$p_{1,S^1}(Y_i) = (p_1(Y_i) + c_{i,1}^2x_i^2 - 2c_{i,2}x_i^2 + 2c_{i,1}x_iz + 3z^2).$$

We can now develop $\int_{X_5(\mathbf{d})} p_1(X_5(\mathbf{d}))\cdot x^3$ into a polynomial in l, which is constant equal to $p_1 \cdot d$ and where $p_1 = 6 + r - \sum_{i=1}^r d_i^2$. This gives us the following system of equations

$$\begin{aligned}
p_1\cdot d &= -a_2(2a_2^2c_{2,1}^2 - 5a_2^2c_{2,2} - a_2^2\int p_1(Y_2) - 3a_2c_{2,1}r_2 + 9r_2^2),\\
0 &= 3(-2a_2^2c_{2,1}^2 + 5a_2^2c_{2,2} + a_2^2\int p_1(Y_2) + 3r_1^2 + 2a_2c_{2,1}r_2 - 3r_2^2),\\
0 &= -3(2a_2c_{2,1}^2 - 5a_2c_{2,2} - a_2\int p_1(Y_2) + c_{1,1}r_1 - c_{2,1}r_2),\\
0 &= 2c_{1,1}^2 - 5c_{1,2} - 2c_{2,1}^2 + 5c_{2,2} + \int p_1(Y_1) + \int p_1(Y_2).
\end{aligned}$$

Furthermore, we know by the Hirzebruch signature formula, that $\int p_1(Y_1) = 3$ and $\int p_1(Y_2) = -3$. Using the above solutions with this equations, we can solve the system in the variable p_1, r_1, r_2 and we have exactly four non-trivial solutions. In particular, we have that

$$p_1 = \frac{6}{a_2^2}, \quad \text{and} \quad d = -a_2^5$$

which forces p_1 to be equal to 6 and $d_1 = \cdots = d_r = 1$. □

For now on, we assume the action is not semi-free around both fixed point components. We decompose again the problem into sub cases. In particular, we argue on the $X_n(\mathbf{d})^{\mathbb{Z}/p}$ fixed point components and their dimensions.

LEMMA 6.2. *Let $X_5(\mathbf{d})$ be a complete intersection with a smooth circle action such that the fixed point components are $X_5(\mathbf{d})^{S^1} = Y_1^4 \cup Y_2^4$. If the action admits a fixed point component $N^8 \subset X_5(\mathbf{d})^{\mathbb{Z}/p}$ for a prime number p, then $\mathbf{d} = (1)$ or (2).*

PROOF. First, one can see that N contains both Y_1 and Y_2. This can be deduced easily, using Corollary 4.7.

Since $\operatorname{sign}(N) = \operatorname{sign}(N^{S^1})$, the signature of N is either 0 or ± 2. Since N is of codimension 2, we can picture the Poincaré-dual class of N, namely $\eta x \in H^2(X_5(\mathbf{d}); \mathbb{Z})$, as a virtual complete interesection in $X_5(\mathbf{d})$ (cf. [28, Chapter 3]). Therefore, N represents a virtual complete intersection in $\mathbb{C}P^{5+r}$ with multidegree $(d_1, \ldots, d_r, |\eta|)$. Thus by Theorem 5.2, we have that $\mathbf{d} = (1)$ or (2). □

We now focus our interest in the case where there exists a fixed point component $N^6 \subset X_5(\mathbf{d})^{\mathbb{Z}/p}$ containing Y_1 and Y_2. Remark first that if there exists a component $N^6 \subset X_5(\mathbf{d})^{\mathbb{Z}/p}$ containing one of the Y_i, then by Corollary 4.7 both fixed point components need to be contained in N. Furthermore, one can see that the normal weights at Y_1 and Y_2 are equal. Indeed, if we denote by $n_{1,j}$'s (respectively $n_{2,j}$'s) the normal weights at Y_1 (resp. Y_2), then we know there exists exactly one of them divisible by p, say $n_{1,j}$ (resp. $n_{2,j}$). Furthermore, if we consider the component $N_{1,j}^6 \subset X_5(\mathbf{d})^{\mathbb{Z}/n_{1,j}}$ containing Y_1, then it follows from

$$Y_1^4 \subset N_{1,j}^6 \subseteq N^6$$

that $N_{1,j} = N$. Thus Y_2 is an S^1-fixed point component at $N_{1,j}$, which implies that $n_{1,j}$ divides $n_{2,j}$. By changing the roles of Y_1 and Y_2 we have that $n_{2,j}$ divides $n_{1,j}$, hence they have to be equal, we write $n_j := n_{1,j} = n_{2,j}$.

Let us now discuss what information one can get using the Atiyah-Bott integration formula on $N_j^6 \subset X_5(\mathbf{d})^{\mathbb{Z}/n_j}$ as above.

LEMMA 6.3. *Let* $N_j^6 \subset X_5(\mathbf{d})^{\mathbb{Z}/n_j}$ *be a connected fixed point component for a prime* p *such that the fixed point set* $N_j^{S^1}$ *consists of two 4-dimensional manifolds* Y_1^4 *and* Y_2^4 *as above. If we write the first Chern class* $c_1(\nu_{i,N_j}) = c_{i,n_j} x_i$*, where* ν_{i,N_j} *is the normal bundle of* Y_i *in* N_j*, then*

$$c_{1,n_j} = \pm c_{2,n_j} \quad \textit{and} \quad r_1 = \pm(r_2 - \frac{a_2 c_{2,n_j}}{n_j}).$$

PROOF. Let us fix the orientation on ν_{i,n_j} via the S^1-action. Thus, we can then fix the orientation on N_j such that it is compatible with the orientation of Y_1 and ν_{1,N_j}. Since the signature of N_j is zero and the signature of Y_1 is equal to $+1$, we have that Y_2 (with the orientation induced by N_j and ν_{2,N_j}) has its signature equal to -1.

To obtain the above equations, it is enough to integrate locally $\int_{N_j^6} x|_{N_j}^3$. This integral is given by the sum of the local datum at the Y_i's. Using the same notation as above, the local datum at Y_i is given by

$$\int_{Y_i} (r_i x_i + (a_i + l) \cdot z)^3 (c_1(\nu_{i,N_j}) + n_j z)^{-1}.$$

If one replaces $c_1(\nu_{i,N_j})$ by $c_{i,n_j} x_i$, we can then develop the above formula and get a polynomial in l. Identifying the coefficient in $l^{>0}$ to be zero gives us the following system of equations

$$\begin{aligned}
\int_{N_j^6} x|_{N_j}^3 &= -\frac{a_2(a_2^2 c_{2,n_j}^2 - 3a_2 c_{2,n_j} n_j r_2 + 3n_j^2 r_2^2)}{n_j^3}, \\
0 &= \frac{3r_1^2}{n_j} - \frac{3(a_2 c_{2,n_j} - n_j r_2)^2}{n_j^3} \\
0 &= -\frac{3c_{1,n_j} r_1}{n_j^2} - \frac{3c_{2,n_j}(a_2 c_{2,n_j} - n_j r_2)}{n_j^3} \\
0 &= \frac{c_{1,n_j}^2}{n_j^3} - \frac{c_{2,n_j}^2}{n_j^3}.
\end{aligned}$$

One can extract the desired equalities out of the last three equations above. □

We now use the Lemma 6.3 to prove the following result.

LEMMA 6.4. *Let* $X_5(\mathbf{d})$ *be a complete intersection with a smooth circle action such that the fixed point components are* $X_5(\mathbf{d})^{S^1} = Y_1^4 \cup Y_2^4$*. If the action admits a fixed point component* $N^6 \subset X_5(\mathbf{d})^{\mathbb{Z}/p}$ *for a prime number* p*, then* $\mathbf{d}$ *is either* (1) *or* (2)*.*

The reader should be aware that we added in the appendix the *Mathematica*® computations of the following proof, cf. Chapter B. These computations can be done by hand, but the detailed *Mathematica*® code can spare some paper, time and computational errors for the reader.

PROOF. In virtue of Lemma 6.1 and Lemma 6.2, we can suppose that the action is not semi-free around the fixed point component and that the normal weights at Y_1 and Y_2 fold into the following three situations:

CASE A. $(n_1, 1, 1)$,

CASE B. $(n_1, n_2, 1)$,

CASE C. (n_1, n_2, n_3).

where the above n_j's are strictly greater than 1 and prime to each other. Furthermore, we denote by N_j^6 the fixed point component of the $\mathbb{Z}/n_j$-action.

The proofs in each of these cases is essentially the same as in the semi-free case and consists on computing locally the integrals

$$\int_{X_5(\mathbf{d})} x^5, \qquad \text{and} \qquad \int_{X_5(\mathbf{d})} p_1(X_5(\mathbf{d})) \cdot x^3.$$

CASE A. The local datum of x^5 at Y_i is given by

$$\int_{Y_i} (r_i x_i + (a_i + l) \cdot z)^5 e_{S^1}(\nu_i)^{-1}$$

where ν_i is the normal bundle of Y_i in $X_5(\mathbf{d})$. Using the splitting principle, we can compute the equivariant Euler class. Since the normal weights at Y_i are $(n_1, 1, 1)$, the normal bundle ν_i of Y_i in $X_5(\mathbf{d})$ splits into $\nu_{i,N_1} \oplus \nu_{i,N_1^\perp}$. Therefore, the Euler class is given by

$$e_{S^1}(\nu_i) = \big(c_1(\nu_{i,N_1}) + n_1 \cdot z\big)\big(c_2(\nu_{i,N_1^\perp}) + c_1(\nu_{i,N_1^\perp}) \cdot z + z^2\big).$$

Let us now replace $c_1(\nu_{i,N_1}) \in H^2(Y_i)$ by $c_{i,n_1} \cdot x_i$, and do the same with $c_k(\nu_{N_1^\perp}) \in H^{2k}(Y_i)$ and replace it by $c_{i,k} \cdot {x_i}^k$. Thus, $e_{S^1}(\nu_i)^{-1}$ is given by the power series expansion in x_i

$$\frac{1}{n_1 z^3} - \left(\frac{c_{i,n_1}}{n_1^2 z^4} + \frac{c_{i,1}}{n_1 z^4}\right) x_i + \left(\frac{c_{i,n_1}^2}{n_1^3 z^5} + \frac{c_{i,1} c_{i,n_1}}{n_1^2 z^5} + \frac{c_{i,1}^2 - c_{i,2}}{n_1 z^5}\right) x_i^2.$$

The above computation allows us to compute locally the integral $\int_{X_5(\mathbf{d})} x^5$. With the above notation, this is a polynomial in l which is constant equal to d. The equations induced by the coefficients in $l^{\leqslant 3}$ give us the following system.

$$\begin{aligned} d = -\frac{a_2^3}{n_1^3}\big(&a_2^2 c_{2,1}^2 n_1^2 + a_2^2 c_{2,1} c_{2,n_1} n_1 - a_2^2 c_{2,2} n_1^2 + a_2^2 c_{2,n_1}^2 \\ &- 5a_2 c_{2,1} n_1^2 r_2 - 5a_2 c_{2,n_1} n_1 r_2 + 10 n_1^2 r_2^2\big), \end{aligned}$$

$$\begin{aligned} 0 = -\frac{5a_2^2}{n_1^3}\big(&a_2^2 c_{2,1}^2 n_1^2 + a_2^2 c_{2,1} c_{2,n_1} n_1 - a_2^2 c_{2,2} n_1^2 + a_2^2 c_{2,n_1}^2 \\ &- 4a_2 c_{2,1} n_1^2 r_2 - 4a_2 c_{2,n_1} n_1 r_2 + 6 n_1^2 r_2^2\big), \end{aligned}$$

$$0 = -\frac{10a_2}{n_1^3}\big(a_2^2c_{2,1}^2n_1^2 + a_2^2c_{2,1}c_{2,n_1}n_1 - a_2^2c_{2,2}n_1^2 + a_2^2c_{2,n_1}^2 \\ - 3a_2c_{2,1}n_1^2r_2 - 3a_2c_{2,n_1}n_1r_2 + 3n_1^2r_2^2\big),$$

$$0 = \frac{10r_1^2}{n_1} - \frac{10}{n_1^3}\big(a_2^2c_{2,1}^2n_1^2 + a_2^2c_{2,1}c_{2,n_1}n_1 - a_2^2c_{2,2}n_1^2 + a_2^2c_{2,n_1}^2 \\ - 2a_2c_{2,1}n_1^2r_2 - 2a_2c_{2,n_1}n_1r_2 + n_1^2r_2^2\big)$$

This system admits two non-trivial solutions in the variables d, $c_{2,2}$, r_1 and r_2, namely

$$d = -\frac{a_2^5(c_{2,1}n_1 + c_{2,n_1})^2}{9n_1^3}, \qquad c_{2,2} = \frac{c_{2,1}^2n_1^2 - c_{2,1}c_{2,n_1}n_1 + c_{2,n_1}^2}{3n_1^2},$$
$$r_2 = \frac{a_2(c_{2,1}n_1 + c_{2,n_1})}{3n_1}, \qquad r_1 = \pm\frac{a_2(c_{2,1}n_1 + c_{2,n_1})}{3n_1}.$$

We now compute locally the integral $\int_{X_5(\mathbf{d})} p_1(X_5(\mathbf{d})) \cdot x^3$. We already computed $e_{S^1}(\nu_i)^{-1}$, thus we only need to compute the equivariant Pontrjagin class at Y_i

$$p_{1,S^1}(Y_i) = (p_1(Y_i) + (y_{i,1} + n_1z)^2 + (y_{i,2} + z)^2 + (y_{i,3} + z)^2) \\ = (p_1(Y_i) + (c_{i,n_1}x_i + n_1z)^2 + c_{i,1}^2x_i^2 + 2c_{i,1}x_iz - 2c_{i,2}x_i^2 + 2z^2).$$

One can now compute $\int_{X_5(\mathbf{d})} p_1(X_5(\mathbf{d})) \cdot x^3$ which is again a polynomial in l. If we look at the constant term in l, we have the equality

$$p_1 \cdot d = \frac{a_2}{n_1^3}\big(- 2a_2^2c_{2,n_1}^2 - a_2^2c_{2,1}^2n_1^2 + 4a_2^2c_{2,2}n_1^2 + a_2^2c_{2,1}c_{2,n_1}n_1^3 \\ - a_2^2c_{2,1}^2n_1^4 + a_2^2c_{2,2}n_1^4 + a_2^2n_1^2\int p_1(Y_2) + 6a_2c_{2,n_1}n_1r_2 \\ - 3a_2c_{2,n_1}n_1^3r_2 + 3a_2c_{2,1}n_1^4r_2 - 6n_1^2r_2^2 - 3n_1^4r_2^2\big).$$

Note that $\int p_1(Y_2) = -3$. If we now use this equation together with the result above and Lemma 6.3, we can obtain

$$p_1 = \frac{c_{1,n_1}^2(4n_1^2 - 1) + 12n_1^2}{a_2^2c_{1,n_1}^2}, \qquad d = -\frac{a_2^5c_{1,n_1}^2}{4n_1^3}$$

which implies that $p_1 = 6 + r - \sum_{i=1}^r d_i^2$ is positive. This implies that $\mathbf{d} = (1)$ or (2).

CASE B. The computations are essentially the same in this case. The normal bundle ν_i of Y_i in $X_5(\mathbf{d})$ splits into $\nu_{i,N_1} \oplus \nu_{i,N_2} \oplus \widetilde{\nu_i}^{\perp}$. Thus the equivariant Euler class is given by

$$e_{S^1}(\nu_i) = \big(c_1(\nu_{i,N_1}) + n_1 \cdot z\big)\big(c_1(\nu_{i,N_2}) + n_2 \cdot z\big)\big(c_1(\widetilde{\nu_i}^{\perp}) + z\big).$$

Let us now replace $c_1(\nu_{i,N_j}) \in H^2(Y_i)$ by $c_{i,n_j} \cdot x_i$, and do the same with $c_1(\widetilde{\nu_i}^{\perp}) \in H^2(Y_i)$ and replace it by $c_{i,1} \cdot x_i$. Thus, $e_{S^1}(\nu_i)^{-1}$ is given by the power series expansion in x_i

$$\begin{aligned}&\frac{1}{n_1 n_2 z^3} - \frac{1}{n_1^2 n_2^2 z^4}(c_{i,n_1} n_2 + c_{i,n_2} n_1 + c_{i,1} n_1 n_2) x_i \\ &+ \frac{1}{n_1^3 n_2^3 z^5}(c_{i,1}^2 n_1^2 n_2^2 + c_{i,1} c_{i,n_1} n_1 n_2^2 + c_{i,1} c_{i,n_2} n_1^2 n_2 \\ &\qquad + c_{i,n_1}^2 n_2^2 + c_{i,n_1} c_{i,n_2} n_1 n_2 + c_{i,n_2}^2 n_1^2) x_i^2.\end{aligned}$$

We can then compute $\int_{X_5(\mathbf{d})} x^5$, which is a polynomial in l that needs to be constant equal to d. The coefficients in $l^{\leqslant 2}$ give us the system of equations

$$\begin{aligned}d = &-\frac{a_2^3}{n_1^3 n_2^3}(a_2^2 c_{2,1}^2 n_1^2 n_2^2 + a_2^2 c_{2,1} c_{2,n_1} n_1 n_2^2 + a_2^2 c_{2,1} c_{2,n_2} n_1^2 n_2 \\ &+ a_2^2 c_{2,n_1}^2 n_2^2 + a_2^2 c_{2,n_1} c_{2,n_2} n_1 n_2 + a_2^2 c_{2,n_2}^2 n_1^2 \\ &- 5a_2 c_{2,1} n_1^2 n_2^2 r_2 - 5a_2 c_{2,n_1} n_1 n_2^2 r_2 \\ &- 5a_2 c_{2,n_2} n_1^2 n_2 r_2 + 10 n_1^2 n_2^2 r_2^2) \\ 0 = &-\frac{5a_2^2}{n_1^3 n_2^3}(a_2^2 c_{2,1}^2 n_1^2 n_2^2 + a_2^2 c_{2,1} c_{2,n_1} n_1 n_2^2 + a_2^2 c_{2,1} c_{2,n_2} n_1^2 n_2 \\ &+ a_2^2 c_{2,n_1}^2 n_2^2 + a_2^2 c_{2,n_1} c_{2,n_2} n_1 n_2 + a_2^2 c_{2,n_2}^2 n_1^2 \\ &- 4a_2 c_{2,1} n_1^2 n_2^2 r_2 - 4a_2 c_{2,n_1} n_1 n_2^2 r_2 \\ &- 4a_2 c_{2,n_2} n_1^2 n_2 r_2 + 6 n_1^2 n_2^2 r_2^2), \\ 0 = &-\frac{10a_2}{n_1^3 n_2^3}(a_2^2 c_{2,1}^2 n_1^2 n_2^2 + a_2^2 c_{2,1} c_{2,n_1} n_1 n_2^2 + a_2^2 c_{2,1} c_{2,n_2} n_1^2 n_2 \\ &+ a_2^2 c_{2,n_1}^2 n_2^2 + a_2^2 c_{2,n_1} c_{2,n_2} n_1 n_2 + a_2^2 c_{2,n_2}^2 n_1^2 \\ &- 3a_2 c_{2,1} n_1^2 n_2^2 r_2 - 3a_2 c_{2,n_1} n_1 n_2^2 r_2 \\ &- 3a_2 c_{2,n_2} n_1^2 n_2 r_2 + 3 n_1^2 n_2^2 r_2^2).\end{aligned}$$

Solving this system in the variables d, c_{2,n_1} and c_{2,n_2} gives us the solutions

$$\begin{aligned}d &= -\frac{a_2^3 r_2^2}{n_1 n_2}, \\ c_{2,n_1} &= -\frac{\sqrt{3}\sqrt{-a_2^2 n_1^2 (r_2 - a_2 c_{2,1})^2} + a_2^2 c_{2,1} n_1 - 3a_2 n_1 r_2}{2a_2^2}, \\ c_{2,n_2} &= \frac{\sqrt{3}\sqrt{-a_2^2 n_2^2 (r_2 - a_2 c_{2,1})^2} - a_2^2 c_{2,1} n_2 + 3a_2 n_2 r_2}{2a_2^2},\end{aligned}$$

or

$$\begin{aligned}d &= -\frac{a_2^3 r_2^2}{n_1 n_2}, \\ c_{2,n_1} &= \frac{\sqrt{3}\sqrt{-a_2^2 n_1^2 (r_2 - a_2 c_{2,1})^2} - a_2^2 c_{2,1} n_1 + 3a_2 n_1 r_2}{2a_2^2}, \\ c_{2,n_2} &= -\frac{\sqrt{3}\sqrt{-a_2^2 n_2^2 (r_2 - a_2 c_{2,1})^2} + a_2^2 c_{2,1} n_2 - 3a_2 n_2 r_2}{2a_2^2},\end{aligned}$$

In both cases, we need to have that $r_2 - a_2 c_{2,1} = 0$ to obtain a solution over the real numbers. Thus, we have

$$c_{2,n_1} = \frac{n_1 r_2}{a_2}, \quad c_{2,n_2} = \frac{n_2 r_2}{a_2}, \quad c_{2,1} = \frac{r_2}{a_2}.$$

One can now compute $\int_{X_5(\mathbf{d})} p_1(X_5(\mathbf{d})) \cdot x^3$ using the Atiyah-Bott integration formula to get a constant polynomial in l equal to $p_1 \cdot d$. This computation is left to the reader and leads to the system of equations below.

$$\begin{aligned}
p_1 \cdot d = -\frac{a_2}{n_1^3 n_2^3} (& a_2^2 c_{2,1}^2 n_1^4 n_2^2 + a_2^2 c_{2,1}^2 n_1^2 n_2^4 - a_2^2 c_{2,1} c_{2,n_1} n_1^3 n_2^2 \\
& + a_2^2 c_{2,1} c_{2,n_1} n_1 n_2^4 - a_2^2 c_{2,1} c_{2,n_1} n_1 n_2^2 + a_2^2 c_{2,1} c_{2,n_2} n_1^4 n_2 \\
& - a_2^2 c_{2,1} c_{2,n_2} n_1^2 n_2^3 - a_2^2 c_{2,1} c_{2,n_2} n_1^2 n_2 + a_2^2 c_{2,n_1}^2 n_2^4 \\
& + a_2^2 c_{2,n_1}^2 n_2^2 - a_2^2 c_{2,n_1} c_{2,n_2} n_1^3 n_2 - a_2^2 c_{2,n_1} c_{2,n_2} n_1 n_2^3 \\
& + a_2^2 c_{2,n_1} c_{2,n_2} n_1 n_2 + a_2^2 c_{2,n_2}^2 n_1^4 + a_2^2 c_{2,n_2}^2 n_1^2 \\
& - a_2^2 n_1^2 n_2^2 \int p_1(Y_2) - 3 a_2 c_{2,1} n_1^4 n_2^2 r_2 - 3 a_2 c_{2,1} n_1^2 n_2^4 r_2 \\
& + 3 a_2 c_{2,1} n_1^2 n_2^2 r_2 + 3 a_2 c_{2,n_1} n_1^3 n_2^2 r_2 - 3 a_2 c_{2,n_1} n_1 n_2^4 r_2 \\
& - 3 a_2 c_{2,n_1} n_1 n_2^2 r_2 - 3 a_2 c_{2,n_2} n_1^4 n_2 r_2 + 3 a_2 c_{2,n_2} n_1^2 n_2^3 r_2 \\
& - 3 a_2 c_{2,n_2} n_1^2 n_2 r_2 + 3 n_1^4 n_2^2 r_2^2 + 3 n_1^2 n_2^4 r_2^2 + 3 n_1^2 n_2^2 r_2^2)
\end{aligned}$$

from which we can deduce using the above equalities that

$$p_1 = \frac{3a_2^2 + r_2^2 \left(n_1^2 + n_2^2 + 1\right)}{a_2^2 r_2^2}.$$

This shows that $p_1 = 6 + r - \sum_{i=1}^{r} d_i^2$ is positive. This implies that $\mathbf{d} = (1)$ or (2). The reader can find in the appendix the *Mathematica*® computations of this case, cf. Chapter B

CASE C. This case can be treated exactly the same way as Case B. One simply needs to adapt the formulae for the equivariant Euler class of the normal bundle at Y_i in order to carry the computations. The reader can find in the appendix the *Mathematica*® computations of this case, cf. Chapter B □

PROOF OF THEOREM D. PART 1. In virtue of the Lemma 6.1, 6.2 and 6.4, there is no other choice that $\mathbf{d}$ is either (1) and $X_5(\mathbf{d})$ is the complex projective space $\mathbb{C}P^5$, or $\mathbf{d} = (2)$ and $X_5(\mathbf{d})$ is a complex quadric. □

6.2 THE CASE WHERE $X_5(\mathbf{d})^{S^1} = Y^4 \cup Z^2 \cup P^0$

As above, we can lift the S^1-action to γ, where γ denotes the restriction to $X_5(\mathbf{d})$ of the dual Hopf line bundle on $\mathbb{C}P^{5+r}$. This allows us to consider the Hopf weights at the fixed point components. Since the lift is not unique, we choose the Hopf weights at Y, Z and P to be respectively l, $a_Z + l$ and $a_P + l$, where $l \in \mathbb{Z}$.

We denote by ν_Y, ν_Z and ν_P the normal bundles of Y, Z and P respectively. We orient these bundles by mean of the S^1-action. The orientation on ν_Y, ν_Z and ν_P are chosen such that their normal weights $n_{Y,j}$, $n_{Z,j}$ and $n_{P,j}$'s are all positive. Furthermore, we choose the orientation on Y, Z and P in order to make it compatible with the orientation of $X_n(\mathbf{d})$ and their respective normal bundles. By Section 3.3, the signature of $X_n(\mathbf{d})$ is the sum of the signature of the fixed points. Thus the signature of Y has to be ± 1, and the signature of P has to be ∓ 1. Up to changing the S^1-action with the inverse S^1-action, let us fix without loss of generality the signature of Y to be equal to $+1$.

We denote by x_Y a fixed generator of $H^2(Y;\mathbb{Z})$ such that $\int_Y x_Y^2 = 1$. Similarly, x_Z denotes the generator of $H^2(Z;\mathbb{Z})$ which satisfies $\int_Z x_Z = 1$. Using this notation, we can write $x|_Y = r_Y x_Y$ and $x|_Z = r_Z x_Z$, where $r_Y, r_Z \in \mathbb{Z}$.

Furthermore, we apply the splitting principle to ν_Y, ν_Z and ν_P. We denote by $\{y_{Y,j} + n_{Y,j} \cdot z\}$, by $\{y_{Z,j} + n_{Z,j} \cdot z\}$, and by $\{n_{P,j} \cdot z\}$ the equivariant normal roots at Y, Z and P respectively.

LEMMA 6.5. *Let $X_5(\mathbf{d})$ be a complete intersection with a smooth circle action such that the fixed point components are $X_5(\mathbf{d})^{S^1} = Y^4 \cup Z^2 \cup P^0$. Then the action is not semi-free around the fixed point components.*

PROOF. Let us assume that the $n_{Y,j}$'s, $n_{Z,j}$'s and the $n_{P,j}$'s are all equal to 1. Thus one can compute the Euler classes of ν_Y and ν_Z,

$$\begin{aligned} e_{S^1}(\nu_Y) &= c_2(\nu_Y) z + c_1(\nu_Y) z^2 + z^3, \\ e_{S^1}(\nu_Z) &= c_1(\nu_Z) z^3 + z^4, \end{aligned}$$

If we now write $c_2(\nu_Y) = c_{Y,2} x_Y^2$, and $c_1(\nu_Y) = c_{Y,1} x_Y$ with $c_{Y,j} \in \mathbb{Z}$, and $c_1(\nu_Z) = c_{Z,1} x_Z$ with $c_{Z,1} \in \mathbb{Z}$, then one can compute $e_{S^1}(\nu_Y)^{-1}$ as a power series in x_Y,

$$e_{S^1}(\nu_Y)^{-1} = \frac{1}{z^3} - \frac{c_{Y,1} x_Y}{z^4} + \frac{(c_{Y,1}^2 - c_{Y,2}) x_Y^2}{z^5}.$$

Similarly, $e_{S^1}(\nu_Z)^{-1}$ is given by the power series in x_Z,

$$e_{S^1}(\nu_Z)^{-1} = \frac{1}{z^4} - \frac{c_{Z,1} x_Z}{z^5}.$$

We now compute locally

$$\int_{X_5(\mathbf{d})} x^5, \quad \int_{X_5(\mathbf{d})} p_1(X_5(\mathbf{d})) \cdot x^3, \quad \text{and} \quad \int_{X_5(\mathbf{d})} s_2(X_5(\mathbf{d})) \cdot x,$$

where $s_2(X_5(\mathbf{d}))$ denotes the sum $\sum_i x_i^4$ and the x_i are the formal roots of the tangent bundle of $X_5(\mathbf{d})$. Note that one could use the second Pontrjagin class $p_2(X_5(\mathbf{d}))$ instead of the class $s_2(X_5(\mathbf{d}))$, but this would make the computations of the local data more complicated.

The local datum of x^5 at Y is given by

$$\begin{aligned}&\int_{Y^4}(r_Y x_Y + l\cdot z)^5\left(\frac{1}{z^3}-\frac{c_{Y,1}x_Y}{z^4}+\frac{(c_{Y,1}^2-c_{Y,2})x_Y^2}{z^5}\right)\\ &= 10r_Y^2 l^3 - 5c_{Y,1}r_Y l^4 + (c_{Y,1}^2-c_{Y,2})l^5.\end{aligned}$$

Similarly, one can compute the local datum of x^5 at Z is

$$\begin{aligned}&\int_{Z^2}(r_Z x_Z + (a_Z+l)\cdot z)^5\left(\frac{1}{z^4}-\frac{c_{Z,1}x_Z}{z^5}\right)\\ &= 5a_Z^4 r_Z - a_Z^5 c_{Z,1} + (20a_Z^3 r_Z - 5a_Z^4 c_{Z,1})\cdot l\\ &+ (30a_Z^2 r_Z - 10a_Z^3 c_{Z,1})\cdot l^2 + (20a_Z r_Z - 10a_Z^2 c_{Z,1})\cdot l^3\\ &+ (5r_Z - 5a_Z c_{Z,1})\cdot l^4 - c_{Z,1}\cdot l^5.\end{aligned}$$

Finally, the local datum of x^5 at P is

$$-(a_P+l)^5 = -a_P^5 - 5a_P^4 l - 10a_P^3 l^2 - 10a_P^2 l^3 - 5a_P l^4 - l^5.$$

Identifying the polynomial in l to be constant and equal to d gives us the system of equations

$$\begin{aligned}d &= -a_P^5 - a_Z^5 c_{Z,1} + 5a_Z^4 r_Z,\\ 0 &= -5\left(a_P^4 + a_Z^4 c_{Z,1} - 4a_Z^3 r_Z\right),\\ 0 &= -10\left(a_P^3 + a_Z^3 c_{Z,1} - 3a_Z^2 r_Z\right),\\ 0 &= -10\left(a_P^2 + a_Z^2 c_{Z,1} - 2a_Z r_Z - r_Y^2\right),\\ 0 &= -5(a_P + a_Z c_{Z,1} + c_{Y,1}r_Y - r_Z),\\ 0 &= c_{Y,1}^2 - c_{Y,2} - c_{Z,1} - 1.\end{aligned}$$

We can now do the same with $p_1(X_5(\mathbf{d}))\cdot x^3$ and obtain the system of equations

$$\begin{aligned}p_1\cdot d &= -5a_P^3 - 2a_Z^3 c_{Z,1} + 12a_Z^2 r_Z,\\ 0 &= -3\left(5a_P^2 + 2a_Z^2 c_{Z,1} - 8a_Z r_Z - 3r_Y^2\right),\\ 0 &= -3(5a_P + 2a_Z c_{Z,1} + c_{Y,1}r_Y - 4r_Z),\\ 0 &= 2c_{Y,1}^2 - 5c_{Y,2} - 2c_{Z,1} + \int p_1(Y) - 5,\end{aligned}$$

where $p_1 = 6 + r - \sum_i d_i^2$.

Let us now integrate $s_2(X_5(\mathbf{d}))\cdot x$ locally. The local datum at Y is given by

$$\int_{Y^4}\left((y_{Y,1}+z)^4 + (y_{Y,2}+z)^4 + (y_{Y,3}+z)^4\right)\cdot(r_Y x_Y + l\cdot z)\cdot e_{S^1}(\nu_Y)^{-1}.$$

By means of symmetric functions, we have the following equality

$$\sum_{i=1}^{3}(y_{Y,i}+z)^4 = 6c_{Y,1}^2 x_Y^2 z^2 - 12c_{Y,2}x_Y^2 z^2 + 4c_{Y,1}x_Y z^3 + 3z^4.$$

Thus, the local datum at Y is $c_{Y,1}r_Y + 5(c_{Y,1}^2 - 3c_{Y,2}) \cdot \mathfrak{l}$. Similarly, the local datum of $s_2(X_5(\mathbf{d})) \cdot x$ at Z is equal to $4r_Z$, and at P is given by $-5a_P - 5\mathfrak{l}$. This gives us the system of equations

$$\int_{X_5(\mathbf{d})} s_2(X_5(\mathbf{d})) \cdot x = -5a_P + c_{Y,1}r_Y + 4r_Z,$$
$$0 = 5(-1 + c_{Y,1}^2 - 3c_{Y,2}).$$

If we now solve the system of equations

$$\begin{aligned}
0 &= c_{Y,1}^2 - c_{Y,2} - c_{Z,1} - 1,\\
0 &= 2c_{Y,1}^2 - 5c_{Y,2} - 2c_{Z,1} + \int p_1(Y) - 5,\\
0 &= 5\left(c_{Y,1}^2 - 3c_{Y,2} - 1\right)
\end{aligned}$$

with $\int p_1(Y) = 3$, we get only two solutions $c_{Y,1} = \pm 1, c_{Y,2} = 0, c_{Z,1} = 0$. This implies that the system of equations

$$\begin{aligned}
d &= -a_P^5 - a_Z^5 c_{Z,1} + 5a_Z^4 r_Z,\\
0 &= -5\left(a_P^4 + a_Z^4 c_{Z,1} - 4a_Z^3 r_Z\right),\\
0 &= -10\left(a_P^3 + a_Z^3 c_{Z,1} - 3a_Z^2 r_Z\right),\\
0 &= -10\left(a_P^2 + a_Z^2 c_{Z,1} - 2a_Z r_Z - r_Y^2\right),\\
0 &= -5(a_P + a_Z c_{Z,1} + c_{Y,1}r_Y - r_Z),\\
0 &= -3\left(5a_P^2 + 2a_Z^2 c_{Z,1} - 8a_Z r_Z - 3r_Y^2\right),\\
0 &= -3(5a_P + 2a_Z c_{Z,1} + c_{Y,1}r_Y - 4r_Z),
\end{aligned}$$

has for solution $d = 0$, $a_P = 0$, $r_Y = 0$, $r_Z = 0$, which is an obvious contradiction. The reader can find in the appendix the *Mathematica*® computations of this case, cf. Chapter B □

LEMMA 6.6. *Let $X_5(\mathbf{d})$ be a complete intersection with a smooth circle action such that the fixed point components are $X_5(\mathbf{d})^{S^1} = Y^4 \cup Z^2 \cup P^0$. If the action admits a fixed point component $N^8 \subset X_5(\mathbf{d})^{\mathbb{Z}/p}$ for a prime number p, then $\mathbf{d} = (1)$ or (2).*

PROOF. First, one can see that N contains both Y and Z, and possibly P. This can be deduced easily, using Corollary 4.7.

Since $\operatorname{sign}(N) = \operatorname{sign}(N^{S^1})$, the signature of N is either ± 1 if it contains only Y and Z. If N contains in addition the isolated fixed point P, then the signature of N is 0 or ± 2. On the other hand, N is of codimension 2, thus one can think about it as a virtual complete intersection in $\mathbb{C}P^{n+r}$ with multidegree $(d_1, \ldots, d_r, |\eta|)$, and where $\eta x \in H^2(X_n(\mathbf{d}); \mathbb{Z})$ is the Poincaré-dual class of N. Thus by Theorem 5.2, we have that $\mathbf{d} = (1)$ or (2) in both cases. □

The next step is now for us to understand what happens if there exists an $N^6 \subset X_5(\mathbf{d})^{\mathbb{Z}/p}$ for a certain prime number p. Using Corollary 4.7 together with Poincaré Duality, there are only two possible situations.

CASE A. N^6 contains Y, Z and P, or

CASE B. N^6 contains Y and P.

We choose the orientation on N^6 by making it compatible with the orientation of Y and ν_Y, the normal bundle of Y in N. In particular, this implies that the normal weights at Y are positive. Furthermore, we fix now the orientation of P (and possibly Z) in N using the just defined orientation of N, together with our convention that the normal weights of their normal bundles in N need to be positive. Using the equivariant signature formula, we can deduce that the orientation of P in N is -1.

For dimensional reasons, N^6 is in fact a connected fixed point component of $X_5(\mathbf{d})^{\mathbb{Z}/n_{Y,1}}$, where $n_{Y,j}$ is the normal weight at Y divisible by p. Let us consider the induced effective $\widetilde{S^1}$-action on N^6, where $\widetilde{S^1} = S^1/(\mathbb{Z}/n_{Y,j})$. By definition, this action is semi-free around Y. One can see that in CASE A, this $\widetilde{S^1}$-action on N^6 can be of two kind, either it is semi-free around all fixed point components, or there exists $M \subset N^{\mathbb{Z}/\widetilde{n}}$ containing Z or P, where we look at $\mathbb{Z}/\widetilde{n}$ as a subgroup of $\widetilde{S^1}$. Note that M is also a connected fixed point component of $X_5(\mathbf{d})^{\mathbb{Z}/n}$ with respect to the original S^1-action on $X_5(\mathbf{d})$ and where $n = n_{Y,j} \cdot \widetilde{n}$

Therefore, let us focus our interest on the special case where there exists a fixed point component $M \subset X_5(\mathbf{d})^{\mathbb{Z}/n}$ containing Z or P, for an integer n. We prove that $M \subset X_5(\mathbf{d})^{\mathbb{Z}/n}$ is 4-dimensional and contains in fact both fixed point components.

By contradiction, assume $M \subset X_5(\mathbf{d})^{\mathbb{Z}/n}$ contains only P. Therefore, we have an orientable manifold M equipped with a smooth S^1-action such that the fixed point set M^{S^1} is equal to P, an isolated fixed point. Using the equivariant signature formula (cf. Section 3.3) or [12, Chapter IV, Corollary 2.3] one gets a contradiction.

On the other hand, assume by contradiction that M contains Z and does not contain P. Since M contains Z strictly, M needs to be of dimension 4 or higher. However, M needs to be contained in N. Thus M is 4-dimensional.

Without loss of generality, we assume that the circle action on M is semi-free around Z, thus if we integrate locally $x|_M^2$ we have that

$$\begin{aligned}\int_{M^4} x|_M{}^2 &= \int_Z (x|_Z + (a_Z + l)\cdot z)^2 \left(\frac{1}{z} - \frac{c_1(\nu_Z)}{z^2}\right)\\ &= \left(-a_Z^2 \int c_1(\nu_Z) + 2a_Z\right) - 2\left(a_Z \int c_1(\nu_Z) - \int x|_Z\right)\cdot l\\ &\quad - \int c_1(\nu_Z)\cdot l^2\end{aligned}$$

Which implies that the first Chern class of the normal bundle $c_1(\nu_Z) = 0$, and $\int x|_Z = 0$ by looking the coefficients in l and l^2. Which is a contradiction to Proposition A (cf. Section 4.2).

LEMMA 6.7. *Let $M^4 \subset X_5(\mathbf{d})^{\mathbb{Z}/n}$ be a connected fixed point component such that the fixed point set M^{S^1} consists of $Z^2 \cup P^0$ as above. Then the normal weights at P and Z are equal to n,*

$$r_Z = \pm\frac{a_P - a_Z}{n}, \qquad \textit{and} \qquad c_1(\nu_Z) = \pm x_Z,$$

where $c_1(\nu_Z)$ is the first Chern class of the normal bundle of Z in M.

PROOF. We choose the orientation on M^4 by making it compatible with the orientation of Z and ν_Z, the normal bundle of Z in M. In particular, this implies that the normal weight at Z is positive. Furthermore, we fix now the orientation of P in M using the just defined orientation of M, together with our convention that the normal weights of the normal bundle in M need to be positive.

We first prove that the normal weights are equal. To show this, we use the Lefschetz fixed point formula for the equivariant signature (cf. Section 3.3)

$$\operatorname{sign}(M) = \mu(\lambda, P) + \mu(\lambda, Z),$$

where $\lambda \in S^1$ is a topological generator. Furthermore, we have that

$$\begin{aligned}
\mu(\lambda, P) &= \int_P \frac{1+\lambda^{-n_{P,1}}}{1-\lambda^{-n_{P,1}}} \cdot \frac{1+\lambda^{-n_{P,2}}}{1-\lambda^{-n_{P,2}}} \\
&= \pm\frac{1+\lambda^{n_{P,1}}}{1-\lambda^{n_{P,1}}} \cdot \frac{1+\lambda^{n_{P,2}}}{1-\lambda^{n_{P,2}}} \\
&= \pm(1+2\lambda^{n_{P,1}}+2\lambda^{2n_{P,1}}+\cdots)(1+2\lambda^{n_{P,2}}+2\lambda^{2n_{P,2}}+\cdots).
\end{aligned}$$

For Z we have a similar power series in λ

$$\begin{aligned}
\mu(\lambda, Z) &= \int_{Z^2} x_{Z,1} \frac{1+e^{-x_{Z,1}}}{1-e^{-x_{Z,1}}} \cdot \frac{1+\lambda^{-n_{Z,1}} e^{-y_{Z,1}}}{1-\lambda^{-n_{Z,1}} e^{-y_{Z,1}}} \\
&= -4\left(\sum_{k\geqslant 1} k\cdot\lambda^{k\cdot n_{Z,1}}\right)\int_Z y_{Z,1},
\end{aligned}$$

where $x_{Z,1}$ is a formal root of the tangent bundle of Z, and $y_{Z,1}$ is a formal root of the normal bundle of Z in M. Since the signature of M does not depend on λ, the two power series need to cancel in all but the constant terms. Thus the coefficient in $\lambda^{n_{Z,1}}$ of the power series $\mu(\lambda, Z)$ needs to cancel with the smallest coefficient in λ in the power series of $\mu(\lambda, P)$. We can then deduce that $n_{Z,1} = n_{P,1} = n_{P,2}$ and $\int_Z y_{Z,1} = \pm 1$, depending on the orientation of P in M.

Let us now deduce the above formulae. The strategy is to integrate locally $x|_M^2$. The local datum of $x|_M^2$ at Z is given by

$$\begin{aligned}
&\int_Z (r_Z x_Z + (a_Z + l)\cdot z)^2 \left(\frac{1}{n\cdot z} - \frac{c_1 x_Z}{n^2\cdot z^2}\right) \\
&= \varepsilon_Z\left(-\frac{a_Z(a_Z c_1 - 2n r_Z)}{n^2} - \frac{2(a_Z c_1 - n r_Z)}{n^2}\cdot l - \frac{c_1}{n^2}\cdot l^2\right)
\end{aligned}$$

where $\varepsilon_Z = \pm 1$ depending on the orientation of Z in M, and $c_1 x_Z = c_1(\nu_Z)$.

Similarly, the local datum of $x|_M^2$ at P is given by

$$\varepsilon_P \left(\frac{a_P^2}{n^2} + \frac{2ap}{n^2} \cdot l + \frac{1}{n^2} \cdot l^2 \right)$$

where $\varepsilon_P = \pm 1$ depending on the orientation of P in M.

Thus,

$$\begin{aligned} \int_N x|_N^2 = &\frac{a_P^2 \varepsilon_P - a_Z^2 c_1 \varepsilon_Z + 2a_Z \varepsilon_Z n r_Z}{n^2} \\ &+ \frac{2(a_P \varepsilon_P - a_Z c_1 \varepsilon_Z + \varepsilon_Z n r_Z)}{n^2} \cdot l + \frac{\varepsilon_P - c_1 \varepsilon_Z}{n^2} \cdot l^2 \end{aligned}$$

Solving now the system

$$\begin{aligned} 0 &= \frac{\varepsilon_P - c_1 \varepsilon_Z}{n^2}, \\ 0 &= \frac{2(a_P \varepsilon_P - a_Z c_1 \varepsilon_Z + \varepsilon_Z n r_Z)}{n^2} \end{aligned}$$

in r_Z and c_1 gives us exactly four solutions, namely

$$r_Z = \pm \frac{a_P - a_Z}{n}, \qquad \text{and} \qquad c_1 = \pm 1. \qquad \square$$

We now focus our attention again on CASE A, i.e. when there exists a fixed point component $N^6 \subset X_5(\mathbf{d})^{\mathbb{Z}/p}$ containing Y, Z and P. First, note that if $n_{Y,1}$ is the normal weight at Y divisible by p, then N^6 is in fact a fixed point component of $X_5(\mathbf{d})^{\mathbb{Z}/n_{Y,1}}$. Therefore, for N^6 to contain also Z and P, the prime number p needs to divide exactly two normal weights at Z and three at P, say $n_{Z,1}, n_{Z,2}, n_{P,1}, n_{P,2}$, and $n_{P,3}$.

As we mentioned early, there are exactly two possible outcomes in the CASE A. Either the induced effective $\widetilde{S^1}$-action on N is semi-free, and all the $n_{Z,1}$, $n_{Z,2}$, $n_{P,1}, n_{P,2}$, and $n_{P,3}$ are equal to $n_{Y,1}$, or there is another intermediate manifold $M^4 \subset N^{\mathbb{Z}/n_{Z,1}}$ containing Z and P. In virtue of Lemma 6.7, we have that $n_{Z,2} = n_{P,2} = n_{P,3}$, $n_{Y,1} = n_{Z,1} = n_{P,1}$ and divides $n_{Z,2}$. Thus, this breaks CASE A into two cases.

Assume the other weights at Y, namely $n_{Y,2}$ and $n_{Y,3}$, are equal to 1. In virtue of Lemma 6.7, there are two possible situation for the remaining weights at Z and P. Either they are all equal to 1, or $n_{Z,3} = n_{P,4} = n_{P,5}$ and $n_{Z,4}$ is equal to 1. This finally breaks CASE A into the following sub cases.

CASE A.1.I The normal weights are

AT Y. $(n_1, 1, 1)$

AT Z. $(n_1, n_1, 1, 1)$

AT P. $(n_1, n_1, n_1, 1, 1)$

CASE A.1.II The normal weights are

AT Y. $(n_1, 1, 1)$

AT Z. $(n_1, n_1, n_2, 1)$

AT P. $(n_1, n_1, n_1, n_2, n_2)$, where $(n_1, n_2) = 1$.

CASE A.2.I The normal weights are

AT Y. $(n_1, 1, 1)$

AT Z. $(n_1 n_2, n_1, 1, 1)$

AT P. $(n_1 n_2, n_1 n_2, n_1, 1, 1)$.

CASE A.2.II The normal weights are

AT Y. $(n_1, 1, 1)$

AT Z. $(n_1 n_2, n_1, n_3, 1)$

AT P. $(n_1 n_2, n_1 n_2, n_1, n_3, n_3)$, where $(n_1 n_2, n_3) = 1$.

Assume now we have $n_{Y,2} > 1$. In virtue of Lemma 6.6, we have without loss of generality that $(n_{Y,1}, n_{Y,2}) = 1$. Thus, we have $N_1^6 \subset X_5(d)^{\mathbb{Z}/n_{Y,1}}$ and $N_2^6 \subset X_5(\mathbf{d})^{\mathbb{Z}/n_{Y,2}}$ containing Y. By the pigeon hole principle, we know that at least one normal weight at P, say $n_{P,1}$, is divisible by both $n_{Y,1}$ and $n_{Y,2}$. As above, this gives us that there exists a connected fixed point component $M^4 \subset X_5(\mathbf{d})^{\mathbb{Z}/n}$ where $n = n_{Y,1} n_{Y,2} \tilde{n}$, and containing Z and P. Therefore, $n_{Z,1} = n_{P,1} = n_{P,2} n$, $n_{Z,2} = n_{P,3} = n_{Y,1}$ and $n_{Z,3} = n_{P,4} = n_{Y,2}$. If we assume $n_{Y,3} = 1$, then the remaining normal weights at Z and P are also equal to 1.

CASE A.3 The normal weights are

AT Y. $(n_1, n_2, 1)$

AT Z. $(n_1 n_2 n_3, n_1, n_2, 1)$

AT P. $(n_1 n_2 n_3, n_1 n_2 n_3, n_1, n_2, 1)$, where $(n_1, n_2) = 1$.

Assume now $n_{Y,3} > 1$ and $(n_{Y,i}, n_{Y,j}) = 1$ for $i \neq j$ by Lemma 6.6. Then we have as above a fixed point component $M^4 \subset X_5(\mathbf{d})^{\mathbb{Z}/n}$ where $n = n_{Y,1} n_{Y,2} \tilde{n}$, and containing Z and P. Thus the normal weights at Z are $n_{Z,1} = n_{Y,1} n_{Y,2} \tilde{n} = n$, $n_{Z,2} = n_{Y,1}$, and $n_{Z,3} = n_{Y,2}$ which leaves no other option that $n_{Z,1} = n_{Y,1} n_{Y,2} n_{Y,3} \overline{n}$ and $n_{Z,4} = n_{Y,3}$.

CASE A.4 The normal weights are

AT Y. (n_1, n_2, n_3)

AT Z. $(n_1 n_2 n_3 n_4, n_1, n_2, n_3)$

AT P. $(n_1 n_2 n_3 n_4, n_1 n_2 n_3 n_4, n_1, n_2, n_3)$, where $(n_i, n_j) = 1$ for $i, j = 1, 2, 3$, and $i \neq j$.

The following lemmas gives us very useful formulae for the above CASE A.

LEMMA 6.8. *Let* $N^6 \subset X_5(\mathbf{d})^{\mathbb{Z}/n_1}$ *be a connected fixed point component for an integer* n_1 *such that the fixed point set* N^{S^1} *consists of* $Y^4 \cup Z^2 \cup P^0$ *as above. If the normal weights are like in the* CASES A.1.*, *then*

$$\begin{aligned} r_Z &= \varepsilon_Z \frac{(1+c_{Y,n_1})(a_P - a_Z)}{n_1}, \\ r_Y &= \frac{a_P - a_Z c_{Y,n_1} - a_Z}{n_1}, \\ c_{Z,n_1} &= \varepsilon_Z (c_{Y,n_1} - 1)(c_{Y,n_1} + 1), \end{aligned}$$

or

$$\begin{aligned} r_Z &= \varepsilon_Z \frac{(1-c_{Y,n_1})(a_P - a_Z)}{n_1}, \\ r_Y &= -\frac{a_P + a_Z c_{Y,n_1} - a_Z}{n_1}, \\ c_{Z,n_1} &= \varepsilon_Z (c_{Y,n_1} - 1)(c_{Y,n_1} + 1), \end{aligned}$$

where $c_{Y,n_1} \cdot x_Y$ *(respectively* $c_{Z,n_1} \cdot x_Z$*) denotes the first Chern class of the normal bundle of* Y *(resp.* Z*) in* N, *and* ε_Z *is equal to* ± 1 *depending on the orientation of* Z *in* N.

PROOF. To obtain the above equations, it is enough to compute locally $\int_{N^6} x|_N^3$ at the fixed point components. In particular, the local datum of $x|_N^3$ at Y is given by

$$\int_{Y^4} (r_Y x_Y + l \cdot z)^3 \left(\frac{1}{n_1 z} - \frac{c_{Y,n_1} x_Y}{n_1^2 z^2} + \frac{c_{Y,n_1}^2 x_Y^2}{n_1^3 z^3} \right)$$

which gives the polynomial in l,

$$\frac{3r_Y^2}{n_1} \cdot l - \frac{3c_{Y,n_1} r_Y}{n_1^2} \cdot l^2 + \frac{c_{Y,n_1}^2}{n_1^3} \cdot l^3.$$

Similarly, we have at Z and P respectively the polynomials

$$\begin{aligned} &-\frac{a_Z^2 \varepsilon_Z (a_Z c_{Z,n_1} - 3n_1 r_Z)}{n_1^3} - \frac{3a_Z \varepsilon_Z (a_Z c_{Z,n_1} - 2n_1 r_Z)}{n_1^3} \cdot l \\ &- \frac{3\varepsilon_Z (a_Z c_{Z,n_1} - n_1 r_Z)}{n_1^3} \cdot l^2 - \frac{c_{Z,n_1} \varepsilon_Z}{n_1^3} \cdot l^3, \end{aligned}$$

and

$$-\frac{a_P^3}{n_1^3} - \frac{3a_P^2}{n_1^3} \cdot l - \frac{3a_P}{n_1^3} \cdot l^2 - \frac{1}{n_1^3} \cdot l^3.$$

This gives us the system

$$
\begin{aligned}
0 &= -\frac{3a_P^2}{n_1^3} - \frac{3a_Z\varepsilon_Z(a_Zc_{Z,n_1} - 2n_1r_Z)}{n_1^3} + \frac{3r_Y^2}{n_1},\\
0 &= -\frac{3a_P}{n_1^3} - \frac{3\varepsilon_Z(a_Zc_{Z,n_1} - n_1r_Z)}{n_1^3} - \frac{3c_{Y,n_1}r_Y}{n_1^2},\\
0 &= \frac{c_{Y,n_1}^2}{n_1^3} - \frac{c_{Z,n_1}\varepsilon_Z}{n_1^3} - \frac{1}{n_1^3}
\end{aligned}
$$

which once solved in the variables r_Z, r_Y and c_{Z,n_1} gives us the desired results. □

Let us now get similar relations in the CASES A.2.∗.

LEMMA 6.9. *Let $N^6 \subset X_5(\mathbf{d})^{\mathbb{Z}/n_1}$ be a connected fixed point component for an integer n_1 such that the fixed point set N^{S^1} consists of $Y^4 \cup Z^2 \cup P^0$ as above. If the normal weights are like in* CASES A.2.∗, *i.e. there exists $M^4 \subset N^{\mathbb{Z}/n_1n_2}$ containing Z and P, then*

$$
\begin{aligned}
&r_Y = \pm\frac{a_P - a_Z}{n_1n_2}, \quad r_Z = \varepsilon_Z^N\frac{a_P - a_Z}{n_1n_2}, \quad \varepsilon_Z^M = -\varepsilon_P^M\varepsilon_Z^N\\
&c_{Y,n_1} = 0, \quad c_{Z,n_1n_2} = -\varepsilon_Z^N, \quad c_{Z,n_1} = 0,
\end{aligned}
$$

or

$$
\begin{aligned}
&r_Y = \pm\frac{a_P + a_Z}{n_1n_2}, \quad r_Z = \varepsilon_Z^N\frac{a_Z - a_P}{n_1n_2}, \quad \varepsilon_Z^M = \varepsilon_P^M\varepsilon_Z^N\\
&c_{Y,n_1} = -\frac{2}{n_2}, \quad c_{Z,n_1n_2} = \varepsilon_Z^N, \quad c_{Z,n_1} = \varepsilon_Z^N\frac{2}{n_2},
\end{aligned}
$$

where $c_{Y,n_1}\cdot x_Y$, $c_{Z,n_1n_2}\cdot x_Z$ and $c_{Z,n_1}\cdot x_Z$ denote respectively the first Chern class of the normal bundle of Y in N, Z in M and its normal complement in N respectively. Furthermore, ε_Z^N, ε_Z^M and ε_P^M are equal to ± 1 depending on the orientations of Z and P in M and N.

PROOF. To prove this result, we have to compute the integrals

$$
\int_N x|_N^3 \quad \text{and} \quad \int_N p_1(N)\cdot x|_N.
$$

The local datum of $x|_N^3$ at Y is given by the formula in the proof of Lemma 6.8. At Z the local datum is like above except that $e_{S^1}(\nu_Z)^{-1}$ is given by

$$
\frac{1}{n_1^2n_2z^2} + \left(-\frac{c_{Z,n_1}}{n_1^3n_2z^3} - \frac{c_{Z,n_1n_2}}{n_1^3n_2^2z^3}\right)\cdot x_Z
$$

where $c_{Z,n_1n_2}\cdot x_Z$ denotes the first Chern class of the normal bundle of Z in $M^4 \subset N^{\mathbb{Z}/n_1n_2}$, and $c_{Z,n_1}\cdot x_Z$ denotes the first Chern class of Z in its complement. The computations give us the following polynomial in l

$$\begin{aligned}
&-\frac{a_Z^2\varepsilon_Z^N(a_Zc_{Z,n_1}n_2+a_Zc_{Z,n_1n_2}-3n_1n_2r_Z)}{n_1^3n_2^2}\\
&-\frac{3a_Z\varepsilon_Z^N(a_Zc_{Z,n_1}n_2+a_Zc_{Z,n_1n_2}-2n_1n_2r_Z)}{n_1^3n_2^2}\cdot l\\
&-\frac{3\varepsilon_Z^N(a_Zc_{Z,n_1}n_2+a_Zc_{Z,n_1n_2}-n_1n_2r_Z)}{n_1^3n_2^2}\cdot l^2\\
&-\frac{\varepsilon_Z^N(c_{Z,n_1}n_2+c_{Z,n_1n_2})}{n_1^3n_2^2}\cdot l^3.
\end{aligned}$$

We are now left to compute the local datum of $x|_N^3$ at P. The later is given by the polynomial

$$-\frac{a_P^3}{n_1^3n_2^2}-\frac{3a_P^2}{n_1^3n_2^2}\cdot l-\frac{3a_P}{n_1^3n_2^2}\cdot l^2-\frac{1}{n_1^3n_2^2}\cdot l^3.$$

By identifying the coefficients in l, l^2 and l^3 to zero gives us the system of equations

$$\begin{aligned}
0&=-\frac{3a_P^2}{n_1^3n_2^2}-\frac{3a_Z\varepsilon_Z^N(a_Zc_{Z,n_1}n_2+a_Zc_{Z,n_1n_2}-2n_1n_2r_Z)}{n_1^3n_2^2}+\frac{3r_Y^2}{n_1},\\
0&=-\frac{3a_P}{n_1^3n_2^2}-\frac{3\varepsilon_Z^N(a_Zc_{Z,n_1}n_2+a_Zc_{Z,n_1n_2}-n_1n_2r_Z)}{n_1^3n_2^2}-\frac{3c_{Y,n_1}r_Y}{n_1^2},\\
0&=\frac{c_{Y,n_1}^2}{n_1^3}-\frac{\varepsilon_Z^N(c_{Z,n_1}n_2+c_{Z,n_1n_2})}{n_1^3n_2^2}-\frac{1}{n_1^3n_2^2}
\end{aligned}$$

In order to have enough equations, one needs to proceed the same way with $\int_N p_1(N)\cdot x|_N$, i.e. compute this integral locally on the fixed point set and identify the coefficient in l to zero. This process gives us the equation

$$\begin{aligned}
0=&-\varepsilon_Z^N\frac{c_{Z,n_1}n_2^3-c_{Z,n_1}n_2-c_{Z,n_1n_2}n_2^2+c_{Z,n_1n_2}}{n_1n_2^2}\\
&+\frac{-n_2^2\int p_1(Y)+2n_2^2+1}{n_1n_2^2}.
\end{aligned}$$

Furthermore, by the signature formula one knows that $\int p_1(Y)=3$.

We complete this set of equations with the one given in Lemma 6.7, since we have an $M^4\subset N^{\mathbb{Z}/n_1n_2}$ containing $Z\cup P$. This system of equations gives us the desired result. □

Let us now get similar relations in the CASES A.3.

LEMMA 6.10. *Let $N_1^6\subset X_5(\mathbf{d})^{\mathbb{Z}/n_i}$ and $N_2^6\subset X_5(\mathbf{d})^{\mathbb{Z}/n_2}$ be connected fixed point components for integers n_1 and n_2, such that the fixed point* *sets* $N_i^{S^1}$ *consists of* $Y^4 \cup Z^2 \cup P^0$ *as above. If the normal weights are like in* CASES A.3, *then*

$$r_Z = \varepsilon_Z^{N_1}\frac{(a_P - a_Z)}{n_1n_2n_3}, \quad r_Y = \pm\frac{a_P - a_Z}{n_1n_2n_3}, \quad c_{Z,n_1n_2n_3} = -\varepsilon_Z^{N_1},$$
$$c_{Z,n_1} = 0, \quad c_{Z,n_2} = 0, \quad c_{Y,n_1} = 0, \quad c_{Y,n_2} = 0,$$
$$\varepsilon_Z^{N_1} = \varepsilon_Z^{N_2} = -\varepsilon_Z^M\varepsilon_P^M,$$

or

$$r_Z = -\varepsilon_Z^{N_1}\frac{(a_P - a_Z)}{n_1n_2n_3}, \quad r_Y = \mp\frac{a_P + a_Z}{n_1n_2n_3}, \quad c_{Z,n_1n_2n_3} = \varepsilon_Z^{N_1},$$
$$c_{Z,n_1} = \frac{2\varepsilon_Z^{N_1}}{n_2n_3}, \quad c_{Z,n_2} = \frac{2\varepsilon_Z^{N_2}}{n_1n_3}, \quad c_{Y,n_1} = \pm\frac{2}{n_2n_3},$$
$$c_{Y,n_2} = \pm\frac{2}{n_1n_3}, \quad \varepsilon_Z^{N_1} = \varepsilon_Z^{N_2} = \varepsilon_Z^M\varepsilon_P^M,$$

PROOF. The above equalities follows by computing

$$\int_{N_i} x|_{N_i}^3 \quad \text{and} \quad \int_{N_i} p_1(N_i)\cdot x|_{N_i}.$$

for $i = 1, 2$ and completing the equations with the one given in Lemma 6.7.

Note that these computations for each N_i are exaclty the same as those in Lemma 6.9. Indeed, the local data of $x|_{N_i}^3$ and $p_1(N_i)\cdot x|_{N_i}$ at Y is given in Lemma 6.8. At Z the local data are like above except that $e_{S^1}(\nu_{Z\subset N_i})^{-1}$ is given by

$$\frac{1}{n_in_1n_2n_3z^2} + \left(-\frac{c_{Z,n_i}}{n_i^2n_1n_2n_3z^3} - \frac{c_{Z,n_1n_2n_3}}{n_in_1^2n_2^2n_3^2z^3}\right)\cdot x_Z$$

where $c_{Z,n_1n_2n_3}\cdot x_Z$ denotes the first Chern class of the normal bundle of Z in $M^4 \subset N_i^{\mathbb{Z}/n_1n_2n_3}$, and $c_{Z,n_i}\cdot x_Z$ denotes the first Chern class of Z in its complement.

Like in Lemma 6.9, we obtain for N_1 the system of equations

$$\begin{aligned}
0 =& -\frac{3a_Z\varepsilon_Z^{N_1}(a_Zc_{Z,n_1}n_2n_3 + a_Zc_{Z,n_1n_2n_3} - 2n_1n_2n_3r_Z)}{n_1^3n_2^2n_3^2}\\
&-\frac{3a_P^2}{n_1^3n_2^2n_3^2} + \frac{3r_Y^2}{n_1},\\
0 =& -\frac{3\varepsilon_Z^{N_1}(a_Zc_{Z,n_1}n_2n_3 + a_Zc_{Z,n_1n_2n_3} - n_1n_2n_3r_Z)}{n_1^3n_2^2n_3^2}\\
&-\frac{3a_P}{n_1^3n_2^2n_3^2} - \frac{3c_{Y,n_1}r_Y}{n_1^2},\\
0 =& \frac{c_{Y,n_1}^2}{n_1^3} - \frac{\varepsilon_Z^{N_1}(c_{Z,n_1}n_2n_3 + c_{Z,n_1n_2n_3})}{n_1^3n_2^2n_3^2} - \frac{1}{n_1^3n_2^2n_3^2}\\
0 =& -\varepsilon_Z^{N_1}\frac{c_{Z,n_1}n_2^3n_3^3 - c_{Z,n_1}n_2n_3 - c_{Z,n_1n_2n_3}n_2^2n_3^2 + c_{Z,n_1n_2n_3}}{n_1n_2^2n_3^2}\\
&+\frac{-n_2^2n_3^2\int p_1(Y) + 2n_2^2n_3^2 + 1}{n_1n_2^2n_3^2}.
\end{aligned}$$

For N_2 we obtain in a similar fashion the following system of equations

$$\begin{aligned}
0 =& -\frac{3a_Z\varepsilon_Z^{N_2}(a_Zc_{Z,n_2}n_1n_3 + a_Zc_{Z,n_1n_2n_3} - 2n_1n_2n_3r_Z)}{n_2^3n_1^2n_3^2} \\
& - \frac{3a_P^2}{n_2^3n_1^2n_3^2} + \frac{3r_Y^2}{n_2}, \\
0 =& -\frac{3\varepsilon_Z^{N_2}(a_Zc_{Z,n_2}n_1n_3 + a_Zc_{Z,n_1n_2n_3} - n_1n_2n_3r_Z)}{n_2^3n_1^2n_3^2} \\
& - \frac{3a_P}{n_2^3n_1^2n_3^2} - \frac{3c_{Y,n_2}r_Y}{n_2^2}, \\
0 =& \frac{c_{Y,n_2}^2}{n_2^3} - \frac{\varepsilon_Z^{N_2}(c_{Z,n_2}n_1n_3 + c_{Z,n_1n_2n_3})}{n_2^3n_1^2n_3^2} - \frac{1}{n_2^3n_1^2n_3^2} \\
0 =& -\varepsilon_Z^{N_2}\frac{c_{Z,n_2}n_1^3n_3^3 - c_{Z,n_2}n_1n_3 - c_{Z,n_1n_2n_3}n_1^2n_3^2 + c_{Z,n_1n_2n_3}}{n_1n_2^2n_3^2} \\
& + \frac{-n_1^2n_3^2\int p_1(Y) + 2n_1^2n_3^2 + 1}{n_1n_2^2n_3^2}.
\end{aligned}$$

Completing the system of equations with the equations given in Lemma 6.7 and solving the system in the variables r_Y, r_Z, c_{Y,n_1}, $c_{Z,n_1n_2n_3}$, c_{Z,n_1}, $\varepsilon_Z^{N_1}$, c_{Z,n_2}, c_{Y,n_2}, $\varepsilon_Z^{N_2}$ gives us the desired solutions. □

Let us now get similar relations in the CASES A.4.

LEMMA 6.11. *Let $N_1^6 \subset X_5(\mathbf{d})^{\mathbb{Z}/n_1}$, $N_2^6 \subset X_5(\mathbf{d})^{\mathbb{Z}/n_2}$ and $N_3^6 \subset X_5(\mathbf{d})^{\mathbb{Z}/n_3}$ be connected fixed point components for integers n_1, n_2 and n_3, such that the fixed point sets $N_i^{S^1}$ consists of $Y^4 \cup Z^2 \cup P^0$ as above. If the normal weights are like in CASES A.4, then*

$$\begin{aligned}
& r_Z = \varepsilon_Z^{N_1}\frac{(a_P - a_Z)}{n_1n_2n_3n_4}, \quad r_Y = \pm\frac{a_P - a_Z}{n_1n_2n_3n_4}, \quad c_{Z,n_1n_2n_3n_4} = -\varepsilon_Z^{N_1}, \\
& c_{Z,n_1} = 0, \quad c_{Z,n_2} = 0, \quad , c_{Z,n_3} = 0 \quad c_{Y,n_1} = 0, \\
& c_{Y,n_2} = 0, c_{Y,n_3} = 0 \quad \varepsilon_Z^{N_1} = \varepsilon_Z^{N_2} = \varepsilon_Z^{N_3} = -\varepsilon_Z^M\varepsilon_P^M,
\end{aligned}$$

or

$$\begin{aligned}
& r_Z = -\varepsilon_Z^{N_1}\frac{(a_P - a_Z)}{n_1n_2n_3n_4}, \quad r_Y = \mp\frac{a_P + a_Z}{n_1n_2n_3n_4}, \quad c_{Z,n_1n_2n_3n_4} = \varepsilon_Z^{N_1}, \\
& c_{Z,n_1} = \frac{2\varepsilon_Z^{N_1}}{n_2n_3n_4}, \quad c_{Z,n_2} = \frac{2\varepsilon_Z^{N_2}}{n_1n_3n_4}, \quad c_{Z,n_3} = \frac{2\varepsilon_Z^{N_3}}{n_1n_2n_4}, \\
& c_{Y,n_1} = \pm\frac{2}{n_2n_3n_4}, c_{Y,n_2} = \pm\frac{2}{n_1n_3n_4}, \quad c_{Y,n_3} = \pm\frac{2}{n_1n_2n_4}, \\
& \varepsilon_Z^{N_1} = \varepsilon_Z^{N_2} = \varepsilon_Z^{N_3} = \varepsilon_Z^M\varepsilon_P^M,
\end{aligned}$$

PROOF. The above equalities follows by computing

$$\int_{N_i} x|_{N_i}^3 \quad \text{and} \quad \int_{N_i} p_1(N_i)\cdot x|_{N_i}.$$

for $i = 1,2,3$ and completing the equations with the one given in Lemma 6.7. These computations are exactly the same as those in the proof of Lemma 6.10. □

We now use Lemmas 6.7, 6.8, 6.9, 6.10 and 6.11 to prove the following result (i.e. CASE A).

LEMMA 6.12. *Let $X_5(\mathbf{d})$ be a complete intersection with a smooth circle action such that the fixed point components are $X_5(\mathbf{d})^{S^1} = Y^4 \cup Z^2 \cup P^0$. If the action admits a fixed point component $N^6 \subset X_5(\mathbf{d})^{\mathbb{Z}/p}$ containing $Y^4 \cup Z^2 \cup P^0$ for a prime number p, then $\mathbf{d} = (1)$ or (2).*

PROOF. To prove this, one needs to compute

$$\int_{X_5(\mathbf{d})} x^5, \quad \text{and} \quad \int_{X_5(\mathbf{d})} p_1(X_5(\mathbf{d})) \cdot x^3$$

locally at Y, Z and P, for CASE A.1.I, CASE A.1.II, CASE A.2.I, CASE A.2.II, CASE A.3 and CASE A.4. Furthermore, we use extensively the Lemmas 6.7, 6.8, 6.9, 6.10 and 6.11.

Let us first remark that in the first four cases, the local datum of x^5 and $p_1(X_5(\mathbf{d})) \cdot x^3$ at Y stays the same since the normal weights at Y are always of the form $(n_1, 1, 1)$. Thus, we compute them for once and for all. One needs first to compute the Euler class of the normal bundle. Since the normal bundle ν_Y splits into $\nu_{Y\subset N}$ and $\nu_{Y\subset N}{}^{\perp}$. This gives us that the Euler class of ν_Y is

$$e_{S^1}(\nu_Y) = (c_{Y,n_1} \cdot x_Y + n_1 \cdot z)\left(c_{Y,2} \cdot x_Y^2 + c_{Y,1} \cdot x_Y \cdot z + z^2\right)$$

where $c_{Y,n_1} \cdot x_Y$ denotes the first Chern class of $\nu_{Y\subset N}$, while $c_{Y,1} \cdot x_Y$ and $c_{Y,2} \cdot x_Y^2$ are the Chern classes of $\nu_{Y\subset N}{}^{\perp}$.

We now compute the local datum of x^5 at Y, which is

$$\begin{aligned}
&\int_{Y^4} (r_Y x_Y + l \cdot z)^5 e_{S^1}(\nu_Y)^{-1} \\
&= \int_{Y^4} (r_Y x_Y + l \cdot z)^5 \cdot \left(\frac{1}{n_1 z^3} + \left(-\frac{c_{Y,1}}{n_1 z^4} - \frac{c_{Y,n_1}}{n_1^2 z^4}\right) \cdot x_Y\right. \\
&\qquad\qquad \left. + \left(\frac{c_{Y,1}^2 - c_{Y,2}}{n_1 z^5} + \frac{c_{Y,1} c_{Y,n_1}}{n_1^2 z^5} + \frac{c_{Y,n_1}^2}{n_1^3 z^5}\right) \cdot x_Y^2\right),
\end{aligned}$$

which gives us the polynomial in l

$$\begin{aligned}
&\frac{10 r_Y^2}{n_1} \cdot l^3 - \frac{5 r_Y (c_{Y,1} n_1 + c_{Y,n_1})}{n_1^2} \cdot l^4 \\
&\frac{c_{Y,1}^2 n_1^2 + c_{Y,1} c_{Y,n_1} n_1 - c_{Y,2} n_1^2 + c_{Y,n_1}^2}{n_1^3} \cdot l^5
\end{aligned}$$

Let us now compute locally $p_1(X_5(\mathbf{d})) \cdot x^3$ at Y, but first one needs to compute $p_{1,S^1}(Y)$ which is given by

$$p_1(Y) + (c_{Y,n_1} x_Y + n_1 z)^2 + \left(c_{Y,1}^2 x_Y^2 + 2 c_{Y,1} x_Y z - 2 c_{Y,2} x_Y^2 + 2z^2\right).$$

Thus the local datum gives us the polynomial in l

$$\begin{aligned}
&\int_{Y^4} p_{1,S^1}(Y)\cdot(r_Y x_Y + l\cdot z)^3 \cdot e_{S^1}(\nu_Y)^{-1} \\
&= \frac{3\left(n_1^2+2\right) r_Y^2}{n_1}\cdot l + \frac{3r_Y\left(-c_{Y,1}n_1^3 + c_{Y,n_1}n_1^2 - 2c_{Y,n_1}\right)}{n_1^2}\cdot l^2 \\
&\quad + \Bigg(\frac{c_{Y,1}^2 n_1^4 + c_{Y,1}^2 n_1^2 - c_{Y,1}c_{Y,n_1}n_1^3}{n_1^3} \\
&\qquad + \frac{-c_{Y,2}n_1^4 - 4c_{Y,2}n_1^2 + 2c_{Y,n_1}^2 + n_1^2\int p_1(Y)}{n_1^3}\Bigg)\cdot l^3
\end{aligned}$$

In the CASE A.3 and CASE A.4, the local data can be computed the same way. The equivariant Euler class of ν_Y are respectively

$$e_{S^1}(\nu_Y) = (c_{Y,n_1}x_Y + n_1 z)\,(c_{Y,n_2}x_Y + n_2 z)\,(c_{Y,1}x_Y + z)$$

and

$$e_{S^1}(\nu_Y) = (c_{Y,n_1}x_Y + n_1 z)\,(c_{Y,n_2}x_Y + n_2 z)\,(c_{Y,n_3}x_Y + n_3 z)\,.$$

The details are left to the reader. However, note that these computations are done in the appendix using *Mathematica*® (cf. Chapter B).

We want now to compute in each of the cases the local datum of x^5 and $p_1(X_5(\mathbf{d}))\cdot x^3$ at Z. Let us first compute the equivariant Euler class of the normal bundle and its inverse in every cases. We then leave the computations of the local datum of x^5 at Z to the reader. Note that one can find the *Mathematica*® computations in the appendix, cf. Chapter B.

CASE A.1.I the normal weights are $(n_1, n_1, 1, 1)$, thus the equivariant Euler class $e_{S^1}(\nu_Z)$ is given by $(c_{Z,n_1}x_Z + n_1 z)\cdot(c_{Z,1}x_Z + z)\cdot(n_1 z)\cdot z$. Then, its inverse is

$$\frac{1}{n_1^2 z^4} - \left(\frac{c_{Z,1}}{n_1^2 z^5} + \frac{c_{Z,n_1}}{n_1^3 z^5}\right)x_Z.$$

CASE A.1.II the normal weights are $(n_1, n_1, n_2, 1)$, thus the equivariant Euler class $e_{S^1}(\nu_Z)$ is given by $(c_{Z,n_1}x_Z + n_1 z)\cdot(c_{Z,n_2}x_Z + n_2 z)\cdot(c_{Z,1}x_Z + z)\cdot n_1 z$. Then, its inverse is

$$\frac{1}{n_1^2 n_2 z^4} - \left(\frac{c_{Z,1}}{n_1^2 n_2 z^5} + \frac{c_{Z,n_1}}{n_1^3 n_2 z^5} + \frac{c_{Z,n_2}}{n_1^2 n_2^2 z^5}\right)x_Z.$$

CASE A.2.I the normal weights are $(n_1, n_1 n_2, 1, 1)$, thus the equivariant Euler class $e_{S^1}(\nu_Z)$ is given by $(c_{Z,n_1}x_Z + n_1 z)\cdot(c_{Z,n_1 n_2}x_Z + n_1 n_2 z)\cdot(c_{Z,1}x_Z + z)\cdot z$. Then its inverse is

$$\frac{1}{n_1^2 n_2 z^4} - \left(\frac{c_{Z,n_1 n_2}}{n_1^3 n_2^2 z^5} + \frac{c_{Z,n_1}}{n_1^3 n_2 z^5} + \frac{c_{Z,1}}{n_1^2 n_2 z^5}\right)x_Z.$$

CASE A.2.II the normal weights are $(n_1, n_1n_2, n_3, 1)$, thus the equivariant Euler class $e_{S^1}(\nu_Z)$ is given by $(c_{Z,n_1}x_Z + n_1z)\cdot(c_{Z,n_1n_2}x_Z + n_1n_2z)\cdot(c_{Z,n_3}x_Z + n_3z)\cdot(c_{Z,1}x_Z + z)$. Then its inverse is

$$\frac{1}{n_1^2n_2n_3z^4} - \left(\frac{c_{Z,1}}{n_1^2n_2n_3z^5} + \frac{c_{Z,n_1}}{n_1^3n_2n_3z^5} + \frac{c_{Z,n_1n_2}}{n_1^3n_2^2n_3z^5} + \frac{c_{Z,n_3}}{n_1^2n_2n_3^2z^5}\right)x_Z.$$

CASE A.3 the normal weights are $(n_1n_2n_3, n_1, n_2, 1)$, thus the equivariant Euler class $e_{S^1}(\nu_Z)$ is given by $(c_{Z,n_1n_2n_3}x_Z + n_1n_2n_3z)\cdot(c_{Z,n_1}x_Z + n_1z)\cdot(c_{Z,n_2}x_Z + n_2z)\cdot(c_{Z,1}x_Z + z)$. Then its inverse is

$$\frac{1}{n_1^2n_2^2n_3z^4} - \left(\frac{c_{Z,1}}{n_1^2n_2^2n_3z^5} + \frac{c_{Z,n_1}}{n_1^3n_2^2n_3z^5} + \frac{c_{Z,n_2}}{n_1^2n_2^3n_3z^5} + \frac{c_{Z,n_1n_2n_3}}{n_1^3n_2^3n_3^2z^5}\right)x_Z.$$

CASE A.4 the normal weights are $(n_1n_2n_3n_4, n_1, n_2, n_3)$, thus the equivariant Euler class $e_{S^1}(\nu_Z)$ is given by $(c_{Z,n_1n_2n_3n_4}x_Z + n_1n_2n_3n_4z)\cdot(c_{Z,n_1}x_Z + n_1z)\cdot(c_{Z,n_2}x_Z + n_2z)\cdot(c_{Z,n_3}x_Z + n_3z)$. Then its inverse is

$$\frac{1}{n_1^2n_2^2n_3^2n_4z^4} - \left(\frac{c_{Z,n_1}}{n_1^3n_2^2n_3^2n_4z^5} + \frac{c_{Z,n_2}}{n_1^2n_2^3n_3^2n_4z^5} + \frac{c_{Z,n_3}}{n_1^2n_2^2n_3^3n_4z^5}\right.$$
$$\left. + \frac{c_{Z,n_1n_2n_3n_4}}{n_1^3n_2^3n_3^3n_4^2z^5}\right)x_Z.$$

We want now to compute locally $p_1(X_5(\mathbf{d}))\cdot x^3$ at Z. Thus one needs to know what the equivariant Pontrjagin class $p_{1,S^1}(Z)$ is in each of the cases. We leave then the computations of the local datum to the reader.

CASE A.1.I The equivariant Pontrjagin class $p_{1,S^1}(Z)$ is given by

$$(c_{Z,1}x_Z + z)^2 + \left(2c_{Z,n_1}n_1x_Zz + 2n_1^2z^2\right) + z^2$$

CASE A.1.II The equivariant Pontrjagin class $p_{1,S^1}(Z)$ is given by

$$(c_{Z,1}x_Z + z)^2 + \left(2c_{Z,n_1}n_1x_Zz + 2n_1^2z^2\right) + (c_{Z,n_2}x_Z + n_2z)^2$$

CASE A.2.I The equivariant Pontrjagin class $p_{1,S^1}(Z)$ is given by

$$(c_{Z,1}x_Z + z)^2 + (c_{Z,n_1}x_Z + n_1z)^2 + (c_{Z,n_1n_2}x_Z + n_1n_2z)^2 + z^2$$

CASE A.2.II The equivariant Pontrjagin class $p_{1,S^1}(Z)$ is given by

$$(c_{Z,1}x_Z + z)^2 + (c_{Z,n_1}x_Z + n_1z)^2$$
$$+ (c_{Z,n_1n_2}x_Z + n_1n_2z)^2 + (c_{Z,n_3}x_Z + n_3z)^2$$

CASE A.3 The equivariant Pontrjagin class $p_{1,S^1}(Z)$ is given by

$$(c_{Z,1}x_Z + z)^2 + (c_{Z,n_1}x_Z + n_1z)^2$$
$$+ (c_{Z,n_2}x_Z + n_2z)^2 + (c_{Z,n_1n_2n_3}x_Z + n_1n_2n_3z)^2$$

CASE A.4 The equivariant Pontrjagin class $p_{1,S^1}(Z)$ is given by

$$\begin{aligned}&(c_{Z,n_1}x_Z+n_1z)^2+(c_{Z,n_2}x_Z+n_2z)^2\\&+(c_{Z,n_3}x_Z+n_3z)^2+(c_{Z,n_1n_2n_3n_4}x_Z+n_1n_2n_3n_4z)^2\end{aligned}$$

Finally, one needs to compute the local datum of x^5 and $p_1(X_5(\mathbf{d}))\cdot x^3$ at P. They are given respectively by

$$-\frac{(a_P+l)^5}{n_{P,1}n_{P,2}\cdots n_{P,5}},\quad\text{and}\quad -\frac{(\sum_i n_{P,i}^2)\cdot(a_P+l)^3}{n_{P,1}n_{P,2}\cdots n_{P,5}}.$$

The later gives us polynomials in l. The coefficients of these polynomials can be easily found by expanding the above equations.

To finish the proof, one needs now to treat each of the cases separately. We have to find a contradiction to the existence of such normal weight structure, or simply show that $\mathbf{d}$ is equal to (1) or (2). We added for the reader the *Mathematica*® computations in each of these cases in the appendix, cf. Chapter B.

CASE A.1.I. We treat this case using the Atiyah-Bott integration formula on x^5 and $p_1(X_5(\mathbf{d}))\cdot x^3$. We extract out of these formulae a system of equations. Furthermore, we complete this system with the equations given by Lemma 6.8. This implies $p_1(X_5(\mathbf{d}))$ to be positive, leaving us with $\mathbf{d}$ equal to (1) or (2).

The integration formula applied on $\int_{X_5(\mathbf{d})} x^5$ gives us a polynomial in l which needs to be constant equal to d. The coefficients in $l^{\leqslant 3}$ give us the system of equations

$$\begin{aligned}
d&=-\frac{a_P^5}{n_1^3}-\frac{a_Z^5c_{Z,1}}{n_1^2}-\frac{a_Z^5c_{Z,n_1}}{n_1^3}+\frac{5a_Z^4r_Z}{n_1^2},\\
0&=-\frac{5a_P^4}{n_1^3}-\frac{5a_Z^4c_{Z,1}}{n_1^2}-\frac{5a_Z^4c_{Z,n_1}}{n_1^3}+\frac{20a_Z^3r_Z}{n_1^2},\\
0&=-\frac{10a_P^3}{n_1^3}-\frac{10a_Z^3c_{Z,1}}{n_1^2}-\frac{10a_Z^3c_{Z,n_1}}{n_1^3}+\frac{30a_Z^2r_Z}{n_1^2},\\
0&=-\frac{10a_P^2}{n_1^3}-\frac{10a_Z^2c_{Z,1}}{n_1^2}-\frac{10a_Z^2c_{Z,n_1}}{n_1^3}+\frac{20a_Zr_Z}{n_1^2}+\frac{10r_Y^2}{n_1}.
\end{aligned}$$

Completing the system with the equations given in Lemma 6.8 gives us four real non-trivial solutions, namely

$$\begin{aligned}
&d=\frac{72a_Z^5}{n_1^3},\quad c_{Z,n_1}=80,\quad c_{Y,n_1}=\pm 9,\quad c_{Z,1}=0,\\
&r_Z=\frac{24a_Z}{n_1},\quad r_Y=\mp\frac{6a_Z}{n_1},\quad a_P=-2a_Z,\quad \varepsilon_Z^N=1,
\end{aligned}$$

and

$$\begin{aligned}
&d=\frac{4a_Z^5}{n_1^3},\quad c_{Z,n_1}=1,\quad c_{Y,n_1}=0,\quad c_{Z,1}=\frac{6}{n_1},\\
&r_Z=\frac{2a_Z}{n_1},\quad r_Y=\mp\frac{2a_Z}{n_1},\quad a_P=-a_Z,\quad \varepsilon_Z^N=-1.
\end{aligned}$$

Furthermore, looking at the constant term given by $\int_{X_5(\mathbf{d})} p_1(X_5(\mathbf{d})) \cdot x^3$ gives us the equation

$$\begin{aligned} p_1 \cdot d = &- \frac{a_P^3\left(3n_1^2+2\right)}{n_1^3} \\ &- \frac{2a_Z^2\left(a_Z c_{Z,1} n_1^3 + a_Z c_{Z,n_1} - 3n_1^3 r_Z - 3n_1 r_Z\right)}{n_1^3} \end{aligned}$$

where $p_1 = 6 - \sum_i d_i^2$. Plugging the above results gives us that

$$p_1 = \frac{7n_1^2}{3a_Z^2}, \quad \text{or} \quad p_1 = \frac{3\left(n_1^2+4\right)}{4a_Z^2}.$$

This implies that $\mathbf{d}$ is equal to (1) or (2) since it needs to be positive. One can in fact go further with these equations and show that there is no real solutions for a_Z and n_1 making $p_1 = 6$ and $\mathbf{d} = (1)$, or $p_1 = 3$ and $\mathbf{d} = (2)$.

CASE A.1.II. We treat this case using the Atiyah-Bott integration formula on x^5. Like in the previous case, we will extract a system of equations. We also use the equations given in Lemmas 6.8 and 6.7 in order to show that the d_i's need to be equal to 0, contradicting the existence of an action with such normal weights.

The integration formula applied on $\int_{X_5(\mathbf{d})} x^5$ gives us a polynomial in l which needs to be constant equal to d. The coefficients in $l^{\leqslant 3}$ give us the system of equations

$$\begin{aligned} d &= -\frac{a_P^5}{n_1^3 n_2^2} - \frac{a_Z^5 c_{Z,1}}{n_1^2 n_2} - \frac{a_Z^5 c_{Z,n_1}}{n_1^3 n_2} - \frac{a_Z^5 c_{Z,n_2}}{n_1^2 n_2^2} + \frac{5a_Z^4 r_Z}{n_1^2 n_2}, \\ 0 &= -\frac{5a_P^4}{n_1^3 n_2^2} - \frac{5a_Z^4 c_{Z,1}}{n_1^2 n_2} - \frac{5a_Z^4 c_{Z,n_1}}{n_1^3 n_2} - \frac{5a_Z^4 c_{Z,n_2}}{n_1^2 n_2^2} + \frac{20a_Z^3 r_Z}{n_1^2 n_2}, \\ 0 &= -\frac{10a_P^3}{n_1^3 n_2^2} - \frac{10a_Z^3 c_{Z,1}}{n_1^2 n_2} - \frac{10a_Z^3 c_{Z,n_1}}{n_1^3 n_2} - \frac{10a_Z^3 c_{Z,n_2}}{n_1^2 n_2^2} + \frac{30a_Z^2 r_Z}{n_1^2 n_2}, \\ 0 &= -\frac{10a_P^2}{n_1^3 n_2^2} - \frac{10a_Z^2 c_{Z,1}}{n_1^2 n_2} - \frac{10a_Z^2 c_{Z,n_1}}{n_1^3 n_2} - \frac{10a_Z^2 c_{Z,n_2}}{n_1^2 n_2^2} + \frac{20a_Z r_Z}{n_1^2 n_2} \\ &\quad + \frac{10r_Y^2}{n_1} \end{aligned}$$

If we now add to this the equations given in Lemmas 6.7 and 6.8, we get a unique solution

$$\begin{gathered} d = 0, \quad r_Z = 0, \quad r_Y = 0, \quad a_P = a_Z, \quad c_{Y,n_1} = 0, \quad c_{Z,n_1} = -\varepsilon_Z^N, \\ c_{Z,n_2} = \varepsilon_P^M \varepsilon_Z^M, \quad c_{Z,1} = -\frac{\varepsilon_P^M \varepsilon_Z^N n_1 + \varepsilon_Z^N \varepsilon_Z^M - \varepsilon_Z^M n_2}{\varepsilon_Z^N \varepsilon_Z^M n_1 n_2}, \end{gathered}$$

contradicting the existence of an action with such normal weights.

CASE A.2.I. We treat this case using the Atiyah-Bott integration formula on x^5 and $p_1(X_5(\mathbf{d})) \cdot x^3$, together with Lemma 6.9. We will have that p_1 needs to be positive, implying that $\mathbf{d}$ is equal to (1) or (2).

As above, we apply the integration formula on $\int_{X_5(\mathbf{d})} x^5$. This gives us a polynomial in l which needs to be constant equal to d. The coefficients in $l^{\leqslant 2}$ give us the system of equations

$$\begin{aligned}
d &= -\frac{a_P^5}{n_1^3 n_2^2} - \frac{a_Z^5 c_{Z,1}}{n_1^2 n_2} - \frac{a_Z^5 c_{Z,n_1}}{n_1^3 n_2} - \frac{a_Z^5 c_{Z,n_1 n_2}}{n_1^3 n_2^2} + \frac{5a_Z^4 r_Z}{n_1^2 n_2},\\
0 &= -\frac{5a_P^4}{n_1^3 n_2^2} - \frac{5a_Z^4 c_{Z,1}}{n_1^2 n_2} - \frac{5a_Z^4 c_{Z,n_1}}{n_1^3 n_2} - \frac{5a_Z^4 c_{Z,n_1 n_2}}{n_1^3 n_2^2} + \frac{20a_Z^3 r_Z}{n_1^2 n_2},\\
0 &= -\frac{10a_P^3}{n_1^3 n_2^2} - \frac{10a_Z^3 c_{Z,1}}{n_1^2 n_2} - \frac{10a_Z^3 c_{Z,n_1}}{n_1^3 n_2} - \frac{10a_Z^3 c_{Z,n_1 n_2}}{n_1^3 n_2^2} + \frac{30a_Z^2 r_Z}{n_1^2 n_2}.
\end{aligned}$$

We can solve this system to get the unique solution

$$\begin{gathered}
d = -\frac{a_P^3 (a_P - a_Z)^2}{n_1^3 n_2^2}, \quad r_Z = -\frac{a_P^3 (a_Z - a_P)}{a_Z^3 n_1 n_2},\\
c_{Z,1} = -\frac{-3a_P^4 + 4a_P^3 a_Z + a_Z^4 c_{Z,n_1} n_2 + a_Z^4 c_{Z,n_1 n_2}}{a_Z^4 n_1 n_2}.
\end{gathered}$$

On the other hand, Lemma 6.9 gives us that $r_Z = \pm(a_P - a_Z)/(n_1 n_2)$. This implies $a_Z = -a_P$ since a_Z cannot be equal to a_P. We can now use the integration formula on $p_1(X_5(\mathbf{d})) \cdot x^3$. This will give us as usually a polynomial in l. Identifying the constant term to $p_1 \cdot d$ gives us an equation, which once solved gives us two solutions for p_1, namely

$$p_1 = \frac{3\left(n_1^2 n_2^2 + 4\right)}{4a_Z^2}, \quad \text{or} \quad p_1 = \frac{3n_1^2 n_2^2 + 4n_1^2 + 8}{4a_Z^2}.$$

This implies that $p_1 = 6 - \sum_i d_i^2$ is positive, thus $\mathbf{d}$ is equal to (1) or (2).

CASE A.2.II. This case is easily treated using simply Lemmas 6.7 and 6.9. These Lemmas give us respectively that

$$r_Z = \pm\frac{a_P - a_Z}{n_1 n_2}, \quad \text{and} \quad r_Z = \pm\frac{a_P - a_Z}{n_3}.$$

Thus $n_1 n_2 = n_3$. On the other hand, we have that $(n_1, n_3) = 1$ and (n_2, n_3). This implies that there is no action on $X_5(\mathbf{d})$ with such normal weights.

CASE A.3. This case is very similar to CASE A.2.I. We use the integration formula together with Lemmas 6.7 and 6.10 to obtain that

$$p_1 = \frac{3\left(5n_1^2 n_2^2 n_3^2 + 4n_1^2 + 4n_2^2\right)}{4a_Z^2}, \quad \text{or} \quad p_1 = \frac{7n_1^2 n_2^2 n_3^2 + 8n_1^2 + 8n_2^2}{4a_Z^2}.$$

This implies that $p_1 = 6 - \sum_i d_i^2$ is positive, thus $\mathbf{d}$ is equal to (1) or (2).

CASE A.4. This case is very similar to CASE A.3. The integration formula together with Lemmas 6.7 and 6.11 gives us

$$p_1 = \frac{3n_1^2n_2^2n_3^2n_4^2 + 4n_1^2 + 4n_2^2 + 4n_3^2}{4a_Z^2}.$$

This leads to the same conclusion as above. □

Let us now focus our interest to the CASE B, i.e. where there exists $N^6 \subset X_5(\mathbf{d})^{\mathbb{Z}/p}$ containing Y and P for a certain prime number p. Therefore, there exists a normal weight at Y, say $n_{Y,1}$, and exactly three normal weights at P say $n_{P,1}$, $n_{P,2}$ and $n_{P,3}$, divisible by p. We prove that they are all equal.

Note first that N^6 is a connected component of $X_5(\mathbf{d})^{\mathbb{Z}/n_{Y,1}}$, thus $n_{Y,1}$ divides the $n_{P,i}$'s. Therefore, let us assume there exists a normal weight $n_{P,i} > n_{Y,1}$. Then there exists a closed smooth and orientable connected component $F \subset X_5(\mathbf{d})^{\mathbb{Z}/n_{P,i}}$ in N and admitting an S^1-action with P as unique isolated fixed point. This contradicts the ending result of Section 3.3. Thus, the normal weights $n_{Y,1}$, $n_{P,1}$, $n_{P,2}$ and $n_{P,3}$ are all equal. Furthermore, by the pigeon hole principle, the other weights at Y need to be equal to 1.

We are now left with two possibilities for the remaining normal weights at Z and P. Either there exists $M^4 \subset X_5(\mathbf{d})^{\mathbb{Z}/n_{Z,1}}$ containing Z and P for a certain integer $n_{Z,1} > 1$, or the actions is semi-free around Z. Thus, we have the following sub cases.

CASE B.1. The normal weights are

AT Y. $(n_1, 1, 1)$

AT Z. $(1, 1, 1, 1)$

AT P. $(n_1, n_1, n_1, 1, 1)$

CASE B.2. The normal weights are

AT Y. $(n_1, 1, 1)$

AT Z. $(n_2, 1, 1, 1)$

AT P. $(n_1, n_1, n_1, n_2, n_2)$ where $(n_1, n_2) = 1$.

Before to treat each of these cases, we discuss the information one can get with a component $N^6 \subset X_5(\mathbf{d})^{\mathbb{Z}/n}$ containing Y and P as above. Let us orient N^6 by means of the orientation of Y and the normal bundle of Y in N. Thus, with this fixed orientation we can assume that in N, the component Y has orientation +1, and P has orientation −1.

LEMMA 6.13. *Let N^6 be an closed smooth oriented manifold equipped with a smooth S^1-action such that the fixed point set consists of $Y^4 \cup P^0$ as above. Then*

$$r_Y = \pm \frac{a_P}{n_1}, \quad \textit{and} \quad c_1(\nu_Y) = \mp x_Y$$

where ν_Y denotes here the normal bundle of Y in N.

PROOF. The proof follows immediately by computing $\int_N x|_N^3$ locally at Y and P. Indeed, the local datum of $x|_N^3$ at Y is given by

$$\frac{3r_Y^2}{n_1} \cdot \iota - \frac{3c_{Y,n_1} r_Y}{n_1^2} \cdot \iota^2 + \frac{c_{Y,n_1}^2}{n_1^3} \cdot \iota^3.$$

Similarly, at P the local datum is given by

$$-\frac{a_P^3}{n_1^3} - \frac{3a_P^2}{n_1^3} \cdot \iota - \frac{3a_P}{n_1^3} \cdot \iota^2 - \frac{1}{n_1^3} \cdot \iota^3.$$

This gives us the system

$$\begin{aligned} 0 &= \frac{3r_Y^2}{n_1} - \frac{3a_P^2}{n_1^3}, \\ 0 &= -\frac{3a_P}{n_1^3} - \frac{3c_{Y,n_1} r_Y}{n_1^2}, \\ 0 &= \frac{c_{Y,n_1}^2}{n_1^3} - \frac{1}{n_1^3}. \end{aligned}$$

One can solve this system to get exactly two solutions in r_Y and c_{Y,n_1} which are the desired equalities. □

LEMMA 6.14. *Let $X_5(\mathbf{d})$ be a complete intersection with a smooth circle action such that the fixed point components are $X_5(\mathbf{d})^{S^1} = Y^4 \cup Z^2 \cup P^0$. If the action admits a fixed point component $N^6 \subset X_5(\mathbf{d})^{\mathbb{Z}/p}$ containing $Y^4 \cup P^0$ for a prime number p, then $\mathbf{d}$ is equal to (1).*

PROOF. To prove this Lemma, we need to compute locally

$$\int_{X_5(\mathbf{d})} x^5, \quad \text{and} \quad \int_{X_5(\mathbf{d})} p_1(X_5(\mathbf{d})) \cdot x^3$$

at Y, Z and P for CASE B.1 and CASE B.2. Furthermore, we use Lemmas 6.7 and 6.13.

Note that the local datum of x^5 and $p_1(X_5(\mathbf{d})) \cdot x^3$ at Y is exactly the same as in the proof of Lemma 6.12 since the normal weights at Y are $(n_1, 1, 1)$. Thus, one needs only to compute the local datum of x^5 and $p_1(X_5(\mathbf{d})) \cdot x^3$ at Z and P. We start this process by computing the equivariant Euler class of ν_Z in both cases.

CASE B.1. The normal weights are $(1,1,1,1)$, thus the equivariant Euler class $e_{S^1}(\nu_Z)$ is given by $(c_{Z,1}x_Z + z)\cdot z^3$. Then, its inverse is

$$\frac{1}{z^4} - \frac{c_{Z,1}}{z^5}\cdot x_Z.$$

CASE B.2. The normal weights are $(n_2,1,1,1)$, thus the equivariant Euler class $e_{S^1}(\nu_Z)$ is given by $(c_{Z,n_2}x_Z + n_2 z)\cdot(c_{Z,1}x_Z + z)\cdot z^2$. Then, its inverse is

$$\frac{1}{n_2 z^4} - \left(\frac{c_{Z,1}}{n_2 z^5} + \frac{c_{Z,n_2}}{n_2^2 z^5}\right) x_Z.$$

We leave the computation of the local datum of x^5 at Z to the reader.

Like in the proof of Lemma 6.12 one needs first compute the equivariant Pontrjagin class $p_{1,S^1}(Z)$ in order to compute the local datum of $p_1(X_5(\mathbf{d}))\cdot x^3$ at Z. We let the reader figure out by himself these computations. These computations can also be found in the appendix, cf. Chapter B. To finish the proof, one needs now to treat the two cases separately.

CASE B.1. We treat this case using the Atiyah-Bott integration formula on x^5 and $p_1(X_5(\mathbf{d}))\cdot x^3$. We will extract a system of equations that we solve using the equations given in Lemma 6.13. This will imply that $\mathbf{d}$ needs to be equal to (1).

Indeed, the Atiyah-Bott integration formula applied on $\int_{X_5(\mathbf{d})} x^5$ gives us the polynomial in l. Since this polynomial needs to be constant equal to d, one gets a system of equations by looking at the coefficients in $l^{\leqslant 3}$

$$\begin{aligned}
d &= -\frac{a_P^5}{n_1^3} - a_Z^5 c_{Z,1} + 5a_Z^4 r_Z\\
0 &= -\frac{5a_P^4}{n_1^3} - 5a_Z^4 c_{Z,1} + 20a_Z^3 r_Z\\
0 &= -\frac{10a_P^3}{n_1^3} - 10a_Z^3 c_{Z,1} + 30a_Z^2 r_Z\\
0 &= -\frac{10a_P^2}{n_1^3} - 10a_Z^2 c_{Z,1} + 20a_Z r_Z + \frac{10r_Y^2}{n_1}.
\end{aligned}$$

Furthermore, we also apply the Atiyah-Bott integration formula to $\int_{X_5(\mathbf{d})} p_1(X_5(\mathbf{d}))x^3$. This will be again a polynomial in l which needs to be constant equal to $p_1\cdot d$, where $p_1 = 6 - \sum d_i^2$. In particular, the constant term gives us that

$$p_1\cdot d = -\frac{a_P^3\left(3n_1^2+2\right)}{n_1^3} - 2a_Z^3 c_{Z,1} + 12a_Z^2 r_Z.$$

Together with Lemma 6.13, one gets only two non-trivial solutions, namely

$$p_1 = \frac{3\left(n_1^2-2\right)}{a_Z^2}, \quad d = -\frac{8a_Z^5}{n_1^3}, \quad c_{Z,1} = \frac{16}{n_1^3},$$
$$c_{Y,n_1} = 1, \quad r_Z = \frac{8a_Z}{n_1^3}, \quad r_Y = \pm\frac{2a_Z}{n_1}, \quad a_P = 2a_Z.$$

Since $c_{Z,1}$ is an integer and n_1 is also an integer strictly greater than 1, we have that $n_1 = 2$ and that $p_1 = 6/a_Z^2$. This let us no other choice than $a_Z = 1$ and therefore $\mathbf{d} = (1)$. We added for the reader the *Mathematica*® computations for this case in the appendix, cf. Chapter B.

CASE B.2. As above, we treat this case using the Atiyah-Bott integration formula on x^5 and $p_1(X_5(\mathbf{d}))\cdot x^3$ together with Lemmas 6.7 and 6.13.

Indeed, if we apply the integration formula to $\int_{X_5(\mathbf{d})} x^5$ we get the system of equations (looking at the coefficients in $l^{\leqslant 3}$)

$$d = -\frac{a_P^5}{n_1^3 n_2^2} - \frac{a_Z^5 c_{Z,1}}{n_2} - \frac{a_Z^5 c_{Z,n_2}}{n_2^2} + \frac{5a_Z^4 r_Z}{n_2},$$
$$0 = -\frac{5a_P^4}{n_1^3 n_2^2} - \frac{5a_Z^4 c_{Z,1}}{n_2} - \frac{5a_Z^4 c_{Z,n_2}}{n_2^2} + \frac{20a_Z^3 r_Z}{n_2},$$
$$0 = -\frac{10a_P^3}{n_1^3 n_2^2} - \frac{10a_Z^3 c_{Z,1}}{n_2} - \frac{10a_Z^3 c_{Z,n_2}}{n_2^2} + \frac{30a_Z^2 r_Z}{n_2},$$
$$0 = -\frac{10a_P^2}{n_1^3 n_2^2} - \frac{10a_Z^2 c_{Z,1}}{n_2} - \frac{10a_Z^2 c_{Z,n_2}}{n_2^2} + \frac{20a_Z r_Z}{n_2} + \frac{10r_Y^2}{n_1},$$

Solving the system in the variables d, $c_{Z,1}$, r_Y, and r_Z, gives us the two solutions

$$d = -\frac{a_P^3(a_P-a_Z)^2}{n_1^3 n_2^2}, \quad c_{Z,1} = -\frac{-3a_P^4 + 4a_P^3 a_Z + a_Z^4 c_{Z,n_2} n_1^3}{a_Z^4 n_1^3 n_2},$$
$$r_Y = \pm\frac{a_P(a_P-a_Z)}{a_Z n_1 n_2}, \quad r_Z = \frac{a_P^3(a_P-a_Z)}{a_Z^3 n_1^3 n_2}.$$

Furthermore, Lemmas 6.7 and 6.13 gives us

$$r_Y = \pm\frac{a_P}{n_1}, \quad \text{and} \quad r_Z = \pm\frac{a_P-a_Z}{n_2}.$$

Thus, this gives us $a_P - a_Z = \pm a_Z n_2$, and $a_P = \pm a_Z n_1$, which implies that $n_2 = n_1 + 1$ or $n_1 = n_2 + 1$, since n_1 and n_2 needs to be positive.

We now apply the integration formula to $\int_{X_5(\mathbf{d})} p_1(X_5(\mathbf{d}))\cdot x^3$ we get the equation

$$p_1 \cdot d = -\frac{a_P^3\left(3n_1^2+2n_2^2\right)}{n_1^3 n_2^2} - a_Z^3 c_{Z,1} n_2 - \frac{a_Z^3 c_{Z,1}}{n_2}$$
$$- \frac{3a_Z^3 c_{Z,n_2}}{n_2^2} + a_Z^3 c_{Z,n_2} + 3a_Z^2 n_2 r_Z + \frac{9a_Z^2 r_Z}{n_2}$$

If we now solve the above system, we get that $p_1 = 6/a_Z^2$ and $d = \pm a_Z^5$ which implies that $a_Z = \pm 1$. The solution reflects in fact the linear situation, i.e. the linear action on $\mathbb{C}P^5$. We added for the reader the *Mathematica*® computations for this case in the appendix, cf. Chapter B. □

We can now assume that for any integer n, the components of $X_5(\mathbf{d})^{\mathbb{Z}/n}$ are of dimension less or equal to 4. Therefore, we consider the case where there exists a prime number p and a component $N^4 \subset X_5(\mathbf{d})^{\mathbb{Z}/p}$. In virtue of Corollary 4.7, N^4 needs to contain Z and P. Note that in this case, the normal weights falls into two situations:

CASE C.1. The normal weights are

AT Y. $(1, 1, 1)$

AT Z. $(n_1, 1, 1, 1)$

AT P. $(n_1, n_1, 1, 1, 1)$.

CASE C.2. The normal weights are

AT Y. $(1, 1, 1)$

AT Z. $(n_1, n_2, 1, 1)$

AT P. $(n_1, n_1, n_2, n_2, 1)$, where $(n_1, n_2) = 1$.

Furthermore, CASE C.2. cannot occur simply by using Lemma 6.7. In this case, it is easy to show that $n_1 = n_2$ contradicting $(n_1, n_2) = 1$.

LEMMA 6.15. *Let $X_5(\mathbf{d})$ be a complete intersection with a smooth circle action such that the fixed point components are $X_5(\mathbf{d})^{S^1} = Y^4 \cup Z^2 \cup P^0$. If the action admits a fixed point component $N^4 \subset X_5(\mathbf{d})^{\mathbb{Z}/p}$ containing $Z^2 \cup P^0$ for a prime number p, then $\mathbf{d}$ is equal to (1).*

PROOF. As above, we prove this lemma using Atiyah-Bott integration formula on $\int_{X_5(\mathbf{d})} x^5$ and $\int_{X_5(\mathbf{d})} p_1(X_5(\mathbf{d})) \cdot x^3$. Note that the local datum at Y has already been computed in the proof of Lemma 6.5. We leave the remaining computations of the local data to the reader. We added for the reader the *Mathematica*® computations for this case in the appendix, cf. Chapter B.

The later computations gives us the following system of equations

$$\begin{aligned}
d &= -\frac{a_P^5}{n_1^2} - \frac{a_Z^5 c_{Z,1}}{n_1} - \frac{a_Z^5 c_{Z,n_1}}{n_1^2} + \frac{5a_Z^4 r_Z}{n_1},\\
0 &= -\frac{5a_P^4}{n_1^2} - \frac{5a_Z^4 c_{Z,1}}{n_1} - \frac{5a_Z^4 c_{Z,n_1}}{n_1^2} + \frac{20a_Z^3 r_Z}{n_1},\\
0 &= -\frac{10a_P^3}{n_1^2} - \frac{10a_Z^3 c_{Z,1}}{n_1} - \frac{10a_Z^3 c_{Z,n_1}}{n_1^2} + \frac{30a_Z^2 r_Z}{n_1},\\
p_1 \cdot d &= -\frac{a_P^3\left(2n_1^2+3\right)}{n_1^2} - a_Z^3 c_{Z,1} n_1 - \frac{a_Z^3 c_{Z,1}}{n_1}\\
&\qquad - \frac{3a_Z^3 c_{Z,n_1}}{n_1^2} + a_Z^3 c_{Z,n_1} + 3a_Z^2 n_1 r_Z + \frac{9a_Z^2 r_Z}{n_1}.
\end{aligned}$$

In addition to that we have equations given by Lemma 6.7. This gives us the solution

$$p_1 = \frac{3\left(n_1^2+4\right)}{4a_Z^2}, \quad d = \frac{4a_Z^5}{n_1^2}, \quad c_{Z,1} = \frac{6}{n_1},$$
$$c_{Z,n_1} = 1, \quad r_Z = \frac{2a_Z}{n_1}, \quad a_P = -a_Z.$$

Since $c_{Z,1}$ needs to be an integer, n_1 is equal to 2, 3 or 6. The above solution makes sense only in the case $n_1 = 2$. Thus, $a_Z = 1$ and $\mathbf{d} = (1)$. In fact, the reader can see that the above solution reflects the situation of a linear action on $\mathbb{C}P^5$. □

PROOF OF THEOREM D. PART 2. In virtue of Lemma 6.5, the action cannot be semi-free around the fixed point component. This implies that there is a fixed point component $N \subset X_5(\mathbf{d})^{\mathbb{Z}/p}$ for a certain prime number p. In virtue of Lemmas 6.6, 6.12, 6.14 and 6.15 we have that in all of these situations $\mathbf{d}$ is equal to (1) or (2). □

6.3 THE CASE WHERE $X_5(\mathbf{d})^{S^1} = Y^4 \cup P_0^0 \cup P_1^0 \cup P_2^0$

Like in the previous section, we lift the S^1-action to γ. The later being the restriction to $X_5(\mathbf{d})$ of the dual Hopf line bundle on $\mathbb{C}P^{5+r}$. We can then consider the Hopf weights at the fixed point components. Since the lift is not unique, we choose the Hopf weights at Y,P_0, P_1 and P_2 to be respectively l, $a_0 + l$, $a_1 + l$ and $a_2 + l$, where $l \in \mathbb{Z}$.

We denote by ν_Y, ν_0, ν_1 and ν_2 the normal bundles of Y, P_0, P_1 and P_2 respectively. We orient these bundles by mean of the S^1-action. The orientation on ν_Y, ν_0, ν_1 and ν_2 are chosen such that their normal weights $n_{Y,j}$, $n_{Z,j}$ and $n_{P,j}$'s are all positive. Furthermore, we choose the orientation on Y, P_0, P_1 and P_2 in order to make it compatible with the orientation of $X_n(\mathbf{d})$ and their respective normal bundles. By Proposition A (cf. Section 4.2) the signature of Y has to be ± 1. Up to changing the S^1-action with the inverse S^1-action, let us fix without loss of generality the signature of Y to be equal to $+1$. Furthermore, since the signature needs to be equal to zero, there is exactly one isolated fixed point P_0, P_1 or P_2 which needs to be positively oriented, while the other needs to be negatively oriented. Let us assume P_0 to be positively oriented and have signature $+1$.

We denote by x_Y a fixed generator of $H^2(Y;\mathbb{Z})$ such that $\int_Y x_Y^2 = 1$. Using this notation, we can write $x|_Y = r_Y x_Y$, where $r_Y \in \mathbb{Z}$.

Furthermore, we apply the splitting principle to ν_Y, ν_0, ν_1 and ν_2. We denote by $\{y_{Y,j} + n_{Y,j} \cdot z\}$ and $\{n_{i,j} \cdot z\}$ the equivariant normal roots at Y, and at the P_i's respectively.

LEMMA 6.16. *Let $X_5(\mathbf{d})$ be a complete intersection with a smooth circle action such that the fixed point components are $X_5(\mathbf{d})^{S^1} = Y^4 \cup P_0^0 \cup P_1^0 \cup P_2^0$. Then the action is not semi-free around the fixed point components.*

PROOF. Let us assume by contradiction all normal weights to be equal to 1. The idea of the proof is to compute

$$\int_{X_5(\mathbf{d})} x^5, \quad \int_{X_5(\mathbf{d})} p_1(X_5(\mathbf{d})) \cdot x^3 \quad \text{and} \quad \int_{X_5(\mathbf{d})} s_2(X_5(\mathbf{d})) \cdot x$$

using the Atiyah-Bott integration formula.

Note that most of the computations have already been done in the proof of Lemma 6.5. Thus, we use the same notations for the Chern classes of Y and leave these computations to the reader. Furthermore, we added for the reader the *Mathematica*® computations for this case in the appendix, cf. Chapter B.

Let us summarize what one should get. First, by integrating $\int_{X_5(\mathbf{d})} x^5$ locally at the fixed point components we get the system of equations

$$\begin{aligned}
d &= a_0^5 - a_1^5 - a_2^5, \\
0 &= 5\left(a_0^4 - a_1^4 - a_2^4\right), \\
0 &= 10\left(a_0^3 - a_1^3 - a_2^3\right), \\
0 &= 10\left(a_0^2 - a_1^2 - a_2^2 + r_Y^2\right), \\
0 &= 5\left(a_0 - a_1 - a_2 - c_{Y,1} r_Y\right), \\
0 &= c_{Y,1}^2 - c_{Y,2} - 1.
\end{aligned}$$

Similarly, by integrating $\int_{X_5(\mathbf{d})} p_1(X_5(\mathbf{d})) \cdot x^3$ locally, one obtains the system of equations

$$\begin{aligned}
p_1 \cdot d &= 5\left(a_0^3 - a_1^3 - a_2^3\right), \\
0 &= 3\left(5a_0^2 - 5a_1^2 - 5a_2^2 + 3r_Y^2\right), \\
0 &= 3\left(5a_0 - 5a_1 - 5a_2 - c_{Y,1} r_Y\right), \\
0 &= 2c_{Y,1}^2 - 5c_{Y,2} + p_1(Y) - 5.
\end{aligned}$$

Finally, by integrating $\int_{X_5(\mathbf{d})} s_2(X_5(\mathbf{d})) \cdot x$ locally, we have the system of equations

$$\begin{aligned}
\int_{X_5(\mathbf{d})} s_2(X_5(\mathbf{d})) \cdot x &= 5a_0 - 5a_1 - 5a_2 + c_{Y,1} r_Y, \\
0 &= 5\left(c_{Y,1}^2 - 3c_{Y,2} - 1\right).
\end{aligned}$$

We will now find a contradiction. First, remark that the equations

$$\begin{aligned}
0 &= c_{Y,1}^2 - c_{Y,2} - 1, \\
0 &= 5\left(c_{Y,1}^2 - 3c_{Y,2} - 1\right),
\end{aligned}$$

induces $c_{Y,1} = \pm 1$, and $c_{Y,2} = 0$. Adding to this the equations

$$\begin{aligned}
0 &= 10\left(a_0^2 - a_1^2 - a_2^2 + r_Y^2\right), \\
0 &= 5\left(a_0 - a_1 - a_2 - c_{Y,1} r_Y\right), \\
0 &= 3\left(5a_0 - 5a_1 - 5a_2 - c_{Y,1} r_Y\right),
\end{aligned}$$

gives us that r_Y is equal to $1/5$ which is an obvious contradiction.

Let us mention that there is multiple ways to play with these equations and find a contradiction. For instance, another way would be to solve the system of equations

$$\begin{aligned}
d &= a_0^5 - a_1^5 - a_2^5, \\
0 &= 5\left(a_0^4 - a_1^4 - a_2^4\right), \\
0 &= 10\left(a_0^3 - a_1^3 - a_2^3\right), \\
0 &= 10\left(a_0^2 - a_1^2 - a_2^2 + r_Y^2\right), \\
0 &= 5\left(a_0 - a_1 - a_2 - c_{Y,1} r_Y\right), \\
0 &= c_{Y,1}^2 - c_{Y,2} - 1, \\
0 &= 5\left(c_{Y,1}^2 - 3c_{Y,2} - 1\right),
\end{aligned}$$

which induces $d = 0$. □

LEMMA 6.17. *Let $X_5(\mathbf{d})$ be a complete intersection with a smooth circle action such that the fixed point components are $X_5(\mathbf{d})^{S^1} = Y^4 \cup P_0^0 \cup P_1^0 \cup P_2^0$. If the action admits a fixed point component $N^8 \subset X_5(\mathbf{d})^{\mathbb{Z}/p}$ for a prime number p, then $\mathbf{d} = (1)$ or (2).*

PROOF. Since $\operatorname{sign}(N) = \operatorname{sign}(N^{S^1})$, the signature of N satisfies

$$|\operatorname{sign}(N)| \leqslant 4.$$

The exact value of the signature depends on which S^1-fixed point components are contained in N and their orientations. Since N is of codimension 2, we can think of the Poincaré-dual class of N, namely $\eta x \in H^2(X_5(\mathbf{d}); \mathbb{Z})$, as a virtual complete interesection in $X_5(\mathbf{d})$ (cf. [28, Chapter 3]). Therefore, N represents a virtual complete intersection in $\mathbb{C}P^{5+r}$ with multidegree $(d_1, \ldots, d_r, |\eta|)$. By [35], in complex dimension four only the complex projective space and the complex quadric have $|\operatorname{sign}(N)| \leqslant 4$. Thus we have that $\mathbf{d} = (1)$ or (2). □

Recall that when we have $N \subset X_5(\mathbf{d})^{\mathbb{Z}/p}$ containing exactly two isolated S^1-fixed point, then their normal weights are equal. This was the statement of Lemma 5.4. We now investigate what happens when we have an $N^4 \subset X_5(\mathbf{d})^{\mathbb{Z}/p}$ containing exactly three fixed point component.

Let us denote by $n_{i,1}$ and $n_{i,2}$ the two normal weights at P_i in N. We orient this normal bundle in a way such that the normal weights are all positive. Furthermore, one can see easily using the equivariant signature formula that N must have signature $\pm 1 = \varepsilon_N$. Making the orientation of the isolated points compatible with N and their respective normal bundle, we get that there is exactly one isolated fixed point, say P_0, which is of orientation $-\varepsilon_N$. The other fixed point, P_1 and P_2 will have orientation ε_N. Let us write ε_i the orientation of P_i.

LEMMA 6.18. *Let N^4 be an closed smooth oriented manifold equipped with a smooth S^1-action such that the fixed point set consists of three isolated fixed point $P_0^0 \cup P_1^0 \cup P_2^0$ with the above orientation. Then $n_{0,1} = n_{1,1}$, $n_{0,2} = n_{2,1}$ and $n_{1,2} = n_{2,2} = n_{0,1} + n_{0,2}$.*

PROOF. To prove this result, we use the equivariant signature formula. Let λ be a topological generator of S^1, thus we have the equality

$$\mathrm{sign}(N) = \mu(\lambda, P_0) + \mu(\lambda, P_1) + \mu(\lambda, P_2),$$

where $\mu(\lambda, P_i)$ is given by

$$\varepsilon_i \cdot \frac{1+\lambda^{-n_{i,1}}}{1-\lambda^{-n_{i,1}}} \cdot \frac{1+\lambda^{-n_{i,2}}}{1-\lambda^{-n_{i,2}}}.$$

If we now develop the above power series in λ, then we have for $\mu(\lambda, P_i)$ the power series

$$\varepsilon_i \cdot (1 + 2\sum_{r\geqslant 1} \lambda^{rn_{i,1}}) \cdot (1 + 2\sum_{s\geqslant 1} \lambda^{sn_{i,2}}).$$

Thus, the above equation gives us the equalities of power series

$$\begin{aligned}
&\sum_{r\geqslant 1} \lambda^{rn_{0,1}} + \sum_{s\geqslant 1} \lambda^{sn_{0,2}} + 2\cdot \sum_{r,s\geqslant 1} \lambda^{rn_{0,1}+sn_{0,2}} \\
&= \sum_{k\geqslant 1} \lambda^{kn_{1,1}} + \sum_{l\geqslant 1} \lambda^{ln_{1,2}} + 2\cdot \sum_{k,l\geqslant 1} \lambda^{kn_{1,1}+ln_{1,2}} \\
&\quad + \sum_{u\geqslant 1} \lambda^{un_{2,1}} + \sum_{v\geqslant 1} \lambda^{vn_{2,2}} + 2\cdot \sum_{u,v\geqslant 1} \lambda^{un_{2,1}+vn_{2,2}}.
\end{aligned}$$

Let us assume without loss of generality that $n_{i,1} \leqslant n_{i,2}$ for any $i = 0,1,2$ and that $n_{1,1} \leqslant n_{2,1}$. Therefore, one can see by looking at the smallest coefficient that $n_{0,1} = n_{1,1}$. Furthermore, we cannot have $n_{0,2} = n_{1,2}$ because then the local datum at P_0 simply cancels with the local datum at P_1, leaving us the local datum at P_2. Thus, $n_{0,2} = n_{2,1}$. If we now rewrite the above equality and clean it up with these relations, we are left with

$$\begin{aligned}
2\cdot \sum_{r,s\geqslant 1} \lambda^{rn_{0,1}+sn_{0,2}} &= \sum_{l\geqslant 1} \lambda^{ln_{1,2}} + 2\cdot \sum_{k,l\geqslant 1} \lambda^{kn_{0,1}+ln_{1,2}} \\
&\quad + \sum_{v\geqslant 1} \lambda^{vn_{2,2}} + 2\cdot \sum_{u,v\geqslant 1} \lambda^{un_{0,2}+vn_{2,2}}.
\end{aligned}$$

Since the coefficients on the left hand side are divisible by 2, we need $n_{1,2}$ to be equal to $n_{2,2}$, thus $n_{1,2} = n_{2,2}$ gives us the equation

$$\sum_{r,s\geqslant 1} \lambda^{rn_{0,1}+sn_{0,2}} = \sum_{k,l\geqslant 1} \lambda^{kn_{0,1}+ln_{1,2}} + \sum_{l\geqslant 1} \lambda^{ln_{1,2}} + \sum_{u,v\geqslant 1} \lambda^{un_{0,2}+vn_{1,2}}.$$

Finally, we can see that $n_{1,2}$ needs to be equal to a linear combination $r \cdot n_{0,1} + s \cdot n_{0,2}$. It is now easy to see that $n_{1,2}$ needs to be equal to $n_{0,1} + n_{0,2}$ in order that all the terms on the left hand side cancels.

The reader can remark that this normal weights structure is similar to the one given by a linear S^1-action on $\mathbb{C}P^2$ admitting three isolated fixed points. □

Together with Lemma 5.4, let us now discuss the normal weight structure at the fixed point components. In virtue of Lemma 6.17, we can exclude the existence of a fixed point component $N^8 \subset X_5(\mathbf{d})^{\mathbb{Z}/n}$ for any integer n. This leaves us with fixed point components $N \subset X_5(\mathbf{d})^{\mathbb{Z}/p}$ of dimension 2, 4 or 6 for a prime number p.

Let us first consider the case where there exists $N^6 \subset X_5(\mathbf{d})^{\mathbb{Z}/p}$. By Corollary 4.7 and Poincaré duality, N contains $Y \cup P_0 \cup P_1 \cup P_2$, or N contains $Y \cup P_i$ for a certain $i = 0, 1, 2$, or N contains exactly two isolated fixed point $P_i \cup P_j$. Similarly, if there exists $N^4 \subset X_5(\mathbf{d})^{\mathbb{Z}/p}$, then either N contains all the isolated fixed point $P_0 \cup P_1 \cup P_2$, or N contains exactly two isolated fixed points $P_i \cup P_j$. Finally, if there exists $N^2 \subset X_5(\mathbf{d})^{\mathbb{Z}/p}$, then N contains exactly two isolated fixed points $P_i \cup P_j$ for $i \neq j$.

One can see that in this situation, a case by case proof might be long and fastidious. Therefore, we will show how one can handle multiple cases into a single case by rearranging the normal weight at the P_i's into our equations.

Let us first ignore all the cases where there is an $N^6 \subset X_5(\mathbf{d})^{\mathbb{Z}/p}$ containing Y. Thus, in virtue of Lemmas 5.4 and 6.18, whenever a normal weight show up at a point P_i, then it needs to show up at another isolated fixed. Furthermore, in the Atiyah-Bott integration formula, we can see that we are often dealing with the product of the normal weights at the P_i's. Therefore, let us introduce the notation $N_{i,j}$ to denote the product of the normal weights shared between P_i and P_j. This notation allows us to write the products

$$\begin{aligned}
n_{0,1} \cdot n_{0,2} \cdots n_{0,5} &= N_{0,1} \cdot N_{0,2}, \\
n_{1,1} \cdot n_{1,2} \cdots n_{1,5} &= N_{0,1} \cdot N_{1,2}, \\
n_{2,1} \cdot n_{2,2} \cdots n_{2,5} &= N_{0,2} \cdot N_{1,2}.
\end{aligned}$$

Let us now consider also the cases where there exists $N^6 \subset X_5(\mathbf{d})^{\mathbb{Z}/p}$ containing Y.

CASE A. $N^{S^1} = Y \cup P_0 \cup P_1 \cup P_2$,

CASE B. $N^{S^1} = Y \cup P_i$ for a certain $i = 0, 1, 2$,

We first discuss CASE A. Let p devide $n_{Y,1} > 1$ the normal weight at Y and let $N_1^6 \subset X_5(\mathbf{d})^{\mathbb{Z}/p}$ be as above in CASE A. Note that N_1 is in fact a component of $X_5(\mathbf{d})^{\mathbb{Z}/n_{Y,1}}$ for dimensional reasons. Assume we have $n_{Y,2} > 1$, then there exists a fixed point component $N_2^6 \subset X_5(\mathbf{d})^{\mathbb{Z}/n_{Y,2}}$. Furthermore, by Lemma 6.17 we have $(n_{Y,1}, n_{Y,2}) = 1$.

We show that $N_2^{S^1} = Y \cup P_0 \cup P_1 \cup P_2$. Thus suppose by contradiction $N_2^{S^1} = Y \cup P_i$ for a certain $i = 0, 1$ or 2. Then by the pigeon hole principle, there is at least one normal weight at P_i divisible by $n_{Y,1}$ and $n_{Y,2}$. Consequently, there is a fixed point component $M \subset X_5(\mathbf{d})^{\mathbb{Z}/n_{Y,1}n_{Y,2}}$ such that $M^{S^1} = P_i$. This gives us our desired contradiction, because no closed manifold admits an isolated fixed point by a smooth S^1-action (cf. Section 3.3).

Therefore, the connected component $N_2^6 \subset X_5(\mathbf{d})^{\mathbb{Z}/n_{Y,2}}$ admits an induced S^1-action such that $N_2^{S^1} = Y \cup P_0 \cup P_1 \cup P_2$. By the pigeon hole principle, there is a fixed point component $M \subset X_5(\mathbf{d})^{\mathbb{Z}/n_{Y,1}n_{Y,2}}$ which admits an S^1-action such that $M^{S^1} = P_0 \cup P_1 \cup P_2$. Note that M needs to be 4-dimensional. This implies that M is like in Lemma 6.18 up to orientation of the fixed points.

The above discussion gives us the following configurations for the normal weights:

CASE A.1 AT Y. $(n_1, 1, 1)$,

AT P_0. $(n_1n_{0,1}, n_1n_{0,2}, n_1n_{0,3}, n_{0,4}, n_{0,5})$,

AT P_1. $(n_1n_{1,1}, n_1n_{1,2}, n_1n_{1,3}, n_{1,4}, n_{1,5})$,

AT P_2. $(n_1n_{2,1}, n_1n_{2,2}, n_1n_{2,3}, n_{2,4}, n_{2,5})$,

CASE A.2 AT Y. $(n_1, n_2, 1)$,

AT P_0. $(n_1n_2n_{0,1}, n_1n_2n_{0,2}, n_1n_{0,3}, n_2n_{0,4}, n_{0,5})$,

AT P_1. $(n_1n_2n_{1,1}, n_1n_2n_{1,2}, n_1n_{1,3}, n_2n_{1,4}, n_{1,5})$,

AT P_2. $(n_1n_2n_{2,1}, n_1n_2n_{2,2}, n_1n_{2,3}, n_2n_{2,4}, n_{2,5})$,

CASE A.3 AT Y. (n_1, n_2, n_3),

AT P_0. $(n_1n_2n_3n_{0,1}, n_1n_2n_3n_{0,2}, n_1n_{0,3}, n_2n_{0,4}, n_3n_{0,5})$,

AT P_1. $(n_1n_2n_3n_{1,1}, n_1n_2n_3n_{1,2}, n_1n_{1,3}, n_2n_{1,4}, n_3n_{1,5})$,

AT P_2. $(n_1n_2n_3n_{2,1}, n_1n_2n_3n_{2,2}, n_1n_{2,3}, n_2n_{2,4}, n_3n_{2,5})$,

where $(n_k, n_l) = 1$ when $k \neq l$, for $k, l = 1, 2, 3$. In virtue of Lemmas 5.4 and 6.18, we use as above the $N_{i,j}$ to denote the product of the $n_{i,k}$'s which are shared between P_i and P_j for $i < j$.

LEMMA 6.19. *Let $X_5(\mathbf{d})$ be a complete intersection with a smooth circle action such that the fixed point components are $X_5(\mathbf{d})^{S^1} = Y^4 \cup P_0^0 \cup P_1^0 \cup P_2^0$. If the action admits a fixed point component $N^6 \subset X_5(\mathbf{d})^{\mathbb{Z}/p}$ containing $Y^4 \cup P_0^0 \cup P_1^0 \cup P_2^0$ for a prime number p, then $\mathbf{d}$ is equal to (1) or (2).*

PROOF. The idea of the proof is to compute

$$\int_{X_5(\mathbf{d})} x^5, \quad \text{and} \quad \int_{X_5(\mathbf{d})} p_1(X_5(\mathbf{d})) \cdot x^3$$

locally at the fixed point using the Atiyah-Bott integration formula. In each of the CASES A.1, A.2 and A.2. Thus, one needs first to compute the local datum of x^5 and $p_1(X_5(\mathbf{d})) \cdot x^3$ at Y. Note that for CASES

A.1, this have been done in the proof of Lemma 6.12. For CASES A.2 and CASES A.3, we compute these local data in a similar way knowing that the equivariant Euler classes of ν_Y the normal bundle of Y in $X_5(\mathbf{d})$ are given by

$$e_{S^1}(\nu_Y) = (c_{Y,n_1}x_Y + n_1 z)(c_{Y,n_2}x_Y + n_2 z)(c_{Y,1}x_Y + z),$$

and

$$e_{S^1}(\nu_Y) = (c_{Y,n_1}x_Y + n_1 z)(c_{Y,n_2}x_Y + n_2 z)(c_{Y,n_3}x_Y + n_3 z)$$

respectively.

Note also that in the proof of Lemma 6.12, we computed the local datum of x^5 and $p_1(X_5(\mathbf{d}))\cdot x^3$ at an isolated fixed point. Thus, using our $N_{i,j}$ notation, the local datum of x^5 and $p_1(X_5(\mathbf{d}))\cdot x^3$ at P_i are given in each of the CASES A.1, A.2 and A.2 by

CASE A.1.

$$\varepsilon_i \frac{(a_i+l)^5}{n_1^3 N_{i,j}N_{i,k}}, \quad \text{and} \quad \varepsilon_i \frac{\Sigma_i^2\cdot(a_i+l)^3}{n_1^3 N_{i,j}N_{i,k}},$$

CASE A.2.

$$\varepsilon_i \frac{(a_i+l)^5}{n_1^3 n_2^3 N_{i,j}N_{i,k}}, \quad \text{and} \quad \varepsilon_i \frac{\Sigma_i^2\cdot(a_i+l)^3}{n_1^3 n_2^3 N_{i,j}N_{i,k}},$$

CASE A.3.

$$\varepsilon_i \frac{(a_i+l)^5}{n_1^3 n_2^3 n_3^3 N_{i,j}N_{i,k}}, \quad \text{and} \quad \varepsilon_i \frac{\Sigma_i^2\cdot(a_i+l)^3}{n_1^3 n_2^3 n_3^3 N_{i,j}N_{i,k}},$$

where $\varepsilon_i = \pm 1$ denotes the orientation of the point P_i, and Σ_i^2 denotes the sum of the squares of the normal weights at P_i. It will be very convenient in our system of equation to consider Σ_i^2 as a single variable for $i = 0, 1, 2$.

Thus for CASE A.1, the integration formula gives us a polynomials in l which needs to be constant equal to d and $p_1 \cdot d$ respectively. We end up then with the system of equations

$$\begin{aligned}
d &= \frac{a_0^5}{N_{0,1}N_{0,2}n_1^3} - \frac{a_1^5}{N_{0,1}n_1^3N_{1,2}} - \frac{a_2^5}{N_{0,2}n_1^3N_{1,2}},\\
0 &= \frac{5a_0^4}{N_{0,1}N_{0,2}n_1^3} - \frac{5a_1^4}{N_{0,1}n_1^3N_{1,2}} - \frac{5a_2^4}{N_{0,2}n_1^3N_{1,2}},\\
0 &= \frac{10a_0^3}{N_{0,1}N_{0,2}n_1^3} - \frac{10a_1^3}{N_{0,1}n_1^3N_{1,2}} - \frac{10a_2^3}{N_{0,2}n_1^3N_{1,2}},\\
0 &= \frac{10a_0^2}{N_{0,1}N_{0,2}n_1^3} - \frac{10a_1^2}{N_{0,1}n_1^3N_{1,2}} - \frac{10a_2^2}{N_{0,2}n_1^3N_{1,2}} + \frac{10r_Y^2}{n_1},\\
0 &= \frac{5a_0}{N_{0,1}N_{0,2}n_1^3} - \frac{5a_1}{N_{0,1}n_1^3N_{1,2}} - \frac{5a_2}{N_{0,2}n_1^3N_{1,2}} - \frac{5r_Y(c_{Y,1}n_1 + c_{Y,n_1})}{n_1^2},
\end{aligned}$$

$$
\begin{aligned}
0 = {} & \frac{c_{Y,1}^2 n_1^2 + c_{Y,1} c_{Y,n_1} n_1 - c_{Y,2} n_1^2 + c_{Y,n_1}^2}{n_1^3} \\
& + \frac{1}{N_{0,1} N_{0,2} n_1^3} - \frac{1}{N_{0,1} n_1^3 N_{1,2}} - \frac{1}{N_{0,2} n_1^3 N_{1,2}}
\end{aligned}
$$

and

$$
\begin{aligned}
p_1 \cdot d = {} & \frac{1}{n_1^3 N_{0,1} N_{0,2} N_{1,2}} \left(a_0^3 N_{1,2} \Sigma_0^2 - a_1^3 N_{0,2} \Sigma_1^2 - a_2^3 N_{0,1} \Sigma_2^2\right), \\
0 = {} & \frac{3}{n_1^3 N_{0,1} N_{0,2} N_{1,2}} \big(a_0^2 N_{1,2} \Sigma_0^2 - a_1^2 N_{0,2} \Sigma_1^2 - a_2^2 N_{0,1} \Sigma_2^2 \\
& + n_1^4 N_{0,1} N_{0,2} N_{1,2} r_Y^2 + 2 n_1^2 N_{0,1} N_{0,2} N_{1,2} r_Y^2\big), \\
0 = {} & -\frac{3}{n_1^3 N_{0,1} N_{0,2} N_{1,2}} \big(-a_0 N_{1,2} \Sigma_0^2 + a_1 N_{0,2} \Sigma_1^2 + a_2 N_{0,1} \Sigma_2^2 \\
& + c_{Y,1} N_{0,1} N_{0,2} n_1^4 N_{1,2} r_Y - c_{Y,n_1} N_{0,1} N_{0,2} n_1^3 N_{1,2} r_Y \\
& + 2 c_{Y,n_1} N_{0,1} N_{0,2} n_1 N_{1,2} r_Y\big), \\
0 = {} & \frac{1}{n_1^3 N_{0,1} N_{0,2} N_{1,2}} \big(c_{Y,1}^2 N_{0,1} N_{0,2} n_1^4 N_{1,2} + c_{Y,1}^2 N_{0,1} N_{0,2} n_1^2 N_{1,2} \\
& - c_{Y,1} c_{Y,n_1} N_{0,1} N_{0,2} n_1^3 N_{1,2} - c_{Y,2} N_{0,1} N_{0,2} n_1^4 N_{1,2} \\
& - 4 c_{Y,2} N_{0,1} N_{0,2} n_1^2 N_{1,2} + 2 c_{Y,n_1}^2 N_{0,1} N_{0,2} N_{1,2} \\
& + N_{0,1} N_{0,2} n_1^2 N_{1,2} \int p_1(Y) - N_{0,1} \Sigma_2^2 - N_{0,2} \Sigma_1^2 \\
& + N_{1,2} \Sigma_0^2\big)
\end{aligned}
$$

Solving this system in the variables p_1, d, $N_{0,1}$, $N_{0,2}$, $N_{1,2}$, Σ_0^2, Σ_1^2, Σ_2^2, $c_{Y,1}$, $c_{Y,2}$, gives us

$$
p_1 = \frac{3}{r_Y^2} + \frac{c_{Y,n_1}^2 (n_1^2 - 1)}{n_1^2 r_Y^2} + \left(\frac{1}{a_0^2} + \frac{1}{a_1^2} + \frac{1}{a_2^2}\right)
$$

which implies that $\mathbf{d}$ is equal to (1) or (2) since p_1 is positive. Furthermore, the reader can find the *Mathematica*® computations for this case in the appendix, cf. Chapter B.

The computations for CASE A.2 and CASE A.3 are similar to those of case CASE A.1. In the CASE A.2, we find that

$$
p_1 = \frac{c_{Y,n_1}^2 + c_{Y,n_2}^2 + c_{Y,1}^2 + 3}{r_Y^2},
$$

and for CASE A.3

$$
p_1 = \frac{c_{Y,n_1}^2 + c_{Y,n_2}^2 + c_{Y,n_3}^2 + 3}{r_Y^2}.
$$

In both cases, we have the same conclusion, i.e. $\mathbf{d}$ is equal to (1) or (2) because p_1 is positive. The reader can find the *Mathematica*® computations in Chapter B. □

Let us now discuss CASE B. For convenience, let us denote by n_0, n_1 and n_2 the normal weights at Y. We write N_i^6 the fixed point component of $X_5(\mathbf{d})^{\mathbb{Z}/n_i}$ containing Y and P_i. It is easy to see that if there is such N_i^6 with a normal weight $n_i > 1$ at Y, then n_i appears as a normal weight at P_i with multiplicity 3. Without loss of generality, there are 5 sub cases with the following normal weights:

CASE B.1. AT Y. $(n_0, 1, 1)$,
AT P_0. $(n_0, n_0, n_0, n_{0,4}, n_{0,5})$,
AT P_1. $(n_{1,1}, n_{1,2}, n_{1,3}, n_{1,4}, n_{1,5})$,
AT P_2. $(n_{2,1}, n_{2,2}, n_{2,3}, n_{2,4}, n_{2,5})$.

CASE B.2. AT Y. $(1, n_1, 1)$,
AT P_0. $(n_{0,1}, n_{0,2}, n_{0,3}, n_{0,4}, n_{0,5})$,
AT P_1. $(n_1, n_1, n_1, n_{1,4}, n_{1,5})$,
AT P_2. $(n_{2,1}, n_{2,2}, n_{2,3}, n_{2,4}, n_{2,5})$.

CASE B.3. AT Y. $(n_0, n_1, 1)$,
AT P_0. $(n_0, n_0, n_0, n_{0,4}, n_{0,5})$,
AT P_1. $(n_1, n_1, n_1, n_{1,4}, n_{1,5})$,
AT P_2. $(n_{2,1}, n_{2,2}, n_{2,3}, n_{2,4}, n_{2,5})$.

CASE B.4. AT Y. $(1, n_1, n_2)$,
AT P_0. $(n_{0,1}, n_{0,2}, n_{0,3}, n_{0,4}, n_{0,5})$,
AT P_1. $(n_1, n_1, n_1, n_{1,4}, n_{1,5})$,
AT P_2. $(n_2, n_2, n_2, n_{2,4}, n_{2,5})$.

CASE B.5. AT Y. (n_0, n_1, n_2),
AT P_0. $(n_0, n_0, n_0, n_{0,4}, n_{0,5})$,
AT P_1. $(n_1, n_1, n_1, n_{1,4}, n_{1,5})$,
AT P_2. $(n_2, n_2, n_2, n_{2,4}, n_{2,5})$.

We use as above the notation $N_{i,j}$ for the product of the normal weights $n_{i,k}$'s which are shared between P_i and P_j.

LEMMA 6.20. *Let $X_5(\mathbf{d})$ be a complete intersection with a smooth circle action such that the fixed point components are $X_5(\mathbf{d})^{S^1} = Y^4 \cup P_0^0 \cup P_1^0 \cup P_2^0$. If the action admits a fixed point component $N^6 \subset X_5(\mathbf{d})^{\mathbb{Z}/p}$ containing $Y^4 \cup P_i^0$ for a certain prime number p and a certain $i = 0, 1, 2$, then $\mathbf{d}$ is equal to (1) or (2).*

PROOF. Like in the proof of Lemma 6.19, we prove that $p_1 > 0$ using the Atiyah-Bott integration formula on

$$\int_{X_5(\mathbf{d})} x^5, \quad \text{and} \quad \int_{X_5(\mathbf{d})} p_1(X_5(\mathbf{d})) \cdot x^3.$$

Furthermore, we use Lemma 6.13 to obtain additional equations which will allow us to solve the system of equations. Note that the reader can find the *Mathematica®* computations for each of these cases in the appendix, cf. Chapter B.

CASE B.1. The local datum at Y is the same as in the proof of Lemmas 6.12 and 6.19. The local datum of x^5 and $p_1(X_5(\mathbf{d})) \cdot x^3$ at P_0 is given by

$$\frac{(a_0+l)^5}{n_0^3 N_{0,1} N_{0,2}}, \quad \text{and} \quad \frac{\Sigma_0^2 \cdot (a_0+l)^3}{n_0^3 N_{0,1} N_{0,2}}.$$

Similarly, the local datum of x^5 and $p_1(X_5(\mathbf{d})) \cdot x^3$ at P_1 and P_2 are given by

$$-\frac{(a_i+l)^5}{N_{0,i} N_{i,j}}, \quad \text{and} \quad -\frac{\Sigma_i^2 \cdot (a_i+l)^3}{N_{0,i} N_{i,j}}.$$

for $i = 1, 2$ and $j \neq 0, i$.

This gives us the systems of equations

$$\begin{aligned}
d &= \frac{a_0^5}{n_0^3 N_{0,1} N_{0,2}} - \frac{a_1^5}{N_{0,1} N_{1,2}} - \frac{a_2^5}{N_{0,2} N_{1,2}}, \\
0 &= \frac{5a_0^4}{n_0^3 N_{0,1} N_{0,2}} - \frac{5a_1^4}{N_{0,1} N_{1,2}} - \frac{5a_2^4}{N_{0,2} N_{1,2}}, \\
0 &= \frac{10a_0^3}{n_0^3 N_{0,1} N_{0,2}} - \frac{10a_1^3}{N_{0,1} N_{1,2}} - \frac{10a_2^3}{N_{0,2} N_{1,2}}, \\
0 &= \frac{10a_0^2}{n_0^3 N_{0,1} N_{0,2}} - \frac{10a_1^2}{N_{0,1} N_{1,2}} - \frac{10a_2^2}{N_{0,2} N_{1,2}} + \frac{10r_Y^2}{n_0}, \\
0 &= \frac{5a_0}{n_0^3 N_{0,1} N_{0,2}} - \frac{5a_1}{N_{0,1} N_{1,2}} - \frac{5a_2}{N_{0,2} N_{1,2}} \\
&\quad - \frac{5r_Y(c_{Y,1} n_0 + c_{Y,n_0})}{n_0^2}, \\
0 &= \frac{c_{Y,1}^2 n_0^2 + c_{Y,1} c_{Y,n_0} n_0 - c_{Y,2} n_0^2 + c_{Y,n_0}^2}{n_0^3} \\
&\quad + \frac{1}{n_0^3 N_{0,1} N_{0,2}} - \frac{1}{N_{0,1} N_{1,2}} - \frac{1}{N_{0,2} N_{1,2}},
\end{aligned}$$

and

$$\begin{aligned}
p_1 \cdot d &= \frac{a_0^3 \Sigma_0^2}{n_0^3 N_{0,1} N_{0,2}} - \frac{a_1^3 \Sigma_1^2}{N_{0,1} N_{1,2}} - \frac{a_2^3 \Sigma_2^2}{N_{0,2} N_{1,2}}, \\
0 &= \frac{3a_0^2 \Sigma_0^2}{n_0^3 N_{0,1} N_{0,2}} - \frac{3a_1^2 \Sigma_1^2}{N_{0,1} N_{1,2}} - \frac{3a_2^2 \Sigma_2^2}{N_{0,2} N_{1,2}} + \frac{3\left(n_0^2+2\right) r_Y^2}{n_0}, \\
0 &= \frac{3a_0 \Sigma_0^2}{n_0^3 N_{0,1} N_{0,2}} - \frac{3a_1 \Sigma_1^2}{N_{0,1} N_{1,2}} - \frac{3a_2 \Sigma_2^2}{N_{0,2} N_{1,2}} \\
&\quad + \frac{3r_Y\left(-c_{Y,1} n_0^3 + c_{Y,n_0} n_0^2 - 2c_{Y,n_0}\right)}{n_0^2},
\end{aligned}$$

$$
\begin{aligned}
0 = \frac{1}{n_0^3} \cdot \big(&c_{Y,1}^2 n_0^4 + c_{Y,1}^2 n_0^2 - c_{Y,1} c_{Y,n_0} n_0^3 - c_{Y,2} n_0^4 - 4c_{Y,2} n_0^2 \\
&+ 2c_{Y,n_0}^2 + n_0^2 \int p_1(Y)\big) + \frac{\Sigma_0^2}{n_0^3 N_{0,1} N_{0,2}} - \frac{\Sigma_1^2}{N_{0,1} N_{1,2}} - \frac{\Sigma_2^2}{N_{0,2} N_{1,2}}
\end{aligned}
$$

Furthermore, we know by Lemma 6.13 that

$$r_Y^2 = \frac{a_0^2}{n_0^2}, \quad \text{and} \quad c_{Y,n_0} = \pm 1.$$

This allows us to solve the above system in the variables p_1, d, n_0, $N_{0,1}$, $N_{0,2}$, $N_{1,2}$, Σ_0^2, Σ_1^2, Σ_2^2, $c_{Y,1}$, $c_{Y,2}$, c_{Y,n_0}, to obtain that

$$p_1 = \frac{4}{r_Y^2} + \frac{1}{a_1^2} + \frac{1}{a_2^2} > 0.$$

CASE B.2. This case is treated in the exact same way with a minor change of sign. Using the Atiyah-Bott integration formula and Lemma 6.13, we obtain as above a system of equation which gives us

$$p_1 = \frac{4}{r_Y^2} + \frac{1}{a_0^2} + \frac{1}{a_2^2} > 0.$$

CASE B.3. Let us compute the local datum of x^5 and $p_1(X_5(\mathbf{d})) \cdot x^3$ at Y. First, one needs to compute the equivariant Euler class of the normal bundle of Y. The later is given by

$$e_{S^1}(\nu_Y) = (c_{Y,n_0} \cdot x_Y + n_0 \cdot z) \cdot (c_{Y,n_1} \cdot x_Y + n_1 \cdot z) \cdot (c_{Y,1} \cdot x_Y + z).$$

Thus, the local datum of x^5 at Y is given by

$$
\begin{aligned}
&\int_{Y^4} (r_Y \cdot x_Y + l \cdot z)^5 \left(\frac{c_{Y,n_0}^2 x_Y^2}{n_0^3 z^3} - \frac{c_{Y,n_0} x_Y}{n_0^2 z^2} + \frac{1}{n_0 z} \right) \\
&\quad \cdot \left(\frac{c_{Y,n_1}^2 x_Y^2}{n_1^3 z^3} - \frac{c_{Y,n_1} x_Y}{n_1^2 z^2} + \frac{1}{n_1 z} \right) \cdot \left(\frac{c_{Y,1}^2 x_Y^2}{z^3} - \frac{c_{Y,1} x_Y}{z^2} + \frac{1}{z} \right).
\end{aligned}
$$

Furthermore, the equivariant first Pontrjagin class of Y, $p_{S^1,1}(Y)$ is given by

$$p_1(Y) + (c_{Y,n_0} x_Y + n_0 z)^2 + (c_{Y,n_1} x_Y + n_1 z)^2 + (c_{Y,1} x_Y + z)^2.$$

We leave the computation of the local datum of $p_1(X_5(\mathbf{d})) \cdot x^3$ to the reader.

Therefore, the Atiyah-Bott integration formula gives us a system of equations which we can solve in the variable p_1, d, n_0, n_1, $N_{0,1}$, $N_{0,2}$, $N_{1,2}$, Σ_0^2, Σ_1^2, Σ_2^2, $c_{Y,1}$, c_{Y,n_0}, c_{Y,n_1}. This is to be done together with the equations given by Lemma 6.13, i.e.

$$r_Y^2 = \frac{a_0^2}{n_0^2} = \frac{a_1^2}{n_1^2}, \quad c_{Y,n_0} = \pm 1, \quad \text{and} \quad c_{Y,n_1} = \pm 1.$$

In particular, the solution for $p_1 = 6 + r - \sum_i d_i^2$ is given by

$$p_1 = \frac{5}{r_Y^2} + \frac{1}{a_2^2} > 0.$$

CASE B.4. This case is treated in the exact same way as CASE B.3. with a minor change of sign. Using the Atiyah-Bott integration formula and Lemma 6.13, we obtain a system of equation which gives us

$$p_1 = \frac{5}{r_Y^2} + \frac{1}{a_0^2} > 0.$$

CASE B.5. This last case is treated as above using the Atiyah-Bott integration formula and Lemma 6.13. The computations gives us

$$p_1 = \frac{6}{r_Y^2} > 0.$$

which again implies that $\mathbf{d}$ is equal to (1) or (2). Note that in this case, it is particularly simple to see that $\mathbf{d}$ cannot be equal to (2). However, the statement that $p_1 > 0$ is enough for our purpose. □

In virtue of Lemmas 6.17, 6.19 and 6.20, we are now left with the case where the action is semi-free around Y.

LEMMA 6.21. *Let $X_5(\mathbf{d})$ be a complete intersection with a smooth circle action such that the fixed point components are $X_5(\mathbf{d})^{S^1} = Y^4 \cup P_0^0 \cup P_1^0 \cup P_2^0$. If the action is semi-free around Y, then $\mathbf{d} = (1)$ or (2).*

PROOF. As above, we treat this case using the Atiyah-Bott integration formula on

$$\int_{X_5(\mathbf{d})} x^5, \quad \text{and} \quad \int_{X_5(\mathbf{d})} p_1(X_5(\mathbf{d})) \cdot x^3.$$

One can see that the local data at Y have been computed in the proof of Lemma 6.16. Furthermore, the local data at the isolated fixed point have also been computed in the proof of Lemma 6.20. This gives us the system of equations

$$\begin{aligned}
d &= \frac{a_0^5}{N_{0,1}N_{0,2}} - \frac{a_1^5}{N_{0,1}N_{1,2}} - \frac{a_2^5}{N_{0,2}N_{1,2}},\\
0 &= \frac{5a_0^4}{N_{0,1}N_{0,2}} - \frac{5a_1^4}{N_{0,1}N_{1,2}} - \frac{5a_2^4}{N_{0,2}N_{1,2}},\\
0 &= \frac{10a_0^3}{N_{0,1}N_{0,2}} - \frac{10a_1^3}{N_{0,1}N_{1,2}} - \frac{10a_2^3}{N_{0,2}N_{1,2}},\\
0 &= \frac{10a_0^2}{N_{0,1}N_{0,2}} - \frac{10a_1^2}{N_{0,1}N_{1,2}} - \frac{10a_2^2}{N_{0,2}N_{1,2}} + 10r_Y^2,\\
0 &= \frac{5a_0}{N_{0,1}N_{0,2}} - \frac{5a_1}{N_{0,1}N_{1,2}} - \frac{5a_2}{N_{0,2}N_{1,2}} - 5c_{Y,1}r_Y,\\
0 &= c_{Y,1}^2 - c_{Y,2} + \frac{1}{N_{0,1}N_{0,2}} - \frac{1}{N_{0,1}N_{1,2}} - \frac{1}{N_{0,2}N_{1,2}},
\end{aligned}$$

$$
\begin{aligned}
p_1 \cdot d &= \frac{a_0^3\Sigma_0^2}{N_{0,1}N_{0,2}} - \frac{a_1^3\Sigma_1^2}{N_{0,1}N_{1,2}} - \frac{a_2^3\Sigma_2^2}{N_{0,2}N_{1,2}}, \\
0 &= \frac{3a_0^2\Sigma_0^2}{N_{0,1}N_{0,2}} - \frac{3a_1^2\Sigma_1^2}{N_{0,1}N_{1,2}} - \frac{3a_2^2\Sigma_2^2}{N_{0,2}N_{1,2}} + 9r_Y^2, \\
0 &= \frac{3a_0\Sigma_0^2}{N_{0,1}N_{0,2}} - \frac{3a_1\Sigma_1^2}{N_{0,1}N_{1,2}} - \frac{3a_2\Sigma_2^2}{N_{0,2}N_{1,2}} - 3c_{Y,1}r_Y, \\
0 &= 2c_{Y,1}^2 - 5c_{Y,2} + \frac{\Sigma_0^2}{N_{0,1}N_{0,2}} - \frac{\Sigma_1^2}{N_{0,1}N_{1,2}} - \frac{\Sigma_2^2}{N_{0,2}N_{1,2}} + \int p_1(Y).
\end{aligned}
$$

Solving this system in the variables p_1, d, $N_{0,1}$, $N_{0,2}$, $N_{1,2}$, $c_{Y,1}$, $c_{Y,2}$, Σ_0^2, Σ_1^2, Σ_2^2, gives us for p_1 the solution

$$
p_1 = \frac{3}{r_Y^2} + \left(\frac{1}{a_0^2} + \frac{1}{a_1^2} + \frac{1}{a_2^2}\right) > 0. \qquad \square
$$

PROOF OF THEOREM D. PART 3. The proof follows from Lemmas 6.17, 6.19, 6.20 and 6.21. $\square$

7 FURTHER DISCUSSION ABOUT S^1-ACTIONS ON COMPLETE INTERSECTIONS

7.1 COMPLETE INTERSECTIONS OF COMPLEX DIMENSION 5

In virtue of Theorem D, it is natural to conjecture the following statement.

CONJECTURE. *Let $X_5(\mathbf{d})$ be a complete intersection equipped with a smooth and effective circle action, then*

$\mathbf{d} = (1)$ *and* $X_5(\mathbf{d}) = \mathbb{C}P^5$, *or*

$\mathbf{d} = (2)$ *and* $X_5(\mathbf{d})$ *is a complex quadric.*

To prove such result, one could use the above tools like in Section 6.1, 6.2 and 6.3 to extend Theorem D to the cases where the fixed point set are given by

7. $X_5(\mathbf{d})^{S^1} = Z_1^2 \cup Z_2^2 \cup Z_3^2$,
8. $X_5(\mathbf{d})^{S^1} = Z_1^2 \cup Z_2^2 \cup P_1^0 \cup P_2^0$, or
9. $X_5(\mathbf{d})^{S^1} = Z^2 \cup P_1^0 \cup P_2^0 \cup P_3^0 \cup P_4^0$.

However, the number of sub cases in the above situations increases dramatically. Indeed, the intermediate manifolds given as $\mathbb{Z}/n$-fixed point components are of many kinds, this yields to many possible configurations of the normal weights. Thus, we did not pursue this strategy. On the other hand, the computations in the semi-free case turned out to be easy and give rise to immediate contradictions similar to those in Lemmas 6.5 and 6.16, i.e. $d = 0$.

EXAMPLE. Let us assume $X_5(\mathbf{d})^{S^1} = Z_1^2 \cup Z_2^2 \cup Z_3^2$. Then using Proposition A (cf. Section 4.2), Corollary 4.7 and Poincaré-Duality, for a prime number p, the intermediate manifold $F^4 \subset X_5(\mathbf{d})^{\mathbb{Z}/p}$ can be of two kind:

IF F^4 CONTAINS $Z_i^2 \cup Z_j^2$. We have $b_2(F) = 2$ and $b_0(F) = b_4(F) = 1$ for $i \neq j$.

IF F^4 CONTAINS $Z_1^2 \cup Z_2^2 \cup Z_3^2$. We have $b_2(F) = 4$ and $b_0(F) = b_4(F) = 1$.

If we consider now $F^6 \subset X_5(\mathbf{d})^{\mathbb{Z}/p}$, then these intermediate manifolds can be also of two kind:

IF F^6 CONTAINS $Z_i^2 \cup Z_j^2$. We have $b_0(F) = b_2(F) = b_4(F) = b_6(F) = 1$ for $i \neq j$.

IF F^6 CONTAINS $Z_1^2 \cup Z_2^2 \cup Z_3^2$. We have $b_2(F) = b_4(F) = 2$ and $b_0(F) = b_6(F) = 1$.

Composing these possible intermediate manifolds with the Z_i's gives a huge amount of possible normal weights configurations.

7.2 OTHER CONJECTURES ABOUT COMPLETE INTERSECTION WITH CIRCLE ACTIONS

So far, we can also think that among complete intersections of dimension greater than 3, only complex projective spaces and complex quadrics admit a smooth non-trivial S^1-action. Thus, we conjectured the following statement.

CONJECTURE. *Let $X_n(\mathbf{d})$ be a complete intersection with $n \geqslant 3$ equipped with a smooth and effective circle action, then*

$\mathbf{d} = (1)$ *and* $X_n(\mathbf{d}) = \mathbb{C}P^n$, *or*

$\mathbf{d} = (2)$ *and* $X_n(\mathbf{d})$ *is a complex quadric.*

This conjecture seems to be difficult to prove in its great generality, since no known obstruction to smooth S^1-action exists yet. Thus, it is also interesting to consider the "weak" conjecture.

CONJECTURE (WEAK). *Let $X_n(\mathbf{d})$ be a complete intersection with $n \geqslant 3$ equipped with a smooth and effective circle action, then $p_1(X_n(\mathbf{d})) > 0$.*

This conjecture is in fact very natural to consider. If one looks in detail the proof of Theorem D, the ending conclusions of Lemmas 6.1, 6.4, 6.12, 6.14, 6.15, 6.19, 6.20 and 6.21 were all that the first Pontrjagin class of the complete intersection had to be positive. The dimension of the complete intersection being 5 yields that $\mathbf{d} = (1)$ or (2). This conjecture is also supported by the main result in the paper of Dessai and Wiemeler [16, Theorem 2.4]. Thus, in terms of *obstruction class to smooth non-trivial S^1-action*, it might be interesting to explore the role of the first Pontrjagin class.

Part III

APPENDIX

A

NOTE ON ORIENTABILITY OF FIXED POINT COMPONENTS

Let X be a closed smooth oriented manifold equipped with a smooth S^1-action. We discuss in this note a criterion to obtain that any connected fixed point component $F \subset X^{\mathbb{Z}/n}$ is orientable for any induced action $\mathbb{Z}/n \subset S^1$.

CRITERION. *If X is totally nonhomologous to zero in X_{S^1} (over $\mathbb{Q}$) and $H^*(X)$ does not have 2-torsion, then any connected component $F \subset X^{\mathbb{Z}/n}$ is orientable.*

Recall that X is totally nonhomologous to zero over $\mathbb{Q}$ in X_{S^1} if and only if the Serre spectral sequence

$$H^*(BS^1; H^*(X;\mathbb{Q})) \Rightarrow H^*(X_{S^1};\mathbb{Q})$$

degenerates at the E_2-term. The reader can find a complete discussion about this in [12, Chapter VII, Theorem 1.6].

LEMMA A.1. *Let X be as above. We consider on X the induced cyclic group action $\mathbb{Z}/2^r \subset S^1$. Then the following equalities hold:*

$$b_{ev}(X^{\mathbb{Z}/2^r}) = \operatorname{rk} H^{ev}(X^{\mathbb{Z}/2^r};\mathbb{F}_2) = \operatorname{rk} H^{ev}(X;\mathbb{F}_2) = b_{ev}(X), \text{ and}$$
$$b_{odd}(X^{\mathbb{Z}/2^r}) = \operatorname{rk} H^{odd}(X^{\mathbb{Z}/2^r};\mathbb{F}_2) = \operatorname{rk} H^{odd}(X;\mathbb{F}_2) = b_{odd}(X).$$

PROOF. Let us fix a prime q large enough such that

- $X^{\mathbb{Z}/q} = X^{S^1}$,
- $X^{\mathbb{Z}/2^r}$ has no q-torsion,
- X^{S^1} has no q-torsion.

It follows by [12, Chapter VII, Theorem 2.2] that for any closed smooth oriented manifold Z with smooth S^1-action

$$\operatorname{rk} H^{ev}(Z^{\mathbb{Z}/2};\mathbb{F}_2) \leqslant \operatorname{rk} H^{ev}(Z;\mathbb{F}_2), \text{ and}$$
$$\operatorname{rk} H^{odd}(Z^{\mathbb{Z}/2};\mathbb{F}_2) \leqslant \operatorname{rk} H^{odd}(Z;\mathbb{F}_2).$$

By induction, one can show that

$$\operatorname{rk} H^{ev}(Z^{\mathbb{Z}/2^r};\mathbb{F}_2) \leqslant \operatorname{rk} H^{ev}(Z;\mathbb{F}_2), \text{ and}$$
$$\operatorname{rk} H^{odd}(Z^{\mathbb{Z}/2^r};\mathbb{F}_2) \leqslant \operatorname{rk} H^{odd}(Z;\mathbb{F}_2).$$

Therefore, we have the following inequalities:

$$\begin{aligned} b_{ev}(X^{S^1}) &= \operatorname{rk} H^{ev}(X^{S^1};\mathbb{F}_q) \\ &= \operatorname{rk} H^{ev}((X^{\mathbb{Z}/2^r})^{\mathbb{Z}/q};\mathbb{F}_q) \\ &\leqslant \operatorname{rk} H^{ev}(X^{\mathbb{Z}/2^r};\mathbb{F}_q) = b_{ev}(X^{\mathbb{Z}/2^r}) \\ &\leqslant \operatorname{rk} H^{ev}(X^{\mathbb{Z}/2^r};\mathbb{F}_2) \\ &\leqslant \operatorname{rk} H^{ev}(X;\mathbb{F}_2) = b_{ev}(X). \end{aligned}$$

The same holds in odd dimensions. Since $b_{ev}(X^{S^1}) = b_{ev}(X)$ and $b_{odd}(X^{S^1}) = b_{odd}(X)$, all the above inequalities are in fact equalities, and this concludes the proof of the lemma. □

LEMMA A.2. *Any fixed point component of $X^{\mathbb{Z}/2^r}$ is orientable.*

PROOF. By Lemma A.1 we know that

$$\begin{aligned} b_{ev}(X^{\mathbb{Z}/2^r}) &= \operatorname{rk} H^{ev}(X^{\mathbb{Z}/2^r};\mathbb{F}_2), \text{ and} \\ b_{odd}(X^{\mathbb{Z}/2^r}) &= \operatorname{rk} H^{odd}(X^{\mathbb{Z}/2^r};\mathbb{F}_2). \end{aligned}$$

This implies in particular that $H^*(X^{\mathbb{Z}/2^r})$ does not have 2-torsion. By [25, Chapter 3, Corollary 3.28] our result follows. □

PROOF OF THE CRITERION. Let F be the fixed point component of the induced $\mathbb{Z}/n \subset S^1$ action. We decomposes the $\mathbb{Z}/n$-action as follow. Write $n = 2^r k$ where k is an odd number, then the connected components of $X^{\mathbb{Z}/2^r}$ are orientable by Lemma A.2. Furthermore, $X^{\mathbb{Z}/2^r}$ is equipped with a $\mathbb{Z}/k$-action such that

$$(X^{\mathbb{Z}/2^r})^{\mathbb{Z}/k} = X^{\mathbb{Z}/n}.$$

This implies that F is a fixed point component of a $\mathbb{Z}/k$-action on a certain orientable connected component $Y \subseteq X^{\mathbb{Z}/2^r}$. Since Y is an orientable manifold and since the $\mathbb{Z}/k$-action induces a complex structure on the normal bundle of F in Y, we have that F is an orientable manifold. □

B
MATHEMATICA® COMPUTATIONS

In this chapter, we explain how to read the following *Mathematica®* computations. To do so, we use the computations of Lemma 6.1 to introduce the *Mathematica®* functions we used. The other computations in Section 6.1 and those in Sections 6.2 and 6.3 are very similar in essence, thus we used the same notations in the *Mathematica®* computations. The version we used is the *Wolfram Mathematica®* 10 running on Mac OS X Yosemite.

In Lemma 6.1, we were first interested to compute the local datum at Y_1 and Y_2 of x^5 in order to apply the Atiyah-Bott integration, i.e.

$$\int_{X_5(\mathbf{d})} x^5 = \mu(x^5, Y_1) + \mu(x^5, Y_2),$$

where $\mu(x^5, Y_i)$ denotes the local datum at Y_i of x^5. Let us explain the *Mathematica®* computations that we did for $\mu(x^5, Y_1)$. The computations for $\mu(x^5, Y_2)$ uses the same *Mathematica®* functions.

Recall that the local datum of x^5 at Y_1 is given by

$$\mu(x^5, Y_1) = \int_{Y_1^4} (r_1 x_1 + l \cdot z)^5 \, e_{S^1}(\nu_1)^{-1}.$$

We computed $e_{S^1}(\nu_1)^{-1}$ by first identifying

$$e_{S^1}(\nu_1) = c_2(\nu_1) z + c_1(\nu_1) z^2 + z^3.$$

Then we used the generator $x_1 \in H^2(Y_1; \mathbb{Z})$ to write $c_2(\nu_1) = c_{1,2} x_1^2$ and $c_1(\nu_1) = c_{1,1} x_1$ with $c_{1,j} \in \mathbb{Z}$. This allows us to consider $e_{S^1}(\nu_1)$ as a polynomial in x_1 with coefficients in the polynomial ring in z. Thus, the inverse $e_{S^1}(\nu_1)^{-1}$ is given by the following power series in x_1

$$\frac{1}{z^3} - \frac{c_{1,1} x_1}{z^4} + \frac{(c_{1,1}^2 - c_{1,2}) x_1^2}{z^5}.$$

To find this using *Mathematica*, we used the function `Series`.

$$\texttt{Series}\left[\left(c12\,x1^2\,t^2\,z + c11\,x1\,t\,z^2 + z^3\right)^{-1}, \{x1, 0, 2\}\right]$$

$$\frac{1}{z^3} - \frac{c11\,t\,x1}{z^4} + \frac{\left(c11^2\,t^2 - c12\,t^2\right)\,x1^2}{z^5} + O[x1]^3$$

This gives us the desired power series up to order 2 in x_1. The higher terms cancel in cohomology, thus are not needed. We will explain later the use of the variable t. So far, the reader should not pay any attention to it.

Note that we used almost the same variables names in our computation. The lower subscripts are not allowed in *Mathematica*, while the upper subscripts are interpreted as powers. In the beginning of each computation, we referenced the variables we use. The reader will easily find his way through the notations.

If one now continues the computations of $\mu(x^5, Y_1)$, then we need to expand the term

$$(r_1 x_1 + l \cdot z)^5.$$

This is done in *Mathematica*® with the function `Expand`.

```
Expand[(r1 x1 t + l z)^5]
```

```
r1^5 t^5 x1^5 + 5 l r1^4 t^4 x1^4 z + 10 l^2 r1^3 t^3 x1^3 z^2 +
 10 l^3 r1^2 t^2 x1^2 z^3 + 5 l^4 r1 t x1 z^4 + l^5 z^5
```

We can now multiply the previous two solutions to get

$$(r_1 x_1 + l \cdot z)^5 \, e_{S^1}(\nu_1)^{-1}.$$

```
Expand[(r1 x1 t + l z)^5]
 Series[(c12 x1^2 t^2 z + c11 x1 t z^2 + z^3)^-1, {x1, 0, 2}]
```

```
l^5 z^2 + (-c11 l^5 t z + 5 l^4 r1 t z) x1 +
 (-5 c11 l^4 r1 t^2 + 10 l^3 r1^2 t^2 + l^5 (c11^2 t^2 - c12 t^2)) x1^2 + O[x1]^3
```

Since the integration $\int_{Y_1}$ considers only the term in degree 2 in x_1, we only need the coefficient of t^2. Thus, we extract it using the *Mathematica*® function `Coefficient`

```
Coefficient[Expand[(r1 x1 t + l z)^5]
  Series[(c12 x1^2 t^2 z + c11 x1 t z^2 + z^3)^-1, {x1, 0, 2}], t, 2]
```

```
c11^2 l^5 x1^2 - c12 l^5 x1^2 - 5 c11 l^4 r1 x1^2 + 10 l^3 r1^2 x1^2
```

We finally need to integrate the variable x_1^2 and extract the coefficients in l. Note that these two operations are independent and can be done in any order. Thus we first extract the coefficients using the function `CoefficientList`.

```
CoefficientList[
 Coefficient[Expand[(r1 x1 t + l z)^5]
   Series[(c12 x1^2 t^2 z + c11 x1 t z^2 + z^3)^-1, {x1, 0, 2}], t, 2], l]
```

```
{0, 0, 0, 10 r1^2 x1^2, -5 c11 r1 x1^2, c11^2 x1^2 - c12 x1^2}
```

Finally, we evaluate `x1`^{2} in 1 using the `/.` function.

```
CoefficientList[
 Coefficient[Expand[(r1 x1 t + l z)^5]
   Series[(c12 x1^2 t^2 z + c11 x1 t z^2 + z^3)^-1, {x1, 0, 2}], t, 2], l] /. x1^2 → 1
```

```
{0, 0, 0, 10 r1^2, -5 c11 r1, c11^2 - c12}
```

The above vector is then stored in the variable μx5Y1. The computations for μx5Y2 are similar.

Computations for Theorem D.

The case where the fixed points are $Y_1{}^4 \cup Y_2{}^4$.

Computations for Lemma 6.1.
The case where the action is semi-free.

```
(******************************
   NOTATIONS
   ri xi denotes the restriction of x to Yi
   ai are the Hopf weight at Yi
   cij xi^j denotes the j-th Chern class of Yi
   μx5Yi denotes the local datum of x^5 at Yi

   We use the variable t to extract the coefficient of order 2 of xi. We
   then only evaluate xi^2 to 1 or -1.
    ******************************)

μx5Y1 =
 Factor[CoefficientList[
    Coefficient[Expand[(r1 x1 t + 1 z)^5]
      Series[(c12 x1^2 t^2 z + c11 x1 t z^2 + z^3)^-1, {x1, 0, 2}], t, 2], 1]] /. x1^2 → 1

{0, 0, 0, 10 r1^2, -5 c11 r1, c11^2 - c12}

μx5Y2 =
 Factor[CoefficientList[
    Coefficient[Expand[(r2 x2 t + (a2 + 1) z)^5]
      Series[(c22 x2^2 t^2 z + c21 x2 t z^2 + z^3)^-1, {x2, 0, 2}], t, 2], 1]] /. x2^2 → -1

{-a2^3 (a2^2 c21^2 - a2^2 c22 - 5 a2 c21 r2 + 10 r2^2),
 -5 a2^2 (a2^2 c21^2 - a2^2 c22 - 4 a2 c21 r2 + 6 r2^2), -10 a2 (a2^2 c21^2 - a2^2 c22 - 3 a2 c21 r2 + 3 r2^2),
 -10 (a2^2 c21^2 - a2^2 c22 - 2 a2 c21 r2 + r2^2), -5 (a2 c21^2 - a2 c22 - c21 r2), -c21^2 + c22}

(******************************
 NOTATIONS
 (yij + z) denotes the formal normal roots at Yi
 μs1x3Yi denotes the local datum of p1(X) x^3 at Yi
  ******************************)

SymmetricReduction[Expand[(y11 t + z)^2 + (y12 t + z)^2 + (y13 t + z)^2],
 {y11, y12, y13}, {c11 x1, c12 x1^2, c13 x1^3}]

{c11^2 t^2 x1^2 - 2 c12 t^2 x1^2 + 2 c11 t x1 z + 3 z^2, 0}

μs1x3Y1 =
 Factor[CoefficientList[
    Coefficient[(pY1 t^2 + c11^2 t^2 x1^2 - 2 c12 t^2 x1^2 + 2 c11 t x1 z + 3 z^2)
      Expand[(r1 x1 t + 1 z)^3] Series[(c12 x1^2 t^2 z + c11 x1 t z^2 + z^3)^-1, {x1, 0, 2}], t, 2],
    1]] /. x1^2 → 1

{0, 9 r1^2, -3 c11 r1, 2 c11^2 - 5 c12 + pY1}
```

```
SymmetricReduction[Expand[(y21 t + z)^2 + (y22 t + z)^2 + (y23 t + z)^2],
 {y21, y22, y23}, {c21 x2, c22 x2^2, c23 x2^3}]
```

$$\left\{c21^2\, t^2\, x2^2 - 2\, c22\, t^2\, x2^2 + 2\, c21\, t\, x2\, z + 3\, z^2,\ 0\right\}$$

```
μs1x3Y2 =
 Factor[CoefficientList[
    Coefficient[(pY2 t^2 + c21^2 t^2 x2^2 - 2 c22 t^2 x2^2 + 2 c21 t x2 z + 3 z^2)
      Expand[(r2 x2 t + (a2 + l) z)^3] Series[(c22 x2^2 t^2 z + c21 x2 t z^2 + z^3)^-1, {x2, 0, 2}],
     t, 2], l]] /. x2^2 → -1
```

$$\left\{a2\left(-2\, a2^2\, c21^2 + 5\, a2^2\, c22 + a2^2\, pY2 + 3\, a2\, c21\, r2 - 9\, r2^2\right),\right.$$
$$3\left(-2\, a2^2\, c21^2 + 5\, a2^2\, c22 + a2^2\, pY2 + 2\, a2\, c21\, r2 - 3\, r2^2\right),$$
$$\left.3\left(-2\, a2\, c21^2 + 5\, a2\, c22 + a2\, pY2 + c21\, r2\right),\ -2\, c21^2 + 5\, c22 + pY2\right\}$$

```
(******************************
 We now sum the local datum at Y1 and Y2
  ******************************)
```

```
Factor[μx5Y1 + μx5Y2]
```

$$\left\{-a2^3\left(a2^2\, c21^2 - a2^2\, c22 - 5\, a2\, c21\, r2 + 10\, r2^2\right),\right.$$
$$-5\, a2^2\left(a2^2\, c21^2 - a2^2\, c22 - 4\, a2\, c21\, r2 + 6\, r2^2\right),\ -10\, a2\left(a2^2\, c21^2 - a2^2\, c22 - 3\, a2\, c21\, r2 + 3\, r2^2\right),$$
$$10\left(-a2^2\, c21^2 + a2^2\, c22 + r1^2 + 2\, a2\, c21\, r2 - r2^2\right),$$
$$\left.-5\left(a2\, c21^2 - a2\, c22 + c11\, r1 - c21\, r2\right),\ c11^2 - c12 - c21^2 + c22\right\}$$

```
Factor[μs1x3Y1 + μs1x3Y2]
```

$$\left\{-a2\left(2\, a2^2\, c21^2 - 5\, a2^2\, c22 - a2^2\, pY2 - 3\, a2\, c21\, r2 + 9\, r2^2\right),\right.$$
$$3\left(-2\, a2^2\, c21^2 + 5\, a2^2\, c22 + a2^2\, pY2 + 3\, r1^2 + 2\, a2\, c21\, r2 - 3\, r2^2\right),$$
$$\left.-3\left(2\, a2\, c21^2 - 5\, a2\, c22 - a2\, pY2 + c11\, r1 - c21\, r2\right),\ 2\, c11^2 - 5\, c12 - 2\, c21^2 + 5\, c22 + pY1 + pY2\right\}$$

```
Solve[{
   d == -a2^3 (a2^2 c21^2 - a2^2 c22 - 5 a2 c21 r2 + 10 r2^2),
   0 == -5 a2^2 (a2^2 c21^2 - a2^2 c22 - 4 a2 c21 r2 + 6 r2^2),
   0 == -10 a2 (a2^2 c21^2 - a2^2 c22 - 3 a2 c21 r2 + 3 r2^2),
   0 == 10 (-a2^2 c21^2 + a2^2 c22 + r1^2 + 2 a2 c21 r2 - r2^2),
   0 == -5 (a2 c21^2 - a2 c22 + c11 r1 - c21 r2),
   0 == c11^2 - c12 - c21^2 + c22
  },
  {d, r2, c11, c12, c21, c22}]
```

```
(******************************
 The coefficients in l gives us a system of equations.
  ******************************)
```

```
Solve[{
  d == -a2^3 (a2^2 c21^2 - a2^2 c22 - 5 a2 c21 r2 + 10 r2^2),
  0 == -5 a2^2 (a2^2 c21^2 - a2^2 c22 - 4 a2 c21 r2 + 6 r2^2),
  0 == -10 a2 (a2^2 c21^2 - a2^2 c22 - 3 a2 c21 r2 + 3 r2^2),
  0 == 10 (-a2^2 c21^2 + a2^2 c22 + r1^2 + 2 a2 c21 r2 - r2^2),
  0 == -5 (a2 c21^2 - a2 c22 + c11 r1 - c21 r2),
  0 == c11^2 - c12 - c21^2 + c22
 },
 {d, r2, c11, c12, c21, c22}]
```

$$\left\{\left\{d\to -a2^3\,r1^2,\ r2\to -r1,\ c11\to -\frac{3\,r1}{a2},\ c12\to \frac{3\,r1^2}{a2^2},\ c21\to -\frac{3\,r1}{a2},\ c22\to \frac{3\,r1^2}{a2^2}\right\},\right.$$
$$\left.\left\{d\to -a2^3\,r1^2,\ r2\to r1,\ c11\to -\frac{3\,r1}{a2},\ c12\to \frac{3\,r1^2}{a2^2},\ c21\to \frac{3\,r1}{a2},\ c22\to \frac{3\,r1^2}{a2^2}\right\}\right\}$$

```
Solve[{
  (*******************************
   The following equations are a copy paste of the equations above
   *******************************)
  d == -a2^3 (a2^2 c21^2 - a2^2 c22 - 5 a2 c21 r2 + 10 r2^2),
  0 == -5 a2^2 (a2^2 c21^2 - a2^2 c22 - 4 a2 c21 r2 + 6 r2^2),
  0 == -10 a2 (a2^2 c21^2 - a2^2 c22 - 3 a2 c21 r2 + 3 r2^2),
  0 == 10 (-a2^2 c21^2 + a2^2 c22 + r1^2 + 2 a2 c21 r2 - r2^2),
  0 == -5 (a2 c21^2 - a2 c22 + c11 r1 - c21 r2),
  0 == c11^2 - c12 - c21^2 + c22,
  (*******************************
   The following equations are given by the integration of p1(X)x^3 locally
   *******************************)
  p d == -a2 (2 a2^2 c21^2 - 5 a2^2 c22 - a2^2 pY2 - 3 a2 c21 r2 + 9 r2^2),
  0 == 3 (-2 a2^2 c21^2 + 5 a2^2 c22 + a2^2 pY2 + 3 r1^2 + 2 a2 c21 r2 - 3 r2^2),
  0 == -3 (2 a2 c21^2 - 5 a2 c22 - a2 pY2 + c11 r1 - c21 r2),
  0 == 2 c11^2 - 5 c12 - 2 c21^2 + 5 c22 + pY1 + pY2,
  pY1 == 3,
  pY2 == -3
 },
 {p, d, pY1, pY2, c21, c22, c11, c12, r1, r2}]
```

$$\{\{d\to 0,\ pY1\to 3,\ pY2\to -3,\ c21\to -1,\ c22\to 1,\ c11\to -1,\ c12\to 1,\ r1\to 0,\ r2\to 0\},$$
$$\{d\to 0,\ pY1\to 3,\ pY2\to -3,\ c21\to -1,\ c22\to 1,\ c11\to 1,\ c12\to 1,\ r1\to 0,\ r2\to 0\},$$
$$\{d\to 0,\ pY1\to 3,\ pY2\to -3,\ c21\to 1,\ c22\to 1,\ c11\to -1,\ c12\to 1,\ r1\to 0,\ r2\to 0\},$$
$$\{d\to 0,\ pY1\to 3,\ pY2\to -3,\ c21\to 1,\ c22\to 1,\ c11\to 1,\ c12\to 1,\ r1\to 0,\ r2\to 0\},$$
$$\left\{p\to \frac{6}{a2^2},\ d\to -a2^5,\ pY1\to 3,\ pY2\to -3,\ c21\to -3,\ c22\to 3,\ c11\to -3,\ c12\to 3,\ r1\to a2,\ r2\to -a2\right\},$$
$$\left\{p\to \frac{6}{a2^2},\ d\to -a2^5,\ pY1\to 3,\ pY2\to -3,\ c21\to -3,\ c22\to 3,\ c11\to 3,\ c12\to 3,\ r1\to -a2,\ r2\to -a2\right\},$$
$$\left\{p\to \frac{6}{a2^2},\ d\to -a2^5,\ pY1\to 3,\ pY2\to -3,\ c21\to 3,\ c22\to 3,\ c11\to -3,\ c12\to 3,\ r1\to a2,\ r2\to a2\right\},$$
$$\left\{p\to \frac{6}{a2^2},\ d\to -a2^5,\ pY1\to 3,\ pY2\to -3,\ c21\to 3,\ c22\to 3,\ c11\to 3,\ c12\to 3,\ r1\to -a2,\ r2\to a2\right\}\}$$

Computations for Lemma 6.4.

The case where there exists N^6 s.t. Fix(N) = $Y_1{}^4 \cup Y_2{}^4$.

Case A where the normal weights are (n1, 1, 1).

```
(*******************************
  NOTATIONS
  ri xi denotes the restriction of x to Yi
  ai are the Hopf weight at Yi
  cij xi^j denotes the j-th Chern class of Yi
  cin1 xi denotes the first Chern class of the normal bundle of Yi in
  the direction of N1
  μx5YiA denotes the local datum of x^5 at Yi

  We use the variable t to extract the coefficient of order 2 of xi. We
  then only evaluate xi^2 to 1 or -1.
   *******************************)
```

```
μx5Y1A =
 Factor[CoefficientList[
    Coefficient[Expand[(r1 x1 t + 1 z)^5] Series[(c1n1 x1 t + n1 z)^-1, {x1, 0, 2}]
      Series[(c12 x1^2 t^2 + c11 x1 t z + z^2)^-1, {x1, 0, 2}], t, 2], 1]] /. x1^2 → 1
```

$$\left\{0,\ 0,\ 0,\ \frac{10\,\mathrm{r1}^2}{\mathrm{n1}},\ -\frac{5\,(\mathrm{c1n1}+\mathrm{c11}\,\mathrm{n1})\,\mathrm{r1}}{\mathrm{n1}^2},\ \frac{\mathrm{c1n1}^2+\mathrm{c11}\,\mathrm{c1n1}\,\mathrm{n1}+\mathrm{c11}^2\,\mathrm{n1}^2-\mathrm{c12}\,\mathrm{n1}^2}{\mathrm{n1}^3}\right\}$$

```
μx5Y2A =
 Factor[CoefficientList[
    Coefficient[Expand[(r2 x2 t + (a2 + 1) z)^5] Series[(c2n1 x2 t + n1 z)^-1, {x2, 0, 2}]
      Series[(c22 x2^2 t^2 + c21 x2 t z + z^2)^-1, {x2, 0, 2}], t, 2], 1]] /. x2^2 → -1
```

$$\Big\{-\frac{1}{\mathrm{n1}^3}\,\mathrm{a2}^3\,\big(\mathrm{a2}^2\,\mathrm{c2n1}^2+\mathrm{a2}^2\,\mathrm{c21}\,\mathrm{c2n1}\,\mathrm{n1}+\mathrm{a2}^2\,\mathrm{c21}^2\,\mathrm{n1}^2-\mathrm{a2}^2\,\mathrm{c22}\,\mathrm{n1}^2-5\,\mathrm{a2}\,\mathrm{c2n1}\,\mathrm{n1}\,\mathrm{r2}-5\,\mathrm{a2}\,\mathrm{c21}\,\mathrm{n1}^2\,\mathrm{r2}+10\,\mathrm{n1}^2\,\mathrm{r2}^2\big),$$
$$-\frac{1}{\mathrm{n1}^3}\,5\,\mathrm{a2}^2\,\big(\mathrm{a2}^2\,\mathrm{c2n1}^2+\mathrm{a2}^2\,\mathrm{c21}\,\mathrm{c2n1}\,\mathrm{n1}+\mathrm{a2}^2\,\mathrm{c21}^2\,\mathrm{n1}^2-\mathrm{a2}^2\,\mathrm{c22}\,\mathrm{n1}^2-4\,\mathrm{a2}\,\mathrm{c2n1}\,\mathrm{n1}\,\mathrm{r2}-4\,\mathrm{a2}\,\mathrm{c21}\,\mathrm{n1}^2\,\mathrm{r2}+6\,\mathrm{n1}^2\,\mathrm{r2}^2\big),$$
$$-\frac{1}{\mathrm{n1}^3}\,10\,\mathrm{a2}\,\big(\mathrm{a2}^2\,\mathrm{c2n1}^2+\mathrm{a2}^2\,\mathrm{c21}\,\mathrm{c2n1}\,\mathrm{n1}+\mathrm{a2}^2\,\mathrm{c21}^2\,\mathrm{n1}^2-\mathrm{a2}^2\,\mathrm{c22}\,\mathrm{n1}^2-3\,\mathrm{a2}\,\mathrm{c2n1}\,\mathrm{n1}\,\mathrm{r2}-3\,\mathrm{a2}\,\mathrm{c21}\,\mathrm{n1}^2\,\mathrm{r2}+3\,\mathrm{n1}^2\,\mathrm{r2}^2\big),$$
$$-\frac{1}{\mathrm{n1}^3}\,10\,\big(\mathrm{a2}^2\,\mathrm{c2n1}^2+\mathrm{a2}^2\,\mathrm{c21}\,\mathrm{c2n1}\,\mathrm{n1}+\mathrm{a2}^2\,\mathrm{c21}^2\,\mathrm{n1}^2-\mathrm{a2}^2\,\mathrm{c22}\,\mathrm{n1}^2-2\,\mathrm{a2}\,\mathrm{c2n1}\,\mathrm{n1}\,\mathrm{r2}-2\,\mathrm{a2}\,\mathrm{c21}\,\mathrm{n1}^2\,\mathrm{r2}+\mathrm{n1}^2\,\mathrm{r2}^2\big),$$
$$-\frac{1}{\mathrm{n1}^3}\,5\,\big(\mathrm{a2}\,\mathrm{c2n1}^2+\mathrm{a2}\,\mathrm{c21}\,\mathrm{c2n1}\,\mathrm{n1}+\mathrm{a2}\,\mathrm{c21}^2\,\mathrm{n1}^2-\mathrm{a2}\,\mathrm{c22}\,\mathrm{n1}^2-\mathrm{c2n1}\,\mathrm{n1}\,\mathrm{r2}-\mathrm{c21}\,\mathrm{n1}^2\,\mathrm{r2}\big),$$
$$-\frac{\mathrm{c2n1}^2+\mathrm{c21}\,\mathrm{c2n1}\,\mathrm{n1}+\mathrm{c21}^2\,\mathrm{n1}^2-\mathrm{c22}\,\mathrm{n1}^2}{\mathrm{n1}^3}\Big\}$$

```
(*******************************
 NOTATIONS
 (yij + z) denotes the formal normal roots at Yi in the direction
 perpedicular to N1
 μs1x3YiA denotes the local datum of p1(X)x^3 at Yi
  *******************************)
```

```
SymmetricReduction[Expand[(y11 t + z)^2 + (y12 t + z)^2], {y11, y12}, {c11 x1, c12 x1^2}]
```

$$\{\mathrm{c11}^2\,t^2\,\mathrm{x1}^2-2\,\mathrm{c12}\,t^2\,\mathrm{x1}^2+2\,\mathrm{c11}\,t\,\mathrm{x1}\,z+2\,z^2,\ 0\}$$

```
μs1x3Y1A =
 Factor[CoefficientList[
   Coefficient[(pY1 t^2 + (c1n1 x1 t + n1 z)^2 + c11^2 t^2 x1^2 - 2 c12 t^2 x1^2 + 2 c11 t x1 z + 2 z^2)
     Expand[(r1 x1 t + 1 z)^3] Series[(c1n1 x1 t + n1 z)^-1, {x1, 0, 2}]
     Series[(c12 x1^2 t^2 + c11 x1 t z + z^2)^-1, {x1, 0, 2}], t, 2], 1]] /. x1^2 → 1
```

$$\left\{0, \frac{3\left(2+n1^2\right) r1^2}{n1}, \frac{3\left(-2\, c1n1+c1n1\, n1^2-c11\, n1^3\right) r1}{n1^2}, \frac{1}{n1^3}\left(2\, c1n1^2+c11^2\, n1^2-4\, c12\, n1^2-c11\, c1n1\, n1^3+c11^2\, n1^4-c12\, n1^4+n1^2\, pY1\right)\right\}$$

```
SymmetricReduction[Expand[(y21 t + z)^2 + (y22 t + z)^2], {y21, y22}, {c21 x2, c22 x2^2}]
```

$$\left\{c21^2\, t^2\, x2^2-2\, c22\, t^2\, x2^2+2\, c21\, t\, x2\, z+2\, z^2, 0\right\}$$

```
μs1x3Y2A =
 Factor[CoefficientList[
   Coefficient[(pY2 t^2 + (c2n1 x2 t + n1 z)^2 + c21^2 t^2 x2^2 - 2 c22 t^2 x2^2 + 2 c21 t x2 z + 2 z^2)
     Expand[(r2 x2 t + (a2 + 1) z)^3] Series[(c2n1 x2 t + n1 z)^-1, {x2, 0, 2}]
     Series[(c22 x2^2 t^2 + c21 x2 t z + z^2)^-1, {x2, 0, 2}], t, 2], 1]] /. x2^2 → -1
```

$$\left\{\frac{1}{n1^3} a2\left(-2\, a2^2\, c2n1^2-a2^2\, c21^2\, n1^2+4\, a2^2\, c22\, n1^2+a2^2\, c21\, c2n1\, n1^3-a2^2\, c21^2\, n1^4+a2^2\, c22\, n1^4+a2^2\, n1^2\, pY2+6\, a2\, c2n1\, n1\, r2-3\, a2\, c2n1\, n1^3\, r2+3\, a2\, c21\, n1^4\, r2-6\, n1^2\, r2^2-3\, n1^4\, r2^2\right),\right.$$
$$\frac{1}{n1^3} 3\left(-2\, a2^2\, c2n1^2-a2^2\, c21^2\, n1^2+4\, a2^2\, c22\, n1^2+a2^2\, c21\, c2n1\, n1^3-a2^2\, c21^2\, n1^4+a2^2\, c22\, n1^4+a2^2\, n1^2\, pY2+4\, a2\, c2n1\, n1\, r2-2\, a2\, c2n1\, n1^3\, r2+2\, a2\, c21\, n1^4\, r2-2\, n1^2\, r2^2-n1^4\, r2^2\right),$$
$$\frac{1}{n1^3} 3\left(-2\, a2\, c2n1^2-a2\, c21^2\, n1^2+4\, a2\, c22\, n1^2+a2\, c21\, c2n1\, n1^3-a2\, c21^2\, n1^4+a2\, c22\, n1^4+a2\, n1^2\, pY2+2\, c2n1\, n1\, r2-c2n1\, n1^3\, r2+c21\, n1^4\, r2\right),$$
$$\left.\frac{1}{n1^3}\left(-2\, c2n1^2-c21^2\, n1^2+4\, c22\, n1^2+c21\, c2n1\, n1^3-c21^2\, n1^4+c22\, n1^4+n1^2\, pY2\right)\right\}$$

```
μx5Y1A + μx5Y2A
```

$$\left\{-\frac{1}{n1^3} a2^3\left(a2^2\, c2n1^2+a2^2\, c21\, c2n1\, n1+a2^2\, c21^2\, n1^2-a2^2\, c22\, n1^2-5\, a2\, c2n1\, n1\, r2-5\, a2\, c21\, n1^2\, r2+10\, n1^2\, r2^2\right),\right.$$
$$-\frac{1}{n1^3} 5\, a2^2\left(a2^2\, c2n1^2+a2^2\, c21\, c2n1\, n1+a2^2\, c21^2\, n1^2-a2^2\, c22\, n1^2-4\, a2\, c2n1\, n1\, r2-4\, a2\, c21\, n1^2\, r2+6\, n1^2\, r2^2\right),$$
$$-\frac{1}{n1^3} 10\, a2\left(a2^2\, c2n1^2+a2^2\, c21\, c2n1\, n1+a2^2\, c21^2\, n1^2-a2^2\, c22\, n1^2-3\, a2\, c2n1\, n1\, r2-3\, a2\, c21\, n1^2\, r2+3\, n1^2\, r2^2\right),$$
$$\frac{10\, r1^2}{n1}-\frac{1}{n1^3} 10\left(a2^2\, c2n1^2+a2^2\, c21\, c2n1\, n1+a2^2\, c21^2\, n1^2-a2^2\, c22\, n1^2-2\, a2\, c2n1\, n1\, r2-2\, a2\, c21\, n1^2\, r2+n1^2\, r2^2\right), -\frac{5\left(c1n1+c11\, n1\right) r1}{n1^2}-$$
$$\frac{1}{n1^3} 5\left(a2\, c2n1^2+a2\, c21\, c2n1\, n1+a2\, c21^2\, n1^2-a2\, c22\, n1^2-c2n1\, n1\, r2-c21\, n1^2\, r2\right),$$
$$\left.\frac{c1n1^2+c11\, c1n1\, n1+c11^2\, n1^2-c12\, n1^2}{n1^3}-\frac{c2n1^2+c21\, c2n1\, n1+c21^2\, n1^2-c22\, n1^2}{n1^3}\right\}$$

μs1x3Y1A + μs1x3Y2A

$$\left\{\frac{1}{n1^3} a2 \left(-2\, a2^2\, c2n1^2 - a2^2\, c21^2\, n1^2 + 4\, a2^2\, c22\, n1^2 + a2^2\, c21\, c2n1\, n1^3 - a2^2\, c21^2\, n1^4 + a2^2\, c22\, n1^4 + a2^2\, n1^2\, pY2 + 6\, a2\, c2n1\, n1\, r2 - 3\, a2\, c2n1\, n1^3\, r2 + 3\, a2\, c21\, n1^4\, r2 - 6\, n1^2\, r2^2 - 3\, n1^4\, r2^2\right),\right.$$
$$\frac{3\left(2+n1^2\right) r1^2}{n1} + \frac{1}{n1^3} 3 \left(-2\, a2^2\, c2n1^2 - a2^2\, c21^2\, n1^2 + 4\, a2^2\, c22\, n1^2 + a2^2\, c21\, c2n1\, n1^3 - a2^2\, c21^2\, n1^4 + a2^2\, c22\, n1^4 + a2^2\, n1^2\, pY2 + 4\, a2\, c2n1\, n1\, r2 - 2\, a2\, c2n1\, n1^3\, r2 + 2\, a2\, c21\, n1^4\, r2 - 2\, n1^2\, r2^2 - n1^4\, r2^2\right), \frac{3\left(-2\, c1n1 + c1n1\, n1^2 - c11\, n1^3\right) r1}{n1^2} +$$
$$\frac{1}{n1^3} 3 \left(-2\, a2\, c2n1^2 - a2\, c21^2\, n1^2 + 4\, a2\, c22\, n1^2 + a2\, c21\, c2n1\, n1^3 - a2\, c21^2\, n1^4 + a2\, c22\, n1^4 + a2\, n1^2\, pY2 + 2\, c2n1\, n1\, r2 - c2n1\, n1^3\, r2 + c21\, n1^4\, r2\right),$$
$$\frac{1}{n1^3}\left(2\, c1n1^2 + c11^2\, n1^2 - 4\, c12\, n1^2 - c11\, c1n1\, n1^3 + c11^2\, n1^4 - c12\, n1^4 + n1^2\, pY1\right) +$$
$$\left.\frac{1}{n1^3}\left(-2\, c2n1^2 - c21^2\, n1^2 + 4\, c22\, n1^2 + c21\, c2n1\, n1^3 - c21^2\, n1^4 + c22\, n1^4 + n1^2\, pY2\right)\right\}$$

Factor[Solve[{

$$\mathbf{d} == -\frac{1}{n1^3} a2^3 \left(a2^2\, c2n1^2 + a2^2\, c21\, c2n1\, n1 + a2^2\, c21^2\, n1^2 - a2^2\, c22\, n1^2 - 5\, a2\, c2n1\, n1\, r2 - 5\, a2\, c21\, n1^2\, r2 + 10\, n1^2\, r2^2\right),$$
$$0 == -\frac{1}{n1^3} 5\, a2^2 \left(a2^2\, c2n1^2 + a2^2\, c21\, c2n1\, n1 + a2^2\, c21^2\, n1^2 - a2^2\, c22\, n1^2 - 4\, a2\, c2n1\, n1\, r2 - 4\, a2\, c21\, n1^2\, r2 + 6\, n1^2\, r2^2\right),$$
$$0 == -\frac{1}{n1^3} 10\, a2 \left(a2^2\, c2n1^2 + a2^2\, c21\, c2n1\, n1 + a2^2\, c21^2\, n1^2 - a2^2\, c22\, n1^2 - 3\, a2\, c2n1\, n1\, r2 - 3\, a2\, c21\, n1^2\, r2 + 3\, n1^2\, r2^2\right),$$
$$0 == \frac{10\, r1^2}{n1} - \frac{1}{n1^3} 10 \left(a2^2\, c2n1^2 + a2^2\, c21\, c2n1\, n1 + a2^2\, c21^2\, n1^2 - a2^2\, c22\, n1^2 - 2\, a2\, c2n1\, n1\, r2 - 2\, a2\, c21\, n1^2\, r2 + n1^2\, r2^2\right)$$

}, {d, c22, r2, r1}]]

$$\left\{\left\{d \to 0,\ c22 \to \frac{c2n1^2 + c21\, c2n1\, n1 + c21^2\, n1^2}{n1^2},\ r2 \to 0,\ r1 \to 0\right\},\right.$$
$$\left\{d \to -\frac{a2^5\left(c2n1 + c21\, n1\right)^2}{9\, n1^3},\ c22 \to \frac{c2n1^2 - c21\, c2n1\, n1 + c21^2\, n1^2}{3\, n1^2},\ r2 \to \frac{a2\left(c2n1 + c21\, n1\right)}{3\, n1},\ r1 \to -\frac{a2\left(c2n1 + c21\, n1\right)}{3\, n1}\right\},\ \left\{d \to -\frac{a2^5\left(c2n1 + c21\, n1\right)^2}{9\, n1^3},\right.$$
$$\left.\left.c22 \to \frac{c2n1^2 - c21\, c2n1\, n1 + c21^2\, n1^2}{3\, n1^2},\ r2 \to \frac{a2\left(c2n1 + c21\, n1\right)}{3\, n1},\ r1 \to \frac{a2\left(c2n1 + c21\, n1\right)}{3\, n1}\right\}\right\}$$

```
Factor[Solve[{
   (******************************
    The following equations are the second and third solutions of the above system
     ******************************)
```

$$d == -\frac{a2^5\,(c2n1 + c21\,n1)^2}{9\,n1^3},\ c22 == \frac{c2n1^2 - c21\,c2n1\,n1 + c21^2\,n1^2}{3\,n1^2},\ r2 == \frac{a2\,(c2n1 + c21\,n1)}{3\,n1},$$

$$r1^2 == r2^2,\ pY1 == 3,\ pY2 == -3,$$

```
   (******************************
     In addition, we have the equation given by integrating locally p1(X)x^3
     ******************************)
```

$$p\,d ==$$

$$\frac{1}{n1^3}\,a2\left(-2\,a2^2\,c2n1^2 - a2^2\,c21^2\,n1^2 + 4\,a2^2\,c22\,n1^2 + a2^2\,c21\,c2n1\,n1^3 - a2^2\,c21^2\,n1^4 + a2^2\,c22\,n1^4 + a2^2\,n1^2\,pY2 + 6\,a2\,c2n1\,n1\,r2 - 3\,a2\,c2n1\,n1^3\,r2 + 3\,a2\,c21\,n1^4\,r2 - 6\,n1^2\,r2^2 - 3\,n1^4\,r2^2\right),$$

```
   (******************************
    Furthermore, the following equations are given by Lemma 6.3
     ******************************)
```

$$0 == \frac{3\,r1^2}{n1} - \frac{3\,(a2\,c2n1 - n1\,r2)^2}{n1^3},$$

$$0 == -\frac{3\,c1n1\,r1}{n1^2} - \frac{3\,c2n1\,(a2\,c2n1 - n1\,r2)}{n1^3},$$

$$0 == \frac{c1n1^2}{n1^3} - \frac{c2n1^2}{n1^3}$$

```
  }, {p, d, pY1, pY2, c2n1, c21, c22, r1, r2}]]
```

$$\left\{\left\{p \to \frac{-c1n1^2 + 12\,n1^2 + 4\,c1n1^2\,n1^2}{a2^2\,c1n1^2},\ d \to -\frac{a2^5\,c1n1^2}{4\,n1^3},\ pY1 \to 3,\ pY2 \to -3,\ c2n1 \to -c1n1,\ c21 \to -\frac{c1n1}{2\,n1},\ c22 \to \frac{c1n1^2}{4\,n1^2},\ r1 \to -\frac{a2\,c1n1}{2\,n1},\ r2 \to -\frac{a2\,c1n1}{2\,n1}\right\},\right.$$

$$\left.\left\{p \to \frac{-c1n1^2 + 12\,n1^2 + 4\,c1n1^2\,n1^2}{a2^2\,c1n1^2},\ d \to -\frac{a2^5\,c1n1^2}{4\,n1^3},\ pY1 \to 3,\ pY2 \to -3,\ c2n1 \to c1n1,\ c21 \to \frac{c1n1}{2\,n1},\ c22 \to \frac{c1n1^2}{4\,n1^2},\ r1 \to -\frac{a2\,c1n1}{2\,n1},\ r2 \to \frac{a2\,c1n1}{2\,n1}\right\}\right\}$$

Computations for Lemma 6.4. The case where there exists $N_1{}^6$ and $N_2{}^6$ s.t. Fix(N_i) = $Y_1{}^4$ U $Y_2{}^4$.

Case B where the normal weights are (n1, n2, 1).

```
(******************************
  NOTATIONS
  ri xi denotes the restriction of x to Yi
  ai are the Hopf weight at Yi
  cij xi^j denotes the j-th Chern class of Yi
  cin1 xi denotes the first Chern class of the normal bundle of Yi in
  the direction of N1
  cin2 xi denotes the first Chern class of the normal bundle of Yi in
  the direction of N2
  μx5YiB denotes the local datum of x^5 at Yi

  We use the variable t to extract the coefficient of order 2 of xi. We
  then only evaluate xi^2 to 1 or -1.
   *****************************)
```

```
μx5Y1B =
 Factor[CoefficientList[
    Coefficient[Expand[(r1 x1 t + l z)^5] Series[(c1n1 x1 t + n1 z)^-1, {x1, 0, 2}]
      Series[(c1n2 x1 t + n2 z)^-1, {x1, 0, 2}] Series[(c11 x1 t + z)^-1, {x1, 0, 2}], t, 2],
    l]] /. x1^2 → 1
```

$$\left\{0,\ 0,\ 0,\ \frac{10\,r1^2}{n1\,n2},\ -\frac{5\,(c1n2\,n1+c1n1\,n2+c11\,n1\,n2)\,r1}{n1^2\,n2^2},\ \frac{c1n2^2\,n1^2+c1n1\,c1n2\,n1\,n2+c11\,c1n2\,n1^2\,n2+c1n1^2\,n2^2+c11\,c1n1\,n1\,n2^2+c11^2\,n1^2\,n2^2}{n1^3\,n2^3}\right\}$$

```
μx5Y2B =
 Factor[CoefficientList[
    Coefficient[Expand[(r2 x2 t + (a2 + 1) z)^5] Series[(c2n1 x2 t + n1 z)^-1, {x2, 0, 2}]
      Series[(c2n2 x2 t + n2 z)^-1, {x2, 0, 2}] Series[(c21 x2 t + z)^-1, {x2, 0, 2}], t, 2],
    1]] /. x2^2 → -1
```

$$
\begin{aligned}
\Big\{ & -\frac{1}{n1^{3}\, n2^{3}}\, a2^{3}\, \big(a2^{2}\, c2n2^{2}\, n1^{2} + a2^{2}\, c2n1\, c2n2\, n1\, n2 + {} \\
& \quad a2^{2}\, c21\, c2n2\, n1^{2}\, n2 + a2^{2}\, c2n1^{2}\, n2^{2} + a2^{2}\, c21\, c2n1\, n1\, n2^{2} + a2^{2}\, c21^{2}\, n1^{2}\, n2^{2} - {} \\
& \quad 5\, a2\, c2n2\, n1^{2}\, n2\, r2 - 5\, a2\, c2n1\, n1\, n2^{2}\, r2 - 5\, a2\, c21\, n1^{2}\, n2^{2}\, r2 + 10\, n1^{2}\, n2^{2}\, r2^{2}\big), \\
& -\frac{1}{n1^{3}\, n2^{3}}\, 5\, a2^{2}\, \big(a2^{2}\, c2n2^{2}\, n1^{2} + a2^{2}\, c2n1\, c2n2\, n1\, n2 + a2^{2}\, c21\, c2n2\, n1^{2}\, n2 + {} \\
& \quad a2^{2}\, c2n1^{2}\, n2^{2} + a2^{2}\, c21\, c2n1\, n1\, n2^{2} + a2^{2}\, c21^{2}\, n1^{2}\, n2^{2} - 4\, a2\, c2n2\, n1^{2}\, n2\, r2 - {} \\
& \quad 4\, a2\, c2n1\, n1\, n2^{2}\, r2 - 4\, a2\, c21\, n1^{2}\, n2^{2}\, r2 + 6\, n1^{2}\, n2^{2}\, r2^{2}\big), \\
& -\frac{1}{n1^{3}\, n2^{3}}\, 10\, a2\, \big(a2^{2}\, c2n2^{2}\, n1^{2} + a2^{2}\, c2n1\, c2n2\, n1\, n2 + a2^{2}\, c21\, c2n2\, n1^{2}\, n2 + {} \\
& \quad a2^{2}\, c2n1^{2}\, n2^{2} + a2^{2}\, c21\, c2n1\, n1\, n2^{2} + a2^{2}\, c21^{2}\, n1^{2}\, n2^{2} - 3\, a2\, c2n2\, n1^{2}\, n2\, r2 - {} \\
& \quad 3\, a2\, c2n1\, n1\, n2^{2}\, r2 - 3\, a2\, c21\, n1^{2}\, n2^{2}\, r2 + 3\, n1^{2}\, n2^{2}\, r2^{2}\big), \\
& -\frac{1}{n1^{3}\, n2^{3}}\, 10\, \big(a2^{2}\, c2n2^{2}\, n1^{2} + a2^{2}\, c2n1\, c2n2\, n1\, n2 + a2^{2}\, c21\, c2n2\, n1^{2}\, n2 + a2^{2}\, c2n1^{2}\, n2^{2} + {} \\
& \quad a2^{2}\, c21\, c2n1\, n1\, n2^{2} + a2^{2}\, c21^{2}\, n1^{2}\, n2^{2} - 2\, a2\, c2n2\, n1^{2}\, n2\, r2 - {} \\
& \quad 2\, a2\, c2n1\, n1\, n2^{2}\, r2 - 2\, a2\, c21\, n1^{2}\, n2^{2}\, r2 + n1^{2}\, n2^{2}\, r2^{2}\big), \\
& -\frac{1}{n1^{3}\, n2^{3}}\, 5\, \big(a2\, c2n2^{2}\, n1^{2} + a2\, c2n1\, c2n2\, n1\, n2 + a2\, c21\, c2n2\, n1^{2}\, n2 + a2\, c2n1^{2}\, n2^{2} + {} \\
& \quad a2\, c21\, c2n1\, n1\, n2^{2} + a2\, c21^{2}\, n1^{2}\, n2^{2} - c2n2\, n1^{2}\, n2\, r2 - c2n1\, n1\, n2^{2}\, r2 - c21\, n1^{2}\, n2^{2}\, r2\big), \\
& -\frac{c2n2^{2}\, n1^{2} + c2n1\, c2n2\, n1\, n2 + c21\, c2n2\, n1^{2}\, n2 + c2n1^{2}\, n2^{2} + c21\, c2n1\, n1\, n2^{2} + c21^{2}\, n1^{2}\, n2^{2}}{n1^{3}\, n2^{3}} \Big\}
\end{aligned}
$$

```
(******************************
 NOTATIONS
 μs1x3YiB denotes the local datum of p1(X)x^3 at Yi
 ******************************)
```

```
μs1x3Y1B =
 Factor[CoefficientList[
    Coefficient[(pY1 t^2 + (c1n1 x1 t + n1 z)^2 + (c1n2 x1 t + n2 z)^2 + (c11 x1 t + z)^2)
      Expand[(r1 x1 t + 1 z)^3] Series[(c1n1 x1 t + n1 z)^-1, {x1, 0, 2}]
      Series[(c1n2 x1 t + n2 z)^-1, {x1, 0, 2}] Series[(c11 x1 t + z)^-1, {x1, 0, 2}], t, 2],
    1]] /. x1^2 → 1
```

$$
\begin{aligned}
\Big\{ & 0,\ \frac{3\,\big(1 + n1^{2} + n2^{2}\big)\, r1^{2}}{n1\, n2}, \\
& -\frac{1}{n1^{2}\, n2^{2}}\, 3\, \big(c1n2\, n1 + c1n2\, n1^{3} + c1n1\, n2 - c11\, n1\, n2 - c1n1\, n1^{2}\, n2 + c11\, n1^{3}\, n2 - {} \\
& \quad c1n2\, n1\, n2^{2} + c1n1\, n2^{3} + c11\, n1\, n2^{3}\big)\, r1, \\
& \frac{1}{n1^{3}\, n2^{3}}\, \big(c1n2^{2}\, n1^{2} + c1n2^{2}\, n1^{4} + c1n1\, c1n2\, n1\, n2 - c11\, c1n2\, n1^{2}\, n2 - c1n1\, c1n2\, n1^{3}\, n2 + {} \\
& \quad c11\, c1n2\, n1^{4}\, n2 + c1n1^{2}\, n2^{2} - c11\, c1n1\, n1\, n2^{2} - c11\, c1n1\, n1^{3}\, n2^{2} + c11^{2}\, n1^{4}\, n2^{2} - {} \\
& \quad c1n1\, c1n2\, n1\, n2^{3} - c11\, c1n2\, n1^{2}\, n2^{3} + c1n1^{2}\, n2^{4} + c11\, c1n1\, n1\, n2^{4} + c11^{2}\, n1^{2}\, n2^{4} + n1^{2}\, n2^{2}\, pY1\big) \Big\}
\end{aligned}
$$

```
μs1x3Y2B =
 Factor[CoefficientList[
    Coefficient[(pY2 t^2 + (c2n1 x2 t + n1 z)^2 + (c2n2 x2 t + n2 z)^2 + (c21 x2 t + z)^2)
      Expand[(r2 x2 t + (a2 + 1) z)^3] Series[(c2n1 x2 t + n1 z)^-1, {x2, 0, 2}]
      Series[(c2n2 x2 t + n2 z)^-1, {x2, 0, 2}] Series[(c21 x2 t + z)^-1, {x2, 0, 2}], t, 2],
    1]] /. x2^2 → -1
```

$$\Big\{\frac{1}{n1^3\, n2^3} a2 \big(-a2^2\, c2n2^2\, n1^2 - a2^2\, c2n2^2\, n1^4 - a2^2\, c2n1\, c2n2\, n1\, n2 + a2^2\, c21\, c2n2\, n1^2\, n2 + a2^2\, c2n1\, c2n2\, n1^3\, n2 - a2^2\, c21\, c2n2\, n1^4\, n2 - a2^2\, c2n1^2\, n2^2 + a2^2\, c21\, c2n1\, n1\, n2^2 + a2^2\, c21\, c2n1\, n1^3\, n2^2 - a2^2\, c21^2\, n1^4\, n2^2 + a2^2\, c2n1\, c2n2\, n1\, n2^3 + a2^2\, c21\, c2n2\, n1^2\, n2^3 - a2^2\, c2n1^2\, n2^4 - a2^2\, c21\, c2n1\, n1\, n2^4 - a2^2\, c21^2\, n1^2\, n2^4 + a2^2\, n1^2\, n2^2\, pY2 + 3\, a2\, c2n2\, n1^2\, n2\, r2 + 3\, a2\, c2n2\, n1^4\, n2\, r2 + 3\, a2\, c2n1\, n1\, n2^2\, r2 - 3\, a2\, c21\, n1^2\, n2^2\, r2 - 3\, a2\, c2n1\, n1^3\, n2^2\, r2 + 3\, a2\, c21\, n1^4\, n2^2\, r2 - 3\, a2\, c2n2\, n1^2\, n2^3\, r2 + 3\, a2\, c2n1\, n1\, n2^4\, r2 + 3\, a2\, c21\, n1^2\, n2^4\, r2 - 3\, n1^2\, n2^2\, r2^2 - 3\, n1^4\, n2^2\, r2^2 - 3\, n1^2\, n2^4\, r2^2\big),$$

$$\frac{1}{n1^3\, n2^3} 3 \big(-a2^2\, c2n2^2\, n1^2 - a2^2\, c2n2^2\, n1^4 - a2^2\, c2n1\, c2n2\, n1\, n2 + a2^2\, c21\, c2n2\, n1^2\, n2 + a2^2\, c2n1\, c2n2\, n1^3\, n2 - a2^2\, c21\, c2n2\, n1^4\, n2 - a2^2\, c2n1^2\, n2^2 + a2^2\, c21\, c2n1\, n1\, n2^2 + a2^2\, c21\, c2n1\, n1^3\, n2^2 - a2^2\, c21^2\, n1^4\, n2^2 + a2^2\, c2n1\, c2n2\, n1\, n2^3 + a2^2\, c21\, c2n2\, n1^2\, n2^3 - a2^2\, c2n1^2\, n2^4 - a2^2\, c21\, c2n1\, n1\, n2^4 - a2^2\, c21^2\, n1^2\, n2^4 + a2^2\, n1^2\, n2^2\, pY2 + 2\, a2\, c2n2\, n1^2\, n2\, r2 + 2\, a2\, c2n2\, n1^4\, n2\, r2 + 2\, a2\, c2n1\, n1\, n2^2\, r2 - 2\, a2\, c21\, n1^2\, n2^2\, r2 - 2\, a2\, c2n1\, n1^3\, n2^2\, r2 + 2\, a2\, c21\, n1^4\, n2^2\, r2 - 2\, a2\, c2n2\, n1^2\, n2^3\, r2 + 2\, a2\, c2n1\, n1\, n2^4\, r2 + 2\, a2\, c21\, n1^2\, n2^4\, r2 - n1^2\, n2^2\, r2^2 - n1^4\, n2^2\, r2^2 - n1^2\, n2^4\, r2^2\big),$$

$$\frac{1}{n1^3\, n2^3} 3 \big(-a2\, c2n2^2\, n1^2 - a2\, c2n2^2\, n1^4 - a2\, c2n1\, c2n2\, n1\, n2 + a2\, c21\, c2n2\, n1^2\, n2 + a2\, c2n1\, c2n2\, n1^3\, n2 - a2\, c21\, c2n2\, n1^4\, n2 - a2\, c2n1^2\, n2^2 + a2\, c21\, c2n1\, n1\, n2^2 + a2\, c21\, c2n1\, n1^3\, n2^2 - a2\, c21^2\, n1^4\, n2^2 + a2\, c2n1\, c2n2\, n1\, n2^3 + a2\, c21\, c2n2\, n1^2\, n2^3 - a2\, c2n1^2\, n2^4 - a2\, c21\, c2n1\, n1\, n2^4 - a2\, c21^2\, n1^2\, n2^4 + a2\, n1^2\, n2^2\, pY2 + c2n2\, n1^2\, n2\, r2 + c2n2\, n1^4\, n2\, r2 + c2n1\, n1\, n2^2\, r2 - c21\, n1^2\, n2^2\, r2 - c2n1\, n1^3\, n2^2\, r2 + c21\, n1^4\, n2^2\, r2 - c2n2\, n1^2\, n2^3\, r2 + c2n1\, n1\, n2^4\, r2 + c21\, n1^2\, n2^4\, r2\big),$$

$$\frac{1}{n1^3\, n2^3} \big(-c2n2^2\, n1^2 - c2n2^2\, n1^4 - c2n1\, c2n2\, n1\, n2 + c21\, c2n2\, n1^2\, n2 + c2n1\, c2n2\, n1^3\, n2 - c21\, c2n2\, n1^4\, n2 - c2n1^2\, n2^2 + c21\, c2n1\, n1\, n2^2 + c21\, c2n1\, n1^3\, n2^2 - c21^2\, n1^4\, n2^2 + c2n1\, c2n2\, n1\, n2^3 + c21\, c2n2\, n1^2\, n2^3 - c2n1^2\, n2^4 - c21\, c2n1\, n1\, n2^4 - c21^2\, n1^2\, n2^4 + n1^2\, n2^2\, pY2\big)\Big\}$$

```
Factor[μx5Y1B + μx5Y2B]
```

$$\left\{-\frac{1}{n1^3\, n2^3}\, a2^3\, \left(a2^2\, c2n2^2\, n1^2 + a2^2\, c2n1\, c2n2\, n1\, n2 + a2^2\, c21\, c2n2\, n1^2\, n2 + a2^2\, c2n1^2\, n2^2 + a2^2\, c21\, c2n1\, n1\, n2^2 + a2^2\, c21^2\, n1^2\, n2^2 - 5\, a2\, c2n2\, n1^2\, n2\, r2 - 5\, a2\, c2n1\, n1\, n2^2\, r2 - 5\, a2\, c21\, n1^2\, n2^2\, r2 + 10\, n1^2\, n2^2\, r2^2\right),\right.$$

$$-\frac{1}{n1^3\, n2^3}\, 5\, a2^2\, \left(a2^2\, c2n2^2\, n1^2 + a2^2\, c2n1\, c2n2\, n1\, n2 + a2^2\, c21\, c2n2\, n1^2\, n2 + a2^2\, c2n1^2\, n2^2 + a2^2\, c21\, c2n1\, n1\, n2^2 + a2^2\, c21^2\, n1^2\, n2^2 - 4\, a2\, c2n2\, n1^2\, n2\, r2 - 4\, a2\, c2n1\, n1\, n2^2\, r2 - 4\, a2\, c21\, n1^2\, n2^2\, r2 + 6\, n1^2\, n2^2\, r2^2\right),$$

$$-\frac{1}{n1^3\, n2^3}\, 10\, a2\, \left(a2^2\, c2n2^2\, n1^2 + a2^2\, c2n1\, c2n2\, n1\, n2 + a2^2\, c21\, c2n2\, n1^2\, n2 + a2^2\, c2n1^2\, n2^2 + a2^2\, c21\, c2n1\, n1\, n2^2 + a2^2\, c21^2\, n1^2\, n2^2 - 3\, a2\, c2n2\, n1^2\, n2\, r2 - 3\, a2\, c2n1\, n1\, n2^2\, r2 - 3\, a2\, c21\, n1^2\, n2^2\, r2 + 3\, n1^2\, n2^2\, r2^2\right),$$

$$\frac{1}{n1^3\, n2^3}\, 10\, \left(-a2^2\, c2n2^2\, n1^2 - a2^2\, c2n1\, c2n2\, n1\, n2 - a2^2\, c21\, c2n2\, n1^2\, n2 - a2^2\, c2n1^2\, n2^2 - a2^2\, c21\, c2n1\, n1\, n2^2 - a2^2\, c21^2\, n1^2\, n2^2 + n1^2\, n2^2\, r1^2 + 2\, a2\, c2n2\, n1^2\, n2\, r2 + 2\, a2\, c2n1\, n1\, n2^2\, r2 + 2\, a2\, c21\, n1^2\, n2^2\, r2 - n1^2\, n2^2\, r2^2\right),$$

$$-\frac{1}{n1^3\, n2^3}\, 5\, \left(a2\, c2n2^2\, n1^2 + a2\, c2n1\, c2n2\, n1\, n2 + a2\, c21\, c2n2\, n1^2\, n2 + a2\, c2n1^2\, n2^2 + a2\, c21\, c2n1\, n1\, n2^2 + a2\, c21^2\, n1^2\, n2^2 + c1n2\, n1^2\, n2\, r1 + c1n1\, n1\, n2^2\, r1 + c11\, n1^2\, n2^2\, r1 - c2n2\, n1^2\, n2\, r2 - c2n1\, n1\, n2^2\, r2 - c21\, n1^2\, n2^2\, r2\right),\ \frac{1}{n1^3\, n2^3}$$

$$\left.\left(c1n2^2\, n1^2 - c2n2^2\, n1^2 + c1n1\, c1n2\, n1\, n2 - c2n1\, c2n2\, n1\, n2 + c11\, c1n2\, n1^2\, n2 - c21\, c2n2\, n1^2\, n2 + c1n1^2\, n2^2 - c2n1^2\, n2^2 + c11\, c1n1\, n1\, n2^2 - c21\, c2n1\, n1\, n2^2 + c11^2\, n1^2\, n2^2 - c21^2\, n1^2\, n2^2\right)\right\}$$

```
Factor[μs1x3Y1B + μs1x3Y2B]
```

$$\Big\{-\frac{1}{n1^3\,n2^3}\,a2\,\big(a2^2\,c2n2^2\,n1^2+a2^2\,c2n2^2\,n1^4+a2^2\,c2n1\,c2n2\,n1\,n2-a2^2\,c21\,c2n2\,n1^2\,n2-a2^2\,c2n1\,c2n2\,n1^3\,n2+a2^2\,c21\,c2n2\,n1^4\,n2+a2^2\,c2n1^2\,n2^2-a2^2\,c21\,c2n1\,n1\,n2^2-a2^2\,c21\,c2n1\,n1^3\,n2^2+a2^2\,c21^2\,n1^4\,n2^2-a2^2\,c2n1\,c2n2\,n1\,n2^3-a2^2\,c21\,c2n2\,n1^2\,n2^3+a2^2\,c2n1^2\,n2^4+a2^2\,c21\,c2n1\,n1\,n2^4+a2^2\,c21^2\,n1^2\,n2^4-a2^2\,n1^2\,n2^2\,pY2-3\,a2\,c2n2\,n1^2\,n2\,r2-3\,a2\,c2n2\,n1^4\,n2\,r2-3\,a2\,c2n1\,n1\,n2^2\,r2+3\,a2\,c21\,n1^2\,n2^2\,r2+3\,a2\,c2n1\,n1^3\,n2^2\,r2-3\,a2\,c21\,n1^4\,n2^2\,r2+3\,a2\,c2n2\,n1^2\,n2^3\,r2-3\,a2\,c2n1\,n1\,n2^4\,r2-3\,a2\,c21\,n1^2\,n2^4\,r2+3\,n1^2\,n2^2\,r2^2+3\,n1^4\,n2^2\,r2^2+3\,n1^2\,n2^4\,r2^2\big),$$

$$\frac{1}{n1^3\,n2^3}\,3\,\big(-a2^2\,c2n2^2\,n1^2-a2^2\,c2n2^2\,n1^4-a2^2\,c2n1\,c2n2\,n1\,n2+a2^2\,c21\,c2n2\,n1^2\,n2+a2^2\,c2n1\,c2n2\,n1^3\,n2-a2^2\,c21\,c2n2\,n1^4\,n2-a2^2\,c2n1^2\,n2^2+a2^2\,c21\,c2n1\,n1\,n2^2+a2^2\,c21\,c2n1\,n1^3\,n2^2-a2^2\,c21^2\,n1^4\,n2^2+a2^2\,c2n1\,c2n2\,n1\,n2^3+a2^2\,c21\,c2n2\,n1^2\,n2^3-a2^2\,c2n1^2\,n2^4-a2^2\,c21\,c2n1\,n1\,n2^4-a2^2\,c21^2\,n1^2\,n2^4+a2^2\,n1^2\,n2^2\,pY2+n1^2\,n2^2\,r1^2+n1^4\,n2^2\,r1^2+n1^2\,n2^4\,r1^2+2\,a2\,c2n2\,n1^2\,n2\,r2+2\,a2\,c2n2\,n1^4\,n2\,r2+2\,a2\,c2n1\,n1\,n2^2\,r2-2\,a2\,c21\,n1^2\,n2^2\,r2-2\,a2\,c2n1\,n1^3\,n2^2\,r2+2\,a2\,c21\,n1^4\,n2^2\,r2-2\,a2\,c2n2\,n1^2\,n2^3\,r2+2\,a2\,c2n1\,n1\,n2^4\,r2+2\,a2\,c21\,n1^2\,n2^4\,r2-n1^2\,n2^2\,r2^2-n1^4\,n2^2\,r2^2-n1^2\,n2^4\,r2^2\big),$$

$$-\frac{1}{n1^3\,n2^3}\,3\,\big(a2\,c2n2^2\,n1^2+a2\,c2n2^2\,n1^4+a2\,c2n1\,c2n2\,n1\,n2-a2\,c21\,c2n2\,n1^2\,n2-a2\,c2n1\,c2n2\,n1^3\,n2+a2\,c21\,c2n2\,n1^4\,n2+a2\,c2n1^2\,n2^2-a2\,c21\,c2n1\,n1\,n2^2-a2\,c21\,c2n1\,n1^3\,n2^2+a2\,c21^2\,n1^4\,n2^2-a2\,c2n1\,c2n2\,n1\,n2^3-a2\,c21\,c2n2\,n1^2\,n2^3+a2\,c2n1^2\,n2^4+a2\,c21\,c2n1\,n1\,n2^4+a2\,c21^2\,n1^2\,n2^4-a2\,n1^2\,n2^2\,pY2+c1n2\,n1^2\,n2\,r1+c1n2\,n1^4\,n2\,r1+c1n1\,n1\,n2^2\,r1-c11\,n1^2\,n2^2\,r1-c1n1\,n1^3\,n2^2\,r1+c11\,n1^4\,n2^2\,r1-c1n2\,n1^2\,n2^3\,r1+c1n1\,n1\,n2^4\,r1+c11\,n1^2\,n2^4\,r1-c2n2\,n1^2\,n2\,r2-c2n2\,n1^4\,n2\,r2-c2n1\,n1\,n2^2\,r2+c21\,n1^2\,n2^2\,r2+c2n1\,n1^3\,n2^2\,r2-c21\,n1^4\,n2^2\,r2+c2n2\,n1^2\,n2^3\,r2-c2n1\,n1\,n2^4\,r2-c21\,n1^2\,n2^4\,r2\big),$$

$$\frac{1}{n1^3\,n2^3}\,\big(c1n2^2\,n1^2-c2n2^2\,n1^2+c1n2^2\,n1^4-c2n2^2\,n1^4+c1n1\,c1n2\,n1\,n2-c2n1\,c2n2\,n1\,n2-c11\,c1n2\,n1^2\,n2+c21\,c2n2\,n1^2\,n2-c1n1\,c1n2\,n1^3\,n2+c2n1\,c2n2\,n1^3\,n2+c11\,c1n2\,n1^4\,n2-c21\,c2n2\,n1^4\,n2+c1n1^2\,n2^2-c2n1^2\,n2^2-c11\,c1n1\,n1\,n2^2+c21\,c2n1\,n1\,n2^2-c11\,c1n1\,n1^3\,n2^2+c21\,c2n1\,n1^3\,n2^2+c11^2\,n1^4\,n2^2-c21^2\,n1^4\,n2^2-c1n1\,c1n2\,n1\,n2^3+c2n1\,c2n2\,n1\,n2^3-c11\,c1n2\,n1^2\,n2^3+c21\,c2n2\,n1^2\,n2^3+c1n1^2\,n2^4-c2n1^2\,n2^4+c11\,c1n1\,n1\,n2^4-c21\,c2n1\,n1\,n2^4+c11^2\,n1^2\,n2^4-c21^2\,n1^2\,n2^4+n1^2\,n2^2\,pY1+n1^2\,n2^2\,pY2\big)\Big\}$$

```
Simplify[Solve[{
   d == -1/(n1^3 n2^3) a2^3 (a2^2 c2n2^2 n1^2 + a2^2 c2n1 c2n2 n1 n2 + a2^2 c21 c2n2 n1^2 n2 +
        a2^2 c2n1^2 n2^2 + a2^2 c21 c2n1 n1 n2^2 + a2^2 c21^2 n1^2 n2^2 - 5 a2 c2n2 n1^2 n2 r2 -
        5 a2 c2n1 n1 n2^2 r2 - 5 a2 c21 n1^2 n2^2 r2 + 10 n1^2 n2^2 r2^2),
   0 ==
    -1/(n1^3 n2^3) 5 a2^2 (a2^2 c2n2^2 n1^2 + a2^2 c2n1 c2n2 n1 n2 + a2^2 c21 c2n2 n1^2 n2 +
        a2^2 c2n1^2 n2^2 + a2^2 c21 c2n1 n1 n2^2 + a2^2 c21^2 n1^2 n2^2 - 4 a2 c2n2 n1^2 n2 r2 -
        4 a2 c2n1 n1 n2^2 r2 - 4 a2 c21 n1^2 n2^2 r2 + 6 n1^2 n2^2 r2^2),
   0 == -1/(n1^3 n2^3) 10 a2
     (a2^2 c2n2^2 n1^2 + a2^2 c2n1 c2n2 n1 n2 + a2^2 c21 c2n2 n1^2 n2 + a2^2 c2n1^2 n2^2 +
        a2^2 c21 c2n1 n1 n2^2 + a2^2 c21^2 n1^2 n2^2 - 3 a2 c2n2 n1^2 n2 r2 - 3 a2 c2n1 n1 n2^2 r2 -
        3 a2 c21 n1^2 n2^2 r2 + 3 n1^2 n2^2 r2^2)
  }, {d, c2n1, c2n2}]]
```

$$\left\{\left\{d \to -\frac{a2^3\, r2^2}{n1\, n2},\ c2n1 \to -\frac{a2^2\, c21\, n1 - 3\, a2\, n1\, r2 + \sqrt{3}\, \sqrt{-a2^2\, n1^2\, (-a2\, c21 + r2)^2}}{2\, a2^2},\right.\right.$$
$$\left.c2n2 \to \frac{n2 \left(-a2^2\, c21\, n1 + 3\, a2\, n1\, r2 + \sqrt{3}\, \sqrt{-a2^2\, n1^2\, (-a2\, c21 + r2)^2}\right)}{2\, a2^2\, n1}\right\},$$
$$\left\{d \to -\frac{a2^3\, r2^2}{n1\, n2},\ c2n1 \to \frac{-a2^2\, c21\, n1 + 3\, a2\, n1\, r2 + \sqrt{3}\, \sqrt{-a2^2\, n1^2\, (-a2\, c21 + r2)^2}}{2\, a2^2},\right.$$
$$\left.\left.c2n2 \to -\frac{n2 \left(a2^2\, c21\, n1 - 3\, a2\, n1\, r2 + \sqrt{3}\, \sqrt{-a2^2\, n1^2\, (-a2\, c21 + r2)^2}\right)}{2\, a2^2\, n1}\right\}\right\}$$

```
(*****************************
 Since the solution needs to be real,
******************************)

Simplify[Solve[{
   d == -1/(n1^3 n2^3) a2^3 (a2^2 c2n2^2 n1^2 + a2^2 c2n1 c2n2 n1 n2 + a2^2 c21 c2n2 n1^2 n2 +
        a2^2 c2n1^2 n2^2 + a2^2 c21 c2n1 n1 n2^2 + a2^2 c21^2 n1^2 n2^2 - 5 a2 c2n2 n1^2 n2 r2 -
        5 a2 c2n1 n1 n2^2 r2 - 5 a2 c21 n1^2 n2^2 r2 + 10 n1^2 n2^2 r2^2),
   0 ==
    -1/(n1^3 n2^3) 5 a2^2 (a2^2 c2n2^2 n1^2 + a2^2 c2n1 c2n2 n1 n2 + a2^2 c21 c2n2 n1^2 n2 +
        a2^2 c2n1^2 n2^2 + a2^2 c21 c2n1 n1 n2^2 + a2^2 c21^2 n1^2 n2^2 - 4 a2 c2n2 n1^2 n2 r2 -
        4 a2 c2n1 n1 n2^2 r2 - 4 a2 c21 n1^2 n2^2 r2 + 6 n1^2 n2^2 r2^2),
   0 == -1/(n1^3 n2^3) 10 a2
     (a2^2 c2n2^2 n1^2 + a2^2 c2n1 c2n2 n1 n2 + a2^2 c21 c2n2 n1^2 n2 + a2^2 c2n1^2 n2^2 +
        a2^2 c21 c2n1 n1 n2^2 + a2^2 c21^2 n1^2 n2^2 - 3 a2 c2n2 n1^2 n2 r2 - 3 a2 c2n1 n1 n2^2 r2 -
        3 a2 c21 n1^2 n2^2 r2 + 3 n1^2 n2^2 r2^2),
   0 == -a2 c21 + r2
  }, {d, c2n1, c2n2, c21}]]
```

$$\left\{\left\{d \to -\frac{a2^3\, r2^2}{n1\, n2},\ c2n1 \to \frac{n1\, r2}{a2},\ c2n2 \to \frac{n2\, r2}{a2},\ c21 \to \frac{r2}{a2}\right\}\right\}$$

```
Simplify[Solve[{
  d == -a2^3 r2^2/(n1 n2), c2n1 == n1 r2/a2, c2n2 == n2 r2/a2, c21 == r2/a2, pY2 == -3,
  p d ==
   -1/(n1^3 n2^3) a2 (a2^2 c2n2^2 n1^2 + a2^2 c2n2^2 n1^4 + a2^2 c2n1 c2n2 n1 n2 - a2^2 c21 c2n2 n1^2 n2 -
      a2^2 c2n1 c2n2 n1^3 n2 + a2^2 c21 c2n2 n1^4 n2 + a2^2 c2n1^2 n2^2 - a2^2 c21 c2n1 n1 n2^2 -
      a2^2 c21 c2n1 n1^3 n2^2 + a2^2 c21^2 n1^4 n2^2 - a2^2 c2n1 c2n2 n1 n2^3 - a2^2 c21 c2n2 n1^2 n2^3 +
      a2^2 c2n1^2 n2^4 + a2^2 c21 c2n1 n1 n2^4 + a2^2 c21^2 n1^2 n2^4 - a2^2 n1^2 n2^2 pY2 -
      3 a2 c2n2 n1^2 n2 r2 - 3 a2 c2n2 n1^4 n2 r2 - 3 a2 c2n1 n1 n2^2 r2 + 3 a2 c21 n1^2 n2^2 r2 +
      3 a2 c2n1 n1^3 n2^2 r2 - 3 a2 c21 n1^4 n2^2 r2 + 3 a2 c2n2 n1^2 n2^3 r2 - 3 a2 c2n1 n1 n2^4 r2 -
      3 a2 c21 n1^2 n2^4 r2 + 3 n1^2 n2^2 r2^2 + 3 n1^4 n2^2 r2^2 + 3 n1^2 n2^4 r2^2)
 }, {p, d, c2n1, c2n2, c21, pY2}]]
```

$$\left\{\left\{p \to \frac{3\, a2^2 + \left(1 + n1^2 + n2^2\right) r2^2}{a2^2\, r2^2},\right.\right.$$
$$\left.\left. d \to -\frac{a2^3\, r2^2}{n1\, n2},\ c2n1 \to \frac{n1\, r2}{a2},\ c2n2 \to \frac{n2\, r2}{a2},\ c21 \to \frac{r2}{a2},\ pY2 \to -3\right\}\right\}$$

Computations for Lemma 6.4.
The case where there exists $N_1{}^6$, $N_2{}^6$ and $N_3{}^6$ s.t. $Fix(N_i) = Y_1{}^4 \cup Y_2{}^4$.

Case C where the normal weights are (n1, n2, n3).

```
(*****************************
  NOTATIONS
  ri xi denotes the restriction of x to Yi
  ai are the Hopf weight at Yi
  cin1 xi denotes the first Chern class of the normal bundle of Yi in
  the direction of N1
  cin2 xi denotes the first Chern class of the normal bundle of Yi in
  the direction of N2
  cin3 xi denotes the first Chern class of the normal bundle of Yi in
  the direction of N3
  µx5YiC denotes the local datum of x^5 at Yi

  We use the variable t to extract the coefficient of order 2 of xi. We
  then only evaluate xi^2 to 1 or -1.
   *****************************)

µx5Y1C =
 Factor[CoefficientList[
    Coefficient[Expand[(r1 x1 t + 1 z)^5] Series[(c1n1 x1 t + n1 z)^-1, {x1, 0, 2}]
      Series[(c1n2 x1 t + n2 z)^-1, {x1, 0, 2}] Series[(c1n3 x1 t + n3 z)^-1, {x1, 0, 2}],
     t, 2], 1]] /. x1^2 → 1
```

$$\left\{0,\ 0,\ 0,\ \frac{10\, r1^2}{n1\, n2\, n3},\ -\frac{5\ (c1n3\, n1\, n2 + c1n2\, n1\, n3 + c1n1\, n2\, n3)\ r1}{n1^2\, n2^2\, n3^2},\right.$$
$$\frac{1}{n1^3\, n2^3\, n3^3}\left(c1n3^2\, n1^2\, n2^2 + c1n2\, c1n3\, n1^2\, n2\, n3 + \right.$$
$$\left.\left. c1n1\, c1n3\, n1\, n2^2\, n3 + c1n2^2\, n1^2\, n3^2 + c1n1\, c1n2\, n1\, n2\, n3^2 + c1n1^2\, n2^2\, n3^2\right)\right\}$$

```
μx5Y2C =
 Factor[CoefficientList[
    Coefficient[Expand[(r2 x2 t + (a2 + l) z)^5] Series[(c2n1 x2 t + n1 z)^-1, {x2, 0, 2}]
      Series[(c2n2 x2 t + n2 z)^-1, {x2, 0, 2}] Series[(c2n3 x2 t + n3 z)^-1, {x2, 0, 2}],
     t, 2], l]] /. x2^2 → -1
```

$$\left\{-\frac{1}{n1^3\ n2^3\ n3^3} a2^3 \left(a2^2\ c2n3^2\ n1^2\ n2^2 + a2^2\ c2n2\ c2n3\ n1^2\ n2\ n3 + a2^2\ c2n1\ c2n3\ n1\ n2^2\ n3 + a2^2\ c2n2^2\ n1^2\ n3^2 + a2^2\ c2n1\ c2n2\ n1\ n2\ n3^2 + a2^2\ c2n1^2\ n2^2\ n3^2 - 5\ a2\ c2n3\ n1^2\ n2^2\ n3\ r2 - 5\ a2\ c2n2\ n1^2\ n2\ n3^2\ r2 - 5\ a2\ c2n1\ n1\ n2^2\ n3^2\ r2 + 10\ n1^2\ n2^2\ n3^2\ r2^2\right),\right.$$

$$-\frac{1}{n1^3\ n2^3\ n3^3} 5\ a2^2 \left(a2^2\ c2n3^2\ n1^2\ n2^2 + a2^2\ c2n2\ c2n3\ n1^2\ n2\ n3 + a2^2\ c2n1\ c2n3\ n1\ n2^2\ n3 + a2^2\ c2n2^2\ n1^2\ n3^2 + a2^2\ c2n1\ c2n2\ n1\ n2\ n3^2 + a2^2\ c2n1^2\ n2^2\ n3^2 - 4\ a2\ c2n3\ n1^2\ n2^2\ n3\ r2 - 4\ a2\ c2n2\ n1^2\ n2\ n3^2\ r2 - 4\ a2\ c2n1\ n1\ n2^2\ n3^2\ r2 + 6\ n1^2\ n2^2\ n3^2\ r2^2\right),$$

$$-\frac{1}{n1^3\ n2^3\ n3^3} 10\ a2 \left(a2^2\ c2n3^2\ n1^2\ n2^2 + a2^2\ c2n2\ c2n3\ n1^2\ n2\ n3 + a2^2\ c2n1\ c2n3\ n1\ n2^2\ n3 + a2^2\ c2n2^2\ n1^2\ n3^2 + a2^2\ c2n1\ c2n2\ n1\ n2\ n3^2 + a2^2\ c2n1^2\ n2^2\ n3^2 - 3\ a2\ c2n3\ n1^2\ n2^2\ n3\ r2 - 3\ a2\ c2n2\ n1^2\ n2\ n3^2\ r2 - 3\ a2\ c2n1\ n1\ n2^2\ n3^2\ r2 + 3\ n1^2\ n2^2\ n3^2\ r2^2\right),$$

$$-\frac{1}{n1^3\ n2^3\ n3^3} 10 \left(a2^2\ c2n3^2\ n1^2\ n2^2 + a2^2\ c2n2\ c2n3\ n1^2\ n2\ n3 + a2^2\ c2n1\ c2n3\ n1\ n2^2\ n3 + a2^2\ c2n2^2\ n1^2\ n3^2 + a2^2\ c2n1\ c2n2\ n1\ n2\ n3^2 + a2^2\ c2n1^2\ n2^2\ n3^2 - 2\ a2\ c2n3\ n1^2\ n2^2\ n3\ r2 - 2\ a2\ c2n2\ n1^2\ n2\ n3^2\ r2 - 2\ a2\ c2n1\ n1\ n2^2\ n3^2\ r2 + n1^2\ n2^2\ n3^2\ r2^2\right),$$

$$-\frac{1}{n1^3\ n2^3\ n3^3} 5 \left(a2\ c2n3^2\ n1^2\ n2^2 + a2\ c2n2\ c2n3\ n1^2\ n2\ n3 + a2\ c2n1\ c2n3\ n1\ n2^2\ n3 + a2\ c2n2^2\ n1^2\ n3^2 + a2\ c2n1\ c2n2\ n1\ n2\ n3^2 + a2\ c2n1^2\ n2^2\ n3^2 - c2n3\ n1^2\ n2^2\ n3\ r2 - c2n2\ n1^2\ n2\ n3^2\ r2 - c2n1\ n1\ n2^2\ n3^2\ r2\right),$$

$$\left.-\frac{1}{n1^3\ n2^3\ n3^3} \left(c2n3^2\ n1^2\ n2^2 + c2n2\ c2n3\ n1^2\ n2\ n3 + c2n1\ c2n3\ n1\ n2^2\ n3 + c2n2^2\ n1^2\ n3^2 + c2n1\ c2n2\ n1\ n2\ n3^2 + c2n1^2\ n2^2\ n3^2\right)\right\}$$

```
(******************************
 NOTATIONS
 μs1x3YiA denotes the local datum of p1(X)x^3 at Yi
  ******************************)

μs1x3Y1C =
 Factor[CoefficientList[
    Coefficient[(pY1 t^2 + (c1n1 x1 t + n1 z)^2 + (c1n2 x1 t + n2 z)^2 + (c1n3 x1 t + n3 z)^2)
      Expand[(r1 x1 t + l z)^3] Series[(c1n1 x1 t + n1 z)^-1, {x1, 0, 2}]
      Series[(c1n2 x1 t + n2 z)^-1, {x1, 0, 2}] Series[(c1n3 x1 t + n3 z)^-1, {x1, 0, 2}],
     t, 2], l]] /. x1^2 → 1
```

$$\left\{0, \frac{3\left(n1^2 + n2^2 + n3^2\right) r1^2}{n1\ n2\ n3},\right.$$

$$-\frac{1}{n1^2\ n2^2\ n3^2} 3 \left(c1n3\ n1^3\ n2 + c1n3\ n1\ n2^3 + c1n2\ n1^3\ n3 - c1n1\ n1^2\ n2\ n3 - c1n2\ n1\ n2^2\ n3 + c1n1\ n2^3\ n3 - c1n3\ n1\ n2\ n3^2 + c1n2\ n1\ n3^3 + c1n1\ n2\ n3^3\right) r1,$$

$$\left.\frac{1}{n1^3\ n2^3\ n3^3} \left(c1n3^2\ n1^4\ n2^2 + c1n3^2\ n1^2\ n2^4 + c1n2\ c1n3\ n1^4\ n2\ n3 - c1n1\ c1n3\ n1^3\ n2^2\ n3 - c1n2\ c1n3\ n1^2\ n2^3\ n3 + c1n1\ c1n3\ n1\ n2^4\ n3 + c1n2^2\ n1^4\ n3^2 - c1n1\ c1n2\ n1^3\ n2\ n3^2 - c1n1\ c1n2\ n1\ n2^3\ n3^2 + c1n1^2\ n2^4\ n3^2 - c1n2\ c1n3\ n1^2\ n2\ n3^3 - c1n1\ c1n3\ n1\ n2^2\ n3^3 + c1n2^2\ n1^2\ n3^4 + c1n1\ c1n2\ n1\ n2\ n3^4 + c1n1^2\ n2^2\ n3^4 + n1^2\ n2^2\ n3^2\ pY1\right)\right\}$$

```
μs1x3Y2C =
  Factor[CoefficientList[
     Coefficient[(pY2 t^2 + (c2n1 x2 t + n1 z)^2 + (c2n2 x2 t + n2 z)^2 + (c2n3 x2 t + n3 z)^2)
       Expand[(r2 x2 t + (a2 + 1) z)^3] Series[(c2n1 x2 t + n1 z)^-1, {x2, 0, 2}]
       Series[(c2n2 x2 t + n2 z)^-1, {x2, 0, 2}] Series[(c2n3 x2 t + n3 z)^-1, {x2, 0, 2}],
      t, 2], 1]] /. x2^2 → -1
```

$$
\begin{aligned}
\Big\{ & \frac{1}{n1^3\,n2^3\,n3^3} \\
& a2\,\big(-a2^2\,c2n3^2\,n1^4\,n2^2 - a2^2\,c2n3^2\,n1^2\,n2^4 - a2^2\,c2n2\,c2n3\,n1^4\,n2\,n3 + a2^2\,c2n1\,c2n3\,n1^3\,n2^2\,n3 + \\
& a2^2\,c2n2\,c2n3\,n1^2\,n2^3\,n3 - a2^2\,c2n1\,c2n3\,n1\,n2^4\,n3 - a2^2\,c2n2^2\,n1^4\,n3^2 + \\
& a2^2\,c2n1\,c2n2\,n1^3\,n2\,n3^2 + a2^2\,c2n1\,c2n2\,n1\,n2^3\,n3^2 - a2^2\,c2n1^2\,n2^4\,n3^2 + \\
& a2^2\,c2n2\,c2n3\,n1^2\,n2\,n3^3 + a2^2\,c2n1\,c2n3\,n1\,n2^2\,n3^3 - a2^2\,c2n2^2\,n1^2\,n3^4 - \\
& a2^2\,c2n1\,c2n2\,n1\,n2\,n3^4 - a2^2\,c2n1^2\,n2^2\,n3^4 + a2^2\,n1^2\,n2^2\,n3^2\,pY2 + \\
& 3\,a2\,c2n3\,n1^4\,n2^2\,n3\,r2 + 3\,a2\,c2n3\,n1^2\,n2^4\,n3\,r2 + 3\,a2\,c2n2\,n1^4\,n2\,n3^2\,r2 - \\
& 3\,a2\,c2n1\,n1^3\,n2^2\,n3^2\,r2 - 3\,a2\,c2n2\,n1^2\,n2^3\,n3^2\,r2 + 3\,a2\,c2n1\,n1\,n2^4\,n3^2\,r2 - \\
& 3\,a2\,c2n3\,n1^2\,n2^2\,n3^3\,r2 + 3\,a2\,c2n2\,n1^2\,n2\,n3^4\,r2 + 3\,a2\,c2n1\,n1\,n2^2\,n3^4\,r2 - \\
& 3\,n1^4\,n2^2\,n3^2\,r2^2 - 3\,n1^2\,n2^4\,n3^2\,r2^2 - 3\,n1^2\,n2^2\,n3^4\,r2^2\big), \\
& \frac{1}{n1^3\,n2^3\,n3^3}\,3\,\big(-a2^2\,c2n3^2\,n1^4\,n2^2 - a2^2\,c2n3^2\,n1^2\,n2^4 - a2^2\,c2n2\,c2n3\,n1^4\,n2\,n3 + \\
& a2^2\,c2n1\,c2n3\,n1^3\,n2^2\,n3 + a2^2\,c2n2\,c2n3\,n1^2\,n2^3\,n3 - a2^2\,c2n1\,c2n3\,n1\,n2^4\,n3 - \\
& a2^2\,c2n2^2\,n1^4\,n3^2 + a2^2\,c2n1\,c2n2\,n1^3\,n2\,n3^2 + a2^2\,c2n1\,c2n2\,n1\,n2^3\,n3^2 - \\
& a2^2\,c2n1^2\,n2^4\,n3^2 + a2^2\,c2n2\,c2n3\,n1^2\,n2\,n3^3 + a2^2\,c2n1\,c2n3\,n1\,n2^2\,n3^3 - \\
& a2^2\,c2n2^2\,n1^2\,n3^4 - a2^2\,c2n1\,c2n2\,n1\,n2\,n3^4 - a2^2\,c2n1^2\,n2^2\,n3^4 + \\
& a2^2\,n1^2\,n2^2\,n3^2\,pY2 + 2\,a2\,c2n3\,n1^4\,n2^2\,n3\,r2 + 2\,a2\,c2n3\,n1^2\,n2^4\,n3\,r2 + \\
& 2\,a2\,c2n2\,n1^4\,n2\,n3^2\,r2 - 2\,a2\,c2n1\,n1^3\,n2^2\,n3^2\,r2 - 2\,a2\,c2n2\,n1^2\,n2^3\,n3^2\,r2 + \\
& 2\,a2\,c2n1\,n1\,n2^4\,n3^2\,r2 - 2\,a2\,c2n3\,n1^2\,n2^2\,n3^3\,r2 + 2\,a2\,c2n2\,n1^2\,n2\,n3^4\,r2 + \\
& 2\,a2\,c2n1\,n1\,n2^2\,n3^4\,r2 - n1^4\,n2^2\,n3^2\,r2^2 - n1^2\,n2^4\,n3^2\,r2^2 - n1^2\,n2^2\,n3^4\,r2^2\big), \\
& \frac{1}{n1^3\,n2^3\,n3^3}\,3\,\big(-a2\,c2n3^2\,n1^4\,n2^2 - a2\,c2n3^2\,n1^2\,n2^4 - a2\,c2n2\,c2n3\,n1^4\,n2\,n3 + \\
& a2\,c2n1\,c2n3\,n1^3\,n2^2\,n3 + a2\,c2n2\,c2n3\,n1^2\,n2^3\,n3 - a2\,c2n1\,c2n3\,n1\,n2^4\,n3 - \\
& a2\,c2n2^2\,n1^4\,n3^2 + a2\,c2n1\,c2n2\,n1^3\,n2\,n3^2 + a2\,c2n1\,c2n2\,n1\,n2^3\,n3^2 - \\
& a2\,c2n1^2\,n2^4\,n3^2 + a2\,c2n2\,c2n3\,n1^2\,n2\,n3^3 + a2\,c2n1\,c2n3\,n1\,n2^2\,n3^3 - a2\,c2n2^2\,n1^2\,n3^4 - \\
& a2\,c2n1\,c2n2\,n1\,n2\,n3^4 - a2\,c2n1^2\,n2^2\,n3^4 + a2\,n1^2\,n2^2\,n3^2\,pY2 + c2n3\,n1^4\,n2^2\,n3\,r2 + \\
& c2n3\,n1^2\,n2^4\,n3\,r2 + c2n2\,n1^4\,n2\,n3^2\,r2 - c2n1\,n1^3\,n2^2\,n3^2\,r2 - c2n2\,n1^2\,n2^3\,n3^2\,r2 + \\
& c2n1\,n1\,n2^4\,n3^2\,r2 - c2n3\,n1^2\,n2^2\,n3^3\,r2 + c2n2\,n1^2\,n2\,n3^4\,r2 + c2n1\,n1\,n2^2\,n3^4\,r2\big), \\
& \frac{1}{n1^3\,n2^3\,n3^3}\,\big(-c2n3^2\,n1^4\,n2^2 - c2n3^2\,n1^2\,n2^4 - c2n2\,c2n3\,n1^4\,n2\,n3 + c2n1\,c2n3\,n1^3\,n2^2\,n3 + \\
& c2n2\,c2n3\,n1^2\,n2^3\,n3 - c2n1\,c2n3\,n1\,n2^4\,n3 - c2n2^2\,n1^4\,n3^2 + c2n1\,c2n2\,n1^3\,n2\,n3^2 + \\
& c2n1\,c2n2\,n1\,n2^3\,n3^2 - c2n1^2\,n2^4\,n3^2 + c2n2\,c2n3\,n1^2\,n2\,n3^3 + c2n1\,c2n3\,n1\,n2^2\,n3^3 - \\
& c2n2^2\,n1^2\,n3^4 - c2n1\,c2n2\,n1\,n2\,n3^4 - c2n1^2\,n2^2\,n3^4 + n1^2\,n2^2\,n3^2\,pY2\big)\Big\}
\end{aligned}
$$

μx5Y1C + μx5Y2C

$$\left\{-\frac{1}{n1^3\ n2^3\ n3^3}\ a2^3\ \left(a2^2\ c2n3^2\ n1^2\ n2^2 + a2^2\ c2n2\ c2n3\ n1^2\ n2\ n3 + a2^2\ c2n1\ c2n3\ n1\ n2^2\ n3 + a2^2\ c2n2^2\ n1^2\ n3^2 + a2^2\ c2n1\ c2n2\ n1\ n2\ n3^2 + a2^2\ c2n1^2\ n2^2\ n3^2 - 5\ a2\ c2n3\ n1^2\ n2^2\ n3\ r2 - 5\ a2\ c2n2\ n1^2\ n2\ n3^2\ r2 - 5\ a2\ c2n1\ n1\ n2^2\ n3^2\ r2 + 10\ n1^2\ n2^2\ n3^2\ r2^2\right),\right.$$

$$-\frac{1}{n1^3\ n2^3\ n3^3}\ 5\ a2^2\ \left(a2^2\ c2n3^2\ n1^2\ n2^2 + a2^2\ c2n2\ c2n3\ n1^2\ n2\ n3 + a2^2\ c2n1\ c2n3\ n1\ n2^2\ n3 + a2^2\ c2n2^2\ n1^2\ n3^2 + a2^2\ c2n1\ c2n2\ n1\ n2\ n3^2 + a2^2\ c2n1^2\ n2^2\ n3^2 - 4\ a2\ c2n3\ n1^2\ n2^2\ n3\ r2 - 4\ a2\ c2n2\ n1^2\ n2\ n3^2\ r2 - 4\ a2\ c2n1\ n1\ n2^2\ n3^2\ r2 + 6\ n1^2\ n2^2\ n3^2\ r2^2\right),$$

$$-\frac{1}{n1^3\ n2^3\ n3^3}\ 10\ a2\ \left(a2^2\ c2n3^2\ n1^2\ n2^2 + a2^2\ c2n2\ c2n3\ n1^2\ n2\ n3 + a2^2\ c2n1\ c2n3\ n1\ n2^2\ n3 + a2^2\ c2n2^2\ n1^2\ n3^2 + a2^2\ c2n1\ c2n2\ n1\ n2\ n3^2 + a2^2\ c2n1^2\ n2^2\ n3^2 - 3\ a2\ c2n3\ n1^2\ n2^2\ n3\ r2 - 3\ a2\ c2n2\ n1^2\ n2\ n3^2\ r2 - 3\ a2\ c2n1\ n1\ n2^2\ n3^2\ r2 + 3\ n1^2\ n2^2\ n3^2\ r2^2\right),$$

$$\frac{10\ r1^2}{n1\ n2\ n3} - \frac{1}{n1^3\ n2^3\ n3^3}\ 10\ \left(a2^2\ c2n3^2\ n1^2\ n2^2 + a2^2\ c2n2\ c2n3\ n1^2\ n2\ n3 + a2^2\ c2n1\ c2n3\ n1\ n2^2\ n3 + a2^2\ c2n2^2\ n1^2\ n3^2 + a2^2\ c2n1\ c2n2\ n1\ n2\ n3^2 + a2^2\ c2n1^2\ n2^2\ n3^2 - 2\ a2\ c2n3\ n1^2\ n2^2\ n3\ r2 - 2\ a2\ c2n2\ n1^2\ n2\ n3^2\ r2 - 2\ a2\ c2n1\ n1\ n2^2\ n3^2\ r2 + n1^2\ n2^2\ n3^2\ r2^2\right),$$

$$-\frac{5\ (c1n3\ n1\ n2 + c1n2\ n1\ n3 + c1n1\ n2\ n3)\ r1}{n1^2\ n2^2\ n3^2} - \frac{1}{n1^3\ n2^3\ n3^3}\ 5\ \left(a2\ c2n3^2\ n1^2\ n2^2 + a2\ c2n2\ c2n3\ n1^2\ n2\ n3 + a2\ c2n1\ c2n3\ n1\ n2^2\ n3 + a2\ c2n2^2\ n1^2\ n3^2 + a2\ c2n1\ c2n2\ n1\ n2\ n3^2 + a2\ c2n1^2\ n2^2\ n3^2 - c2n3\ n1^2\ n2^2\ n3\ r2 - c2n2\ n1^2\ n2\ n3^2\ r2 - c2n1\ n1\ n2^2\ n3^2\ r2\right),$$

$$\frac{1}{n1^3\ n2^3\ n3^3}\ \left(c1n3^2\ n1^2\ n2^2 + c1n2\ c1n3\ n1^2\ n2\ n3 + c1n1\ c1n3\ n1\ n2^2\ n3 + c1n2^2\ n1^2\ n3^2 + c1n1\ c1n2\ n1\ n2\ n3^2 + c1n1^2\ n2^2\ n3^2\right) - \frac{1}{n1^3\ n2^3\ n3^3}\ \left(c2n3^2\ n1^2\ n2^2 + c2n2\ c2n3\ n1^2\ n2\ n3 + c2n1\ c2n3\ n1\ n2^2\ n3 + c2n2^2\ n1^2\ n3^2 + c2n1\ c2n2\ n1\ n2\ n3^2 + c2n1^2\ n2^2\ n3^2\right)\right\}$$

```
Factor[μs1x3Y1C + μs1x3Y2C]
```

$$\Big\{-\frac{1}{n1^3\, n2^3\, n3^3} a2 \big(a2^2\, c2n3^2\, n1^4\, n2^2 + a2^2\, c2n3^2\, n1^2\, n2^4 + a2^2\, c2n2\, c2n3\, n1^4\, n2\, n3 - a2^2\, c2n1\, c2n3\, n1^3\, n2^2\, n3 - a2^2\, c2n2\, c2n3\, n1^2\, n2^3\, n3 + a2^2\, c2n1\, c2n3\, n1\, n2^4\, n3 + a2^2\, c2n2^2\, n1^4\, n3^2 - a2^2\, c2n1\, c2n2\, n1^3\, n2\, n3^2 - a2^2\, c2n1\, c2n2\, n1\, n2^3\, n3^2 + a2^2\, c2n1^2\, n2^4\, n3^2 - a2^2\, c2n2\, c2n3\, n1^2\, n2\, n3^3 - a2^2\, c2n1\, c2n3\, n1\, n2^2\, n3^3 + a2^2\, c2n2^2\, n1^2\, n3^4 + a2^2\, c2n1\, c2n2\, n1\, n2\, n3^4 + a2^2\, c2n1^2\, n2^2\, n3^4 - a2^2\, n1^2\, n2^2\, n3^2\, pY2 - 3\, a2\, c2n3\, n1^4\, n2^2\, n3\, r2 - 3\, a2\, c2n3\, n1^2\, n2^4\, n3\, r2 - 3\, a2\, c2n2\, n1^4\, n2\, n3^2\, r2 + 3\, a2\, c2n1\, n1^3\, n2^2\, n3^2\, r2 + 3\, a2\, c2n2\, n1^2\, n2^3\, n3^2\, r2 - 3\, a2\, c2n1\, n1\, n2^4\, n3^2\, r2 + 3\, a2\, c2n3\, n1^2\, n2^2\, n3^3\, r2 - 3\, a2\, c2n2\, n1^2\, n2\, n3^4\, r2 - 3\, a2\, c2n1\, n1\, n2^2\, n3^4\, r2 + 3\, n1^4\, n2^2\, n3^2\, r2^2 + 3\, n1^2\, n2^4\, n3^2\, r2^2 + 3\, n1^2\, n2^2\, n3^4\, r2^2\big),$$

$$\frac{1}{n1^3\, n2^3\, n3^3} 3 \big(-a2^2\, c2n3^2\, n1^4\, n2^2 - a2^2\, c2n3^2\, n1^2\, n2^4 - a2^2\, c2n2\, c2n3\, n1^4\, n2\, n3 + a2^2\, c2n1\, c2n3\, n1^3\, n2^2\, n3 + a2^2\, c2n2\, c2n3\, n1^2\, n2^3\, n3 - a2^2\, c2n1\, c2n3\, n1\, n2^4\, n3 - a2^2\, c2n2^2\, n1^4\, n3^2 + a2^2\, c2n1\, c2n2\, n1^3\, n2\, n3^2 + a2^2\, c2n1\, c2n2\, n1\, n2^3\, n3^2 - a2^2\, c2n1^2\, n2^4\, n3^2 + a2^2\, c2n2\, c2n3\, n1^2\, n2\, n3^3 + a2^2\, c2n1\, c2n3\, n1\, n2^2\, n3^3 - a2^2\, c2n2^2\, n1^2\, n3^4 - a2^2\, c2n1\, c2n2\, n1\, n2\, n3^4 - a2^2\, c2n1^2\, n2^2\, n3^4 + a2^2\, n1^2\, n2^2\, n3^2\, pY2 + n1^4\, n2^2\, n3^2\, r1^2 + n1^2\, n2^4\, n3^2\, r1^2 + n1^2\, n2^2\, n3^4\, r1^2 + 2\, a2\, c2n3\, n1^4\, n2^2\, n3\, r2 + 2\, a2\, c2n3\, n1^2\, n2^4\, n3\, r2 + 2\, a2\, c2n2\, n1^4\, n2\, n3^2\, r2 - 2\, a2\, c2n1\, n1^3\, n2^2\, n3^2\, r2 - 2\, a2\, c2n2\, n1^2\, n2^3\, n3^2\, r2 + 2\, a2\, c2n1\, n1\, n2^4\, n3^2\, r2 - 2\, a2\, c2n3\, n1^2\, n2^2\, n3^3\, r2 + 2\, a2\, c2n2\, n1^2\, n2\, n3^4\, r2 + 2\, a2\, c2n1\, n1\, n2^2\, n3^4\, r2 - n1^4\, n2^2\, n3^2\, r2^2 - n1^2\, n2^4\, n3^2\, r2^2 - n1^2\, n2^2\, n3^4\, r2^2\big),$$

$$-\frac{1}{n1^3\, n2^3\, n3^3} 3 \big(a2\, c2n3^2\, n1^4\, n2^2 + a2\, c2n3^2\, n1^2\, n2^4 + a2\, c2n2\, c2n3\, n1^4\, n2\, n3 - a2\, c2n1\, c2n3\, n1^3\, n2^2\, n3 - a2\, c2n2\, c2n3\, n1^2\, n2^3\, n3 + a2\, c2n1\, c2n3\, n1\, n2^4\, n3 + a2\, c2n2^2\, n1^4\, n3^2 - a2\, c2n1\, c2n2\, n1^3\, n2\, n3^2 - a2\, c2n1\, c2n2\, n1\, n2^3\, n3^2 + a2\, c2n1^2\, n2^4\, n3^2 - a2\, c2n2\, c2n3\, n1^2\, n2\, n3^3 - a2\, c2n1\, c2n3\, n1\, n2^2\, n3^3 + a2\, c2n2^2\, n1^2\, n3^4 + a2\, c2n1\, c2n2\, n1\, n2\, n3^4 + a2\, c2n1^2\, n2^2\, n3^4 - a2\, n1^2\, n2^2\, n3^2\, pY2 + c1n3\, n1^4\, n2^2\, n3\, r1 + c1n3\, n1^2\, n2^4\, n3\, r1 + c1n2\, n1^4\, n2\, n3^2\, r1 - c1n1\, n1^3\, n2^2\, n3^2\, r1 - c1n2\, n1^2\, n2^3\, n3^2\, r1 + c1n1\, n1\, n2^4\, n3^2\, r1 - c1n3\, n1^2\, n2^2\, n3^3\, r1 + c1n2\, n1^2\, n2\, n3^4\, r1 + c1n1\, n1\, n2^2\, n3^4\, r1 - c2n3\, n1^4\, n2^2\, n3\, r2 - c2n3\, n1^2\, n2^4\, n3\, r2 - c2n2\, n1^4\, n2\, n3^2\, r2 + c2n1\, n1^3\, n2^2\, n3^2\, r2 + c2n2\, n1^2\, n2^3\, n3^2\, r2 - c2n1\, n1\, n2^4\, n3^2\, r2 + c2n3\, n1^2\, n2^2\, n3^3\, r2 - c2n2\, n1^2\, n2\, n3^4\, r2 - c2n1\, n1\, n2^2\, n3^4\, r2\big),$$

$$\frac{1}{n1^3\, n2^3\, n3^3} \big(c1n3^2\, n1^4\, n2^2 - c2n3^2\, n1^4\, n2^2 + c1n3^2\, n1^2\, n2^4 - c2n3^2\, n1^2\, n2^4 + c1n2\, c1n3\, n1^4\, n2\, n3 - c2n2\, c2n3\, n1^4\, n2\, n3 - c1n1\, c1n3\, n1^3\, n2^2\, n3 + c2n1\, c2n3\, n1^3\, n2^2\, n3 - c1n2\, c1n3\, n1^2\, n2^3\, n3 + c2n2\, c2n3\, n1^2\, n2^3\, n3 + c1n1\, c1n3\, n1\, n2^4\, n3 - c2n1\, c2n3\, n1\, n2^4\, n3 + c1n2^2\, n1^4\, n3^2 - c2n2^2\, n1^4\, n3^2 - c1n1\, c1n2\, n1^3\, n2\, n3^2 + c2n1\, c2n2\, n1^3\, n2\, n3^2 - c1n1\, c1n2\, n1\, n2^3\, n3^2 + c2n1\, c2n2\, n1\, n2^3\, n3^2 + c1n1^2\, n2^4\, n3^2 - c2n1^2\, n2^4\, n3^2 - c1n2\, c1n3\, n1^2\, n2\, n3^3 + c2n2\, c2n3\, n1^2\, n2\, n3^3 - c1n1\, c1n3\, n1\, n2^2\, n3^3 + c2n1\, c2n3\, n1\, n2^2\, n3^3 + c1n2^2\, n1^2\, n3^4 - c2n2^2\, n1^2\, n3^4 + c1n1\, c1n2\, n1\, n2\, n3^4 - c2n1\, c2n2\, n1\, n2\, n3^4 + c1n1^2\, n2^2\, n3^4 - c2n1^2\, n2^2\, n3^4 + n1^2\, n2^2\, n3^2\, pY1 + n1^2\, n2^2\, n3^2\, pY2\big)\Big\}$$

```
Factor[Solve[{
   d == - 1/(n1^3 n2^3 n3^3) a2^3
      (a2^2 c2n3^2 n1^2 n2^2 + a2^2 c2n2 c2n3 n1^2 n2 n3 + a2^2 c2n1 c2n3 n1 n2^2 n3 +
        a2^2 c2n2^2 n1^2 n3^2 + a2^2 c2n1 c2n2 n1 n2 n3^2 + a2^2 c2n1^2 n2^2 n3^2 -
        5 a2 c2n3 n1^2 n2^2 n3 r2 - 5 a2 c2n2 n1^2 n2 n3^2 r2 - 5 a2 c2n1 n1 n2^2 n3^2 r2 +
        10 n1^2 n2^2 n3^2 r2^2),
   0 == - 1/(n1^3 n2^3 n3^3) 5 a2^2
      (a2^2 c2n3^2 n1^2 n2^2 + a2^2 c2n2 c2n3 n1^2 n2 n3 + a2^2 c2n1 c2n3 n1 n2^2 n3 +
        a2^2 c2n2^2 n1^2 n3^2 + a2^2 c2n1 c2n2 n1 n2 n3^2 + a2^2 c2n1^2 n2^2 n3^2 -
        4 a2 c2n3 n1^2 n2^2 n3 r2 - 4 a2 c2n2 n1^2 n2 n3^2 r2 - 4 a2 c2n1 n1 n2^2 n3^2 r2 +
        6 n1^2 n2^2 n3^2 r2^2),
   0 == - 1/(n1^3 n2^3 n3^3) 10 a2
      (a2^2 c2n3^2 n1^2 n2^2 + a2^2 c2n2 c2n3 n1^2 n2 n3 + a2^2 c2n1 c2n3 n1 n2^2 n3 +
        a2^2 c2n2^2 n1^2 n3^2 + a2^2 c2n1 c2n2 n1 n2 n3^2 + a2^2 c2n1^2 n2^2 n3^2 -
        3 a2 c2n3 n1^2 n2^2 n3 r2 - 3 a2 c2n2 n1^2 n2 n3^2 r2 - 3 a2 c2n1 n1 n2^2 n3^2 r2 +
        3 n1^2 n2^2 n3^2 r2^2)
  }, {d, c2n1, c2n2}]]
```

$$\Bigg\{\Big\{d \to -\frac{a2^3\, r2^2}{n1\, n2\, n3},\ c2n1 \to -\frac{a2^2\, c2n3\, n1\, n3 - 3\, a2\, n1\, n3^2\, r2 + \sqrt{3}\,\sqrt{-a2^2\, n1^2\, n3^2\, (a2\, c2n3 - n3\, r2)^2}}{2\, a2^2\, n3^2},$$
$$c2n2 \to -\frac{n2\left(a2^2\, c2n3\, n1\, n3 - 3\, a2\, n1\, n3^2\, r2 - \sqrt{3}\,\sqrt{-a2^2\, n1^2\, n3^2\, (a2\, c2n3 - n3\, r2)^2}\right)}{2\, a2^2\, n1\, n3^2}\Big\},$$
$$\Big\{d \to -\frac{a2^3\, r2^2}{n1\, n2\, n3},\ c2n1 \to -\frac{a2^2\, c2n3\, n1\, n3 - 3\, a2\, n1\, n3^2\, r2 - \sqrt{3}\,\sqrt{-a2^2\, n1^2\, n3^2\, (a2\, c2n3 - n3\, r2)^2}}{2\, a2^2\, n3^2},$$
$$c2n2 \to -\frac{n2\left(a2^2\, c2n3\, n1\, n3 - 3\, a2\, n1\, n3^2\, r2 + \sqrt{3}\,\sqrt{-a2^2\, n1^2\, n3^2\, (a2\, c2n3 - n3\, r2)^2}\right)}{2\, a2^2\, n1\, n3^2}\Big\}\Bigg\}$$

```
(******************************
 As in Case B, the solution cannot be complex thus (a2 c2n3-n3 r2) == 0
   ******************************)
```

```
Factor[Solve[{
   d == -1/(n1^3 n2^3 n3^3) a2^3
     (a2^2 c2n3^2 n1^2 n2^2 + a2^2 c2n2 c2n3 n1^2 n2 n3 + a2^2 c2n1 c2n3 n1 n2^2 n3 +
       a2^2 c2n2^2 n1^2 n3^2 + a2^2 c2n1 c2n2 n1 n2 n3^2 + a2^2 c2n1^2 n2^2 n3^2 -
       5 a2 c2n3 n1^2 n2^2 n3 r2 - 5 a2 c2n2 n1^2 n2 n3^2 r2 - 5 a2 c2n1 n1 n2^2 n3^2 r2 +
       10 n1^2 n2^2 n3^2 r2^2),
   0 == -1/(n1^3 n2^3 n3^3) 5 a2^2
     (a2^2 c2n3^2 n1^2 n2^2 + a2^2 c2n2 c2n3 n1^2 n2 n3 + a2^2 c2n1 c2n3 n1 n2^2 n3 +
       a2^2 c2n2^2 n1^2 n3^2 + a2^2 c2n1 c2n2 n1 n2 n3^2 + a2^2 c2n1^2 n2^2 n3^2 -
       4 a2 c2n3 n1^2 n2^2 n3 r2 - 4 a2 c2n2 n1^2 n2 n3^2 r2 - 4 a2 c2n1 n1 n2^2 n3^2 r2 +
       6 n1^2 n2^2 n3^2 r2^2),
   0 == -1/(n1^3 n2^3 n3^3) 10 a2
     (a2^2 c2n3^2 n1^2 n2^2 + a2^2 c2n2 c2n3 n1^2 n2 n3 + a2^2 c2n1 c2n3 n1 n2^2 n3 +
       a2^2 c2n2^2 n1^2 n3^2 + a2^2 c2n1 c2n2 n1 n2 n3^2 + a2^2 c2n1^2 n2^2 n3^2 -
       3 a2 c2n3 n1^2 n2^2 n3 r2 - 3 a2 c2n2 n1^2 n2 n3^2 r2 - 3 a2 c2n1 n1 n2^2 n3^2 r2 +
       3 n1^2 n2^2 n3^2 r2^2),
   0 == a2 c2n3 - n3 r2
  }, {d, c2n1, c2n2, c2n3}]]
```

$$\left\{\left\{d \to -\frac{a2^3\, r2^2}{n1\, n2\, n3},\ c2n1 \to \frac{n1\, r2}{a2},\ c2n2 \to \frac{n2\, r2}{a2},\ c2n3 \to \frac{n3\, r2}{a2}\right\}\right\}$$

```
Factor[Solve[{
   d == -(a2^3 r2^2)/(n1 n2 n3), c2n1 == (n1 r2)/a2, c2n2 == (n2 r2)/a2, c2n3 == (n3 r2)/a2, pY2 == -3,
   p d ==
    -1/(n1^3 n2^3 n3^3) a2 (a2^2 c2n3^2 n1^4 n2^2 + a2^2 c2n3^2 n1^2 n2^4 + a2^2 c2n2 c2n3 n1^4 n2 n3 -
       a2^2 c2n1 c2n3 n1^3 n2^2 n3 - a2^2 c2n2 c2n3 n1^2 n2^3 n3 + a2^2 c2n1 c2n3 n1 n2^4 n3 +
       a2^2 c2n2^2 n1^4 n3^2 - a2^2 c2n1 c2n2 n1^3 n2 n3^2 - a2^2 c2n1 c2n2 n1 n2^3 n3^2 +
       a2^2 c2n1^2 n2^4 n3^2 - a2^2 c2n2 c2n3 n1^2 n2 n3^3 - a2^2 c2n1 c2n3 n1 n2^2 n3^3 +
       a2^2 c2n2^2 n1^2 n3^4 + a2^2 c2n1 c2n2 n1 n2 n3^4 + a2^2 c2n1^2 n2^2 n3^4 - a2^2 n1^2 n2^2 n3^2 pY2 -
       3 a2 c2n3 n1^4 n2^2 n3 r2 - 3 a2 c2n3 n1^2 n2^4 n3 r2 - 3 a2 c2n2 n1^4 n2 n3^2 r2 +
       3 a2 c2n1 n1^3 n2^2 n3^2 r2 + 3 a2 c2n2 n1^2 n2^3 n3^2 r2 - 3 a2 c2n1 n1 n2^4 n3^2 r2 +
       3 a2 c2n3 n1^2 n2^2 n3^3 r2 - 3 a2 c2n2 n1^2 n2 n3^4 r2 - 3 a2 c2n1 n1 n2^2 n3^4 r2 +
       3 n1^4 n2^2 n3^2 r2^2 + 3 n1^2 n2^4 n3^2 r2^2 + 3 n1^2 n2^2 n3^4 r2^2)
  }, {p, d, c2n1, c2n2, c2n3, pY2}]]
```

$$\left\{\left\{p \to \frac{3\, a2^2 + n1^2\, r2^2 + n2^2\, r2^2 + n3^2\, r2^2}{a2^2\, r2^2},\right.\right.$$
$$\left.\left. d \to -\frac{a2^3\, r2^2}{n1\, n2\, n3},\ c2n1 \to \frac{n1\, r2}{a2},\ c2n2 \to \frac{n2\, r2}{a2},\ c2n3 \to \frac{n3\, r2}{a2},\ pY2 \to -3\right\}\right\}$$

Computations for Theorem D.

The case where the fixed points are $Y^4 \cup Z^2 \cup P^0$.

Computations for Lemma 6.5.
The case where the action is semi-free.

```
(******************************
  NOTATIONS
  rY xY denotes the restriction of x to Y
  rZ xZ denotes the restriction of x to Z
  aZ denotes the Hopf weight at Z
  aP denotes the Hopf weight at P
  cYj xY^j denotes the j-th Chern class of the normal bundle of Y
  cZ1 xZ denotes the first Chern class of the normal bundle of Z
  μx5Y denotes the local datum of x^5 at Y
  μx5Z denotes the local datum of x^5 at Z
  μx5P denotes the local datum of x^5 at P

  We use the variable t to extract the coefficient of order 2 of xY and
  of order 1 of xZ. We then evaluate xY^2 and xZ to 1.
  The orientation of the point P is -1, this explains the negative sign.
  Once evaluated, we extract all the coefficients in l into a list.
  ******************************)

μx5Y =
 Factor[CoefficientList[
    Coefficient[Expand[(rY xY t + l z)^5]
      Series[(cY2 xY^2 t^2 z + cY1 xY t z^2 + z^3)^-1, {xY, 0, 2}], t, 2], l]] /. xY^2 → 1
```

$\{0, 0, 0, 10\,rY^2, -5\,cY1\,rY, cY1^2 - cY2\}$

```
μx5Z =
 CoefficientList[
   Coefficient[Expand[(rZ xZ t + (aZ + l) z)^5] Series[(cZ1 xZ t + z)^-1, {xZ, 0, 1}] z^-3,
    t, 1], l] /. xZ → 1
```

$\{-aZ^5\,cZ1 + 5\,aZ^4\,rZ, -5\,aZ^4\,cZ1 + 20\,aZ^3\,rZ,$
$-10\,aZ^3\,cZ1 + 30\,aZ^2\,rZ, -10\,aZ^2\,cZ1 + 20\,aZ\,rZ, -5\,aZ\,cZ1 + 5\,rZ, -cZ1\}$

```
μx5P = CoefficientList[Expand[-((aP + l) z)^5] z^-5, l]
```

$\{-aP^5, -5\,aP^4, -10\,aP^3, -10\,aP^2, -5\,aP, -1\}$

```
(******************************
 NOTATIONS
 μs1x3Y denotes the local datum of p1(X)x^3 at Y
 μs1x3Z denotes the local datum of p1(X)x^3 at Z
 μs1x3P denotes the local datum of p1(X)x^3 at P
  ******************************)

SymmetricReduction[(yY1 t + z)^2 + (yY2 t + z)^2 + (yY3 t + z)^2 , {yY1, yY2, yY3},
 {cY1 xY , cY2 xY^2, cY3 xY^3}]
```

$\{cY1^2\,t^2\,xY^2 - 2\,cY2\,t^2\,xY^2 + 2\,cY1\,t\,xY\,z + 3\,z^2, 0\}$

```
μs1x3Y =
 Factor[CoefficientList[
    Coefficient[(pY1 t^2 + cY1^2 t^2 xY^2 - 2 cY2 t^2 xY^2 + 2 cY1 t xY z + 3 z^2)
      Expand[(rY xY t + l z)^3] Series[(cY2 xY^2 t^2 z + cY1 xY t z^2 + z^3)^-1, {xY, 0, 2}], t, 2],
    l]] /. xY^2 → 1
```

$\{0, 9\,rY^2, -3\,cY1\,rY, 2\,cY1^2 - 5\,cY2 + pY1\}$

```
μs1x3Z =
 CoefficientList[
   Coefficient[((cZ1 xZ t + z)^2 + 3 z^2) Expand[(rZ xZ t + (aZ + l) z)^3]
     Series[(cZ1 xZ t + z)^-1, {xZ, 0, 1}] z^-3, t, 1], l] /. xZ → 1
```

$\{-2\,aZ^3\,cZ1 + 12\,aZ^2\,rZ, -6\,aZ^2\,cZ1 + 24\,aZ\,rZ, -6\,aZ\,cZ1 + 12\,rZ, -2\,cZ1\}$

```
μs1x3P = CoefficientList[-5 z^2 Expand[((aP + l) z)^3] z^-5, l]
```

$\{-5\,aP^3, -15\,aP^2, -15\,aP, -5\}$

```
(*****************************
 NOTATIONS
 μs2xY denotes the local datum of s2(X)x at Y
 μs2xZ denotes the local datum of s2(X)x at Z
 μs2xP denotes the local datum of s2(X)x at P
  *****************************)

SymmetricReduction[Expand[(yY1 t + z)^4 + (yY2 t + z)^4 + (yY3 t + z)^4], {yY1, yY2, yY3},
 {cY1 xY, cY2 xY^2 , cY3 xY^3}]
```

$\{cY1^4\,t^4\,xY^4 - 4\,cY1^2\,cY2\,t^4\,xY^4 + 2\,cY2^2\,t^4\,xY^4 + 4\,cY1\,cY3\,t^4\,xY^4 + 4\,cY1^3\,t^3\,xY^3\,z - 12\,cY1\,cY2\,t^3\,xY^3\,z + 12\,cY3\,t^3\,xY^3\,z + 6\,cY1^2\,t^2\,xY^2\,z^2 - 12\,cY2\,t^2\,xY^2\,z^2 + 4\,cY1\,t\,xY\,z^3 + 3\,z^4, 0\}$

```
μs2xY =
 Factor[CoefficientList[
    Coefficient[(6 cY1^2 t^2 xY^2 z^2 - 12 cY2 t^2 xY^2 z^2 + 4 cY1 t xY z^3 + 3 z^4) (rY xY t + l z)
      Series[(cY2 xY^2 t^2 z + cY1 xY t z^2 + z^3)^-1, {xY, 0, 2}], t, 2], l]] /. xY^2 → 1
```

$\{cY1\,rY, 5\,(cY1^2 - 3\,cY2)\}$

```
μs2xZ =
 CoefficientList[
   Coefficient[((cZ1 xZ t + z)^4 + 3 z^4) (rZ xZ t + (aZ + l) z)
     Series[(cZ1 xZ t + z)^-1, {xZ, 0, 1}] z^-3, t, 1], l, 2] /. xZ → 1
```

$\{4\,rZ, 0\}$

```
μs2xP = CoefficientList[-5 z^4 ((aP + l) z) z^-5, l]
```

$\{-5\,aP, -5\}$

```
Factor[μx5Y + μx5Z + μx5P]
```

$\{-aP^5 - aZ^5\,cZ1 + 5\,aZ^4\,rZ, -5\,(aP^4 + aZ^4\,cZ1 - 4\,aZ^3\,rZ), -10\,(aP^3 + aZ^3\,cZ1 - 3\,aZ^2\,rZ), -10\,(aP^2 + aZ^2\,cZ1 - rY^2 - 2\,aZ\,rZ), -5\,(aP + aZ\,cZ1 + cY1\,rY - rZ), -1 + cY1^2 - cY2 - cZ1\}$

```
Factor[μs1x3Y + μs1x3Z + μs1x3P]
```

$\{-5\,aP^3 - 2\,aZ^3\,cZ1 + 12\,aZ^2\,rZ, -3\,(5\,aP^2 + 2\,aZ^2\,cZ1 - 3\,rY^2 - 8\,aZ\,rZ), -3\,(5\,aP + 2\,aZ\,cZ1 + cY1\,rY - 4\,rZ), -5 + 2\,cY1^2 - 5\,cY2 - 2\,cZ1 + pY1\}$

```
Factor[μs2xY + μs2xZ + μs2xP]
{-5 aP + cY1 rY + 4 rZ, 5 (-1 + cY1^2 - 3 cY2)}

Solve[{
  (******************************
   The following equations are given respectively by the above computations of x^5,
  p1(X) x^3 and s2(X) x locally
    ******************************)
  0 == -1 + cY1^2 - cY2 - cZ1,
  0 == -5 + 2 cY1^2 - 5 cY2 - 2 cZ1 + pY1,
  0 == 5 (-1 + cY1^2 - 3 cY2),
  pY1 == 3
 },
 {cY1, cY2, cZ1, pY1}]

{{cY1 → -1, cY2 → 0, cZ1 → 0, pY1 → 3}, {cY1 → 1, cY2 → 0, cZ1 → 0, pY1 → 3}}

Solve[{
  0 == -1 + cY1^2 - cY2 - cZ1,
  0 == -5 + 2 cY1^2 - 5 cY2 - 2 cZ1 + pY1,
  0 == 5 (-1 + cY1^2 - 3 cY2),
  pY1 == 3,
  (******************************
   The following equations are given by the above computations of x^5 locally
    ******************************)
  d == -aP^5 - aZ^5 cZ1 + 5 aZ^4 rZ,
  0 == -5 (aP^4 + aZ^4 cZ1 - 4 aZ^3 rZ),
  0 == -10 (aP^3 + aZ^3 cZ1 - 3 aZ^2 rZ),
  0 == -10 (aP^2 + aZ^2 cZ1 - rY^2 - 2 aZ rZ),
  0 == -5 (aP + aZ cZ1 + cY1 rY - rZ),
  (******************************
   The following equations are given by the above computations of p1(X) x^3 locally
    ******************************)
  0 == -3 (5 aP^2 + 2 aZ^2 cZ1 - 3 rY^2 - 8 aZ rZ),
  0 == -3 (5 aP + 2 aZ cZ1 + cY1 rY - 4 rZ)
 },
 {d, aP, rY, rZ, cY1, cY2, cZ1, pY1}]

{{d → 0, aP → 0, rY → 0, rZ → 0, cY1 → -1, cY2 → 0, cZ1 → 0, pY1 → 3},
 {d → 0, aP → 0, rY → 0, rZ → 0, cY1 → 1, cY2 → 0, cZ1 → 0, pY1 → 3}}
```

Computations for Lemma 6.10.
The case where there exists N^6 s.t. Fix $(N) = Y^4 \cup Z^2 \cup P^0$.
Case A.1.i where the other normal weights are trivial.

```
(******************************
  NOTATIONS
  rY xY denotes the restriction of x to Y
  rZ xZ denotes the restriction of x to Z
  aZ denotes the Hopf weight at Z
  aP denotes the Hopf weight at P
  cYn1 xY denotes the first Chern class of the normal bundle of Y in the
  direction of N
  cZn1 xZ denotes the first Chern class of the normal bundle of Z in the
  direction of N
  cYj xY^j denotes the j-th Chern class of the normal bundle of Y in the
  remaining directions
  cZ1 xZ denotes the first Chern class of the normal bundle of Z in the
  remaining directions
  μx5YA1i denotes the local datum of x^5 at Y
  μx5ZA1i denotes the local datum of x^5 at Z
  μx5PA1i denotes the local datum of x^5 at P

  We use the variable t to extract the coefficient of order 2 of xY and
  of order 1 of xZ. We then evaluate xY^2 and xZ to 1.
  The orientation of the point P is -1, this explains the negative sign.
  Once evaluated, we extract all the coefficients in l into a list.
  ******************************)
```

```
μx5YA1i =
 Factor[CoefficientList[
    Coefficient[Expand[(rY xY t + l z)^5] Series[(cYn1 xY t + n1 z)^-1, {xY, 0, 2}]
      Series[(cY2 xY^2 t^2 + cY1 xY t z + z^2)^-1, {xY, 0, 2}], t, 2], l]] /. xY^2 → 1
```

$$\left\{0,\ 0,\ 0,\ \frac{10\ rY^2}{n1},\ -\frac{5\ (cYn1 + cY1\ n1)\ rY}{n1^2},\ \frac{cYn1^2 + cY1\ cYn1\ n1 + cY1^2\ n1^2 - cY2\ n1^2}{n1^3}\right\}$$

```
SymmetricReduction[Expand[(yZ1 t + n1 z) (yZ2 t + n1 z)], {yZ1, yZ2},
 {cZn1 xZ, cZn2 xZ^2}]
```

$$\left\{cZn2\ t^2\ xZ^2 + cZn1\ n1\ t\ xZ\ z + n1^2\ z^2,\ 0\right\}$$

```
μx5ZA1i =
 CoefficientList[
   Coefficient[Expand[(rZ xZ t + (aZ + l) z)^5]
     Series[(cZn1 n1 t xZ z + n1^2 z^2)^-1, {xZ, 0, 1}] Series[(cZ1 xZ t + z)^-1, {xZ, 0, 1}] z^-1,
    t, 1], l] /. xZ → 1
```

$$\left\{-\frac{aZ^5\ cZn1}{n1^3} - \frac{aZ^5\ cZ1}{n1^2} + \frac{5\ aZ^4\ rZ}{n1^2},\ -\frac{5\ aZ^4\ cZn1}{n1^3} - \frac{5\ aZ^4\ cZ1}{n1^2} + \frac{20\ aZ^3\ rZ}{n1^2},\right.$$
$$-\frac{10\ aZ^3\ cZn1}{n1^3} - \frac{10\ aZ^3\ cZ1}{n1^2} + \frac{30\ aZ^2\ rZ}{n1^2},\ -\frac{10\ aZ^2\ cZn1}{n1^3} - \frac{10\ aZ^2\ cZ1}{n1^2} + \frac{20\ aZ\ rZ}{n1^2},$$
$$\left.-\frac{5\ aZ\ cZn1}{n1^3} - \frac{5\ aZ\ cZ1}{n1^2} + \frac{5\ rZ}{n1^2},\ -\frac{cZn1}{n1^3} - \frac{cZ1}{n1^2}\right\}$$

```
μx5PA1i = CoefficientList[Expand[-((aP + l) z)^5] (n1 z)^-3 (z)^-2, l]
```

$$\left\{-\frac{aP^5}{n1^3},\ -\frac{5\ aP^4}{n1^3},\ -\frac{10\ aP^3}{n1^3},\ -\frac{10\ aP^2}{n1^3},\ -\frac{5\ aP}{n1^3},\ -\frac{1}{n1^3}\right\}$$

```
(*****************************
 NOTATIONS
 μs1x3YA1i denotes the local datum of p1(X)x^3 at Y
 μs1x3ZA1i denotes the local datum of p1(X)x^3 at Z
 μs1x3PA1i denotes the local datum of p1(X)x^3 at P
  *****************************)

SymmetricReduction[Expand[(yY2 t + z)^2 + (yY3 t + z)^2], {yY2, yY3}, {cY1 xY, cY2 xY^2}]
```

$$\left\{cY1^2\, t^2\, xY^2 - 2\, cY2\, t^2\, xY^2 + 2\, cY1\, t\, xY\, z + 2\, z^2,\ 0\right\}$$

```
μs1x3YA1i =
 Factor[CoefficientList[
    Coefficient[
     Expand[pY1 t^2 + (cYn1 xY t + n1 z)^2 + (cY1^2 t^2 xY^2 - 2 cY2 t^2 xY^2 + 2 cY1 t xY z + 2 z^2)]
      (rY xY t + l z)^3 Series[(cYn1 xY t + n1 z)^-1, {xY, 0, 2}]
      Series[(cY2 xY^2 t^2 + cY1 xY t z + z^2)^-1, {xY, 0, 2}], t, 2], l]] /. xY^2 → 1
```

$$\left\{0,\ \frac{3\left(2+n1^2\right) rY^2}{n1},\ \frac{3\left(-2\, cYn1 + cYn1\, n1^2 - cY1\, n1^3\right) rY}{n1^2},\ \frac{2\, cYn1^2 + cY1^2\, n1^2 - 4\, cY2\, n1^2 - cY1\, cYn1\, n1^3 + cY1^2\, n1^4 - cY2\, n1^4 + n1^2\, pY1}{n1^3}\right\}$$

```
SymmetricReduction[Expand[(yZ1 t + n1 z)^2 + (yZ2 t + n1 z)^2], {yZ1, yZ2},
 {cZn1 xZ, cZn2 xZ^2}]
```

$$\left\{cZn1^2\, t^2\, xZ^2 - 2\, cZn2\, t^2\, xZ^2 + 2\, cZn1\, n1\, t\, xZ\, z + 2\, n1^2\, z^2,\ 0\right\}$$

```
μs1x3ZA1i =
 Factor[CoefficientList[
    Coefficient[Expand[(2 cZn1 n1 t xZ z + 2 n1^2 z^2) + (cZ1 xZ t + z)^2 + z^2]
      (rZ xZ t + (aZ + l) z)^3 Series[(cZn1 n1 t xZ z + n1^2 z^2)^-1, {xZ, 0, 1}]
      Series[(cZ1 xZ t + z)^-1, {xZ, 0, 1}] z^-1, t, 1], l]] /. xZ → 1
```

$$\left\{-\frac{2\, aZ^2\left(aZ\, cZn1 + aZ\, cZ1\, n1^3 - 3\, n1\, rZ - 3\, n1^3\, rZ\right)}{n1^3},\ -\frac{6\, aZ\left(aZ\, cZn1 + aZ\, cZ1\, n1^3 - 2\, n1\, rZ - 2\, n1^3\, rZ\right)}{n1^3},\ -\frac{6\left(aZ\, cZn1 + aZ\, cZ1\, n1^3 - n1\, rZ - n1^3\, rZ\right)}{n1^3},\ -\frac{2\left(cZn1 + cZ1\, n1^3\right)}{n1^3}\right\}$$

```
μs1x3PA1i =
 Factor[CoefficientList[((n1 z)^2 + (n1 z)^2 + (n1 z)^2 + z^2 + z^2) Expand[-((aP + l) z)^3]
    (n1 z)^-3 (z)^-2, l]]
```

$$\left\{-\frac{aP^3\left(2+3\, n1^2\right)}{n1^3},\ -\frac{3\, aP^2\left(2+3\, n1^2\right)}{n1^3},\ -\frac{3\, aP\left(2+3\, n1^2\right)}{n1^3},\ -\frac{2+3\, n1^2}{n1^3}\right\}$$

μx5YA1i + μx5ZA1i + μx5PA1i

$$\left\{-\frac{aP^5}{n1^3}-\frac{aZ^5\,cZn1}{n1^3}-\frac{aZ^5\,cZ1}{n1^2}+\frac{5\,aZ^4\,rZ}{n1^2},\ -\frac{5\,aP^4}{n1^3}-\frac{5\,aZ^4\,cZn1}{n1^3}-\frac{5\,aZ^4\,cZ1}{n1^2}+\frac{20\,aZ^3\,rZ}{n1^2},\right.$$
$$-\frac{10\,aP^3}{n1^3}-\frac{10\,aZ^3\,cZn1}{n1^3}-\frac{10\,aZ^3\,cZ1}{n1^2}+\frac{30\,aZ^2\,rZ}{n1^2},$$
$$-\frac{10\,aP^2}{n1^3}-\frac{10\,aZ^2\,cZn1}{n1^3}-\frac{10\,aZ^2\,cZ1}{n1^2}+\frac{10\,rY^2}{n1}+\frac{20\,aZ\,rZ}{n1^2},$$
$$-\frac{5\,aP}{n1^3}-\frac{5\,aZ\,cZn1}{n1^3}-\frac{5\,aZ\,cZ1}{n1^2}-\frac{5\,(cYn1+cY1\,n1)\,rY}{n1^2}+\frac{5\,rZ}{n1^2},$$
$$\left.-\frac{1}{n1^3}-\frac{cZn1}{n1^3}-\frac{cZ1}{n1^2}+\frac{cYn1^2+cY1\,cYn1\,n1+cY1^2\,n1^2-cY2\,n1^2}{n1^3}\right\}$$

μs1x3YA1i + μs1x3ZA1i + μs1x3PA1i

$$\left\{-\frac{aP^3\,(2+3\,n1^2)}{n1^3}-\frac{2\,aZ^2\,(aZ\,cZn1+aZ\,cZ1\,n1^3-3\,n1\,rZ-3\,n1^3\,rZ)}{n1^3},\right.$$
$$-\frac{3\,aP^2\,(2+3\,n1^2)}{n1^3}+\frac{3\,(2+n1^2)\,rY^2}{n1}-\frac{6\,aZ\,(aZ\,cZn1+aZ\,cZ1\,n1^3-2\,n1\,rZ-2\,n1^3\,rZ)}{n1^3},$$
$$-\frac{3\,aP\,(2+3\,n1^2)}{n1^3}+\frac{3\,(-2\,cYn1+cYn1\,n1^2-cY1\,n1^3)\,rY}{n1^2}-$$
$$\frac{6\,(aZ\,cZn1+aZ\,cZ1\,n1^3-n1\,rZ-n1^3\,rZ)}{n1^3},\ -\frac{2+3\,n1^2}{n1^3}-\frac{2\,(cZn1+cZ1\,n1^3)}{n1^3}+$$
$$\left.\frac{2\,cYn1^2+cY1^2\,n1^2-4\,cY2\,n1^2-cY1\,cYn1\,n1^3+cY1^2\,n1^4-cY2\,n1^4+n1^2\,pY1}{n1^3}\right\}$$

Factor[Solve[{

(*******************************
The following equations are given by the above computations of x^5 locally
*******************************)

$$d == -\frac{aP^5}{n1^3}-\frac{aZ^5\,cZn1}{n1^3}-\frac{aZ^5\,cZ1}{n1^2}+\frac{5\,aZ^4\,rZ}{n1^2},$$
$$0 == -\frac{5\,aP^4}{n1^3}-\frac{5\,aZ^4\,cZn1}{n1^3}-\frac{5\,aZ^4\,cZ1}{n1^2}+\frac{20\,aZ^3\,rZ}{n1^2},$$
$$0 == -\frac{10\,aP^3}{n1^3}-\frac{10\,aZ^3\,cZn1}{n1^3}-\frac{10\,aZ^3\,cZ1}{n1^2}+\frac{30\,aZ^2\,rZ}{n1^2},$$
$$0 == -\frac{10\,aP^2}{n1^3}-\frac{10\,aZ^2\,cZn1}{n1^3}-\frac{10\,aZ^2\,cZ1}{n1^2}+\frac{10\,rY^2}{n1}+\frac{20\,aZ\,rZ}{n1^2},$$

(*******************************
The following equations are given by the Lemma 6.8
eZN denotes the orientation of Z in N
*******************************)

$$0 == -\frac{3\,aP^2}{n1^3}+\frac{3\,rY^2}{n1}-\frac{3\,aZ\,eZN\,(aZ\,cZn1-2\,n1\,rZ)}{n1^3},$$
$$0 == -\frac{3\,aP}{n1^3}-\frac{3\,cYn1\,rY}{n1^2}-\frac{3\,eZN\,(aZ\,cZn1-n1\,rZ)}{n1^3},$$
$$0 == -\frac{1}{n1^3}+\frac{cYn1^2}{n1^3}-\frac{cZn1\,eZN}{n1^3},$$
$$eZN^2 == 1$$

},

{d, cZn1, cYn1, cZ1, rZ, rY, aP, eZN}]]

$$\{\{d\to 0,\ cZn1\to 0,\ cYn1\to -1,\ cZ1\to 0,\ rZ\to 0,\ rY\to 0,\ aP\to 0,\ eZN\to -1\},$$
$$\{d\to 0,\ cZn1\to 0,\ cYn1\to 1,\ cZ1\to 0,\ rZ\to 0,\ rY\to 0,\ aP\to 0,\ eZN\to -1\},$$
$$\{d\to 0,\ cZn1\to 0,\ cYn1\to -1,\ cZ1\to 0,\ rZ\to 0,\ rY\to 0,\ aP\to 0,\ eZN\to 1\},$$
$$\{d\to 0,\ cZn1\to 0,\ cYn1\to 1,\ cZ1\to 0,\ rZ\to 0,\ rY\to 0,\ aP\to 0,\ eZN\to 1\},$$
$$\left\{d\to \frac{72\ aZ^5}{n1^3},\ cZn1\to 80,\ cYn1\to 9,\ cZ1\to 0,\ rZ\to \frac{24\ aZ}{n1},\ rY\to -\frac{6\ aZ}{n1},\ aP\to -2\ aZ,\ eZN\to 1\right\},$$
$$\left\{d\to \frac{72\ aZ^5}{n1^3},\ cZn1\to 80,\ cYn1\to -9,\ cZ1\to 0,\ rZ\to \frac{24\ aZ}{n1},\ rY\to \frac{6\ aZ}{n1},\ aP\to -2\ aZ,\ eZN\to 1\right\},$$
$$\left\{d\to \frac{4\ aZ^5}{n1^3},\ cZn1\to 1,\ cYn1\to 0,\ cZ1\to \frac{6}{n1},\ rZ\to \frac{2\ aZ}{n1},\ rY\to -\frac{2\ aZ}{n1},\ aP\to -aZ,\ eZN\to -1\right\},$$
$$\left\{d\to \frac{4\ aZ^5}{n1^3},\ cZn1\to 1,\ cYn1\to 0,\ cZ1\to \frac{6}{n1},\ rZ\to \frac{2\ aZ}{n1},\ rY\to \frac{2\ aZ}{n1},\ aP\to -aZ,\ eZN\to -1\right\},$$
$$\{d\to 0,\ cZn1\to -1,\ cYn1\to 0,\ cZ1\to 0,\ rZ\to 0,\ rY\to 0,\ aP\to aZ,\ eZN\to 1\},$$
$$\left\{d\to 0,\ cZn1\to 1,\ cYn1\to 0,\ cZ1\to -\frac{2}{n1},\ rZ\to 0,\ rY\to 0,\ aP\to aZ,\ eZN\to -1\right\},$$
$$\left\{d\to \frac{2\,i\left(i+\sqrt{7}\right)aZ^5}{n1^3},\ cZn1\to \frac{1}{2}\left(1+3\,i\sqrt{7}\right),\ cYn1\to \frac{1}{2}\left(3-i\sqrt{7}\right),\right.$$
$$\left.cZ1\to \frac{11+i\sqrt{7}}{n1},\ rZ\to \frac{\left(3+i\sqrt{7}\right)aZ}{n1},\ rY\to -\frac{2\ aZ}{n1},\ aP\to \frac{1}{2}\left(1-i\sqrt{7}\right)aZ,\ eZN\to -1\right\},$$
$$\left\{d\to \frac{2\,i\left(i+\sqrt{7}\right)aZ^5}{n1^3},\ cZn1\to \frac{1}{2}\left(1+3\,i\sqrt{7}\right),\ cYn1\to \frac{1}{2}\,i\left(3\,i+\sqrt{7}\right),\right.$$
$$\left.cZ1\to \frac{11+i\sqrt{7}}{n1},\ rZ\to \frac{\left(3+i\sqrt{7}\right)aZ}{n1},\ rY\to \frac{2\ aZ}{n1},\ aP\to \frac{1}{2}\left(1-i\sqrt{7}\right)aZ,\ eZN\to -1\right\},$$
$$\left\{d\to -\frac{2\,i\left(-i+\sqrt{7}\right)aZ^5}{n1^3},\ cZn1\to \frac{1}{2}\left(1-3\,i\sqrt{7}\right),\ cYn1\to \frac{1}{2}\left(3+i\sqrt{7}\right),\right.$$
$$\left.cZ1\to \frac{11-i\sqrt{7}}{n1},\ rZ\to \frac{\left(3-i\sqrt{7}\right)aZ}{n1},\ rY\to -\frac{2\ aZ}{n1},\ aP\to \frac{1}{2}\left(1+i\sqrt{7}\right)aZ,\ eZN\to -1\right\},$$
$$\left\{d\to -\frac{2\,i\left(-i+\sqrt{7}\right)aZ^5}{n1^3},\ cZn1\to \frac{1}{2}\left(1-3\,i\sqrt{7}\right),\ cYn1\to -\frac{1}{2}\,i\left(-3\,i+\sqrt{7}\right),\right.$$
$$\left.cZ1\to \frac{11-i\sqrt{7}}{n1},\ rZ\to \frac{\left(3-i\sqrt{7}\right)aZ}{n1},\ rY\to \frac{2\ aZ}{n1},\ aP\to \frac{1}{2}\left(1+i\sqrt{7}\right)aZ,\ eZN\to -1\right\}\}$$

```
Factor[Solve[{
   (******************************
    Since there are many solutions to the above system of equations, we simply
    copy it instead of copying the solutions and compute in each of the cases

    The following equations are given by the above computations of x^5 locally.
     ******************************)
   d == - aP^5/n1^3 - (aZ^5 cZn1)/n1^3 - (aZ^5 cZ1)/n1^2 + (5 aZ^4 rZ)/n1^2,
   0 == - (5 aP^4)/n1^3 - (5 aZ^4 cZn1)/n1^3 - (5 aZ^4 cZ1)/n1^2 + (20 aZ^3 rZ)/n1^2,
   0 == - (10 aP^3)/n1^3 - (10 aZ^3 cZn1)/n1^3 - (10 aZ^3 cZ1)/n1^2 + (30 aZ^2 rZ)/n1^2,
   0 == - (10 aP^2)/n1^3 - (10 aZ^2 cZn1)/n1^3 - (10 aZ^2 cZ1)/n1^2 + (10 rY^2)/n1 + (20 aZ rZ)/n1^2,
   (******************************
    The following equations are given by the Lemma 7.8
    eZN denotes the orientation of Z in N
     ******************************)
   0 == - (3 aP^2)/n1^3 + (3 rY^2)/n1 - (3 aZ eZN (aZ cZn1 - 2 n1 rZ))/n1^3,
   0 == - (3 aP)/n1^3 - (3 cYn1 rY)/n1^2 - (3 eZN (aZ cZn1 - n1 rZ))/n1^3,
   0 == - 1/n1^3 + cYn1^2/n1^3 - (cZn1 eZN)/n1^3,
   eZN^2 == 1,
   (******************************
    The following equations are given by the above computations of p1(X)x^3 locally
     ******************************)
   p d == - (aP^3 (2 + 3 n1^2))/n1^3 - (2 aZ^2 (aZ cZn1 + aZ cZ1 n1^3 - 3 n1 rZ - 3 n1^3 rZ))/n1^3
  }, {p, d, cYn1, cZn1, cZ1, rZ, rY, aP, eZN}]]
```

$$\Big\{\{d\to 0,\ cYn1\to -1,\ cZn1\to 0,\ cZ1\to 0,\ rZ\to 0,\ rY\to 0,\ aP\to 0,\ eZN\to -1\},$$
$$\{d\to 0,\ cYn1\to 1,\ cZn1\to 0,\ cZ1\to 0,\ rZ\to 0,\ rY\to 0,\ aP\to 0,\ eZN\to -1\},$$
$$\{d\to 0,\ cYn1\to -1,\ cZn1\to 0,\ cZ1\to 0,\ rZ\to 0,\ rY\to 0,\ aP\to 0,\ eZN\to 1\},$$
$$\{d\to 0,\ cYn1\to 1,\ cZn1\to 0,\ cZ1\to 0,\ rZ\to 0,\ rY\to 0,\ aP\to 0,\ eZN\to 1\},$$
$$\Big\{p\to \frac{7\,n1^2}{3\,aZ^2},\ d\to \frac{72\,aZ^5}{n1^3},\ cYn1\to 9,\ cZn1\to 80,\ cZ1\to 0,\ rZ\to \frac{24\,aZ}{n1},$$
$$rY\to -\frac{6\,aZ}{n1},\ aP\to -2\,aZ,\ eZN\to 1\Big\},\ \Big\{p\to \frac{7\,n1^2}{3\,aZ^2},\ d\to \frac{72\,aZ^5}{n1^3},\ cYn1\to -9,$$
$$cZn1\to 80,\ cZ1\to 0,\ rZ\to \frac{24\,aZ}{n1},\ rY\to \frac{6\,aZ}{n1},\ aP\to -2\,aZ,\ eZN\to 1\Big\},$$
$$\Big\{p\to \frac{3\,(4+n1^2)}{4\,aZ^2},\ d\to \frac{4\,aZ^5}{n1^3},\ cYn1\to 0,\ cZn1\to 1,\ cZ1\to \frac{6}{n1},\ rZ\to \frac{2\,aZ}{n1},$$
$$rY\to -\frac{2\,aZ}{n1},\ aP\to -aZ,\ eZN\to -1\Big\},\ \Big\{p\to \frac{3\,(4+n1^2)}{4\,aZ^2},\ d\to \frac{4\,aZ^5}{n1^3},\ cYn1\to 0,$$
$$cZn1\to 1,\ cZ1\to \frac{6}{n1},\ rZ\to \frac{2\,aZ}{n1},\ rY\to \frac{2\,aZ}{n1},\ aP\to -aZ,\ eZN\to -1\Big\},$$
$$\Big\{p\to \frac{-4-12\,i\sqrt{7}+7\,n1^2-3\,i\sqrt{7}\,n1^2}{8\,aZ^2},\ d\to \frac{2\,i\,(i+\sqrt{7})\,aZ^5}{n1^3},\ cYn1\to \frac{1}{2}\,(3-i\sqrt{7}),$$
$$cZn1\to \frac{1}{2}\,(1+3\,i\sqrt{7}),\ cZ1\to \frac{11+i\sqrt{7}}{n1},\ rZ\to \frac{(3+i\sqrt{7})\,aZ}{n1},\ rY\to -\frac{2\,aZ}{n1},$$
$$aP\to \frac{1}{2}\,(1-i\sqrt{7})\,aZ,\ eZN\to -1\Big\},\ \Big\{p\to \frac{-4-12\,i\sqrt{7}+7\,n1^2-3\,i\sqrt{7}\,n1^2}{8\,aZ^2},$$
$$d\to \frac{2\,i\,(i+\sqrt{7})\,aZ^5}{n1^3},\ cYn1\to \frac{1}{2}\,i\,(3\,i+\sqrt{7}),\ cZn1\to \frac{1}{2}\,(1+3\,i\sqrt{7}),$$
$$cZ1\to \frac{11+i\sqrt{7}}{n1},\ rZ\to \frac{(3+i\sqrt{7})\,aZ}{n1},\ rY\to \frac{2\,aZ}{n1},\ aP\to \frac{1}{2}\,(1-i\sqrt{7})\,aZ,\ eZN\to -1\Big\},$$
$$\Big\{p\to \frac{-4+12\,i\sqrt{7}+7\,n1^2+3\,i\sqrt{7}\,n1^2}{8\,aZ^2},\ d\to -\frac{2\,i\,(-i+\sqrt{7})\,aZ^5}{n1^3},\ cYn1\to \frac{1}{2}\,(3+i\sqrt{7}),$$
$$cZn1\to \frac{1}{2}\,(1-3\,i\sqrt{7}),\ cZ1\to \frac{11-i\sqrt{7}}{n1},\ rZ\to \frac{(3-i\sqrt{7})\,aZ}{n1},\ rY\to -\frac{2\,aZ}{n1},$$
$$aP\to \frac{1}{2}\,(1+i\sqrt{7})\,aZ,\ eZN\to -1\Big\},\ \Big\{p\to \frac{-4+12\,i\sqrt{7}+7\,n1^2+3\,i\sqrt{7}\,n1^2}{8\,aZ^2},$$
$$d\to -\frac{2\,i\,(-i+\sqrt{7})\,aZ^5}{n1^3},\ cYn1\to -\frac{1}{2}\,i\,(-3\,i+\sqrt{7}),\ cZn1\to \frac{1}{2}\,(1-3\,i\sqrt{7}),$$
$$cZ1\to \frac{11-i\sqrt{7}}{n1},\ rZ\to \frac{(3-i\sqrt{7})\,aZ}{n1},\ rY\to \frac{2\,aZ}{n1},\ aP\to \frac{1}{2}\,(1+i\sqrt{7})\,aZ,\ eZN\to -1\Big\}\Big\}$$

Computations for Lemma 6.10.
The case where there exists N^6 s.t. Fix$(N) = Y^4 \cup Z^2 \cup P^0$.
Case A.1.ii where there exists M (not contained in N) containing Z and P.

```
(*******************************
  NOTATIONS
  rY xY denotes the restriction of x to Y
  rZ xZ denotes the restriction of x to Z
  aZ denotes the Hopf weight at Z
  aP denotes the Hopf weight at P
  cZn2 xZ denotes the first Chern class of the normal bundle of Z in the
  direction of M
  cZn1 xZ denotes the first Chern class of the normal bundle of Z in the
  direction of N
  cZ1 xZ denotes the first Chern class of the normal bundle of Z in the
  remaining directions
  cYn1 xY denotes the first Chern class of the normal bundle of Y in the
  direction of N
  cYj xY^j denotes the j-th Chern class of the normal bundle of Y in the
  remaining directions
  μx5YA1ii denotes the local datum of x^5 at Y
  μx5ZA1ii denotes the local datum of x^5 at Z
  μx5PA1ii denotes the local datum of x^5 at P

  We use the variable t to extract the coefficient of order 2 of xY and
  of order 1 of xZ. We then evaluate xY^2 and xZ to 1.
  The orientation of the point P is -1, this explains the negative sign.
  Once evaluated, we extract all the coefficients in l into a list.
  *******************************)
```

```
μx5YA1ii =
 Factor[CoefficientList[
    Coefficient[Expand[(rY xY t + l z)^5] Series[(cYn1 xY t + n1 z)^-1, {xY, 0, 2}]
      Series[(cY2 xY^2 t^2 + cY1 xY t z + z^2)^-1, {xY, 0, 2}], t, 2], l]] /. xY^2 → 1
```

$$\left\{0,\ 0,\ 0,\ \frac{10\,\text{rY}^2}{\text{n1}},\ -\frac{5\,(\text{cYn1}+\text{cY1 n1})\,\text{rY}}{\text{n1}^2},\ \frac{\text{cYn1}^2+\text{cY1 cYn1 n1}+\text{cY1}^2\,\text{n1}^2-\text{cY2 n1}^2}{\text{n1}^3}\right\}$$

```
SymmetricReduction[Expand[(yZ1 t + n1 z) (yZ2 t + n1 z)], {yZ1, yZ2},
 {cZn1 xZ, cZ2n1 xZ^2}]
```

$$\left\{\text{cZ2n1}\,t^2\,\text{xZ}^2+\text{cZn1 n1}\,t\,\text{xZ}\,z+\text{n1}^2\,z^2,\ 0\right\}$$

```
μx5ZA1ii =
 CoefficientList[
   Coefficient[Expand[(rZ xZ t + (aZ + l) z)^5]
     Series[(cZn1 n1 t xZ z + n1^2 z^2)^-1, {xZ, 0, 1}] Series[(cZn2 xZ t + n2 z)^-1, {xZ, 0, 1}]
     Series[(cZ1 xZ t + z)^-1, {xZ, 0, 1}], t, 1], l] /. xZ → 1
```

$$\Big\{-\frac{\text{aZ}^5\,\text{cZn2}}{\text{n1}^2\,\text{n2}^2}-\frac{\text{aZ}^5\,\text{cZn1}}{\text{n1}^3\,\text{n2}}-\frac{\text{aZ}^5\,\text{cZ1}}{\text{n1}^2\,\text{n2}}+\frac{5\,\text{aZ}^4\,\text{rZ}}{\text{n1}^2\,\text{n2}},\ -\frac{5\,\text{aZ}^4\,\text{cZn2}}{\text{n1}^2\,\text{n2}^2}-\frac{5\,\text{aZ}^4\,\text{cZn1}}{\text{n1}^3\,\text{n2}}-\frac{5\,\text{aZ}^4\,\text{cZ1}}{\text{n1}^2\,\text{n2}}+\frac{20\,\text{aZ}^3\,\text{rZ}}{\text{n1}^2\,\text{n2}},$$
$$-\frac{10\,\text{aZ}^3\,\text{cZn2}}{\text{n1}^2\,\text{n2}^2}-\frac{10\,\text{aZ}^3\,\text{cZn1}}{\text{n1}^3\,\text{n2}}-\frac{10\,\text{aZ}^3\,\text{cZ1}}{\text{n1}^2\,\text{n2}}+\frac{30\,\text{aZ}^2\,\text{rZ}}{\text{n1}^2\,\text{n2}},$$
$$-\frac{10\,\text{aZ}^2\,\text{cZn2}}{\text{n1}^2\,\text{n2}^2}-\frac{10\,\text{aZ}^2\,\text{cZn1}}{\text{n1}^3\,\text{n2}}-\frac{10\,\text{aZ}^2\,\text{cZ1}}{\text{n1}^2\,\text{n2}}+\frac{20\,\text{aZ rZ}}{\text{n1}^2\,\text{n2}},$$
$$-\frac{5\,\text{aZ cZn2}}{\text{n1}^2\,\text{n2}^2}-\frac{5\,\text{aZ cZn1}}{\text{n1}^3\,\text{n2}}-\frac{5\,\text{aZ cZ1}}{\text{n1}^2\,\text{n2}}+\frac{5\,\text{rZ}}{\text{n1}^2\,\text{n2}},\ -\frac{\text{cZn2}}{\text{n1}^2\,\text{n2}^2}-\frac{\text{cZn1}}{\text{n1}^3\,\text{n2}}-\frac{\text{cZ1}}{\text{n1}^2\,\text{n2}}\Big\}$$

```
μx5PA1ii = CoefficientList[Expand[- ((aP + 1) z)^5] (n1 z)^-3 (n2 z)^-2, 1]
```

$$\left\{-\frac{aP^5}{n1^3\,n2^2},\ -\frac{5\,aP^4}{n1^3\,n2^2},\ -\frac{10\,aP^3}{n1^3\,n2^2},\ -\frac{10\,aP^2}{n1^3\,n2^2},\ -\frac{5\,aP}{n1^3\,n2^2},\ -\frac{1}{n1^3\,n2^2}\right\}$$

```
(******************************
 NOTATIONS
 μs1x3YA1ii denotes the local datum of p1(X)x^3 at Y
 μs1x3ZA1ii denotes the local datum of p1(X)x^3 at Z
 μs1x3PA1ii denotes the local datum of p1(X)x^3 at P
  ******************************)
```

```
SymmetricReduction[Expand[(yY2 t + z)^2 + (yY3 t + z)^2], {yY2, yY3}, {cY1 xY, cY2 xY^2}]
```

$$\left\{cY1^2\,t^2\,xY^2 - 2\,cY2\,t^2\,xY^2 + 2\,cY1\,t\,xY\,z + 2\,z^2,\ 0\right\}$$

```
μs1x3YA1ii =
 Factor[CoefficientList[
    Coefficient[
     Expand[pY1 t^2 + (cYn1 xY t + n1 z)^2 + (cY1^2 t^2 xY^2 - 2 cY2 t^2 xY^2 + 2 cY1 t xY z + 2 z^2)]
      (rY xY t + 1 z)^3 Series[(cYn1 xY t + n1 z)^-1, {xY, 0, 2}]
      Series[(cY2 xY^2 t^2 + cY1 xY t z + z^2)^-1, {xY, 0, 2}], t, 2], 1]] /. xY^2 → 1
```

$$\left\{0,\ \frac{3\left(2+n1^2\right)rY^2}{n1},\ \frac{3\left(-2\,cYn1+cYn1\,n1^2-cY1\,n1^3\right)rY}{n1^2},\right.$$
$$\left.\frac{2\,cYn1^2+cY1^2\,n1^2-4\,cY2\,n1^2-cY1\,cYn1\,n1^3+cY1^2\,n1^4-cY2\,n1^4+n1^2\,pY1}{n1^3}\right\}$$

```
SymmetricReduction[Expand[(yZ1 t + n1 z)^2 + (yZ2 t + n1 z)^2], {yZ1, yZ2},
 {cZn1 xZ, cz2n2 xZ^2}]
```

$$\left\{-2\,cz2n2\,t^2\,xZ^2 + cZn1^2\,t^2\,xZ^2 + 2\,cZn1\,n1\,t\,xZ\,z + 2\,n1^2\,z^2,\ 0\right\}$$

```
μs1x3ZA1ii =
 Factor[CoefficientList[
    Coefficient[Expand[(2 cZn1 n1 t xZ z + 2 n1^2 z^2) + (cZn2 xZ t + n2 z)^2 + (cZ1 xZ t + z)^2]
      (rZ xZ t + (aZ + 1) z)^3 Series[(cZn1 n1 t xZ z + n1^2 z^2)^-1, {xZ, 0, 1}]
      Series[(cZn2 xZ t + n2 z)^-1, {xZ, 0, 1}] Series[(cZ1 xZ t + z)^-1, {xZ, 0, 1}], t, 1],
    1]] /. xZ → 1
```

$$\left\{-\frac{1}{n1^3\,n2^2}\,aZ^2\left(aZ\,cZn2\,n1+2\,aZ\,cZn2\,n1^3+aZ\,cZn1\,n2-aZ\,cZ1\,n1\,n2+2\,aZ\,cZ1\,n1^3\,n2-aZ\,cZn2\,n1\,n2^2+\right.\right.$$
$$\left.aZ\,cZn1\,n2^3+aZ\,cZ1\,n1\,n2^3-3\,n1\,n2\,rZ-6\,n1^3\,n2\,rZ-3\,n1\,n2^3\,rZ\right),$$
$$-\frac{1}{n1^3\,n2^2}3\,aZ\left(aZ\,cZn2\,n1+2\,aZ\,cZn2\,n1^3+aZ\,cZn1\,n2-aZ\,cZ1\,n1\,n2+2\,aZ\,cZ1\,n1^3\,n2-\right.$$
$$\left.aZ\,cZn2\,n1\,n2^2+aZ\,cZn1\,n2^3+aZ\,cZ1\,n1\,n2^3-2\,n1\,n2\,rZ-4\,n1^3\,n2\,rZ-2\,n1\,n2^3\,rZ\right),$$
$$-\frac{1}{n1^3\,n2^2}3\left(aZ\,cZn2\,n1+2\,aZ\,cZn2\,n1^3+aZ\,cZn1\,n2-aZ\,cZ1\,n1\,n2+2\,aZ\,cZ1\,n1^3\,n2-\right.$$
$$\left.aZ\,cZn2\,n1\,n2^2+aZ\,cZn1\,n2^3+aZ\,cZ1\,n1\,n2^3-n1\,n2\,rZ-2\,n1^3\,n2\,rZ-n1\,n2^3\,rZ\right),$$
$$\left.-\frac{cZn2\,n1+2\,cZn2\,n1^3+cZn1\,n2-cZ1\,n1\,n2+2\,cZ1\,n1^3\,n2-cZn2\,n1\,n2^2+cZn1\,n2^3+cZ1\,n1\,n2^3}{n1^3\,n2^2}\right\}$$

```
μs1x3PA1ii =
 Factor[CoefficientList[((n1 z)^2 + (n1 z)^2 + (n1 z)^2 + (n2 z)^2 + (n2 z)^2)
    Expand[-((aP + l) z)^3] (n1 z)^-3 (n2 z)^-2, l]]
```

$$\left\{-\frac{aP^3\,(3\,n1^2+2\,n2^2)}{n1^3\,n2^2},\ -\frac{3\,aP^2\,(3\,n1^2+2\,n2^2)}{n1^3\,n2^2},\ -\frac{3\,aP\,(3\,n1^2+2\,n2^2)}{n1^3\,n2^2},\ -\frac{3\,n1^2+2\,n2^2}{n1^3\,n2^2}\right\}$$

```
μx5YA1ii + μx5ZA1ii + μx5PA1ii
```

$$\begin{aligned}\Bigg\{&-\frac{aP^5}{n1^3\,n2^2}-\frac{aZ^5\,cZn2}{n1^2\,n2^2}-\frac{aZ^5\,cZn1}{n1^3\,n2}-\frac{aZ^5\,cZ1}{n1^2\,n2}+\frac{5\,aZ^4\,rZ}{n1^2\,n2},\\
&-\frac{5\,aP^4}{n1^3\,n2^2}-\frac{5\,aZ^4\,cZn2}{n1^2\,n2^2}-\frac{5\,aZ^4\,cZn1}{n1^3\,n2}-\frac{5\,aZ^4\,cZ1}{n1^2\,n2}+\frac{20\,aZ^3\,rZ}{n1^2\,n2},\\
&-\frac{10\,aP^3}{n1^3\,n2^2}-\frac{10\,aZ^3\,cZn2}{n1^2\,n2^2}-\frac{10\,aZ^3\,cZn1}{n1^3\,n2}-\frac{10\,aZ^3\,cZ1}{n1^2\,n2}+\frac{30\,aZ^2\,rZ}{n1^2\,n2},\\
&-\frac{10\,aP^2}{n1^3\,n2^2}-\frac{10\,aZ^2\,cZn2}{n1^2\,n2^2}-\frac{10\,aZ^2\,cZn1}{n1^3\,n2}-\frac{10\,aZ^2\,cZ1}{n1^2\,n2}+\frac{10\,rY^2}{n1}+\frac{20\,aZ\,rZ}{n1^2\,n2},\\
&-\frac{5\,aP}{n1^3\,n2^2}-\frac{5\,aZ\,cZn2}{n1^2\,n2^2}-\frac{5\,aZ\,cZn1}{n1^3\,n2}-\frac{5\,aZ\,cZ1}{n1^2\,n2}-\frac{5\,(cYn1+cY1\,n1)\,rY}{n1^2}+\frac{5\,rZ}{n1^2\,n2},\\
&\frac{cYn1^2+cY1\,cYn1\,n1+cY1^2\,n1^2-cY2\,n1^2}{n1^3}-\frac{1}{n1^3\,n2^2}-\frac{cZn2}{n1^2\,n2^2}-\frac{cZn1}{n1^3\,n2}-\frac{cZ1}{n1^2\,n2}\Bigg\}\end{aligned}$$

```
μs1x3YA1ii + μs1x3ZA1ii + μs1x3PA1ii
```

$$\begin{aligned}\Bigg\{&-\frac{aP^3\,(3\,n1^2+2\,n2^2)}{n1^3\,n2^2}-\frac{1}{n1^3\,n2^2}\\
&\quad aZ^2\,\big(aZ\,cZn2\,n1+2\,aZ\,cZn2\,n1^3+aZ\,cZn1\,n2-aZ\,cZ1\,n1\,n2+2\,aZ\,cZ1\,n1^3\,n2-\\
&\qquad aZ\,cZn2\,n1\,n2^2+aZ\,cZn1\,n2^3+aZ\,cZ1\,n1\,n2^3-3\,n1\,n2\,rZ-6\,n1^3\,n2\,rZ-3\,n1\,n2^3\,rZ\big),\\
&-\frac{3\,aP^2\,(3\,n1^2+2\,n2^2)}{n1^3\,n2^2}+\frac{3\,(2+n1^2)\,rY^2}{n1}-\frac{1}{n1^3\,n2^2}\\
&\quad 3\,aZ\,\big(aZ\,cZn2\,n1+2\,aZ\,cZn2\,n1^3+aZ\,cZn1\,n2-aZ\,cZ1\,n1\,n2+2\,aZ\,cZ1\,n1^3\,n2-\\
&\qquad aZ\,cZn2\,n1\,n2^2+aZ\,cZn1\,n2^3+aZ\,cZ1\,n1\,n2^3-2\,n1\,n2\,rZ-4\,n1^3\,n2\,rZ-2\,n1\,n2^3\,rZ\big),\\
&-\frac{3\,aP\,(3\,n1^2+2\,n2^2)}{n1^3\,n2^2}+\frac{3\,(-2\,cYn1+cYn1\,n1^2-cY1\,n1^3)\,rY}{n1^2}-\frac{1}{n1^3\,n2^2}\\
&\quad 3\,\big(aZ\,cZn2\,n1+2\,aZ\,cZn2\,n1^3+aZ\,cZn1\,n2-aZ\,cZ1\,n1\,n2+2\,aZ\,cZ1\,n1^3\,n2-aZ\,cZn2\,n1\,n2^2+\\
&\qquad aZ\,cZn1\,n2^3+aZ\,cZ1\,n1\,n2^3-n1\,n2\,rZ-2\,n1^3\,n2\,rZ-n1\,n2^3\,rZ\big),\ -\frac{3\,n1^2+2\,n2^2}{n1^3\,n2^2}-\\
&\quad \frac{cZn2\,n1+2\,cZn2\,n1^3+cZn1\,n2-cZ1\,n1\,n2+2\,cZ1\,n1^3\,n2-cZn2\,n1\,n2^2+cZn1\,n2^3+cZ1\,n1\,n2^3}{n1^3\,n2^2}+\\
&\quad \frac{2\,cYn1^2+cY1^2\,n1^2-4\,cY2\,n1^2-cY1\,cYn1\,n1^3+cY1^2\,n1^4-cY2\,n1^4+n1^2\,pY1}{n1^3}\Bigg\}\end{aligned}$$

```
Factor[Solve[{
  (******************************
   The following equations are given by the above computations of x^5 locally
   ******************************)
```

$$d == -\frac{aP^5}{n1^3\,n2^2} - \frac{aZ^5\,cZn2}{n1^2\,n2^2} - \frac{aZ^5\,cZn1}{n1^3\,n2} - \frac{aZ^5\,cZ1}{n1^2\,n2} + \frac{5\,aZ^4\,rZ}{n1^2\,n2},$$

$$0 == -\frac{5\,aP^4}{n1^3\,n2^2} - \frac{5\,aZ^4\,cZn2}{n1^2\,n2^2} - \frac{5\,aZ^4\,cZn1}{n1^3\,n2} - \frac{5\,aZ^4\,cZ1}{n1^2\,n2} + \frac{20\,aZ^3\,rZ}{n1^2\,n2},$$

$$0 == -\frac{10\,aP^3}{n1^3\,n2^2} - \frac{10\,aZ^3\,cZn2}{n1^2\,n2^2} - \frac{10\,aZ^3\,cZn1}{n1^3\,n2} - \frac{10\,aZ^3\,cZ1}{n1^2\,n2} + \frac{30\,aZ^2\,rZ}{n1^2\,n2},$$

$$0 == -\frac{10\,aP^2}{n1^3\,n2^2} - \frac{10\,aZ^2\,cZn2}{n1^2\,n2^2} - \frac{10\,aZ^2\,cZn1}{n1^3\,n2} - \frac{10\,aZ^2\,cZ1}{n1^2\,n2} + \frac{10\,rY^2}{n1} + \frac{20\,aZ\,rZ}{n1^2\,n2},$$

```
  (******************************
   The following equations are given by Lemma 6.8
   ******************************)
```

$$0 == -\frac{3\,aP^2}{n1^3} + \frac{3\,rY^2}{n1} - \frac{3\,aZ\,eZN\,(aZ\,cZn1 - 2\,n1\,rZ)}{n1^3},$$

$$0 == -\frac{3\,aP}{n1^3} - \frac{3\,cYn1\,rY}{n1^2} - \frac{3\,eZN\,(aZ\,cZn1 - n1\,rZ)}{n1^3},$$

$$0 == -\frac{1}{n1^3} + \frac{cYn1^2}{n1^3} - \frac{cZn1\,eZN}{n1^3},$$

```
  (******************************
   The following equations are given by Lemma 6.7
   ******************************)
```

$$0 == \frac{2\,(aP\,ePM - aZ\,cZn2\,eZM + eZM\,n2\,rZ)}{n2^2},$$

$$0 == \frac{ePM - cZn2\,eZM}{n2^2}$$

```
 },
 {d, cZn1, cZn2, cZ1, cYn1, rZ, rY, aP}]]
```

$$\left\{\left\{d \to 0,\ cZn1 \to -\frac{1}{eZN},\ cZn2 \to \frac{ePM}{eZM},\right.\right.$$

$$\left.\left. cZ1 \to -\frac{eZM\,eZN + ePM\,eZN\,n1 - eZM\,n2}{eZM\,eZN\,n1\,n2},\ cYn1 \to 0,\ rZ \to 0,\ rY \to 0,\ aP \to aZ\right\}\right\}$$

Computations for Lemma 6.10.
The case where there exists N^6 s.t. Fix $(N) = Y^4 \cup Z^2 \cup P^0$ and M^4 in N^6 s.t. Fix$(M) = Z^2 \cup P^0$

Case A.2.i where the other normal weights are trivial.

```
(******************************
  NOTATIONS
  rY xY denotes the restriction of x to Y
  rZ xZ denotes the restriction of x to Z
  aZ denotes the Hopf weight at Z
  aP denotes the Hopf weight at P
  cZn12 xZ denotes the first Chern class of the normal bundle of Z in
  the direction of M
  cZn1 xZ denotes the first Chern class of the normal bundle of Z in the
  remaining direction in N
  cZ1 xZ denotes the first Chern class of the normal bundle of Z in the
  remaining directions
  cYn1 xY denotes the first Chern class of the normal bundle of Y in the
  direction of N
  cYj xY^j denotes the j-th Chern class of the normal bundle of Y in the
  remaining directions
  μx5YA2i denotes the local datum of x^5 at Y
  μx5ZA2i denotes the local datum of x^5 at Z
  μx5PA2i denotes the local datum of x^5 at P

  We use the variable t to extract the coefficient of order 2 of xY and
  of order 1 of xZ. We then evaluate xY^2 and xZ to 1.
  The orientation of the point P is -1, this explains the negative sign.
  Once evaluated, we extract all the coefficients in l into a list.
  ******************************)

μx5YA2i =
 Factor[CoefficientList[
    Coefficient[Expand[(rY xY t + l z)^5] Series[(cYn1 xY t + n1 z)^-1, {xY, 0, 2}]
      Series[(cY2 xY^2 t^2 + cY1 xY t z + z^2)^-1, {xY, 0, 2}], t, 2], l]] /. xY^2 → 1
```

$$\left\{0,\ 0,\ 0,\ \frac{10\ \text{rY}^2}{\text{n1}},\ -\frac{5\ (\text{cYn1}+\text{cY1 n1})\ \text{rY}}{\text{n1}^2},\ \frac{\text{cYn1}^2+\text{cY1 cYn1 n1}+\text{cY1}^2\ \text{n1}^2-\text{cY2 n1}^2}{\text{n1}^3}\right\}$$

```
μx5ZA2i =
 CoefficientList[
   Coefficient[Expand[(rZ xZ t + (aZ + l) z)^5] Series[(cZn1 t xZ + n1 z)^-1, {xZ, 0, 1}]
     Series[(cZn12 xZ t + n1 n2 z)^-1, {xZ, 0, 1}] Series[(cZ1 xZ t + z)^-1, {xZ, 0, 1}] z^-1,
    t, 1], l] /. xZ → 1
```

$$\left\{-\frac{\text{aZ}^5\ \text{cZn12}}{\text{n1}^3\ \text{n2}^2}-\frac{\text{aZ}^5\ \text{cZn1}}{\text{n1}^3\ \text{n2}}-\frac{\text{aZ}^5\ \text{cZ1}}{\text{n1}^2\ \text{n2}}+\frac{5\ \text{aZ}^4\ \text{rZ}}{\text{n1}^2\ \text{n2}},\ -\frac{5\ \text{aZ}^4\ \text{cZn12}}{\text{n1}^3\ \text{n2}^2}-\frac{5\ \text{aZ}^4\ \text{cZn1}}{\text{n1}^3\ \text{n2}}-\frac{5\ \text{aZ}^4\ \text{cZ1}}{\text{n1}^2\ \text{n2}}+\frac{20\ \text{aZ}^3\ \text{rZ}}{\text{n1}^2\ \text{n2}},\right.$$
$$-\frac{10\ \text{aZ}^3\ \text{cZn12}}{\text{n1}^3\ \text{n2}^2}-\frac{10\ \text{aZ}^3\ \text{cZn1}}{\text{n1}^3\ \text{n2}}-\frac{10\ \text{aZ}^3\ \text{cZ1}}{\text{n1}^2\ \text{n2}}+\frac{30\ \text{aZ}^2\ \text{rZ}}{\text{n1}^2\ \text{n2}},$$
$$-\frac{10\ \text{aZ}^2\ \text{cZn12}}{\text{n1}^3\ \text{n2}^2}-\frac{10\ \text{aZ}^2\ \text{cZn1}}{\text{n1}^3\ \text{n2}}-\frac{10\ \text{aZ}^2\ \text{cZ1}}{\text{n1}^2\ \text{n2}}+\frac{20\ \text{aZ rZ}}{\text{n1}^2\ \text{n2}},$$
$$\left.-\frac{5\ \text{aZ cZn12}}{\text{n1}^3\ \text{n2}^2}-\frac{5\ \text{aZ cZn1}}{\text{n1}^3\ \text{n2}}-\frac{5\ \text{aZ cZ1}}{\text{n1}^2\ \text{n2}}+\frac{5\ \text{rZ}}{\text{n1}^2\ \text{n2}},\ -\frac{\text{cZn12}}{\text{n1}^3\ \text{n2}^2}-\frac{\text{cZn1}}{\text{n1}^3\ \text{n2}}-\frac{\text{cZ1}}{\text{n1}^2\ \text{n2}}\right\}$$

```
μx5PA2i = CoefficientList[Expand[-((aP + l) z)^5] (n1 z)^-1 (n1 n2 z)^-2 (z)^-2, l]
```

$$\left\{-\frac{aP^5}{n1^3\, n2^2}, -\frac{5\, aP^4}{n1^3\, n2^2}, -\frac{10\, aP^3}{n1^3\, n2^2}, -\frac{10\, aP^2}{n1^3\, n2^2}, -\frac{5\, aP}{n1^3\, n2^2}, -\frac{1}{n1^3\, n2^2}\right\}$$

```
(******************************
 NOTATIONS
 μs1x3YA2i denotes the local datum of p1(X)x^3 at Y
 μs1x3ZA2i denotes the local datum of p1(X)x^3 at Z
 μs1x3PA2i denotes the local datum of p1(X)x^3 at P
 ******************************)
```

```
SymmetricReduction[Expand[(yY2 t + z)^2 + (yY3 t + z)^2], {yY2, yY3}, {cY1 xY, cY2 xY^2}]
```

$$\left\{cY1^2\, t^2\, xY^2 - 2\, cY2\, t^2\, xY^2 + 2\, cY1\, t\, xY\, z + 2\, z^2, 0\right\}$$

```
μs1x3YA2i =
 Factor[CoefficientList[
    Coefficient[
     Expand[pY1 t^2 + (cYn1 xY t + n1 z)^2 + (cY1^2 t^2 xY^2 - 2 cY2 t^2 xY^2 + 2 cY1 t xY z + 2 z^2)]
      (rY xY t + l z)^3 Series[(cYn1 xY t + n1 z)^-1, {xY, 0, 2}]
      Series[(cY2 xY^2 t^2 + cY1 xY t z + z^2)^-1, {xY, 0, 2}], t, 2], l]] /. xY^2 → 1
```

$$\left\{0, \frac{3\left(2 + n1^2\right) rY^2}{n1}, \frac{3\left(-2\, cYn1 + cYn1\, n1^2 - cY1\, n1^3\right) rY}{n1^2}, \frac{2\, cYn1^2 + cY1^2\, n1^2 - 4\, cY2\, n1^2 - cY1\, cYn1\, n1^3 + cY1^2\, n1^4 - cY2\, n1^4 + n1^2\, pY1}{n1^3}\right\}$$

```
μs1x3ZA2i =
 Factor[CoefficientList[
    Coefficient[Expand[(cZn1 xZ t + n1 z)^2 + (cZn12 xZ t + n1 n2 z)^2 + (cZ1 xZ t + z)^2 + z^2]
      (rZ xZ t + (aZ + l) z)^3 Series[(cZn1 t xZ + n1 z)^-1, {xZ, 0, 1}]
      Series[(cZn12 xZ t + n1 n2 z)^-1, {xZ, 0, 1}] Series[(cZ1 xZ t + z)^-1, {xZ, 0, 1}] z^-1,
     t, 1], l]] /. xZ → 1
```

$$\left\{\frac{1}{n1^3\, n2^2} aZ^2 \left(-2\, aZ\, cZn12 - aZ\, cZn12\, n1^2 - 2\, aZ\, cZn1\, n2 + aZ\, cZn1\, n1^2\, n2 - aZ\, cZ1\, n1^3\, n2 + aZ\, cZn12\, n1^2\, n2^2 - aZ\, cZn1\, n1^2\, n2^3 - aZ\, cZ1\, n1^3\, n2^3 + 6\, n1\, n2\, rZ + 3\, n1^3\, n2\, rZ + 3\, n1^3\, n2^3\, rZ\right),\right.$$
$$\frac{1}{n1^3\, n2^2} 3\, aZ \left(-2\, aZ\, cZn12 - aZ\, cZn12\, n1^2 - 2\, aZ\, cZn1\, n2 + aZ\, cZn1\, n1^2\, n2 - aZ\, cZ1\, n1^3\, n2 + aZ\, cZn12\, n1^2\, n2^2 - aZ\, cZn1\, n1^2\, n2^3 - aZ\, cZ1\, n1^3\, n2^3 + 4\, n1\, n2\, rZ + 2\, n1^3\, n2\, rZ + 2\, n1^3\, n2^3\, rZ\right),$$
$$\frac{1}{n1^3\, n2^2} 3 \left(-2\, aZ\, cZn12 - aZ\, cZn12\, n1^2 - 2\, aZ\, cZn1\, n2 + aZ\, cZn1\, n1^2\, n2 - aZ\, cZ1\, n1^3\, n2 + aZ\, cZn12\, n1^2\, n2^2 - aZ\, cZn1\, n1^2\, n2^3 - aZ\, cZ1\, n1^3\, n2^3 + 2\, n1\, n2\, rZ + n1^3\, n2\, rZ + n1^3\, n2^3\, rZ\right),$$
$$\left.\frac{1}{n1^3\, n2^2} \left(-2\, cZn12 - cZn12\, n1^2 - 2\, cZn1\, n2 + cZn1\, n1^2\, n2 - cZ1\, n1^3\, n2 + cZn12\, n1^2\, n2^2 - cZn1\, n1^2\, n2^3 - cZ1\, n1^3\, n2^3\right)\right\}$$

```
μs1x3PA2i =
 Factor[CoefficientList[((n1 z)^2 + (n1 n2 z)^2 + (n1 n2 z)^2 + z^2 + z^2) Expand[-((aP + l) z)^3]
    (n1 z)^-1 (n1 n2 z)^-2 (z)^-2, l]]
```

$$\left\{-\frac{aP^3\,(2+n1^2+2\,n1^2\,n2^2)}{n1^3\,n2^2},\ -\frac{3\,aP^2\,(2+n1^2+2\,n1^2\,n2^2)}{n1^3\,n2^2},\ -\frac{3\,aP\,(2+n1^2+2\,n1^2\,n2^2)}{n1^3\,n2^2},\ -\frac{2+n1^2+2\,n1^2\,n2^2}{n1^3\,n2^2}\right\}$$

```
μx5YA2i + μx5ZA2i + μx5PA2i
```

$$\left\{-\frac{aP^5}{n1^3\,n2^2}-\frac{aZ^5\,cZn12}{n1^3\,n2^2}-\frac{aZ^5\,cZn1}{n1^3\,n2}-\frac{aZ^5\,cZ1}{n1^2\,n2}+\frac{5\,aZ^4\,rZ}{n1^2\,n2},\right.$$
$$-\frac{5\,aP^4}{n1^3\,n2^2}-\frac{5\,aZ^4\,cZn12}{n1^3\,n2^2}-\frac{5\,aZ^4\,cZn1}{n1^3\,n2}-\frac{5\,aZ^4\,cZ1}{n1^2\,n2}+\frac{20\,aZ^3\,rZ}{n1^2\,n2},$$
$$-\frac{10\,aP^3}{n1^3\,n2^2}-\frac{10\,aZ^3\,cZn12}{n1^3\,n2^2}-\frac{10\,aZ^3\,cZn1}{n1^3\,n2}-\frac{10\,aZ^3\,cZ1}{n1^2\,n2}+\frac{30\,aZ^2\,rZ}{n1^2\,n2},$$
$$-\frac{10\,aP^2}{n1^3\,n2^2}-\frac{10\,aZ^2\,cZn12}{n1^3\,n2^2}-\frac{10\,aZ^2\,cZn1}{n1^3\,n2}-\frac{10\,aZ^2\,cZ1}{n1^2\,n2}+\frac{10\,rY^2}{n1}+\frac{20\,aZ\,rZ}{n1^2\,n2},$$
$$-\frac{5\,aP}{n1^3\,n2^2}-\frac{5\,aZ\,cZn12}{n1^3\,n2^2}-\frac{5\,aZ\,cZn1}{n1^3\,n2}-\frac{5\,aZ\,cZ1}{n1^2\,n2}-\frac{5\,(cYn1+cY1\,n1)\,rY}{n1^2}+\frac{5\,rZ}{n1^2\,n2},$$
$$\left.\frac{cYn1^2+cY1\,cYn1\,n1+cY1^2\,n1^2-cY2\,n1^2}{n1^3}-\frac{1}{n1^3\,n2^2}-\frac{cZn12}{n1^3\,n2^2}-\frac{cZn1}{n1^3\,n2}-\frac{cZ1}{n1^2\,n2}\right\}$$

```
μs1x3YA2i + μs1x3ZA2i + μs1x3PA2i
```

$$\left\{-\frac{aP^3\,(2+n1^2+2\,n1^2\,n2^2)}{n1^3\,n2^2}+\frac{1}{n1^3\,n2^2}\right.$$
$$aZ^2\,(-2\,aZ\,cZn12-aZ\,cZn12\,n1^2-2\,aZ\,cZn1\,n2+aZ\,cZn1\,n1^2\,n2-aZ\,cZ1\,n1^3\,n2+$$
$$aZ\,cZn12\,n1^2\,n2^2-aZ\,cZn1\,n1^2\,n2^3-aZ\,cZ1\,n1^3\,n2^3+6\,n1\,n2\,rZ+3\,n1^3\,n2\,rZ+3\,n1^3\,n2^3\,rZ),$$
$$-\frac{3\,aP^2\,(2+n1^2+2\,n1^2\,n2^2)}{n1^3\,n2^2}+\frac{3\,(2+n1^2)\,rY^2}{n1}+\frac{1}{n1^3\,n2^2}$$
$$3\,aZ\,(-2\,aZ\,cZn12-aZ\,cZn12\,n1^2-2\,aZ\,cZn1\,n2+aZ\,cZn1\,n1^2\,n2-aZ\,cZ1\,n1^3\,n2+$$
$$aZ\,cZn12\,n1^2\,n2^2-aZ\,cZn1\,n1^2\,n2^3-aZ\,cZ1\,n1^3\,n2^3+4\,n1\,n2\,rZ+2\,n1^3\,n2\,rZ+2\,n1^3\,n2^3\,rZ),$$
$$-\frac{3\,aP\,(2+n1^2+2\,n1^2\,n2^2)}{n1^3\,n2^2}+\frac{3\,(-2\,cYn1+cYn1\,n1^2-cY1\,n1^3)\,rY}{n1^2}+\frac{1}{n1^3\,n2^2}$$
$$3\,(-2\,aZ\,cZn12-aZ\,cZn12\,n1^2-2\,aZ\,cZn1\,n2+aZ\,cZn1\,n1^2\,n2-aZ\,cZ1\,n1^3\,n2+$$
$$aZ\,cZn12\,n1^2\,n2^2-aZ\,cZn1\,n1^2\,n2^3-aZ\,cZ1\,n1^3\,n2^3+2\,n1\,n2\,rZ+n1^3\,n2\,rZ+n1^3\,n2^3\,rZ),$$
$$-\frac{2+n1^2+2\,n1^2\,n2^2}{n1^3\,n2^2}+\frac{1}{n1^3\,n2^2}(-2\,cZn12-cZn12\,n1^2-2\,cZn1\,n2+cZn1\,n1^2\,n2-$$
$$cZ1\,n1^3\,n2+cZn12\,n1^2\,n2^2-cZn1\,n1^2\,n2^3-cZ1\,n1^3\,n2^3)+$$
$$\left.\frac{2\,cYn1^2+cY1^2\,n1^2-4\,cY2\,n1^2-cY1\,cYn1\,n1^3+cY1^2\,n1^4-cY2\,n1^4+n1^2\,pY1}{n1^3}\right\}$$

```
Factor[Solve[{
  (*******************************
   The following equations are given by the above computations of x^5 locally
   *******************************)
```

$$d == -\frac{aP^5}{n1^3\ n2^2} - \frac{aZ^5\ cZn12}{n1^3\ n2^2} - \frac{aZ^5\ cZn1}{n1^3\ n2} - \frac{aZ^5\ cZ1}{n1^2\ n2} + \frac{5\ aZ^4\ rZ}{n1^2\ n2},$$

$$0 == -\frac{5\ aP^4}{n1^3\ n2^2} - \frac{5\ aZ^4\ cZn12}{n1^3\ n2^2} - \frac{5\ aZ^4\ cZn1}{n1^3\ n2} - \frac{5\ aZ^4\ cZ1}{n1^2\ n2} + \frac{20\ aZ^3\ rZ}{n1^2\ n2},$$

$$0 == -\frac{10\ aP^3}{n1^3\ n2^2} - \frac{10\ aZ^3\ cZn12}{n1^3\ n2^2} - \frac{10\ aZ^3\ cZn1}{n1^3\ n2} - \frac{10\ aZ^3\ cZ1}{n1^2\ n2} + \frac{30\ aZ^2\ rZ}{n1^2\ n2}$$

```
  },
  {d, cZ1, rZ}]]
```

$$\left\{\left\{d \to -\frac{aP^3\ (aP - aZ)^2}{n1^3\ n2^2},\ cZ1 \to -\frac{-3\ aP^4 + 4\ aP^3\ aZ + aZ^4\ cZn12 + aZ^4\ cZn1\ n2}{aZ^4\ n1\ n2},\ rZ \to -\frac{aP^3\ (-aP + aZ)}{aZ^3\ n1\ n2}\right\}\right\}$$

```
Solve[{
  (*******************************
   The following equations are given by Lemma 6.9
   *******************************)
```

$$0 == -\frac{3\ aP^2}{n1^3\ n2^2} + \frac{3\ rY^2}{n1} - \frac{3\ aZ\ eZN\ (aZ\ cZn12 + aZ\ cZn1\ n2 - 2\ n1\ n2\ rZ)}{n1^3\ n2^2},$$

$$0 == -\frac{3\ aP}{n1^3\ n2^2} - \frac{3\ cYn1\ rY}{n1^2} - \frac{3\ eZN\ (aZ\ cZn12 + aZ\ cZn1\ n2 - n1\ n2\ rZ)}{n1^3\ n2^2},$$

$$0 == \frac{cYn1^2}{n1^3} - \frac{1}{n1^3\ n2^2} - \frac{eZN\ (cZn12 + cZn1\ n2)}{n1^3\ n2^2},$$

$$0 == -\frac{1 + cZn12\ eZN - cZn1\ eZN\ n2 + 2\ n2^2 - cZn12\ eZN\ n2^2 + cZn1\ eZN\ n2^3 - n2^2\ pY1}{n1\ n2^2},$$

$$pY1 == 3,$$

```
  (*******************************
   The following equations are given by Lemma 6.7
   *******************************)
```

$$0 == \frac{2\ aP\ ePM}{n1^2\ n2^2} - \frac{2\ eZM\ (aZ\ cZn12 - n1\ n2\ rZ)}{n1^2\ n2^2},$$

$$0 == \frac{ePM}{n1^2\ n2^2} - \frac{cZn12\ eZM}{n1^2\ n2^2}$$

```
  },
  {rY, rZ, pY1, rY, cYn1, cZn12, cZn1, eZM}]
```

$$\left\{\left\{rY \to \frac{aP - aZ}{n1\ n2},\ rZ \to \frac{aP - aZ}{eZN\ n1\ n2},\ pY1 \to 3,\ cYn1 \to 0,\ cZn12 \to -\frac{1}{eZN},\ cZn1 \to 0,\ eZM \to -ePM\ eZN\right\},\right.$$

$$\left\{rY \to \frac{-aP + aZ}{n1\ n2},\ rZ \to \frac{aP - aZ}{eZN\ n1\ n2},\ pY1 \to 3,\ cYn1 \to 0,\ cZn12 \to -\frac{1}{eZN},\ cZn1 \to 0,\ eZM \to -ePM\ eZN\right\},$$

$$\left\{rY \to -\frac{aP + aZ}{n1\ n2},\ rZ \to \frac{-aP + aZ}{eZN\ n1\ n2},\ pY1 \to 3,\ cYn1 \to \frac{2}{n2},\ cZn12 \to \frac{1}{eZN},\ cZn1 \to \frac{2}{eZN\ n2},\ eZM \to ePM\ eZN\right\},\ \left\{rY \to \frac{aP + aZ}{n1\ n2},\ rZ \to \frac{-aP + aZ}{eZN\ n1\ n2},\right.$$

$$\left.pY1 \to 3,\ cYn1 \to -\frac{2}{n2},\ cZn12 \to \frac{1}{eZN},\ cZn1 \to \frac{2}{eZN\ n2},\ eZM \to ePM\ eZN\right\}\right\}$$

```
(******************************
 It is now interesting to compare the values of rZ
  ******************************)
```

Factor[Solve[{

$$-\frac{aP - aZ}{n1\ n2} == -\frac{aP^3\ (-aP + aZ)}{aZ^3\ n1\ n2}$$

}, {aP}]]

$$\left\{\{aP \to -aZ\},\ \{aP \to aZ\},\ \left\{aP \to \frac{1}{2}\left(1 - i\sqrt{3}\right) aZ\right\},\ \left\{aP \to \frac{1}{2}\left(1 + i\sqrt{3}\right) aZ\right\}\right\}$$

```
(******************************
 Since aP is not equal to aZ and needs to be an integer, we have that aP = -aZ
   ******************************)
```

```
Factor[Solve[{
  (*******************************
   The following equations are give by the above computations
   *******************************)
  aP == -aZ, d == -(aP^3 (aP - aZ)^2)/(n1^3 n2^2), cZ1 == -(-3 aP^4 + 4 aP^3 aZ + aZ^4 cZn12 + aZ^4 cZn1 n2)/(aZ^4 n1 n2),
  rZ == -(aP^3 (-aP + aZ))/(aZ^3 n1 n2),
  (*******************************
   The following equations are given by the above computations of p1(X)x^3 locally
   *******************************)
  p d == -(aP^3 (2 + n1^2 + 2 n1^2 n2^2))/(n1^3 n2^2) +
    1/(n1^3 n2^2)
    aZ^2 (-2 aZ cZn12 - aZ cZn12 n1^2 - 2 aZ cZn1 n2 + aZ cZn1 n1^2 n2 - aZ cZ1 n1^3 n2 +
        aZ cZn12 n1^2 n2^2 - aZ cZn1 n1^2 n2^3 - aZ cZ1 n1^3 n2^3 + 6 n1 n2 rZ + 3 n1^3 n2 rZ +
        3 n1^3 n2^3 rZ),
  (*******************************
   The following equations are given by Lemma 6.9
   *******************************)
  0 == -(3 aP^2)/(n1^3 n2^2) + (3 rY^2)/n1 - (3 aZ eZN (aZ cZn12 + aZ cZn1 n2 - 2 n1 n2 rZ))/(n1^3 n2^2),
  0 == -(3 aP)/(n1^3 n2^2) - (3 cYn1 rY)/n1^2 - (3 eZN (aZ cZn12 + aZ cZn1 n2 - n1 n2 rZ))/(n1^3 n2^2),
  0 == cYn1^2/n1^3 - 1/(n1^3 n2^2) - (eZN (cZn12 + cZn1 n2))/(n1^3 n2^2),
  0 == -(1 + cZn12 eZN - cZn1 eZN n2 + 2 n2^2 - cZn12 eZN n2^2 + cZn1 eZN n2^3 - n2^2 pY1)/(n1 n2^2),
  pY1 == 3, eZN^2 == 1,
  (*******************************
   The following equations are given by Lemma 6.7
   *******************************)
  0 == (2 aP ePM)/(n1^2 n2^2) - (2 eZM (aZ cZn12 - n1 n2 rZ))/(n1^2 n2^2),
  0 == ePM/(n1^2 n2^2) - (cZn12 eZM)/(n1^2 n2^2),
  eZM^2 == 1, ePM^2 == 1
 },
 {p, d, cZ1, cZn12, cZn1, cYn1, rZ, rY, aP, eZM, ePM, eZN, pY1}]]
```

$$\Big\{\Big\{p \to \frac{3\,(4+n1^2\,n2^2)}{4\,aZ^2},\ d \to \frac{4\,aZ^5}{n1^3\,n2^2},\ cZ1 \to \frac{6}{n1\,n2},\ cZn12 \to 1,\ cZn1 \to 0,\ cYn1 \to 0,\ rZ \to \frac{2\,aZ}{n1\,n2},\ rY \to -\frac{2\,aZ}{n1\,n2},\ aP \to -aZ,\ eZM \to -1,\ ePM \to -1,\ eZN \to -1,\ pY1 \to 3\Big\},$$

$$\Big\{p \to \frac{3\,(4+n1^2\,n2^2)}{4\,aZ^2},\ d \to \frac{4\,aZ^5}{n1^3\,n2^2},\ cZ1 \to \frac{6}{n1\,n2},\ cZn12 \to 1,\ cZn1 \to 0,\ cYn1 \to 0,\ rZ \to \frac{2\,aZ}{n1\,n2},\ rY \to -\frac{2\,aZ}{n1\,n2},\ aP \to -aZ,\ eZM \to 1,\ ePM \to 1,\ eZN \to -1,\ pY1 \to 3\Big\},$$

$$\Big\{p \to \frac{3\,(4+n1^2\,n2^2)}{4\,aZ^2},\ d \to \frac{4\,aZ^5}{n1^3\,n2^2},\ cZ1 \to \frac{6}{n1\,n2},\ cZn12 \to 1,\ cZn1 \to 0,\ cYn1 \to 0,\ rZ \to \frac{2\,aZ}{n1\,n2},\ rY \to \frac{2\,aZ}{n1\,n2},\ aP \to -aZ,\ eZM \to -1,\ ePM \to -1,\ eZN \to -1,\ pY1 \to 3\Big\},$$

$$\Big\{p \to \frac{3\,(4+n1^2\,n2^2)}{4\,aZ^2},\ d \to \frac{4\,aZ^5}{n1^3\,n2^2},\ cZ1 \to \frac{6}{n1\,n2},\ cZn12 \to 1,\ cZn1 \to 0,\ cYn1 \to 0,\ rZ \to \frac{2\,aZ}{n1\,n2},\ rY \to \frac{2\,aZ}{n1\,n2},\ aP \to -aZ,\ eZM \to 1,\ ePM \to 1,\ eZN \to -1,\ pY1 \to 3\Big\},$$

$$\Big\{p \to \frac{8+4\,n1^2+3\,n1^2\,n2^2}{4\,aZ^2},\ d \to \frac{4\,aZ^5}{n1^3\,n2^2},\ cZ1 \to \frac{4}{n1\,n2},\ cZn12 \to 1,\ cZn1 \to \frac{2}{n2},\ cYn1 \to -\frac{2}{n2},\ rZ \to \frac{2\,aZ}{n1\,n2},\ rY \to 0,\ aP \to -aZ,\ eZM \to -1,\ ePM \to -1,\ eZN \to 1,\ pY1 \to 3\Big\},$$

$$\Big\{p \to \frac{8+4\,n1^2+3\,n1^2\,n2^2}{4\,aZ^2},\ d \to \frac{4\,aZ^5}{n1^3\,n2^2},\ cZ1 \to \frac{4}{n1\,n2},\ cZn12 \to 1,\ cZn1 \to \frac{2}{n2},\ cYn1 \to -\frac{2}{n2},\ rZ \to \frac{2\,aZ}{n1\,n2},\ rY \to 0,\ aP \to -aZ,\ eZM \to 1,\ ePM \to 1,\ eZN \to 1,\ pY1 \to 3\Big\},$$

$$\Big\{p \to \frac{8+4\,n1^2+3\,n1^2\,n2^2}{4\,aZ^2},\ d \to \frac{4\,aZ^5}{n1^3\,n2^2},\ cZ1 \to \frac{4}{n1\,n2},\ cZn12 \to 1,\ cZn1 \to \frac{2}{n2},\ cYn1 \to \frac{2}{n2},\ rZ \to \frac{2\,aZ}{n1\,n2},\ rY \to 0,\ aP \to -aZ,\ eZM \to -1,\ ePM \to -1,\ eZN \to 1,\ pY1 \to 3\Big\},$$

$$\Big\{p \to \frac{8+4\,n1^2+3\,n1^2\,n2^2}{4\,aZ^2},\ d \to \frac{4\,aZ^5}{n1^3\,n2^2},\ cZ1 \to \frac{4}{n1\,n2},\ cZn12 \to 1,\ cZn1 \to \frac{2}{n2},\ cYn1 \to \frac{2}{n2},\ rZ \to \frac{2\,aZ}{n1\,n2},\ rY \to 0,\ aP \to -aZ,\ eZM \to 1,\ ePM \to 1,\ eZN \to 1,\ pY1 \to 3\Big\}\Big\}$$

Computations for Lemma 6.10.
The case where there exists N_i^6 s.t. Fix(N_i) = Y^4 U Z^2 U P^0 and M^4 in N_1^6 and N_2^6 s.t. Fix(M) = Z^2 U P^0

Case A.3

```
(*****************************
  NOTATIONS
  rY xY denotes the restriction of x to Y
  rZ xZ denotes the restriction of x to Z
  aZ denotes the Hopf weight at Z
  aP denotes the Hopf weight at P
  cZn123 xZ denotes the first Chern class of the normal bundle of Z in
  the direction of M
  cZn1 xZ denotes the first Chern class of the normal bundle of Z in the
  remaining direction in N1
  cZn2 xZ denotes the first Chern class of the normal bundle of Z in the
  remaining direction in N2
  cZ1 xZ denotes the first Chern class of the normal bundle of Z in the
  remaining directions
  cYn1 xY denotes the first Chern class of the normal bundle of Y in the
  direction of N1
  cYn2 xY denotes the first Chern class of the normal bundle of Y in the
  direction of N2
  cY1 xY denotes the first Chern class of the normal bundle of Y in the
  remaining directions
  μx5YA3 denotes the local datum of x^5 at Y
  μx5ZA3 denotes the local datum of x^5 at Z
  μx5PA3 denotes the local datum of x^5 at P

  We use the variable t to extract the coefficient of order 2 of xY and
  of order 1 of xZ. We then evaluate xY^2 and xZ to 1.
  The orientation of the point P is -1, this explains the negative sign.
  Once evaluated, we extract all the coefficients in l into a list.
  *****************************)
```

```
μx5YA3 =
 Factor[CoefficientList[
    Coefficient[Expand[(rY xY t + l z)^5] Series[(cYn1 xY t + n1 z)^-1, {xY, 0, 2}]
      Series[(cYn2 xY t + n2 z)^-1, {xY, 0, 2}] Series[(cY1 xY t + z)^-1, {xY, 0, 2}], t, 2],
    l]] /. xY^2 → 1
```

$$\left\{0,\ 0,\ 0,\ \frac{10\,\text{rY}^2}{\text{n1}\,\text{n2}},\ -\frac{5\,(\text{cYn2}\,\text{n1}+\text{cYn1}\,\text{n2}+\text{cY1}\,\text{n1}\,\text{n2})\,\text{rY}}{\text{n1}^2\,\text{n2}^2},\ \frac{\text{cYn2}^2\,\text{n1}^2+\text{cYn1}\,\text{cYn2}\,\text{n1}\,\text{n2}+\text{cY1}\,\text{cYn2}\,\text{n1}^2\,\text{n2}+\text{cYn1}^2\,\text{n2}^2+\text{cY1}\,\text{cYn1}\,\text{n1}\,\text{n2}^2+\text{cY1}^2\,\text{n1}^2\,\text{n2}^2}{\text{n1}^3\,\text{n2}^3}\right\}$$

```
μx5ZA3 =
 CoefficientList[
   Coefficient[Expand[(rZ xZ t + (aZ + 1) z)^5] Series[(cZn1 t xZ + n1 z)^-1, {xZ, 0, 1}]
     Series[(cZn123 xZ t + n1 n2 n3 z)^-1, {xZ, 0, 1}]
     Series[(cZn2 t xZ + n2 z)^-1, {xZ, 0, 1}] Series[(cZ1 xZ t + z)^-1, {xZ, 0, 1}], t, 1],
   1] /. xZ → 1
```

$$\left\{-\frac{aZ^5\,cZn123}{n1^3\,n2^3\,n3^2}-\frac{aZ^5\,cZn2}{n1^2\,n2^3\,n3}-\frac{aZ^5\,cZn1}{n1^3\,n2^2\,n3}-\frac{aZ^5\,cZ1}{n1^2\,n2^2\,n3}+\frac{5\,aZ^4\,rZ}{n1^2\,n2^2\,n3},\right.$$
$$-\frac{5\,aZ^4\,cZn123}{n1^3\,n2^3\,n3^2}-\frac{5\,aZ^4\,cZn2}{n1^2\,n2^3\,n3}-\frac{5\,aZ^4\,cZn1}{n1^3\,n2^2\,n3}-\frac{5\,aZ^4\,cZ1}{n1^2\,n2^2\,n3}+\frac{20\,aZ^3\,rZ}{n1^2\,n2^2\,n3},$$
$$-\frac{10\,aZ^3\,cZn123}{n1^3\,n2^3\,n3^2}-\frac{10\,aZ^3\,cZn2}{n1^2\,n2^3\,n3}-\frac{10\,aZ^3\,cZn1}{n1^3\,n2^2\,n3}-\frac{10\,aZ^3\,cZ1}{n1^2\,n2^2\,n3}+\frac{30\,aZ^2\,rZ}{n1^2\,n2^2\,n3},$$
$$-\frac{10\,aZ^2\,cZn123}{n1^3\,n2^3\,n3^2}-\frac{10\,aZ^2\,cZn2}{n1^2\,n2^3\,n3}-\frac{10\,aZ^2\,cZn1}{n1^3\,n2^2\,n3}-\frac{10\,aZ^2\,cZ1}{n1^2\,n2^2\,n3}+\frac{20\,aZ\,rZ}{n1^2\,n2^2\,n3},$$
$$-\frac{5\,aZ\,cZn123}{n1^3\,n2^3\,n3^2}-\frac{5\,aZ\,cZn2}{n1^2\,n2^3\,n3}-\frac{5\,aZ\,cZn1}{n1^3\,n2^2\,n3}-\frac{5\,aZ\,cZ1}{n1^2\,n2^2\,n3}+\frac{5\,rZ}{n1^2\,n2^2\,n3},$$
$$\left.-\frac{cZn123}{n1^3\,n2^3\,n3^2}-\frac{cZn2}{n1^2\,n2^3\,n3}-\frac{cZn1}{n1^3\,n2^2\,n3}-\frac{cZ1}{n1^2\,n2^2\,n3}\right\}$$

```
μx5PA3 = CoefficientList[Expand[-((aP + 1) z)^5] (n1 n2 n3 z)^-2 (n1 z)^-1 (n2 z)^-1 z^-1, 1]
```

$$\left\{-\frac{aP^5}{n1^3\,n2^3\,n3^2},\,-\frac{5\,aP^4}{n1^3\,n2^3\,n3^2},\,-\frac{10\,aP^3}{n1^3\,n2^3\,n3^2},\,-\frac{10\,aP^2}{n1^3\,n2^3\,n3^2},\,-\frac{5\,aP}{n1^3\,n2^3\,n3^2},\,-\frac{1}{n1^3\,n2^3\,n3^2}\right\}$$

```
(*****************************
 NOTATIONS
 μs1x3YA3 denotes the local datum of p1(X)x^3 at Y
 μs1x3ZA3 denotes the local datum of p1(X)x^3 at Z
 μs1x3PA3 denotes the local datum of p1(X)x^3 at P
  *****************************)
```

```
μs1x3YA3 =
 Factor[CoefficientList[
    Coefficient[Expand[pY1 t^2 + (cYn1 xY t + n1 z)^2 + (cYn2 xY t + n2 z)^2 + (cY1 xY t + z)^2]
       (rY xY t + 1 z)^3 Series[(cYn1 xY t + n1 z)^-1, {xY, 0, 2}]
       Series[(cYn2 xY t + n2 z)^-1, {xY, 0, 2}] Series[(cY1 xY t + z)^-1, {xY, 0, 2}], t, 2],
    1]] /. xY^2 → 1
```

$$\left\{0,\,\frac{3\left(1+n1^2+n2^2\right)rY^2}{n1\,n2},\right.$$
$$-\frac{1}{n1^2\,n2^2}3\left(cYn2\,n1+cYn2\,n1^3+cYn1\,n2-cY1\,n1\,n2-cYn1\,n1^2\,n2+cY1\,n1^3\,n2-\right.$$
$$\left.cYn2\,n1\,n2^2+cYn1\,n2^3+cY1\,n1\,n2^3\right)rY,$$
$$\frac{1}{n1^3\,n2^3}\left(cYn2^2\,n1^2+cYn2^2\,n1^4+cYn1\,cYn2\,n1\,n2-cY1\,cYn2\,n1^2\,n2-cYn1\,cYn2\,n1^3\,n2+\right.$$
$$cY1\,cYn2\,n1^4\,n2+cYn1^2\,n2^2-cY1\,cYn1\,n1\,n2^2-cY1\,cYn1\,n1^3\,n2^2+cY1^2\,n1^4\,n2^2-$$
$$\left.\left.cYn1\,cYn2\,n1\,n2^3-cY1\,cYn2\,n1^2\,n2^3+cYn1^2\,n2^4+cY1\,cYn1\,n1\,n2^4+cY1^2\,n1^2\,n2^4+n1^2\,n2^2\,pY1\right)\right\}$$

```
μs1x3ZA3 =
 Factor[CoefficientList[
   Coefficient[
    Expand[(cZn1 xZ t + n1 z)^2 + (cZn123 xZ t + n1 n2 n3 z)^2 + (cZn2 xZ t + n2 z)^2 +
       (cZ1 xZ t + z)^2] (rZ xZ t + (aZ + 1) z)^3 Series[(cZn1 t xZ + n1 z)^-1, {xZ, 0, 1}]
     Series[(cZn123 xZ t + n1 n2 n3 z)^-1, {xZ, 0, 1}]
     Series[(cZn2 t xZ + n2 z)^-1, {xZ, 0, 1}] Series[(cZ1 xZ t + z)^-1, {xZ, 0, 1}], t, 1],
   1]] /. xZ → 1
```

$$\Big\{\frac{1}{n1^3\, n2^3\, n3^2}\, aZ^2 \big(-aZ\, cZn123 - aZ\, cZn123\, n1^2 - aZ\, cZn123\, n2^2 - aZ\, cZn2\, n1\, n3 - aZ\, cZn2\, n1^3\, n3 - aZ\, cZn1\, n2\, n3 + aZ\, cZ1\, n1\, n2\, n3 + aZ\, cZn1\, n1^2\, n2\, n3 - aZ\, cZ1\, n1^3\, n2\, n3 + aZ\, cZn2\, n1\, n2^2\, n3 - aZ\, cZn1\, n2^3\, n3 - aZ\, cZ1\, n1\, n2^3\, n3 + aZ\, cZn123\, n1^2\, n2^2\, n3^2 - aZ\, cZn2\, n1^3\, n2^2\, n3^3 - aZ\, cZn1\, n1^2\, n2^3\, n3^3 - aZ\, cZ1\, n1^3\, n2^3\, n3^3 + 3\, n1\, n2\, n3\, rZ + 3\, n1^3\, n2\, n3\, rZ + 3\, n1\, n2^3\, n3\, rZ + 3\, n1^3\, n2^3\, n3^3\, rZ\big),$$

$$\frac{1}{n1^3\, n2^3\, n3^2}\, 3\, aZ \big(-aZ\, cZn123 - aZ\, cZn123\, n1^2 - aZ\, cZn123\, n2^2 - aZ\, cZn2\, n1\, n3 - aZ\, cZn2\, n1^3\, n3 - aZ\, cZn1\, n2\, n3 + aZ\, cZ1\, n1\, n2\, n3 + aZ\, cZn1\, n1^2\, n2\, n3 - aZ\, cZ1\, n1^3\, n2\, n3 + aZ\, cZn2\, n1\, n2^2\, n3 - aZ\, cZn1\, n2^3\, n3 - aZ\, cZ1\, n1\, n2^3\, n3 + aZ\, cZn123\, n1^2\, n2^2\, n3^2 - aZ\, cZn2\, n1^3\, n2^2\, n3^3 - aZ\, cZn1\, n1^2\, n2^3\, n3^3 - aZ\, cZ1\, n1^3\, n2^3\, n3^3 + 2\, n1\, n2\, n3\, rZ + 2\, n1^3\, n2\, n3\, rZ + 2\, n1\, n2^3\, n3\, rZ + 2\, n1^3\, n2^3\, n3^3\, rZ\big),$$

$$\frac{1}{n1^3\, n2^3\, n3^2}\, 3 \big(-aZ\, cZn123 - aZ\, cZn123\, n1^2 - aZ\, cZn123\, n2^2 - aZ\, cZn2\, n1\, n3 - aZ\, cZn2\, n1^3\, n3 - aZ\, cZn1\, n2\, n3 + aZ\, cZ1\, n1\, n2\, n3 + aZ\, cZn1\, n1^2\, n2\, n3 - aZ\, cZ1\, n1^3\, n2\, n3 + aZ\, cZn2\, n1\, n2^2\, n3 - aZ\, cZn1\, n2^3\, n3 - aZ\, cZ1\, n1\, n2^3\, n3 + aZ\, cZn123\, n1^2\, n2^2\, n3^2 - aZ\, cZn2\, n1^3\, n2^2\, n3^3 - aZ\, cZn1\, n1^2\, n2^3\, n3^3 - aZ\, cZ1\, n1^3\, n2^3\, n3^3 + n1\, n2\, n3\, rZ + n1^3\, n2\, n3\, rZ + n1\, n2^3\, n3\, rZ + n1^3\, n2^3\, n3^3\, rZ\big),$$

$$\frac{1}{n1^3\, n2^3\, n3^2} \big(-cZn123 - cZn123\, n1^2 - cZn123\, n2^2 - cZn2\, n1\, n3 - cZn2\, n1^3\, n3 - cZn1\, n2\, n3 + cZ1\, n1\, n2\, n3 + cZn1\, n1^2\, n2\, n3 - cZ1\, n1^3\, n2\, n3 + cZn2\, n1\, n2^2\, n3 - cZn1\, n2^3\, n3 - cZ1\, n1\, n2^3\, n3 + cZn123\, n1^2\, n2^2\, n3^2 - cZn2\, n1^3\, n2^2\, n3^3 - cZn1\, n1^2\, n2^3\, n3^3 - cZ1\, n1^3\, n2^3\, n3^3\big)\Big\}$$

```
μs1x3PA3 =
 Factor[CoefficientList[((n1 n2 n3 z)^2 + (n1 n2 n3 z)^2 + (n1 z)^2 + (n2 z)^2 + z^2)
   Expand[-((aP + 1) z)^3] (n1 n2 n3 z)^-2 (n1 z)^-1 (n2 z)^-1 z^-1, 1]]
```

$$\Big\{-\frac{aP^3\,(1 + n1^2 + n2^2 + 2\, n1^2\, n2^2\, n3^2)}{n1^3\, n2^3\, n3^2},\ -\frac{3\, aP^2\,(1 + n1^2 + n2^2 + 2\, n1^2\, n2^2\, n3^2)}{n1^3\, n2^3\, n3^2},\ -\frac{3\, aP\,(1 + n1^2 + n2^2 + 2\, n1^2\, n2^2\, n3^2)}{n1^3\, n2^3\, n3^2},\ -\frac{1 + n1^2 + n2^2 + 2\, n1^2\, n2^2\, n3^2}{n1^3\, n2^3\, n3^2}\Big\}$$

μx5YA3 + μx5ZA3 + μx5PA3

$$\left\{-\frac{aP^5}{n1^3\,n2^3\,n3^2}-\frac{aZ^5\,cZn123}{n1^3\,n2^3\,n3^2}-\frac{aZ^5\,cZn2}{n1^2\,n2^3\,n3}-\frac{aZ^5\,cZn1}{n1^3\,n2^2\,n3}-\frac{aZ^5\,cZ1}{n1^2\,n2^2\,n3}+\frac{5\,aZ^4\,rZ}{n1^2\,n2^2\,n3},\right.$$

$$-\frac{5\,aP^4}{n1^3\,n2^3\,n3^2}-\frac{5\,aZ^4\,cZn123}{n1^3\,n2^3\,n3^2}-\frac{5\,aZ^4\,cZn2}{n1^2\,n2^3\,n3}-\frac{5\,aZ^4\,cZn1}{n1^3\,n2^2\,n3}-\frac{5\,aZ^4\,cZ1}{n1^2\,n2^2\,n3}+\frac{20\,aZ^3\,rZ}{n1^2\,n2^2\,n3},$$

$$-\frac{10\,aP^3}{n1^3\,n2^3\,n3^2}-\frac{10\,aZ^3\,cZn123}{n1^3\,n2^3\,n3^2}-\frac{10\,aZ^3\,cZn2}{n1^2\,n2^3\,n3}-\frac{10\,aZ^3\,cZn1}{n1^3\,n2^2\,n3}-\frac{10\,aZ^3\,cZ1}{n1^2\,n2^2\,n3}+\frac{30\,aZ^2\,rZ}{n1^2\,n2^2\,n3},$$

$$-\frac{10\,aP^2}{n1^3\,n2^3\,n3^2}-\frac{10\,aZ^2\,cZn123}{n1^3\,n2^3\,n3^2}-\frac{10\,aZ^2\,cZn2}{n1^2\,n2^3\,n3}-\frac{10\,aZ^2\,cZn1}{n1^3\,n2^2\,n3}-\frac{10\,aZ^2\,cZ1}{n1^2\,n2^2\,n3}+\frac{10\,rY^2}{n1\,n2}+\frac{20\,aZ\,rZ}{n1^2\,n2^2\,n3},$$

$$-\frac{5\,aP}{n1^3\,n2^3\,n3^2}-\frac{5\,aZ\,cZn123}{n1^3\,n2^3\,n3^2}-\frac{5\,aZ\,cZn2}{n1^2\,n2^3\,n3}-\frac{5\,aZ\,cZn1}{n1^3\,n2^2\,n3}-$$

$$\frac{5\,aZ\,cZ1}{n1^2\,n2^2\,n3}-\frac{5\,(cYn2\,n1+cYn1\,n2+cY1\,n1\,n2)\,rY}{n1^2\,n2^2}+\frac{5\,rZ}{n1^2\,n2^2\,n3},$$

$$\frac{cYn2^2\,n1^2+cYn1\,cYn2\,n1\,n2+cY1\,cYn2\,n1^2\,n2+cYn1^2\,n2^2+cY1\,cYn1\,n1\,n2^2+cY1^2\,n1^2\,n2^2}{n1^3\,n2^3}-$$

$$\left.\frac{1}{n1^3\,n2^3\,n3^2}-\frac{cZn123}{n1^3\,n2^3\,n3^2}-\frac{cZn2}{n1^2\,n2^3\,n3}-\frac{cZn1}{n1^3\,n2^2\,n3}-\frac{cZ1}{n1^2\,n2^2\,n3}\right\}$$

μs1x3YA3 + μs1x3ZA3 + μs1x3PA3

$$
\begin{aligned}
\Big\{ & -\frac{aP^3\left(1+n1^2+n2^2+2\,n1^2\,n2^2\,n3^2\right)}{n1^3\,n2^3\,n3^2}+ \\
& \frac{1}{n1^3\,n2^3\,n3^2}\,aZ^2\,\big(-aZ\,cZn123-aZ\,cZn123\,n1^2-aZ\,cZn123\,n2^2-aZ\,cZn2\,n1\,n3- \\
& \quad aZ\,cZn2\,n1^3\,n3-aZ\,cZn1\,n2\,n3+aZ\,cZ1\,n1\,n2\,n3+aZ\,cZn1\,n1^2\,n2\,n3- \\
& \quad aZ\,cZ1\,n1^3\,n2\,n3+aZ\,cZn2\,n1\,n2^2\,n3-aZ\,cZn1\,n2^3\,n3-aZ\,cZ1\,n1\,n2^3\,n3+ \\
& \quad aZ\,cZn123\,n1^2\,n2^2\,n3^2-aZ\,cZn2\,n1^3\,n2^2\,n3^3-aZ\,cZn1\,n1^2\,n2^3\,n3^3-aZ\,cZ1\,n1^3\,n2^3\,n3^3+ \\
& \quad 3\,n1\,n2\,n3\,rZ+3\,n1^3\,n2\,n3\,rZ+3\,n1\,n2^3\,n3\,rZ+3\,n1^3\,n2^3\,n3^3\,rZ\big), \\
& -\frac{3\,aP^2\left(1+n1^2+n2^2+2\,n1^2\,n2^2\,n3^2\right)}{n1^3\,n2^3\,n3^2}+\frac{3\left(1+n1^2+n2^2\right)rY^2}{n1\,n2}+ \\
& \frac{1}{n1^3\,n2^3\,n3^2}\,3\,aZ \\
& \quad \big(-aZ\,cZn123-aZ\,cZn123\,n1^2-aZ\,cZn123\,n2^2-aZ\,cZn2\,n1\,n3-aZ\,cZn2\,n1^3\,n3-aZ\,cZn1\,n2\,n3+ \\
& \quad aZ\,cZ1\,n1\,n2\,n3+aZ\,cZn1\,n1^2\,n2\,n3-aZ\,cZ1\,n1^3\,n2\,n3+aZ\,cZn2\,n1\,n2^2\,n3-aZ\,cZn1\,n2^3\,n3- \\
& \quad aZ\,cZ1\,n1\,n2^3\,n3+aZ\,cZn123\,n1^2\,n2^2\,n3^2-aZ\,cZn2\,n1^3\,n2^2\,n3^3-aZ\,cZn1\,n1^2\,n2^3\,n3^3- \\
& \quad aZ\,cZ1\,n1^3\,n2^3\,n3^3+2\,n1\,n2\,n3\,rZ+2\,n1^3\,n2\,n3\,rZ+2\,n1\,n2^3\,n3\,rZ+2\,n1^3\,n2^3\,n3^3\,rZ\big), \\
& -\frac{3\,aP\left(1+n1^2+n2^2+2\,n1^2\,n2^2\,n3^2\right)}{n1^3\,n2^3\,n3^2}-\frac{1}{n1^2\,n2^2}3\,\big(cYn2\,n1+cYn2\,n1^3+cYn1\,n2-cY1\,n1\,n2- \\
& \quad cYn1\,n1^2\,n2+cY1\,n1^3\,n2-cYn2\,n1\,n2^2+cYn1\,n2^3+cY1\,n1\,n2^3\big)\,rY+\frac{1}{n1^3\,n2^3\,n3^2} \\
& \quad 3\,\big(-aZ\,cZn123-aZ\,cZn123\,n1^2-aZ\,cZn123\,n2^2-aZ\,cZn2\,n1\,n3-aZ\,cZn2\,n1^3\,n3-aZ\,cZn1\,n2\,n3+ \\
& \quad aZ\,cZ1\,n1\,n2\,n3+aZ\,cZn1\,n1^2\,n2\,n3-aZ\,cZ1\,n1^3\,n2\,n3+aZ\,cZn2\,n1\,n2^2\,n3-aZ\,cZn1\,n2^3\,n3- \\
& \quad aZ\,cZ1\,n1\,n2^3\,n3+aZ\,cZn123\,n1^2\,n2^2\,n3^2-aZ\,cZn2\,n1^3\,n2^2\,n3^3-aZ\,cZn1\,n1^2\,n2^3\,n3^3- \\
& \quad aZ\,cZ1\,n1^3\,n2^3\,n3^3+n1\,n2\,n3\,rZ+n1^3\,n2\,n3\,rZ+n1\,n2^3\,n3\,rZ+n1^3\,n2^3\,n3^3\,rZ\big), \\
& -\frac{1+n1^2+n2^2+2\,n1^2\,n2^2\,n3^2}{n1^3\,n2^3\,n3^2}+\frac{1}{n1^3\,n2^3\,n3^2}\,\big(-cZn123-cZn123\,n1^2-cZn123\,n2^2- \\
& \quad cZn2\,n1\,n3-cZn2\,n1^3\,n3-cZn1\,n2\,n3+cZ1\,n1\,n2\,n3+cZn1\,n1^2\,n2\,n3- \\
& \quad cZ1\,n1^3\,n2\,n3+cZn2\,n1\,n2^2\,n3-cZn1\,n2^3\,n3-cZ1\,n1\,n2^3\,n3+ \\
& \quad cZn123\,n1^2\,n2^2\,n3^2-cZn2\,n1^3\,n2^2\,n3^3-cZn1\,n1^2\,n2^3\,n3^3-cZ1\,n1^3\,n2^3\,n3^3\big)+ \\
& \frac{1}{n1^3\,n2^3}\,\big(cYn2^2\,n1^2+cYn2^2\,n1^4+cYn1\,cYn2\,n1\,n2-cY1\,cYn2\,n1^2\,n2-cYn1\,cYn2\,n1^3\,n2+ \\
& \quad cY1\,cYn2\,n1^4\,n2+cYn1^2\,n2^2-cY1\,cYn1\,n1\,n2^2-cY1\,cYn1\,n1^3\,n2^2+cY1^2\,n1^4\,n2^2-cYn1\,cYn2 \\
& \quad n1\,n2^3-cY1\,cYn2\,n1^2\,n2^3+cYn1^2\,n2^4+cY1\,cYn1\,n1\,n2^4+cY1^2\,n1^2\,n2^4+n1^2\,n2^2\,pY1\big)\Big\}
\end{aligned}
$$

```
Factor[Solve[{
    (*******************************
     The following equations are given by the above computations of x^5 locally
     *******************************)
```

$$d == -\frac{aP^5}{n1^3\,n2^3\,n3^2} - \frac{aZ^5\,cZn123}{n1^3\,n2^3\,n3^2} - \frac{aZ^5\,cZn2}{n1^2\,n2^3\,n3} - \frac{aZ^5\,cZn1}{n1^3\,n2^2\,n3} - \frac{aZ^5\,cZ1}{n1^2\,n2^2\,n3} + \frac{5\,aZ^4\,rZ}{n1^2\,n2^2\,n3},$$

$$0 == -\frac{5\,aP^4}{n1^3\,n2^3\,n3^2} - \frac{5\,aZ^4\,cZn123}{n1^3\,n2^3\,n3^2} - \frac{5\,aZ^4\,cZn2}{n1^2\,n2^3\,n3} - \frac{5\,aZ^4\,cZn1}{n1^3\,n2^2\,n3} - \frac{5\,aZ^4\,cZ1}{n1^2\,n2^2\,n3} + \frac{20\,aZ^3\,rZ}{n1^2\,n2^2\,n3},$$

$$0 == -\frac{10\,aP^3}{n1^3\,n2^3\,n3^2} - \frac{10\,aZ^3\,cZn123}{n1^3\,n2^3\,n3^2} - \frac{10\,aZ^3\,cZn2}{n1^2\,n2^3\,n3} - \frac{10\,aZ^3\,cZn1}{n1^3\,n2^2\,n3} - \frac{10\,aZ^3\,cZ1}{n1^2\,n2^2\,n3} + \frac{30\,aZ^2\,rZ}{n1^2\,n2^2\,n3}$$

```
   },
   {d, cZ1, rZ}]]
```

$$\left\{\left\{d \to -\frac{aP^3\,(aP - aZ)^2}{n1^3\,n2^3\,n3^2},\right.\right.$$

$$\left.\left. cZ1 \to -\frac{-3\,aP^4 + 4\,aP^3\,aZ + aZ^4\,cZn123 + aZ^4\,cZn2\,n1\,n3 + aZ^4\,cZn1\,n2\,n3}{aZ^4\,n1\,n2\,n3},\ rZ \to -\frac{aP^3\,(-aP + aZ)}{aZ^3\,n1\,n2\,n3}\right\}\right\}$$

```
Solve[{
  (*******************************
   The following equations are given by Lemma 6.10 for N1
    *******************************)
```

$$0 == -\frac{3\,aP^2}{n1^3\,n2^2\,n3^2} + \frac{3\,rY^2}{n1} - \frac{3\,aZ\,eZN1\,(aZ\,cZn123 + aZ\,cZn1\,n2\,n3 - 2\,n1\,n2\,n3\,rZ)}{n1^3\,n2^2\,n3^2},$$

$$0 == -\frac{3\,aP}{n1^3\,n2^2\,n3^2} - \frac{3\,cYn1\,rY}{n1^2} - \frac{3\,eZN1\,(aZ\,cZn123 + aZ\,cZn1\,n2\,n3 - n1\,n2\,n3\,rZ)}{n1^3\,n2^2\,n3^2},$$

$$0 == \frac{cYn1^2}{n1^3} - \frac{1}{n1^3\,n2^2\,n3^2} - \frac{eZN1\,(cZn123 + cZn1\,n2\,n3)}{n1^3\,n2^2\,n3^2},$$

$$0 == -\frac{1}{n1\,n2^2\,n3^2}\left(1 + cZn123\,eZN1 - cZn1\,eZN1\,n2\,n3 + 2\,n2^2\,n3^2 - cZn123\,eZN1\,n2^2\,n3^2 + cZn1\,eZN1\,n2^3\,n3^3 - n2^2\,n3^2\,pY1\right),$$

$$pY1 == 3,$$

```
  (*******************************
   The following equations are given by Lemma 6.7 for M
    *******************************)
```

$$0 == \frac{2\,aP\,ePM}{n1^2\,n2^2\,n3^2} - \frac{2\,eZM\,(aZ\,cZn123 - n1\,n2\,n3\,rZ)}{n1^2\,n2^2\,n3^2},$$

$$0 == \frac{ePM}{n1^2\,n2^2\,n3^2} - \frac{cZn123\,eZM}{n1^2\,n2^2\,n3^2}$$

```
 },
 {rY, rZ, pY1, rY, cYn1, cZn123, cZn1, eZM}]
```

$$\left\{\left\{rY \to \frac{aP - aZ}{n1\,n2\,n3},\ rZ \to \frac{aP - aZ}{eZN1\,n1\,n2\,n3},\ pY1 \to 3,\ cYn1 \to 0,\ cZn123 \to -\frac{1}{eZN1},\ cZn1 \to 0,\ eZM \to -ePM\,eZN1\right\},\ \left\{rY \to \frac{-aP + aZ}{n1\,n2\,n3},\ rZ \to \frac{aP - aZ}{eZN1\,n1\,n2\,n3},\ pY1 \to 3,\ cYn1 \to 0,\ cZn123 \to -\frac{1}{eZN1},\ cZn1 \to 0,\ eZM \to -ePM\,eZN1\right\},\ \left\{rY \to -\frac{aP + aZ}{n1\,n2\,n3},\ rZ \to \frac{-aP + aZ}{eZN1\,n1\,n2\,n3},\ pY1 \to 3,\ cYn1 \to \frac{2}{n2\,n3},\ cZn123 \to \frac{1}{eZN1},\ cZn1 \to \frac{2}{eZN1\,n2\,n3},\ eZM \to ePM\,eZN1\right\},\ \left\{rY \to \frac{aP + aZ}{n1\,n2\,n3},\ rZ \to \frac{-aP + aZ}{eZN1\,n1\,n2\,n3},\ pY1 \to 3,\ cYn1 \to -\frac{2}{n2\,n3},\ cZn123 \to \frac{1}{eZN1},\ cZn1 \to \frac{2}{eZN1\,n2\,n3},\ eZM \to ePM\,eZN1\right\}\right\}$$

```
(*******************************
 It is now interesting to compare the values of rZ
  *******************************)
```

```
Factor[Solve[{
```

$$-\frac{aP^3\,(-aP + aZ)}{aZ^3\,n1\,n2\,n3} == \frac{-aP + aZ}{n1\,n2\,n3}$$

```
 }, {aP}]]
```

$$\left\{\{aP \to -aZ\},\ \{aP \to aZ\},\ \left\{aP \to \frac{1}{2}\left(1 - i\sqrt{3}\right)aZ\right\},\ \left\{aP \to \frac{1}{2}\left(1 + i\sqrt{3}\right)aZ\right\}\right\}$$

```
(*******************************
 Since aP is not equal to aZ and needs to be an integer, we have that aP = -aZ
   *******************************)
```

```
Factor[Solve[{
```

```
(******************************
 The following equations are give by the above computations
  ******************************)
```

$$aP == -aZ,\ d == -\frac{aP^3\,(aP - aZ)^2}{n1^3\,n2^3\,n3^2},$$

$$cZ1 == \frac{-3\,aP^4 + 4\,aP^3\,aZ + aZ^4\,cZn123 + aZ^4\,cZn2\,n1\,n3 + aZ^4\,cZn1\,n2\,n3}{aZ^4\,n1\,n2\,n3},\ rZ == -\frac{aP^3\,(-aP + aZ)}{aZ^3\,n1\,n2\,n3},$$

```
(******************************
 The following equations are given by the above computations of p1(X)x^3 locally
  ******************************)
```

$$p\,d\ == -\frac{aP^3\left(1 + n1^2 + n2^2 + 2\,n1^2\,n2^2\,n3^2\right)}{n1^3\,n2^3\,n3^2} +$$

$$\frac{1}{n1^3\,n2^3\,n3^2}\,aZ^2\,\big(-aZ\,cZn123 - aZ\,cZn123\,n1^2 - aZ\,cZn123\,n2^2 - aZ\,cZn2\,n1\,n3 -$$
$$aZ\,cZn2\,n1^3\,n3 - aZ\,cZn1\,n2\,n3 + aZ\,cZ1\,n1\,n2\,n3 + aZ\,cZn1\,n1^2\,n2\,n3 -$$
$$aZ\,cZ1\,n1^3\,n2\,n3 + aZ\,cZn2\,n1\,n2^2\,n3 - aZ\,cZn1\,n2^3\,n3 - aZ\,cZ1\,n1\,n2^3\,n3 +$$
$$aZ\,cZn123\,n1^2\,n2^2\,n3^2 - aZ\,cZn2\,n1^3\,n2^2\,n3^3 - aZ\,cZn1\,n1^2\,n2^3\,n3^3 -$$
$$aZ\,cZ1\,n1^3\,n2^3\,n3^3 + 3\,n1\,n2\,n3\,rZ + 3\,n1^3\,n2\,n3\,rZ + 3\,n1\,n2^3\,n3\,rZ + 3\,n1^3\,n2^3\,n3^3\,rZ\big),$$

```
(******************************
 The following equations are given by Lemma 6.10 for N1
  ******************************)
```

$$0 == -\frac{3\,aP^2}{n1^3\,n2^2\,n3^2} + \frac{3\,rY^2}{n1} - \frac{3\,aZ\,eZN1\,(aZ\,cZn123 + aZ\,cZn1\,n2\,n3 - 2\,n1\,n2\,n3\,rZ)}{n1^3\,n2^2\,n3^2},$$

$$0 == -\frac{3\,aP}{n1^3\,n2^2\,n3^2} - \frac{3\,cYn1\,rY}{n1^2} - \frac{3\,eZN1\,(aZ\,cZn123 + aZ\,cZn1\,n2\,n3 - n1\,n2\,n3\,rZ)}{n1^3\,n2^2\,n3^2},$$

$$0 == \frac{cYn1^2}{n1^3} - \frac{1}{n1^3\,n2^2\,n3^2} - \frac{eZN1\,(cZn123 + cZn1\,n2\,n3)}{n1^3\,n2^2\,n3^2},$$

$$0 ==$$
$$-\frac{1}{n1\,n2^2\,n3^2}\big(1 + cZn123\,eZN1 - cZn1\,eZN1\,n2\,n3 + 2\,n2^2\,n3^2 - cZn123\,eZN1\,n2^2\,n3^2 +$$
$$cZn1\,eZN1\,n2^3\,n3^3 - n2^2\,n3^2\,pY1\big),$$

$$pY1 == 3,\ eZN1^2 == 1,$$

```
(******************************
 The following equations are given by Lemma 6.10 for N2
  ******************************)
```

$$0 == -\frac{3\,aP^2}{n1^2\,n2^3\,n3^2} + \frac{3\,rY^2}{n2} - \frac{3\,aZ\,eZN2\,(aZ\,cZn123 + aZ\,cZn2\,n1\,n3 - 2\,n1\,n2\,n3\,rZ)}{n1^2\,n2^3\,n3^2},$$

$$0 == -\frac{3\,aP}{n1^2\,n2^3\,n3^2} - \frac{3\,cYn2\,rY}{n2^2} - \frac{3\,eZN2\,(aZ\,cZn123 + aZ\,cZn2\,n1\,n3 - n1\,n2\,n3\,rZ)}{n1^2\,n2^3\,n3^2},$$

$$0 == \frac{cYn2^2}{n2^3} - \frac{1}{n1^2\,n2^3\,n3^2} - \frac{eZN2\,(cZn123 + cZn2\,n1\,n3)}{n1^2\,n2^3\,n3^2},$$

$$0 ==$$
$$-\frac{1}{n1^2\,n2\,n3^2}\big(1 + cZn123\,eZN2 - cZn2\,eZN2\,n1\,n3 + 2\,n1^2\,n3^2 - cZn123\,eZN2\,n1^2\,n3^2 +$$
$$cZn2\,eZN2\,n1^3\,n3^3 - n1^2\,n3^2\,pY1\big),$$

$$eZN2^2 == 1,$$

```
(******************************
 The following equations are given by Lemma 6.7
  ******************************)
```

$$0 == \frac{2\,aP\,ePM}{n1^2\,n2^2\,n3^2} - \frac{2\,eZM\,(aZ\,cZn123 - n1\,n2\,n3\,rZ)}{n1^2\,n2^2\,n3^2},$$

$$0 == \frac{ePM}{n1^2\,n2^2\,n3^2} - \frac{cZn123\,eZM}{n1^2\,n2^2\,n3^2},$$

$$eZM^2 == 1,\ ePM^2 == 1$$

$$\Big\},$$

$$\{p,\ d,\ cZ1,\ cZn123,\ cZn1,\ cZn2,\ cYn1,\ cYn2,\ rZ,\ rY,\ aP,\ eZM,\ ePM,\ eZN1,\ eZN2,\ pY1\}\Big]\Big]$$

$$\Big\{\Big\{p\to\frac{3\left(4\,n1^2+4\,n2^2+5\,n1^2\,n2^2\,n3^2\right)}{4\,aZ^2},\ d\to\frac{4\,aZ^5}{n1^3\,n2^3\,n3^2},\ cZ1\to-\frac{6}{n1\,n2\,n3},$$
$$cZn123\to1,\ cZn1\to0,\ cZn2\to0,\ cYn1\to0,\ cYn2\to0,\ rZ\to\frac{2\,aZ}{n1\,n2\,n3},$$
$$rY\to-\frac{2\,aZ}{n1\,n2\,n3},\ aP\to-aZ,\ eZM\to-1,\ ePM\to-1,\ eZN1\to-1,\ eZN2\to-1,\ pY1\to3\Big\},$$

$$\Big\{p\to\frac{3\left(4\,n1^2+4\,n2^2+5\,n1^2\,n2^2\,n3^2\right)}{4\,aZ^2},\ d\to\frac{4\,aZ^5}{n1^3\,n2^3\,n3^2},\ cZ1\to-\frac{6}{n1\,n2\,n3},$$
$$cZn123\to1,\ cZn1\to0,\ cZn2\to0,\ cYn1\to0,\ cYn2\to0,\ rZ\to\frac{2\,aZ}{n1\,n2\,n3},$$
$$rY\to-\frac{2\,aZ}{n1\,n2\,n3},\ aP\to-aZ,\ eZM\to1,\ ePM\to1,\ eZN1\to-1,\ eZN2\to-1,\ pY1\to3\Big\},$$

$$\Big\{p\to\frac{3\left(4\,n1^2+4\,n2^2+5\,n1^2\,n2^2\,n3^2\right)}{4\,aZ^2},\ d\to\frac{4\,aZ^5}{n1^3\,n2^3\,n3^2},\ cZ1\to-\frac{6}{n1\,n2\,n3},$$
$$cZn123\to1,\ cZn1\to0,\ cZn2\to0,\ cYn1\to0,\ cYn2\to0,\ rZ\to\frac{2\,aZ}{n1\,n2\,n3},$$
$$rY\to\frac{2\,aZ}{n1\,n2\,n3},\ aP\to-aZ,\ eZM\to-1,\ ePM\to-1,\ eZN1\to-1,\ eZN2\to-1,\ pY1\to3\Big\},$$

$$\Big\{p\to\frac{3\left(4\,n1^2+4\,n2^2+5\,n1^2\,n2^2\,n3^2\right)}{4\,aZ^2},\ d\to\frac{4\,aZ^5}{n1^3\,n2^3\,n3^2},\ cZ1\to-\frac{6}{n1\,n2\,n3},$$
$$cZn123\to1,\ cZn1\to0,\ cZn2\to0,\ cYn1\to0,\ cYn2\to0,\ rZ\to\frac{2\,aZ}{n1\,n2\,n3},$$
$$rY\to\frac{2\,aZ}{n1\,n2\,n3},\ aP\to-aZ,\ eZM\to1,\ ePM\to1,\ eZN1\to-1,\ eZN2\to-1,\ pY1\to3\Big\},$$

$$\Big\{p\to\frac{8\,n1^2+8\,n2^2+7\,n1^2\,n2^2\,n3^2}{4\,aZ^2},\ d\to\frac{4\,aZ^5}{n1^3\,n2^3\,n3^2},\ cZ1\to-\frac{2}{n1\,n2\,n3},\ cZn123\to1,$$
$$cZn1\to\frac{2}{n2\,n3},\ cZn2\to\frac{2}{n1\,n3},\ cYn1\to-\frac{2}{n2\,n3},\ cYn2\to-\frac{2}{n1\,n3},\ rZ\to\frac{2\,aZ}{n1\,n2\,n3},$$
$$rY\to0,\ aP\to-aZ,\ eZM\to-1,\ ePM\to-1,\ eZN1\to1,\ eZN2\to1,\ pY1\to3\Big\},$$

$$\Big\{p\to\frac{8\,n1^2+8\,n2^2+7\,n1^2\,n2^2\,n3^2}{4\,aZ^2},\ d\to\frac{4\,aZ^5}{n1^3\,n2^3\,n3^2},\ cZ1\to-\frac{2}{n1\,n2\,n3},\ cZn123\to1,$$
$$cZn1\to\frac{2}{n2\,n3},\ cZn2\to\frac{2}{n1\,n3},\ cYn1\to-\frac{2}{n2\,n3},\ cYn2\to-\frac{2}{n1\,n3},\ rZ\to\frac{2\,aZ}{n1\,n2\,n3},$$
$$rY\to0,\ aP\to-aZ,\ eZM\to1,\ ePM\to1,\ eZN1\to1,\ eZN2\to1,\ pY1\to3\Big\},$$

$$\Big\{p\to\frac{8\,n1^2+8\,n2^2+7\,n1^2\,n2^2\,n3^2}{4\,aZ^2},\ d\to\frac{4\,aZ^5}{n1^3\,n2^3\,n3^2},\ cZ1\to-\frac{2}{n1\,n2\,n3},\ cZn123\to1,$$
$$cZn1\to\frac{2}{n2\,n3},\ cZn2\to\frac{2}{n1\,n3},\ cYn1\to-\frac{2}{n2\,n3},\ cYn2\to\frac{2}{n1\,n3},\ rZ\to\frac{2\,aZ}{n1\,n2\,n3},$$
$$rY\to0,\ aP\to-aZ,\ eZM\to-1,\ ePM\to-1,\ eZN1\to1,\ eZN2\to1,\ pY1\to3\Big\},$$

$$\Big\{p\to\frac{8\,n1^2+8\,n2^2+7\,n1^2\,n2^2\,n3^2}{4\,aZ^2},\ d\to\frac{4\,aZ^5}{n1^3\,n2^3\,n3^2},\ cZ1\to-\frac{2}{n1\,n2\,n3},\ cZn123\to1,$$
$$cZn1\to\frac{2}{n2\,n3},\ cZn2\to\frac{2}{n1\,n3},\ cYn1\to-\frac{2}{n2\,n3},\ cYn2\to\frac{2}{n1\,n3},\ rZ\to\frac{2\,aZ}{n1\,n2\,n3},$$
$$rY\to0,\ aP\to-aZ,\ eZM\to1,\ ePM\to1,\ eZN1\to1,\ eZN2\to1,\ pY1\to3\Big\},$$

$$\Big\{p\to\frac{8\,n1^2+8\,n2^2+7\,n1^2\,n2^2\,n3^2}{4\,aZ^2},\ d\to\frac{4\,aZ^5}{n1^3\,n2^3\,n3^2},\ cZ1\to-\frac{2}{n1\,n2\,n3},\ cZn123\to1,$$
$$cZn1\to\frac{2}{n2\,n3},\ cZn2\to\frac{2}{n1\,n3},\ cYn1\to\frac{2}{n2\,n3},\ cYn2\to-\frac{2}{n1\,n3},\ rZ\to\frac{2\,aZ}{n1\,n2\,n3},$$

$\mathrm{rY} \to 0,\ \mathrm{aP} \to -\mathrm{aZ},\ \mathrm{eZM} \to -1,\ \mathrm{ePM} \to -1,\ \mathrm{eZN1} \to 1,\ \mathrm{eZN2} \to 1,\ \mathrm{pY1} \to 3\Big\},$

$\Big\{\mathrm{p} \to \frac{8\,\mathrm{n1}^2 + 8\,\mathrm{n2}^2 + 7\,\mathrm{n1}^2\,\mathrm{n2}^2\,\mathrm{n3}^2}{4\,\mathrm{aZ}^2},\ \mathrm{d} \to \frac{4\,\mathrm{aZ}^5}{\mathrm{n1}^3\,\mathrm{n2}^3\,\mathrm{n3}^2},\ \mathrm{cZ1} \to -\frac{2}{\mathrm{n1}\,\mathrm{n2}\,\mathrm{n3}},\ \mathrm{cZn123} \to 1,$

$\mathrm{cZn1} \to \frac{2}{\mathrm{n2}\,\mathrm{n3}},\ \mathrm{cZn2} \to \frac{2}{\mathrm{n1}\,\mathrm{n3}},\ \mathrm{cYn1} \to \frac{2}{\mathrm{n2}\,\mathrm{n3}},\ \mathrm{cYn2} \to -\frac{2}{\mathrm{n1}\,\mathrm{n3}},\ \mathrm{rZ} \to \frac{2\,\mathrm{aZ}}{\mathrm{n1}\,\mathrm{n2}\,\mathrm{n3}},$

$\mathrm{rY} \to 0,\ \mathrm{aP} \to -\mathrm{aZ},\ \mathrm{eZM} \to 1,\ \mathrm{ePM} \to 1,\ \mathrm{eZN1} \to 1,\ \mathrm{eZN2} \to 1,\ \mathrm{pY1} \to 3\Big\},$

$\Big\{\mathrm{p} \to \frac{8\,\mathrm{n1}^2 + 8\,\mathrm{n2}^2 + 7\,\mathrm{n1}^2\,\mathrm{n2}^2\,\mathrm{n3}^2}{4\,\mathrm{aZ}^2},\ \mathrm{d} \to \frac{4\,\mathrm{aZ}^5}{\mathrm{n1}^3\,\mathrm{n2}^3\,\mathrm{n3}^2},\ \mathrm{cZ1} \to -\frac{2}{\mathrm{n1}\,\mathrm{n2}\,\mathrm{n3}},\ \mathrm{cZn123} \to 1,$

$\mathrm{cZn1} \to \frac{2}{\mathrm{n2}\,\mathrm{n3}},\ \mathrm{cZn2} \to \frac{2}{\mathrm{n1}\,\mathrm{n3}},\ \mathrm{cYn1} \to \frac{2}{\mathrm{n2}\,\mathrm{n3}},\ \mathrm{cYn2} \to \frac{2}{\mathrm{n1}\,\mathrm{n3}},\ \mathrm{rZ} \to \frac{2\,\mathrm{aZ}}{\mathrm{n1}\,\mathrm{n2}\,\mathrm{n3}},$

$\mathrm{rY} \to 0,\ \mathrm{aP} \to -\mathrm{aZ},\ \mathrm{eZM} \to -1,\ \mathrm{ePM} \to -1,\ \mathrm{eZN1} \to 1,\ \mathrm{eZN2} \to 1,\ \mathrm{pY1} \to 3\Big\},$

$\Big\{\mathrm{p} \to \frac{8\,\mathrm{n1}^2 + 8\,\mathrm{n2}^2 + 7\,\mathrm{n1}^2\,\mathrm{n2}^2\,\mathrm{n3}^2}{4\,\mathrm{aZ}^2},\ \mathrm{d} \to \frac{4\,\mathrm{aZ}^5}{\mathrm{n1}^3\,\mathrm{n2}^3\,\mathrm{n3}^2},\ \mathrm{cZ1} \to -\frac{2}{\mathrm{n1}\,\mathrm{n2}\,\mathrm{n3}},\ \mathrm{cZn123} \to 1,$

$\mathrm{cZn1} \to \frac{2}{\mathrm{n2}\,\mathrm{n3}},\ \mathrm{cZn2} \to \frac{2}{\mathrm{n1}\,\mathrm{n3}},\ \mathrm{cYn1} \to \frac{2}{\mathrm{n2}\,\mathrm{n3}},\ \mathrm{cYn2} \to \frac{2}{\mathrm{n1}\,\mathrm{n3}},\ \mathrm{rZ} \to \frac{2\,\mathrm{aZ}}{\mathrm{n1}\,\mathrm{n2}\,\mathrm{n3}},$

$\mathrm{rY} \to 0,\ \mathrm{aP} \to -\mathrm{aZ},\ \mathrm{eZM} \to 1,\ \mathrm{ePM} \to 1,\ \mathrm{eZN1} \to 1,\ \mathrm{eZN2} \to 1,\ \mathrm{pY1} \to 3\Big\}\Big\}$

Computations for Lemma 6.10.
The case where there exists N_i^6 s.t. Fix(N_i) = $Y^4 \cup Z^2 \cup P^0$ and M^4 in N_1^6, N_2^6 and N_3^6 s.t. Fix(M) = $Z^2 \cup P^0$

Case A.4

```
(******************************
 NOTATIONS
 rY xY denotes the restriction of x to Y
 rZ xZ denotes the restriction of x to Z
 aZ denotes the Hopf weight at Z
 aP denotes the Hopf weight at P
 cZn1234 xZ denotes the first Chern class of the normal bundle of Z in
 the direction of M
 cZn1 xZ denotes the first Chern class of the normal bundle of Z in the
 remaining direction in N1
 cZn2 xZ denotes the first Chern class of the normal bundle of Z in the
 remaining direction in N2
 cZn3 xZ denotes the first Chern class of the normal bundle of Z in the
 remaining direction in N3
 cYn1 xY denotes the first Chern class of the normal bundle of Y in the
 direction of N1
 cYn2 xY denotes the first Chern class of the normal bundle of Y in the
 direction of N2
 cYn3 xY denotes the first Chern class of the normal bundle of Y in the
 direction of N3
 μx5YA4 denotes the local datum of x^5 at Y
 μx5ZA4 denotes the local datum of x^5 at Z
 μx5PA4 denotes the local datum of x^5 at P

 We use the variable t to extract the coefficient of order 2 of xY and
 of order 1 of xZ. We then evaluate xY^2 and xZ to 1.
 The orientation of the point P is -1, this explains the negative sign.
 Once evaluated, we extract all the coefficients in l into a list.
 ******************************)
```

```
μx5YA4 =
 Factor[CoefficientList[
    Coefficient[Expand[(rY xY t + l z)^5] Series[(cYn1 xY t + n1 z)^-1, {xY, 0, 2}]
      Series[(cYn2 xY t + n2 z)^-1, {xY, 0, 2}] Series[(cYn3 xY t + n3 z)^-1, {xY, 0, 2}],
     t, 2], l]] /. xY^2 → 1
```

$$\left\{0,\,0,\,0,\,\frac{10\,rY^2}{n1\,n2\,n3},\,-\frac{5\,(cYn3\,n1\,n2+cYn2\,n1\,n3+cYn1\,n2\,n3)\,rY}{n1^2\,n2^2\,n3^2},\right.$$
$$\left.\frac{1}{n1^3\,n2^3\,n3^3}\left(cYn3^2\,n1^2\,n2^2+cYn2\,cYn3\,n1^2\,n2\,n3+cYn1\,cYn3\,n1\,n2^2\,n3+cYn2^2\,n1^2\,n3^2+cYn1\,cYn2\,n1\,n2\,n3^2+cYn1^2\,n2^2\,n3^2\right)\right\}$$

```
μx5ZA4 =
 CoefficientList[
   Coefficient[Expand[(rZ xZ t + (aZ + l) z)^5]
      Series[(cZn1234 xZ t + n1 n2 n3 n4 z)^-1, {xZ, 0, 1}]
      Series[(cZn1 t xZ + n1 z)^-1, {xZ, 0, 1}] Series[(cZn2 t xZ + n2 z)^-1, {xZ, 0, 1}]
      Series[(cZn3 xZ t + n3 z)^-1, {xZ, 0, 1}], t, 1], l] /. xZ → 1
```

$$\left\{-\frac{aZ^5\,cZn1234}{n1^3\,n2^3\,n3^3\,n4^2}-\frac{aZ^5\,cZn3}{n1^2\,n2^2\,n3^3\,n4}-\frac{aZ^5\,cZn2}{n1^2\,n2^3\,n3^2\,n4}-\frac{aZ^5\,cZn1}{n1^3\,n2^2\,n3^2\,n4}+\frac{5\,aZ^4\,rZ}{n1^2\,n2^2\,n3^2\,n4},\right.$$
$$-\frac{5\,aZ^4\,cZn1234}{n1^3\,n2^3\,n3^3\,n4^2}-\frac{5\,aZ^4\,cZn3}{n1^2\,n2^2\,n3^3\,n4}-\frac{5\,aZ^4\,cZn2}{n1^2\,n2^3\,n3^2\,n4}-\frac{5\,aZ^4\,cZn1}{n1^3\,n2^2\,n3^2\,n4}+\frac{20\,aZ^3\,rZ}{n1^2\,n2^2\,n3^2\,n4},$$
$$-\frac{10\,aZ^3\,cZn1234}{n1^3\,n2^3\,n3^3\,n4^2}-\frac{10\,aZ^3\,cZn3}{n1^2\,n2^2\,n3^3\,n4}-\frac{10\,aZ^3\,cZn2}{n1^2\,n2^3\,n3^2\,n4}-\frac{10\,aZ^3\,cZn1}{n1^3\,n2^2\,n3^2\,n4}+\frac{30\,aZ^2\,rZ}{n1^2\,n2^2\,n3^2\,n4},$$
$$-\frac{10\,aZ^2\,cZn1234}{n1^3\,n2^3\,n3^3\,n4^2}-\frac{10\,aZ^2\,cZn3}{n1^2\,n2^2\,n3^3\,n4}-\frac{10\,aZ^2\,cZn2}{n1^2\,n2^3\,n3^2\,n4}-\frac{10\,aZ^2\,cZn1}{n1^3\,n2^2\,n3^2\,n4}+\frac{20\,aZ\,rZ}{n1^2\,n2^2\,n3^2\,n4},$$
$$-\frac{5\,aZ\,cZn1234}{n1^3\,n2^3\,n3^3\,n4^2}-\frac{5\,aZ\,cZn3}{n1^2\,n2^2\,n3^3\,n4}-\frac{5\,aZ\,cZn2}{n1^2\,n2^3\,n3^2\,n4}-\frac{5\,aZ\,cZn1}{n1^3\,n2^2\,n3^2\,n4}+\frac{5\,rZ}{n1^2\,n2^2\,n3^2\,n4},$$
$$\left.-\frac{cZn1234}{n1^3\,n2^3\,n3^3\,n4^2}-\frac{cZn3}{n1^2\,n2^2\,n3^3\,n4}-\frac{cZn2}{n1^2\,n2^3\,n3^2\,n4}-\frac{cZn1}{n1^3\,n2^2\,n3^2\,n4}\right\}$$

```
μx5PA4 = CoefficientList[Expand[-((aP + l) z)^5] (n1 n2 n3 n4 z)^-2 (n1 z)^-1 (n2 z)^-1 (n3 z)^-1,
   l]
```

$$\left\{-\frac{aP^5}{n1^3\,n2^3\,n3^3\,n4^2},\,-\frac{5\,aP^4}{n1^3\,n2^3\,n3^3\,n4^2},\,-\frac{10\,aP^3}{n1^3\,n2^3\,n3^3\,n4^2},\right.$$
$$\left.-\frac{10\,aP^2}{n1^3\,n2^3\,n3^3\,n4^2},\,-\frac{5\,aP}{n1^3\,n2^3\,n3^3\,n4^2},\,-\frac{1}{n1^3\,n2^3\,n3^3\,n4^2}\right\}$$

```
(******************************
 NOTATIONS
 μs1x3YA4 denotes the local datum of p1(X)x^3 at Y
 μs1x3ZA4 denotes the local datum of p1(X)x^3 at Z
 μs1x3PA4 denotes the local datum of p1(X)x^3 at P
  ******************************)
```

```
μs1x3YA4 =
 Factor[CoefficientList[
    Coefficient[
     Expand[pY1 t^2 + (cYn1 xY t + n1 z)^2 + (cYn2 xY t + n2 z)^2 + (cYn3 xY t + n3 z)^2]
      (rY xY t + l z)^3 Series[(cYn1 xY t + n1 z)^-1, {xY, 0, 2}]
      Series[(cYn2 xY t + n2 z)^-1, {xY, 0, 2}] Series[(cYn3 xY t + n3 z)^-1, {xY, 0, 2}],
     t, 2], l]] /. xY^2 → 1
```

$$\Big\{0,\ \frac{3\left(n1^2+n2^2+n3^2\right) rY^2}{n1\, n2\, n3},$$

$$-\frac{1}{n1^2\, n2^2\, n3^2} 3 \big(cYn3\, n1^3\, n2 + cYn3\, n1\, n2^3 + cYn2\, n1^3\, n3 - cYn1\, n1^2\, n2\, n3 - cYn2\, n1\, n2^2\, n3 + cYn1\, n2^3\, n3 - cYn3\, n1\, n2\, n3^2 + cYn2\, n1\, n3^3 + cYn1\, n2\, n3^3\big)\, rY,$$

$$\frac{1}{n1^3\, n2^3\, n3^3} \big(cYn3^2\, n1^4\, n2^2 + cYn3^2\, n1^2\, n2^4 + cYn2\, cYn3\, n1^4\, n2\, n3 - cYn1\, cYn3\, n1^3\, n2^2\, n3 - cYn2\, cYn3\, n1^2\, n2^3\, n3 + cYn1\, cYn3\, n1\, n2^4\, n3 + cYn2^2\, n1^4\, n3^2 - cYn1\, cYn2\, n1^3\, n2\, n3^2 - cYn1\, cYn2\, n1\, n2^3\, n3^2 + cYn1^2\, n2^4\, n3^2 - cYn2\, cYn3\, n1^2\, n2\, n3^3 - cYn1\, cYn3\, n1\, n2^2\, n3^3 + cYn2^2\, n1^2\, n3^4 + cYn1\, cYn2\, n1\, n2\, n3^4 + cYn1^2\, n2^2\, n3^4 + n1^2\, n2^2\, n3^2\, pY1\big)\Big\}$$

```
μs1x3ZA4 =
 Factor[CoefficientList[
    Coefficient[
     Expand[(cZn1234 xZ t + n1 n2 n3 n4 z)^2 + (cZn1 xZ t + n1 z)^2 + (cZn2 xZ t + n2 z)^2 +
        (cZn3 xZ t + n3 z)^2] (rZ xZ t + (aZ + l) z)^3 Series[(cZn1 t xZ + n1 z)^-1, {xZ, 0, 1}]
      Series[(cZn1234 xZ t + n1 n2 n3 n4 z)^-1, {xZ, 0, 1}]
      Series[(cZn2 t xZ + n2 z)^-1, {xZ, 0, 1}] Series[(cZn3 xZ t + n3 z)^-1, {xZ, 0, 1}],
     t, 1], l]] /. xZ → 1
```

$$\Big\{\frac{1}{n1^3\, n2^3\, n3^3\, n4^2} aZ^2 \big(-aZ\, cZn1234\, n1^2 - aZ\, cZn1234\, n2^2 - aZ\, cZn1234\, n3^2 - aZ\, cZn3\, n1^3\, n2\, n4 - aZ\, cZn3\, n1\, n2^3\, n4 - aZ\, cZn2\, n1^3\, n3\, n4 + aZ\, cZn1\, n1^2\, n2\, n3\, n4 + aZ\, cZn2\, n1\, n2^2\, n3\, n4 - aZ\, cZn1\, n2^3\, n3\, n4 + aZ\, cZn3\, n1\, n2\, n3^2\, n4 - aZ\, cZn2\, n1\, n3^3\, n4 - aZ\, cZn1\, n2\, n3^3\, n4 + aZ\, cZn1234\, n1^2\, n2^2\, n3^2\, n4^2 - aZ\, cZn3\, n1^3\, n2^3\, n3^2\, n4^3 - aZ\, cZn2\, n1^3\, n2^2\, n3^3\, n4^3 - aZ\, cZn1\, n1^2\, n2^3\, n3^3\, n4^3 + 3\, n1^3\, n2\, n3\, n4\, rZ + 3\, n1\, n2^3\, n3\, n4\, rZ + 3\, n1\, n2\, n3^3\, n4\, rZ + 3\, n1^3\, n2^3\, n3^3\, n4^3\, rZ\big),$$

$$\frac{1}{n1^3\, n2^3\, n3^3\, n4^2} 3\, aZ \big(-aZ\, cZn1234\, n1^2 - aZ\, cZn1234\, n2^2 - aZ\, cZn1234\, n3^2 - aZ\, cZn3\, n1^3\, n2\, n4 - aZ\, cZn3\, n1\, n2^3\, n4 - aZ\, cZn2\, n1^3\, n3\, n4 + aZ\, cZn1\, n1^2\, n2\, n3\, n4 + aZ\, cZn2\, n1\, n2^2\, n3\, n4 - aZ\, cZn1\, n2^3\, n3\, n4 + aZ\, cZn3\, n1\, n2\, n3^2\, n4 - aZ\, cZn2\, n1\, n3^3\, n4 - aZ\, cZn1\, n2\, n3^3\, n4 + aZ\, cZn1234\, n1^2\, n2^2\, n3^2\, n4^2 - aZ\, cZn3\, n1^3\, n2^3\, n3^2\, n4^3 - aZ\, cZn2\, n1^3\, n2^2\, n3^3\, n4^3 - aZ\, cZn1\, n1^2\, n2^3\, n3^3\, n4^3 + 2\, n1^3\, n2\, n3\, n4\, rZ + 2\, n1\, n2^3\, n3\, n4\, rZ + 2\, n1\, n2\, n3^3\, n4\, rZ + 2\, n1^3\, n2^3\, n3^3\, n4^3\, rZ\big),$$

$$\frac{1}{n1^3\, n2^3\, n3^3\, n4^2} 3 \big(-aZ\, cZn1234\, n1^2 - aZ\, cZn1234\, n2^2 - aZ\, cZn1234\, n3^2 - aZ\, cZn3\, n1^3\, n2\, n4 - aZ\, cZn3\, n1\, n2^3\, n4 - aZ\, cZn2\, n1^3\, n3\, n4 + aZ\, cZn1\, n1^2\, n2\, n3\, n4 + aZ\, cZn2\, n1\, n2^2\, n3\, n4 - aZ\, cZn1\, n2^3\, n3\, n4 + aZ\, cZn3\, n1\, n2\, n3^2\, n4 - aZ\, cZn2\, n1\, n3^3\, n4 - aZ\, cZn1\, n2\, n3^3\, n4 + aZ\, cZn1234\, n1^2\, n2^2\, n3^2\, n4^2 - aZ\, cZn3\, n1^3\, n2^3\, n3^2\, n4^3 - aZ\, cZn2\, n1^3\, n2^2\, n3^3\, n4^3 - aZ\, cZn1\, n1^2\, n2^3\, n3^3\, n4^3 + n1^3\, n2\, n3\, n4\, rZ + n1\, n2^3\, n3\, n4\, rZ + n1\, n2\, n3^3\, n4\, rZ + n1^3\, n2^3\, n3^3\, n4^3\, rZ\big),$$

$$\frac{1}{n1^3\, n2^3\, n3^3\, n4^2} \big(-cZn1234\, n1^2 - cZn1234\, n2^2 - cZn1234\, n3^2 - cZn3\, n1^3\, n2\, n4 - cZn3\, n1\, n2^3\, n4 - cZn2\, n1^3\, n3\, n4 + cZn1\, n1^2\, n2\, n3\, n4 + cZn2\, n1\, n2^2\, n3\, n4 - cZn1\, n2^3\, n3\, n4 + cZn3\, n1\, n2\, n3^2\, n4 - cZn2\, n1\, n3^3\, n4 - cZn1\, n2\, n3^3\, n4 + cZn1234\, n1^2\, n2^2\, n3^2\, n4^2 - cZn3\, n1^3\, n2^3\, n3^2\, n4^3 - cZn2\, n1^3\, n2^2\, n3^3\, n4^3 - cZn1\, n1^2\, n2^3\, n3^3\, n4^3\big)\Big\}$$

```
μs1x3PA4 =
  Factor[CoefficientList[((n1 n2 n3 n4 z)^2 + (n1 n2 n3 n4 z)^2 + (n1 z)^2 + (n2 z)^2 + (n3 z)^2)
     Expand[-((aP + 1) z)^3] (n1 n2 n3 n4 z)^-2 (n1 z)^-1 (n2 z)^-1 (n3 z)^-1, 1]]
```

$$\left\{-\frac{aP^3\left(n1^2+n2^2+n3^2+2\,n1^2\,n2^2\,n3^2\,n4^2\right)}{n1^3\,n2^3\,n3^3\,n4^2},\ -\frac{3\,aP^2\left(n1^2+n2^2+n3^2+2\,n1^2\,n2^2\,n3^2\,n4^2\right)}{n1^3\,n2^3\,n3^3\,n4^2},\ -\frac{3\,aP\left(n1^2+n2^2+n3^2+2\,n1^2\,n2^2\,n3^2\,n4^2\right)}{n1^3\,n2^3\,n3^3\,n4^2},\ -\frac{n1^2+n2^2+n3^2+2\,n1^2\,n2^2\,n3^2\,n4^2}{n1^3\,n2^3\,n3^3\,n4^2}\right\}$$

```
μx5YA4 + μx5ZA4 + μx5PA4
```

$$\left\{-\frac{aP^5}{n1^3\,n2^3\,n3^3\,n4^2}-\frac{aZ^5\,cZn1234}{n1^3\,n2^3\,n3^3\,n4^2}-\frac{aZ^5\,cZn3}{n1^2\,n2^2\,n3^3\,n4}-\frac{aZ^5\,cZn2}{n1^2\,n2^3\,n3^2\,n4}-\frac{aZ^5\,cZn1}{n1^3\,n2^2\,n3^2\,n4}+\frac{5\,aZ^4\,rZ}{n1^2\,n2^2\,n3^2\,n4},\right.$$
$$-\frac{5\,aP^4}{n1^3\,n2^3\,n3^3\,n4^2}-\frac{5\,aZ^4\,cZn1234}{n1^3\,n2^3\,n3^3\,n4^2}-\frac{5\,aZ^4\,cZn3}{n1^2\,n2^2\,n3^3\,n4}-\frac{5\,aZ^4\,cZn2}{n1^2\,n2^3\,n3^2\,n4}-\frac{5\,aZ^4\,cZn1}{n1^3\,n2^2\,n3^2\,n4}+\frac{20\,aZ^3\,rZ}{n1^2\,n2^2\,n3^2\,n4},$$
$$-\frac{10\,aP^3}{n1^3\,n2^3\,n3^3\,n4^2}-\frac{10\,aZ^3\,cZn1234}{n1^3\,n2^3\,n3^3\,n4^2}-\frac{10\,aZ^3\,cZn3}{n1^2\,n2^2\,n3^3\,n4}-\frac{10\,aZ^3\,cZn2}{n1^2\,n2^3\,n3^2\,n4}-\frac{10\,aZ^3\,cZn1}{n1^3\,n2^2\,n3^2\,n4}+\frac{30\,aZ^2\,rZ}{n1^2\,n2^2\,n3^2\,n4},$$
$$-\frac{10\,aP^2}{n1^3\,n2^3\,n3^3\,n4^2}-\frac{10\,aZ^2\,cZn1234}{n1^3\,n2^3\,n3^3\,n4^2}-\frac{10\,aZ^2\,cZn3}{n1^2\,n2^2\,n3^3\,n4}-\frac{10\,aZ^2\,cZn2}{n1^2\,n2^3\,n3^2\,n4}-\frac{10\,aZ^2\,cZn1}{n1^3\,n2^2\,n3^2\,n4}+\frac{10\,rY^2}{n1\,n2\,n3}+\frac{20\,aZ\,rZ}{n1^2\,n2^2\,n3^2\,n4},$$
$$-\frac{5\,aP}{n1^3\,n2^3\,n3^3\,n4^2}-\frac{5\,aZ\,cZn1234}{n1^3\,n2^3\,n3^3\,n4^2}-\frac{5\,aZ\,cZn3}{n1^2\,n2^2\,n3^3\,n4}-\frac{5\,aZ\,cZn2}{n1^2\,n2^3\,n3^2\,n4}-\frac{5\,aZ\,cZn1}{n1^3\,n2^2\,n3^2\,n4}-\frac{5\,(cYn3\,n1\,n2+cYn2\,n1\,n3+cYn1\,n2\,n3)\,rY}{n1^2\,n2^2\,n3^2}+\frac{5\,rZ}{n1^2\,n2^2\,n3^2\,n4},$$
$$\frac{1}{n1^3\,n2^3\,n3^3}\left(cYn3^2\,n1^2\,n2^2+cYn2\,cYn3\,n1^2\,n2\,n3+cYn1\,cYn3\,n1\,n2^2\,n3+cYn2^2\,n1^2\,n3^2+cYn1\,cYn2\,n1\,n2\,n3^2+cYn1^2\,n2^2\,n3^2\right)-\frac{1}{n1^3\,n2^3\,n3^3\,n4^2}-$$
$$\left.\frac{cZn1234}{n1^3\,n2^3\,n3^3\,n4^2}-\frac{cZn3}{n1^2\,n2^2\,n3^3\,n4}-\frac{cZn2}{n1^2\,n2^3\,n3^2\,n4}-\frac{cZn1}{n1^3\,n2^2\,n3^2\,n4}\right\}$$

μs1x3YA4 + μs1x3ZA4 + μs1x3PA4

$$\Big\{-\frac{aP^3\,(n1^2+n2^2+n3^2+2\,n1^2\,n2^2\,n3^2\,n4^2)}{n1^3\,n2^3\,n3^3\,n4^2}+\frac{1}{n1^3\,n2^3\,n3^3\,n4^2}\,aZ^2\,\big(-aZ\,cZn1234\,n1^2-aZ\,cZn1234\,n2^2-aZ\,cZn1234\,n3^2-aZ\,cZn3\,n1^3\,n2\,n4-aZ\,cZn3\,n1\,n2^3\,n4-aZ\,cZn2\,n1^3\,n3\,n4+aZ\,cZn1\,n1^2\,n2\,n3\,n4+aZ\,cZn2\,n1\,n2^2\,n3\,n4-aZ\,cZn1\,n2^3\,n3\,n4+aZ\,cZn3\,n1\,n2\,n3^2\,n4-aZ\,cZn2\,n1\,n3^3\,n4-aZ\,cZn1\,n2\,n3^3\,n4+aZ\,cZn1234\,n1^2\,n2^2\,n3^2\,n4^2-aZ\,cZn3\,n1^3\,n2^3\,n3^2\,n4^3-aZ\,cZn2\,n1^3\,n2^2\,n3^3\,n4^3-aZ\,cZn1\,n1^2\,n2^3\,n3^3\,n4^3+3\,n1^3\,n2\,n3\,n4\,rZ+3\,n1\,n2^3\,n3\,n4\,rZ+3\,n1\,n2\,n3^3\,n4\,rZ+3\,n1^3\,n2^3\,n3^3\,n4^3\,rZ\big),$$

$$-\frac{3\,aP^2\,(n1^2+n2^2+n3^2+2\,n1^2\,n2^2\,n3^2\,n4^2)}{n1^3\,n2^3\,n3^3\,n4^2}+\frac{3\,(n1^2+n2^2+n3^2)\,rY^2}{n1\,n2\,n3}+\frac{1}{n1^3\,n2^3\,n3^3\,n4^2}\,3\,aZ\,\big(-aZ\,cZn1234\,n1^2-aZ\,cZn1234\,n2^2-aZ\,cZn1234\,n3^2-aZ\,cZn3\,n1^3\,n2\,n4-aZ\,cZn3\,n1\,n2^3\,n4-aZ\,cZn2\,n1^3\,n3\,n4+aZ\,cZn1\,n1^2\,n2\,n3\,n4+aZ\,cZn2\,n1\,n2^2\,n3\,n4-aZ\,cZn1\,n2^3\,n3\,n4+aZ\,cZn3\,n1\,n2\,n3^2\,n4-aZ\,cZn2\,n1\,n3^3\,n4-aZ\,cZn1\,n2\,n3^3\,n4+aZ\,cZn1234\,n1^2\,n2^2\,n3^2\,n4^2-aZ\,cZn3\,n1^3\,n2^3\,n3^2\,n4^3-aZ\,cZn2\,n1^3\,n2^2\,n3^3\,n4^3-aZ\,cZn1\,n1^2\,n2^3\,n3^3\,n4^3+2\,n1^3\,n2\,n3\,n4\,rZ+2\,n1\,n2^3\,n3\,n4\,rZ+2\,n1\,n2\,n3^3\,n4\,rZ+2\,n1^3\,n2^3\,n3^3\,n4^3\,rZ\big),$$

$$-\frac{3\,aP\,(n1^2+n2^2+n3^2+2\,n1^2\,n2^2\,n3^2\,n4^2)}{n1^3\,n2^3\,n3^3\,n4^2}-\frac{1}{n1^2\,n2^2\,n3^2}\,3\,\big(cYn3\,n1^3\,n2+cYn3\,n1\,n2^3+cYn2\,n1^3\,n3-cYn1\,n1^2\,n2\,n3-cYn2\,n1\,n2^2\,n3+cYn1\,n2^3\,n3-cYn3\,n1\,n2\,n3^2+cYn2\,n1\,n3^3+cYn1\,n2\,n3^3\big)\,rY+\frac{1}{n1^3\,n2^3\,n3^3\,n4^2}\,3\,\big(-aZ\,cZn1234\,n1^2-aZ\,cZn1234\,n2^2-aZ\,cZn1234\,n3^2-aZ\,cZn3\,n1^3\,n2\,n4-aZ\,cZn3\,n1\,n2^3\,n4-aZ\,cZn2\,n1^3\,n3\,n4+aZ\,cZn1\,n1^2\,n2\,n3\,n4+aZ\,cZn2\,n1\,n2^2\,n3\,n4-aZ\,cZn1\,n2^3\,n3\,n4+aZ\,cZn3\,n1\,n2\,n3^2\,n4-aZ\,cZn2\,n1\,n3^3\,n4-aZ\,cZn1\,n2\,n3^3\,n4+aZ\,cZn1234\,n1^2\,n2^2\,n3^2\,n4^2-aZ\,cZn3\,n1^3\,n2^3\,n3^2\,n4^3-aZ\,cZn2\,n1^3\,n2^2\,n3^3\,n4^3-aZ\,cZn1\,n1^2\,n2^3\,n3^3\,n4^3+n1^3\,n2\,n3\,n4\,rZ+n1\,n2^3\,n3\,n4\,rZ+n1\,n2\,n3^3\,n4\,rZ+n1^3\,n2^3\,n3^3\,n4^3\,rZ\big),$$

$$-\frac{n1^2+n2^2+n3^2+2\,n1^2\,n2^2\,n3^2\,n4^2}{n1^3\,n2^3\,n3^3\,n4^2}+\frac{1}{n1^3\,n2^3\,n3^3\,n4^2}\,\big(-cZn1234\,n1^2-cZn1234\,n2^2-cZn1234\,n3^2-cZn3\,n1^3\,n2\,n4-cZn3\,n1\,n2^3\,n4-cZn2\,n1^3\,n3\,n4+cZn1\,n1^2\,n2\,n3\,n4+cZn2\,n1\,n2^2\,n3\,n4-cZn1\,n2^3\,n3\,n4+cZn3\,n1\,n2\,n3^2\,n4-cZn2\,n1\,n3^3\,n4-cZn1\,n2\,n3^3\,n4+cZn1234\,n1^2\,n2^2\,n3^2\,n4^2-cZn3\,n1^3\,n2^3\,n3^2\,n4^3-cZn2\,n1^3\,n2^2\,n3^3\,n4^3-cZn1\,n1^2\,n2^3\,n3^3\,n4^3\big)+\frac{1}{n1^3\,n2^3\,n3^3}\,\big(cYn3^2\,n1^4\,n2^2+cYn3^2\,n1^2\,n2^4+cYn2\,cYn3\,n1^4\,n2\,n3-cYn1\,cYn3\,n1^3\,n2^2\,n3-cYn2\,cYn3\,n1^2\,n2^3\,n3+cYn1\,cYn3\,n1\,n2^4\,n3+cYn2^2\,n1^4\,n3^2-cYn1\,cYn2\,n1^3\,n2\,n3^2-cYn1\,cYn2\,n1\,n2^3\,n3^2+cYn1^2\,n2^4\,n3^2-cYn2\,cYn3\,n1^2\,n2\,n3^3-cYn1\,cYn3\,n1\,n2^2\,n3^3+cYn2^2\,n1^2\,n3^4+cYn1\,cYn2\,n1\,n2\,n3^4+cYn1^2\,n2^2\,n3^4+n1^2\,n2^2\,n3^2\,pY1\big)\Big\}$$

```
Factor[Solve[{
  (*******************************
   The following equations are given by the above computations of x^5 locally
   *******************************)
  d == -aP^5/(n1^3 n2^3 n3^3 n4^2) - aZ^5 cZn1234/(n1^3 n2^3 n3^3 n4^2) - aZ^5 cZn3/(n1^2 n2^2 n3^3 n4) - aZ^5 cZn2/(n1^2 n2^3 n3^2 n4) -
    aZ^5 cZn1/(n1^3 n2^2 n3^2 n4) + 5 aZ^4 rZ/(n1^2 n2^2 n3^2 n4),
  0 == -5 aP^4/(n1^3 n2^3 n3^3 n4^2) - 5 aZ^4 cZn1234/(n1^3 n2^3 n3^3 n4^2) - 5 aZ^4 cZn3/(n1^2 n2^2 n3^3 n4) - 5 aZ^4 cZn2/(n1^2 n2^3 n3^2 n4) -
    5 aZ^4 cZn1/(n1^3 n2^2 n3^2 n4) + 20 aZ^3 rZ/(n1^2 n2^2 n3^2 n4),
  0 == -10 aP^3/(n1^3 n2^3 n3^3 n4^2) - 10 aZ^3 cZn1234/(n1^3 n2^3 n3^3 n4^2) - 10 aZ^3 cZn3/(n1^2 n2^2 n3^3 n4) - 10 aZ^3 cZn2/(n1^2 n2^3 n3^2 n4) -
    10 aZ^3 cZn1/(n1^3 n2^2 n3^2 n4) + 30 aZ^2 rZ/(n1^2 n2^2 n3^2 n4)
 },
 {d, cZn1, rZ}]]
```

$$\left\{\left\{d \to -\frac{aP^3\,(aP - aZ)^2}{n1^3\,n2^3\,n3^3\,n4^2},\right.\right.$$

$$cZn1 \to -\frac{-3\,aP^4 + 4\,aP^3\,aZ + aZ^4\,cZn1234 + aZ^4\,cZn3\,n1\,n2\,n4 + aZ^4\,cZn2\,n1\,n3\,n4}{aZ^4\,n2\,n3\,n4},$$

$$\left.\left.rZ \to -\frac{aP^3\,(-aP + aZ)}{aZ^3\,n1\,n2\,n3\,n4}\right\}\right\}$$

```
Solve[{
  (******************************
   The following equations are given by Lemma 6.11 for N1
   ******************************)
```

$$0 == -\frac{3\,aP^2}{n1^3\,n2^2\,n3^2\,n4^2} + \frac{3\,rY^2}{n1} - \frac{3\,aZ\,eZN1\,(aZ\,cZn1234 + aZ\,cZn1\,n2\,n3\,n4 - 2\,n1\,n2\,n3\,n4\,rZ)}{n1^3\,n2^2\,n3^2\,n4^2},$$

$$0 == -\frac{3\,aP}{n1^3\,n2^2\,n3^2\,n4^2} - \frac{3\,cYn1\,rY}{n1^2} - \frac{3\,eZN1\,(aZ\,cZn1234 + aZ\,cZn1\,n2\,n3\,n4 - n1\,n2\,n3\,n4\,rZ)}{n1^3\,n2^2\,n3^2\,n4^2},$$

$$0 == \frac{cYn1^2}{n1^3} - \frac{1}{n1^3\,n2^2\,n3^2\,n4^2} - \frac{eZN1\,(cZn1234 + cZn1\,n2\,n3\,n4)}{n1^3\,n2^2\,n3^2\,n4^2},$$

$$0 == -\frac{1}{n1\,n2^2\,n3^2\,n4^2}\left(1 + cZn1234\,eZN1 - cZn1\,eZN1\,n2\,n3\,n4 + 2\,n2^2\,n3^2\,n4^2 - cZn1234\,eZN1\,n2^2\,n3^2\,n4^2 + cZn1\,eZN1\,n2^3\,n3^3\,n4^3 - n2^2\,n3^2\,n4^2\,pY1\right),$$

$$pY1 == 3,$$

```
  (******************************
   The following equations are given by Lemma 6.7 for M
   ******************************)
```

$$0 == \frac{2\,aP\,ePM}{n1^2\,n2^2\,n3^2\,n4^2} - \frac{2\,eZM\,(aZ\,cZn1234 - n1\,n2\,n3\,n4\,rZ)}{n1^2\,n2^2\,n3^2\,n4^2},$$

$$0 == \frac{ePM}{n1^2\,n2^2\,n3^2\,n4^2} - \frac{cZn1234\,eZM}{n1^2\,n2^2\,n3^2\,n4^2}$$

```
 },
 {rY, rZ, pY1, cYn1, cZn1234, cZn1, eZM}]
```

$$\Big\{\Big\{rY \to \frac{aP - aZ}{n1\,n2\,n3\,n4},\ rZ \to \frac{aP - aZ}{eZN1\,n1\,n2\,n3\,n4},\ pY1 \to 3,\ cYn1 \to 0,\ cZn1234 \to -\frac{1}{eZN1},\ cZn1 \to 0,\ eZM \to -ePM\,eZN1\Big\},$$

$$\Big\{rY \to \frac{-aP + aZ}{n1\,n2\,n3\,n4},\ rZ \to \frac{aP - aZ}{eZN1\,n1\,n2\,n3\,n4},\ pY1 \to 3,\ cYn1 \to 0,\ cZn1234 \to -\frac{1}{eZN1},\ cZn1 \to 0,\ eZM \to -ePM\,eZN1\Big\},$$

$$\Big\{rY \to -\frac{aP + aZ}{n1\,n2\,n3\,n4},\ rZ \to \frac{-aP + aZ}{eZN1\,n1\,n2\,n3\,n4},\ pY1 \to 3,\ cYn1 \to \frac{2}{n2\,n3\,n4},\ cZn1234 \to \frac{1}{eZN1},\ cZn1 \to \frac{2}{eZN1\,n2\,n3\,n4},\ eZM \to ePM\,eZN1\Big\},$$

$$\Big\{rY \to \frac{aP + aZ}{n1\,n2\,n3\,n4},\ rZ \to \frac{-aP + aZ}{eZN1\,n1\,n2\,n3\,n4},\ pY1 \to 3,\ cYn1 \to -\frac{2}{n2\,n3\,n4},\ cZn1234 \to \frac{1}{eZN1},\ cZn1 \to \frac{2}{eZN1\,n2\,n3\,n4},\ eZM \to ePM\,eZN1\Big\}\Big\}$$

```
(******************************
 It is now interesting to compare the values of rZ
 ******************************)
```

```
Factor[Solve[{
```

$$-\frac{aP^3\,(-aP + aZ)}{aZ^3\,n1\,n2\,n3\,n4} == -\frac{aP - aZ}{n1\,n2\,n3\,n4}$$

```
 }, {aP}]]
```

$$\Big\{\{aP \to -aZ\},\ \{aP \to aZ\},\ \Big\{aP \to \frac{1}{2}\left(1 - i\sqrt{3}\right) aZ\Big\},\ \Big\{aP \to \frac{1}{2}\left(1 + i\sqrt{3}\right) aZ\Big\}\Big\}$$

```
(******************************
 Since aP is not equal to aZ and needs to be an integer, we have that aP = -aZ
  ******************************)

Factor[Solve[{
   (******************************
    The following equations are give by the above computations
     ******************************)
   aP == -aZ, d == -(aP^3 (aP - aZ)^2)/(n1^3 n2^3 n3^3 n4^2),
   cZn1 == -(-3 aP^4 + 4 aP^3 aZ + aZ^4 cZn1234 + aZ^4 cZn3 n1 n2 n4 + aZ^4 cZn2 n1 n3 n4)/(aZ^4 n2 n3 n4),
   rZ == -(aP^3 (-aP + aZ))/(aZ^3 n1 n2 n3 n4),
   (******************************
    The following equations are given by the above computations of p1(X)x^3 locally
     ******************************)
   p d == -(aP^3 (n1^2 + n2^2 + n3^2 + 2 n1^2 n2^2 n3^2 n4^2))/(n1^3 n2^3 n3^3 n4^2) +
     1/(n1^3 n2^3 n3^3 n4^2) aZ^2
      (-aZ cZn1234 n1^2 - aZ cZn1234 n2^2 - aZ cZn1234 n3^2 - aZ cZn3 n1^3 n2 n4 -
        aZ cZn3 n1 n2^3 n4 - aZ cZn2 n1^3 n3 n4 + aZ cZn1 n1^2 n2 n3 n4 + aZ cZn2 n1 n2^2 n3 n4 -
        aZ cZn1 n2^3 n3 n4 + aZ cZn3 n1 n2 n3^2 n4 - aZ cZn2 n1 n3^3 n4 - aZ cZn1 n2 n3^3 n4 +
        aZ cZn1234 n1^2 n2^2 n3^2 n4^2 - aZ cZn3 n1^3 n2^3 n3^2 n4^3 - aZ cZn2 n1^3 n2^2 n3^3 n4^3 -
        aZ cZn1 n1^2 n2^3 n3^3 n4^3 + 3 n1^3 n2 n3 n4 rZ + 3 n1 n2^3 n3 n4 rZ + 3 n1 n2 n3^3 n4 rZ +
        3 n1^3 n2^3 n3^3 n4^3 rZ),
   (******************************
    The following equations are given by Lemma 6.11 for N1
     ******************************)
   0 == -(3 aP^2)/(n1^3 n2^2 n3^2 n4^2) + (3 rY^2)/n1 - (3 aZ eZN1 (aZ cZn1234 + aZ cZn1 n2 n3 n4 - 2 n1 n2 n3 n4 rZ))/(n1^3 n2^2 n3^2 n4^2),
   0 == -(3 aP)/(n1^3 n2^2 n3^2 n4^2) - (3 cYn1 rY)/n1^2 - (3 eZN1 (aZ cZn1234 + aZ cZn1 n2 n3 n4 - n1 n2 n3 n4 rZ))/(n1^3 n2^2 n3^2 n4^2),
   0 == cYn1^2/n1^3 - 1/(n1^3 n2^2 n3^2 n4^2) - (eZN1 (cZn1234 + cZn1 n2 n3 n4))/(n1^3 n2^2 n3^2 n4^2),
   0 == -1/(n1 n2^2 n3^2 n4^2)
     (1 + cZn1234 eZN1 - cZn1 eZN1 n2 n3 n4 + 2 n2^2 n3^2 n4^2 - cZn1234 eZN1 n2^2 n3^2 n4^2 +
       cZn1 eZN1 n2^3 n3^3 n4^3 - n2^2 n3^2 n4^2 pY1),
   pY1 == 3, eZN1^2 == 1,
   (******************************
    The following equations are given by Lemma 6.11 for N2
     ******************************)
   0 == -(3 aP^2)/(n1^2 n2^3 n3^2 n4^2) + (3 rY^2)/n2 - (3 aZ eZN2 (aZ cZn1234 + aZ cZn2 n1 n3 n4 - 2 n1 n2 n3 n4 rZ))/(n1^2 n2^3 n3^2 n4^2),
   0 == -(3 aP)/(n1^2 n2^3 n3^2 n4^2) - (3 cYn2 rY)/n2^2 - (3 eZN2 (aZ cZn1234 + aZ cZn2 n1 n3 n4 - n1 n2 n3 n4 rZ))/(n1^2 n2^3 n3^2 n4^2),
   0 == cYn2^2/n2^3 - 1/(n1^2 n2^3 n3^2 n4^2) - (eZN2 (cZn1234 + cZn2 n1 n3 n4))/(n1^2 n2^3 n3^2 n4^2),
   0 == -1/(n1^2 n2 n3^2 n4^2)
     (1 + cZn1234 eZN2 - cZn2 eZN2 n1 n3 n4 + 2 n1^2 n3^2 n4^2 - cZn1234 eZN2 n1^2 n3^2 n4^2 +
       cZn2 eZN2 n1^3 n3^3 n4^3 - n1^2 n3^2 n4^2 pY1),
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$$eZN2^2 == 1,$$

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(*******************************
 The following equations are given by Lemma 6.11 for N3
 *******************************)
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$$0 == -\frac{3\, aP^2}{n1^2\, n2^2\, n3^3\, n4^2} + \frac{3\, rY^2}{n3} - \frac{3\, aZ\, eZN3\, (aZ\, cZn1234 + aZ\, cZn3\, n1\, n2\, n4 - 2\, n1\, n2\, n3\, n4\, rZ)}{n1^2\, n2^2\, n3^3\, n4^2},$$

$$0 == -\frac{3\, aP}{n1^2\, n2^2\, n3^3\, n4^2} - \frac{3\, cYn3\, rY}{n3^2} - \frac{3\, eZN3\, (aZ\, cZn1234 + aZ\, cZn3\, n1\, n2\, n4 - n1\, n2\, n3\, n4\, rZ)}{n1^2\, n2^2\, n3^3\, n4^2},$$

$$0 == \frac{cYn3^2}{n3^3} - \frac{1}{n1^2\, n2^2\, n3^3\, n4^2} - \frac{eZN3\, (cZn1234 + cZn3\, n1\, n2\, n4)}{n1^2\, n2^2\, n3^3\, n4^2},$$

$$0 == -\frac{1}{n1^2\, n2^2\, n3\, n4^2} \left(1 + cZn1234\, eZN3 - cZn3\, eZN3\, n1\, n2\, n4 + 2\, n1^2\, n2^2\, n4^2 - cZn1234\, eZN3\, n1^2\, n2^2\, n4^2 + cZn3\, eZN3\, n1^3\, n2^3\, n4^3 - n1^2\, n2^2\, n4^2\, pY1\right),$$

$$eZN2^2 == 1,$$

```
(*******************************
 The following equations are given by Lemma 6.7 for M
 *******************************)
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$$0 == \frac{2\, aP\, ePM}{n1^2\, n2^2\, n3^2\, n4^2} - \frac{2\, eZM\, (aZ\, cZn1234 - n1\, n2\, n3\, n4\, rZ)}{n1^2\, n2^2\, n3^2\, n4^2},$$

$$0 == \frac{ePM}{n1^2\, n2^2\, n3^2\, n4^2} - \frac{cZn1234\, eZM}{n1^2\, n2^2\, n3^2\, n4^2},$$

$$eZM^2 == 1,\ ePM^2 == 1$$

},

{p, d, cZn1234, cZn1, cZn2, cZn3, cYn1, cYn2, cYn3, rZ, rY, aP, eZM, ePM, eZN1, eZN2, eZN3, pY1}]]

$$\left\{\left\{p \to \frac{4\, n1^2 + 4\, n2^2 + 4\, n3^2 + 3\, n1^2\, n2^2\, n3^2\, n4^2}{4\, aZ^2},\ d \to \frac{4\, aZ^5}{n1^3\, n2^3\, n3^3\, n4^2},\ cZn1234 \to 1,\ cZn1 \to \frac{2}{n2\, n3\, n4},\ cZn2 \to \frac{2}{n1\, n3\, n4},\ cZn3 \to \frac{2}{n1\, n2\, n4},\ cYn1 \to -\frac{2}{n2\, n3\, n4},\ cYn2 \to -\frac{2}{n1\, n3\, n4},\ cYn3 \to -\frac{2}{n1\, n2\, n4},\ rZ \to \frac{2\, aZ}{n1\, n2\, n3\, n4},\ rY \to 0,\ aP \to -aZ,\ eZM \to -1,\ ePM \to -1,\ eZN1 \to 1,\ eZN2 \to 1,\ eZN3 \to 1,\ pY1 \to 3\right\},\right.$$

$$\left\{p \to \frac{4\, n1^2 + 4\, n2^2 + 4\, n3^2 + 3\, n1^2\, n2^2\, n3^2\, n4^2}{4\, aZ^2},\ d \to \frac{4\, aZ^5}{n1^3\, n2^3\, n3^3\, n4^2},\ cZn1234 \to 1,\ cZn1 \to \frac{2}{n2\, n3\, n4},\ cZn2 \to \frac{2}{n1\, n3\, n4},\ cZn3 \to \frac{2}{n1\, n2\, n4},\ cYn1 \to -\frac{2}{n2\, n3\, n4},\ cYn2 \to -\frac{2}{n1\, n3\, n4},\ cYn3 \to -\frac{2}{n1\, n2\, n4},\ rZ \to \frac{2\, aZ}{n1\, n2\, n3\, n4},\ rY \to 0,\ aP \to -aZ,\ eZM \to 1,\ ePM \to 1,\ eZN1 \to 1,\ eZN2 \to 1,\ eZN3 \to 1,\ pY1 \to 3\right\},$$

$$\left\{p \to \frac{4\, n1^2 + 4\, n2^2 + 4\, n3^2 + 3\, n1^2\, n2^2\, n3^2\, n4^2}{4\, aZ^2},\ d \to \frac{4\, aZ^5}{n1^3\, n2^3\, n3^3\, n4^2},\ cZn1234 \to 1,\ cZn1 \to \frac{2}{n2\, n3\, n4},\ cZn2 \to \frac{2}{n1\, n3\, n4},\ cZn3 \to \frac{2}{n1\, n2\, n4},\ cYn1 \to -\frac{2}{n2\, n3\, n4},\ cYn2 \to -\frac{2}{n1\, n3\, n4},\ cYn3 \to \frac{2}{n1\, n2\, n4},\ rZ \to \frac{2\, aZ}{n1\, n2\, n3\, n4},\ rY \to 0,\ aP \to -aZ,\ eZM \to -1,\ ePM \to -1,\ eZN1 \to 1,\ eZN2 \to 1,\ eZN3 \to 1,\ pY1 \to 3\right\},$$

$$\left\{p \to \frac{4\, n1^2 + 4\, n2^2 + 4\, n3^2 + 3\, n1^2\, n2^2\, n3^2\, n4^2}{4\, aZ^2},\ d \to \frac{4\, aZ^5}{n1^3\, n2^3\, n3^3\, n4^2},\ cZn1234 \to 1,\ cZn1 \to \frac{2}{n2\, n3\, n4},\ cZn2 \to \frac{2}{n1\, n3\, n4},\ cZn3 \to \frac{2}{n1\, n2\, n4},\ cYn1 \to -\frac{2}{n2\, n3\, n4},\right.$$

$$\mathrm{cYn2}\to -\frac{2}{\mathrm{n1\,n3\,n4}},\ \mathrm{cYn3}\to \frac{2}{\mathrm{n1\,n2\,n4}},\ \mathrm{rZ}\to \frac{2\,\mathrm{aZ}}{\mathrm{n1\,n2\,n3\,n4}},\ \mathrm{rY}\to 0,$$
$$\mathrm{aP}\to -\mathrm{aZ},\ \mathrm{eZM}\to 1,\ \mathrm{ePM}\to 1,\ \mathrm{eZN1}\to 1,\ \mathrm{eZN2}\to 1,\ \mathrm{eZN3}\to 1,\ \mathrm{pY1}\to 3\Big\},$$

$$\Big\{\mathrm{p}\to \frac{4\,\mathrm{n1}^2+4\,\mathrm{n2}^2+4\,\mathrm{n3}^2+3\,\mathrm{n1}^2\,\mathrm{n2}^2\,\mathrm{n3}^2\,\mathrm{n4}^2}{4\,\mathrm{aZ}^2},\ \mathrm{d}\to \frac{4\,\mathrm{aZ}^5}{\mathrm{n1}^3\,\mathrm{n2}^3\,\mathrm{n3}^3\,\mathrm{n4}^2},\ \mathrm{cZn1234}\to 1,$$
$$\mathrm{cZn1}\to \frac{2}{\mathrm{n2\,n3\,n4}},\ \mathrm{cZn2}\to \frac{2}{\mathrm{n1\,n3\,n4}},\ \mathrm{cZn3}\to \frac{2}{\mathrm{n1\,n2\,n4}},\ \mathrm{cYn1}\to -\frac{2}{\mathrm{n2\,n3\,n4}},$$
$$\mathrm{cYn2}\to \frac{2}{\mathrm{n1\,n3\,n4}},\ \mathrm{cYn3}\to -\frac{2}{\mathrm{n1\,n2\,n4}},\ \mathrm{rZ}\to \frac{2\,\mathrm{aZ}}{\mathrm{n1\,n2\,n3\,n4}},\ \mathrm{rY}\to 0,$$
$$\mathrm{aP}\to -\mathrm{aZ},\ \mathrm{eZM}\to -1,\ \mathrm{ePM}\to -1,\ \mathrm{eZN1}\to 1,\ \mathrm{eZN2}\to 1,\ \mathrm{eZN3}\to 1,\ \mathrm{pY1}\to 3\Big\},$$

$$\Big\{\mathrm{p}\to \frac{4\,\mathrm{n1}^2+4\,\mathrm{n2}^2+4\,\mathrm{n3}^2+3\,\mathrm{n1}^2\,\mathrm{n2}^2\,\mathrm{n3}^2\,\mathrm{n4}^2}{4\,\mathrm{aZ}^2},\ \mathrm{d}\to \frac{4\,\mathrm{aZ}^5}{\mathrm{n1}^3\,\mathrm{n2}^3\,\mathrm{n3}^3\,\mathrm{n4}^2},\ \mathrm{cZn1234}\to 1,$$
$$\mathrm{cZn1}\to \frac{2}{\mathrm{n2\,n3\,n4}},\ \mathrm{cZn2}\to \frac{2}{\mathrm{n1\,n3\,n4}},\ \mathrm{cZn3}\to \frac{2}{\mathrm{n1\,n2\,n4}},\ \mathrm{cYn1}\to -\frac{2}{\mathrm{n2\,n3\,n4}},$$
$$\mathrm{cYn2}\to \frac{2}{\mathrm{n1\,n3\,n4}},\ \mathrm{cYn3}\to -\frac{2}{\mathrm{n1\,n2\,n4}},\ \mathrm{rZ}\to \frac{2\,\mathrm{aZ}}{\mathrm{n1\,n2\,n3\,n4}},\ \mathrm{rY}\to 0,$$
$$\mathrm{aP}\to -\mathrm{aZ},\ \mathrm{eZM}\to 1,\ \mathrm{ePM}\to 1,\ \mathrm{eZN1}\to 1,\ \mathrm{eZN2}\to 1,\ \mathrm{eZN3}\to 1,\ \mathrm{pY1}\to 3\Big\},$$

$$\Big\{\mathrm{p}\to \frac{4\,\mathrm{n1}^2+4\,\mathrm{n2}^2+4\,\mathrm{n3}^2+3\,\mathrm{n1}^2\,\mathrm{n2}^2\,\mathrm{n3}^2\,\mathrm{n4}^2}{4\,\mathrm{aZ}^2},\ \mathrm{d}\to \frac{4\,\mathrm{aZ}^5}{\mathrm{n1}^3\,\mathrm{n2}^3\,\mathrm{n3}^3\,\mathrm{n4}^2},\ \mathrm{cZn1234}\to 1,$$
$$\mathrm{cZn1}\to \frac{2}{\mathrm{n2\,n3\,n4}},\ \mathrm{cZn2}\to \frac{2}{\mathrm{n1\,n3\,n4}},\ \mathrm{cZn3}\to \frac{2}{\mathrm{n1\,n2\,n4}},\ \mathrm{cYn1}\to -\frac{2}{\mathrm{n2\,n3\,n4}},$$
$$\mathrm{cYn2}\to \frac{2}{\mathrm{n1\,n3\,n4}},\ \mathrm{cYn3}\to \frac{2}{\mathrm{n1\,n2\,n4}},\ \mathrm{rZ}\to \frac{2\,\mathrm{aZ}}{\mathrm{n1\,n2\,n3\,n4}},\ \mathrm{rY}\to 0,\ \mathrm{aP}\to -\mathrm{aZ},$$
$$\mathrm{eZM}\to -1,\ \mathrm{ePM}\to -1,\ \mathrm{eZN1}\to 1,\ \mathrm{eZN2}\to 1,\ \mathrm{eZN3}\to 1,\ \mathrm{pY1}\to 3\Big\},$$

$$\Big\{\mathrm{p}\to \frac{4\,\mathrm{n1}^2+4\,\mathrm{n2}^2+4\,\mathrm{n3}^2+3\,\mathrm{n1}^2\,\mathrm{n2}^2\,\mathrm{n3}^2\,\mathrm{n4}^2}{4\,\mathrm{aZ}^2},\ \mathrm{d}\to \frac{4\,\mathrm{aZ}^5}{\mathrm{n1}^3\,\mathrm{n2}^3\,\mathrm{n3}^3\,\mathrm{n4}^2},\ \mathrm{cZn1234}\to 1,$$
$$\mathrm{cZn1}\to \frac{2}{\mathrm{n2\,n3\,n4}},\ \mathrm{cZn2}\to \frac{2}{\mathrm{n1\,n3\,n4}},\ \mathrm{cZn3}\to \frac{2}{\mathrm{n1\,n2\,n4}},\ \mathrm{cYn1}\to -\frac{2}{\mathrm{n2\,n3\,n4}},$$
$$\mathrm{cYn2}\to \frac{2}{\mathrm{n1\,n3\,n4}},\ \mathrm{cYn3}\to \frac{2}{\mathrm{n1\,n2\,n4}},\ \mathrm{rZ}\to \frac{2\,\mathrm{aZ}}{\mathrm{n1\,n2\,n3\,n4}},\ \mathrm{rY}\to 0,$$
$$\mathrm{aP}\to -\mathrm{aZ},\ \mathrm{eZM}\to 1,\ \mathrm{ePM}\to 1,\ \mathrm{eZN1}\to 1,\ \mathrm{eZN2}\to 1,\ \mathrm{eZN3}\to 1,\ \mathrm{pY1}\to 3\Big\},$$

$$\Big\{\mathrm{p}\to \frac{4\,\mathrm{n1}^2+4\,\mathrm{n2}^2+4\,\mathrm{n3}^2+3\,\mathrm{n1}^2\,\mathrm{n2}^2\,\mathrm{n3}^2\,\mathrm{n4}^2}{4\,\mathrm{aZ}^2},\ \mathrm{d}\to \frac{4\,\mathrm{aZ}^5}{\mathrm{n1}^3\,\mathrm{n2}^3\,\mathrm{n3}^3\,\mathrm{n4}^2},\ \mathrm{cZn1234}\to 1,$$
$$\mathrm{cZn1}\to \frac{2}{\mathrm{n2\,n3\,n4}},\ \mathrm{cZn2}\to \frac{2}{\mathrm{n1\,n3\,n4}},\ \mathrm{cZn3}\to \frac{2}{\mathrm{n1\,n2\,n4}},\ \mathrm{cYn1}\to \frac{2}{\mathrm{n2\,n3\,n4}},$$
$$\mathrm{cYn2}\to -\frac{2}{\mathrm{n1\,n3\,n4}},\ \mathrm{cYn3}\to -\frac{2}{\mathrm{n1\,n2\,n4}},\ \mathrm{rZ}\to \frac{2\,\mathrm{aZ}}{\mathrm{n1\,n2\,n3\,n4}},\ \mathrm{rY}\to 0,$$
$$\mathrm{aP}\to -\mathrm{aZ},\ \mathrm{eZM}\to -1,\ \mathrm{ePM}\to -1,\ \mathrm{eZN1}\to 1,\ \mathrm{eZN2}\to 1,\ \mathrm{eZN3}\to 1,\ \mathrm{pY1}\to 3\Big\},$$

$$\Big\{\mathrm{p}\to \frac{4\,\mathrm{n1}^2+4\,\mathrm{n2}^2+4\,\mathrm{n3}^2+3\,\mathrm{n1}^2\,\mathrm{n2}^2\,\mathrm{n3}^2\,\mathrm{n4}^2}{4\,\mathrm{aZ}^2},\ \mathrm{d}\to \frac{4\,\mathrm{aZ}^5}{\mathrm{n1}^3\,\mathrm{n2}^3\,\mathrm{n3}^3\,\mathrm{n4}^2},\ \mathrm{cZn1234}\to 1,$$
$$\mathrm{cZn1}\to \frac{2}{\mathrm{n2\,n3\,n4}},\ \mathrm{cZn2}\to \frac{2}{\mathrm{n1\,n3\,n4}},\ \mathrm{cZn3}\to \frac{2}{\mathrm{n1\,n2\,n4}},\ \mathrm{cYn1}\to \frac{2}{\mathrm{n2\,n3\,n4}},$$
$$\mathrm{cYn2}\to -\frac{2}{\mathrm{n1\,n3\,n4}},\ \mathrm{cYn3}\to -\frac{2}{\mathrm{n1\,n2\,n4}},\ \mathrm{rZ}\to \frac{2\,\mathrm{aZ}}{\mathrm{n1\,n2\,n3\,n4}},\ \mathrm{rY}\to 0,$$
$$\mathrm{aP}\to -\mathrm{aZ},\ \mathrm{eZM}\to 1,\ \mathrm{ePM}\to 1,\ \mathrm{eZN1}\to 1,\ \mathrm{eZN2}\to 1,\ \mathrm{eZN3}\to 1,\ \mathrm{pY1}\to 3\Big\},$$

$$\Big\{\mathrm{p}\to \frac{4\,\mathrm{n1}^2+4\,\mathrm{n2}^2+4\,\mathrm{n3}^2+3\,\mathrm{n1}^2\,\mathrm{n2}^2\,\mathrm{n3}^2\,\mathrm{n4}^2}{4\,\mathrm{aZ}^2},\ \mathrm{d}\to \frac{4\,\mathrm{aZ}^5}{\mathrm{n1}^3\,\mathrm{n2}^3\,\mathrm{n3}^3\,\mathrm{n4}^2},\ \mathrm{cZn1234}\to 1,$$
$$\mathrm{cZn1}\to \frac{2}{\mathrm{n2\,n3\,n4}},\ \mathrm{cZn2}\to \frac{2}{\mathrm{n1\,n3\,n4}},\ \mathrm{cZn3}\to \frac{2}{\mathrm{n1\,n2\,n4}},\ \mathrm{cYn1}\to \frac{2}{\mathrm{n2\,n3\,n4}},$$
$$\mathrm{cYn2}\to -\frac{2}{\mathrm{n1\,n3\,n4}},\ \mathrm{cYn3}\to \frac{2}{\mathrm{n1\,n2\,n4}},\ \mathrm{rZ}\to \frac{2\,\mathrm{aZ}}{\mathrm{n1\,n2\,n3\,n4}},\ \mathrm{rY}\to 0,$$

$$\text{aP}\to -\text{aZ},\ \text{eZM}\to -1,\ \text{ePM}\to -1,\ \text{eZN1}\to 1,\ \text{eZN2}\to 1,\ \text{eZN3}\to 1,\ \text{pY1}\to 3\Big\},$$

$$\Big\{\text{p}\to \frac{4\,\text{n1}^2+4\,\text{n2}^2+4\,\text{n3}^2+3\,\text{n1}^2\,\text{n2}^2\,\text{n3}^2\,\text{n4}^2}{4\,\text{aZ}^2},\ \text{d}\to \frac{4\,\text{aZ}^5}{\text{n1}^3\,\text{n2}^3\,\text{n3}^3\,\text{n4}^2},\ \text{cZn1234}\to 1,$$
$$\text{cZn1}\to \frac{2}{\text{n2 n3 n4}},\ \text{cZn2}\to \frac{2}{\text{n1 n3 n4}},\ \text{cZn3}\to \frac{2}{\text{n1 n2 n4}},\ \text{cYn1}\to \frac{2}{\text{n2 n3 n4}},$$
$$\text{cYn2}\to -\frac{2}{\text{n1 n3 n4}},\ \text{cYn3}\to \frac{2}{\text{n1 n2 n4}},\ \text{rZ}\to \frac{2\,\text{aZ}}{\text{n1 n2 n3 n4}},\ \text{rY}\to 0,$$
$$\text{aP}\to -\text{aZ},\ \text{eZM}\to 1,\ \text{ePM}\to 1,\ \text{eZN1}\to 1,\ \text{eZN2}\to 1,\ \text{eZN3}\to 1,\ \text{pY1}\to 3\Big\},$$

$$\Big\{\text{p}\to \frac{4\,\text{n1}^2+4\,\text{n2}^2+4\,\text{n3}^2+3\,\text{n1}^2\,\text{n2}^2\,\text{n3}^2\,\text{n4}^2}{4\,\text{aZ}^2},\ \text{d}\to \frac{4\,\text{aZ}^5}{\text{n1}^3\,\text{n2}^3\,\text{n3}^3\,\text{n4}^2},\ \text{cZn1234}\to 1,$$
$$\text{cZn1}\to \frac{2}{\text{n2 n3 n4}},\ \text{cZn2}\to \frac{2}{\text{n1 n3 n4}},\ \text{cZn3}\to \frac{2}{\text{n1 n2 n4}},\ \text{cYn1}\to \frac{2}{\text{n2 n3 n4}},$$
$$\text{cYn2}\to \frac{2}{\text{n1 n3 n4}},\ \text{cYn3}\to -\frac{2}{\text{n1 n2 n4}},\ \text{rZ}\to \frac{2\,\text{aZ}}{\text{n1 n2 n3 n4}},\ \text{rY}\to 0,$$
$$\text{aP}\to -\text{aZ},\ \text{eZM}\to -1,\ \text{ePM}\to -1,\ \text{eZN1}\to 1,\ \text{eZN2}\to 1,\ \text{eZN3}\to 1,\ \text{pY1}\to 3\Big\},$$

$$\Big\{\text{p}\to \frac{4\,\text{n1}^2+4\,\text{n2}^2+4\,\text{n3}^2+3\,\text{n1}^2\,\text{n2}^2\,\text{n3}^2\,\text{n4}^2}{4\,\text{aZ}^2},\ \text{d}\to \frac{4\,\text{aZ}^5}{\text{n1}^3\,\text{n2}^3\,\text{n3}^3\,\text{n4}^2},\ \text{cZn1234}\to 1,$$
$$\text{cZn1}\to \frac{2}{\text{n2 n3 n4}},\ \text{cZn2}\to \frac{2}{\text{n1 n3 n4}},\ \text{cZn3}\to \frac{2}{\text{n1 n2 n4}},\ \text{cYn1}\to \frac{2}{\text{n2 n3 n4}},$$
$$\text{cYn2}\to \frac{2}{\text{n1 n3 n4}},\ \text{cYn3}\to -\frac{2}{\text{n1 n2 n4}},\ \text{rZ}\to \frac{2\,\text{aZ}}{\text{n1 n2 n3 n4}},\ \text{rY}\to 0,$$
$$\text{aP}\to -\text{aZ},\ \text{eZM}\to 1,\ \text{ePM}\to 1,\ \text{eZN1}\to 1,\ \text{eZN2}\to 1,\ \text{eZN3}\to 1,\ \text{pY1}\to 3\Big\},$$

$$\Big\{\text{p}\to \frac{4\,\text{n1}^2+4\,\text{n2}^2+4\,\text{n3}^2+3\,\text{n1}^2\,\text{n2}^2\,\text{n3}^2\,\text{n4}^2}{4\,\text{aZ}^2},\ \text{d}\to \frac{4\,\text{aZ}^5}{\text{n1}^3\,\text{n2}^3\,\text{n3}^3\,\text{n4}^2},\ \text{cZn1234}\to 1,$$
$$\text{cZn1}\to \frac{2}{\text{n2 n3 n4}},\ \text{cZn2}\to \frac{2}{\text{n1 n3 n4}},\ \text{cZn3}\to \frac{2}{\text{n1 n2 n4}},\ \text{cYn1}\to \frac{2}{\text{n2 n3 n4}},$$
$$\text{cYn2}\to \frac{2}{\text{n1 n3 n4}},\ \text{cYn3}\to \frac{2}{\text{n1 n2 n4}},\ \text{rZ}\to \frac{2\,\text{aZ}}{\text{n1 n2 n3 n4}},\ \text{rY}\to 0,\ \text{aP}\to -\text{aZ},$$
$$\text{eZM}\to -1,\ \text{ePM}\to -1,\ \text{eZN1}\to 1,\ \text{eZN2}\to 1,\ \text{eZN3}\to 1,\ \text{pY1}\to 3\Big\},$$

$$\Big\{\text{p}\to \frac{4\,\text{n1}^2+4\,\text{n2}^2+4\,\text{n3}^2+3\,\text{n1}^2\,\text{n2}^2\,\text{n3}^2\,\text{n4}^2}{4\,\text{aZ}^2},\ \text{d}\to \frac{4\,\text{aZ}^5}{\text{n1}^3\,\text{n2}^3\,\text{n3}^3\,\text{n4}^2},\ \text{cZn1234}\to 1,$$
$$\text{cZn1}\to \frac{2}{\text{n2 n3 n4}},\ \text{cZn2}\to \frac{2}{\text{n1 n3 n4}},\ \text{cZn3}\to \frac{2}{\text{n1 n2 n4}},\ \text{cYn1}\to \frac{2}{\text{n2 n3 n4}},$$
$$\text{cYn2}\to \frac{2}{\text{n1 n3 n4}},\ \text{cYn3}\to \frac{2}{\text{n1 n2 n4}},\ \text{rZ}\to \frac{2\,\text{aZ}}{\text{n1 n2 n3 n4}},\ \text{rY}\to 0,$$
$$\text{aP}\to -\text{aZ},\ \text{eZM}\to 1,\ \text{ePM}\to 1,\ \text{eZN1}\to 1,\ \text{eZN2}\to 1,\ \text{eZN3}\to 1,\ \text{pY1}\to 3\Big\}\Big\}$$

Computations for Lemma 6.14.
The case there exists N^6 s.t. Fix $(N) = Y^4 \cup P^0$.

Case B.1 where the other normal weights are trivial.

```
(******************************
  NOTATIONS
  rY xY denotes the restriction of x to Y
  rZ xZ denotes the restriction of x to Z
  aZ denotes the Hopf weight at Z
  aP denotes the Hopf weight at P
  cYn1 xY denotes the first Chern class of the normal bundle of Y in the
  direction of N
  cYj xY^j denotes the j-th Chern class of the normal bundle of Y in the
  remaining directions
  cZ1 xZ denotes the first Chern class of the normal bundle of Z
  μx5YB1 denotes the local datum of x^5 at Y
  μx5ZB1 denotes the local datum of x^5 at Z
  μx5PB1 denotes the local datum of x^5 at P

  We use the variable t to extract the coefficient of order 2 of xY and
  of order 1 of xZ. We then evaluate xY^2 and xZ to 1.
  The orientation of the point P is -1, this explains the negative sign.
  Once evaluated, we extract all the coefficients in l into a list.
  ******************************)

μx5YB1 =
 Factor[CoefficientList[
   Coefficient[Expand[(rY xY t + l z)^5] Series[(cYn1 xY t + n1 z)^-1, {xY, 0, 2}]
     Series[(cY2 xY^2 t^2 + cY1 xY t z + z^2)^-1, {xY, 0, 2}], t, 2], l]] /. xY^2 → 1
```

$$\left\{0,\ 0,\ 0,\ \frac{10\,\mathrm{rY}^2}{\mathrm{n1}},\ -\frac{5\,(\mathrm{cYn1}+\mathrm{cY1}\,\mathrm{n1})\,\mathrm{rY}}{\mathrm{n1}^2},\ \frac{\mathrm{cYn1}^2+\mathrm{cY1}\,\mathrm{cYn1}\,\mathrm{n1}+\mathrm{cY1}^2\,\mathrm{n1}^2-\mathrm{cY2}\,\mathrm{n1}^2}{\mathrm{n1}^3}\right\}$$

```
μx5ZB1 =
 CoefficientList[
   Coefficient[Expand[(rZ xZ t + (aZ + l) z)^5] Series[(cZ1 xZ t + z)^-1, {xZ, 0, 1}] z^-3,
    t, 1], l] /. xZ → 1
```

$$\{-\mathrm{aZ}^5\,\mathrm{cZ1}+5\,\mathrm{aZ}^4\,\mathrm{rZ},\ -5\,\mathrm{aZ}^4\,\mathrm{cZ1}+20\,\mathrm{aZ}^3\,\mathrm{rZ},$$
$$-10\,\mathrm{aZ}^3\,\mathrm{cZ1}+30\,\mathrm{aZ}^2\,\mathrm{rZ},\ -10\,\mathrm{aZ}^2\,\mathrm{cZ1}+20\,\mathrm{aZ}\,\mathrm{rZ},\ -5\,\mathrm{aZ}\,\mathrm{cZ1}+5\,\mathrm{rZ},\ -\mathrm{cZ1}\}$$

```
μx5PB1 = CoefficientList[Expand[-((aP + l) z)^5] z^-1 z^-1 (n1 z)^-1 (n1 z)^-1 (n1 z)^-1, l]
```

$$\left\{-\frac{\mathrm{aP}^5}{\mathrm{n1}^3},\ -\frac{5\,\mathrm{aP}^4}{\mathrm{n1}^3},\ -\frac{10\,\mathrm{aP}^3}{\mathrm{n1}^3},\ -\frac{10\,\mathrm{aP}^2}{\mathrm{n1}^3},\ -\frac{5\,\mathrm{aP}}{\mathrm{n1}^3},\ -\frac{1}{\mathrm{n1}^3}\right\}$$

```
(******************************
 NOTATIONS
 μs1x3YB1 denotes the local datum of p1(X)x^3 at Y
 μs1x3ZB1 denotes the local datum of p1(X)x^3 at Z
 μs1x3PB1 denotes the local datum of p1(X)x^3 at P
  ******************************)

SymmetricReduction[Expand[(cYn1 xY t + n1 z)^2 + (yY2 t + z)^2 + (yY3 t + z)^2],
 {yY2, yY3}, {cY1 xY, cY2 xY^2}]
```

$$\{\mathrm{cY1}^2\,t^2\,\mathrm{xY}^2-2\,\mathrm{cY2}\,t^2\,\mathrm{xY}^2+\mathrm{cYn1}^2\,t^2\,\mathrm{xY}^2+2\,\mathrm{cY1}\,t\,\mathrm{xY}\,z+2\,\mathrm{cYn1}\,\mathrm{n1}\,t\,\mathrm{xY}\,z+2\,z^2+\mathrm{n1}^2\,z^2,\ 0\}$$

```
μs1x3YB1 =
 Factor[CoefficientList[
     Coefficient[
      (pY1 t^2 + cY1^2 t^2 xY^2 - 2 cY2 t^2 xY^2 + cYn1^2 t^2 xY^2 + 2 cY1 t xY z + 2 cYn1 n1 t xY z +
          2 z^2 + n1^2 z^2) Expand[(rY xY t + l z)^3] Series[(cYn1 xY t + n1 z)^-1, {xY, 0, 2}]
        Series[(cY2 xY^2 t^2 + cY1 xY t z + z^2)^-1, {xY, 0, 2}], t, 2], l]] /. xY^2 → 1
```

$$\left\{0,\ \frac{3\left(2+\mathrm{n1}^2\right)\mathrm{rY}^2}{\mathrm{n1}},\ \frac{3\left(-2\,\mathrm{cYn1}+\mathrm{cYn1}\,\mathrm{n1}^2-\mathrm{cY1}\,\mathrm{n1}^3\right)\mathrm{rY}}{\mathrm{n1}^2},\ \frac{2\,\mathrm{cYn1}^2+\mathrm{cY1}^2\,\mathrm{n1}^2-4\,\mathrm{cY2}\,\mathrm{n1}^2-\mathrm{cY1}\,\mathrm{cYn1}\,\mathrm{n1}^3+\mathrm{cY1}^2\,\mathrm{n1}^4-\mathrm{cY2}\,\mathrm{n1}^4+\mathrm{n1}^2\,\mathrm{pY1}}{\mathrm{n1}^3}\right\}$$

```
μs1x3ZB1 =
 CoefficientList[
   Coefficient[((cZ1 xZ t + z)^2 + 3 z^2) Expand[(rZ xZ t + (aZ + l) z)^3]
      Series[(cZ1 xZ t + z)^-1, {xZ, 0, 1}] z^-3, t, 1], l] /. xZ → 1
```

$$\left\{-2\,\mathrm{aZ}^3\,\mathrm{cZ1}+12\,\mathrm{aZ}^2\,\mathrm{rZ},\ -6\,\mathrm{aZ}^2\,\mathrm{cZ1}+24\,\mathrm{aZ}\,\mathrm{rZ},\ -6\,\mathrm{aZ}\,\mathrm{cZ1}+12\,\mathrm{rZ},\ -2\,\mathrm{cZ1}\right\}$$

```
μs1x3PB1 =
 Factor[CoefficientList[(z^2 + z^2 + (n1 z)^2 + (n1 z)^2 + (n1 z)^2) Expand[-((aP + l) z)^3]
     z^-1 z^-1 (n1 z)^-1 (n1 z)^-1 (n1 z)^-1, l]]
```

$$\left\{-\frac{\mathrm{aP}^3\left(2+3\,\mathrm{n1}^2\right)}{\mathrm{n1}^3},\ -\frac{3\,\mathrm{aP}^2\left(2+3\,\mathrm{n1}^2\right)}{\mathrm{n1}^3},\ -\frac{3\,\mathrm{aP}\left(2+3\,\mathrm{n1}^2\right)}{\mathrm{n1}^3},\ -\frac{2+3\,\mathrm{n1}^2}{\mathrm{n1}^3}\right\}$$

```
μx5YB1 + μx5ZB1 + μx5PB1
```

$$\left\{-\mathrm{aZ}^5\,\mathrm{cZ1}-\frac{\mathrm{aP}^5}{\mathrm{n1}^3}+5\,\mathrm{aZ}^4\,\mathrm{rZ},\ -5\,\mathrm{aZ}^4\,\mathrm{cZ1}-\frac{5\,\mathrm{aP}^4}{\mathrm{n1}^3}+20\,\mathrm{aZ}^3\,\mathrm{rZ},\ -10\,\mathrm{aZ}^3\,\mathrm{cZ1}-\frac{10\,\mathrm{aP}^3}{\mathrm{n1}^3}+30\,\mathrm{aZ}^2\,\mathrm{rZ},\right.$$
$$-10\,\mathrm{aZ}^2\,\mathrm{cZ1}-\frac{10\,\mathrm{aP}^2}{\mathrm{n1}^3}+\frac{10\,\mathrm{rY}^2}{\mathrm{n1}}+20\,\mathrm{aZ}\,\mathrm{rZ},\ -5\,\mathrm{aZ}\,\mathrm{cZ1}-\frac{5\,\mathrm{aP}}{\mathrm{n1}^3}-\frac{5\left(\mathrm{cYn1}+\mathrm{cY1}\,\mathrm{n1}\right)\mathrm{rY}}{\mathrm{n1}^2}+5\,\mathrm{rZ},$$
$$\left.-\mathrm{cZ1}-\frac{1}{\mathrm{n1}^3}+\frac{\mathrm{cYn1}^2+\mathrm{cY1}\,\mathrm{cYn1}\,\mathrm{n1}+\mathrm{cY1}^2\,\mathrm{n1}^2-\mathrm{cY2}\,\mathrm{n1}^2}{\mathrm{n1}^3}\right\}$$

```
μs1x3YB1 + μs1x3ZB1 + μs1x3PB1
```

$$\left\{-2\,\mathrm{aZ}^3\,\mathrm{cZ1}-\frac{\mathrm{aP}^3\left(2+3\,\mathrm{n1}^2\right)}{\mathrm{n1}^3}+12\,\mathrm{aZ}^2\,\mathrm{rZ},\right.$$
$$-6\,\mathrm{aZ}^2\,\mathrm{cZ1}-\frac{3\,\mathrm{aP}^2\left(2+3\,\mathrm{n1}^2\right)}{\mathrm{n1}^3}+\frac{3\left(2+\mathrm{n1}^2\right)\mathrm{rY}^2}{\mathrm{n1}}+24\,\mathrm{aZ}\,\mathrm{rZ},$$
$$-6\,\mathrm{aZ}\,\mathrm{cZ1}-\frac{3\,\mathrm{aP}\left(2+3\,\mathrm{n1}^2\right)}{\mathrm{n1}^3}+\frac{3\left(-2\,\mathrm{cYn1}+\mathrm{cYn1}\,\mathrm{n1}^2-\mathrm{cY1}\,\mathrm{n1}^3\right)\mathrm{rY}}{\mathrm{n1}^2}+12\,\mathrm{rZ},$$
$$\left.-2\,\mathrm{cZ1}-\frac{2+3\,\mathrm{n1}^2}{\mathrm{n1}^3}+\frac{2\,\mathrm{cYn1}^2+\mathrm{cY1}^2\,\mathrm{n1}^2-4\,\mathrm{cY2}\,\mathrm{n1}^2-\mathrm{cY1}\,\mathrm{cYn1}\,\mathrm{n1}^3+\mathrm{cY1}^2\,\mathrm{n1}^4-\mathrm{cY2}\,\mathrm{n1}^4+\mathrm{n1}^2\,\mathrm{pY1}}{\mathrm{n1}^3}\right\}$$

```
Factor[Solve[{
   (*******************************
    The following equations are given by the above computations of x^5 locally
    *******************************)
   d == -aZ^5 cZ1 - aP^5/n1^3 + 5 aZ^4 rZ,
   0 == -5 aZ^4 cZ1 - 5 aP^4/n1^3 + 20 aZ^3 rZ,
   0 == -10 aZ^3 cZ1 - 10 aP^3/n1^3 + 30 aZ^2 rZ,
   0 == -10 aZ^2 cZ1 - 10 aP^2/n1^3 + 10 rY^2/n1 + 20 aZ rZ,
   (*******************************
    The following equation is given by the above computations of p1(X) x^3 locally
    *******************************)
   p d == -2 aZ^3 cZ1 - aP^3 (2 + 3 n1^2)/n1^3 + 12 aZ^2 rZ,
   (*******************************
    The following equations are given by Lemma 6.13
    *******************************)
   0 == -3 aP^2/n1^3 + 3 rY^2/n1,
   0 == -3 aP/n1^3 - 3 cYn1 rY/n1^2,
   0 == -1/n1^3 + cYn1^2/n1^3
  }, {p, d, cZ1, cYn1, rZ, rY, aP, n1}]]
```

$$\left\{\{d\to 0,\ cZ1\to 0,\ cYn1\to -1,\ rZ\to 0,\ rY\to 0,\ aP\to 0\},\right.$$
$$\{d\to 0,\ cZ1\to 0,\ cYn1\to 1,\ rZ\to 0,\ rY\to 0,\ aP\to 0\},$$
$$\left\{p\to \frac{3\left(-2+n1^2\right)}{aZ^2},\ d\to -\frac{8\,aZ^5}{n1^3},\ cZ1\to \frac{16}{n1^3},\ cYn1\to 1,\ rZ\to \frac{8\,aZ}{n1^3},\ rY\to -\frac{2\,aZ}{n1},\ aP\to 2\,aZ\right\},$$
$$\left.\left\{p\to \frac{3\left(-2+n1^2\right)}{aZ^2},\ d\to -\frac{8\,aZ^5}{n1^3},\ cZ1\to \frac{16}{n1^3},\ cYn1\to -1,\ rZ\to \frac{8\,aZ}{n1^3},\ rY\to \frac{2\,aZ}{n1},\ aP\to 2\,aZ\right\}\right\}$$

Computations for Lemma 6.14.
Case where there exists N^6 s.t. Fix $(N) = Y^4\, U\, P^0$.
Case B.2 where the other normal weights are not trivial.

```
(*******************************
  NOTATIONS
  rY xY denotes the restriction of x to Y
  rZ xZ denotes the restriction of x to Z
  aZ denotes the Hopf weight at Z
  aP denotes the Hopf weight at P
  cYn1 xY denotes the first Chern class of the normal bundle of Y in the
  direction of N
  cYj xY^j denotes the j-th Chern class of the normal bundle of Y in the
  remaining directions
  cZn2 xZ denotes the first Chern class of the normal bundle of Z in the
  direction of M
  cZ1 xZ denotes the first Chern class of the normal bundle of Z in the
  remaining directions
  μx5YB2 denotes the local datum of x^5 at Y
  μx5ZB2 denotes the local datum of x^5 at Z
  μx5PB2 denotes the local datum of x^5 at P

  We use the variable t to extract the coefficient of order 2 of xY and
  of order 1 of xZ. We then evaluate xY^2 and xZ to 1.
  The orientation of the point P is -1, this explains the negative sign.
  Once evaluated, we extract all the coefficients in l into a list.
  *******************************)
```

```
μx5YB2 =
 Factor[CoefficientList[
    Coefficient[Expand[(rY xY t + l z)^5] Series[(cYn1 xY t + n1 z)^-1, {xY, 0, 2}]
      Series[(cY2 xY^2 t^2 + cY1 xY t z + z^2)^-1, {xY, 0, 2}], t, 2], l]] /. xY^2 → 1
```

$$\left\{0,\ 0,\ 0,\ \frac{10\,\text{rY}^2}{\text{n1}},\ -\frac{5\,(\text{cYn1}+\text{cY1}\,\text{n1})\,\text{rY}}{\text{n1}^2},\ \frac{\text{cYn1}^2+\text{cY1}\,\text{cYn1}\,\text{n1}+\text{cY1}^2\,\text{n1}^2-\text{cY2}\,\text{n1}^2}{\text{n1}^3}\right\}$$

```
Simplify[Series[(cZn2 xZ t + n2 z)^-1, {xZ, 0, 1}] Series[(cZ1 xZ t + z)^-1, {xZ, 0, 1}] z^-2]
```

$$\frac{1}{\text{n2}\,z^4}-\frac{(\text{cZn2}+\text{cZ1}\,\text{n2})\,t\,\text{xZ}}{\text{n2}^2\,z^5}+O[\text{xZ}]^2$$

```
μx5ZB2 =
 CoefficientList[
   Coefficient[Expand[(rZ xZ t + (aZ + l) z)^5] Series[(cZn2 xZ t + n2 z)^-1, {xZ, 0, 1}]
     Series[(cZ1 xZ t + z)^-1, {xZ, 0, 1}] z^-2, t, 1], l] /. xZ → 1
```

$$\left\{-\frac{\text{aZ}^5\,\text{cZn2}}{\text{n2}^2}-\frac{\text{aZ}^5\,\text{cZ1}}{\text{n2}}+\frac{5\,\text{aZ}^4\,\text{rZ}}{\text{n2}},\ -\frac{5\,\text{aZ}^4\,\text{cZn2}}{\text{n2}^2}-\frac{5\,\text{aZ}^4\,\text{cZ1}}{\text{n2}}+\frac{20\,\text{aZ}^3\,\text{rZ}}{\text{n2}},\ -\frac{10\,\text{aZ}^3\,\text{cZn2}}{\text{n2}^2}-\frac{10\,\text{aZ}^3\,\text{cZ1}}{\text{n2}}+\frac{30\,\text{aZ}^2\,\text{rZ}}{\text{n2}},\ -\frac{10\,\text{aZ}^2\,\text{cZn2}}{\text{n2}^2}-\frac{10\,\text{aZ}^2\,\text{cZ1}}{\text{n2}}+\frac{20\,\text{aZ}\,\text{rZ}}{\text{n2}},\ -\frac{5\,\text{aZ}\,\text{cZn2}}{\text{n2}^2}-\frac{5\,\text{aZ}\,\text{cZ1}}{\text{n2}}+\frac{5\,\text{rZ}}{\text{n2}},\ -\frac{\text{cZn2}}{\text{n2}^2}-\frac{\text{cZ1}}{\text{n2}}\right\}$$

```
μx5PB2 = CoefficientList[Expand[-((aP + l) z)^5] (n1 z)^-1 (n1 z)^-1 (n1 z)^-1 (n2 z)^-1 (n2 z)^-1,
  l]
```

$$\left\{-\frac{\text{aP}^5}{\text{n1}^3\,\text{n2}^2},\ -\frac{5\,\text{aP}^4}{\text{n1}^3\,\text{n2}^2},\ -\frac{10\,\text{aP}^3}{\text{n1}^3\,\text{n2}^2},\ -\frac{10\,\text{aP}^2}{\text{n1}^3\,\text{n2}^2},\ -\frac{5\,\text{aP}}{\text{n1}^3\,\text{n2}^2},\ -\frac{1}{\text{n1}^3\,\text{n2}^2}\right\}$$

```
(*****************************
NOTATIONS
μs1x3YB2 denotes the local datum of p1(X)x^3 at Y
μs1x3ZB2 denotes the local datum of p1(X)x^3 at Z
μs1x3PB2 denotes the local datum of p1(X)x^3 at P
*****************************)

SymmetricReduction[Expand[(cYn1 xY t + n1 z)^2 + (yY2 t + z)^2 + (yY3 t + z)^2],
 {yY2, yY3}, {cY1 xY, cY2 xY^2}]
```

$$\left\{cY1^2\, t^2\, xY^2 - 2\, cY2\, t^2\, xY^2 + cYn1^2\, t^2\, xY^2 + 2\, cY1\, t\, xY\, z + 2\, cYn1\, n1\, t\, xY\, z + 2\, z^2 + n1^2\, z^2,\ 0\right\}$$

```
μs1x3YB2 =
 Factor[CoefficientList[
   Coefficient[
    (pY1 xY^2 t^2 + cY1^2 t^2 xY^2 - 2 cY2 t^2 xY^2 + cYn1^2 t^2 xY^2 + 2 cY1 t xY z + 2 cYn1 n1 t xY z +
       2 z^2 + n1^2 z^2) Expand[(rY xY t + l z)^3] Series[(cYn1 xY t + n1 z)^-1, {xY, 0, 2}]
     Series[(cY2 xY^2 t^2 + cY1 xY t z + z^2)^-1, {xY, 0, 2}], t, 2], l]] /. xY^2 → 1
```

$$\left\{0,\ \frac{3\left(2+n1^2\right) rY^2}{n1},\ \frac{3\left(-2\, cYn1 + cYn1\, n1^2 - cY1\, n1^3\right) rY}{n1^2},\ \frac{2\, cYn1^2 + cY1^2\, n1^2 - 4\, cY2\, n1^2 - cY1\, cYn1\, n1^3 + cY1^2\, n1^4 - cY2\, n1^4 + n1^2\, pY1}{n1^3}\right\}$$

```
μs1x3ZB2 =
 CoefficientList[
  Coefficient[((cZn2 xZ t + n2 z)^2 + (cZ1 xZ t + z)^2 + 2 z^2) Expand[(rZ xZ t + (aZ + l) z)^3]
    Series[(cZn2 xZ t + n2 z)^-1, {xZ, 0, 1}] Series[(cZ1 xZ t + z)^-1, {xZ, 0, 1}] z^-2,
   t, 1], l] /. xZ → 1
```

$$\left\{aZ^3\, cZn2 - \frac{3\, aZ^3\, cZn2}{n2^2} - \frac{aZ^3\, cZ1}{n2} - aZ^3\, cZ1\, n2 + \frac{9\, aZ^2\, rZ}{n2} + 3\, aZ^2\, n2\, rZ,\right.$$
$$3\, aZ^2\, cZn2 - \frac{9\, aZ^2\, cZn2}{n2^2} - \frac{3\, aZ^2\, cZ1}{n2} - 3\, aZ^2\, cZ1\, n2 + \frac{18\, aZ\, rZ}{n2} + 6\, aZ\, n2\, rZ,$$
$$\left.3\, aZ\, cZn2 - \frac{9\, aZ\, cZn2}{n2^2} - \frac{3\, aZ\, cZ1}{n2} - 3\, aZ\, cZ1\, n2 + \frac{9\, rZ}{n2} + 3\, n2\, rZ,\ cZn2 - \frac{3\, cZn2}{n2^2} - \frac{cZ1}{n2} - cZ1\, n2\right\}$$

```
μs1x3PB2 =
 Factor[CoefficientList[((n2 z)^2 + (n2 z)^2 + (n1 z)^2 + (n1 z)^2 + (n1 z)^2)
   Expand[-((aP + l) z)^3] (n2 z)^-1 (n2 z)^-1 (n1 z)^-1 (n1 z)^-1 (n1 z)^-1, l]]
```

$$\left\{-\frac{aP^3\left(3\, n1^2 + 2\, n2^2\right)}{n1^3\, n2^2},\ -\frac{3\, aP^2\left(3\, n1^2 + 2\, n2^2\right)}{n1^3\, n2^2},\ -\frac{3\, aP\left(3\, n1^2 + 2\, n2^2\right)}{n1^3\, n2^2},\ -\frac{3\, n1^2 + 2\, n2^2}{n1^3\, n2^2}\right\}$$

```
μx5YB2 + μx5ZB2 + μx5PB2
```

$$\left\{-\frac{aZ^5\, cZn2}{n2^2} - \frac{aP^5}{n1^3\, n2^2} - \frac{aZ^5\, cZ1}{n2} + \frac{5\, aZ^4\, rZ}{n2},\right.$$
$$-\frac{5\, aZ^4\, cZn2}{n2^2} - \frac{5\, aP^4}{n1^3\, n2^2} - \frac{5\, aZ^4\, cZ1}{n2} + \frac{20\, aZ^3\, rZ}{n2},\ -\frac{10\, aZ^3\, cZn2}{n2^2} - \frac{10\, aP^3}{n1^3\, n2^2} - \frac{10\, aZ^3\, cZ1}{n2} + \frac{30\, aZ^2\, rZ}{n2},$$
$$-\frac{10\, aZ^2\, cZn2}{n2^2} - \frac{10\, aP^2}{n1^3\, n2^2} - \frac{10\, aZ^2\, cZ1}{n2} + \frac{10\, rY^2}{n1} + \frac{20\, aZ\, rZ}{n2},$$
$$-\frac{5\, aZ\, cZn2}{n2^2} - \frac{5\, aP}{n1^3\, n2^2} - \frac{5\, aZ\, cZ1}{n2} - \frac{5\left(cYn1 + cY1\, n1\right) rY}{n1^2} + \frac{5\, rZ}{n2},$$
$$\left.\frac{cYn1^2 + cY1\, cYn1\, n1 + cY1^2\, n1^2 - cY2\, n1^2}{n1^3} - \frac{cZn2}{n2^2} - \frac{1}{n1^3\, n2^2} - \frac{cZ1}{n2}\right\}$$

```
μs1x3YB2 + μs1x3ZB2 + μs1x3PB2
```

$$\left\{aZ^3\,cZn2-\frac{3\,aZ^3\,cZn2}{n2^2}-\frac{aZ^3\,cZ1}{n2}-aZ^3\,cZ1\,n2-\frac{aP^3\left(3\,n1^2+2\,n2^2\right)}{n1^3\,n2^2}+\frac{9\,aZ^2\,rZ}{n2}+3\,aZ^2\,n2\,rZ,\right.$$
$$3\,aZ^2\,cZn2-\frac{9\,aZ^2\,cZn2}{n2^2}-\frac{3\,aZ^2\,cZ1}{n2}-3\,aZ^2\,cZ1\,n2-\frac{3\,aP^2\left(3\,n1^2+2\,n2^2\right)}{n1^3\,n2^2}+\frac{3\left(2+n1^2\right)rY^2}{n1}+$$
$$\frac{18\,aZ\,rZ}{n2}+6\,aZ\,n2\,rZ,\ 3\,aZ\,cZn2-\frac{9\,aZ\,cZn2}{n2^2}-\frac{3\,aZ\,cZ1}{n2}-3\,aZ\,cZ1\,n2-\frac{3\,aP\left(3\,n1^2+2\,n2^2\right)}{n1^3\,n2^2}+$$
$$\frac{3\left(-2\,cYn1+cYn1\,n1^2-cY1\,n1^3\right)rY}{n1^2}+\frac{9\,rZ}{n2}+3\,n2\,rZ,\ cZn2-\frac{3\,cZn2}{n2^2}-\frac{cZ1}{n2}-cZ1\,n2-$$
$$\left.\frac{3\,n1^2+2\,n2^2}{n1^3\,n2^2}+\frac{2\,cYn1^2+cY1^2\,n1^2-4\,cY2\,n1^2-cY1\,cYn1\,n1^3+cY1^2\,n1^4-cY2\,n1^4+n1^2\,pY1}{n1^3}\right\}$$

```
Factor[Solve[{
  (******************************
   The following equations are given by Lemma 6.7
   ******************************)
```

$$0==\frac{2\,(aP\,ePM-aZ\,cZn2\,eZM+eZM\,n2\,rZ)}{n2^2},$$
$$0==\frac{ePM-cZn2\,eZM}{n2^2},$$

```
  (******************************
   The following equations are given by Lemma 6.13
   ******************************)
```

$$0==-\frac{3\,aP^2}{n1^3}+\frac{3\,rY^2}{n1},$$
$$0==-\frac{3\,aP}{n1^3}-\frac{3\,cYn1\,rY}{n1^2},$$
$$0==-\frac{1}{n1^3}+\frac{cYn1^2}{n1^3}$$

```
  }, {rY, rZ, cZn2, cYn1}]]
```

$$\left\{\left\{rY\to-\frac{aP}{n1},\ rZ\to-\frac{(aP-aZ)\,ePM}{eZM\,n2},\ cZn2\to\frac{ePM}{eZM},\ cYn1\to 1\right\},\right.$$
$$\left.\left\{rY\to\frac{aP}{n1},\ rZ\to-\frac{(aP-aZ)\,ePM}{eZM\,n2},\ cZn2\to\frac{ePM}{eZM},\ cYn1\to -1\right\}\right\}$$

```
Factor[Solve[{
  (******************************
   The following equations are given by the above computations of x^5 locally
   ******************************)
```

$$d == -\frac{aZ^5\, cZn2}{n2^2} - \frac{aP^5}{n1^3\, n2^2} - \frac{aZ^5\, cZ1}{n2} + \frac{5\, aZ^4\, rZ}{n2},$$

$$0 == -\frac{5\, aZ^4\, cZn2}{n2^2} - \frac{5\, aP^4}{n1^3\, n2^2} - \frac{5\, aZ^4\, cZ1}{n2} + \frac{20\, aZ^3\, rZ}{n2},$$

$$0 == -\frac{10\, aZ^3\, cZn2}{n2^2} - \frac{10\, aP^3}{n1^3\, n2^2} - \frac{10\, aZ^3\, cZ1}{n2} + \frac{30\, aZ^2\, rZ}{n2},$$

$$0 == -\frac{10\, aZ^2\, cZn2}{n2^2} - \frac{10\, aP^2}{n1^3\, n2^2} - \frac{10\, aZ^2\, cZ1}{n2} + \frac{10\, rY^2}{n1} + \frac{20\, aZ\, rZ}{n2}$$

```
}, {d, cZ1, rY, rZ}]]
```

$$\left\{\left\{d \to -\frac{aP^3\,(aP - aZ)^2}{n1^3\, n2^2},\ cZ1 \to -\frac{-3\, aP^4 + 4\, aP^3\, aZ + aZ^4\, cZn2\, n1^3}{aZ^4\, n1^3\, n2},\ rY \to -\frac{aP\,(aP - aZ)}{aZ\, n1\, n2},\ rZ \to \frac{aP^3\,(aP - aZ)}{aZ^3\, n1^3\, n2}\right\},\ \left\{d \to -\frac{aP^3\,(aP - aZ)^2}{n1^3\, n2^2},\ cZ1 \to -\frac{-3\, aP^4 + 4\, aP^3\, aZ + aZ^4\, cZn2\, n1^3}{aZ^4\, n1^3\, n2},\ rY \to \frac{aP\,(aP - aZ)}{aZ\, n1\, n2},\ rZ \to \frac{aP^3\,(aP - aZ)}{aZ^3\, n1^3\, n2}\right\}\right\}$$

```
(******************************
  Comparing the values of rZ and rY allows us to obtain that n1 ==
 n2-1 or n1 == n2+1. We write n1 == n2 + u, where u^2 == 1
   ******************************)
```

```
Factor[Solve[{
   n1 == (n2 + u), u^2 == 1,
   (*******************************
    The following equations are given by the above computations of p1(X) x^3 locally
     *******************************)
   p d == aZ^3 cZn2 - (3 aZ^3 cZn2)/n2^2 - (aZ^3 cZ1)/n2 - aZ^3 cZ1 n2 - (aP^3 (3 n1^2 + 2 n2^2))/(n1^3 n2^2) + (9 aZ^2 rZ)/n2 +
     3 aZ^2 n2 rZ,
   (*******************************
    The following equations are given by the above computations of x^5 locally
     *******************************)
   d == -(aZ^5 cZn2)/n2^2 - aP^5/(n1^3 n2^2) - (aZ^5 cZ1)/n2 + (5 aZ^4 rZ)/n2,
   0 == -(5 aZ^4 cZn2)/n2^2 - (5 aP^4)/(n1^3 n2^2) - (5 aZ^4 cZ1)/n2 + (20 aZ^3 rZ)/n2,
   0 == -(10 aZ^3 cZn2)/n2^2 - (10 aP^3)/(n1^3 n2^2) - (10 aZ^3 cZ1)/n2 + (30 aZ^2 rZ)/n2,
   0 == -(10 aZ^2 cZn2)/n2^2 - (10 aP^2)/(n1^3 n2^2) - (10 aZ^2 cZ1)/n2 + (10 rY^2)/n1 + (20 aZ rZ)/n2,
   (*******************************
    The following equations are given by Lemma 6.7
     *******************************)

   0 == (2 (aP ePM - aZ cZn2 eZM + eZM n2 rZ))/n2^2,
   0 == (ePM - cZn2 eZM)/n2^2,
   ePM^2 == 1, eZM^2 == 1,
   (*******************************
    The following equations are given by Lemma 6.13
     *******************************)
   0 == -(3 aP^2)/n1^3 + (3 rY^2)/n1,
   0 == -(3 aP)/n1^3 - (3 cYn1 rY)/n1^2,
   0 == -1/n1^3 + cYn1^2/n1^3
  }, {p, d, cZ1, cYn1, cZn2, rY, rZ, eZM, ePM, aP, n1, u}]]
```

$$\left\{\left\{p \to \frac{6}{aZ^2}, d \to aZ^5, cZ1 \to 3, cYn1 \to -1, cZn2 \to 1, rY \to -aZ, rZ \to aZ, eZM \to -1, ePM \to -1, aP \to -aZ\,(-1+n2), n1 \to -1+n2, u \to -1\right\},\right.$$
$$\left\{p \to \frac{6}{aZ^2}, d \to aZ^5, cZ1 \to 3, cYn1 \to -1, cZn2 \to 1, rY \to -aZ, rZ \to aZ, eZM \to 1, ePM \to 1, aP \to -aZ\,(-1+n2), n1 \to -1+n2, u \to -1\right\},$$
$$\left\{p \to \frac{6}{aZ^2}, d \to aZ^5, cZ1 \to 3, cYn1 \to 1, cZn2 \to 1, rY \to aZ, rZ \to aZ, eZM \to -1, ePM \to -1, aP \to -aZ\,(-1+n2), n1 \to -1+n2, u \to -1\right\},$$
$$\left\{p \to \frac{6}{aZ^2}, d \to aZ^5, cZ1 \to 3, cYn1 \to 1, cZn2 \to 1, rY \to aZ, rZ \to aZ, eZM \to 1, ePM \to 1, aP \to -aZ\,(-1+n2), n1 \to -1+n2, u \to -1\right\},$$
$$\left\{p \to \frac{6}{aZ^2}, d \to -aZ^5, cZ1 \to 3, cYn1 \to 1, cZn2 \to -1, rY \to -aZ, rZ \to aZ, eZM \to -1, ePM \to 1, aP \to aZ\,(1+n2), n1 \to 1+n2, u \to 1\right\}, \left\{p \to \frac{6}{aZ^2}, d \to -aZ^5, cZ1 \to 3, cYn1 \to 1, cZn2 \to -1, rY \to -aZ, rZ \to aZ, eZM \to 1, ePM \to -1, aP \to aZ\,(1+n2), n1 \to 1+n2, u \to 1\right\},$$
$$\left\{p \to \frac{6}{aZ^2}, d \to -aZ^5, cZ1 \to 3, cYn1 \to -1, cZn2 \to -1, rY \to aZ, rZ \to aZ, eZM \to -1, ePM \to 1, aP \to aZ\,(1+n2), n1 \to 1+n2, u \to 1\right\}, \left.\left\{p \to \frac{6}{aZ^2}, d \to -aZ^5, cZ1 \to 3, cYn1 \to -1, cZn2 \to -1, rY \to aZ, rZ \to aZ, eZM \to 1, ePM \to -1, aP \to aZ\,(1+n2), n1 \to 1+n2, u \to 1\right\}\right\}$$

Computations for Lemma 6.15.
Case where there exists N^4 s.t. Fix(N) = $Z^2 \cup P^0$.

Case C where the other normal weights are trivial.

```
(******************************
  NOTATIONS
  rY xY denotes the restriction of x to Y
  rZ xZ denotes the restriction of x to Z
  aZ denotes the Hopf weight at Z
  aP denotes the Hopf weight at P
  cYj xY^j denotes the j-th Chern class of the normal bundle of Y
  cZn1 xZ denotes the first Chern class of the normal bundle of Z in the
  direction of N
  cZ1 xZ denotes the first Chern class of the normal bundle of Z in the
  remaining directions
  μx5YC denotes the local datum of x^5 at Y
  μx5ZC denotes the local datum of x^5 at Z
  μx5PC denotes the local datum of x^5 at P

  We use the variable t to extract the coefficient of order 2 of xY and
  of order 1 of xZ. We then evaluate xY^2 and xZ to 1.
  The orientation of the point P is -1, this explains the negative sign.
  Once evaluated, we extract all the coefficients in l into a list.
  ******************************)
```

```
μx5YC =
 Factor[CoefficientList[
    Coefficient[Expand[(rY xY t + l z)^5]
      Series[(cY2 xY^2 t^2 z + cY1 xY t z^2 + z^3)^-1, {xY, 0, 2}], t, 2], l]] /. xY^2 → 1
```

$$\{0, 0, 0, 10\,rY^2, -5\,cY1\,rY, cY1^2 - cY2\}$$

```
μx5ZC =
 CoefficientList[
   Coefficient[Expand[(rZ xZ t + (aZ + l) z)^5] Series[(cZn1 xZ t + n1 z)^-1, {xZ, 0, 1}]
     Series[(cZ1 xZ t + z)^-1, {xZ, 0, 1}] z^-2, t, 1], l] /. xZ → 1
```

$$\left\{-\frac{aZ^5\,cZn1}{n1^2} - \frac{aZ^5\,cZ1}{n1} + \frac{5\,aZ^4\,rZ}{n1}, -\frac{5\,aZ^4\,cZn1}{n1^2} - \frac{5\,aZ^4\,cZ1}{n1} + \frac{20\,aZ^3\,rZ}{n1},\right.$$
$$-\frac{10\,aZ^3\,cZn1}{n1^2} - \frac{10\,aZ^3\,cZ1}{n1} + \frac{30\,aZ^2\,rZ}{n1}, -\frac{10\,aZ^2\,cZn1}{n1^2} - \frac{10\,aZ^2\,cZ1}{n1} + \frac{20\,aZ\,rZ}{n1},$$
$$\left.-\frac{5\,aZ\,cZn1}{n1^2} - \frac{5\,aZ\,cZ1}{n1} + \frac{5\,rZ}{n1}, -\frac{cZn1}{n1^2} - \frac{cZ1}{n1}\right\}$$

```
μx5PC = CoefficientList[Expand[-((aP + l) z)^5] z^-1 z^-1 z^-1 (n1 z)^-1 (n1 z)^-1, l]
```

$$\left\{-\frac{aP^5}{n1^2}, -\frac{5\,aP^4}{n1^2}, -\frac{10\,aP^3}{n1^2}, -\frac{10\,aP^2}{n1^2}, -\frac{5\,aP}{n1^2}, -\frac{1}{n1^2}\right\}$$

```
(*******************************
 NOTATIONS
 μs1x3YC denotes the local datum of p1(X)x^3 at Y
 μs1x3ZC denotes the local datum of p1(X)x^3 at Z
 μs1x3PC denotes the local datum of p1(X)x^3 at P
  *******************************)

SymmetricReduction[Expand[(yY1 t + z)^2 + (yY2 t + z)^2 + (yY3 t + z)^2],
 {yY1, yY2, yY3}, {cY1 xY, cY2 xY^2, cY3 xY^3}]
```

$$\{cY1^2\,t^2\,xY^2 - 2\,cY2\,t^2\,xY^2 + 2\,cY1\,t\,xY\,z + 3\,z^2, 0\}$$

```
μs1x3YC =
 Factor[CoefficientList[
    Coefficient[(pY1 xY^2 t^2 + cY1^2 t^2 xY^2 - 2 cY2 t^2 xY^2 + 2 cY1 t xY z + 3 z^2)
      Expand[(rY xY t + l z)^3] Series[(cY2 xY^2 t^2 z + cY1 xY t z^2 + z^3)^-1, {xY, 0, 2}], t, 2],
    l]] /. xY^2 → 1
```

$$\{0, 9\,rY^2, -3\,cY1\,rY, 2\,cY1^2 - 5\,cY2 + pY1\}$$

```
μs1x3ZC =
 CoefficientList[
   Coefficient[((cZn1 xZ t + n1 z)^2 + (cZ1 xZ t + z)^2 + 2 z^2) Expand[(rZ xZ t + (aZ + l) z)^3]
     Series[(cZn1 xZ t + n1 z)^-1, {xZ, 0, 1}] Series[(cZ1 xZ t + z)^-1, {xZ, 0, 1}] z^-2,
    t, 1], l] /. xZ → 1
```

$$\left\{aZ^3\,cZn1 - \frac{3\,aZ^3\,cZn1}{n1^2} - \frac{aZ^3\,cZ1}{n1} - aZ^3\,cZ1\,n1 + \frac{9\,aZ^2\,rZ}{n1} + 3\,aZ^2\,n1\,rZ,\right.$$
$$3\,aZ^2\,cZn1 - \frac{9\,aZ^2\,cZn1}{n1^2} - \frac{3\,aZ^2\,cZ1}{n1} - 3\,aZ^2\,cZ1\,n1 + \frac{18\,aZ\,rZ}{n1} + 6\,aZ\,n1\,rZ,$$
$$\left.3\,aZ\,cZn1 - \frac{9\,aZ\,cZn1}{n1^2} - \frac{3\,aZ\,cZ1}{n1} - 3\,aZ\,cZ1\,n1 + \frac{9\,rZ}{n1} + 3\,n1\,rZ, cZn1 - \frac{3\,cZn1}{n1^2} - \frac{cZ1}{n1} - cZ1\,n1\right\}$$

```
μs1x3PC =
  Factor[CoefficientList[(z^2 + z^2 + z^2 + (n1 z)^2 + (n1 z)^2) Expand[-((aP + 1) z)^3]
     z^-1 z^-1 z^-1 (n1 z)^-1 (n1 z)^-1, 1]]
```

$$\left\{-\frac{aP^3 \left(3+2\, n1^2\right)}{n1^2}, -\frac{3\, aP^2 \left(3+2\, n1^2\right)}{n1^2}, -\frac{3\, aP \left(3+2\, n1^2\right)}{n1^2}, -\frac{3+2\, n1^2}{n1^2}\right\}$$

```
μx5YC + μx5ZC + μx5PC
```

$$\left\{-\frac{aP^5}{n1^2}-\frac{aZ^5\, cZn1}{n1^2}-\frac{aZ^5\, cZ1}{n1}+\frac{5\, aZ^4\, rZ}{n1}, -\frac{5\, aP^4}{n1^2}-\frac{5\, aZ^4\, cZn1}{n1^2}-\frac{5\, aZ^4\, cZ1}{n1}+\frac{20\, aZ^3\, rZ}{n1},\right.$$
$$-\frac{10\, aP^3}{n1^2}-\frac{10\, aZ^3\, cZn1}{n1^2}-\frac{10\, aZ^3\, cZ1}{n1}+\frac{30\, aZ^2\, rZ}{n1},$$
$$-\frac{10\, aP^2}{n1^2}-\frac{10\, aZ^2\, cZn1}{n1^2}-\frac{10\, aZ^2\, cZ1}{n1}+10\, rY^2+\frac{20\, aZ\, rZ}{n1},$$
$$\left.-\frac{5\, aP}{n1^2}-\frac{5\, aZ\, cZn1}{n1^2}-\frac{5\, aZ\, cZ1}{n1}-5\, cY1\, rY+\frac{5\, rZ}{n1}, cY1^2-cY2-\frac{1}{n1^2}-\frac{cZn1}{n1^2}-\frac{cZ1}{n1}\right\}$$

```
μs1x3YC + μs1x3ZC + μs1x3PC
```

$$\left\{aZ^3\, cZn1-\frac{3\, aZ^3\, cZn1}{n1^2}-\frac{aZ^3\, cZ1}{n1}-aZ^3\, cZ1\, n1-\frac{aP^3\left(3+2\, n1^2\right)}{n1^2}+\frac{9\, aZ^2\, rZ}{n1}+3\, aZ^2\, n1\, rZ,\right.$$
$$3\, aZ^2\, cZn1-\frac{9\, aZ^2\, cZn1}{n1^2}-\frac{3\, aZ^2\, cZ1}{n1}-3\, aZ^2\, cZ1\, n1-$$
$$\frac{3\, aP^2\left(3+2\, n1^2\right)}{n1^2}+9\, rY^2+\frac{18\, aZ\, rZ}{n1}+6\, aZ\, n1\, rZ,$$
$$3\, aZ\, cZn1-\frac{9\, aZ\, cZn1}{n1^2}-\frac{3\, aZ\, cZ1}{n1}-3\, aZ\, cZ1\, n1-\frac{3\, aP\left(3+2\, n1^2\right)}{n1^2}-3\, cY1\, rY+\frac{9\, rZ}{n1}+3\, n1\, rZ,$$
$$\left.2\, cY1^2-5\, cY2+cZn1-\frac{3\, cZn1}{n1^2}-\frac{cZ1}{n1}-cZ1\, n1-\frac{3+2\, n1^2}{n1^2}+pY1\right\}$$

```
Factor[Solve[{
   (*******************************
    The following equations are given by Lemma 6.7
    *******************************)
   0 == 2 (aP ePN - aZ cZn1 eZN + eZN n1 rZ) / n1^2,
   0 == (ePN - cZn1 eZN) / n1^2
  },
  {cZn1, rZ}]]
```

$$\left\{\left\{cZn1\to\frac{ePN}{eZN}, rZ\to-\frac{(aP-aZ)\, ePN}{eZN\, n1}\right\}\right\}$$

$$
\begin{aligned}
&\mathbf{Factor}\Big[\mathbf{Solve}\Big[\Big\{\\
&\quad (\text{*******************************}\\
&\quad\ \text{The following equations are given by the above computations of } x^5 \text{ locally}\\
&\quad\ \ \text{*******************************})\\
&\quad d == -\frac{aP^5}{n1^2} - \frac{aZ^5\, cZn1}{n1^2} - \frac{aZ^5\, cZ1}{n1} + \frac{5\, aZ^4\, rZ}{n1},\\
&\quad 0 == -\frac{5\, aP^4}{n1^2} - \frac{5\, aZ^4\, cZn1}{n1^2} - \frac{5\, aZ^4\, cZ1}{n1} + \frac{20\, aZ^3\, rZ}{n1},\\
&\quad 0 == -\frac{10\, aP^3}{n1^2} - \frac{10\, aZ^3\, cZn1}{n1^2} - \frac{10\, aZ^3\, cZ1}{n1} + \frac{30\, aZ^2\, rZ}{n1},\\
&\quad (\text{*******************************}\\
&\quad\ \text{The following equations are given by the above computations of } p1(X)\, x^3 \text{ locally}\\
&\quad\ \ \text{*******************************})\\
&\quad p\, d == aZ^3\, cZn1 - \frac{3\, aZ^3\, cZn1}{n1^2} - \frac{aZ^3\, cZ1}{n1} - aZ^3\, cZ1\, n1 - \frac{aP^3\left(3 + 2\, n1^2\right)}{n1^2} + \frac{9\, aZ^2\, rZ}{n1} + 3\, aZ^2\, n1\, rZ,\\
&\quad (\text{*******************************}\\
&\quad\ \text{The following equations are given by Lemma 6.7}\\
&\quad\ \ \text{*******************************})\\
&\quad 0 == \frac{2\,(aP\, ePN - aZ\, cZn1\, eZN + eZN\, n1\, rZ)}{n1^2},\\
&\quad 0 == \frac{ePN - cZn1\, eZN}{n1^2},\\
&\quad ePN^2 == 1,\ eZN^2 == 1\\
&\ \Big\},\\
&\ \{p,\ d,\ cZ1,\ cZn1,\ rZ,\ aP,\ ePN,\ eZN\}\Big]\Big]
\end{aligned}
$$

$$\Big\{\Big\{p\to\frac{3\,(4+n1^2)}{4\,aZ^2},\ d\to\frac{4\,aZ^5}{n1^2},\ cZ1\to\frac{6}{n1},\ cZn1\to 1,\ rZ\to\frac{2\,aZ}{n1},\ aP\to -aZ,\ ePN\to -1,\ eZN\to -1\Big\},$$

$$\Big\{p\to\frac{3\,(4+n1^2)}{4\,aZ^2},\ d\to\frac{4\,aZ^5}{n1^2},\ cZ1\to\frac{6}{n1},\ cZn1\to 1,\ rZ\to\frac{2\,aZ}{n1},\ aP\to -aZ,\ ePN\to 1,\ eZN\to 1\Big\},$$

$$\Big\{p\to\frac{3\,\left(4+4\,(-1)^{1/3}+4\,(-1)^{2/3}-n1^2+(-1)^{1/3}\,n1^2+3\,(-1)^{2/3}\,n1^2\right)}{4\,aZ^2},$$
$$d\to\frac{\left(-1+(-1)^{1/3}\right)^2 aZ^5}{n1^2},\ cZ1\to-\frac{3\,\left(-1+(-1)^{1/3}\right)}{n1},\ cZn1\to 1,$$
$$rZ\to-\frac{\left(-1+(-1)^{1/3}\right)aZ}{n1},\ aP\to(-1)^{1/3}\,aZ,\ ePN\to -1,\ eZN\to -1\Big\},$$

$$\Big\{p\to\frac{3\,\left(4+4\,(-1)^{1/3}+4\,(-1)^{2/3}-n1^2+(-1)^{1/3}\,n1^2+3\,(-1)^{2/3}\,n1^2\right)}{4\,aZ^2},\ d\to\frac{\left(-1+(-1)^{1/3}\right)^2 aZ^5}{n1^2},$$
$$cZ1\to-\frac{3\,\left(-1+(-1)^{1/3}\right)}{n1},\ cZn1\to 1,\ rZ\to-\frac{\left(-1+(-1)^{1/3}\right)aZ}{n1},\ aP\to(-1)^{1/3}\,aZ,\ ePN\to 1,$$
$$eZN\to 1\Big\},\ \Big\{p\to-\frac{3\,\left(-4+4\,(-1)^{1/3}+4\,(-1)^{2/3}+n1^2+3\,(-1)^{1/3}\,n1^2+(-1)^{2/3}\,n1^2\right)}{4\,aZ^2},$$
$$d\to\frac{\left(1-(-1)^{1/3}+2\,(-1)^{2/3}\right)aZ^5}{n1^2},\ cZ1\to\frac{3\,\left(1+(-1)^{2/3}\right)}{n1},\ cZn1\to 1,$$
$$rZ\to\frac{\left(1+(-1)^{2/3}\right)aZ}{n1},\ aP\to-(-1)^{2/3}\,aZ,\ ePN\to -1,\ eZN\to -1\Big\},$$

$$\Big\{p\to-\frac{3\,\left(-4+4\,(-1)^{1/3}+4\,(-1)^{2/3}+n1^2+3\,(-1)^{1/3}\,n1^2+(-1)^{2/3}\,n1^2\right)}{4\,aZ^2},$$
$$d\to\frac{\left(1-(-1)^{1/3}+2\,(-1)^{2/3}\right)aZ^5}{n1^2},\ cZ1\to\frac{3\,\left(1+(-1)^{2/3}\right)}{n1},\ cZn1\to 1,$$
$$rZ\to\frac{\left(1+(-1)^{2/3}\right)aZ}{n1},\ aP\to-(-1)^{2/3}\,aZ,\ ePN\to 1,\ eZN\to 1\Big\},$$

$$\Big\{p\to\frac{6-2\,i\,\sqrt{3}+n1^2-i\,\sqrt{3}\,n1^2}{2\,aZ^2},\ d\to-\frac{3\,i\,\left(-i+\sqrt{3}\right)aZ^5}{2\,n1^2},\ cZ1\to-\frac{3\,i\,\left(-3\,i+\sqrt{3}\right)}{2\,n1},$$
$$cZn1\to -1,\ rZ\to-\frac{i\,\left(-3\,i+\sqrt{3}\right)aZ}{2\,n1},\ aP\to-\frac{1}{2}\,i\,\left(-i+\sqrt{3}\right)aZ,\ ePN\to 1,\ eZN\to -1\Big\},$$

$$\Big\{p\to\frac{6-2\,i\,\sqrt{3}+n1^2-i\,\sqrt{3}\,n1^2}{2\,aZ^2},\ d\to-\frac{3\,i\,\left(-i+\sqrt{3}\right)aZ^5}{2\,n1^2},\ cZ1\to-\frac{3\,i\,\left(-3\,i+\sqrt{3}\right)}{2\,n1},$$
$$cZn1\to -1,\ rZ\to-\frac{i\,\left(-3\,i+\sqrt{3}\right)aZ}{2\,n1},\ aP\to-\frac{1}{2}\,i\,\left(-i+\sqrt{3}\right)aZ,\ ePN\to -1,\ eZN\to 1\Big\},$$

$$\Big\{p\to\frac{6+2\,i\,\sqrt{3}+n1^2+i\,\sqrt{3}\,n1^2}{2\,aZ^2},\ d\to\frac{3\,i\,\left(i+\sqrt{3}\right)aZ^5}{2\,n1^2},\ cZ1\to\frac{3\,i\,\left(3\,i+\sqrt{3}\right)}{2\,n1},$$
$$cZn1\to -1,\ rZ\to\frac{i\,\left(3\,i+\sqrt{3}\right)aZ}{2\,n1},\ aP\to\frac{1}{2}\,i\,\left(i+\sqrt{3}\right)aZ,\ ePN\to 1,\ eZN\to -1\Big\},$$

$$\Big\{p\to\frac{6+2\,i\,\sqrt{3}+n1^2+i\,\sqrt{3}\,n1^2}{2\,aZ^2},\ d\to\frac{3\,i\,\left(i+\sqrt{3}\right)aZ^5}{2\,n1^2},\ cZ1\to\frac{3\,i\,\left(3\,i+\sqrt{3}\right)}{2\,n1},$$
$$cZn1\to -1,\ rZ\to\frac{i\,\left(3\,i+\sqrt{3}\right)aZ}{2\,n1},\ aP\to\frac{1}{2}\,i\,\left(i+\sqrt{3}\right)aZ,\ ePN\to -1,\ eZN\to 1\Big\}\Big\}$$

Computations for Theorem D.

The case where the fixed points are $Y^4 \cup P_0 \cup P_1 \cup P_2$.

Computations for Lemma 6.16.

The case where the action is semi-free.

```
(******************************
  NOTATIONS
  rY xY denotes the restriction of x to Y
  ai denotes the Hopf weight at Pi
  cYj xY^j denotes the j-th Chern class of the normal bundle of Y
  µx5Y denotes the local datum of x^5 at Y
  µx5Pi denotes the local datum of x^5 at Pi

  We use the variable t to extract the coefficient of order 2 of xY. We
  then evaluate xY^2 to 1.
  The orientation of the point P0 is 1 while those from P1 and P2 are -
 1. This explains the signs at their local datum
  Once evaluated, we extract all the coefficients in l into a list.
  ******************************)

µx5Y =
 Factor[CoefficientList[
    Coefficient[Expand[(rY xY t + l z)^5]
      Series[(cY2 xY^2 t^2 z + cY1 xY t z^2 + z^3)^-1, {xY, 0, 2}], t, 2], l]] /. xY^2 → 1

{0, 0, 0, 10 rY^2, -5 cY1 rY, cY1^2 - cY2}

µx5P0 = CoefficientList[Expand[((a0 + l) z)^5] z^-5, l]

{a0^5, 5 a0^4, 10 a0^3, 10 a0^2, 5 a0, 1}

µx5P1 = CoefficientList[Expand[-((a1 + l) z)^5] z^-5, l]

{-a1^5, -5 a1^4, -10 a1^3, -10 a1^2, -5 a1, -1}

µx5P2 = CoefficientList[Expand[-((a2 + l) z)^5] z^-5, l]

{-a2^5, -5 a2^4, -10 a2^3, -10 a2^2, -5 a2, -1}

(******************************
 NOTATIONS
 µs1x3Y denotes the local datum of p1(X)x^3 at Y
 µs1x3Pi denotes the local datum of p1(X)x^3 at Pi
  ******************************)

SymmetricReduction[(yY1 t + z)^2 + (yY2 t + z)^2 + (yY3 t + z)^2 , {yY1, yY2, yY3},
 {cY1 xY , cY2 xY^2, cy3 xY^3}]

{cY1^2 t^2 xY^2 - 2 cY2 t^2 xY^2 + 2 cY1 t xY z + 3 z^2, 0}
```

```
μs1x3Y =
 Factor[CoefficientList[
    Coefficient[(pY1 t^2 + cY1^2 t^2 xY^2 - 2 cY2 t^2 xY^2 + 2 cY1 t xY z + 3 z^2)
      Expand[(rY xY t + l z)^3] Series[(cY2 xY^2 t^2 z + cY1 xY t z^2 + z^3)^-1, {xY, 0, 2}], t, 2],
    l]] /. xY^2 → 1
```

$\{0, 9\,rY^2, -3\,cY1\,rY, 2\,cY1^2 - 5\,cY2 + pY1\}$

```
μs1x3P0 = CoefficientList[((z)^2 + (z)^2 + (z)^2 + (z)^2 + (z)^2) Expand[((a0 + l) z)^3] z^-5, l]
```

$\{5\,a0^3, 15\,a0^2, 15\,a0, 5\}$

```
μs1x3P1 = CoefficientList[((z)^2 + (z)^2 + (z)^2 + (z)^2 + (z)^2) Expand[-((a1 + l) z)^3] z^-5, l]
```

$\{-5\,a1^3, -15\,a1^2, -15\,a1, -5\}$

```
μs1x3P2 = CoefficientList[((z)^2 + (z)^2 + (z)^2 + (z)^2 + (z)^2) Expand[-((a2 + l) z)^3] z^-5, l]
```

$\{-5\,a2^3, -15\,a2^2, -15\,a2, -5\}$

```
(******************************
 NOTATIONS
 μs2xY denotes the local datum of s2(X)x at Y
 μs2xPi denotes the local datum of s2(X)x at Pi
  ******************************)

SymmetricReduction[(yY1 t + z)^4 + (yY2 t + z)^4 + (yY3 t + z)^4 , {yY1, yY2, yY3},
 {cY1 xY , cY2 xY^2, cy3 xY^3}]
```

$\{cY1^4\,t^4\,xY^4 - 4\,cY1^2\,cY2\,t^4\,xY^4 + 2\,cY2^2\,t^4\,xY^4 + 4\,cY1\,cy3\,t^4\,xY^4 + 4\,cY1^3\,t^3\,xY^3\,z - 12\,cY1\,cY2\,t^3\,xY^3\,z + 12\,cy3\,t^3\,xY^3\,z + 6\,cY1^2\,t^2\,xY^2\,z^2 - 12\,cY2\,t^2\,xY^2\,z^2 + 4\,cY1\,t\,xY\,z^3 + 3\,z^4, 0\}$

```
μs2xY =
 Factor[CoefficientList[
    Coefficient[(6 cY1^2 t^2 xY^2 z^2 - 12 cY2 t^2 xY^2 z^2 + 4 cY1 t xY z^3 + 3 z^4) (rY xY t + l z)
      Series[(cY2 xY^2 t^2 z + cY1 xY t z^2 + z^3)^-1, {xY, 0, 2}], t, 2], l]] /. xY^2 → 1
```

$\{cY1\,rY, 5\,(cY1^2 - 3\,cY2)\}$

```
μs2xP0 = CoefficientList[((z)^4 + (z)^4 + (z)^4 + (z)^4 + (z)^4) Expand[((a0 + l) z)] z^-5, l]
```

$\{5\,a0, 5\}$

```
μs2xP1 = CoefficientList[((z)^4 + (z)^4 + (z)^4 + (z)^4 + (z)^4) Expand[-((a1 + l) z)] z^-5, l]
```

$\{-5\,a1, -5\}$

```
μs2xP2 = CoefficientList[((z)^4 + (z)^4 + (z)^4 + (z)^4 + (z)^4) Expand[-((a2 + l) z)] z^-5, l]
```

$\{-5\,a2, -5\}$

```
Factor[μx5Y + μx5P0 + μx5P1 + μx5P2]
```

$\{a0^5 - a1^5 - a2^5, 5\,(a0^4 - a1^4 - a2^4), 10\,(a0^3 - a1^3 - a2^3), 10\,(a0^2 - a1^2 - a2^2 + rY^2), 5\,(a0 - a1 - a2 - cY1\,rY), -1 + cY1^2 - cY2\}$

```
Factor[μs1x3Y + μs1x3P0 + μs1x3P1 + μs1x3P2]
```

$\{5\,(a0^3 - a1^3 - a2^3), 3\,(5\,a0^2 - 5\,a1^2 - 5\,a2^2 + 3\,rY^2), 3\,(5\,a0 - 5\,a1 - 5\,a2 - cY1\,rY), -5 + 2\,cY1^2 - 5\,cY2 + pY1\}$

```
Factor[μs2xY + μs2xP0 + μs2xP1 + μs2xP2]
```

$\{5\,a0 - 5\,a1 - 5\,a2 + cY1\,rY,\ 5\,(-1 + cY1^2 - 3\,cY2)\}$

```
(*******************************
 One way to get a contradiction
  *******************************)

Solve[{
   (*******************************
    The following equations are given by the above computations of x^5 locally
     *******************************)
   (*d== a0^5-a1^5-a2^5,
   0==5 (a0^4-a1^4-a2^4),
   0==10 (a0^3-a1^3-a2^3),*)
   0 == 10 (a0^2 - a1^2 - a2^2 + rY^2),
   0 == 5 (a0 - a1 - a2 - cY1 rY),
   0 == -1 + cY1^2 - cY2,
   (*******************************
    The following equations are given by the above computations of p1(X)x^3 locally
     *******************************)
   (*p d == 5 (a0^3-a1^3-a2^3),
   0== 3 (3+5 a0^2-5 a1^2-5 a2^2),*)
   0 == 3 (5 a0 - 5 a1 - 5 a2 - cY1),
   0 == -5 + 2 cY1^2 - 5 cY2 + pY1,
   (*******************************
    The following equation is given by the above computations of s2(X)x locally
     *******************************)
   0 == 5 (-1 + cY1^2 - 3 cY2)
  },
  {cY1, cY2, rY, pY1, a0, a1}]
```

$$\left\{\left\{cY1 \to -1,\ cY2 \to 0,\ rY \to \frac{1}{5},\ pY1 \to 3,\ a0 \to a2,\ a1 \to \frac{1}{5}\right\},\right.$$
$$\left.\left\{cY1 \to 1,\ cY2 \to 0,\ rY \to \frac{1}{5},\ pY1 \to 3,\ a0 \to a2,\ a1 \to -\frac{1}{5}\right\}\right\}$$

```
(*******************************
 Another way to get a contradiction
  *******************************)
```

```
Solve[{
  (******************************
   The following equations are given by the above computations of x^5 locally
    ******************************)
  d == a0^5 - a1^5 - a2^5,
  0 == 5 (a0^4 - a1^4 - a2^4),
  0 == 10 (a0^3 - a1^3 - a2^3),
  0 == 10 (a0^2 - a1^2 - a2^2 + rY^2),
  0 == 5 (a0 - a1 - a2 - cY1 rY),
  0 == -1 + cY1^2 - cY2,
  (******************************
   The following equations are given by the above computations of p1(X)x^3 locally
    ******************************)
  (*p d == 5 (a0^3-a1^3-a2^3),
  0== 3 (3+5 a0^2-5 a1^2-5 a2^2),
  0== 3 (5 a0-5 a1-5 a2-cY1),
  0== -5+2 cY1^2-5 cY2+pY1,*)
  (******************************
   The following equation is given by the above computations of s2(X)x locally
    ******************************)
  0 == 5 (-1 + cY1^2 - 3 cY2)
 },
 {d, cY1, cY2, rY, a1, a2}]

{{d → 0, cY1 → -1, cY2 → 0, rY → 0, a1 → 0, a2 → a0},
 {d → 0, cY1 → 1, cY2 → 0, rY → 0, a1 → 0, a2 → a0},
 {d → 0, cY1 → -1, cY2 → 0, rY → 0, a1 → a0, a2 → 0},
 {d → 0, cY1 → 1, cY2 → 0, rY → 0, a1 → a0, a2 → 0}}
```

Computations for Lemma 6.19. Case A.1 where there exists N^6 s.t. Fix(N) = $Y^4 \cup P_0 \cup P_1 \cup P_2$.

```
(******************************
  NOTATIONS
  rY xY denotes the restriction of x to Y
  ai denotes the Hopf weight at Pi
  Nij denotes the product of the normal weights shared between Pi and Pj
  Sni is the sum of the square of the normal weights at Pi
  cYn1 xY denotes the first Chern class of the normal bundle of Y in the
  direction of N
  cYj xY^j denotes the j-th Chern class of the normal bundle of Y in the
  remaining directions
  μx5YA denotes the local datum of x^5 at Y
  μx5PiA denotes the local datum of x^5 at Pi

  We use the variable t to extract the coefficient of order 2 of xY. We
  then evaluate xY^2 to 1.
  The orientation of the point P0 is 1 while those from P1 and P2 are -
 1. This explains the signs at their local datum
  Once evaluated, we extract all the coefficients in l into a list.
  ******************************)
```

```
μx5YA1 =
 Factor[CoefficientList[
    Coefficient[Expand[(rY xY t + l z)^5]
      Series[(cY2 xY^2 t^2 + cY1 xY t z + z^2)^-1, {xY, 0, 2}]
      Series[(cYn1 xY t + n1 z)^-1, {xY, 0, 2}], t, 2], l]] /. xY^2 → 1
```

$$\left\{0,\ 0,\ 0,\ \frac{10\ rY^2}{n1},\ -\frac{5\ (cYn1 + cY1\ n1)\ rY}{n1^2},\ \frac{cYn1^2 + cY1\ cYn1\ n1 + cY1^2\ n1^2 - cY2\ n1^2}{n1^3}\right\}$$

```
μx5P0A1 = CoefficientList[Expand[((a0 + l) z)^5] n1^-3 N01^-1 N02^-1 z^-5, l]
```

$$\left\{\frac{a0^5}{N01\ N02\ n1^3},\ \frac{5\ a0^4}{N01\ N02\ n1^3},\ \frac{10\ a0^3}{N01\ N02\ n1^3},\ \frac{10\ a0^2}{N01\ N02\ n1^3},\ \frac{5\ a0}{N01\ N02\ n1^3},\ \frac{1}{N01\ N02\ n1^3}\right\}$$

```
μx5P1A1 = CoefficientList[Expand[-((a1 + l) z)^5] n1^-3 N01^-1 N12^-1 z^-5, l]
```

$$\left\{-\frac{a1^5}{N01\ n1^3\ N12},\ -\frac{5\ a1^4}{N01\ n1^3\ N12},\ -\frac{10\ a1^3}{N01\ n1^3\ N12},\ -\frac{10\ a1^2}{N01\ n1^3\ N12},\ -\frac{5\ a1}{N01\ n1^3\ N12},\ -\frac{1}{N01\ n1^3\ N12}\right\}$$

```
μx5P2A1 = CoefficientList[Expand[-((a2 + l) z)^5] n1^-3 N02^-1 N12^-1 z^-5, l]
```

$$\left\{-\frac{a2^5}{N02\ n1^3\ N12},\ -\frac{5\ a2^4}{N02\ n1^3\ N12},\ -\frac{10\ a2^3}{N02\ n1^3\ N12},\ -\frac{10\ a2^2}{N02\ n1^3\ N12},\ -\frac{5\ a2}{N02\ n1^3\ N12},\ -\frac{1}{N02\ n1^3\ N12}\right\}$$

```
(******************************
 NOTATIONS
 μs1x3YA denotes the local datum of p1(X)x^3 at Y
 μs1x3PiA denotes the local datum of p1(X)x^3 at Pi
  ******************************)

SymmetricReduction[Expand[(yY2 t + z)^2 + (yY3 t + z)^2], {yY2, yY3}, {cY1 xY, cY2 xY^2}]
```

$$\left\{cY1^2\ t^2\ xY^2 - 2\ cY2\ t^2\ xY^2 + 2\ cY1\ t\ xY\ z + 2\ z^2,\ 0\right\}$$

```
μs1x3YA1 =
 Factor[CoefficientList[
    Coefficient[
     Expand[pY1 t^2 + (cYn1 xY t + n1 z)^2 + cY1^2 t^2 xY^2 - 2 cY2 t^2 xY^2 + 2 cY1 t xY z + 2 z^2]
      (rY xY t + l z)^3 Series[(cY2 xY^2 t^2 + cY1 xY t z + z^2)^-1, {xY, 0, 2}]
      Series[(cYn1 xY t + n1 z)^-1, {xY, 0, 2}], t, 2], l]] /. xY^2 → 1
```

$$\left\{0,\ \frac{3\ \left(2 + n1^2\right)\ rY^2}{n1},\ \frac{3\ \left(-2\ cYn1 + cYn1\ n1^2 - cY1\ n1^3\right)\ rY}{n1^2},\ \frac{2\ cYn1^2 + cY1^2\ n1^2 - 4\ cY2\ n1^2 - cY1\ cYn1\ n1^3 + cY1^2\ n1^4 - cY2\ n1^4 + n1^2\ pY1}{n1^3}\right\}$$

```
μs1x3P0A1 = Factor[CoefficientList[Sn0 z^2 ((a0 + l) z)^3 n1^-3 N01^-1 N02^-1 z^-5, l]]
```

$$\left\{\frac{a0^3\ Sn0}{N01\ N02\ n1^3},\ \frac{3\ a0^2\ Sn0}{N01\ N02\ n1^3},\ \frac{3\ a0\ Sn0}{N01\ N02\ n1^3},\ \frac{Sn0}{N01\ N02\ n1^3}\right\}$$

```
μs1x3P1A1 = Factor[CoefficientList[- Sn1 z^2 ((a1 + l) z)^3 n1^-3 N01^-1 N12^-1 z^-5, l]]
```

$$\left\{-\frac{a1^3\ Sn1}{N01\ n1^3\ N12},\ -\frac{3\ a1^2\ Sn1}{N01\ n1^3\ N12},\ -\frac{3\ a1\ Sn1}{N01\ n1^3\ N12},\ -\frac{Sn1}{N01\ n1^3\ N12}\right\}$$

```
μs1x3P2A1 = Factor[CoefficientList[- Sn2 z^2 ((a2 + 1) z)^3 n1^-3 N02^-1 N12^-1 z^-5, 1]]
```

$$\left\{-\frac{a2^3\,Sn2}{N02\,n1^3\,N12},\ -\frac{3\,a2^2\,Sn2}{N02\,n1^3\,N12},\ -\frac{3\,a2\,Sn2}{N02\,n1^3\,N12},\ -\frac{Sn2}{N02\,n1^3\,N12}\right\}$$

```
μx5YA1 + μx5P0A1 + μx5P1A1 + μx5P2A1
```

$$\left\{\frac{a0^5}{N01\,N02\,n1^3}-\frac{a1^5}{N01\,n1^3\,N12}-\frac{a2^5}{N02\,n1^3\,N12},\ \frac{5\,a0^4}{N01\,N02\,n1^3}-\frac{5\,a1^4}{N01\,n1^3\,N12}-\frac{5\,a2^4}{N02\,n1^3\,N12},\right.$$
$$\frac{10\,a0^3}{N01\,N02\,n1^3}-\frac{10\,a1^3}{N01\,n1^3\,N12}-\frac{10\,a2^3}{N02\,n1^3\,N12},\ \frac{10\,a0^2}{N01\,N02\,n1^3}-\frac{10\,a1^2}{N01\,n1^3\,N12}-\frac{10\,a2^2}{N02\,n1^3\,N12}+\frac{10\,rY^2}{n1},$$
$$\frac{5\,a0}{N01\,N02\,n1^3}-\frac{5\,a1}{N01\,n1^3\,N12}-\frac{5\,a2}{N02\,n1^3\,N12}-\frac{5\,(cYn1+cY1\,n1)\,rY}{n1^2},$$
$$\left.\frac{1}{N01\,N02\,n1^3}+\frac{cYn1^2+cY1\,cYn1\,n1+cY1^2\,n1^2-cY2\,n1^2}{n1^3}-\frac{1}{N01\,n1^3\,N12}-\frac{1}{N02\,n1^3\,N12}\right\}$$

```
Factor[μs1x3YA1 + μs1x3P0A1 + μs1x3P1A1 + μs1x3P2A1]
```

$$\left\{\frac{a0^3\,N12\,Sn0-a1^3\,N02\,Sn1-a2^3\,N01\,Sn2}{N01\,N02\,n1^3\,N12},\right.$$
$$\frac{3\left(2\,N01\,N02\,n1^2\,N12\,rY^2+N01\,N02\,n1^4\,N12\,rY^2+a0^2\,N12\,Sn0-a1^2\,N02\,Sn1-a2^2\,N01\,Sn2\right)}{N01\,N02\,n1^3\,N12},$$
$$-\frac{1}{N01\,N02\,n1^3\,N12}\,3\,\big(2\,cYn1\,N01\,N02\,n1\,N12\,rY-cYn1\,N01\,N02\,n1^3\,N12\,rY+$$
$$cY1\,N01\,N02\,n1^4\,N12\,rY-a0\,N12\,Sn0+a1\,N02\,Sn1+a2\,N01\,Sn2\big),$$
$$\frac{1}{N01\,N02\,n1^3\,N12}\,\big(2\,cYn1^2\,N01\,N02\,N12+cY1^2\,N01\,N02\,n1^2\,N12-4\,cY2\,N01\,N02\,n1^2\,N12-$$
$$cY1\,cYn1\,N01\,N02\,n1^3\,N12+cY1^2\,N01\,N02\,n1^4\,N12-cY2\,N01\,N02\,n1^4\,N12+$$
$$\left.N01\,N02\,n1^2\,N12\,pY1+N12\,Sn0-N02\,Sn1-N01\,Sn2\big)\right\}$$

```
Factor[Solve[{
  d == a0^5/(N01 N02 n1^3) - a1^5/(N01 n1^3 N12) - a2^5/(N02 n1^3 N12),
  0 == 5 a0^4/(N01 N02 n1^3) - 5 a1^4/(N01 n1^3 N12) - 5 a2^4/(N02 n1^3 N12),
  0 == 10 a0^3/(N01 N02 n1^3) - 10 a1^3/(N01 n1^3 N12) - 10 a2^3/(N02 n1^3 N12),
  0 == 10 a0^2/(N01 N02 n1^3) - 10 a1^2/(N01 n1^3 N12) - 10 a2^2/(N02 n1^3 N12) + 10 rY^2/n1,
  0 == 5 a0/(N01 N02 n1^3) - 5 a1/(N01 n1^3 N12) - 5 a2/(N02 n1^3 N12) - 5 (cYn1 + cY1 n1) rY/n1^2,
  0 == 1/(N01 N02 n1^3) + (cYn1^2 + cY1 cYn1 n1 + cY1^2 n1^2 - cY2 n1^2)/n1^3 - 1/(N01 n1^3 N12) - 1/(N02 n1^3 N12),
  p d == (a0^3 N12 Sn0 - a1^3 N02 Sn1 - a2^3 N01 Sn2)/(N01 N02 n1^3 N12),
  0 ==
   3 (2 N01 N02 n1^2 N12 rY^2 + N01 N02 n1^4 N12 rY^2 + a0^2 N12 Sn0 - a1^2 N02 Sn1 - a2^2 N01 Sn2)/(N01 N02 n1^3 N12),
  0 == -1/(N01 N02 n1^3 N12) 3
    (2 cYn1 N01 N02 n1 N12 rY - cYn1 N01 N02 n1^3 N12 rY + cY1 N01 N02 n1^4 N12 rY -
      a0 N12 Sn0 + a1 N02 Sn1 + a2 N01 Sn2),
  0 == 1/(N01 N02 n1^3 N12)
    (2 cYn1^2 N01 N02 N12 + cY1^2 N01 N02 n1^2 N12 - 4 cY2 N01 N02 n1^2 N12 -
      cY1 cYn1 N01 N02 n1^3 N12 + cY1^2 N01 N02 n1^4 N12 - cY2 N01 N02 n1^4 N12 +
      N01 N02 n1^2 N12 pY1 + N12 Sn0 - N02 Sn1 - N01 Sn2),
  pY1 == 3
 }, {p, d, N01, N02, N12, Sn0, Sn1, Sn2, pY1, cY1, cY2}]]
```

$$\Big\{\Big\{p\to\frac{1}{a0^2\,a1^2\,a2^2\,n1^2\,rY^2}\,(-a0^2\,a1^2\,a2^2\,cYn1^2+3\,a0^2\,a1^2\,a2^2\,n1^2+a0^2\,a1^2\,a2^2\,cYn1^2\,n1^2+a0^2\,a1^2\,n1^2\,rY^2+a0^2\,a2^2\,n1^2\,rY^2+a1^2\,a2^2\,n1^2\,rY^2),\ d\to-\frac{a0\,a1\,a2\,rY^2}{n1},$$

$$N01\to-\frac{a0\,(a0-a1)\,a1}{a2^2\,n1\,rY},\ N02\to\frac{a0\,(a0-a2)\,a2}{a1^2\,n1\,rY},\ N12\to\frac{a1\,(a1-a2)\,a2}{a0^2\,n1\,rY},$$

$$Sn0\to\frac{1}{a1^2\,a2^2\,n1^2\,rY^2}\,(-a0^2\,a1^2\,a2^2\,cYn1^2+3\,a0^2\,a1^2\,a2^2\,n1^2+a0^2\,a1^2\,a2^2\,cYn1^2\,n1^2-2\,a0\,a1^2\,a2^2\,cYn1\,n1\,rY+2\,a0\,a1^2\,a2^2\,cYn1\,n1^3\,rY+a0^2\,a1^2\,n1^2\,rY^2-2\,a0\,a1^2\,a2\,n1^2\,rY^2+a0^2\,a2^2\,n1^2\,rY^2-2\,a0\,a1\,a2^2\,n1^2\,rY^2+a1^2\,a2^2\,n1^2\,rY^2+a1^2\,a2^2\,n1^4\,rY^2),$$

$$Sn1\to\frac{1}{a0^2\,a2^2\,n1^2\,rY^2}\,(-a0^2\,a1^2\,a2^2\,cYn1^2+3\,a0^2\,a1^2\,a2^2\,n1^2+a0^2\,a1^2\,a2^2\,cYn1^2\,n1^2-2\,a0^2\,a1\,a2^2\,cYn1\,n1\,rY+2\,a0^2\,a1\,a2^2\,cYn1\,n1^3\,rY+a0^2\,a1^2\,n1^2\,rY^2-2\,a0^2\,a1\,a2\,n1^2\,rY^2+a0^2\,a2^2\,n1^2\,rY^2-2\,a0\,a1\,a2^2\,n1^2\,rY^2+a1^2\,a2^2\,n1^2\,rY^2+a0^2\,a2^2\,n1^4\,rY^2),$$

$$Sn2\to\frac{1}{a0^2\,a1^2\,n1^2\,rY^2}\,(-a0^2\,a1^2\,a2^2\,cYn1^2+3\,a0^2\,a1^2\,a2^2\,n1^2+a0^2\,a1^2\,a2^2\,cYn1^2\,n1^2-2\,a0^2\,a1^2\,a2\,cYn1\,n1\,rY+2\,a0^2\,a1^2\,a2\,cYn1\,n1^3\,rY+a0^2\,a1^2\,n1^2\,rY^2-2\,a0^2\,a1\,a2\,n1^2\,rY^2-2\,a0\,a1^2\,a2\,n1^2\,rY^2+a0^2\,a2^2\,n1^2\,rY^2+a1^2\,a2^2\,n1^2\,rY^2+a0^2\,a1^2\,n1^4\,rY^2),$$

$$pY1\to3,\ cY1\to-\frac{a0\,a1\,a2\,cYn1+a0\,a1\,n1\,rY+a0\,a2\,n1\,rY+a1\,a2\,n1\,rY}{a0\,a1\,a2\,n1},$$

$$cY2\to\frac{1}{a0\,a1\,a2\,n1^2}\,(a0\,a1\,a2\,cYn1^2+a0\,a1\,cYn1\,n1\,rY+a0\,a2\,cYn1\,n1\,rY+a1\,a2\,cYn1\,n1\,rY+a0\,n1^2\,rY^2+a1\,n1^2\,rY^2+a2\,n1^2\,rY^2)\Big\},$$

$$\Big\{p\to\frac{1}{a0^2\,a1^2\,a2^2\,n1^2\,rY^2}\,(-a0^2\,a1^2\,a2^2\,cYn1^2+3\,a0^2\,a1^2\,a2^2\,n1^2+a0^2\,a1^2\,a2^2\,cYn1^2\,n1^2+a0^2\,a1^2\,n1^2\,rY^2+a0^2\,a2^2\,n1^2\,rY^2+a1^2\,a2^2\,n1^2\,rY^2),$$

$$d\to-\frac{a0\,a1\,a2\,rY^2}{n1},\ N01\to\frac{a0\,(a0-a1)\,a1}{a2^2\,n1\,rY},\ N02\to-\frac{a0\,(a0-a2)\,a2}{a1^2\,n1\,rY},$$

$$N12\to-\frac{a1\,(a1-a2)\,a2}{a0^2\,n1\,rY},$$

$$Sn0\to\frac{1}{a1^2\,a2^2\,n1^2\,rY^2}\,(-a0^2\,a1^2\,a2^2\,cYn1^2+3\,a0^2\,a1^2\,a2^2\,n1^2+a0^2\,a1^2\,a2^2\,cYn1^2\,n1^2-2\,a0\,a1^2\,a2^2\,cYn1\,n1\,rY+2\,a0\,a1^2\,a2^2\,cYn1\,n1^3\,rY+a0^2\,a1^2\,n1^2\,rY^2-2\,a0\,a1^2\,a2\,n1^2\,rY^2+a0^2\,a2^2\,n1^2\,rY^2-2\,a0\,a1\,a2^2\,n1^2\,rY^2+a1^2\,a2^2\,n1^2\,rY^2+a1^2\,a2^2\,n1^4\,rY^2),$$

$$Sn1\to\frac{1}{a0^2\,a2^2\,n1^2\,rY^2}\,(-a0^2\,a1^2\,a2^2\,cYn1^2+3\,a0^2\,a1^2\,a2^2\,n1^2+a0^2\,a1^2\,a2^2\,cYn1^2\,n1^2-2\,a0^2\,a1\,a2^2\,cYn1\,n1\,rY+2\,a0^2\,a1\,a2^2\,cYn1\,n1^3\,rY+a0^2\,a1^2\,n1^2\,rY^2-2\,a0^2\,a1\,a2\,n1^2\,rY^2+a0^2\,a2^2\,n1^2\,rY^2-2\,a0\,a1\,a2^2\,n1^2\,rY^2+a1^2\,a2^2\,n1^2\,rY^2+a0^2\,a2^2\,n1^4\,rY^2),$$

$$Sn2\to\frac{1}{a0^2\,a1^2\,n1^2\,rY^2}\,(-a0^2\,a1^2\,a2^2\,cYn1^2+3\,a0^2\,a1^2\,a2^2\,n1^2+a0^2\,a1^2\,a2^2\,cYn1^2\,n1^2-2\,a0^2\,a1^2\,a2\,cYn1\,n1\,rY+2\,a0^2\,a1^2\,a2\,cYn1\,n1^3\,rY+a0^2\,a1^2\,n1^2\,rY^2-2\,a0^2\,a1\,a2\,n1^2\,rY^2-2\,a0\,a1^2\,a2\,n1^2\,rY^2+a0^2\,a2^2\,n1^2\,rY^2+a1^2\,a2^2\,n1^2\,rY^2+a0^2\,a1^2\,n1^4\,rY^2),$$

$$pY1\to3,\ cY1\to-\frac{a0\,a1\,a2\,cYn1+a0\,a1\,n1\,rY+a0\,a2\,n1\,rY+a1\,a2\,n1\,rY}{a0\,a1\,a2\,n1},$$

$$cY2\to\frac{1}{a0\,a1\,a2\,n1^2}\,(a0\,a1\,a2\,cYn1^2+a0\,a1\,cYn1\,n1\,rY+a0\,a2\,cYn1\,n1\,rY+a1\,a2\,cYn1\,n1\,rY+a0\,n1^2\,rY^2+a1\,n1^2\,rY^2+a2\,n1^2\,rY^2)\Big\}\Big\}$$

Computations for Lemma 6.19.
Case A.2 where there exists N_i^6 s.t. Fix(N_i) = $Y^4 \cup P_0 \cup P_1 \cup$

P_2.

```
(******************************
 NOTATIONS
 rY xY denotes the restriction of x to Y
 ai denotes the Hopf weight at Pi
 Nij denotes the product of the normal weights shared between Pi and Pj
 Sni is the sum of the square of the normal weights at Pi
 cYn1 xY denotes the first Chern class of the normal bundle of Y in the
 direction of N1
 cYn2 xY denotes the first Chern class of the normal bundle of Y in the
 direction of N2
 cY1 xY denotes the first Chern class of the normal bundle of Y in the
 remaining directions
 μx5YA denotes the local datum of x^5 at Y
 μx5PiA denotes the local datum of x^5 at Pi

 We use the variable t to extract the coefficient of order 2 of xY. We
 then evaluate xY^2 to 1.
 The orientation of the point P0 is 1 while those from P1 and P2 are -
1. This explains the signs at their local datum
 Once evaluated, we extract all the coefficients in 1 into a list.
 ******************************)
```

```
μx5YA2 =
 Factor[CoefficientList[
   Coefficient[Expand[(rY xY t + 1 z)^5] Series[(cYn1 xY t + n1 z)^-1, {xY, 0, 2}]
     Series[(cYn2 xY t + n2 z)^-1, {xY, 0, 2}] Series[(cY1 xY t + z)^-1, {xY, 0, 2}], t, 2],
   1]] /. xY^2 → 1
```

$$\left\{0,\ 0,\ 0,\ \frac{10\, rY^2}{n1\, n2},\ -\frac{5\,(cYn2\, n1 + cYn1\, n2 + cY1\, n1\, n2)\, rY}{n1^2\, n2^2},\ \frac{cYn2^2\, n1^2 + cYn1\, cYn2\, n1\, n2 + cY1\, cYn2\, n1^2\, n2 + cYn1^2\, n2^2 + cY1\, cYn1\, n1\, n2^2 + cY1^2\, n1^2\, n2^2}{n1^3\, n2^3}\right\}$$

```
μx5P0A2 = CoefficientList[Expand[((a0 + 1) z)^5] n2^-3 n1^-3 N01^-1 N02^-1 z^-5, 1]
```

$$\left\{\frac{a0^5}{N01\, N02\, n1^3\, n2^3},\ \frac{5\, a0^4}{N01\, N02\, n1^3\, n2^3},\ \frac{10\, a0^3}{N01\, N02\, n1^3\, n2^3},\ \frac{10\, a0^2}{N01\, N02\, n1^3\, n2^3},\ \frac{5\, a0}{N01\, N02\, n1^3\, n2^3},\ \frac{1}{N01\, N02\, n1^3\, n2^3}\right\}$$

```
μx5P1A2 = CoefficientList[Expand[-((a1 + 1) z)^5] n1^-3 n2^-3 N01^-1 N12^-1 z^-5, 1]
```

$$\left\{-\frac{a1^5}{N01\, n1^3\, N12\, n2^3},\ -\frac{5\, a1^4}{N01\, n1^3\, N12\, n2^3},\ -\frac{10\, a1^3}{N01\, n1^3\, N12\, n2^3},\ -\frac{10\, a1^2}{N01\, n1^3\, N12\, n2^3},\ -\frac{5\, a1}{N01\, n1^3\, N12\, n2^3},\ -\frac{1}{N01\, n1^3\, N12\, n2^3}\right\}$$

```
μx5P2A2 = CoefficientList[Expand[-((a2 + 1) z)^5] n1^-3 n2^-3 N02^-1 N12^-1 z^-5, 1]
```

$$\left\{-\frac{a2^5}{N02\, n1^3\, N12\, n2^3},\ -\frac{5\, a2^4}{N02\, n1^3\, N12\, n2^3},\ -\frac{10\, a2^3}{N02\, n1^3\, N12\, n2^3},\ -\frac{10\, a2^2}{N02\, n1^3\, N12\, n2^3},\ -\frac{5\, a2}{N02\, n1^3\, N12\, n2^3},\ -\frac{1}{N02\, n1^3\, N12\, n2^3}\right\}$$

```
(******************************
 NOTATIONS
 μs1x3YA2 denotes the local datum of p1(X)x^3 at Y
 μs1x3PiA2 denotes the local datum of p1(X)x^3 at Pi
 ******************************)
```

```
SymmetricReduction[Expand[(yY3 t + z)^2], {yY3}, {cY1 xY}]
```

$$\left\{cY1^2\, t^2\, xY^2 + 2\, cY1\, t\, xY\, z + z^2,\ 0\right\}$$

```
μs1x3YA2 =
 Factor[CoefficientList[
    Coefficient[
     Expand[pY1 t^2 + (cYn1 xY t + n1 z)^2 + (cYn2 xY t + n2 z)^2 + cY1^2 t^2 xY^2 + 2 cY1 t xY z + z^2]
      (rY xY t + 1 z)^3 Series[(cYn1 xY t + n1 z)^-1, {xY, 0, 2}]
      Series[(cYn2 xY t + n2 z)^-1, {xY, 0, 2}] Series[(cY1 xY t + z)^-1, {xY, 0, 2}], t, 2],
    1]] /. xY^2 → 1
```

$$\left\{0,\ \frac{3\left(1+n1^2+n2^2\right) rY^2}{n1\, n2},\right.$$
$$-\frac{1}{n1^2\, n2^2}\, 3\left(cYn2\, n1 + cYn2\, n1^3 + cYn1\, n2 - cY1\, n1\, n2 - cYn1\, n1^2\, n2 + cY1\, n1^3\, n2 - cYn2\, n1\, n2^2 + cYn1\, n2^3 + cY1\, n1\, n2^3\right) rY,$$
$$\frac{1}{n1^3\, n2^3}\left(cYn2^2\, n1^2 + cYn2^2\, n1^4 + cYn1\, cYn2\, n1\, n2 - cY1\, cYn2\, n1^2\, n2 - cYn1\, cYn2\, n1^3\, n2 + cY1\, cYn2\, n1^4\, n2 + cYn1^2\, n2^2 - cY1\, cYn1\, n1\, n2^2 - cY1\, cYn1\, n1^3\, n2^2 + cY1^2\, n1^4\, n2^2 - cYn1\, cYn2\, n1\, n2^3 - cY1\, cYn2\, n1^2\, n2^3 + cYn1^2\, n2^4 + cY1\, cYn1\, n1\, n2^4 + cY1^2\, n1^2\, n2^4 + n1^2\, n2^2\, pY1\right)\left.\right\}$$

```
μs1x3P0A2 = Factor[CoefficientList[Sn0 z^2 ((a0 + 1) z)^3 n1^-3 n2^-3 N01^-1 N02^-1 z^-5, 1]]
```

$$\left\{\frac{a0^3\, Sn0}{N01\, N02\, n1^3\, n2^3},\ \frac{3\, a0^2\, Sn0}{N01\, N02\, n1^3\, n2^3},\ \frac{3\, a0\, Sn0}{N01\, N02\, n1^3\, n2^3},\ \frac{Sn0}{N01\, N02\, n1^3\, n2^3}\right\}$$

```
μs1x3P1A2 = Factor[CoefficientList[- Sn1 z^2 ((a1 + 1) z)^3 n1^-3 n2^-3 N01^-1 N12^-1 z^-5, 1]]
```

$$\left\{-\frac{a1^3\, Sn1}{N01\, n1^3\, N12\, n2^3},\ -\frac{3\, a1^2\, Sn1}{N01\, n1^3\, N12\, n2^3},\ -\frac{3\, a1\, Sn1}{N01\, n1^3\, N12\, n2^3},\ -\frac{Sn1}{N01\, n1^3\, N12\, n2^3}\right\}$$

```
μs1x3P2A2 = Factor[CoefficientList[- Sn2 z^2 ((a2 + 1) z)^3 n1^-3 n2^-3 N02^-1 N12^-1 z^-5, 1]]
```

$$\left\{-\frac{a2^3\, Sn2}{N02\, n1^3\, N12\, n2^3},\ -\frac{3\, a2^2\, Sn2}{N02\, n1^3\, N12\, n2^3},\ -\frac{3\, a2\, Sn2}{N02\, n1^3\, N12\, n2^3},\ -\frac{Sn2}{N02\, n1^3\, N12\, n2^3}\right\}$$

μx5YA2 + μx5P0A2 + μx5P1A2 + μx5P2A2

$$\left\{\frac{a0^5}{N01\,N02\,n1^3\,n2^3}-\frac{a1^5}{N01\,n1^3\,N12\,n2^3}-\frac{a2^5}{N02\,n1^3\,N12\,n2^3},\right.$$
$$\frac{5\,a0^4}{N01\,N02\,n1^3\,n2^3}-\frac{5\,a1^4}{N01\,n1^3\,N12\,n2^3}-\frac{5\,a2^4}{N02\,n1^3\,N12\,n2^3},$$
$$\frac{10\,a0^3}{N01\,N02\,n1^3\,n2^3}-\frac{10\,a1^3}{N01\,n1^3\,N12\,n2^3}-\frac{10\,a2^3}{N02\,n1^3\,N12\,n2^3},$$
$$\frac{10\,a0^2}{N01\,N02\,n1^3\,n2^3}-\frac{10\,a1^2}{N01\,n1^3\,N12\,n2^3}-\frac{10\,a2^2}{N02\,n1^3\,N12\,n2^3}+\frac{10\,rY^2}{n1\,n2},$$
$$\frac{5\,a0}{N01\,N02\,n1^3\,n2^3}-\frac{5\,a1}{N01\,n1^3\,N12\,n2^3}-\frac{5\,a2}{N02\,n1^3\,N12\,n2^3}-\frac{5\,(cYn2\,n1+cYn1\,n2+cY1\,n1\,n2)\,rY}{n1^2\,n2^2},$$
$$\frac{1}{N01\,N02\,n1^3\,n2^3}-\frac{1}{N01\,n1^3\,N12\,n2^3}-\frac{1}{N02\,n1^3\,N12\,n2^3}+$$
$$\left.\frac{cYn2^2\,n1^2+cYn1\,cYn2\,n1\,n2+cY1\,cYn2\,n1^2\,n2+cYn1^2\,n2^2+cY1\,cYn1\,n1\,n2^2+cY1^2\,n1^2\,n2^2}{n1^3\,n2^3}\right\}$$

Factor[μs1x3YA2 + μs1x3P0A2 + μs1x3P1A2 + μs1x3P2A2]

$$\left\{\frac{a0^3\,N12\,Sn0-a1^3\,N02\,Sn1-a2^3\,N01\,Sn2}{N01\,N02\,n1^3\,N12\,n2^3},\right.$$
$$\frac{1}{N01\,N02\,n1^3\,N12\,n2^3}\,3\,\big(N01\,N02\,n1^2\,N12\,n2^2\,rY^2+N01\,N02\,n1^4\,N12\,n2^2\,rY^2+$$
$$N01\,N02\,n1^2\,N12\,n2^4\,rY^2+a0^2\,N12\,Sn0-a1^2\,N02\,Sn1-a2^2\,N01\,Sn2\big),$$
$$-\frac{1}{N01\,N02\,n1^3\,N12\,n2^3}\,3\,\big(cYn2\,N01\,N02\,n1^2\,N12\,n2\,rY+cYn2\,N01\,N02\,n1^4\,N12\,n2\,rY+$$
$$cYn1\,N01\,N02\,n1\,N12\,n2^2\,rY-cY1\,N01\,N02\,n1^2\,N12\,n2^2\,rY-cYn1\,N01\,N02\,n1^3\,N12\,n2^2\,rY+$$
$$cY1\,N01\,N02\,n1^4\,N12\,n2^2\,rY-cYn2\,N01\,N02\,n1^2\,N12\,n2^3\,rY+cYn1\,N01\,N02\,n1\,N12\,n2^4\,rY+$$
$$cY1\,N01\,N02\,n1^2\,N12\,n2^4\,rY-a0\,N12\,Sn0+a1\,N02\,Sn1+a2\,N01\,Sn2\big),$$
$$\frac{1}{N01\,N02\,n1^3\,N12\,n2^3}\,\big(cYn2^2\,N01\,N02\,n1^2\,N12+cYn2^2\,N01\,N02\,n1^4\,N12+$$
$$cYn1\,cYn2\,N01\,N02\,n1\,N12\,n2-cY1\,cYn2\,N01\,N02\,n1^2\,N12\,n2-cYn1\,cYn2\,N01\,N02\,n1^3\,N12\,n2+$$
$$cY1\,cYn2\,N01\,N02\,n1^4\,N12\,n2+cYn1^2\,N01\,N02\,N12\,n2^2-cY1\,cYn1\,N01\,N02\,n1\,N12\,n2^2-$$
$$cY1\,cYn1\,N01\,N02\,n1^3\,N12\,n2^2+cY1^2\,N01\,N02\,n1^4\,N12\,n2^2-cYn1\,cYn2\,N01\,N02\,n1\,N12\,n2^3-$$
$$cY1\,cYn2\,N01\,N02\,n1^2\,N12\,n2^3+cYn1^2\,N01\,N02\,N12\,n2^4+cY1\,cYn1\,N01\,N02\,n1\,N12\,n2^4+$$
$$\left.cY1^2\,N01\,N02\,n1^2\,N12\,n2^4+N01\,N02\,n1^2\,N12\,n2^2\,pY1+N12\,Sn0-N02\,Sn1-N01\,Sn2\big)\right\}$$

```
Factor[Solve[{
  d == a0^5/(N01 N02 n1^3 n2^3) - a1^5/(N01 n1^3 N12 n2^3) - a2^5/(N02 n1^3 N12 n2^3),
  0 == 5 a0^4/(N01 N02 n1^3 n2^3) - 5 a1^4/(N01 n1^3 N12 n2^3) - 5 a2^4/(N02 n1^3 N12 n2^3),
  0 == 10 a0^3/(N01 N02 n1^3 n2^3) - 10 a1^3/(N01 n1^3 N12 n2^3) - 10 a2^3/(N02 n1^3 N12 n2^3),
  0 == 10 a0^2/(N01 N02 n1^3 n2^3) - 10 a1^2/(N01 n1^3 N12 n2^3) - 10 a2^2/(N02 n1^3 N12 n2^3) + 10 rY^2/(n1 n2),
  0 == 5 a0/(N01 N02 n1^3 n2^3) - 5 a1/(N01 n1^3 N12 n2^3) - 5 a2/(N02 n1^3 N12 n2^3) -
    5 (cYn2 n1 + cYn1 n2 + cY1 n1 n2) rY/(n1^2 n2^2),
  0 == 1/(N01 N02 n1^3 n2^3) - 1/(N01 n1^3 N12 n2^3) - 1/(N02 n1^3 N12 n2^3) +
    (cYn2^2 n1^2 + cYn1 cYn2 n1 n2 + cY1 cYn2 n1^2 n2 + cYn1^2 n2^2 + cY1 cYn1 n1 n2^2 + cY1^2 n1^2 n2^2)/(n1^3 n2^3),
  p d == (a0^3 N12 Sn0 - a1^3 N02 Sn1 - a2^3 N01 Sn2)/(N01 N02 n1^3 N12 n2^3),
  0 == 1/(N01 N02 n1^3 N12 n2^3)
    3 (N01 N02 n1^2 N12 n2^2 rY^2 + N01 N02 n1^4 N12 n2^2 rY^2 + N01 N02 n1^2 N12 n2^4 rY^2 +
      a0^2 N12 Sn0 - a1^2 N02 Sn1 - a2^2 N01 Sn2),
  0 == -1/(N01 N02 n1^3 N12 n2^3) 3
    (cYn2 N01 N02 n1^2 N12 n2 rY + cYn2 N01 N02 n1^4 N12 n2 rY + cYn1 N01 N02 n1 N12 n2^2 rY -
      cY1 N01 N02 n1^2 N12 n2^2 rY - cYn1 N01 N02 n1^3 N12 n2^2 rY + cY1 N01 N02 n1^4 N12 n2^2 rY -
      cYn2 N01 N02 n1^2 N12 n2^3 rY + cYn1 N01 N02 n1 N12 n2^4 rY + cY1 N01 N02 n1^2 N12 n2^4 rY -
      a0 N12 Sn0 + a1 N02 Sn1 + a2 N01 Sn2),
  0 == 1/(N01 N02 n1^3 N12 n2^3)
    (cYn2^2 N01 N02 n1^2 N12 + cYn2^2 N01 N02 n1^4 N12 + cYn1 cYn2 N01 N02 n1 N12 n2 -
      cY1 cYn2 N01 N02 n1^2 N12 n2 - cYn1 cYn2 N01 N02 n1^3 N12 n2 + cY1 cYn2 N01 N02 n1^4 N12 n2 +
      cYn1^2 N01 N02 N12 n2^2 - cY1 cYn1 N01 N02 n1 N12 n2^2 - cY1 cYn1 N01 N02 n1^3 N12 n2^2 +
      cY1^2 N01 N02 n1^4 N12 n2^2 - cYn1 cYn2 N01 N02 n1 N12 n2^3 - cY1 cYn2 N01 N02 n1^2 N12 n2^3 +
      cYn1^2 N01 N02 N12 n2^4 + cY1 cYn1 N01 N02 n1 N12 n2^4 + cY1^2 N01 N02 n1^2 N12 n2^4 +
      N01 N02 n1^2 N12 n2^2 pY1 + N12 Sn0 - N02 Sn1 - N01 Sn2),
  pY1 == 3
 }, {p, d, N01, N02, N12, Sn0, Sn1, Sn2, pY1, n1, n2}]]
```

$$\Big\{\Big\{p \to \frac{3 + cY1^2 + cYn1^2 + cYn2^2}{rY^2},$$

$$d \to -\frac{rY^2\,(a0\,a1\,a2\,cY1^2 + a0\,a1\,cY1\,rY + a0\,a2\,cY1\,rY + a1\,a2\,cY1\,rY + a0\,rY^2 + a1\,rY^2 + a2\,rY^2)}{cYn1\,cYn2},$$

$$N01 \to -\frac{(a0 - a1)\,(a0\,a1\,a2\,cY1^2 + a0\,a1\,cY1\,rY + a0\,a2\,cY1\,rY + a1\,a2\,cY1\,rY + a0\,rY^2 + a1\,rY^2 + a2\,rY^2)}{a2^3\,cYn1\,cYn2\,rY},$$

$$N02 \to \frac{(a0 - a2)\,(a0\,a1\,a2\,cY1^2 + a0\,a1\,cY1\,rY + a0\,a2\,cY1\,rY + a1\,a2\,cY1\,rY + a0\,rY^2 + a1\,rY^2 + a2\,rY^2)}{a1^3\,cYn1\,cYn2\,rY},$$

$N12 \to$

$$\frac{(a1-a2)\left(a0\ a1\ a2\ cY1^{2}+a0\ a1\ cY1\ rY+a0\ a2\ cY1\ rY+a1\ a2\ cY1\ rY+a0\ rY^{2}+a1\ rY^{2}+a2\ rY^{2}\right)}{a0^{3}\ cYn1\ cYn2\ rY},$$

$$\begin{aligned}
Sn0 \to \Big(&6\ a0^{4}\ a1^{2}\ a2^{2}\ cY1^{4}\ cYn1+2\ a0^{4}\ a1^{2}\ a2^{2}\ cY1^{6}\ cYn1+2\ a0^{4}\ a1^{2}\ a2^{2}\ cY1^{4}\ cYn1^{3}+\\
&2\ a0^{4}\ a1^{2}\ a2^{2}\ cY1^{4}\ cYn1\ cYn2^{2}+12\ a0^{4}\ a1^{2}\ a2\ cY1^{3}\ cYn1\ rY+12\ a0^{4}\ a1\ a2^{2}\ cY1^{3}\ cYn1\ rY+\\
&12\ a0^{3}\ a1^{2}\ a2^{2}\ cY1^{3}\ cYn1\ rY+4\ a0^{4}\ a1^{2}\ a2\ cY1^{5}\ cYn1\ rY+4\ a0^{4}\ a1\ a2^{2}\ cY1^{5}\ cYn1\ rY+\\
&8\ a0^{3}\ a1^{2}\ a2^{2}\ cY1^{5}\ cYn1\ rY+4\ a0^{4}\ a1^{2}\ a2\ cY1^{3}\ cYn1^{3}\ rY+4\ a0^{4}\ a1\ a2^{2}\ cY1^{3}\ cYn1^{3}\ rY+\\
&2\ a0^{3}\ a1^{2}\ a2^{2}\ cY1^{3}\ cYn1^{3}\ rY+4\ a0^{4}\ a1^{2}\ a2\ cY1^{3}\ cYn1\ cYn2^{2}\ rY+\\
&4\ a0^{4}\ a1\ a2^{2}\ cY1^{3}\ cYn1\ cYn2^{2}\ rY+2\ a0^{3}\ a1^{2}\ a2^{2}\ cY1^{3}\ cYn1\ cYn2^{2}\ rY+6\ a0^{4}\ a1^{2}\ cY1^{2}\ cYn1\ rY^{2}+\\
&24\ a0^{4}\ a1\ a2\ cY1^{2}\ cYn1\ rY^{2}+24\ a0^{3}\ a1^{2}\ a2\ cY1^{2}\ cYn1\ rY^{2}+6\ a0^{4}\ a2^{2}\ cY1^{2}\ cYn1\ rY^{2}+\\
&24\ a0^{3}\ a1\ a2^{2}\ cY1^{2}\ cYn1\ rY^{2}+6\ a0^{2}\ a1^{2}\ a2^{2}\ cY1^{2}\ cYn1\ rY^{2}+2\ a0^{4}\ a1^{2}\ cY1^{4}\ cYn1\ rY^{2}+\\
&8\ a0^{4}\ a1\ a2\ cY1^{4}\ cYn1\ rY^{2}+16\ a0^{3}\ a1^{2}\ a2\ cY1^{4}\ cYn1\ rY^{2}+2\ a0^{4}\ a2^{2}\ cY1^{4}\ cYn1\ rY^{2}+\\
&16\ a0^{3}\ a1\ a2^{2}\ cY1^{4}\ cYn1\ rY^{2}+12\ a0^{2}\ a1^{2}\ a2^{2}\ cY1^{4}\ cYn1\ rY^{2}+2\ a0^{4}\ a1^{2}\ cY1^{2}\ cYn1^{3}\ rY^{2}+\\
&8\ a0^{4}\ a1\ a2\ cY1^{2}\ cYn1^{3}\ rY^{2}+4\ a0^{3}\ a1^{2}\ a2\ cY1^{2}\ cYn1^{3}\ rY^{2}+2\ a0^{4}\ a2^{2}\ cY1^{2}\ cYn1^{3}\ rY^{2}+\\
&4\ a0^{3}\ a1\ a2^{2}\ cY1^{2}\ cYn1^{3}\ rY^{2}-3\ a0^{2}\ a1^{2}\ a2^{2}\ cY1^{2}\ cYn1^{3}\ rY^{2}+2\ a0^{4}\ a1^{2}\ cY1^{2}\ cYn1\ cYn2^{2}\ rY^{2}+\\
&8\ a0^{4}\ a1\ a2\ cY1^{2}\ cYn1\ cYn2^{2}\ rY^{2}+4\ a0^{3}\ a1^{2}\ a2\ cY1^{2}\ cYn1\ cYn2^{2}\ rY^{2}+\\
&2\ a0^{4}\ a2^{2}\ cY1^{2}\ cYn1\ cYn2^{2}\ rY^{2}+4\ a0^{3}\ a1\ a2^{2}\ cY1^{2}\ cYn1\ cYn2^{2}\ rY^{2}-\\
&3\ a0^{2}\ a1^{2}\ a2^{2}\ cY1^{2}\ cYn1\ cYn2^{2}\ rY^{2}+12\ a0^{4}\ a1\ cY1\ cYn1\ rY^{3}+12\ a0^{3}\ a1^{2}\ cY1\ cYn1\ rY^{3}+\\
&12\ a0^{4}\ a2\ cY1\ cYn1\ rY^{3}+36\ a0^{3}\ a1\ a2\ cY1\ cYn1\ rY^{3}+12\ a0^{2}\ a1^{2}\ a2\ cY1\ cYn1\ rY^{3}+\\
&12\ a0^{3}\ a2^{2}\ cY1\ cYn1\ rY^{3}+12\ a0^{2}\ a1\ a2^{2}\ cY1\ cYn1\ rY^{3}+4\ a0^{4}\ a1\ cY1^{3}\ cYn1\ rY^{3}+\\
&8\ a0^{3}\ a1^{2}\ cY1^{3}\ cYn1\ rY^{3}+4\ a0^{4}\ a2\ cY1^{3}\ cYn1\ rY^{3}+28\ a0^{3}\ a1\ a2\ cY1^{3}\ cYn1\ rY^{3}+\\
&24\ a0^{2}\ a1^{2}\ a2\ cY1^{3}\ cYn1\ rY^{3}+8\ a0^{3}\ a2^{2}\ cY1^{3}\ cYn1\ rY^{3}+24\ a0^{2}\ a1\ a2^{2}\ cY1^{3}\ cYn1\ rY^{3}+\\
&8\ a0\ a1^{2}\ a2^{2}\ cY1^{3}\ cYn1\ rY^{3}+4\ a0^{4}\ a1\ cY1\ cYn1^{3}\ rY^{3}+2\ a0^{3}\ a1^{2}\ cY1\ cYn1^{3}\ rY^{3}+\\
&4\ a0^{4}\ a2\ cY1\ cYn1^{3}\ rY^{3}+6\ a0^{3}\ a1\ a2\ cY1\ cYn1^{3}\ rY^{3}-2\ a0^{2}\ a1^{2}\ a2\ cY1\ cYn1^{3}\ rY^{3}+\\
&2\ a0^{3}\ a2^{2}\ cY1\ cYn1^{3}\ rY^{3}-2\ a0^{2}\ a1\ a2^{2}\ cY1\ cYn1^{3}\ rY^{3}-2\ a0\ a1^{2}\ a2^{2}\ cY1\ cYn1^{3}\ rY^{3}+\\
&4\ a0^{4}\ a1\ cY1\ cYn1\ cYn2^{2}\ rY^{3}+2\ a0^{3}\ a1^{2}\ cY1\ cYn1\ cYn2^{2}\ rY^{3}+4\ a0^{4}\ a2\ cY1\ cYn1\ cYn2^{2}\ rY^{3}+\\
&6\ a0^{3}\ a1\ a2\ cY1\ cYn1\ cYn2^{2}\ rY^{3}-2\ a0^{2}\ a1^{2}\ a2\ cY1\ cYn1\ cYn2^{2}\ rY^{3}+\\
&2\ a0^{3}\ a2^{2}\ cY1\ cYn1\ cYn2^{2}\ rY^{3}-2\ a0^{2}\ a1\ a2^{2}\ cY1\ cYn1\ cYn2^{2}\ rY^{3}-\\
&2\ a0\ a1^{2}\ a2^{2}\ cY1\ cYn1\ cYn2^{2}\ rY^{3}+6\ a0^{4}\ cYn1\ rY^{4}+12\ a0^{3}\ a1\ cYn1\ rY^{4}+6\ a0^{2}\ a1^{2}\ cYn1\ rY^{4}+\\
&12\ a0^{3}\ a2\ cYn1\ rY^{4}+12\ a0^{2}\ a1\ a2\ cYn1\ rY^{4}+6\ a0^{2}\ a2^{2}\ cYn1\ rY^{4}+2\ a0^{4}\ cY1^{2}\ cYn1\ rY^{4}+\\
&12\ a0^{3}\ a1\ cY1^{2}\ cYn1\ rY^{4}+12\ a0^{2}\ a1^{2}\ cY1^{2}\ cYn1\ rY^{4}+12\ a0^{3}\ a2\ cY1^{2}\ cYn1\ rY^{4}+\\
&36\ a0^{2}\ a1\ a2\ cY1^{2}\ cYn1\ rY^{4}+16\ a0\ a1^{2}\ a2\ cY1^{2}\ cYn1\ rY^{4}+12\ a0^{2}\ a2^{2}\ cY1^{2}\ cYn1\ rY^{4}+\\
&16\ a0\ a1\ a2^{2}\ cY1^{2}\ cYn1\ rY^{4}+2\ a1^{2}\ a2^{2}\ cY1^{2}\ cYn1\ rY^{4}+2\ a0^{4}\ cYn1^{3}\ rY^{4}+2\ a0^{3}\ a1\ cYn1^{3}\ rY^{4}+\\
&a0^{2}\ a1^{2}\ cYn1^{3}\ rY^{4}+2\ a0^{3}\ a2\ cYn1^{3}\ rY^{4}-2\ a0^{2}\ a1\ a2\ cYn1^{3}\ rY^{4}-2\ a0\ a1^{2}\ a2\ cYn1^{3}\ rY^{4}+\\
&a0^{2}\ a2^{2}\ cYn1^{3}\ rY^{4}-2\ a0\ a1\ a2^{2}\ cYn1^{3}\ rY^{4}+a1^{2}\ a2^{2}\ cYn1^{3}\ rY^{4}+2\ a0^{4}\ cYn1\ cYn2^{2}\ rY^{4}+\\
&2\ a0^{3}\ a1\ cYn1\ cYn2^{2}\ rY^{4}+a0^{2}\ a1^{2}\ cYn1\ cYn2^{2}\ rY^{4}+2\ a0^{3}\ a2\ cYn1\ cYn2^{2}\ rY^{4}-\\
&2\ a0^{2}\ a1\ a2\ cYn1\ cYn2^{2}\ rY^{4}-2\ a0\ a1^{2}\ a2\ cYn1\ cYn2^{2}\ rY^{4}+a0^{2}\ a2^{2}\ cYn1\ cYn2^{2}\ rY^{4}-\\
&2\ a0\ a1\ a2^{2}\ cYn1\ cYn2^{2}\ rY^{4}+a1^{2}\ a2^{2}\ cYn1\ cYn2^{2}\ rY^{4}+4\ a0^{3}\ cY1\ cYn1\ rY^{5}+\\
&12\ a0^{2}\ a1\ cY1\ cYn1\ rY^{5}+8\ a0\ a1^{2}\ cY1\ cYn1\ rY^{5}+12\ a0^{2}\ a2\ cY1\ cYn1\ rY^{5}+\\
&20\ a0\ a1\ a2\ cY1\ cYn1\ rY^{5}+4\ a1^{2}\ a2\ cY1\ cYn1\ rY^{5}+8\ a0\ a2^{2}\ cY1\ cYn1\ rY^{5}+\\
&4\ a1\ a2^{2}\ cY1\ cYn1\ rY^{5}+2\ a0^{2}\ cYn1\ rY^{6}+4\ a0\ a1\ cYn1\ rY^{6}+2\ a1^{2}\ cYn1\ rY^{6}+\\
&4\ a0\ a2\ cYn1\ rY^{6}+4\ a1\ a2\ cYn1\ rY^{6}+2\ a2^{2}\ cYn1\ rY^{6}-2\ a0^{2}\ a1\ a2\ cY1^{2}\ cYn1^{2}\\
&\quad rY\sqrt{\Big((a0\ a1\ a2\ cY1\ cYn1+a0\ a1\ cYn1\ rY+a0\ a2\ cYn1\ rY+a1\ a2\ cYn1\ rY)^{2}-}\\
&\qquad 4\ a0\ a1\ a2\ cYn1^{2}\ \big(a0\ a1\ a2\ cY1^{2}+a0\ a1\ cY1\ rY+a0\ a2\ cY1\ rY+\\
&\qquad\quad a1\ a2\ cY1\ rY+a0\ rY^{2}+a1\ rY^{2}+a2\ rY^{2}\big)\Big)+2\ a0^{2}\ a1\ a2\ cY1^{2}\ cYn2^{2}\\
&\quad rY\sqrt{\Big((a0\ a1\ a2\ cY1\ cYn1+a0\ a1\ cYn1\ rY+a0\ a2\ cYn1\ rY+a1\ a2\ cYn1\ rY)^{2}-}\\
&\qquad 4\ a0\ a1\ a2\ cYn1^{2}\ \big(a0\ a1\ a2\ cY1^{2}+a0\ a1\ cY1\ rY+a0\ a2\ cY1\ rY+\\
&\qquad\quad a1\ a2\ cY1\ rY+a0\ rY^{2}+a1\ rY^{2}+a2\ rY^{2}\big)\Big)-2\ a0^{2}\ a1\ cY1\ cYn1^{2}\\
&\quad rY^{2}\sqrt{\Big((a0\ a1\ a2\ cY1\ cYn1+a0\ a1\ cYn1\ rY+a0\ a2\ cYn1\ rY+a1\ a2\ cYn1\ rY)^{2}-}\\
&\qquad 4\ a0\ a1\ a2\ cYn1^{2}\ \big(a0\ a1\ a2\ cY1^{2}+a0\ a1\ cY1\ rY+a0\ a2\ cY1\ rY+\\
&\qquad\quad a1\ a2\ cY1\ rY+a0\ rY^{2}+a1\ rY^{2}+a2\ rY^{2}\big)\Big)-2\ a0^{2}\ a2\ cY1\ cYn1^{2}\\
&\quad rY^{2}\sqrt{\Big((a0\ a1\ a2\ cY1\ cYn1+a0\ a1\ cYn1\ rY+a0\ a2\ cYn1\ rY+a1\ a2\ cYn1\ rY)^{2}-}\\
&\qquad 4\ a0\ a1\ a2\ cYn1^{2}\ \big(a0\ a1\ a2\ cY1^{2}+a0\ a1\ cY1\ rY+a0\ a2\ cY1\ rY+\\
&\qquad\quad a1\ a2\ cY1\ rY+a0\ rY^{2}+a1\ rY^{2}+a2\ rY^{2}\big)\Big)-a0\ a1\ a2\ cY1\ cYn1^{2}\\
&\quad rY^{2}\sqrt{\Big((a0\ a1\ a2\ cY1\ cYn1+a0\ a1\ cYn1\ rY+a0\ a2\ cYn1\ rY+a1\ a2\ cYn1\ rY)^{2}-}\\
&\qquad 4\ a0\ a1\ a2\ cYn1^{2}\ \big(a0\ a1\ a2\ cY1^{2}+a0\ a1\ cY1\ rY+a0\ a2\ cY1\ rY+
\end{aligned}$$

$a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2\big)\big) + 2\ a0^2\ a1\ cY1\ cYn2^2$
$rY^2\ \sqrt{\big(}(a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 -$
$4\ a0\ a1\ a2\ cYn1^2\ \big(a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + a0\ a2\ cY1\ rY +$
$a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2\big)\big) + 2\ a0^2\ a2\ cY1\ cYn2^2$
$rY^2\ \sqrt{\big(}(a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 -$
$4\ a0\ a1\ a2\ cYn1^2\ \big(a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + a0\ a2\ cY1\ rY +$
$a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2\big)\big) + a0\ a1\ a2\ cY1\ cYn2^2$
$rY^2\ \sqrt{\big(}(a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 -$
$4\ a0\ a1\ a2\ cYn1^2\ \big(a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY +$
$a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2\big)\big) -$
$2\ a0^2\ cYn1^2\ rY^3\ \sqrt{\big(}(a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 -$
$4\ a0\ a1\ a2\ cYn1^2\ \big(a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY +$
$a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2\big)\big) -$
$a0\ a1\ cYn1^2\ rY^3\ \sqrt{\big(}(a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 -$
$4\ a0\ a1\ a2\ cYn1^2\ \big(a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY +$
$a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2\big)\big) -$
$a0\ a2\ cYn1^2\ rY^3\ \sqrt{\big(}(a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 -$
$4\ a0\ a1\ a2\ cYn1^2\ \big(a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY +$
$a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2\big)\big) +$
$a1\ a2\ cYn1^2\ rY^3\ \sqrt{\big(}(a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 -$
$4\ a0\ a1\ a2\ cYn1^2\ \big(a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY +$
$a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2\big)\big) +$
$2\ a0^2\ cYn2^2\ rY^3\ \sqrt{\big(}(a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 -$
$4\ a0\ a1\ a2\ cYn1^2\ \big(a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY +$
$a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2\big)\big) +$
$a0\ a1\ cYn2^2\ rY^3\ \sqrt{\big(}(a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 -$
$4\ a0\ a1\ a2\ cYn1^2\ \big(a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY +$
$a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2\big)\big) +$
$a0\ a2\ cYn2^2\ rY^3\ \sqrt{\big(}(a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 -$
$4\ a0\ a1\ a2\ cYn1^2\ \big(a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY +$
$a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2\big)\big) -$
$a1\ a2\ cYn2^2\ rY^3\ \sqrt{\big(}(a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 -$
$4\ a0\ a1\ a2\ cYn1^2\ \big(a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + a0\ a2\ cY1\ rY +$
$a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2\big)\big)\big) \big/ \big(2\ cYn1\ rY^2$
$\big(a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2\big)^2\big),$
$Sn1 \to \big(6\ a0^2\ a1^4\ a2^2\ cY1^4\ cYn1 + 2\ a0^2\ a1^4\ a2^2\ cY1^6\ cYn1 + 2\ a0^2\ a1^4\ a2^2\ cY1^4\ cYn1^3 +$
$2\ a0^2\ a1^4\ a2^2\ cY1^4\ cYn1\ cYn2^2 + 12\ a0^2\ a1^4\ a2\ cY1^3\ cYn1\ rY +$
$12\ a0^2\ a1^3\ a2^2\ cY1^3\ cYn1\ rY + 12\ a0\ a1^4\ a2^2\ cY1^3\ cYn1\ rY +$
$4\ a0^2\ a1^4\ a2\ cY1^5\ cYn1\ rY + 8\ a0^2\ a1^3\ a2^2\ cY1^5\ cYn1\ rY +$
$4\ a0\ a1^4\ a2^2\ cY1^5\ cYn1\ rY + 4\ a0^2\ a1^4\ a2\ cY1^3\ cYn1^3\ rY +$
$2\ a0^2\ a1^3\ a2^2\ cY1^3\ cYn1^3\ rY + 4\ a0\ a1^4\ a2^2\ cY1^3\ cYn1^3\ rY +$
$4\ a0^2\ a1^4\ a2\ cY1^3\ cYn1\ cYn2^2\ rY + 2\ a0^2\ a1^3\ a2^2\ cY1^3\ cYn1\ cYn2^2\ rY +$
$4\ a0\ a1^4\ a2^2\ cY1^3\ cYn1\ cYn2^2\ rY + 6\ a0^2\ a1^4\ cY1^2\ cYn1\ rY^2 +$
$24\ a0^2\ a1^3\ a2\ cY1^2\ cYn1\ rY^2 + 24\ a0\ a1^4\ a2\ cY1^2\ cYn1\ rY^2 +$
$6\ a0^2\ a1^2\ a2^2\ cY1^2\ cYn1\ rY^2 + 24\ a0\ a1^3\ a2^2\ cY1^2\ cYn1\ rY^2 +$
$6\ a1^4\ a2^2\ cY1^2\ cYn1\ rY^2 + 2\ a0^2\ a1^4\ cY1^4\ cYn1\ rY^2 +$
$16\ a0^2\ a1^3\ a2\ cY1^4\ cYn1\ rY^2 + 8\ a0\ a1^4\ a2\ cY1^4\ cYn1\ rY^2 +$
$12\ a0^2\ a1^2\ a2^2\ cY1^4\ cYn1\ rY^2 + 16\ a0\ a1^3\ a2^2\ cY1^4\ cYn1\ rY^2 +$
$2\ a1^4\ a2^2\ cY1^4\ cYn1\ rY^2 + 2\ a0^2\ a1^4\ cY1^2\ cYn1^3\ rY^2 +$
$4\ a0^2\ a1^3\ a2\ cY1^2\ cYn1^3\ rY^2 + 8\ a0\ a1^4\ a2\ cY1^2\ cYn1^3\ rY^2 -$
$3\ a0^2\ a1^2\ a2^2\ cY1^2\ cYn1^3\ rY^2 + 4\ a0\ a1^3\ a2^2\ cY1^2\ cYn1^3\ rY^2 +$
$2\ a1^4\ a2^2\ cY1^2\ cYn1^3\ rY^2 + 2\ a0^2\ a1^4\ cY1^2\ cYn1\ cYn2^2\ rY^2 +$

$$
\begin{aligned}
&4\,a0^2\,a1^3\,a2\,cY1^2\,cYn1\,cYn2^2\,rY^2+8\,a0\,a1^4\,a2\,cY1^2\,cYn1\,cYn2^2\,rY^2-\\
&3\,a0^2\,a1^2\,a2^2\,cY1^2\,cYn1\,cYn2^2\,rY^2+4\,a0\,a1^3\,a2^2\,cY1^2\,cYn1\,cYn2^2\,rY^2+\\
&2\,a1^4\,a2^2\,cY1^2\,cYn1\,cYn2^2\,rY^2+12\,a0^2\,a1^3\,cY1\,cYn1\,rY^3+\\
&12\,a0\,a1^4\,cY1\,cYn1\,rY^3+12\,a0^2\,a1^2\,a2\,cY1\,cYn1\,rY^3+\\
&36\,a0\,a1^3\,a2\,cY1\,cYn1\,rY^3+12\,a1^4\,a2\,cY1\,cYn1\,rY^3+\\
&12\,a0\,a1^2\,a2^2\,cY1\,cYn1\,rY^3+12\,a1^3\,a2^2\,cY1\,cYn1\,rY^3+\\
&8\,a0^2\,a1^3\,cY1^3\,cYn1\,rY^3+4\,a0\,a1^4\,cY1^3\,cYn1\,rY^3+\\
&24\,a0^2\,a1^2\,a2\,cY1^3\,cYn1\,rY^3+28\,a0\,a1^3\,a2\,cY1^3\,cYn1\,rY^3+\\
&4\,a1^4\,a2\,cY1^3\,cYn1\,rY^3+8\,a0^2\,a1\,a2^2\,cY1^3\,cYn1\,rY^3+\\
&24\,a0\,a1^2\,a2^2\,cY1^3\,cYn1\,rY^3+8\,a1^3\,a2^2\,cY1^3\,cYn1\,rY^3+\\
&2\,a0^2\,a1^3\,cY1\,cYn1^3\,rY^3+4\,a0\,a1^4\,cY1\,cYn1^3\,rY^3-\\
&2\,a0^2\,a1^2\,a2\,cY1\,cYn1^3\,rY^3+6\,a0\,a1^3\,a2\,cY1\,cYn1^3\,rY^3+\\
&4\,a1^4\,a2\,cY1\,cYn1^3\,rY^3-2\,a0^2\,a1\,a2^2\,cY1\,cYn1^3\,rY^3-\\
&2\,a0\,a1^2\,a2^2\,cY1\,cYn1^3\,rY^3+2\,a1^3\,a2^2\,cY1\,cYn1^3\,rY^3+\\
&2\,a0^2\,a1^3\,cY1\,cYn1\,cYn2^2\,rY^3+4\,a0\,a1^4\,cY1\,cYn1\,cYn2^2\,rY^3-\\
&2\,a0^2\,a1^2\,a2\,cY1\,cYn1\,cYn2^2\,rY^3+6\,a0\,a1^3\,a2\,cY1\,cYn1\,cYn2^2\,rY^3+\\
&4\,a1^4\,a2\,cY1\,cYn1\,cYn2^2\,rY^3-2\,a0^2\,a1\,a2^2\,cY1\,cYn1\,cYn2^2\,rY^3-\\
&2\,a0\,a1^2\,a2^2\,cY1\,cYn1\,cYn2^2\,rY^3+2\,a1^3\,a2^2\,cY1\,cYn1\,cYn2^2\,rY^3+\\
&6\,a0^2\,a1^2\,cYn1\,rY^4+12\,a0\,a1^3\,cYn1\,rY^4+6\,a1^4\,cYn1\,rY^4+\\
&12\,a0\,a1^2\,a2\,cYn1\,rY^4+12\,a1^3\,a2\,cYn1\,rY^4+6\,a1^2\,a2^2\,cYn1\,rY^4+\\
&12\,a0^2\,a1^2\,cY1^2\,cYn1\,rY^4+12\,a0\,a1^3\,cY1^2\,cYn1\,rY^4+2\,a1^4\,cY1^2\,cYn1\,rY^4+\\
&16\,a0^2\,a1\,a2\,cY1^2\,cYn1\,rY^4+36\,a0\,a1^2\,a2\,cY1^2\,cYn1\,rY^4+\\
&12\,a1^3\,a2\,cY1^2\,cYn1\,rY^4+2\,a0^2\,a2^2\,cY1^2\,cYn1\,rY^4+\\
&16\,a0\,a1\,a2^2\,cY1^2\,cYn1\,rY^4+12\,a1^2\,a2^2\,cY1^2\,cYn1\,rY^4+\\
&a0^2\,a1^2\,cYn1^3\,rY^4+2\,a0\,a1^3\,cYn1^3\,rY^4+2\,a1^4\,cYn1^3\,rY^4-\\
&2\,a0^2\,a1\,a2\,cYn1^3\,rY^4-2\,a0\,a1^2\,a2\,cYn1^3\,rY^4+2\,a1^3\,a2\,cYn1^3\,rY^4+\\
&a0^2\,a2^2\,cYn1^3\,rY^4-2\,a0\,a1\,a2^2\,cYn1^3\,rY^4+a1^2\,a2^2\,cYn1^3\,rY^4+\\
&a0^2\,a1^2\,cYn1\,cYn2^2\,rY^4+2\,a0\,a1^3\,cYn1\,cYn2^2\,rY^4+2\,a1^4\,cYn1\,cYn2^2\,rY^4-\\
&2\,a0^2\,a1\,a2\,cYn1\,cYn2^2\,rY^4-2\,a0\,a1^2\,a2\,cYn1\,cYn2^2\,rY^4+\\
&2\,a1^3\,a2\,cYn1\,cYn2^2\,rY^4+a0^2\,a2^2\,cYn1\,cYn2^2\,rY^4-2\,a0\,a1\,a2^2\,cYn1\,cYn2^2\,rY^4+\\
&a1^2\,a2^2\,cYn1\,cYn2^2\,rY^4+8\,a0^2\,a1\,cY1\,cYn1\,rY^5+12\,a0\,a1^2\,cY1\,cYn1\,rY^5+\\
&4\,a1^3\,cY1\,cYn1\,rY^5+4\,a0^2\,a2\,cY1\,cYn1\,rY^5+20\,a0\,a1\,a2\,cY1\,cYn1\,rY^5+\\
&12\,a1^2\,a2\,cY1\,cYn1\,rY^5+4\,a0\,a2^2\,cY1\,cYn1\,rY^5+8\,a1\,a2^2\,cY1\,cYn1\,rY^5+\\
&2\,a0^2\,cYn1\,rY^6+4\,a0\,a1\,cYn1\,rY^6+2\,a1^2\,cYn1\,rY^6+4\,a0\,a2\,cYn1\,rY^6+\\
&4\,a1\,a2\,cYn1\,rY^6+2\,a2^2\,cYn1\,rY^6-2\,a0\,a1^2\,a2\,cY1^2\,cYn1^2\,rY\\
&\quad\sqrt{\Big((a0\,a1\,a2\,cY1\,cYn1+a0\,a1\,cYn1\,rY+a0\,a2\,cYn1\,rY+a1\,a2\,cYn1\,rY)^2-}\\
&\qquad 4\,a0\,a1\,a2\,cYn1^2\,\big(a0\,a1\,a2\,cY1^2+a0\,a1\,cY1\,rY+a0\,a2\,cY1\,rY+\\
&\qquad\quad a1\,a2\,cY1\,rY+a0\,rY^2+a1\,rY^2+a2\,rY^2\big)\Big)+2\,a0\,a1^2\,a2\,cY1^2\,cYn2^2\\
&\quad rY\,\sqrt{\Big((a0\,a1\,a2\,cY1\,cYn1+a0\,a1\,cYn1\,rY+a0\,a2\,cYn1\,rY+a1\,a2\,cYn1\,rY)^2-}\\
&\qquad 4\,a0\,a1\,a2\,cYn1^2\,\big(a0\,a1\,a2\,cY1^2+a0\,a1\,cY1\,rY+a0\,a2\,cY1\,rY+\\
&\qquad\quad a1\,a2\,cY1\,rY+a0\,rY^2+a1\,rY^2+a2\,rY^2\big)\Big)-2\,a0\,a1^2\,cY1\,cYn1^2\\
&\quad rY^2\,\sqrt{\Big((a0\,a1\,a2\,cY1\,cYn1+a0\,a1\,cYn1\,rY+a0\,a2\,cYn1\,rY+a1\,a2\,cYn1\,rY)^2-}\\
&\qquad 4\,a0\,a1\,a2\,cYn1^2\,\big(a0\,a1\,a2\,cY1^2+a0\,a1\,cY1\,rY+a0\,a2\,cY1\,rY+\\
&\qquad\quad a1\,a2\,cY1\,rY+a0\,rY^2+a1\,rY^2+a2\,rY^2\big)\Big)-a0\,a1\,a2\,cY1\,cYn1^2\\
&\quad rY^2\,\sqrt{\Big((a0\,a1\,a2\,cY1\,cYn1+a0\,a1\,cYn1\,rY+a0\,a2\,cYn1\,rY+a1\,a2\,cYn1\,rY)^2-}\\
&\qquad 4\,a0\,a1\,a2\,cYn1^2\,\big(a0\,a1\,a2\,cY1^2+a0\,a1\,cY1\,rY+a0\,a2\,cY1\,rY+\\
&\qquad\quad a1\,a2\,cY1\,rY+a0\,rY^2+a1\,rY^2+a2\,rY^2\big)\Big)-2\,a1^2\,a2\,cY1\,cYn1^2\\
&\quad rY^2\,\sqrt{\Big((a0\,a1\,a2\,cY1\,cYn1+a0\,a1\,cYn1\,rY+a0\,a2\,cYn1\,rY+a1\,a2\,cYn1\,rY)^2-}\\
&\qquad 4\,a0\,a1\,a2\,cYn1^2\,\big(a0\,a1\,a2\,cY1^2+a0\,a1\,cY1\,rY+a0\,a2\,cY1\,rY+\\
&\qquad\quad a1\,a2\,cY1\,rY+a0\,rY^2+a1\,rY^2+a2\,rY^2\big)\Big)+2\,a0\,a1^2\,cY1\,cYn2^2\\
&\quad rY^2\,\sqrt{\Big((a0\,a1\,a2\,cY1\,cYn1+a0\,a1\,cYn1\,rY+a0\,a2\,cYn1\,rY+a1\,a2\,cYn1\,rY)^2-}\\
&\qquad 4\,a0\,a1\,a2\,cYn1^2\,\big(a0\,a1\,a2\,cY1^2+a0\,a1\,cY1\,rY+a0\,a2\,cY1\,rY+\\
&\qquad\quad a1\,a2\,cY1\,rY+a0\,rY^2+a1\,rY^2+a2\,rY^2\big)\Big)+a0\,a1\,a2\,cY1\,cYn2^2\\
&\quad rY^2\,\sqrt{\Big((a0\,a1\,a2\,cY1\,cYn1+a0\,a1\,cYn1\,rY+a0\,a2\,cYn1\,rY+a1\,a2\,cYn1\,rY)^2-}
\end{aligned}
$$

$4\ a0\ a1\ a2\ cYn1^2\ (a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + a0\ a2\ cY1\ rY +$

$a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2)) + 2\ a1^2\ a2\ cY1\ cYn2^2$

$rY^2 \surd ((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 -$

$4\ a0\ a1\ a2\ cYn1^2\ (a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY +$

$a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2)) -$

$a0\ a1\ cYn1^2\ rY^3 \surd ((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 -$

$4\ a0\ a1\ a2\ cYn1^2\ (a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY +$

$a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2)) -$

$2\ a1^2\ cYn1^2\ rY^3 \surd ((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 -$

$4\ a0\ a1\ a2\ cYn1^2\ (a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY +$

$a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2)) +$

$a0\ a2\ cYn1^2\ rY^3 \surd ((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 -$

$4\ a0\ a1\ a2\ cYn1^2\ (a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY +$

$a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2)) -$

$a1\ a2\ cYn1^2\ rY^3 \surd ((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 -$

$4\ a0\ a1\ a2\ cYn1^2\ (a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY +$

$a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2)) +$

$a0\ a1\ cYn2^2\ rY^3 \surd ((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 -$

$4\ a0\ a1\ a2\ cYn1^2\ (a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY +$

$a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2)) +$

$2\ a1^2\ cYn2^2\ rY^3 \surd ((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 -$

$4\ a0\ a1\ a2\ cYn1^2\ (a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY +$

$a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2)) -$

$a0\ a2\ cYn2^2\ rY^3 \surd ((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 -$

$4\ a0\ a1\ a2\ cYn1^2\ (a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY +$

$a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2)) +$

$a1\ a2\ cYn2^2\ rY^3 \surd ((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 -$

$4\ a0\ a1\ a2\ cYn1^2\ (a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + a0\ a2\ cY1\ rY +$

$a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2)))) / (2\ cYn1\ rY^2$

$(a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2)^2),$

$Sn2 \to (6\ a0^2\ a1^2\ a2^4\ cY1^4\ cYn1 + 2\ a0^2\ a1^2\ a2^4\ cY1^6\ cYn1 + 2\ a0^2\ a1^2\ a2^4\ cY1^4\ cYn1^3 +$

$2\ a0^2\ a1^2\ a2^4\ cY1^4\ cYn1\ cYn2^2 + 12\ a0^2\ a1^2\ a2^3\ cY1^3\ cYn1\ rY +$

$12\ a0^2\ a1\ a2^4\ cY1^3\ cYn1\ rY + 12\ a0\ a1^2\ a2^4\ cY1^3\ cYn1\ rY +$

$8\ a0^2\ a1^2\ a2^3\ cY1^5\ cYn1\ rY + 4\ a0^2\ a1\ a2^4\ cY1^5\ cYn1\ rY +$

$4\ a0\ a1^2\ a2^4\ cY1^5\ cYn1\ rY + 2\ a0^2\ a1^2\ a2^3\ cY1^3\ cYn1^3\ rY +$

$4\ a0^2\ a1\ a2^4\ cY1^3\ cYn1^3\ rY + 4\ a0\ a1^2\ a2^4\ cY1^3\ cYn1^3\ rY +$

$2\ a0^2\ a1^2\ a2^3\ cY1^3\ cYn1\ cYn2^2\ rY + 4\ a0^2\ a1\ a2^4\ cY1^3\ cYn1\ cYn2^2\ rY +$

$4\ a0\ a1^2\ a2^4\ cY1^3\ cYn1\ cYn2^2\ rY + 6\ a0^2\ a1^2\ a2^2\ cY1^2\ cYn1\ rY^2 +$

$24\ a0^2\ a1\ a2^3\ cY1^2\ cYn1\ rY^2 + 24\ a0\ a1^2\ a2^3\ cY1^2\ cYn1\ rY^2 +$

$6\ a0^2\ a2^4\ cY1^2\ cYn1\ rY^2 + 24\ a0\ a1\ a2^4\ cY1^2\ cYn1\ rY^2 +$

$6\ a1^2\ a2^4\ cY1^2\ cYn1\ rY^2 + 12\ a0^2\ a1^2\ a2^2\ cY1^4\ cYn1\ rY^2 +$

$16\ a0^2\ a1\ a2^3\ cY1^4\ cYn1\ rY^2 + 16\ a0\ a1^2\ a2^3\ cY1^4\ cYn1\ rY^2 +$

$2\ a0^2\ a2^4\ cY1^4\ cYn1\ rY^2 + 8\ a0\ a1\ a2^4\ cY1^4\ cYn1\ rY^2 +$

$2\ a1^2\ a2^4\ cY1^4\ cYn1\ rY^2 - 3\ a0^2\ a1^2\ a2^2\ cY1^2\ cYn1^3\ rY^2 +$

$4\ a0^2\ a1\ a2^3\ cY1^2\ cYn1^3\ rY^2 + 4\ a0\ a1^2\ a2^3\ cY1^2\ cYn1^3\ rY^2 +$

$2\ a0^2\ a2^4\ cY1^2\ cYn1^3\ rY^2 + 8\ a0\ a1\ a2^4\ cY1^2\ cYn1^3\ rY^2 +$

$2\ a1^2\ a2^4\ cY1^2\ cYn1^3\ rY^2 - 3\ a0^2\ a1^2\ a2^2\ cY1^2\ cYn1\ cYn2^2\ rY^2 +$

$4\ a0^2\ a1\ a2^3\ cY1^2\ cYn1\ cYn2^2\ rY^2 + 4\ a0\ a1^2\ a2^3\ cY1^2\ cYn1\ cYn2^2\ rY^2 +$

$2\ a0^2\ a2^4\ cY1^2\ cYn1\ cYn2^2\ rY^2 + 8\ a0\ a1\ a2^4\ cY1^2\ cYn1\ cYn2^2\ rY^2 +$

$2\ a1^2\ a2^4\ cY1^2\ cYn1\ cYn2^2\ rY^2 + 12\ a0^2\ a1\ a2^2\ cY1\ cYn1\ rY^3 +$

$12\ a0\ a1^2\ a2^2\ cY1\ cYn1\ rY^3 + 12\ a0^2\ a2^3\ cY1\ cYn1\ rY^3 +$

$36\ a0\ a1\ a2^3\ cY1\ cYn1\ rY^3 + 12\ a1^2\ a2^3\ cY1\ cYn1\ rY^3 +$

$12\ a0\ a2^4\ cY1\ cYn1\ rY^3 + 12\ a1\ a2^4\ cY1\ cYn1\ rY^3 +$

$8\ a0^2\ a1^2\ a2\ cY1^3\ cYn1\ rY^3 + 24\ a0^2\ a1\ a2^2\ cY1^3\ cYn1\ rY^3 +$
$24\ a0\ a1^2\ a2^2\ cY1^3\ cYn1\ rY^3 + 8\ a0^2\ a2^3\ cY1^3\ cYn1\ rY^3 +$
$28\ a0\ a1\ a2^3\ cY1^3\ cYn1\ rY^3 + 8\ a1^2\ a2^3\ cY1^3\ cYn1\ rY^3 +$
$4\ a0\ a2^4\ cY1^3\ cYn1\ rY^3 + 4\ a1\ a2^4\ cY1^3\ cYn1\ rY^3 -$
$2\ a0^2\ a1^2\ a2\ cY1\ cYn1^3\ rY^3 - 2\ a0^2\ a1\ a2^2\ cY1\ cYn1^3\ rY^3 -$
$2\ a0\ a1^2\ a2^2\ cY1\ cYn1^3\ rY^3 + 2\ a0^2\ a2^3\ cY1\ cYn1^3\ rY^3 +$
$6\ a0\ a1\ a2^3\ cY1\ cYn1^3\ rY^3 + 2\ a1^2\ a2^3\ cY1\ cYn1^3\ rY^3 +$
$4\ a0\ a2^4\ cY1\ cYn1^3\ rY^3 + 4\ a1\ a2^4\ cY1\ cYn1^3\ rY^3 -$
$2\ a0^2\ a1^2\ a2\ cY1\ cYn1\ cYn2^2\ rY^3 - 2\ a0^2\ a1\ a2^2\ cY1\ cYn1\ cYn2^2\ rY^3 -$
$2\ a0\ a1^2\ a2^2\ cY1\ cYn1\ cYn2^2\ rY^3 + 2\ a0^2\ a2^3\ cY1\ cYn1\ cYn2^2\ rY^3 +$
$6\ a0\ a1\ a2^3\ cY1\ cYn1\ cYn2^2\ rY^3 + 2\ a1^2\ a2^3\ cY1\ cYn1\ cYn2^2\ rY^3 +$
$4\ a0\ a2^4\ cY1\ cYn1\ cYn2^2\ rY^3 + 4\ a1\ a2^4\ cY1\ cYn1\ cYn2^2\ rY^3 +$
$6\ a0^2\ a2^2\ cYn1\ rY^4 + 12\ a0\ a1\ a2^2\ cYn1\ rY^4 + 6\ a1^2\ a2^2\ cYn1\ rY^4 +$
$12\ a0\ a2^3\ cYn1\ rY^4 + 12\ a1\ a2^3\ cYn1\ rY^4 + 6\ a2^4\ cYn1\ rY^4 +$
$2\ a0^2\ a1^2\ cY1^2\ cYn1\ rY^4 + 16\ a0^2\ a1\ a2\ cY1^2\ cYn1\ rY^4 +$
$16\ a0\ a1^2\ a2\ cY1^2\ cYn1\ rY^4 + 12\ a0^2\ a2^2\ cY1^2\ cYn1\ rY^4 +$
$36\ a0\ a1\ a2^2\ cY1^2\ cYn1\ rY^4 + 12\ a1^2\ a2^2\ cY1^2\ cYn1\ rY^4 +$
$12\ a0\ a2^3\ cY1^2\ cYn1\ rY^4 + 12\ a1\ a2^3\ cY1^2\ cYn1\ rY^4 +$
$2\ a2^4\ cY1^2\ cYn1\ rY^4 + a0^2\ a1^2\ cYn1^3\ rY^4 - 2\ a0^2\ a1\ a2\ cYn1^3\ rY^4 -$
$2\ a0\ a1^2\ a2\ cYn1^3\ rY^4 + a0^2\ a2^2\ cYn1^3\ rY^4 - 2\ a0\ a1\ a2^2\ cYn1^3\ rY^4 +$
$a1^2\ a2^2\ cYn1^3\ rY^4 + 2\ a0\ a2^3\ cYn1^3\ rY^4 + 2\ a1\ a2^3\ cYn1^3\ rY^4 +$
$2\ a2^4\ cYn1^3\ rY^4 + a0^2\ a1^2\ cYn1\ cYn2^2\ rY^4 - 2\ a0^2\ a1\ a2\ cYn1\ cYn2^2\ rY^4 -$
$2\ a0\ a1^2\ a2\ cYn1\ cYn2^2\ rY^4 + a0^2\ a2^2\ cYn1\ cYn2^2\ rY^4 -$
$2\ a0\ a1\ a2^2\ cYn1\ cYn2^2\ rY^4 + a1^2\ a2^2\ cYn1\ cYn2^2\ rY^4 +$
$2\ a0\ a2^3\ cYn1\ cYn2^2\ rY^4 + 2\ a1\ a2^3\ cYn1\ cYn2^2\ rY^4 +$
$2\ a2^4\ cYn1\ cYn2^2\ rY^4 + 4\ a0^2\ a1\ cY1\ cYn1\ rY^5 + 4\ a0\ a1^2\ cY1\ cYn1\ rY^5 +$
$8\ a0^2\ a2\ cY1\ cYn1\ rY^5 + 20\ a0\ a1\ a2\ cY1\ cYn1\ rY^5 +$
$8\ a1^2\ a2\ cY1\ cYn1\ rY^5 + 12\ a0\ a2^2\ cY1\ cYn1\ rY^5 +$
$12\ a1\ a2^2\ cY1\ cYn1\ rY^5 + 4\ a2^3\ cY1\ cYn1\ rY^5 + 2\ a0^2\ cYn1\ rY^6 +$
$4\ a0\ a1\ cYn1\ rY^6 + 2\ a1^2\ cYn1\ rY^6 + 4\ a0\ a2\ cYn1\ rY^6 +$
$4\ a1\ a2\ cYn1\ rY^6 + 2\ a2^2\ cYn1\ rY^6 - 2\ a0\ a1\ a2^2\ cY1^2\ cYn1^2\ rY$
$\sqrt{\Big(}(a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 -$
$4\ a0\ a1\ a2\ cYn1^2\ \big(a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + a0\ a2\ cY1\ rY +$
$a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2\big)\Big) + 2\ a0\ a1\ a2^2\ cY1^2\ cYn2^2$
$rY\ \sqrt{\Big(}(a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 -$
$4\ a0\ a1\ a2\ cYn1^2\ \big(a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + a0\ a2\ cY1\ rY +$
$a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2\big)\Big) - a0\ a1\ a2\ cY1\ cYn1^2$
$rY^2\ \sqrt{\Big(}(a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 -$
$4\ a0\ a1\ a2\ cYn1^2\ \big(a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + a0\ a2\ cY1\ rY +$
$a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2\big)\Big) - 2\ a0\ a2^2\ cY1\ cYn1^2$
$rY^2\ \sqrt{\Big(}(a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 -$
$4\ a0\ a1\ a2\ cYn1^2\ \big(a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + a0\ a2\ cY1\ rY +$
$a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2\big)\Big) - 2\ a1\ a2^2\ cY1\ cYn1^2$
$rY^2\ \sqrt{\Big(}(a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 -$
$4\ a0\ a1\ a2\ cYn1^2\ \big(a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + a0\ a2\ cY1\ rY +$
$a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2\big)\Big) + a0\ a1\ a2\ cY1\ cYn2^2$
$rY^2\ \sqrt{\Big(}(a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 -$
$4\ a0\ a1\ a2\ cYn1^2\ \big(a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + a0\ a2\ cY1\ rY +$
$a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2\big)\Big) + 2\ a0\ a2^2\ cY1\ cYn2^2$
$rY^2\ \sqrt{\Big(}(a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 -$
$4\ a0\ a1\ a2\ cYn1^2\ \big(a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + a0\ a2\ cY1\ rY +$
$a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2\big)\Big) + 2\ a1\ a2^2\ cY1\ cYn2^2$
$rY^2\ \sqrt{\Big(}(a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 -$
$4\ a0\ a1\ a2\ cYn1^2\ \big(a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY +$

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            a0 a2 cY1 rY + a1 a2 cY1 rY + a0 rY^2 + a1 rY^2 + a2 rY^2)) +
    a0 a1 cYn1^2 rY^3 √((a0 a1 a2 cY1 cYn1 + a0 a1 cYn1 rY + a0 a2 cYn1 rY + a1 a2 cYn1 rY)^2 -
         4 a0 a1 a2 cYn1^2 (a0 a1 a2 cY1^2 + a0 a1 cY1 rY +
            a0 a2 cY1 rY + a1 a2 cY1 rY + a0 rY^2 + a1 rY^2 + a2 rY^2)) -
    a0 a2 cYn1^2 rY^3 √((a0 a1 a2 cY1 cYn1 + a0 a1 cYn1 rY + a0 a2 cYn1 rY + a1 a2 cYn1 rY)^2 -
         4 a0 a1 a2 cYn1^2 (a0 a1 a2 cY1^2 + a0 a1 cY1 rY +
            a0 a2 cY1 rY + a1 a2 cY1 rY + a0 rY^2 + a1 rY^2 + a2 rY^2)) -
    a1 a2 cYn1^2 rY^3 √((a0 a1 a2 cY1 cYn1 + a0 a1 cYn1 rY + a0 a2 cYn1 rY + a1 a2 cYn1 rY)^2 -
         4 a0 a1 a2 cYn1^2 (a0 a1 a2 cY1^2 + a0 a1 cY1 rY +
            a0 a2 cY1 rY + a1 a2 cY1 rY + a0 rY^2 + a1 rY^2 + a2 rY^2)) -
    2 a2^2 cYn1^2 rY^3 √((a0 a1 a2 cY1 cYn1 + a0 a1 cYn1 rY + a0 a2 cYn1 rY + a1 a2 cYn1 rY)^2 -
         4 a0 a1 a2 cYn1^2 (a0 a1 a2 cY1^2 + a0 a1 cY1 rY +
            a0 a2 cY1 rY + a1 a2 cY1 rY + a0 rY^2 + a1 rY^2 + a2 rY^2)) -
    a0 a1 cYn2^2 rY^3 √((a0 a1 a2 cY1 cYn1 + a0 a1 cYn1 rY + a0 a2 cYn1 rY + a1 a2 cYn1 rY)^2 -
         4 a0 a1 a2 cYn1^2 (a0 a1 a2 cY1^2 + a0 a1 cY1 rY +
            a0 a2 cY1 rY + a1 a2 cY1 rY + a0 rY^2 + a1 rY^2 + a2 rY^2)) +
    a0 a2 cYn2^2 rY^3 √((a0 a1 a2 cY1 cYn1 + a0 a1 cYn1 rY + a0 a2 cYn1 rY + a1 a2 cYn1 rY)^2 -
         4 a0 a1 a2 cYn1^2 (a0 a1 a2 cY1^2 + a0 a1 cY1 rY +
            a0 a2 cY1 rY + a1 a2 cY1 rY + a0 rY^2 + a1 rY^2 + a2 rY^2)) +
    a1 a2 cYn2^2 rY^3 √((a0 a1 a2 cY1 cYn1 + a0 a1 cYn1 rY + a0 a2 cYn1 rY + a1 a2 cYn1 rY)^2 -
         4 a0 a1 a2 cYn1^2 (a0 a1 a2 cY1^2 + a0 a1 cY1 rY +
            a0 a2 cY1 rY + a1 a2 cY1 rY + a0 rY^2 + a1 rY^2 + a2 rY^2)) +
    2 a2^2 cYn2^2 rY^3 √((a0 a1 a2 cY1 cYn1 + a0 a1 cYn1 rY + a0 a2 cYn1 rY + a1 a2 cYn1 rY)^2 -
         4 a0 a1 a2 cYn1^2 (a0 a1 a2 cY1^2 + a0 a1 cY1 rY + a0 a2 cY1 rY +
            a1 a2 cY1 rY + a0 rY^2 + a1 rY^2 + a2 rY^2))) / (2 cYn1 rY^2
    (a0 a1 a2 cY1^2 + a0 a1 cY1 rY + a0 a2 cY1 rY + a1 a2 cY1 rY + a0 rY^2 + a1 rY^2 + a2 rY^2)^2),
pY1 → 3, n1 → -((a0 a1 a2 cY1 cYn1 + a0 a1 cYn1 rY + a0 a2 cYn1 rY + a1 a2 cYn1 rY +
      √((a0 a1 a2 cY1 cYn1 + a0 a1 cYn1 rY + a0 a2 cYn1 rY + a1 a2 cYn1 rY)^2 -
         4 a0 a1 a2 cYn1^2 (a0 a1 a2 cY1^2 + a0 a1 cY1 rY +
            a0 a2 cY1 rY + a1 a2 cY1 rY + a0 rY^2 + a1 rY^2 + a2 rY^2))) /
    (2 (a0 a1 a2 cY1^2 + a0 a1 cY1 rY + a0 a2 cY1 rY + a1 a2 cY1 rY + a0 rY^2 + a1 rY^2 + a2 rY^2))),
n2 → -((cYn2 (a0 a1 a2 cY1 cYn1 + a0 a1 cYn1 rY + a0 a2 cYn1 rY + a1 a2 cYn1 rY -
       √((a0 a1 a2 cY1 cYn1 + a0 a1 cYn1 rY + a0 a2 cYn1 rY + a1 a2 cYn1 rY)^2 -
          4 a0 a1 a2 cYn1^2 (a0 a1 a2 cY1^2 + a0 a1 cY1 rY +
             a0 a2 cY1 rY + a1 a2 cY1 rY + a0 rY^2 + a1 rY^2 + a2 rY^2)))) /
    (2 cYn1 (a0 a1 a2 cY1^2 + a0 a1 cY1 rY + a0 a2 cY1 rY + a1 a2 cY1 rY + a0 rY^2 +
          a1 rY^2 + a2 rY^2)))}, {p → (3 + cY1^2 + cYn1^2 + cYn2^2) / rY^2,
d → -(rY^2 (a0 a1 a2 cY1^2 + a0 a1 cY1 rY + a0 a2 cY1 rY + a1 a2 cY1 rY + a0 rY^2 + a1 rY^2 + a2 rY^2)) / (cYn1 cYn2),
N01 →
  ((a0 - a1) (a0 a1 a2 cY1^2 + a0 a1 cY1 rY + a0 a2 cY1 rY + a1 a2 cY1 rY + a0 rY^2 + a1 rY^2 + a2 rY^2)) / (a2^3 cYn1 cYn2 rY),
N02 →
  -((a0 - a2) (a0 a1 a2 cY1^2 + a0 a1 cY1 rY + a0 a2 cY1 rY + a1 a2 cY1 rY + a0 rY^2 + a1 rY^2 + a2 rY^2)) / (a1^3 cYn1 cYn2 rY)
,
N12 →
  -((a1 - a2) (a0 a1 a2 cY1^2 + a0 a1 cY1 rY + a0 a2 cY1 rY + a1 a2 cY1 rY + a0 rY^2 + a1 rY^2 + a2 rY^2)) / (a0^3 cYn1 cYn2 rY)
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$, \mathrm{Sn0} \to \big(6\, a0^4\, a1^2\, a2^2\, cY1^4\, cYn1 + 2\, a0^4\, a1^2\, a2^2\, cY1^6\, cYn1 +$

$2\, a0^4\, a1^2\, a2^2\, cY1^4\, cYn1^3 + 2\, a0^4\, a1^2\, a2^2\, cY1^4\, cYn1\, cYn2^2 + 12\, a0^4\, a1^2\, a2\, cY1^3\, cYn1\, rY +$

$12\, a0^4\, a1\, a2^2\, cY1^3\, cYn1\, rY + 12\, a0^3\, a1^2\, a2^2\, cY1^3\, cYn1\, rY + 4\, a0^4\, a1^2\, a2\, cY1^5\, cYn1\, rY +$

$4\, a0^4\, a1\, a2^2\, cY1^5\, cYn1\, rY + 8\, a0^3\, a1^2\, a2^2\, cY1^5\, cYn1\, rY + 4\, a0^4\, a1^2\, a2\, cY1^3\, cYn1^3\, rY +$

$4\, a0^4\, a1\, a2^2\, cY1^3\, cYn1^3\, rY + 2\, a0^3\, a1^2\, a2^2\, cY1^3\, cYn1^3\, rY + 4\, a0^4\, a1^2\, a2\, cY1^3\, cYn1\, cYn2^2\, rY +$

$4\, a0^4\, a1\, a2^2\, cY1^3\, cYn1\, cYn2^2\, rY + 2\, a0^3\, a1^2\, a2^2\, cY1^3\, cYn1\, cYn2^2\, rY + 6\, a0^4\, a1^2\, cY1^2\, cYn1\, rY^2 +$

$24\, a0^4\, a1\, a2\, cY1^2\, cYn1\, rY^2 + 24\, a0^3\, a1^2\, a2\, cY1^2\, cYn1\, rY^2 + 6\, a0^4\, a2^2\, cY1^2\, cYn1\, rY^2 +$

$24\, a0^3\, a1\, a2^2\, cY1^2\, cYn1\, rY^2 + 6\, a0^2\, a1^2\, a2^2\, cY1^2\, cYn1\, rY^2 + 2\, a0^4\, a1^2\, cY1^4\, cYn1\, rY^2 +$

$8\, a0^4\, a1\, a2\, cY1^4\, cYn1\, rY^2 + 16\, a0^3\, a1^2\, a2\, cY1^4\, cYn1\, rY^2 + 2\, a0^4\, a2^2\, cY1^4\, cYn1\, rY^2 +$

$16\, a0^3\, a1\, a2^2\, cY1^4\, cYn1\, rY^2 + 12\, a0^2\, a1^2\, a2^2\, cY1^4\, cYn1\, rY^2 + 2\, a0^4\, a1^2\, cY1^2\, cYn1^3\, rY^2 +$

$8\, a0^4\, a1\, a2\, cY1^2\, cYn1^3\, rY^2 + 4\, a0^3\, a1^2\, a2\, cY1^2\, cYn1^3\, rY^2 + 2\, a0^4\, a2^2\, cY1^2\, cYn1^3\, rY^2 +$

$4\, a0^3\, a1\, a2^2\, cY1^2\, cYn1^3\, rY^2 - 3\, a0^2\, a1^2\, a2^2\, cY1^2\, cYn1^3\, rY^2 + 2\, a0^4\, a1^2\, cY1^2\, cYn1\, cYn2^2\, rY^2 +$

$8\, a0^4\, a1\, a2\, cY1^2\, cYn1\, cYn2^2\, rY^2 + 4\, a0^3\, a1^2\, a2\, cY1^2\, cYn1\, cYn2^2\, rY^2 +$

$2\, a0^4\, a2^2\, cY1^2\, cYn1\, cYn2^2\, rY^2 + 4\, a0^3\, a1\, a2^2\, cY1^2\, cYn1\, cYn2^2\, rY^2 -$

$3\, a0^2\, a1^2\, a2^2\, cY1^2\, cYn1\, cYn2^2\, rY^2 + 12\, a0^4\, a1\, cY1\, cYn1\, rY^3 + 12\, a0^3\, a1^2\, cY1\, cYn1\, rY^3 +$

$12\, a0^4\, a2\, cY1\, cYn1\, rY^3 + 36\, a0^3\, a1\, a2\, cY1\, cYn1\, rY^3 + 12\, a0^2\, a1^2\, a2\, cY1\, cYn1\, rY^3 +$

$12\, a0^3\, a2^2\, cY1\, cYn1\, rY^3 + 12\, a0^2\, a1\, a2^2\, cY1\, cYn1\, rY^3 + 4\, a0^4\, a1\, cY1^3\, cYn1\, rY^3 +$

$8\, a0^3\, a1^2\, cY1^3\, cYn1\, rY^3 + 4\, a0^4\, a2\, cY1^3\, cYn1\, rY^3 + 28\, a0^3\, a1\, a2\, cY1^3\, cYn1\, rY^3 +$

$24\, a0^2\, a1^2\, a2\, cY1^3\, cYn1\, rY^3 + 8\, a0^3\, a2^2\, cY1^3\, cYn1\, rY^3 + 24\, a0^2\, a1\, a2^2\, cY1^3\, cYn1\, rY^3 +$

$8\, a0\, a1^2\, a2^2\, cY1^3\, cYn1\, rY^3 + 4\, a0^4\, a1\, cY1\, cYn1^3\, rY^3 + 2\, a0^3\, a1^2\, cY1\, cYn1^3\, rY^3 +$

$4\, a0^4\, a2\, cY1\, cYn1^3\, rY^3 + 6\, a0^3\, a1\, a2\, cY1\, cYn1^3\, rY^3 - 2\, a0^2\, a1^2\, a2\, cY1\, cYn1^3\, rY^3 +$

$2\, a0^3\, a2^2\, cY1\, cYn1^3\, rY^3 - 2\, a0^2\, a1\, a2^2\, cY1\, cYn1^3\, rY^3 - 2\, a0\, a1^2\, a2^2\, cY1\, cYn1^3\, rY^3 +$

$4\, a0^4\, a1\, cY1\, cYn1\, cYn2^2\, rY^3 + 2\, a0^3\, a1^2\, cY1\, cYn1\, cYn2^2\, rY^3 + 4\, a0^4\, a2\, cY1\, cYn1\, cYn2^2\, rY^3 +$

$6\, a0^3\, a1\, a2\, cY1\, cYn1\, cYn2^2\, rY^3 - 2\, a0^2\, a1^2\, a2\, cY1\, cYn1\, cYn2^2\, rY^3 +$

$2\, a0^3\, a2^2\, cY1\, cYn1\, cYn2^2\, rY^3 - 2\, a0^2\, a1\, a2^2\, cY1\, cYn1\, cYn2^2\, rY^3 -$

$2\, a0\, a1^2\, a2^2\, cY1\, cYn1\, cYn2^2\, rY^3 + 6\, a0^4\, cYn1\, rY^4 + 12\, a0^3\, a1\, cYn1\, rY^4 + 6\, a0^2\, a1^2\, cYn1\, rY^4 +$

$12\, a0^3\, a2\, cYn1\, rY^4 + 12\, a0^2\, a1\, a2\, cYn1\, rY^4 + 6\, a0^2\, a2^2\, cYn1\, rY^4 + 2\, a0^4\, cY1^2\, cYn1\, rY^4 +$

$12\, a0^3\, a1\, cY1^2\, cYn1\, rY^4 + 12\, a0^2\, a1^2\, cY1^2\, cYn1\, rY^4 + 12\, a0^3\, a2\, cY1^2\, cYn1\, rY^4 +$

$36\, a0^2\, a1\, a2\, cY1^2\, cYn1\, rY^4 + 16\, a0\, a1^2\, a2\, cY1^2\, cYn1\, rY^4 + 12\, a0^2\, a2^2\, cY1^2\, cYn1\, rY^4 +$

$16\, a0\, a1\, a2^2\, cY1^2\, cYn1\, rY^4 + 2\, a1^2\, a2^2\, cY1^2\, cYn1\, rY^4 + 2\, a0^4\, cYn1^3\, rY^4 + 2\, a0^3\, a1\, cYn1^3\, rY^4 +$

$a0^2\, a1^2\, cYn1^3\, rY^4 + 2\, a0^3\, a2\, cYn1^3\, rY^4 - 2\, a0^2\, a1\, a2\, cYn1^3\, rY^4 - 2\, a0\, a1^2\, a2\, cYn1^3\, rY^4 +$

$a0^2\, a2^2\, cYn1^3\, rY^4 - 2\, a0\, a1\, a2^2\, cYn1^3\, rY^4 + a1^2\, a2^2\, cYn1^3\, rY^4 + 2\, a0^4\, cYn1\, cYn2^2\, rY^4 +$

$2\, a0^3\, a1\, cYn1\, cYn2^2\, rY^4 + a0^2\, a1^2\, cYn1\, cYn2^2\, rY^4 + 2\, a0^3\, a2\, cYn1\, cYn2^2\, rY^4 -$

$2\, a0^2\, a1\, a2\, cYn1\, cYn2^2\, rY^4 - 2\, a0\, a1^2\, a2\, cYn1\, cYn2^2\, rY^4 + a0^2\, a2^2\, cYn1\, cYn2^2\, rY^4 -$

$2\, a0\, a1\, a2^2\, cYn1\, cYn2^2\, rY^4 + a1^2\, a2^2\, cYn1\, cYn2^2\, rY^4 + 4\, a0^3\, cY1\, cYn1\, rY^5 +$

$12\, a0^2\, a1\, cY1\, cYn1\, rY^5 + 8\, a0\, a1^2\, cY1\, cYn1\, rY^5 + 12\, a0^2\, a2\, cY1\, cYn1\, rY^5 +$

$20\, a0\, a1\, a2\, cY1\, cYn1\, rY^5 + 4\, a1^2\, a2\, cY1\, cYn1\, rY^5 + 8\, a0\, a2^2\, cY1\, cYn1\, rY^5 +$

$4\, a1\, a2^2\, cY1\, cYn1\, rY^5 + 2\, a0^2\, cYn1\, rY^6 + 4\, a0\, a1\, cYn1\, rY^6 + 2\, a1^2\, cYn1\, rY^6 +$

$4\, a0\, a2\, cYn1\, rY^6 + 4\, a1\, a2\, cYn1\, rY^6 + 2\, a2^2\, cYn1\, rY^6 - 2\, a0^2\, a1\, a2\, cY1^2\, cYn1^2$

$rY \sqrt{\Big((a0\, a1\, a2\, cY1\, cYn1 + a0\, a1\, cYn1\, rY + a0\, a2\, cYn1\, rY + a1\, a2\, cYn1\, rY)^2 -}$

$4\, a0\, a1\, a2\, cYn1^2\, \big(a0\, a1\, a2\, cY1^2 + a0\, a1\, cY1\, rY + a0\, a2\, cY1\, rY +$

$a1\, a2\, cY1\, rY + a0\, rY^2 + a1\, rY^2 + a2\, rY^2\big)\Big) + 2\, a0^2\, a1\, a2\, cY1^2\, cYn2^2$

$rY \sqrt{\Big((a0\, a1\, a2\, cY1\, cYn1 + a0\, a1\, cYn1\, rY + a0\, a2\, cYn1\, rY + a1\, a2\, cYn1\, rY)^2 -}$

$4\, a0\, a1\, a2\, cYn1^2\, \big(a0\, a1\, a2\, cY1^2 + a0\, a1\, cY1\, rY + a0\, a2\, cY1\, rY +$

$a1\, a2\, cY1\, rY + a0\, rY^2 + a1\, rY^2 + a2\, rY^2\big)\Big) - 2\, a0^2\, a1\, cY1\, cYn1^2$

$rY^2 \sqrt{\Big((a0\, a1\, a2\, cY1\, cYn1 + a0\, a1\, cYn1\, rY + a0\, a2\, cYn1\, rY + a1\, a2\, cYn1\, rY)^2 -}$

$4\, a0\, a1\, a2\, cYn1^2\, \big(a0\, a1\, a2\, cY1^2 + a0\, a1\, cY1\, rY + a0\, a2\, cY1\, rY +$

$a1\, a2\, cY1\, rY + a0\, rY^2 + a1\, rY^2 + a2\, rY^2\big)\Big) - 2\, a0^2\, a2\, cY1\, cYn1^2$

$rY^2 \sqrt{\Big((a0\, a1\, a2\, cY1\, cYn1 + a0\, a1\, cYn1\, rY + a0\, a2\, cYn1\, rY + a1\, a2\, cYn1\, rY)^2 -}$

$4\, a0\, a1\, a2\, cYn1^2\, \big(a0\, a1\, a2\, cY1^2 + a0\, a1\, cY1\, rY + a0\, a2\, cY1\, rY +$

$a1\, a2\, cY1\, rY + a0\, rY^2 + a1\, rY^2 + a2\, rY^2\big)\Big) - a0\, a1\, a2\, cY1\, cYn1^2$

$rY^2 \sqrt{\Big((a0\, a1\, a2\, cY1\, cYn1 + a0\, a1\, cYn1\, rY + a0\, a2\, cYn1\, rY + a1\, a2\, cYn1\, rY)^2 -}$

$4\, a0\, a1\, a2\, cYn1^2\, \big(a0\, a1\, a2\, cY1^2 + a0\, a1\, cY1\, rY + a0\, a2\, cY1\, rY +$

$a1\, a2\, cY1\, rY + a0\, rY^2 + a1\, rY^2 + a2\, rY^2\big)\Big) + 2\, a0^2\, a1\, cY1\, cYn2^2$

$rY^2 \sqrt{\Big((a0\, a1\, a2\, cY1\, cYn1 + a0\, a1\, cYn1\, rY + a0\, a2\, cYn1\, rY + a1\, a2\, cYn1\, rY)^2 -}$

$4\ a0\ a1\ a2\ cYn1^2\ (a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + a0\ a2\ cY1\ rY +$
$a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2)) + 2\ a0^2\ a2\ cY1\ cYn2^2$
$rY^2 \sqrt{(} (a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 -$
$4\ a0\ a1\ a2\ cYn1^2\ (a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + a0\ a2\ cY1\ rY +$
$a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2)) + a0\ a1\ a2\ cY1\ cYn2^2$
$rY^2 \sqrt{(} (a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 -$
$4\ a0\ a1\ a2\ cYn1^2\ (a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY +$
$a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2)) -$
$2\ a0^2\ cYn1^2\ rY^3 \sqrt{(} (a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 -$
$4\ a0\ a1\ a2\ cYn1^2\ (a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY +$
$a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2)) -$
$a0\ a1\ cYn1^2\ rY^3 \sqrt{(} (a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 -$
$4\ a0\ a1\ a2\ cYn1^2\ (a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY +$
$a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2)) -$
$a0\ a2\ cYn1^2\ rY^3 \sqrt{(} (a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 -$
$4\ a0\ a1\ a2\ cYn1^2\ (a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY +$
$a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2)) +$
$a1\ a2\ cYn1^2\ rY^3 \sqrt{(} (a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 -$
$4\ a0\ a1\ a2\ cYn1^2\ (a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY +$
$a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2)) +$
$2\ a0^2\ cYn2^2\ rY^3 \sqrt{(} (a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 -$
$4\ a0\ a1\ a2\ cYn1^2\ (a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY +$
$a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2)) +$
$a0\ a1\ cYn2^2\ rY^3 \sqrt{(} (a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 -$
$4\ a0\ a1\ a2\ cYn1^2\ (a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY +$
$a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2)) +$
$a0\ a2\ cYn2^2\ rY^3 \sqrt{(} (a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 -$
$4\ a0\ a1\ a2\ cYn1^2\ (a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY +$
$a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2)) -$
$a1\ a2\ cYn2^2\ rY^3 \sqrt{(} (a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 -$
$4\ a0\ a1\ a2\ cYn1^2\ (a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + a0\ a2\ cY1\ rY +$
$a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2)))\ /\ (2\ cYn1\ rY^2$
$(a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2)^2),$
$Sn1 \to (6\ a0^2\ a1^4\ a2^2\ cY1^4\ cYn1 + 2\ a0^2\ a1^4\ a2^2\ cY1^6\ cYn1 + 2\ a0^2\ a1^4\ a2^2\ cY1^4\ cYn1^3 +$
$2\ a0^2\ a1^4\ a2^2\ cY1^4\ cYn1\ cYn2^2 + 12\ a0^2\ a1^4\ a2\ cY1^3\ cYn1\ rY +$
$12\ a0^2\ a1^3\ a2^2\ cY1^3\ cYn1\ rY + 12\ a0\ a1^4\ a2^2\ cY1^3\ cYn1\ rY +$
$4\ a0^2\ a1^4\ a2\ cY1^5\ cYn1\ rY + 8\ a0^2\ a1^3\ a2^2\ cY1^5\ cYn1\ rY +$
$4\ a0\ a1^4\ a2^2\ cY1^5\ cYn1\ rY + 4\ a0^2\ a1^4\ a2\ cY1^3\ cYn1^3\ rY +$
$2\ a0^2\ a1^3\ a2^2\ cY1^3\ cYn1^3\ rY + 4\ a0\ a1^4\ a2^2\ cY1^3\ cYn1^3\ rY +$
$4\ a0^2\ a1^4\ a2\ cY1^3\ cYn1\ cYn2^2\ rY + 2\ a0^2\ a1^3\ a2^2\ cY1^3\ cYn1\ cYn2^2\ rY +$
$4\ a0\ a1^4\ a2^2\ cY1^3\ cYn1\ cYn2^2\ rY + 6\ a0^2\ a1^4\ cY1^2\ cYn1\ rY^2 +$
$24\ a0^2\ a1^3\ a2\ cY1^2\ cYn1\ rY^2 + 24\ a0\ a1^4\ a2\ cY1^2\ cYn1\ rY^2 +$
$6\ a0^2\ a1^2\ a2^2\ cY1^2\ cYn1\ rY^2 + 24\ a0\ a1^3\ a2^2\ cY1^2\ cYn1\ rY^2 +$
$6\ a1^4\ a2^2\ cY1^2\ cYn1\ rY^2 + 2\ a0^2\ a1^4\ cY1^4\ cYn1\ rY^2 +$
$16\ a0^2\ a1^3\ a2\ cY1^4\ cYn1\ rY^2 + 8\ a0\ a1^4\ a2\ cY1^4\ cYn1\ rY^2 +$
$12\ a0^2\ a1^2\ a2^2\ cY1^4\ cYn1\ rY^2 + 16\ a0\ a1^3\ a2^2\ cY1^4\ cYn1\ rY^2 +$
$2\ a1^4\ a2^2\ cY1^4\ cYn1\ rY^2 + 2\ a0^2\ a1^4\ cY1^2\ cYn1^3\ rY^2 +$
$4\ a0^2\ a1^3\ a2\ cY1^2\ cYn1^3\ rY^2 + 8\ a0\ a1^4\ a2\ cY1^2\ cYn1^3\ rY^2 -$
$3\ a0^2\ a1^2\ a2^2\ cY1^2\ cYn1^3\ rY^2 + 4\ a0\ a1^3\ a2^2\ cY1^2\ cYn1^3\ rY^2 +$
$2\ a1^4\ a2^2\ cY1^2\ cYn1^3\ rY^2 + 2\ a0^2\ a1^4\ cY1^2\ cYn1\ cYn2^2\ rY^2 +$
$4\ a0^2\ a1^3\ a2\ cY1^2\ cYn1\ cYn2^2\ rY^2 + 8\ a0\ a1^4\ a2\ cY1^2\ cYn1\ cYn2^2\ rY^2 -$
$3\ a0^2\ a1^2\ a2^2\ cY1^2\ cYn1\ cYn2^2\ rY^2 + 4\ a0\ a1^3\ a2^2\ cY1^2\ cYn1\ cYn2^2\ rY^2 +$

$2\,a1^4\,a2^2\,cY1^2\,cYn1\,cYn2^2\,rY^2 + 12\,a0^2\,a1^3\,cY1\,cYn1\,rY^3 +$
$12\,a0\,a1^4\,cY1\,cYn1\,rY^3 + 12\,a0^2\,a1^2\,a2\,cY1\,cYn1\,rY^3 +$
$36\,a0\,a1^3\,a2\,cY1\,cYn1\,rY^3 + 12\,a1^4\,a2\,cY1\,cYn1\,rY^3 +$
$12\,a0\,a1^2\,a2^2\,cY1\,cYn1\,rY^3 + 12\,a1^3\,a2^2\,cY1\,cYn1\,rY^3 +$
$8\,a0^2\,a1^3\,cY1^3\,cYn1\,rY^3 + 4\,a0\,a1^4\,cY1^3\,cYn1\,rY^3 +$
$24\,a0^2\,a1^2\,a2\,cY1^3\,cYn1\,rY^3 + 28\,a0\,a1^3\,a2\,cY1^3\,cYn1\,rY^3 +$
$4\,a1^4\,a2\,cY1^3\,cYn1\,rY^3 + 8\,a0^2\,a1\,a2^2\,cY1^3\,cYn1\,rY^3 +$
$24\,a0\,a1^2\,a2^2\,cY1^3\,cYn1\,rY^3 + 8\,a1^3\,a2^2\,cY1^3\,cYn1\,rY^3 +$
$2\,a0^2\,a1^3\,cY1\,cYn1^3\,rY^3 + 4\,a0\,a1^4\,cY1\,cYn1^3\,rY^3 -$
$2\,a0^2\,a1^2\,a2\,cY1\,cYn1^3\,rY^3 + 6\,a0\,a1^3\,a2\,cY1\,cYn1^3\,rY^3 +$
$4\,a1^4\,a2\,cY1\,cYn1^3\,rY^3 - 2\,a0^2\,a1\,a2^2\,cY1\,cYn1^3\,rY^3 -$
$2\,a0\,a1^2\,a2^2\,cY1\,cYn1^3\,rY^3 + 2\,a1^3\,a2^2\,cY1\,cYn1^3\,rY^3 +$
$2\,a0^2\,a1^3\,cY1\,cYn1\,cYn2^2\,rY^3 + 4\,a0\,a1^4\,cY1\,cYn1\,cYn2^2\,rY^3 -$
$2\,a0^2\,a1^2\,a2\,cY1\,cYn1\,cYn2^2\,rY^3 + 6\,a0\,a1^3\,a2\,cY1\,cYn1\,cYn2^2\,rY^3 +$
$4\,a1^4\,a2\,cY1\,cYn1\,cYn2^2\,rY^3 - 2\,a0^2\,a1\,a2^2\,cY1\,cYn1\,cYn2^2\,rY^3 -$
$2\,a0\,a1^2\,a2^2\,cY1\,cYn1\,cYn2^2\,rY^3 + 2\,a1^3\,a2^2\,cY1\,cYn1\,cYn2^2\,rY^3 +$
$6\,a0^2\,a1^2\,cYn1\,rY^4 + 12\,a0\,a1^3\,cYn1\,rY^4 + 6\,a1^4\,cYn1\,rY^4 +$
$12\,a0\,a1^2\,a2\,cYn1\,rY^4 + 12\,a1^3\,a2\,cYn1\,rY^4 + 6\,a1^2\,a2^2\,cYn1\,rY^4 +$
$12\,a0^2\,a1^2\,cY1^2\,cYn1\,rY^4 + 12\,a0\,a1^3\,cY1^2\,cYn1\,rY^4 + 2\,a1^4\,cY1^2\,cYn1\,rY^4 +$
$16\,a0^2\,a1\,a2\,cY1^2\,cYn1\,rY^4 + 36\,a0\,a1^2\,a2\,cY1^2\,cYn1\,rY^4 +$
$12\,a1^3\,a2\,cY1^2\,cYn1\,rY^4 + 2\,a0^2\,a2^2\,cY1^2\,cYn1\,rY^4 +$
$16\,a0\,a1\,a2^2\,cY1^2\,cYn1\,rY^4 + 12\,a1^2\,a2^2\,cY1^2\,cYn1\,rY^4 +$
$a0^2\,a1^2\,cYn1^3\,rY^4 + 2\,a0\,a1^3\,cYn1^3\,rY^4 + 2\,a1^4\,cYn1^3\,rY^4 -$
$2\,a0^2\,a1\,a2\,cYn1^3\,rY^4 - 2\,a0\,a1^2\,a2\,cYn1^3\,rY^4 + 2\,a1^3\,a2\,cYn1^3\,rY^4 +$
$a0^2\,a2^2\,cYn1^3\,rY^4 - 2\,a0\,a1\,a2^2\,cYn1^3\,rY^4 + a1^2\,a2^2\,cYn1^3\,rY^4 +$
$a0^2\,a1^2\,cYn1\,cYn2^2\,rY^4 + 2\,a0\,a1^3\,cYn1\,cYn2^2\,rY^4 + 2\,a1^4\,cYn1\,cYn2^2\,rY^4 -$
$2\,a0^2\,a1\,a2\,cYn1\,cYn2^2\,rY^4 - 2\,a0\,a1^2\,a2\,cYn1\,cYn2^2\,rY^4 +$
$2\,a1^3\,a2\,cYn1\,cYn2^2\,rY^4 + a0^2\,a2^2\,cYn1\,cYn2^2\,rY^4 -$
$2\,a0\,a1\,a2^2\,cYn1\,cYn2^2\,rY^4 + a1^2\,a2^2\,cYn1\,cYn2^2\,rY^4 +$
$8\,a0^2\,a1\,cY1\,cYn1\,rY^5 + 12\,a0\,a1^2\,cY1\,cYn1\,rY^5 + 4\,a1^3\,cY1\,cYn1\,rY^5 +$
$4\,a0^2\,a2\,cY1\,cYn1\,rY^5 + 20\,a0\,a1\,a2\,cY1\,cYn1\,rY^5 + 12\,a1^2\,a2\,cY1\,cYn1\,rY^5 +$
$4\,a0\,a2^2\,cY1\,cYn1\,rY^5 + 8\,a1\,a2^2\,cY1\,cYn1\,rY^5 + 2\,a0^2\,cYn1\,rY^6 +$
$4\,a0\,a1\,cYn1\,rY^6 + 2\,a1^2\,cYn1\,rY^6 + 4\,a0\,a2\,cYn1\,rY^6 +$
$4\,a1\,a2\,cYn1\,rY^6 + 2\,a2^2\,cYn1\,rY^6 - 2\,a0\,a1^2\,a2\,cY1^2\,cYn1^2\,rY$
$\sqrt{\Big((a0\,a1\,a2\,cY1\,cYn1 + a0\,a1\,cYn1\,rY + a0\,a2\,cYn1\,rY + a1\,a2\,cYn1\,rY)^2 -}$
$4\,a0\,a1\,a2\,cYn1^2\,\big(a0\,a1\,a2\,cY1^2 + a0\,a1\,cY1\,rY + a0\,a2\,cY1\,rY +$
$a1\,a2\,cY1\,rY + a0\,rY^2 + a1\,rY^2 + a2\,rY^2\big)\Big) + 2\,a0\,a1^2\,a2\,cY1^2\,cYn2^2$
$rY\,\sqrt{\Big((a0\,a1\,a2\,cY1\,cYn1 + a0\,a1\,cYn1\,rY + a0\,a2\,cYn1\,rY + a1\,a2\,cYn1\,rY)^2 -}$
$4\,a0\,a1\,a2\,cYn1^2\,\big(a0\,a1\,a2\,cY1^2 + a0\,a1\,cY1\,rY + a0\,a2\,cY1\,rY +$
$a1\,a2\,cY1\,rY + a0\,rY^2 + a1\,rY^2 + a2\,rY^2\big)\Big) - 2\,a0\,a1^2\,cY1\,cYn1^2$
$rY^2\,\sqrt{\Big((a0\,a1\,a2\,cY1\,cYn1 + a0\,a1\,cYn1\,rY + a0\,a2\,cYn1\,rY + a1\,a2\,cYn1\,rY)^2 -}$
$4\,a0\,a1\,a2\,cYn1^2\,\big(a0\,a1\,a2\,cY1^2 + a0\,a1\,cY1\,rY + a0\,a2\,cY1\,rY +$
$a1\,a2\,cY1\,rY + a0\,rY^2 + a1\,rY^2 + a2\,rY^2\big)\Big) - a0\,a1\,a2\,cY1\,cYn1^2$
$rY^2\,\sqrt{\Big((a0\,a1\,a2\,cY1\,cYn1 + a0\,a1\,cYn1\,rY + a0\,a2\,cYn1\,rY + a1\,a2\,cYn1\,rY)^2 -}$
$4\,a0\,a1\,a2\,cYn1^2\,\big(a0\,a1\,a2\,cY1^2 + a0\,a1\,cY1\,rY + a0\,a2\,cY1\,rY +$
$a1\,a2\,cY1\,rY + a0\,rY^2 + a1\,rY^2 + a2\,rY^2\big)\Big) - 2\,a1^2\,a2\,cY1\,cYn1^2$
$rY^2\,\sqrt{\Big((a0\,a1\,a2\,cY1\,cYn1 + a0\,a1\,cYn1\,rY + a0\,a2\,cYn1\,rY + a1\,a2\,cYn1\,rY)^2 -}$
$4\,a0\,a1\,a2\,cYn1^2\,\big(a0\,a1\,a2\,cY1^2 + a0\,a1\,cY1\,rY + a0\,a2\,cY1\,rY +$
$a1\,a2\,cY1\,rY + a0\,rY^2 + a1\,rY^2 + a2\,rY^2\big)\Big) + 2\,a0\,a1^2\,cY1\,cYn2^2$
$rY^2\,\sqrt{\Big((a0\,a1\,a2\,cY1\,cYn1 + a0\,a1\,cYn1\,rY + a0\,a2\,cYn1\,rY + a1\,a2\,cYn1\,rY)^2 -}$
$4\,a0\,a1\,a2\,cYn1^2\,\big(a0\,a1\,a2\,cY1^2 + a0\,a1\,cY1\,rY + a0\,a2\,cY1\,rY +$
$a1\,a2\,cY1\,rY + a0\,rY^2 + a1\,rY^2 + a2\,rY^2\big)\Big) + a0\,a1\,a2\,cY1\,cYn2^2$
$rY^2\,\sqrt{\Big((a0\,a1\,a2\,cY1\,cYn1 + a0\,a1\,cYn1\,rY + a0\,a2\,cYn1\,rY + a1\,a2\,cYn1\,rY)^2 -}$
$4\,a0\,a1\,a2\,cYn1^2\,\big(a0\,a1\,a2\,cY1^2 + a0\,a1\,cY1\,rY + a0\,a2\,cY1\,rY +$

$$\text{a1 a2 cY1 rY} + \text{a0 rY}^2 + \text{a1 rY}^2 + \text{a2 rY}^2\big)\big) + 2\ \text{a1}^2\ \text{a2 cY1 cYn2}^2$$
$$\text{rY}^2 \sqrt{\big(}\big((\text{a0 a1 a2 cY1 cYn1} + \text{a0 a1 cYn1 rY} + \text{a0 a2 cYn1 rY} + \text{a1 a2 cYn1 rY})^2 -$$
$$4\ \text{a0 a1 a2 cYn1}^2\ \big(\text{a0 a1 a2 cY1}^2 + \text{a0 a1 cY1 rY} +$$
$$\text{a0 a2 cY1 rY} + \text{a1 a2 cY1 rY} + \text{a0 rY}^2 + \text{a1 rY}^2 + \text{a2 rY}^2\big)\big) -$$
$$\text{a0 a1 cYn1}^2\ \text{rY}^3 \sqrt{\big(}\big((\text{a0 a1 a2 cY1 cYn1} + \text{a0 a1 cYn1 rY} + \text{a0 a2 cYn1 rY} + \text{a1 a2 cYn1 rY})^2 -$$
$$4\ \text{a0 a1 a2 cYn1}^2\ \big(\text{a0 a1 a2 cY1}^2 + \text{a0 a1 cY1 rY} +$$
$$\text{a0 a2 cY1 rY} + \text{a1 a2 cY1 rY} + \text{a0 rY}^2 + \text{a1 rY}^2 + \text{a2 rY}^2\big)\big) -$$
$$2\ \text{a1}^2\ \text{cYn1}^2\ \text{rY}^3 \sqrt{\big(}\big((\text{a0 a1 a2 cY1 cYn1} + \text{a0 a1 cYn1 rY} + \text{a0 a2 cYn1 rY} + \text{a1 a2 cYn1 rY})^2 -$$
$$4\ \text{a0 a1 a2 cYn1}^2\ \big(\text{a0 a1 a2 cY1}^2 + \text{a0 a1 cY1 rY} +$$
$$\text{a0 a2 cY1 rY} + \text{a1 a2 cY1 rY} + \text{a0 rY}^2 + \text{a1 rY}^2 + \text{a2 rY}^2\big)\big) +$$
$$\text{a0 a2 cYn1}^2\ \text{rY}^3 \sqrt{\big(}\big((\text{a0 a1 a2 cY1 cYn1} + \text{a0 a1 cYn1 rY} + \text{a0 a2 cYn1 rY} + \text{a1 a2 cYn1 rY})^2 -$$
$$4\ \text{a0 a1 a2 cYn1}^2\ \big(\text{a0 a1 a2 cY1}^2 + \text{a0 a1 cY1 rY} +$$
$$\text{a0 a2 cY1 rY} + \text{a1 a2 cY1 rY} + \text{a0 rY}^2 + \text{a1 rY}^2 + \text{a2 rY}^2\big)\big) -$$
$$\text{a1 a2 cYn1}^2\ \text{rY}^3 \sqrt{\big(}\big((\text{a0 a1 a2 cY1 cYn1} + \text{a0 a1 cYn1 rY} + \text{a0 a2 cYn1 rY} + \text{a1 a2 cYn1 rY})^2 -$$
$$4\ \text{a0 a1 a2 cYn1}^2\ \big(\text{a0 a1 a2 cY1}^2 + \text{a0 a1 cY1 rY} +$$
$$\text{a0 a2 cY1 rY} + \text{a1 a2 cY1 rY} + \text{a0 rY}^2 + \text{a1 rY}^2 + \text{a2 rY}^2\big)\big) +$$
$$\text{a0 a1 cYn2}^2\ \text{rY}^3 \sqrt{\big(}\big((\text{a0 a1 a2 cY1 cYn1} + \text{a0 a1 cYn1 rY} + \text{a0 a2 cYn1 rY} + \text{a1 a2 cYn1 rY})^2 -$$
$$4\ \text{a0 a1 a2 cYn1}^2\ \big(\text{a0 a1 a2 cY1}^2 + \text{a0 a1 cY1 rY} +$$
$$\text{a0 a2 cY1 rY} + \text{a1 a2 cY1 rY} + \text{a0 rY}^2 + \text{a1 rY}^2 + \text{a2 rY}^2\big)\big) +$$
$$2\ \text{a1}^2\ \text{cYn2}^2\ \text{rY}^3 \sqrt{\big(}\big((\text{a0 a1 a2 cY1 cYn1} + \text{a0 a1 cYn1 rY} + \text{a0 a2 cYn1 rY} + \text{a1 a2 cYn1 rY})^2 -$$
$$4\ \text{a0 a1 a2 cYn1}^2\ \big(\text{a0 a1 a2 cY1}^2 + \text{a0 a1 cY1 rY} +$$
$$\text{a0 a2 cY1 rY} + \text{a1 a2 cY1 rY} + \text{a0 rY}^2 + \text{a1 rY}^2 + \text{a2 rY}^2\big)\big) -$$
$$\text{a0 a2 cYn2}^2\ \text{rY}^3 \sqrt{\big(}\big((\text{a0 a1 a2 cY1 cYn1} + \text{a0 a1 cYn1 rY} + \text{a0 a2 cYn1 rY} + \text{a1 a2 cYn1 rY})^2 -$$
$$4\ \text{a0 a1 a2 cYn1}^2\ \big(\text{a0 a1 a2 cY1}^2 + \text{a0 a1 cY1 rY} +$$
$$\text{a0 a2 cY1 rY} + \text{a1 a2 cY1 rY} + \text{a0 rY}^2 + \text{a1 rY}^2 + \text{a2 rY}^2\big)\big) +$$
$$\text{a1 a2 cYn2}^2\ \text{rY}^3 \sqrt{\big(}\big((\text{a0 a1 a2 cY1 cYn1} + \text{a0 a1 cYn1 rY} + \text{a0 a2 cYn1 rY} + \text{a1 a2 cYn1 rY})^2 -$$
$$4\ \text{a0 a1 a2 cYn1}^2\ \big(\text{a0 a1 a2 cY1}^2 + \text{a0 a1 cY1 rY} + \text{a0 a2 cY1 rY} +$$
$$\text{a1 a2 cY1 rY} + \text{a0 rY}^2 + \text{a1 rY}^2 + \text{a2 rY}^2\big)\big)\big) \big/ \big(2\ \text{cYn1 rY}^2$$
$$\big(\text{a0 a1 a2 cY1}^2 + \text{a0 a1 cY1 rY} + \text{a0 a2 cY1 rY} + \text{a1 a2 cY1 rY} + \text{a0 rY}^2 + \text{a1 rY}^2 + \text{a2 rY}^2\big)^2\big),$$
$$\text{Sn2} \to \big(6\ \text{a0}^2\ \text{a1}^2\ \text{a2}^4\ \text{cY1}^4\ \text{cYn1} + 2\ \text{a0}^2\ \text{a1}^2\ \text{a2}^4\ \text{cY1}^6\ \text{cYn1} + 2\ \text{a0}^2\ \text{a1}^2\ \text{a2}^4\ \text{cY1}^4\ \text{cYn1}^3 +$$
$$2\ \text{a0}^2\ \text{a1}^2\ \text{a2}^4\ \text{cY1}^4\ \text{cYn1 cYn2}^2 + 12\ \text{a0}^2\ \text{a1}^2\ \text{a2}^3\ \text{cY1}^3\ \text{cYn1 rY} +$$
$$12\ \text{a0}^2\ \text{a1 a2}^4\ \text{cY1}^3\ \text{cYn1 rY} + 12\ \text{a0 a1}^2\ \text{a2}^4\ \text{cY1}^3\ \text{cYn1 rY} +$$
$$8\ \text{a0}^2\ \text{a1}^2\ \text{a2}^3\ \text{cY1}^5\ \text{cYn1 rY} + 4\ \text{a0}^2\ \text{a1 a2}^4\ \text{cY1}^5\ \text{cYn1 rY} +$$
$$4\ \text{a0 a1}^2\ \text{a2}^4\ \text{cY1}^5\ \text{cYn1 rY} + 2\ \text{a0}^2\ \text{a1}^2\ \text{a2}^3\ \text{cY1}^3\ \text{cYn1}^3\ \text{rY} +$$
$$4\ \text{a0}^2\ \text{a1 a2}^4\ \text{cY1}^3\ \text{cYn1}^3\ \text{rY} + 4\ \text{a0 a1}^2\ \text{a2}^4\ \text{cY1}^3\ \text{cYn1}^3\ \text{rY} +$$
$$2\ \text{a0}^2\ \text{a1}^2\ \text{a2}^3\ \text{cY1}^3\ \text{cYn1 cYn2}^2\ \text{rY} + 4\ \text{a0}^2\ \text{a1 a2}^4\ \text{cY1}^3\ \text{cYn1 cYn2}^2\ \text{rY} +$$
$$4\ \text{a0 a1}^2\ \text{a2}^4\ \text{cY1}^3\ \text{cYn1 cYn2}^2\ \text{rY} + 6\ \text{a0}^2\ \text{a1}^2\ \text{a2}^2\ \text{cY1}^2\ \text{cYn1 rY}^2 +$$
$$24\ \text{a0}^2\ \text{a1 a2}^3\ \text{cY1}^2\ \text{cYn1 rY}^2 + 24\ \text{a0 a1}^2\ \text{a2}^3\ \text{cY1}^2\ \text{cYn1 rY}^2 +$$
$$6\ \text{a0}^2\ \text{a2}^4\ \text{cY1}^2\ \text{cYn1 rY}^2 + 24\ \text{a0 a1 a2}^4\ \text{cY1}^2\ \text{cYn1 rY}^2 +$$
$$6\ \text{a1}^2\ \text{a2}^4\ \text{cY1}^2\ \text{cYn1 rY}^2 + 12\ \text{a0}^2\ \text{a1}^2\ \text{a2}^2\ \text{cY1}^4\ \text{cYn1 rY}^2 +$$
$$16\ \text{a0}^2\ \text{a1 a2}^3\ \text{cY1}^4\ \text{cYn1 rY}^2 + 16\ \text{a0 a1}^2\ \text{a2}^3\ \text{cY1}^4\ \text{cYn1 rY}^2 +$$
$$2\ \text{a0}^2\ \text{a2}^4\ \text{cY1}^4\ \text{cYn1 rY}^2 + 8\ \text{a0 a1 a2}^4\ \text{cY1}^4\ \text{cYn1 rY}^2 +$$
$$2\ \text{a1}^2\ \text{a2}^4\ \text{cY1}^4\ \text{cYn1 rY}^2 - 3\ \text{a0}^2\ \text{a1}^2\ \text{a2}^2\ \text{cY1}^2\ \text{cYn1}^3\ \text{rY}^2 +$$
$$4\ \text{a0}^2\ \text{a1 a2}^3\ \text{cY1}^2\ \text{cYn1}^3\ \text{rY}^2 + 4\ \text{a0 a1}^2\ \text{a2}^3\ \text{cY1}^2\ \text{cYn1}^3\ \text{rY}^2 +$$
$$2\ \text{a0}^2\ \text{a2}^4\ \text{cY1}^2\ \text{cYn1}^3\ \text{rY}^2 + 8\ \text{a0 a1 a2}^4\ \text{cY1}^2\ \text{cYn1}^3\ \text{rY}^2 +$$
$$2\ \text{a1}^2\ \text{a2}^4\ \text{cY1}^2\ \text{cYn1}^3\ \text{rY}^2 - 3\ \text{a0}^2\ \text{a1}^2\ \text{a2}^2\ \text{cY1}^2\ \text{cYn1 cYn2}^2\ \text{rY}^2 +$$
$$4\ \text{a0}^2\ \text{a1 a2}^3\ \text{cY1}^2\ \text{cYn1 cYn2}^2\ \text{rY}^2 + 4\ \text{a0 a1}^2\ \text{a2}^3\ \text{cY1}^2\ \text{cYn1 cYn2}^2\ \text{rY}^2 +$$
$$2\ \text{a0}^2\ \text{a2}^4\ \text{cY1}^2\ \text{cYn1 cYn2}^2\ \text{rY}^2 + 8\ \text{a0 a1 a2}^4\ \text{cY1}^2\ \text{cYn1 cYn2}^2\ \text{rY}^2 +$$
$$2\ \text{a1}^2\ \text{a2}^4\ \text{cY1}^2\ \text{cYn1 cYn2}^2\ \text{rY}^2 + 12\ \text{a0}^2\ \text{a1 a2}^2\ \text{cY1 cYn1 rY}^3 +$$
$$12\ \text{a0 a1}^2\ \text{a2}^2\ \text{cY1 cYn1 rY}^3 + 12\ \text{a0}^2\ \text{a2}^3\ \text{cY1 cYn1 rY}^3 +$$
$$36\ \text{a0 a1 a2}^3\ \text{cY1 cYn1 rY}^3 + 12\ \text{a1}^2\ \text{a2}^3\ \text{cY1 cYn1 rY}^3 +$$
$$12\ \text{a0 a2}^4\ \text{cY1 cYn1 rY}^3 + 12\ \text{a1 a2}^4\ \text{cY1 cYn1 rY}^3 +$$
$$8\ \text{a0}^2\ \text{a1}^2\ \text{a2 cY1}^3\ \text{cYn1 rY}^3 + 24\ \text{a0}^2\ \text{a1 a2}^2\ \text{cY1}^3\ \text{cYn1 rY}^3 +$$

$24\,a0\,a1^2\,a2^2\,cY1^3\,cYn1\,rY^3+8\,a0^2\,a2^3\,cY1^3\,cYn1\,rY^3+$
$28\,a0\,a1\,a2^3\,cY1^3\,cYn1\,rY^3+8\,a1^2\,a2^3\,cY1^3\,cYn1\,rY^3+$
$4\,a0\,a2^4\,cY1^3\,cYn1\,rY^3+4\,a1\,a2^4\,cY1^3\,cYn1\,rY^3-$
$2\,a0^2\,a1^2\,a2\,cY1\,cYn1^3\,rY^3-2\,a0^2\,a1\,a2^2\,cY1\,cYn1^3\,rY^3-$
$2\,a0\,a1^2\,a2^2\,cY1\,cYn1^3\,rY^3+2\,a0^2\,a2^3\,cY1\,cYn1^3\,rY^3+$
$6\,a0\,a1\,a2^3\,cY1\,cYn1^3\,rY^3+2\,a1^2\,a2^3\,cY1\,cYn1^3\,rY^3+$
$4\,a0\,a2^4\,cY1\,cYn1^3\,rY^3+4\,a1\,a2^4\,cY1\,cYn1^3\,rY^3-$
$2\,a0^2\,a1^2\,a2\,cY1\,cYn1\,cYn2^2\,rY^3-2\,a0^2\,a1\,a2^2\,cY1\,cYn1\,cYn2^2\,rY^3-$
$2\,a0\,a1^2\,a2^2\,cY1\,cYn1\,cYn2^2\,rY^3+2\,a0^2\,a2^3\,cY1\,cYn1\,cYn2^2\,rY^3+$
$6\,a0\,a1\,a2^3\,cY1\,cYn1\,cYn2^2\,rY^3+2\,a1^2\,a2^3\,cY1\,cYn1\,cYn2^2\,rY^3+$
$4\,a0\,a2^4\,cY1\,cYn1\,cYn2^2\,rY^3+4\,a1\,a2^4\,cY1\,cYn1\,cYn2^2\,rY^3+$
$6\,a0^2\,a2^2\,cYn1\,rY^4+12\,a0\,a1\,a2^2\,cYn1\,rY^4+6\,a1^2\,a2^2\,cYn1\,rY^4+$
$12\,a0\,a2^3\,cYn1\,rY^4+12\,a1\,a2^3\,cYn1\,rY^4+6\,a2^4\,cYn1\,rY^4+$
$2\,a0^2\,a1^2\,cY1^2\,cYn1\,rY^4+16\,a0^2\,a1\,a2\,cY1^2\,cYn1\,rY^4+$
$16\,a0\,a1^2\,a2\,cY1^2\,cYn1\,rY^4+12\,a0^2\,a2^2\,cY1^2\,cYn1\,rY^4+$
$36\,a0\,a1\,a2^2\,cY1^2\,cYn1\,rY^4+12\,a1^2\,a2^2\,cY1^2\,cYn1\,rY^4+$
$12\,a0\,a2^3\,cY1^2\,cYn1\,rY^4+12\,a1\,a2^3\,cY1^2\,cYn1\,rY^4+$
$2\,a2^4\,cY1^2\,cYn1\,rY^4+a0^2\,a1^2\,cYn1^3\,rY^4-2\,a0^2\,a1\,a2\,cYn1^3\,rY^4-$
$2\,a0\,a1^2\,a2\,cYn1^3\,rY^4+a0^2\,a2^2\,cYn1^3\,rY^4-$
$2\,a0\,a1\,a2^2\,cYn1^3\,rY^4+a1^2\,a2^2\,cYn1^3\,rY^4+2\,a0\,a2^3\,cYn1^3\,rY^4+$
$2\,a1\,a2^3\,cYn1^3\,rY^4+2\,a2^4\,cYn1^3\,rY^4+a0^2\,a1^2\,cYn1\,cYn2^2\,rY^4-$
$2\,a0^2\,a1\,a2\,cYn1\,cYn2^2\,rY^4-2\,a0\,a1^2\,a2\,cYn1\,cYn2^2\,rY^4+$
$a0^2\,a2^2\,cYn1\,cYn2^2\,rY^4-2\,a0\,a1\,a2^2\,cYn1\,cYn2^2\,rY^4+$
$a1^2\,a2^2\,cYn1\,cYn2^2\,rY^4+2\,a0\,a2^3\,cYn1\,cYn2^2\,rY^4+$
$2\,a1\,a2^3\,cYn1\,cYn2^2\,rY^4+2\,a2^4\,cYn1\,cYn2^2\,rY^4+$
$4\,a0^2\,a1\,cY1\,cYn1\,rY^5+4\,a0\,a1^2\,cY1\,cYn1\,rY^5+$
$8\,a0^2\,a2\,cY1\,cYn1\,rY^5+20\,a0\,a1\,a2\,cY1\,cYn1\,rY^5+$
$8\,a1^2\,a2\,cY1\,cYn1\,rY^5+12\,a0\,a2^2\,cY1\,cYn1\,rY^5+$
$12\,a1\,a2^2\,cY1\,cYn1\,rY^5+4\,a2^3\,cY1\,cYn1\,rY^5+2\,a0^2\,cYn1\,rY^6+$
$4\,a0\,a1\,cYn1\,rY^6+2\,a1^2\,cYn1\,rY^6+4\,a0\,a2\,cYn1\,rY^6+$
$4\,a1\,a2\,cYn1\,rY^6+2\,a2^2\,cYn1\,rY^6-2\,a0\,a1\,a2^2\,cY1^2\,cYn1^2\,rY$
$\sqrt{\Big((a0\,a1\,a2\,cY1\,cYn1+a0\,a1\,cYn1\,rY+a0\,a2\,cYn1\,rY+a1\,a2\,cYn1\,rY)^2-}$
$4\,a0\,a1\,a2\,cYn1^2\,\big(a0\,a1\,a2\,cY1^2+a0\,a1\,cY1\,rY+a0\,a2\,cY1\,rY+$
$a1\,a2\,cY1\,rY+a0\,rY^2+a1\,rY^2+a2\,rY^2\big)\Big)}+2\,a0\,a1\,a2^2\,cY1^2\,cYn2^2$
$rY\sqrt{\Big((a0\,a1\,a2\,cY1\,cYn1+a0\,a1\,cYn1\,rY+a0\,a2\,cYn1\,rY+a1\,a2\,cYn1\,rY)^2-}$
$4\,a0\,a1\,a2\,cYn1^2\,\big(a0\,a1\,a2\,cY1^2+a0\,a1\,cY1\,rY+a0\,a2\,cY1\,rY+$
$a1\,a2\,cY1\,rY+a0\,rY^2+a1\,rY^2+a2\,rY^2\big)\Big)}-a0\,a1\,a2\,cY1\,cYn1^2$
$rY^2\sqrt{\Big((a0\,a1\,a2\,cY1\,cYn1+a0\,a1\,cYn1\,rY+a0\,a2\,cYn1\,rY+a1\,a2\,cYn1\,rY)^2-}$
$4\,a0\,a1\,a2\,cYn1^2\,\big(a0\,a1\,a2\,cY1^2+a0\,a1\,cY1\,rY+a0\,a2\,cY1\,rY+$
$a1\,a2\,cY1\,rY+a0\,rY^2+a1\,rY^2+a2\,rY^2\big)\Big)}-2\,a0\,a2^2\,cY1\,cYn1^2$
$rY^2\sqrt{\Big((a0\,a1\,a2\,cY1\,cYn1+a0\,a1\,cYn1\,rY+a0\,a2\,cYn1\,rY+a1\,a2\,cYn1\,rY)^2-}$
$4\,a0\,a1\,a2\,cYn1^2\,\big(a0\,a1\,a2\,cY1^2+a0\,a1\,cY1\,rY+a0\,a2\,cY1\,rY+$
$a1\,a2\,cY1\,rY+a0\,rY^2+a1\,rY^2+a2\,rY^2\big)\Big)}-2\,a1\,a2^2\,cY1\,cYn1^2$
$rY^2\sqrt{\Big((a0\,a1\,a2\,cY1\,cYn1+a0\,a1\,cYn1\,rY+a0\,a2\,cYn1\,rY+a1\,a2\,cYn1\,rY)^2-}$
$4\,a0\,a1\,a2\,cYn1^2\,\big(a0\,a1\,a2\,cY1^2+a0\,a1\,cY1\,rY+a0\,a2\,cY1\,rY+$
$a1\,a2\,cY1\,rY+a0\,rY^2+a1\,rY^2+a2\,rY^2\big)\Big)}+a0\,a1\,a2\,cY1\,cYn2^2$
$rY^2\sqrt{\Big((a0\,a1\,a2\,cY1\,cYn1+a0\,a1\,cYn1\,rY+a0\,a2\,cYn1\,rY+a1\,a2\,cYn1\,rY)^2-}$
$4\,a0\,a1\,a2\,cYn1^2\,\big(a0\,a1\,a2\,cY1^2+a0\,a1\,cY1\,rY+a0\,a2\,cY1\,rY+$
$a1\,a2\,cY1\,rY+a0\,rY^2+a1\,rY^2+a2\,rY^2\big)\Big)}+2\,a0\,a2^2\,cY1\,cYn2^2$
$rY^2\sqrt{\Big((a0\,a1\,a2\,cY1\,cYn1+a0\,a1\,cYn1\,rY+a0\,a2\,cYn1\,rY+a1\,a2\,cYn1\,rY)^2-}$
$4\,a0\,a1\,a2\,cYn1^2\,\big(a0\,a1\,a2\,cY1^2+a0\,a1\,cY1\,rY+a0\,a2\,cY1\,rY+$
$a1\,a2\,cY1\,rY+a0\,rY^2+a1\,rY^2+a2\,rY^2\big)\Big)}+2\,a1\,a2^2\,cY1\,cYn2^2$
$rY^2\sqrt{\Big((a0\,a1\,a2\,cY1\,cYn1+a0\,a1\,cYn1\,rY+a0\,a2\,cYn1\,rY+a1\,a2\,cYn1\,rY)^2-}$
$4\,a0\,a1\,a2\,cYn1^2\,\big(a0\,a1\,a2\,cY1^2+a0\,a1\,cY1\,rY+$

$$a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2)) +$$
$$a0\ a1\ cYn1^2\ rY^3 \sqrt{((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 -}$$
$$4\ a0\ a1\ a2\ cYn1^2\ (a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY +$$
$$a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2)) -$$
$$a0\ a2\ cYn1^2\ rY^3 \sqrt{((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 -}$$
$$4\ a0\ a1\ a2\ cYn1^2\ (a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY +$$
$$a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2)) -$$
$$a1\ a2\ cYn1^2\ rY^3 \sqrt{((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 -}$$
$$4\ a0\ a1\ a2\ cYn1^2\ (a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY +$$
$$a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2)) -$$
$$2\ a2^2\ cYn1^2\ rY^3 \sqrt{((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 -}$$
$$4\ a0\ a1\ a2\ cYn1^2\ (a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY +$$
$$a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2)) -$$
$$a0\ a1\ cYn2^2\ rY^3 \sqrt{((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 -}$$
$$4\ a0\ a1\ a2\ cYn1^2\ (a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY +$$
$$a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2)) +$$
$$a0\ a2\ cYn2^2\ rY^3 \sqrt{((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 -}$$
$$4\ a0\ a1\ a2\ cYn1^2\ (a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY +$$
$$a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2)) +$$
$$a1\ a2\ cYn2^2\ rY^3 \sqrt{((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 -}$$
$$4\ a0\ a1\ a2\ cYn1^2\ (a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY +$$
$$a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2)) +$$
$$2\ a2^2\ cYn2^2\ rY^3 \sqrt{((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 -}$$
$$4\ a0\ a1\ a2\ cYn1^2\ (a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + a0\ a2\ cY1\ rY +$$
$$a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2)))\ /\ (2\ cYn1\ rY^2$$
$$(a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2)^2),$$
$$pY1 \to 3,\ n1 \to -((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY +$$
$$\sqrt{((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 -}$$
$$4\ a0\ a1\ a2\ cYn1^2\ (a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY +$$
$$a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2)))\ /$$
$$(2\ (a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2))),$$
$$n2 \to -((cYn2\ (a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY -$$
$$\sqrt{((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 -}$$
$$4\ a0\ a1\ a2\ cYn1^2\ (a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY +$$
$$a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2))))\ /$$
$$(2\ cYn1\ (a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 +$$
$$a1\ rY^2 + a2\ rY^2)))\},\ \{p \to \frac{3 + cY1^2 + cYn1^2 + cYn2^2}{rY^2},$$
$$d \to -\frac{rY^2\ (a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2)}{cYn1\ cYn2},$$
$$N01 \to$$
$$-\frac{(a0 - a1)\ (a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2)}{a2^3\ cYn1\ cYn2\ rY}$$
$$,$$
$$N02 \to$$
$$\frac{(a0 - a2)\ (a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2)}{a1^3\ cYn1\ cYn2\ rY},$$
$$N12 \to$$
$$\frac{(a1 - a2)\ (a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2)}{a0^3\ cYn1\ cYn2\ rY},$$

$\text{Sn0} \to \Big(6\, a0^4\, a1^2\, a2^2\, cY1^4\, cYn1 + 2\, a0^4\, a1^2\, a2^2\, cY1^6\, cYn1 + 2\, a0^4\, a1^2\, a2^2\, cY1^4\, cYn1^3 +$

$2\, a0^4\, a1^2\, a2^2\, cY1^4\, cYn1\, cYn2^2 + 12\, a0^4\, a1^2\, a2\, cY1^3\, cYn1\, rY + 12\, a0^4\, a1\, a2^2\, cY1^3\, cYn1\, rY +$

$12\, a0^3\, a1^2\, a2^2\, cY1^3\, cYn1\, rY + 4\, a0^4\, a1^2\, a2\, cY1^5\, cYn1\, rY + 4\, a0^4\, a1\, a2^2\, cY1^5\, cYn1\, rY +$

$8\, a0^3\, a1^2\, a2^2\, cY1^5\, cYn1\, rY + 4\, a0^4\, a1^2\, a2\, cY1^3\, cYn1^3\, rY + 4\, a0^4\, a1\, a2^2\, cY1^3\, cYn1^3\, rY +$

$2\, a0^3\, a1^2\, a2^2\, cY1^3\, cYn1^3\, rY + 4\, a0^4\, a1^2\, a2\, cY1^3\, cYn1\, cYn2^2\, rY +$

$4\, a0^4\, a1\, a2^2\, cY1^3\, cYn1\, cYn2^2\, rY + 2\, a0^3\, a1^2\, a2^2\, cY1^3\, cYn1\, cYn2^2\, rY + 6\, a0^4\, a1^2\, cY1^2\, cYn1\, rY^2 +$

$24\, a0^4\, a1\, a2\, cY1^2\, cYn1\, rY^2 + 24\, a0^3\, a1^2\, a2\, cY1^2\, cYn1\, rY^2 + 6\, a0^4\, a2^2\, cY1^2\, cYn1\, rY^2 +$

$24\, a0^3\, a1\, a2^2\, cY1^2\, cYn1\, rY^2 + 6\, a0^2\, a1^2\, a2^2\, cY1^2\, cYn1\, rY^2 + 2\, a0^4\, a1^2\, cY1^4\, cYn1\, rY^2 +$

$8\, a0^4\, a1\, a2\, cY1^4\, cYn1\, rY^2 + 16\, a0^3\, a1^2\, a2\, cY1^4\, cYn1\, rY^2 + 2\, a0^4\, a2^2\, cY1^4\, cYn1\, rY^2 +$

$16\, a0^3\, a1\, a2^2\, cY1^4\, cYn1\, rY^2 + 12\, a0^2\, a1^2\, a2^2\, cY1^4\, cYn1\, rY^2 + 2\, a0^4\, a1^2\, cY1^2\, cYn1^3\, rY^2 +$

$8\, a0^4\, a1\, a2\, cY1^2\, cYn1^3\, rY^2 + 4\, a0^3\, a1^2\, a2\, cY1^2\, cYn1^3\, rY^2 + 2\, a0^4\, a2^2\, cY1^2\, cYn1^3\, rY^2 +$

$4\, a0^3\, a1\, a2^2\, cY1^2\, cYn1^3\, rY^2 - 3\, a0^2\, a1^2\, a2^2\, cY1^2\, cYn1^3\, rY^2 + 2\, a0^4\, a1^2\, cY1^2\, cYn1\, cYn2^2\, rY^2 +$

$8\, a0^4\, a1\, a2\, cY1^2\, cYn1\, cYn2^2\, rY^2 + 4\, a0^3\, a1^2\, a2\, cY1^2\, cYn1\, cYn2^2\, rY^2 +$

$2\, a0^4\, a2^2\, cY1^2\, cYn1\, cYn2^2\, rY^2 + 4\, a0^3\, a1\, a2^2\, cY1^2\, cYn1\, cYn2^2\, rY^2 -$

$3\, a0^2\, a1^2\, a2^2\, cY1^2\, cYn1\, cYn2^2\, rY^2 + 12\, a0^4\, a1\, cY1\, cYn1\, rY^3 + 12\, a0^3\, a1^2\, cY1\, cYn1\, rY^3 +$

$12\, a0^4\, a2\, cY1\, cYn1\, rY^3 + 36\, a0^3\, a1\, a2\, cY1\, cYn1\, rY^3 + 12\, a0^2\, a1^2\, a2\, cY1\, cYn1\, rY^3 +$

$12\, a0^3\, a2^2\, cY1\, cYn1\, rY^3 + 12\, a0^2\, a1\, a2^2\, cY1\, cYn1\, rY^3 + 4\, a0^4\, a1\, cY1^3\, cYn1\, rY^3 +$

$8\, a0^3\, a1^2\, cY1^3\, cYn1\, rY^3 + 4\, a0^4\, a2\, cY1^3\, cYn1\, rY^3 + 28\, a0^3\, a1\, a2\, cY1^3\, cYn1\, rY^3 +$

$24\, a0^2\, a1^2\, a2\, cY1^3\, cYn1\, rY^3 + 8\, a0^3\, a2^2\, cY1^3\, cYn1\, rY^3 + 24\, a0^2\, a1\, a2^2\, cY1^3\, cYn1\, rY^3 +$

$8\, a0\, a1^2\, a2^2\, cY1^3\, cYn1\, rY^3 + 4\, a0^4\, a1\, cY1\, cYn1^3\, rY^3 + 2\, a0^3\, a1^2\, cY1\, cYn1^3\, rY^3 +$

$4\, a0^4\, a2\, cY1\, cYn1^3\, rY^3 + 6\, a0^3\, a1\, a2\, cY1\, cYn1^3\, rY^3 - 2\, a0^2\, a1^2\, a2\, cY1\, cYn1^3\, rY^3 +$

$2\, a0^3\, a2^2\, cY1\, cYn1^3\, rY^3 - 2\, a0^2\, a1\, a2^2\, cY1\, cYn1^3\, rY^3 - 2\, a0\, a1^2\, a2^2\, cY1\, cYn1^3\, rY^3 +$

$4\, a0^4\, a1\, cY1\, cYn1\, cYn2^2\, rY^3 + 2\, a0^3\, a1^2\, cY1\, cYn1\, cYn2^2\, rY^3 + 4\, a0^4\, a2\, cY1\, cYn1\, cYn2^2\, rY^3 +$

$6\, a0^3\, a1\, a2\, cY1\, cYn1\, cYn2^2\, rY^3 - 2\, a0^2\, a1^2\, a2\, cY1\, cYn1\, cYn2^2\, rY^3 +$

$2\, a0^3\, a2^2\, cY1\, cYn1\, cYn2^2\, rY^3 - 2\, a0^2\, a1\, a2^2\, cY1\, cYn1\, cYn2^2\, rY^3 -$

$2\, a0\, a1^2\, a2^2\, cY1\, cYn1\, cYn2^2\, rY^3 + 6\, a0^4\, cYn1\, rY^4 + 12\, a0^3\, a1\, cYn1\, rY^4 + 6\, a0^2\, a1^2\, cYn1\, rY^4 +$

$12\, a0^3\, a2\, cYn1\, rY^4 + 12\, a0^2\, a1\, a2\, cYn1\, rY^4 + 6\, a0^2\, a2^2\, cYn1\, rY^4 + 2\, a0^4\, cY1^2\, cYn1\, rY^4 +$

$12\, a0^3\, a1\, cY1^2\, cYn1\, rY^4 + 12\, a0^2\, a1^2\, cY1^2\, cYn1\, rY^4 + 12\, a0^3\, a2\, cY1^2\, cYn1\, rY^4 +$

$36\, a0^2\, a1\, a2\, cY1^2\, cYn1\, rY^4 + 16\, a0\, a1^2\, a2\, cY1^2\, cYn1\, rY^4 + 12\, a0^2\, a2^2\, cY1^2\, cYn1\, rY^4 +$

$16\, a0\, a1\, a2^2\, cY1^2\, cYn1\, rY^4 + 2\, a1^2\, a2^2\, cY1^2\, cYn1\, rY^4 + 2\, a0^4\, cYn1^3\, rY^4 + 2\, a0^3\, a1\, cYn1^3\, rY^4 +$

$a0^2\, a1^2\, cYn1^3\, rY^4 + 2\, a0^3\, a2\, cYn1^3\, rY^4 - 2\, a0^2\, a1\, a2\, cYn1^3\, rY^4 - 2\, a0\, a1^2\, a2\, cYn1^3\, rY^4 +$

$a0^2\, a2^2\, cYn1^3\, rY^4 - 2\, a0\, a1\, a2^2\, cYn1^3\, rY^4 + a1^2\, a2^2\, cYn1^3\, rY^4 + 2\, a0^4\, cYn1\, cYn2^2\, rY^4 +$

$2\, a0^3\, a1\, cYn1\, cYn2^2\, rY^4 + a0^2\, a1^2\, cYn1\, cYn2^2\, rY^4 + 2\, a0^3\, a2\, cYn1\, cYn2^2\, rY^4 -$

$2\, a0^2\, a1\, a2\, cYn1\, cYn2^2\, rY^4 - 2\, a0\, a1^2\, a2\, cYn1\, cYn2^2\, rY^4 + a0^2\, a2^2\, cYn1\, cYn2^2\, rY^4 -$

$2\, a0\, a1\, a2^2\, cYn1\, cYn2^2\, rY^4 + a1^2\, a2^2\, cYn1\, cYn2^2\, rY^4 + 4\, a0^3\, cY1\, cYn1\, rY^5 +$

$12\, a0^2\, a1\, cY1\, cYn1\, rY^5 + 8\, a0\, a1^2\, cY1\, cYn1\, rY^5 + 12\, a0^2\, a2\, cY1\, cYn1\, rY^5 +$

$20\, a0\, a1\, a2\, cY1\, cYn1\, rY^5 + 4\, a1^2\, a2\, cY1\, cYn1\, rY^5 + 8\, a0\, a2^2\, cY1\, cYn1\, rY^5 +$

$4\, a1\, a2^2\, cY1\, cYn1\, rY^5 + 2\, a0^2\, cYn1\, rY^6 + 4\, a0\, a1\, cYn1\, rY^6 + 2\, a1^2\, cYn1\, rY^6 +$

$4\, a0\, a2\, cYn1\, rY^6 + 4\, a1\, a2\, cYn1\, rY^6 + 2\, a2^2\, cYn1\, rY^6 + 2\, a0^2\, a1\, a2\, cY1^2\, cYn1^2$

$rY \sqrt{}\Big((a0\, a1\, a2\, cY1\, cYn1 + a0\, a1\, cYn1\, rY + a0\, a2\, cYn1\, rY + a1\, a2\, cYn1\, rY)^2 -$

$4\, a0\, a1\, a2\, cYn1^2\, \big(a0\, a1\, a2\, cY1^2 + a0\, a1\, cY1\, rY + a0\, a2\, cY1\, rY +$

$a1\, a2\, cY1\, rY + a0\, rY^2 + a1\, rY^2 + a2\, rY^2\big)\Big) - 2\, a0^2\, a1\, a2\, cY1^2\, cYn2^2$

$rY \sqrt{}\Big((a0\, a1\, a2\, cY1\, cYn1 + a0\, a1\, cYn1\, rY + a0\, a2\, cYn1\, rY + a1\, a2\, cYn1\, rY)^2 -$

$4\, a0\, a1\, a2\, cYn1^2\, \big(a0\, a1\, a2\, cY1^2 + a0\, a1\, cY1\, rY + a0\, a2\, cY1\, rY +$

$a1\, a2\, cY1\, rY + a0\, rY^2 + a1\, rY^2 + a2\, rY^2\big)\Big) + 2\, a0^2\, a1\, cY1\, cYn1^2$

$rY^2 \sqrt{}\Big((a0\, a1\, a2\, cY1\, cYn1 + a0\, a1\, cYn1\, rY + a0\, a2\, cYn1\, rY + a1\, a2\, cYn1\, rY)^2 -$

$4\, a0\, a1\, a2\, cYn1^2\, \big(a0\, a1\, a2\, cY1^2 + a0\, a1\, cY1\, rY + a0\, a2\, cY1\, rY +$

$a1\, a2\, cY1\, rY + a0\, rY^2 + a1\, rY^2 + a2\, rY^2\big)\Big) + 2\, a0^2\, a2\, cY1\, cYn1^2$

$rY^2 \sqrt{}\Big((a0\, a1\, a2\, cY1\, cYn1 + a0\, a1\, cYn1\, rY + a0\, a2\, cYn1\, rY + a1\, a2\, cYn1\, rY)^2 -$

$4\, a0\, a1\, a2\, cYn1^2\, \big(a0\, a1\, a2\, cY1^2 + a0\, a1\, cY1\, rY + a0\, a2\, cY1\, rY +$

$a1\, a2\, cY1\, rY + a0\, rY^2 + a1\, rY^2 + a2\, rY^2\big)\Big) + a0\, a1\, a2\, cY1\, cYn1^2$

$rY^2 \sqrt{}\Big((a0\, a1\, a2\, cY1\, cYn1 + a0\, a1\, cYn1\, rY + a0\, a2\, cYn1\, rY + a1\, a2\, cYn1\, rY)^2 -$

$4\, a0\, a1\, a2\, cYn1^2\, \big(a0\, a1\, a2\, cY1^2 + a0\, a1\, cY1\, rY + a0\, a2\, cY1\, rY +$

$a1\, a2\, cY1\, rY + a0\, rY^2 + a1\, rY^2 + a2\, rY^2\big)\Big) - 2\, a0^2\, a1\, cY1\, cYn2^2$

$rY^2 \sqrt{}\Big((a0\, a1\, a2\, cY1\, cYn1 + a0\, a1\, cYn1\, rY + a0\, a2\, cYn1\, rY + a1\, a2\, cYn1\, rY)^2 -$

$$4 a0 a1 a2 cYn1^2 (a0 a1 a2 cY1^2 + a0 a1 cY1 rY + a0 a2 cY1 rY +$$
$$a1 a2 cY1 rY + a0 rY^2 + a1 rY^2 + a2 rY^2)) - 2 a0^2 a2 cY1 cYn2^2$$
$$rY^2 \sqrt{(}(a0 a1 a2 cY1 cYn1 + a0 a1 cYn1 rY + a0 a2 cYn1 rY + a1 a2 cYn1 rY)^2 -$$
$$4 a0 a1 a2 cYn1^2 (a0 a1 a2 cY1^2 + a0 a1 cY1 rY + a0 a2 cY1 rY +$$
$$a1 a2 cY1 rY + a0 rY^2 + a1 rY^2 + a2 rY^2)) - a0 a1 a2 cY1 cYn2^2$$
$$rY^2 \sqrt{(}(a0 a1 a2 cY1 cYn1 + a0 a1 cYn1 rY + a0 a2 cYn1 rY + a1 a2 cYn1 rY)^2 -$$
$$4 a0 a1 a2 cYn1^2 (a0 a1 a2 cY1^2 + a0 a1 cY1 rY +$$
$$a0 a2 cY1 rY + a1 a2 cY1 rY + a0 rY^2 + a1 rY^2 + a2 rY^2)) +$$
$$2 a0^2 cYn1^2 rY^3 \sqrt{(}(a0 a1 a2 cY1 cYn1 + a0 a1 cYn1 rY + a0 a2 cYn1 rY + a1 a2 cYn1 rY)^2 -$$
$$4 a0 a1 a2 cYn1^2 (a0 a1 a2 cY1^2 + a0 a1 cY1 rY +$$
$$a0 a2 cY1 rY + a1 a2 cY1 rY + a0 rY^2 + a1 rY^2 + a2 rY^2)) +$$
$$a0 a1 cYn1^2 rY^3 \sqrt{(}(a0 a1 a2 cY1 cYn1 + a0 a1 cYn1 rY + a0 a2 cYn1 rY + a1 a2 cYn1 rY)^2 -$$
$$4 a0 a1 a2 cYn1^2 (a0 a1 a2 cY1^2 + a0 a1 cY1 rY +$$
$$a0 a2 cY1 rY + a1 a2 cY1 rY + a0 rY^2 + a1 rY^2 + a2 rY^2)) +$$
$$a0 a2 cYn1^2 rY^3 \sqrt{(}(a0 a1 a2 cY1 cYn1 + a0 a1 cYn1 rY + a0 a2 cYn1 rY + a1 a2 cYn1 rY)^2 -$$
$$4 a0 a1 a2 cYn1^2 (a0 a1 a2 cY1^2 + a0 a1 cY1 rY +$$
$$a0 a2 cY1 rY + a1 a2 cY1 rY + a0 rY^2 + a1 rY^2 + a2 rY^2)) -$$
$$a1 a2 cYn1^2 rY^3 \sqrt{(}(a0 a1 a2 cY1 cYn1 + a0 a1 cYn1 rY + a0 a2 cYn1 rY + a1 a2 cYn1 rY)^2 -$$
$$4 a0 a1 a2 cYn1^2 (a0 a1 a2 cY1^2 + a0 a1 cY1 rY +$$
$$a0 a2 cY1 rY + a1 a2 cY1 rY + a0 rY^2 + a1 rY^2 + a2 rY^2)) -$$
$$2 a0^2 cYn2^2 rY^3 \sqrt{(}(a0 a1 a2 cY1 cYn1 + a0 a1 cYn1 rY + a0 a2 cYn1 rY + a1 a2 cYn1 rY)^2 -$$
$$4 a0 a1 a2 cYn1^2 (a0 a1 a2 cY1^2 + a0 a1 cY1 rY +$$
$$a0 a2 cY1 rY + a1 a2 cY1 rY + a0 rY^2 + a1 rY^2 + a2 rY^2)) -$$
$$a0 a1 cYn2^2 rY^3 \sqrt{(}(a0 a1 a2 cY1 cYn1 + a0 a1 cYn1 rY + a0 a2 cYn1 rY + a1 a2 cYn1 rY)^2 -$$
$$4 a0 a1 a2 cYn1^2 (a0 a1 a2 cY1^2 + a0 a1 cY1 rY +$$
$$a0 a2 cY1 rY + a1 a2 cY1 rY + a0 rY^2 + a1 rY^2 + a2 rY^2)) -$$
$$a0 a2 cYn2^2 rY^3 \sqrt{(}(a0 a1 a2 cY1 cYn1 + a0 a1 cYn1 rY + a0 a2 cYn1 rY + a1 a2 cYn1 rY)^2 -$$
$$4 a0 a1 a2 cYn1^2 (a0 a1 a2 cY1^2 + a0 a1 cY1 rY +$$
$$a0 a2 cY1 rY + a1 a2 cY1 rY + a0 rY^2 + a1 rY^2 + a2 rY^2)) +$$
$$a1 a2 cYn2^2 rY^3 \sqrt{(}(a0 a1 a2 cY1 cYn1 + a0 a1 cYn1 rY + a0 a2 cYn1 rY + a1 a2 cYn1 rY)^2 -$$
$$4 a0 a1 a2 cYn1^2 (a0 a1 a2 cY1^2 + a0 a1 cY1 rY + a0 a2 cY1 rY +$$
$$a1 a2 cY1 rY + a0 rY^2 + a1 rY^2 + a2 rY^2))) / (2 cYn1 rY^2$$
$$(a0 a1 a2 cY1^2 + a0 a1 cY1 rY + a0 a2 cY1 rY + a1 a2 cY1 rY + a0 rY^2 + a1 rY^2 + a2 rY^2)^2),$$
$$Sn1 \to (6 a0^2 a1^4 a2^2 cY1^4 cYn1 + 2 a0^2 a1^4 a2^2 cY1^6 cYn1 + 2 a0^2 a1^4 a2^2 cY1^4 cYn1^3 +$$
$$2 a0^2 a1^4 a2^2 cY1^4 cYn1 cYn2^2 + 12 a0^2 a1^4 a2 cY1^3 cYn1 rY +$$
$$12 a0^2 a1^3 a2^2 cY1^3 cYn1 rY + 12 a0 a1^4 a2^2 cY1^3 cYn1 rY +$$
$$4 a0^2 a1^4 a2 cY1^5 cYn1 rY + 8 a0^2 a1^3 a2^2 cY1^5 cYn1 rY +$$
$$4 a0 a1^4 a2^2 cY1^5 cYn1 rY + 4 a0^2 a1^4 a2 cY1^3 cYn1^3 rY +$$
$$2 a0^2 a1^3 a2^2 cY1^3 cYn1^3 rY + 4 a0 a1^4 a2^2 cY1^3 cYn1^3 rY +$$
$$4 a0^2 a1^4 a2 cY1^3 cYn1 cYn2^2 rY + 2 a0^2 a1^3 a2^2 cY1^3 cYn1 cYn2^2 rY +$$
$$4 a0 a1^4 a2^2 cY1^3 cYn1 cYn2^2 rY + 6 a0^2 a1^4 cY1^2 cYn1 rY^2 +$$
$$24 a0^2 a1^3 a2 cY1^2 cYn1 rY^2 + 24 a0 a1^4 a2 cY1^2 cYn1 rY^2 +$$
$$6 a0^2 a1^2 a2^2 cY1^2 cYn1 rY^2 + 24 a0 a1^3 a2^2 cY1^2 cYn1 rY^2 +$$
$$6 a1^4 a2^2 cY1^2 cYn1 rY^2 + 2 a0^2 a1^4 cY1^4 cYn1 rY^2 +$$
$$16 a0^2 a1^3 a2 cY1^4 cYn1 rY^2 + 8 a0 a1^4 a2 cY1^4 cYn1 rY^2 +$$
$$12 a0^2 a1^2 a2^2 cY1^4 cYn1 rY^2 + 16 a0 a1^3 a2^2 cY1^4 cYn1 rY^2 +$$
$$2 a1^4 a2^2 cY1^4 cYn1 rY^2 + 2 a0^2 a1^4 cY1^2 cYn1^3 rY^2 +$$
$$4 a0^2 a1^3 a2 cY1^2 cYn1^3 rY^2 + 8 a0 a1^4 a2 cY1^2 cYn1^3 rY^2 -$$
$$3 a0^2 a1^2 a2^2 cY1^2 cYn1^3 rY^2 + 4 a0 a1^3 a2^2 cY1^2 cYn1^3 rY^2 +$$
$$2 a1^4 a2^2 cY1^2 cYn1^3 rY^2 + 2 a0^2 a1^4 cY1^2 cYn1 cYn2^2 rY^2 +$$
$$4 a0^2 a1^3 a2 cY1^2 cYn1 cYn2^2 rY^2 + 8 a0 a1^4 a2 cY1^2 cYn1 cYn2^2 rY^2 -$$
$$3 a0^2 a1^2 a2^2 cY1^2 cYn1 cYn2^2 rY^2 + 4 a0 a1^3 a2^2 cY1^2 cYn1 cYn2^2 rY^2 +$$

$$
\begin{aligned}
&2\ a1^4\ a2^2\ cY1^2\ cYn1\ cYn2^2\ rY^2 + 12\ a0^2\ a1^3\ cY1\ cYn1\ rY^3 + \\
&12\ a0\ a1^4\ cY1\ cYn1\ rY^3 + 12\ a0^2\ a1^2\ a2\ cY1\ cYn1\ rY^3 + \\
&36\ a0\ a1^3\ a2\ cY1\ cYn1\ rY^3 + 12\ a1^4\ a2\ cY1\ cYn1\ rY^3 + \\
&12\ a0\ a1^2\ a2^2\ cY1\ cYn1\ rY^3 + 12\ a1^3\ a2^2\ cY1\ cYn1\ rY^3 + \\
&8\ a0^2\ a1^3\ cY1^3\ cYn1\ rY^3 + 4\ a0\ a1^4\ cY1^3\ cYn1\ rY^3 + \\
&24\ a0^2\ a1^2\ a2\ cY1^3\ cYn1\ rY^3 + 28\ a0\ a1^3\ a2\ cY1^3\ cYn1\ rY^3 + \\
&4\ a1^4\ a2\ cY1^3\ cYn1\ rY^3 + 8\ a0^2\ a1\ a2^2\ cY1^3\ cYn1\ rY^3 + \\
&24\ a0\ a1^2\ a2^2\ cY1^3\ cYn1\ rY^3 + 8\ a1^3\ a2^2\ cY1^3\ cYn1\ rY^3 + \\
&2\ a0^2\ a1^3\ cY1\ cYn1^3\ rY^3 + 4\ a0\ a1^4\ cY1\ cYn1^3\ rY^3 - \\
&2\ a0^2\ a1^2\ a2\ cY1\ cYn1^3\ rY^3 + 6\ a0\ a1^3\ a2\ cY1\ cYn1^3\ rY^3 + \\
&4\ a1^4\ a2\ cY1\ cYn1^3\ rY^3 - 2\ a0^2\ a1\ a2^2\ cY1\ cYn1^3\ rY^3 - \\
&2\ a0\ a1^2\ a2^2\ cY1\ cYn1^3\ rY^3 + 2\ a1^3\ a2^2\ cY1\ cYn1^3\ rY^3 + \\
&2\ a0^2\ a1^3\ cY1\ cYn1\ cYn2^2\ rY^3 + 4\ a0\ a1^4\ cY1\ cYn1\ cYn2^2\ rY^3 - \\
&2\ a0^2\ a1^2\ a2\ cY1\ cYn1\ cYn2^2\ rY^3 + 6\ a0\ a1^3\ a2\ cY1\ cYn1\ cYn2^2\ rY^3 + \\
&4\ a1^4\ a2\ cY1\ cYn1\ cYn2^2\ rY^3 - 2\ a0^2\ a1\ a2^2\ cY1\ cYn1\ cYn2^2\ rY^3 - \\
&2\ a0\ a1^2\ a2^2\ cY1\ cYn1\ cYn2^2\ rY^3 + 2\ a1^3\ a2^2\ cY1\ cYn1\ cYn2^2\ rY^3 + \\
&6\ a0^2\ a1^2\ cYn1\ rY^4 + 12\ a0\ a1^3\ cYn1\ rY^4 + 6\ a1^4\ cYn1\ rY^4 + \\
&12\ a0\ a1^2\ a2\ cYn1\ rY^4 + 12\ a1^3\ a2\ cYn1\ rY^4 + 6\ a1^2\ a2^2\ cYn1\ rY^4 + \\
&12\ a0^2\ a1^2\ cY1^2\ cYn1\ rY^4 + 12\ a0\ a1^3\ cY1^2\ cYn1\ rY^4 + 2\ a1^4\ cY1^2\ cYn1\ rY^4 + \\
&16\ a0^2\ a1\ a2\ cY1^2\ cYn1\ rY^4 + 36\ a0\ a1^2\ a2\ cY1^2\ cYn1\ rY^4 + \\
&12\ a1^3\ a2\ cY1^2\ cYn1\ rY^4 + 2\ a0^2\ a2^2\ cY1^2\ cYn1\ rY^4 + \\
&16\ a0\ a1\ a2^2\ cY1^2\ cYn1\ rY^4 + 12\ a1^2\ a2^2\ cY1^2\ cYn1\ rY^4 + \\
&a0^2\ a1^2\ cYn1^3\ rY^4 + 2\ a0\ a1^3\ cYn1^3\ rY^4 + 2\ a1^4\ cYn1^3\ rY^4 - \\
&2\ a0^2\ a1\ a2\ cYn1^3\ rY^4 - 2\ a0\ a1^2\ a2\ cYn1^3\ rY^4 + 2\ a1^3\ a2\ cYn1^3\ rY^4 + \\
&a0^2\ a2^2\ cYn1^3\ rY^4 - 2\ a0\ a1\ a2^2\ cYn1^3\ rY^4 + a1^2\ a2^2\ cYn1^3\ rY^4 + \\
&a0^2\ a1^2\ cYn1\ cYn2^2\ rY^4 + 2\ a0\ a1^3\ cYn1\ cYn2^2\ rY^4 + 2\ a1^4\ cYn1\ cYn2^2\ rY^4 - \\
&2\ a0^2\ a1\ a2\ cYn1\ cYn2^2\ rY^4 - 2\ a0\ a1^2\ a2\ cYn1\ cYn2^2\ rY^4 + \\
&2\ a1^3\ a2\ cYn1\ cYn2^2\ rY^4 + a0^2\ a2^2\ cYn1\ cYn2^2\ rY^4 - 2\ a0\ a1\ a2^2\ cYn1\ cYn2^2\ rY^4 + \\
&a1^2\ a2^2\ cYn1\ cYn2^2\ rY^4 + 8\ a0^2\ a1\ cY1\ cYn1\ rY^5 + 12\ a0\ a1^2\ cY1\ cYn1\ rY^5 + \\
&4\ a1^3\ cY1\ cYn1\ rY^5 + 4\ a0^2\ a2\ cY1\ cYn1\ rY^5 + 20\ a0\ a1\ a2\ cY1\ cYn1\ rY^5 + \\
&12\ a1^2\ a2\ cY1\ cYn1\ rY^5 + 4\ a0\ a2^2\ cY1\ cYn1\ rY^5 + 8\ a1\ a2^2\ cY1\ cYn1\ rY^5 + \\
&2\ a0^2\ cYn1\ rY^6 + 4\ a0\ a1\ cYn1\ rY^6 + 2\ a1^2\ cYn1\ rY^6 + 4\ a0\ a2\ cYn1\ rY^6 + \\
&4\ a1\ a2\ cYn1\ rY^6 + 2\ a2^2\ cYn1\ rY^6 + 2\ a0\ a1^2\ a2\ cY1^2\ cYn1^2\ rY \\
&\quad \surd\big((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 - \\
&\qquad 4\ a0\ a1\ a2\ cYn1^2\ \big(a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + a0\ a2\ cY1\ rY + \\
&\qquad\quad a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2\big)\big) - 2\ a0\ a1^2\ a2\ cY1^2\ cYn2^2 \\
&\quad rY\ \surd\big((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 - \\
&\qquad 4\ a0\ a1\ a2\ cYn1^2\ \big(a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + a0\ a2\ cY1\ rY + \\
&\qquad\quad a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2\big)\big) + 2\ a0\ a1^2\ cY1\ cYn1^2 \\
&\quad rY^2\ \surd\big((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 - \\
&\qquad 4\ a0\ a1\ a2\ cYn1^2\ \big(a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + a0\ a2\ cY1\ rY + \\
&\qquad\quad a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2\big)\big) + a0\ a1\ a2\ cY1\ cYn1^2 \\
&\quad rY^2\ \surd\big((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 - \\
&\qquad 4\ a0\ a1\ a2\ cYn1^2\ \big(a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + a0\ a2\ cY1\ rY + \\
&\qquad\quad a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2\big)\big) + 2\ a1^2\ a2\ cY1\ cYn1^2 \\
&\quad rY^2\ \surd\big((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 - \\
&\qquad 4\ a0\ a1\ a2\ cYn1^2\ \big(a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + a0\ a2\ cY1\ rY + \\
&\qquad\quad a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2\big)\big) - 2\ a0\ a1^2\ cY1\ cYn2^2 \\
&\quad rY^2\ \surd\big((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 - \\
&\qquad 4\ a0\ a1\ a2\ cYn1^2\ \big(a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + a0\ a2\ cY1\ rY + \\
&\qquad\quad a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2\big)\big) - a0\ a1\ a2\ cY1\ cYn2^2 \\
&\quad rY^2\ \surd\big((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 - \\
&\qquad 4\ a0\ a1\ a2\ cYn1^2\ \big(a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + a0\ a2\ cY1\ rY + \\
&\qquad\quad a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2\big)\big) - 2\ a1^2\ a2\ cY1\ cYn2^2
\end{aligned}
$$

$$rY^2 \sqrt{\left((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 - 4\ a0\ a1\ a2\ cYn1^2 \left(a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2\right)\right)} +$$

$$a0\ a1\ cYn1^2\ rY^3 \sqrt{\left((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 - 4\ a0\ a1\ a2\ cYn1^2 \left(a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2\right)\right)} +$$

$$2\ a1^2\ cYn1^2\ rY^3 \sqrt{\left((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 - 4\ a0\ a1\ a2\ cYn1^2 \left(a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2\right)\right)} -$$

$$a0\ a2\ cYn1^2\ rY^3 \sqrt{\left((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 - 4\ a0\ a1\ a2\ cYn1^2 \left(a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2\right)\right)} +$$

$$a1\ a2\ cYn1^2\ rY^3 \sqrt{\left((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 - 4\ a0\ a1\ a2\ cYn1^2 \left(a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2\right)\right)} -$$

$$a0\ a1\ cYn2^2\ rY^3 \sqrt{\left((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 - 4\ a0\ a1\ a2\ cYn1^2 \left(a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2\right)\right)} -$$

$$2\ a1^2\ cYn2^2\ rY^3 \sqrt{\left((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 - 4\ a0\ a1\ a2\ cYn1^2 \left(a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2\right)\right)} +$$

$$a0\ a2\ cYn2^2\ rY^3 \sqrt{\left((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 - 4\ a0\ a1\ a2\ cYn1^2 \left(a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2\right)\right)} -$$

$$a1\ a2\ cYn2^2\ rY^3 \sqrt{\left((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 - 4\ a0\ a1\ a2\ cYn1^2 \left(a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2\right)\right)}\Bigg) \Big/ \Big(2\ cYn1\ rY^2 \left(a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2\right)^2\Big),$$

$$\begin{aligned}
Sn2 \to \Big(&6\ a0^2\ a1^2\ a2^4\ cY1^4\ cYn1 + 2\ a0^2\ a1^2\ a2^4\ cY1^6\ cYn1 + 2\ a0^2\ a1^2\ a2^4\ cY1^4\ cYn1^3 + \\
&2\ a0^2\ a1^2\ a2^4\ cY1^4\ cYn1\ cYn2^2 + 12\ a0^2\ a1^2\ a2^3\ cY1^3\ cYn1\ rY + \\
&12\ a0^2\ a1\ a2^4\ cY1^3\ cYn1\ rY + 12\ a0\ a1^2\ a2^4\ cY1^3\ cYn1\ rY + \\
&8\ a0^2\ a1^2\ a2^3\ cY1^5\ cYn1\ rY + 4\ a0^2\ a1\ a2^4\ cY1^5\ cYn1\ rY + \\
&4\ a0\ a1^2\ a2^4\ cY1^5\ cYn1\ rY + 2\ a0^2\ a1^2\ a2^3\ cY1^3\ cYn1^3\ rY + \\
&4\ a0^2\ a1\ a2^4\ cY1^3\ cYn1^3\ rY + 4\ a0\ a1^2\ a2^4\ cY1^3\ cYn1^3\ rY + \\
&2\ a0^2\ a1^2\ a2^3\ cY1^3\ cYn1\ cYn2^2\ rY + 4\ a0^2\ a1\ a2^4\ cY1^3\ cYn1\ cYn2^2\ rY + \\
&4\ a0\ a1^2\ a2^4\ cY1^3\ cYn1\ cYn2^2\ rY + 6\ a0^2\ a1^2\ a2^2\ cY1^2\ cYn1\ rY^2 + \\
&24\ a0^2\ a1\ a2^3\ cY1^2\ cYn1\ rY^2 + 24\ a0\ a1^2\ a2^3\ cY1^2\ cYn1\ rY^2 + \\
&6\ a0^2\ a2^4\ cY1^2\ cYn1\ rY^2 + 24\ a0\ a1\ a2^4\ cY1^2\ cYn1\ rY^2 + \\
&6\ a1^2\ a2^4\ cY1^2\ cYn1\ rY^2 + 12\ a0^2\ a1^2\ a2^2\ cY1^4\ cYn1\ rY^2 + \\
&16\ a0^2\ a1\ a2^3\ cY1^4\ cYn1\ rY^2 + 16\ a0\ a1^2\ a2^3\ cY1^4\ cYn1\ rY^2 + \\
&2\ a0^2\ a2^4\ cY1^4\ cYn1\ rY^2 + 8\ a0\ a1\ a2^4\ cY1^4\ cYn1\ rY^2 + \\
&2\ a1^2\ a2^4\ cY1^4\ cYn1\ rY^2 - 3\ a0^2\ a1^2\ a2^2\ cY1^2\ cYn1^3\ rY^2 + \\
&4\ a0^2\ a1\ a2^3\ cY1^2\ cYn1^3\ rY^2 + 4\ a0\ a1^2\ a2^3\ cY1^2\ cYn1^3\ rY^2 + \\
&2\ a0^2\ a2^4\ cY1^2\ cYn1^3\ rY^2 + 8\ a0\ a1\ a2^4\ cY1^2\ cYn1^3\ rY^2 + \\
&2\ a1^2\ a2^4\ cY1^2\ cYn1^3\ rY^2 - 3\ a0^2\ a1^2\ a2^2\ cY1^2\ cYn1\ cYn2^2\ rY^2 + \\
&4\ a0^2\ a1\ a2^3\ cY1^2\ cYn1\ cYn2^2\ rY^2 + 4\ a0\ a1^2\ a2^3\ cY1^2\ cYn1\ cYn2^2\ rY^2 + \\
&2\ a0^2\ a2^4\ cY1^2\ cYn1\ cYn2^2\ rY^2 + 8\ a0\ a1\ a2^4\ cY1^2\ cYn1\ cYn2^2\ rY^2 + \\
&2\ a1^2\ a2^4\ cY1^2\ cYn1\ cYn2^2\ rY^2 + 12\ a0^2\ a1\ a2^2\ cY1\ cYn1\ rY^3 + \\
&12\ a0\ a1^2\ a2^2\ cY1\ cYn1\ rY^3 + 12\ a0^2\ a2^3\ cY1\ cYn1\ rY^3 + \\
&36\ a0\ a1\ a2^3\ cY1\ cYn1\ rY^3 + 12\ a1^2\ a2^3\ cY1\ cYn1\ rY^3 + \\
&12\ a0\ a2^4\ cY1\ cYn1\ rY^3 + 12\ a1\ a2^4\ cY1\ cYn1\ rY^3 + \\
&8\ a0^2\ a1^2\ a2\ cY1^3\ cYn1\ rY^3 + 24\ a0^2\ a1\ a2^2\ cY1^3\ cYn1\ rY^3 + \\
&24\ a0\ a1^2\ a2^2\ cY1^3\ cYn1\ rY^3 + 8\ a0^2\ a2^3\ cY1^3\ cYn1\ rY^3 +
\end{aligned}$$

$$
\begin{aligned}
&28\, a0\, a1\, a2^{3}\, cY1^{3}\, cYn1\, rY^{3} + 8\, a1^{2}\, a2^{3}\, cY1^{3}\, cYn1\, rY^{3} +\\
&4\, a0\, a2^{4}\, cY1^{3}\, cYn1\, rY^{3} + 4\, a1\, a2^{4}\, cY1^{3}\, cYn1\, rY^{3} -\\
&2\, a0^{2}\, a1^{2}\, a2\, cY1\, cYn1^{3}\, rY^{3} - 2\, a0^{2}\, a1\, a2^{2}\, cY1\, cYn1^{3}\, rY^{3} -\\
&2\, a0\, a1^{2}\, a2^{2}\, cY1\, cYn1^{3}\, rY^{3} + 2\, a0^{2}\, a2^{3}\, cY1\, cYn1^{3}\, rY^{3} +\\
&6\, a0\, a1\, a2^{3}\, cY1\, cYn1^{3}\, rY^{3} + 2\, a1^{2}\, a2^{3}\, cY1\, cYn1^{3}\, rY^{3} +\\
&4\, a0\, a2^{4}\, cY1\, cYn1^{3}\, rY^{3} + 4\, a1\, a2^{4}\, cY1\, cYn1^{3}\, rY^{3} -\\
&2\, a0^{2}\, a1^{2}\, a2\, cY1\, cYn1\, cYn2^{2}\, rY^{3} - 2\, a0^{2}\, a1\, a2^{2}\, cY1\, cYn1\, cYn2^{2}\, rY^{3} -\\
&2\, a0\, a1^{2}\, a2^{2}\, cY1\, cYn1\, cYn2^{2}\, rY^{3} + 2\, a0^{2}\, a2^{3}\, cY1\, cYn1\, cYn2^{2}\, rY^{3} +\\
&6\, a0\, a1\, a2^{3}\, cY1\, cYn1\, cYn2^{2}\, rY^{3} + 2\, a1^{2}\, a2^{3}\, cY1\, cYn1\, cYn2^{2}\, rY^{3} +\\
&4\, a0\, a2^{4}\, cY1\, cYn1\, cYn2^{2}\, rY^{3} + 4\, a1\, a2^{4}\, cY1\, cYn1\, cYn2^{2}\, rY^{3} +\\
&6\, a0^{2}\, a2^{2}\, cYn1\, rY^{4} + 12\, a0\, a1\, a2^{2}\, cYn1\, rY^{4} + 6\, a1^{2}\, a2^{2}\, cYn1\, rY^{4} +\\
&12\, a0\, a2^{3}\, cYn1\, rY^{4} + 12\, a1\, a2^{3}\, cYn1\, rY^{4} + 6\, a2^{4}\, cYn1\, rY^{4} +\\
&2\, a0^{2}\, a1^{2}\, cY1^{2}\, cYn1\, rY^{4} + 16\, a0^{2}\, a1\, a2\, cY1^{2}\, cYn1\, rY^{4} +\\
&16\, a0\, a1^{2}\, a2\, cY1^{2}\, cYn1\, rY^{4} + 12\, a0^{2}\, a2^{2}\, cY1^{2}\, cYn1\, rY^{4} +\\
&36\, a0\, a1\, a2^{2}\, cY1^{2}\, cYn1\, rY^{4} + 12\, a1^{2}\, a2^{2}\, cY1^{2}\, cYn1\, rY^{4} +\\
&12\, a0\, a2^{3}\, cY1^{2}\, cYn1\, rY^{4} + 12\, a1\, a2^{3}\, cY1^{2}\, cYn1\, rY^{4} +\\
&2\, a2^{4}\, cY1^{2}\, cYn1\, rY^{4} + a0^{2}\, a1^{2}\, cYn1^{3}\, rY^{4} - 2\, a0^{2}\, a1\, a2\, cYn1^{3}\, rY^{4} -\\
&2\, a0\, a1^{2}\, a2\, cYn1^{3}\, rY^{4} + a0^{2}\, a2^{2}\, cYn1^{3}\, rY^{4} - 2\, a0\, a1\, a2^{2}\, cYn1^{3}\, rY^{4} +\\
&a1^{2}\, a2^{2}\, cYn1^{3}\, rY^{4} + 2\, a0\, a2^{3}\, cYn1^{3}\, rY^{4} + 2\, a1\, a2^{3}\, cYn1^{3}\, rY^{4} +\\
&2\, a2^{4}\, cYn1^{3}\, rY^{4} + a0^{2}\, a1^{2}\, cYn1\, cYn2^{2}\, rY^{4} - 2\, a0^{2}\, a1\, a2\, cYn1\, cYn2^{2}\, rY^{4} -\\
&2\, a0\, a1^{2}\, a2\, cYn1\, cYn2^{2}\, rY^{4} + a0^{2}\, a2^{2}\, cYn1\, cYn2^{2}\, rY^{4} -\\
&2\, a0\, a1\, a2^{2}\, cYn1\, cYn2^{2}\, rY^{4} + a1^{2}\, a2^{2}\, cYn1\, cYn2^{2}\, rY^{4} +\\
&2\, a0\, a2^{3}\, cYn1\, cYn2^{2}\, rY^{4} + 2\, a1\, a2^{3}\, cYn1\, cYn2^{2}\, rY^{4} +\\
&2\, a2^{4}\, cYn1\, cYn2^{2}\, rY^{4} + 4\, a0^{2}\, a1\, cY1\, cYn1\, rY^{5} + 4\, a0\, a1^{2}\, cY1\, cYn1\, rY^{5} +\\
&8\, a0^{2}\, a2\, cY1\, cYn1\, rY^{5} + 20\, a0\, a1\, a2\, cY1\, cYn1\, rY^{5} +\\
&8\, a1^{2}\, a2\, cY1\, cYn1\, rY^{5} + 12\, a0\, a2^{2}\, cY1\, cYn1\, rY^{5} +\\
&12\, a1\, a2^{2}\, cY1\, cYn1\, rY^{5} + 4\, a2^{3}\, cY1\, cYn1\, rY^{5} + 2\, a0^{2}\, cYn1\, rY^{6} +\\
&4\, a0\, a1\, cYn1\, rY^{6} + 2\, a1^{2}\, cYn1\, rY^{6} + 4\, a0\, a2\, cYn1\, rY^{6} +\\
&4\, a1\, a2\, cYn1\, rY^{6} + 2\, a2^{2}\, cYn1\, rY^{6} + 2\, a0\, a1\, a2^{2}\, cY1^{2}\, cYn1^{2}\, rY\\
&\quad \sqrt{}\Big((a0\, a1\, a2\, cY1\, cYn1 + a0\, a1\, cYn1\, rY + a0\, a2\, cYn1\, rY + a1\, a2\, cYn1\, rY)^{2} -\\
&\qquad 4\, a0\, a1\, a2\, cYn1^{2}\, \big(a0\, a1\, a2\, cY1^{2} + a0\, a1\, cY1\, rY + a0\, a2\, cY1\, rY +\\
&\qquad\quad a1\, a2\, cY1\, rY + a0\, rY^{2} + a1\, rY^{2} + a2\, rY^{2}\big)\Big) - 2\, a0\, a1\, a2^{2}\, cY1^{2}\, cYn2^{2}\\
&\quad rY \sqrt{}\Big((a0\, a1\, a2\, cY1\, cYn1 + a0\, a1\, cYn1\, rY + a0\, a2\, cYn1\, rY + a1\, a2\, cYn1\, rY)^{2} -\\
&\qquad 4\, a0\, a1\, a2\, cYn1^{2}\, \big(a0\, a1\, a2\, cY1^{2} + a0\, a1\, cY1\, rY + a0\, a2\, cY1\, rY +\\
&\qquad\quad a1\, a2\, cY1\, rY + a0\, rY^{2} + a1\, rY^{2} + a2\, rY^{2}\big)\Big) + a0\, a1\, a2\, cY1\, cYn1^{2}\\
&\quad rY^{2} \sqrt{}\Big((a0\, a1\, a2\, cY1\, cYn1 + a0\, a1\, cYn1\, rY + a0\, a2\, cYn1\, rY + a1\, a2\, cYn1\, rY)^{2} -\\
&\qquad 4\, a0\, a1\, a2\, cYn1^{2}\, \big(a0\, a1\, a2\, cY1^{2} + a0\, a1\, cY1\, rY + a0\, a2\, cY1\, rY +\\
&\qquad\quad a1\, a2\, cY1\, rY + a0\, rY^{2} + a1\, rY^{2} + a2\, rY^{2}\big)\Big) + 2\, a0\, a2^{2}\, cY1\, cYn1^{2}\\
&\quad rY^{2} \sqrt{}\Big((a0\, a1\, a2\, cY1\, cYn1 + a0\, a1\, cYn1\, rY + a0\, a2\, cYn1\, rY + a1\, a2\, cYn1\, rY)^{2} -\\
&\qquad 4\, a0\, a1\, a2\, cYn1^{2}\, \big(a0\, a1\, a2\, cY1^{2} + a0\, a1\, cY1\, rY + a0\, a2\, cY1\, rY +\\
&\qquad\quad a1\, a2\, cY1\, rY + a0\, rY^{2} + a1\, rY^{2} + a2\, rY^{2}\big)\Big) + 2\, a1\, a2^{2}\, cY1\, cYn1^{2}\\
&\quad rY^{2} \sqrt{}\Big((a0\, a1\, a2\, cY1\, cYn1 + a0\, a1\, cYn1\, rY + a0\, a2\, cYn1\, rY + a1\, a2\, cYn1\, rY)^{2} -\\
&\qquad 4\, a0\, a1\, a2\, cYn1^{2}\, \big(a0\, a1\, a2\, cY1^{2} + a0\, a1\, cY1\, rY + a0\, a2\, cY1\, rY +\\
&\qquad\quad a1\, a2\, cY1\, rY + a0\, rY^{2} + a1\, rY^{2} + a2\, rY^{2}\big)\Big) - a0\, a1\, a2\, cY1\, cYn2^{2}\\
&\quad rY^{2} \sqrt{}\Big((a0\, a1\, a2\, cY1\, cYn1 + a0\, a1\, cYn1\, rY + a0\, a2\, cYn1\, rY + a1\, a2\, cYn1\, rY)^{2} -\\
&\qquad 4\, a0\, a1\, a2\, cYn1^{2}\, \big(a0\, a1\, a2\, cY1^{2} + a0\, a1\, cY1\, rY + a0\, a2\, cY1\, rY +\\
&\qquad\quad a1\, a2\, cY1\, rY + a0\, rY^{2} + a1\, rY^{2} + a2\, rY^{2}\big)\Big) - 2\, a0\, a2^{2}\, cY1\, cYn2^{2}\\
&\quad rY^{2} \sqrt{}\Big((a0\, a1\, a2\, cY1\, cYn1 + a0\, a1\, cYn1\, rY + a0\, a2\, cYn1\, rY + a1\, a2\, cYn1\, rY)^{2} -\\
&\qquad 4\, a0\, a1\, a2\, cYn1^{2}\, \big(a0\, a1\, a2\, cY1^{2} + a0\, a1\, cY1\, rY + a0\, a2\, cY1\, rY +\\
&\qquad\quad a1\, a2\, cY1\, rY + a0\, rY^{2} + a1\, rY^{2} + a2\, rY^{2}\big)\Big) - 2\, a1\, a2^{2}\, cY1\, cYn2^{2}\\
&\quad rY^{2} \sqrt{}\Big((a0\, a1\, a2\, cY1\, cYn1 + a0\, a1\, cYn1\, rY + a0\, a2\, cYn1\, rY + a1\, a2\, cYn1\, rY)^{2} -\\
&\qquad 4\, a0\, a1\, a2\, cYn1^{2}\, \big(a0\, a1\, a2\, cY1^{2} + a0\, a1\, cY1\, rY +\\
&\qquad\quad a0\, a2\, cY1\, rY + a1\, a2\, cY1\, rY + a0\, rY^{2} + a1\, rY^{2} + a2\, rY^{2}\big)\Big) -\\
&a0\, a1\, cYn1^{2}\, rY^{3} \sqrt{}\Big((a0\, a1\, a2\, cY1\, cYn1 + a0\, a1\, cYn1\, rY + a0\, a2\, cYn1\, rY + a1\, a2\, cYn1\, rY)^{2} -
\end{aligned}
$$

$$
\begin{array}{l}
\quad 4\ a0\ a1\ a2\ cYn1^2\ (a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + \\
\qquad a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2)) + \\
a0\ a2\ cYn1^2\ rY^3\ \surd((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 - \\
\quad 4\ a0\ a1\ a2\ cYn1^2\ (a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + \\
\qquad a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2)) + \\
a1\ a2\ cYn1^2\ rY^3\ \surd((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 - \\
\quad 4\ a0\ a1\ a2\ cYn1^2\ (a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + \\
\qquad a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2)) + \\
2\ a2^2\ cYn1^2\ rY^3\ \surd((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 - \\
\quad 4\ a0\ a1\ a2\ cYn1^2\ (a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + \\
\qquad a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2)) + \\
a0\ a1\ cYn2^2\ rY^3\ \surd((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 - \\
\quad 4\ a0\ a1\ a2\ cYn1^2\ (a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + \\
\qquad a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2)) - \\
a0\ a2\ cYn2^2\ rY^3\ \surd((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 - \\
\quad 4\ a0\ a1\ a2\ cYn1^2\ (a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + \\
\qquad a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2)) - \\
a1\ a2\ cYn2^2\ rY^3\ \surd((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 - \\
\quad 4\ a0\ a1\ a2\ cYn1^2\ (a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + \\
\qquad a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2)) - \\
2\ a2^2\ cYn2^2\ rY^3\ \surd((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 - \\
\quad 4\ a0\ a1\ a2\ cYn1^2\ (a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + a0\ a2\ cY1\ rY + \\
\qquad a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2)))\ /\ (2\ cYn1\ rY^2 \\
(a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2)^2),
\end{array}
$$

$$
\begin{array}{l}
pY1 \to 3,\ n1 \to -((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY - \\
\quad \surd((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 - \\
\qquad 4\ a0\ a1\ a2\ cYn1^2\ (a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + \\
\qquad\quad a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2)))\ / \\
\quad (2\ (a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2))), \\
n2 \to -((cYn2\ (a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY + \\
\quad \surd((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 - \\
\qquad 4\ a0\ a1\ a2\ cYn1^2\ (a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + \\
\qquad\quad a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2))))\ / \\
\quad (2\ cYn1\ (a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + \\
\qquad a1\ rY^2 + a2\ rY^2)))\},\ \{p \to \dfrac{3 + cY1^2 + cYn1^2 + cYn2^2}{rY^2},
\end{array}
$$

$$
d \to -\frac{rY^2\ (a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2)}{cYn1\ cYn2},
$$

$$
N01 \to \frac{(a0 - a1)\ (a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2)}{a2^3\ cYn1\ cYn2\ rY},
$$

$$
N02 \to -\frac{(a0 - a2)\ (a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2)}{a1^3\ cYn1\ cYn2\ rY}
$$

,

$$
N12 \to -\frac{(a1 - a2)\ (a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2)}{a0^3\ cYn1\ cYn2\ rY}
$$

$$
\begin{array}{l}
,\ Sn0 \to (6\ a0^4\ a1^2\ a2^2\ cY1^4\ cYn1 + 2\ a0^4\ a1^2\ a2^2\ cY1^6\ cYn1 + \\
\quad 2\ a0^4\ a1^2\ a2^2\ cY1^4\ cYn1^3 + 2\ a0^4\ a1^2\ a2^2\ cY1^4\ cYn1\ cYn2^2 + 12\ a0^4\ a1^2\ a2\ cY1^3\ cYn1\ rY + \\
\quad 12\ a0^4\ a1\ a2^2\ cY1^3\ cYn1\ rY + 12\ a0^3\ a1^2\ a2^2\ cY1^3\ cYn1\ rY + 4\ a0^4\ a1^2\ a2\ cY1^5\ cYn1\ rY +
\end{array}
$$

$4\,a0^4\,a1\,a2^2\,cY1^5\,cYn1\,rY + 8\,a0^3\,a1^2\,a2^2\,cY1^5\,cYn1\,rY + 4\,a0^4\,a1^2\,a2\,cY1^3\,cYn1^3\,rY +$
$4\,a0^4\,a1\,a2^2\,cY1^3\,cYn1^3\,rY + 2\,a0^3\,a1^2\,a2^2\,cY1^3\,cYn1^3\,rY + 4\,a0^4\,a1^2\,a2\,cY1^3\,cYn1\,cYn2^2\,rY +$
$4\,a0^4\,a1\,a2^2\,cY1^3\,cYn1\,cYn2^2\,rY + 2\,a0^3\,a1^2\,a2^2\,cY1^3\,cYn1\,cYn2^2\,rY + 6\,a0^4\,a1^2\,cY1^2\,cYn1\,rY^2 +$
$24\,a0^4\,a1\,a2\,cY1^2\,cYn1\,rY^2 + 24\,a0^3\,a1^2\,a2\,cY1^2\,cYn1\,rY^2 + 6\,a0^4\,a2^2\,cY1^2\,cYn1\,rY^2 +$
$24\,a0^3\,a1\,a2^2\,cY1^2\,cYn1\,rY^2 + 6\,a0^2\,a1^2\,a2^2\,cY1^2\,cYn1\,rY^2 + 2\,a0^4\,a1^2\,cY1^4\,cYn1\,rY^2 +$
$8\,a0^4\,a1\,a2\,cY1^4\,cYn1\,rY^2 + 16\,a0^3\,a1^2\,a2\,cY1^4\,cYn1\,rY^2 + 2\,a0^4\,a2^2\,cY1^4\,cYn1\,rY^2 +$
$16\,a0^3\,a1\,a2^2\,cY1^4\,cYn1\,rY^2 + 12\,a0^2\,a1^2\,a2^2\,cY1^4\,cYn1\,rY^2 + 2\,a0^4\,a1^2\,cY1^2\,cYn1^3\,rY^2 +$
$8\,a0^4\,a1\,a2\,cY1^2\,cYn1^3\,rY^2 + 4\,a0^3\,a1^2\,a2\,cY1^2\,cYn1^3\,rY^2 + 2\,a0^4\,a2^2\,cY1^2\,cYn1^3\,rY^2 +$
$4\,a0^3\,a1\,a2^2\,cY1^2\,cYn1^3\,rY^2 - 3\,a0^2\,a1^2\,a2^2\,cY1^2\,cYn1^3\,rY^2 + 2\,a0^4\,a1^2\,cY1^2\,cYn1\,cYn2^2\,rY^2 +$
$8\,a0^4\,a1\,a2\,cY1^2\,cYn1\,cYn2^2\,rY^2 + 4\,a0^3\,a1^2\,a2\,cY1^2\,cYn1\,cYn2^2\,rY^2 +$
$2\,a0^4\,a2^2\,cY1^2\,cYn1\,cYn2^2\,rY^2 + 4\,a0^3\,a1\,a2^2\,cY1^2\,cYn1\,cYn2^2\,rY^2 -$
$3\,a0^2\,a1^2\,a2^2\,cY1^2\,cYn1\,cYn2^2\,rY^2 + 12\,a0^4\,a1\,cY1\,cYn1\,rY^3 + 12\,a0^3\,a1^2\,cY1\,cYn1\,rY^3 +$
$12\,a0^4\,a2\,cY1\,cYn1\,rY^3 + 36\,a0^3\,a1\,a2\,cY1\,cYn1\,rY^3 + 12\,a0^2\,a1^2\,a2\,cY1\,cYn1\,rY^3 +$
$12\,a0^3\,a2^2\,cY1\,cYn1\,rY^3 + 12\,a0^2\,a1\,a2^2\,cY1\,cYn1\,rY^3 + 4\,a0^4\,a1\,cY1^3\,cYn1\,rY^3 +$
$8\,a0^3\,a1^2\,cY1^3\,cYn1\,rY^3 + 4\,a0^4\,a2\,cY1^3\,cYn1\,rY^3 + 28\,a0^3\,a1\,a2\,cY1^3\,cYn1\,rY^3 +$
$24\,a0^2\,a1^2\,a2\,cY1^3\,cYn1\,rY^3 + 8\,a0^3\,a2^2\,cY1^3\,cYn1\,rY^3 + 24\,a0^2\,a1\,a2^2\,cY1^3\,cYn1\,rY^3 +$
$8\,a0\,a1^2\,a2^2\,cY1^3\,cYn1\,rY^3 + 4\,a0^4\,a1\,cY1\,cYn1^3\,rY^3 + 2\,a0^3\,a1^2\,cY1\,cYn1^3\,rY^3 +$
$4\,a0^4\,a2\,cY1\,cYn1^3\,rY^3 + 6\,a0^3\,a1\,a2\,cY1\,cYn1^3\,rY^3 - 2\,a0^2\,a1^2\,a2\,cY1\,cYn1^3\,rY^3 +$
$2\,a0^3\,a2^2\,cY1\,cYn1^3\,rY^3 - 2\,a0^2\,a1\,a2^2\,cY1\,cYn1^3\,rY^3 - 2\,a0\,a1^2\,a2^2\,cY1\,cYn1^3\,rY^3 +$
$4\,a0^4\,a1\,cY1\,cYn1\,cYn2^2\,rY^3 + 2\,a0^3\,a1^2\,cY1\,cYn1\,cYn2^2\,rY^3 + 4\,a0^4\,a2\,cY1\,cYn1\,cYn2^2\,rY^3 +$
$6\,a0^3\,a1\,a2\,cY1\,cYn1\,cYn2^2\,rY^3 - 2\,a0^2\,a1^2\,a2\,cY1\,cYn1\,cYn2^2\,rY^3 +$
$2\,a0^3\,a2^2\,cY1\,cYn1\,cYn2^2\,rY^3 - 2\,a0^2\,a1\,a2^2\,cY1\,cYn1\,cYn2^2\,rY^3 -$
$2\,a0\,a1^2\,a2^2\,cY1\,cYn1\,cYn2^2\,rY^3 + 6\,a0^4\,cYn1\,rY^4 + 12\,a0^3\,a1\,cYn1\,rY^4 + 6\,a0^2\,a1^2\,cYn1\,rY^4 +$
$12\,a0^3\,a2\,cYn1\,rY^4 + 12\,a0^2\,a1\,a2\,cYn1\,rY^4 + 6\,a0^2\,a2^2\,cYn1\,rY^4 + 2\,a0^4\,cY1^2\,cYn1\,rY^4 +$
$12\,a0^3\,a1\,cY1^2\,cYn1\,rY^4 + 12\,a0^2\,a1^2\,cY1^2\,cYn1\,rY^4 + 12\,a0^3\,a2\,cY1^2\,cYn1\,rY^4 +$
$36\,a0^2\,a1\,a2\,cY1^2\,cYn1\,rY^4 + 16\,a0\,a1^2\,a2\,cY1^2\,cYn1\,rY^4 + 12\,a0^2\,a2^2\,cY1^2\,cYn1\,rY^4 +$
$16\,a0\,a1\,a2^2\,cY1^2\,cYn1\,rY^4 + 2\,a1^2\,a2^2\,cY1^2\,cYn1\,rY^4 + 2\,a0^4\,cYn1^3\,rY^4 + 2\,a0^3\,a1\,cYn1^3\,rY^4 +$
$a0^2\,a1^2\,cYn1^3\,rY^4 + 2\,a0^3\,a2\,cYn1^3\,rY^4 - 2\,a0^2\,a1\,a2\,cYn1^3\,rY^4 - 2\,a0\,a1^2\,a2\,cYn1^3\,rY^4 +$
$a0^2\,a2^2\,cYn1^3\,rY^4 - 2\,a0\,a1\,a2^2\,cYn1^3\,rY^4 + a1^2\,a2^2\,cYn1^3\,rY^4 + 2\,a0^4\,cYn1\,cYn2^2\,rY^4 +$
$2\,a0^3\,a1\,cYn1\,cYn2^2\,rY^4 + a0^2\,a1^2\,cYn1\,cYn2^2\,rY^4 + 2\,a0^3\,a2\,cYn1\,cYn2^2\,rY^4 -$
$2\,a0^2\,a1\,a2\,cYn1\,cYn2^2\,rY^4 - 2\,a0\,a1^2\,a2\,cYn1\,cYn2^2\,rY^4 + a0^2\,a2^2\,cYn1\,cYn2^2\,rY^4 -$
$2\,a0\,a1\,a2^2\,cYn1\,cYn2^2\,rY^4 + a1^2\,a2^2\,cYn1\,cYn2^2\,rY^4 + 4\,a0^3\,cY1\,cYn1\,rY^5 +$
$12\,a0^2\,a1\,cY1\,cYn1\,rY^5 + 8\,a0\,a1^2\,cY1\,cYn1\,rY^5 + 12\,a0^2\,a2\,cY1\,cYn1\,rY^5 +$
$20\,a0\,a1\,a2\,cY1\,cYn1\,rY^5 + 4\,a1^2\,a2\,cY1\,cYn1\,rY^5 + 8\,a0\,a2^2\,cY1\,cYn1\,rY^5 +$
$4\,a1\,a2^2\,cY1\,cYn1\,rY^5 + 2\,a0^2\,cYn1\,rY^6 + 4\,a0\,a1\,cYn1\,rY^6 + 2\,a1^2\,cYn1\,rY^6 +$
$4\,a0\,a2\,cYn1\,rY^6 + 4\,a1\,a2\,cYn1\,rY^6 + 2\,a2^2\,cYn1\,rY^6 + 2\,a0^2\,a1\,a2\,cY1^2\,cYn1^2$
$rY\,\sqrt{\left((a0\,a1\,a2\,cY1\,cYn1 + a0\,a1\,cYn1\,rY + a0\,a2\,cYn1\,rY + a1\,a2\,cYn1\,rY)^2 -\right.}$
$4\,a0\,a1\,a2\,cYn1^2\,\left(a0\,a1\,a2\,cY1^2 + a0\,a1\,cY1\,rY + a0\,a2\,cY1\,rY +\right.$
$\left.\left.a1\,a2\,cY1\,rY + a0\,rY^2 + a1\,rY^2 + a2\,rY^2\right)\right) - 2\,a0^2\,a1\,a2\,cY1^2\,cYn2^2$
$rY\,\sqrt{\left((a0\,a1\,a2\,cY1\,cYn1 + a0\,a1\,cYn1\,rY + a0\,a2\,cYn1\,rY + a1\,a2\,cYn1\,rY)^2 -\right.}$
$4\,a0\,a1\,a2\,cYn1^2\,\left(a0\,a1\,a2\,cY1^2 + a0\,a1\,cY1\,rY + a0\,a2\,cY1\,rY +\right.$
$\left.\left.a1\,a2\,cY1\,rY + a0\,rY^2 + a1\,rY^2 + a2\,rY^2\right)\right) + 2\,a0^2\,a1\,cY1\,cYn1^2$
$rY^2\,\sqrt{\left((a0\,a1\,a2\,cY1\,cYn1 + a0\,a1\,cYn1\,rY + a0\,a2\,cYn1\,rY + a1\,a2\,cYn1\,rY)^2 -\right.}$
$4\,a0\,a1\,a2\,cYn1^2\,\left(a0\,a1\,a2\,cY1^2 + a0\,a1\,cY1\,rY + a0\,a2\,cY1\,rY +\right.$
$\left.\left.a1\,a2\,cY1\,rY + a0\,rY^2 + a1\,rY^2 + a2\,rY^2\right)\right) + 2\,a0^2\,a2\,cY1\,cYn1^2$
$rY^2\,\sqrt{\left((a0\,a1\,a2\,cY1\,cYn1 + a0\,a1\,cYn1\,rY + a0\,a2\,cYn1\,rY + a1\,a2\,cYn1\,rY)^2 -\right.}$
$4\,a0\,a1\,a2\,cYn1^2\,\left(a0\,a1\,a2\,cY1^2 + a0\,a1\,cY1\,rY + a0\,a2\,cY1\,rY +\right.$
$\left.\left.a1\,a2\,cY1\,rY + a0\,rY^2 + a1\,rY^2 + a2\,rY^2\right)\right) + a0\,a1\,a2\,cY1\,cYn1^2$
$rY^2\,\sqrt{\left((a0\,a1\,a2\,cY1\,cYn1 + a0\,a1\,cYn1\,rY + a0\,a2\,cYn1\,rY + a1\,a2\,cYn1\,rY)^2 -\right.}$
$4\,a0\,a1\,a2\,cYn1^2\,\left(a0\,a1\,a2\,cY1^2 + a0\,a1\,cY1\,rY + a0\,a2\,cY1\,rY +\right.$
$\left.\left.a1\,a2\,cY1\,rY + a0\,rY^2 + a1\,rY^2 + a2\,rY^2\right)\right) - 2\,a0^2\,a1\,cY1\,cYn2^2$
$rY^2\,\sqrt{\left((a0\,a1\,a2\,cY1\,cYn1 + a0\,a1\,cYn1\,rY + a0\,a2\,cYn1\,rY + a1\,a2\,cYn1\,rY)^2 -\right.}$
$4\,a0\,a1\,a2\,cYn1^2\,\left(a0\,a1\,a2\,cY1^2 + a0\,a1\,cY1\,rY + a0\,a2\,cY1\,rY +\right.$
$\left.\left.a1\,a2\,cY1\,rY + a0\,rY^2 + a1\,rY^2 + a2\,rY^2\right)\right) - 2\,a0^2\,a2\,cY1\,cYn2^2$
$rY^2\,\sqrt{\left((a0\,a1\,a2\,cY1\,cYn1 + a0\,a1\,cYn1\,rY + a0\,a2\,cYn1\,rY + a1\,a2\,cYn1\,rY)^2 -\right.}$

$$4\, a0\, a1\, a2\, cYn1^2\, (a0\, a1\, a2\, cY1^2 + a0\, a1\, cY1\, rY + a0\, a2\, cY1\, rY +$$
$$a1\, a2\, cY1\, rY + a0\, rY^2 + a1\, rY^2 + a2\, rY^2)) - a0\, a1\, a2\, cY1\, cYn2^2$$
$$rY^2 \surd((a0\, a1\, a2\, cY1\, cYn1 + a0\, a1\, cYn1\, rY + a0\, a2\, cYn1\, rY + a1\, a2\, cYn1\, rY)^2 -$$
$$4\, a0\, a1\, a2\, cYn1^2\, (a0\, a1\, a2\, cY1^2 + a0\, a1\, cY1\, rY +$$
$$a0\, a2\, cY1\, rY + a1\, a2\, cY1\, rY + a0\, rY^2 + a1\, rY^2 + a2\, rY^2)) +$$
$$2\, a0^2\, cYn1^2\, rY^3 \surd((a0\, a1\, a2\, cY1\, cYn1 + a0\, a1\, cYn1\, rY + a0\, a2\, cYn1\, rY + a1\, a2\, cYn1\, rY)^2 -$$
$$4\, a0\, a1\, a2\, cYn1^2\, (a0\, a1\, a2\, cY1^2 + a0\, a1\, cY1\, rY +$$
$$a0\, a2\, cY1\, rY + a1\, a2\, cY1\, rY + a0\, rY^2 + a1\, rY^2 + a2\, rY^2)) +$$
$$a0\, a1\, cYn1^2\, rY^3 \surd((a0\, a1\, a2\, cY1\, cYn1 + a0\, a1\, cYn1\, rY + a0\, a2\, cYn1\, rY + a1\, a2\, cYn1\, rY)^2 -$$
$$4\, a0\, a1\, a2\, cYn1^2\, (a0\, a1\, a2\, cY1^2 + a0\, a1\, cY1\, rY +$$
$$a0\, a2\, cY1\, rY + a1\, a2\, cY1\, rY + a0\, rY^2 + a1\, rY^2 + a2\, rY^2)) +$$
$$a0\, a2\, cYn1^2\, rY^3 \surd((a0\, a1\, a2\, cY1\, cYn1 + a0\, a1\, cYn1\, rY + a0\, a2\, cYn1\, rY + a1\, a2\, cYn1\, rY)^2 -$$
$$4\, a0\, a1\, a2\, cYn1^2\, (a0\, a1\, a2\, cY1^2 + a0\, a1\, cY1\, rY +$$
$$a0\, a2\, cY1\, rY + a1\, a2\, cY1\, rY + a0\, rY^2 + a1\, rY^2 + a2\, rY^2)) -$$
$$a1\, a2\, cYn1^2\, rY^3 \surd((a0\, a1\, a2\, cY1\, cYn1 + a0\, a1\, cYn1\, rY + a0\, a2\, cYn1\, rY + a1\, a2\, cYn1\, rY)^2 -$$
$$4\, a0\, a1\, a2\, cYn1^2\, (a0\, a1\, a2\, cY1^2 + a0\, a1\, cY1\, rY +$$
$$a0\, a2\, cY1\, rY + a1\, a2\, cY1\, rY + a0\, rY^2 + a1\, rY^2 + a2\, rY^2)) -$$
$$2\, a0^2\, cYn2^2\, rY^3 \surd((a0\, a1\, a2\, cY1\, cYn1 + a0\, a1\, cYn1\, rY + a0\, a2\, cYn1\, rY + a1\, a2\, cYn1\, rY)^2 -$$
$$4\, a0\, a1\, a2\, cYn1^2\, (a0\, a1\, a2\, cY1^2 + a0\, a1\, cY1\, rY +$$
$$a0\, a2\, cY1\, rY + a1\, a2\, cY1\, rY + a0\, rY^2 + a1\, rY^2 + a2\, rY^2)) -$$
$$a0\, a1\, cYn2^2\, rY^3 \surd((a0\, a1\, a2\, cY1\, cYn1 + a0\, a1\, cYn1\, rY + a0\, a2\, cYn1\, rY + a1\, a2\, cYn1\, rY)^2 -$$
$$4\, a0\, a1\, a2\, cYn1^2\, (a0\, a1\, a2\, cY1^2 + a0\, a1\, cY1\, rY +$$
$$a0\, a2\, cY1\, rY + a1\, a2\, cY1\, rY + a0\, rY^2 + a1\, rY^2 + a2\, rY^2)) -$$
$$a0\, a2\, cYn2^2\, rY^3 \surd((a0\, a1\, a2\, cY1\, cYn1 + a0\, a1\, cYn1\, rY + a0\, a2\, cYn1\, rY + a1\, a2\, cYn1\, rY)^2 -$$
$$4\, a0\, a1\, a2\, cYn1^2\, (a0\, a1\, a2\, cY1^2 + a0\, a1\, cY1\, rY +$$
$$a0\, a2\, cY1\, rY + a1\, a2\, cY1\, rY + a0\, rY^2 + a1\, rY^2 + a2\, rY^2)) +$$
$$a1\, a2\, cYn2^2\, rY^3 \surd((a0\, a1\, a2\, cY1\, cYn1 + a0\, a1\, cYn1\, rY + a0\, a2\, cYn1\, rY + a1\, a2\, cYn1\, rY)^2 -$$
$$4\, a0\, a1\, a2\, cYn1^2\, (a0\, a1\, a2\, cY1^2 + a0\, a1\, cY1\, rY + a0\, a2\, cY1\, rY +$$
$$a1\, a2\, cY1\, rY + a0\, rY^2 + a1\, rY^2 + a2\, rY^2)))\, /\, (2\, cYn1\, rY^2$$
$$(a0\, a1\, a2\, cY1^2 + a0\, a1\, cY1\, rY + a0\, a2\, cY1\, rY + a1\, a2\, cY1\, rY + a0\, rY^2 + a1\, rY^2 + a2\, rY^2)^2),$$
$$Sn1 \to (6\, a0^2\, a1^4\, a2^2\, cY1^4\, cYn1 + 2\, a0^2\, a1^4\, a2^2\, cY1^6\, cYn1 + 2\, a0^2\, a1^4\, a2^2\, cY1^4\, cYn1^3 +$$
$$2\, a0^2\, a1^4\, a2^2\, cY1^4\, cYn1\, cYn2^2 + 12\, a0^2\, a1^4\, a2\, cY1^3\, cYn1\, rY +$$
$$12\, a0^2\, a1^3\, a2^2\, cY1^3\, cYn1\, rY + 12\, a0\, a1^4\, a2^2\, cY1^3\, cYn1\, rY +$$
$$4\, a0^2\, a1^4\, a2\, cY1^5\, cYn1\, rY + 8\, a0^2\, a1^3\, a2^2\, cY1^5\, cYn1\, rY +$$
$$4\, a0\, a1^4\, a2^2\, cY1^5\, cYn1\, rY + 4\, a0^2\, a1^4\, a2\, cY1^3\, cYn1^3\, rY +$$
$$2\, a0^2\, a1^3\, a2^2\, cY1^3\, cYn1^3\, rY + 4\, a0\, a1^4\, a2^2\, cY1^3\, cYn1^3\, rY +$$
$$4\, a0^2\, a1^4\, a2\, cY1^3\, cYn1\, cYn2^2\, rY + 2\, a0^2\, a1^3\, a2^2\, cY1^3\, cYn1\, cYn2^2\, rY +$$
$$4\, a0\, a1^4\, a2^2\, cY1^3\, cYn1\, cYn2^2\, rY + 6\, a0^2\, a1^4\, cY1^2\, cYn1\, rY^2 +$$
$$24\, a0^2\, a1^3\, a2\, cY1^2\, cYn1\, rY^2 + 24\, a0\, a1^4\, a2\, cY1^2\, cYn1\, rY^2 +$$
$$6\, a0^2\, a1^2\, a2^2\, cY1^2\, cYn1\, rY^2 + 24\, a0\, a1^3\, a2^2\, cY1^2\, cYn1\, rY^2 +$$
$$6\, a1^4\, a2^2\, cY1^2\, cYn1\, rY^2 + 2\, a0^2\, a1^4\, cY1^4\, cYn1\, rY^2 +$$
$$16\, a0^2\, a1^3\, a2\, cY1^4\, cYn1\, rY^2 + 8\, a0\, a1^4\, a2\, cY1^4\, cYn1\, rY^2 +$$
$$12\, a0^2\, a1^2\, a2^2\, cY1^4\, cYn1\, rY^2 + 16\, a0\, a1^3\, a2^2\, cY1^4\, cYn1\, rY^2 +$$
$$2\, a1^4\, a2^2\, cY1^4\, cYn1\, rY^2 + 2\, a0^2\, a1^4\, cY1^2\, cYn1^3\, rY^2 +$$
$$4\, a0^2\, a1^3\, a2\, cY1^2\, cYn1^3\, rY^2 + 8\, a0\, a1^4\, a2\, cY1^2\, cYn1^3\, rY^2 -$$
$$3\, a0^2\, a1^2\, a2^2\, cY1^2\, cYn1^3\, rY^2 + 4\, a0\, a1^3\, a2^2\, cY1^2\, cYn1^3\, rY^2 +$$
$$2\, a1^4\, a2^2\, cY1^2\, cYn1^3\, rY^2 + 2\, a0^2\, a1^4\, cY1^2\, cYn1\, cYn2^2\, rY^2 +$$
$$4\, a0^2\, a1^3\, a2\, cY1^2\, cYn1\, cYn2^2\, rY^2 + 8\, a0\, a1^4\, a2\, cY1^2\, cYn1\, cYn2^2\, rY^2 -$$
$$3\, a0^2\, a1^2\, a2^2\, cY1^2\, cYn1\, cYn2^2\, rY^2 + 4\, a0\, a1^3\, a2^2\, cY1^2\, cYn1\, cYn2^2\, rY^2 +$$
$$2\, a1^4\, a2^2\, cY1^2\, cYn1\, cYn2^2\, rY^2 + 12\, a0^2\, a1^3\, cY1\, cYn1\, rY^3 +$$
$$12\, a0\, a1^4\, cY1\, cYn1\, rY^3 + 12\, a0^2\, a1^2\, a2\, cY1\, cYn1\, rY^3 +$$
$$36\, a0\, a1^3\, a2\, cY1\, cYn1\, rY^3 + 12\, a1^4\, a2\, cY1\, cYn1\, rY^3 +$$
$$12\, a0\, a1^2\, a2^2\, cY1\, cYn1\, rY^3 + 12\, a1^3\, a2^2\, cY1\, cYn1\, rY^3 +$$

$$8\, a0^2\, a1^3\, cY1^3\, cYn1\, rY^3 + 4\, a0\, a1^4\, cY1^3\, cYn1\, rY^3 +$$
$$24\, a0^2\, a1^2\, a2\, cY1^3\, cYn1\, rY^3 + 28\, a0\, a1^3\, a2\, cY1^3\, cYn1\, rY^3 +$$
$$4\, a1^4\, a2\, cY1^3\, cYn1\, rY^3 + 8\, a0^2\, a1\, a2^2\, cY1^3\, cYn1\, rY^3 +$$
$$24\, a0\, a1^2\, a2^2\, cY1^3\, cYn1\, rY^3 + 8\, a1^3\, a2^2\, cY1^3\, cYn1\, rY^3 +$$
$$2\, a0^2\, a1^3\, cY1\, cYn1^3\, rY^3 + 4\, a0\, a1^4\, cY1\, cYn1^3\, rY^3 -$$
$$2\, a0^2\, a1^2\, a2\, cY1\, cYn1^3\, rY^3 + 6\, a0\, a1^3\, a2\, cY1\, cYn1^3\, rY^3 +$$
$$4\, a1^4\, a2\, cY1\, cYn1^3\, rY^3 - 2\, a0^2\, a1\, a2^2\, cY1\, cYn1^3\, rY^3 -$$
$$2\, a0\, a1^2\, a2^2\, cY1\, cYn1^3\, rY^3 + 2\, a1^3\, a2^2\, cY1\, cYn1^3\, rY^3 +$$
$$2\, a0^2\, a1^3\, cY1\, cYn1\, cYn2^2\, rY^3 + 4\, a0\, a1^4\, cY1\, cYn1\, cYn2^2\, rY^3 -$$
$$2\, a0^2\, a1^2\, a2\, cY1\, cYn1\, cYn2^2\, rY^3 + 6\, a0\, a1^3\, a2\, cY1\, cYn1\, cYn2^2\, rY^3 +$$
$$4\, a1^4\, a2\, cY1\, cYn1\, cYn2^2\, rY^3 - 2\, a0^2\, a1\, a2^2\, cY1\, cYn1\, cYn2^2\, rY^3 -$$
$$2\, a0\, a1^2\, a2^2\, cY1\, cYn1\, cYn2^2\, rY^3 + 2\, a1^3\, a2^2\, cY1\, cYn1\, cYn2^2\, rY^3 +$$
$$6\, a0^2\, a1^2\, cYn1\, rY^4 + 12\, a0\, a1^3\, cYn1\, rY^4 + 6\, a1^4\, cYn1\, rY^4 +$$
$$12\, a0\, a1^2\, a2\, cYn1\, rY^4 + 12\, a1^3\, a2\, cYn1\, rY^4 + 6\, a1^2\, a2^2\, cYn1\, rY^4 +$$
$$12\, a0^2\, a1^2\, cY1^2\, cYn1\, rY^4 + 12\, a0\, a1^3\, cY1^2\, cYn1\, rY^4 + 2\, a1^4\, cY1^2\, cYn1\, rY^4 +$$
$$16\, a0^2\, a1\, a2\, cY1^2\, cYn1\, rY^4 + 36\, a0\, a1^2\, a2\, cY1^2\, cYn1\, rY^4 +$$
$$12\, a1^3\, a2\, cY1^2\, cYn1\, rY^4 + 2\, a0^2\, a2^2\, cY1^2\, cYn1\, rY^4 +$$
$$16\, a0\, a1\, a2^2\, cY1^2\, cYn1\, rY^4 + 12\, a1^2\, a2^2\, cY1^2\, cYn1\, rY^4 +$$
$$a0^2\, a1^2\, cYn1^3\, rY^4 + 2\, a0\, a1^3\, cYn1^3\, rY^4 + 2\, a1^4\, cYn1^3\, rY^4 -$$
$$2\, a0^2\, a1\, a2\, cYn1^3\, rY^4 - 2\, a0\, a1^2\, a2\, cYn1^3\, rY^4 + 2\, a1^3\, a2\, cYn1^3\, rY^4 +$$
$$a0^2\, a2^2\, cYn1^3\, rY^4 - 2\, a0\, a1\, a2^2\, cYn1^3\, rY^4 + a1^2\, a2^2\, cYn1^3\, rY^4 +$$
$$a0^2\, a1^2\, cYn1\, cYn2^2\, rY^4 + 2\, a0\, a1^3\, cYn1\, cYn2^2\, rY^4 + 2\, a1^4\, cYn1\, cYn2^2\, rY^4 -$$
$$2\, a0^2\, a1\, a2\, cYn1\, cYn2^2\, rY^4 - 2\, a0\, a1^2\, a2\, cYn1\, cYn2^2\, rY^4 +$$
$$2\, a1^3\, a2\, cYn1\, cYn2^2\, rY^4 + a0^2\, a2^2\, cYn1\, cYn2^2\, rY^4 -$$
$$2\, a0\, a1\, a2^2\, cYn1\, cYn2^2\, rY^4 + a1^2\, a2^2\, cYn1\, cYn2^2\, rY^4 +$$
$$8\, a0^2\, a1\, cY1\, cYn1\, rY^5 + 12\, a0\, a1^2\, cY1\, cYn1\, rY^5 + 4\, a1^3\, cY1\, cYn1\, rY^5 +$$
$$4\, a0^2\, a2\, cY1\, cYn1\, rY^5 + 20\, a0\, a1\, a2\, cY1\, cYn1\, rY^5 + 12\, a1^2\, a2\, cY1\, cYn1\, rY^5 +$$
$$4\, a0\, a2^2\, cY1\, cYn1\, rY^5 + 8\, a1\, a2^2\, cY1\, cYn1\, rY^5 + 2\, a0^2\, cYn1\, rY^6 +$$
$$4\, a0\, a1\, cYn1\, rY^6 + 2\, a1^2\, cYn1\, rY^6 + 4\, a0\, a2\, cYn1\, rY^6 +$$
$$4\, a1\, a2\, cYn1\, rY^6 + 2\, a2^2\, cYn1\, rY^6 + 2\, a0\, a1^2\, a2\, cY1^2\, cYn1^2\, rY$$
$$\sqrt{\left((a0\, a1\, a2\, cY1\, cYn1 + a0\, a1\, cYn1\, rY + a0\, a2\, cYn1\, rY + a1\, a2\, cYn1\, rY)^2 - 4\, a0\, a1\, a2\, cYn1^2 \left(a0\, a1\, a2\, cY1^2 + a0\, a1\, cY1\, rY + a0\, a2\, cY1\, rY + a1\, a2\, cY1\, rY + a0\, rY^2 + a1\, rY^2 + a2\, rY^2\right)\right)} - 2\, a0\, a1^2\, a2\, cY1^2\, cYn2^2$$
$$rY \sqrt{\left((a0\, a1\, a2\, cY1\, cYn1 + a0\, a1\, cYn1\, rY + a0\, a2\, cYn1\, rY + a1\, a2\, cYn1\, rY)^2 - 4\, a0\, a1\, a2\, cYn1^2 \left(a0\, a1\, a2\, cY1^2 + a0\, a1\, cY1\, rY + a0\, a2\, cY1\, rY + a1\, a2\, cY1\, rY + a0\, rY^2 + a1\, rY^2 + a2\, rY^2\right)\right)} + 2\, a0\, a1^2\, cY1\, cYn1^2$$
$$rY^2 \sqrt{\left((a0\, a1\, a2\, cY1\, cYn1 + a0\, a1\, cYn1\, rY + a0\, a2\, cYn1\, rY + a1\, a2\, cYn1\, rY)^2 - 4\, a0\, a1\, a2\, cYn1^2 \left(a0\, a1\, a2\, cY1^2 + a0\, a1\, cY1\, rY + a0\, a2\, cY1\, rY + a1\, a2\, cY1\, rY + a0\, rY^2 + a1\, rY^2 + a2\, rY^2\right)\right)} + a0\, a1\, a2\, cY1\, cYn1^2$$
$$rY^2 \sqrt{\left((a0\, a1\, a2\, cY1\, cYn1 + a0\, a1\, cYn1\, rY + a0\, a2\, cYn1\, rY + a1\, a2\, cYn1\, rY)^2 - 4\, a0\, a1\, a2\, cYn1^2 \left(a0\, a1\, a2\, cY1^2 + a0\, a1\, cY1\, rY + a0\, a2\, cY1\, rY + a1\, a2\, cY1\, rY + a0\, rY^2 + a1\, rY^2 + a2\, rY^2\right)\right)} + 2\, a1^2\, a2\, cY1\, cYn1^2$$
$$rY^2 \sqrt{\left((a0\, a1\, a2\, cY1\, cYn1 + a0\, a1\, cYn1\, rY + a0\, a2\, cYn1\, rY + a1\, a2\, cYn1\, rY)^2 - 4\, a0\, a1\, a2\, cYn1^2 \left(a0\, a1\, a2\, cY1^2 + a0\, a1\, cY1\, rY + a0\, a2\, cY1\, rY + a1\, a2\, cY1\, rY + a0\, rY^2 + a1\, rY^2 + a2\, rY^2\right)\right)} - 2\, a0\, a1^2\, cY1\, cYn2^2$$
$$rY^2 \sqrt{\left((a0\, a1\, a2\, cY1\, cYn1 + a0\, a1\, cYn1\, rY + a0\, a2\, cYn1\, rY + a1\, a2\, cYn1\, rY)^2 - 4\, a0\, a1\, a2\, cYn1^2 \left(a0\, a1\, a2\, cY1^2 + a0\, a1\, cY1\, rY + a0\, a2\, cY1\, rY + a1\, a2\, cY1\, rY + a0\, rY^2 + a1\, rY^2 + a2\, rY^2\right)\right)} - a0\, a1\, a2\, cY1\, cYn2^2$$
$$rY^2 \sqrt{\left((a0\, a1\, a2\, cY1\, cYn1 + a0\, a1\, cYn1\, rY + a0\, a2\, cYn1\, rY + a1\, a2\, cYn1\, rY)^2 - 4\, a0\, a1\, a2\, cYn1^2 \left(a0\, a1\, a2\, cY1^2 + a0\, a1\, cY1\, rY + a0\, a2\, cY1\, rY + a1\, a2\, cY1\, rY + a0\, rY^2 + a1\, rY^2 + a2\, rY^2\right)\right)} - 2\, a1^2\, a2\, cY1\, cYn2^2$$
$$rY^2 \sqrt{\left((a0\, a1\, a2\, cY1\, cYn1 + a0\, a1\, cYn1\, rY + a0\, a2\, cYn1\, rY + a1\, a2\, cYn1\, rY)^2 - 4\, a0\, a1\, a2\, cYn1^2 \left(a0\, a1\, a2\, cY1^2 + a0\, a1\, cY1\, rY + a0\, a2\, cY1\, rY + a1\, a2\, cY1\, rY + a0\, rY^2 + a1\, rY^2 + a2\, rY^2\right)\right)} +$$

$$a0\ a1\ cYn1^2\ rY^3 \sqrt{\Big((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 -}$$
$$4\ a0\ a1\ a2\ cYn1^2\ \big(a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY +$$
$$a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2\big)\Big) +$$
$$2\ a1^2\ cYn1^2\ rY^3 \sqrt{\Big((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 -}$$
$$4\ a0\ a1\ a2\ cYn1^2\ \big(a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY +$$
$$a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2\big)\Big) -$$
$$a0\ a2\ cYn1^2\ rY^3 \sqrt{\Big((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 -}$$
$$4\ a0\ a1\ a2\ cYn1^2\ \big(a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY +$$
$$a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2\big)\Big) +$$
$$a1\ a2\ cYn1^2\ rY^3 \sqrt{\Big((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 -}$$
$$4\ a0\ a1\ a2\ cYn1^2\ \big(a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY +$$
$$a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2\big)\Big) -$$
$$a0\ a1\ cYn2^2\ rY^3 \sqrt{\Big((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 -}$$
$$4\ a0\ a1\ a2\ cYn1^2\ \big(a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY +$$
$$a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2\big)\Big) -$$
$$2\ a1^2\ cYn2^2\ rY^3 \sqrt{\Big((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 -}$$
$$4\ a0\ a1\ a2\ cYn1^2\ \big(a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY +$$
$$a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2\big)\Big) +$$
$$a0\ a2\ cYn2^2\ rY^3 \sqrt{\Big((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 -}$$
$$4\ a0\ a1\ a2\ cYn1^2\ \big(a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY +$$
$$a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2\big)\Big) -$$
$$a1\ a2\ cYn2^2\ rY^3 \sqrt{\Big((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^2 -}$$
$$4\ a0\ a1\ a2\ cYn1^2\ \big(a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + a0\ a2\ cY1\ rY +$$
$$a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2\big)\Big)\Big) \Big/ \Big(2\ cYn1\ rY^2$$
$$\big(a0\ a1\ a2\ cY1^2 + a0\ a1\ cY1\ rY + a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^2 + a1\ rY^2 + a2\ rY^2\big)^2\Big),$$
$$Sn2 \to \big(6\ a0^2\ a1^2\ a2^4\ cY1^4\ cYn1 + 2\ a0^2\ a1^2\ a2^4\ cY1^6\ cYn1 + 2\ a0^2\ a1^2\ a2^4\ cY1^4\ cYn1^3 +$$
$$2\ a0^2\ a1^2\ a2^4\ cY1^4\ cYn1\ cYn2^2 + 12\ a0^2\ a1^2\ a2^3\ cY1^3\ cYn1\ rY +$$
$$12\ a0^2\ a1\ a2^4\ cY1^3\ cYn1\ rY + 12\ a0\ a1^2\ a2^4\ cY1^3\ cYn1\ rY +$$
$$8\ a0^2\ a1^2\ a2^3\ cY1^5\ cYn1\ rY + 4\ a0^2\ a1\ a2^4\ cY1^5\ cYn1\ rY +$$
$$4\ a0\ a1^2\ a2^4\ cY1^5\ cYn1\ rY + 2\ a0^2\ a1^2\ a2^3\ cY1^3\ cYn1^3\ rY +$$
$$4\ a0^2\ a1\ a2^4\ cY1^3\ cYn1^3\ rY + 4\ a0\ a1^2\ a2^4\ cY1^3\ cYn1^3\ rY +$$
$$2\ a0^2\ a1^2\ a2^3\ cY1^3\ cYn1\ cYn2^2\ rY + 4\ a0^2\ a1\ a2^4\ cY1^3\ cYn1\ cYn2^2\ rY +$$
$$4\ a0\ a1^2\ a2^4\ cY1^3\ cYn1\ cYn2^2\ rY + 6\ a0^2\ a1^2\ a2^2\ cY1^2\ cYn1\ rY^2 +$$
$$24\ a0^2\ a1\ a2^3\ cY1^2\ cYn1\ rY^2 + 24\ a0\ a1^2\ a2^3\ cY1^2\ cYn1\ rY^2 +$$
$$6\ a0^2\ a2^4\ cY1^2\ cYn1\ rY^2 + 24\ a0\ a1\ a2^4\ cY1^2\ cYn1\ rY^2 +$$
$$6\ a1^2\ a2^4\ cY1^2\ cYn1\ rY^2 + 12\ a0^2\ a1^2\ a2^2\ cY1^4\ cYn1\ rY^2 +$$
$$16\ a0^2\ a1\ a2^3\ cY1^4\ cYn1\ rY^2 + 16\ a0\ a1^2\ a2^3\ cY1^4\ cYn1\ rY^2 +$$
$$2\ a0^2\ a2^4\ cY1^4\ cYn1\ rY^2 + 8\ a0\ a1\ a2^4\ cY1^4\ cYn1\ rY^2 +$$
$$2\ a1^2\ a2^4\ cY1^4\ cYn1\ rY^2 - 3\ a0^2\ a1^2\ a2^2\ cY1^2\ cYn1^3\ rY^2 +$$
$$4\ a0^2\ a1\ a2^3\ cY1^2\ cYn1^3\ rY^2 + 4\ a0\ a1^2\ a2^3\ cY1^2\ cYn1^3\ rY^2 +$$
$$2\ a0^2\ a2^4\ cY1^2\ cYn1^3\ rY^2 + 8\ a0\ a1\ a2^4\ cY1^2\ cYn1^3\ rY^2 +$$
$$2\ a1^2\ a2^4\ cY1^2\ cYn1^3\ rY^2 - 3\ a0^2\ a1^2\ a2^2\ cY1^2\ cYn1\ cYn2^2\ rY^2 +$$
$$4\ a0^2\ a1\ a2^3\ cY1^2\ cYn1\ cYn2^2\ rY^2 + 4\ a0\ a1^2\ a2^3\ cY1^2\ cYn1\ cYn2^2\ rY^2 +$$
$$2\ a0^2\ a2^4\ cY1^2\ cYn1\ cYn2^2\ rY^2 + 8\ a0\ a1\ a2^4\ cY1^2\ cYn1\ cYn2^2\ rY^2 +$$
$$2\ a1^2\ a2^4\ cY1^2\ cYn1\ cYn2^2\ rY^2 + 12\ a0^2\ a1\ a2^2\ cY1\ cYn1\ rY^3 +$$
$$12\ a0\ a1^2\ a2^2\ cY1\ cYn1\ rY^3 + 12\ a0^2\ a2^3\ cY1\ cYn1\ rY^3 +$$
$$36\ a0\ a1\ a2^3\ cY1\ cYn1\ rY^3 + 12\ a1^2\ a2^3\ cY1\ cYn1\ rY^3 +$$
$$12\ a0\ a2^4\ cY1\ cYn1\ rY^3 + 12\ a1\ a2^4\ cY1\ cYn1\ rY^3 +$$
$$8\ a0^2\ a1^2\ a2\ cY1^3\ cYn1\ rY^3 + 24\ a0^2\ a1\ a2^2\ cY1^3\ cYn1\ rY^3 +$$
$$24\ a0\ a1^2\ a2^2\ cY1^3\ cYn1\ rY^3 + 8\ a0^2\ a2^3\ cY1^3\ cYn1\ rY^3 +$$
$$28\ a0\ a1\ a2^3\ cY1^3\ cYn1\ rY^3 + 8\ a1^2\ a2^3\ cY1^3\ cYn1\ rY^3 +$$
$$4\ a0\ a2^4\ cY1^3\ cYn1\ rY^3 + 4\ a1\ a2^4\ cY1^3\ cYn1\ rY^3 -$$
$$2\ a0^2\ a1^2\ a2\ cY1\ cYn1^3\ rY^3 - 2\ a0^2\ a1\ a2^2\ cY1\ cYn1^3\ rY^3 -$$

$$2\ a0\ a1^{2}\ a2^{2}\ cY1\ cYn1^{3}\ rY^{3} + 2\ a0^{2}\ a2^{3}\ cY1\ cYn1^{3}\ rY^{3} +$$
$$6\ a0\ a1\ a2^{3}\ cY1\ cYn1^{3}\ rY^{3} + 2\ a1^{2}\ a2^{3}\ cY1\ cYn1^{3}\ rY^{3} +$$
$$4\ a0\ a2^{4}\ cY1\ cYn1^{3}\ rY^{3} + 4\ a1\ a2^{4}\ cY1\ cYn1^{3}\ rY^{3} -$$
$$2\ a0^{2}\ a1^{2}\ a2\ cY1\ cYn1\ cYn2^{2}\ rY^{3} - 2\ a0^{2}\ a1\ a2^{2}\ cY1\ cYn1\ cYn2^{2}\ rY^{3} -$$
$$2\ a0\ a1^{2}\ a2^{2}\ cY1\ cYn1\ cYn2^{2}\ rY^{3} + 2\ a0^{2}\ a2^{3}\ cY1\ cYn1\ cYn2^{2}\ rY^{3} +$$
$$6\ a0\ a1\ a2^{3}\ cY1\ cYn1\ cYn2^{2}\ rY^{3} + 2\ a1^{2}\ a2^{3}\ cY1\ cYn1\ cYn2^{2}\ rY^{3} +$$
$$4\ a0\ a2^{4}\ cY1\ cYn1\ cYn2^{2}\ rY^{3} + 4\ a1\ a2^{4}\ cY1\ cYn1\ cYn2^{2}\ rY^{3} +$$
$$6\ a0^{2}\ a2^{2}\ cYn1\ rY^{4} + 12\ a0\ a1\ a2^{2}\ cYn1\ rY^{4} + 6\ a1^{2}\ a2^{2}\ cYn1\ rY^{4} +$$
$$12\ a0\ a2^{3}\ cYn1\ rY^{4} + 12\ a1\ a2^{3}\ cYn1\ rY^{4} + 6\ a2^{4}\ cYn1\ rY^{4} +$$
$$2\ a0^{2}\ a1^{2}\ cY1^{2}\ cYn1\ rY^{4} + 16\ a0^{2}\ a1\ a2\ cY1^{2}\ cYn1\ rY^{4} +$$
$$16\ a0\ a1^{2}\ a2\ cY1^{2}\ cYn1\ rY^{4} + 12\ a0^{2}\ a2^{2}\ cY1^{2}\ cYn1\ rY^{4} +$$
$$36\ a0\ a1\ a2^{2}\ cY1^{2}\ cYn1\ rY^{4} + 12\ a1^{2}\ a2^{2}\ cY1^{2}\ cYn1\ rY^{4} +$$
$$12\ a0\ a2^{3}\ cY1^{2}\ cYn1\ rY^{4} + 12\ a1\ a2^{3}\ cY1^{2}\ cYn1\ rY^{4} +$$
$$2\ a2^{4}\ cY1^{2}\ cYn1\ rY^{4} + a0^{2}\ a1^{2}\ cYn1^{3}\ rY^{4} - 2\ a0^{2}\ a1\ a2\ cYn1^{3}\ rY^{4} -$$
$$2\ a0\ a1^{2}\ a2\ cYn1^{3}\ rY^{4} + a0^{2}\ a2^{2}\ cYn1^{3}\ rY^{4} -$$
$$2\ a0\ a1\ a2^{2}\ cYn1^{3}\ rY^{4} + a1^{2}\ a2^{2}\ cYn1^{3}\ rY^{4} + 2\ a0\ a2^{3}\ cYn1^{3}\ rY^{4} +$$
$$2\ a1\ a2^{3}\ cYn1^{3}\ rY^{4} + 2\ a2^{4}\ cYn1^{3}\ rY^{4} + a0^{2}\ a1^{2}\ cYn1\ cYn2^{2}\ rY^{4} -$$
$$2\ a0^{2}\ a1\ a2\ cYn1\ cYn2^{2}\ rY^{4} - 2\ a0\ a1^{2}\ a2\ cYn1\ cYn2^{2}\ rY^{4} +$$
$$a0^{2}\ a2^{2}\ cYn1\ cYn2^{2}\ rY^{4} - 2\ a0\ a1\ a2^{2}\ cYn1\ cYn2^{2}\ rY^{4} +$$
$$a1^{2}\ a2^{2}\ cYn1\ cYn2^{2}\ rY^{4} + 2\ a0\ a2^{3}\ cYn1\ cYn2^{2}\ rY^{4} +$$
$$2\ a1\ a2^{3}\ cYn1\ cYn2^{2}\ rY^{4} + 2\ a2^{4}\ cYn1\ cYn2^{2}\ rY^{4} +$$
$$4\ a0^{2}\ a1\ cY1\ cYn1\ rY^{5} + 4\ a0\ a1^{2}\ cY1\ cYn1\ rY^{5} +$$
$$8\ a0^{2}\ a2\ cY1\ cYn1\ rY^{5} + 20\ a0\ a1\ a2\ cY1\ cYn1\ rY^{5} +$$
$$8\ a1^{2}\ a2\ cY1\ cYn1\ rY^{5} + 12\ a0\ a2^{2}\ cY1\ cYn1\ rY^{5} +$$
$$12\ a1\ a2^{2}\ cY1\ cYn1\ rY^{5} + 4\ a2^{3}\ cY1\ cYn1\ rY^{5} + 2\ a0^{2}\ cYn1\ rY^{6} +$$
$$4\ a0\ a1\ cYn1\ rY^{6} + 2\ a1^{2}\ cYn1\ rY^{6} + 4\ a0\ a2\ cYn1\ rY^{6} +$$
$$4\ a1\ a2\ cYn1\ rY^{6} + 2\ a2^{2}\ cYn1\ rY^{6} + 2\ a0\ a1\ a2^{2}\ cY1^{2}\ cYn1^{2}\ rY$$
$$\surd\Big((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^{2} -$$
$$4\ a0\ a1\ a2\ cYn1^{2}\ \big(a0\ a1\ a2\ cY1^{2} + a0\ a1\ cY1\ rY + a0\ a2\ cY1\ rY +$$
$$a1\ a2\ cY1\ rY + a0\ rY^{2} + a1\ rY^{2} + a2\ rY^{2}\big)\Big) - 2\ a0\ a1\ a2^{2}\ cY1^{2}\ cYn2^{2}$$
$$rY\ \surd\Big((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^{2} -$$
$$4\ a0\ a1\ a2\ cYn1^{2}\ \big(a0\ a1\ a2\ cY1^{2} + a0\ a1\ cY1\ rY + a0\ a2\ cY1\ rY +$$
$$a1\ a2\ cY1\ rY + a0\ rY^{2} + a1\ rY^{2} + a2\ rY^{2}\big)\Big) + a0\ a1\ a2\ cY1\ cYn1^{2}$$
$$rY^{2}\ \surd\Big((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^{2} -$$
$$4\ a0\ a1\ a2\ cYn1^{2}\ \big(a0\ a1\ a2\ cY1^{2} + a0\ a1\ cY1\ rY + a0\ a2\ cY1\ rY +$$
$$a1\ a2\ cY1\ rY + a0\ rY^{2} + a1\ rY^{2} + a2\ rY^{2}\big)\Big) + 2\ a0\ a2^{2}\ cY1\ cYn1^{2}$$
$$rY^{2}\ \surd\Big((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^{2} -$$
$$4\ a0\ a1\ a2\ cYn1^{2}\ \big(a0\ a1\ a2\ cY1^{2} + a0\ a1\ cY1\ rY + a0\ a2\ cY1\ rY +$$
$$a1\ a2\ cY1\ rY + a0\ rY^{2} + a1\ rY^{2} + a2\ rY^{2}\big)\Big) + 2\ a1\ a2^{2}\ cY1\ cYn1^{2}$$
$$rY^{2}\ \surd\Big((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^{2} -$$
$$4\ a0\ a1\ a2\ cYn1^{2}\ \big(a0\ a1\ a2\ cY1^{2} + a0\ a1\ cY1\ rY + a0\ a2\ cY1\ rY +$$
$$a1\ a2\ cY1\ rY + a0\ rY^{2} + a1\ rY^{2} + a2\ rY^{2}\big)\Big) - a0\ a1\ a2\ cY1\ cYn2^{2}$$
$$rY^{2}\ \surd\Big((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^{2} -$$
$$4\ a0\ a1\ a2\ cYn1^{2}\ \big(a0\ a1\ a2\ cY1^{2} + a0\ a1\ cY1\ rY + a0\ a2\ cY1\ rY +$$
$$a1\ a2\ cY1\ rY + a0\ rY^{2} + a1\ rY^{2} + a2\ rY^{2}\big)\Big) - 2\ a0\ a2^{2}\ cY1\ cYn2^{2}$$
$$rY^{2}\ \surd\Big((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^{2} -$$
$$4\ a0\ a1\ a2\ cYn1^{2}\ \big(a0\ a1\ a2\ cY1^{2} + a0\ a1\ cY1\ rY + a0\ a2\ cY1\ rY +$$
$$a1\ a2\ cY1\ rY + a0\ rY^{2} + a1\ rY^{2} + a2\ rY^{2}\big)\Big) - 2\ a1\ a2^{2}\ cY1\ cYn2^{2}$$
$$rY^{2}\ \surd\Big((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^{2} -$$
$$4\ a0\ a1\ a2\ cYn1^{2}\ \big(a0\ a1\ a2\ cY1^{2} + a0\ a1\ cY1\ rY +$$
$$a0\ a2\ cY1\ rY + a1\ a2\ cY1\ rY + a0\ rY^{2} + a1\ rY^{2} + a2\ rY^{2}\big)\Big) -$$
$$a0\ a1\ cYn1^{2}\ rY^{3}\ \surd\Big((a0\ a1\ a2\ cY1\ cYn1 + a0\ a1\ cYn1\ rY + a0\ a2\ cYn1\ rY + a1\ a2\ cYn1\ rY)^{2} -$$
$$4\ a0\ a1\ a2\ cYn1^{2}\ \big(a0\ a1\ a2\ cY1^{2} + a0\ a1\ cY1\ rY +$$

```
            a0 a2 cY1 rY + a1 a2 cY1 rY + a0 rY^2 + a1 rY^2 + a2 rY^2)) +
      a0 a2 cYn1^2 rY^3 √((a0 a1 a2 cY1 cYn1 + a0 a1 cYn1 rY + a0 a2 cYn1 rY + a1 a2 cYn1 rY)^2 -
          4 a0 a1 a2 cYn1^2 (a0 a1 a2 cY1^2 + a0 a1 cY1 rY +
            a0 a2 cY1 rY + a1 a2 cY1 rY + a0 rY^2 + a1 rY^2 + a2 rY^2)) +
      a1 a2 cYn1^2 rY^3 √((a0 a1 a2 cY1 cYn1 + a0 a1 cYn1 rY + a0 a2 cYn1 rY + a1 a2 cYn1 rY)^2 -
          4 a0 a1 a2 cYn1^2 (a0 a1 a2 cY1^2 + a0 a1 cY1 rY +
            a0 a2 cY1 rY + a1 a2 cY1 rY + a0 rY^2 + a1 rY^2 + a2 rY^2)) +
      2 a2^2 cYn1^2 rY^3 √((a0 a1 a2 cY1 cYn1 + a0 a1 cYn1 rY + a0 a2 cYn1 rY + a1 a2 cYn1 rY)^2 -
          4 a0 a1 a2 cYn1^2 (a0 a1 a2 cY1^2 + a0 a1 cY1 rY +
            a0 a2 cY1 rY + a1 a2 cY1 rY + a0 rY^2 + a1 rY^2 + a2 rY^2)) +
      a0 a1 cYn2^2 rY^3 √((a0 a1 a2 cY1 cYn1 + a0 a1 cYn1 rY + a0 a2 cYn1 rY + a1 a2 cYn1 rY)^2 -
          4 a0 a1 a2 cYn1^2 (a0 a1 a2 cY1^2 + a0 a1 cY1 rY +
            a0 a2 cY1 rY + a1 a2 cY1 rY + a0 rY^2 + a1 rY^2 + a2 rY^2)) -
      a0 a2 cYn2^2 rY^3 √((a0 a1 a2 cY1 cYn1 + a0 a1 cYn1 rY + a0 a2 cYn1 rY + a1 a2 cYn1 rY)^2 -
          4 a0 a1 a2 cYn1^2 (a0 a1 a2 cY1^2 + a0 a1 cY1 rY +
            a0 a2 cY1 rY + a1 a2 cY1 rY + a0 rY^2 + a1 rY^2 + a2 rY^2)) -
      a1 a2 cYn2^2 rY^3 √((a0 a1 a2 cY1 cYn1 + a0 a1 cYn1 rY + a0 a2 cYn1 rY + a1 a2 cYn1 rY)^2 -
          4 a0 a1 a2 cYn1^2 (a0 a1 a2 cY1^2 + a0 a1 cY1 rY +
            a0 a2 cY1 rY + a1 a2 cY1 rY + a0 rY^2 + a1 rY^2 + a2 rY^2)) -
      2 a2^2 cYn2^2 rY^3 √((a0 a1 a2 cY1 cYn1 + a0 a1 cYn1 rY + a0 a2 cYn1 rY + a1 a2 cYn1 rY)^2 -
          4 a0 a1 a2 cYn1^2 (a0 a1 a2 cY1^2 + a0 a1 cY1 rY + a0 a2 cY1 rY +
            a1 a2 cY1 rY + a0 rY^2 + a1 rY^2 + a2 rY^2))) / (2 cYn1 rY^2
      (a0 a1 a2 cY1^2 + a0 a1 cY1 rY + a0 a2 cY1 rY + a1 a2 cY1 rY + a0 rY^2 + a1 rY^2 + a2 rY^2)^2),
  pY1 → 3, n1 → -((a0 a1 a2 cY1 cYn1 + a0 a1 cYn1 rY + a0 a2 cYn1 rY + a1 a2 cYn1 rY -
        √((a0 a1 a2 cY1 cYn1 + a0 a1 cYn1 rY + a0 a2 cYn1 rY + a1 a2 cYn1 rY)^2 -
          4 a0 a1 a2 cYn1^2 (a0 a1 a2 cY1^2 + a0 a1 cY1 rY +
            a0 a2 cY1 rY + a1 a2 cY1 rY + a0 rY^2 + a1 rY^2 + a2 rY^2))) /
      (2 (a0 a1 a2 cY1^2 + a0 a1 cY1 rY + a0 a2 cY1 rY + a1 a2 cY1 rY + a0 rY^2 + a1 rY^2 + a2 rY^2))),
  n2 → -((cYn2 (a0 a1 a2 cY1 cYn1 + a0 a1 cYn1 rY + a0 a2 cYn1 rY + a1 a2 cYn1 rY +
          √((a0 a1 a2 cY1 cYn1 + a0 a1 cYn1 rY + a0 a2 cYn1 rY + a1 a2 cYn1 rY)^2 -
            4 a0 a1 a2 cYn1^2 (a0 a1 a2 cY1^2 + a0 a1 cY1 rY + a0 a2 cY1 rY +
              a1 a2 cY1 rY + a0 rY^2 + a1 rY^2 + a2 rY^2)))) / (2 cYn1
        (a0 a1 a2 cY1^2 + a0 a1 cY1 rY + a0 a2 cY1 rY + a1 a2 cY1 rY + a0 rY^2 + a1 rY^2 + a2 rY^2)))}}
```

Computations for Lemma 6.19.
Case A.3 where there exists N_i^6 s.t. Fix(N_i) = $Y^4 \cup P_0 \cup P_1 \cup P_2$.

```
(******************************
  NOTATIONS
  rY xY denotes the restriction of x to Y
  ai denotes the Hopf weight at Pi
  Nij denotes the product of the normal weights shared between Pi and Pj
  Sni is the sum of the square of the normal weights at Pi
  cYni xY denotes the first Chern class of the normal bundle of Y in the
  direction of Ni
  µx5YA denotes the local datum of x^5 at Y
  µx5PiA denotes the local datum of x^5 at Pi

  We use the variable t to extract the coefficient of order 2 of xY. We
  then evaluate xY^2 to 1.
  The orientation of the point P0 is 1 while those from P1 and P2 are -
 1. This explains the signs at their local datum
  Once evaluated, we extract all the coefficients in l into a list.
  ******************************)
```

```
µx5YA3 =
 Factor[CoefficientList[
    Coefficient[Expand[(rY xY t + l z)^5] Series[(cYn1 xY t + n1 z)^-1, {xY, 0, 2}]
      Series[(cYn2 xY t + n2 z)^-1, {xY, 0, 2}] Series[(cYn3 xY t + n3 z)^-1, {xY, 0, 2}],
     t, 2], l]] /. xY^2 → 1
```

$$\left\{0,\ 0,\ 0,\ \frac{10\ \mathrm{rY}^2}{\mathrm{n1\ n2\ n3}},\ -\frac{5\ (\mathrm{cYn3\ n1\ n2 + cYn2\ n1\ n3 + cYn1\ n2\ n3})\ \mathrm{rY}}{\mathrm{n1}^2\ \mathrm{n2}^2\ \mathrm{n3}^2},\ \frac{1}{\mathrm{n1}^3\ \mathrm{n2}^3\ \mathrm{n3}^3}\left(\mathrm{cYn3}^2\ \mathrm{n1}^2\ \mathrm{n2}^2 + \mathrm{cYn2\ cYn3\ n1}^2\ \mathrm{n2\ n3} + \mathrm{cYn1\ cYn3\ n1\ n2}^2\ \mathrm{n3} + \mathrm{cYn2}^2\ \mathrm{n1}^2\ \mathrm{n3}^2 + \mathrm{cYn1\ cYn2\ n1\ n2\ n3}^2 + \mathrm{cYn1}^2\ \mathrm{n2}^2\ \mathrm{n3}^2\right)\right\}$$

```
µx5P0A3 = CoefficientList[Expand[((a0 + l) z)^5] n1^-3 n2^-3 n3^-3 N01^-1 N02^-1 z^-5, l]
```

$$\left\{\frac{\mathrm{a0}^5}{\mathrm{N01\ N02\ n1}^3\ \mathrm{n2}^3\ \mathrm{n3}^3},\ \frac{5\ \mathrm{a0}^4}{\mathrm{N01\ N02\ n1}^3\ \mathrm{n2}^3\ \mathrm{n3}^3},\ \frac{10\ \mathrm{a0}^3}{\mathrm{N01\ N02\ n1}^3\ \mathrm{n2}^3\ \mathrm{n3}^3},\ \frac{10\ \mathrm{a0}^2}{\mathrm{N01\ N02\ n1}^3\ \mathrm{n2}^3\ \mathrm{n3}^3},\ \frac{5\ \mathrm{a0}}{\mathrm{N01\ N02\ n1}^3\ \mathrm{n2}^3\ \mathrm{n3}^3},\ \frac{1}{\mathrm{N01\ N02\ n1}^3\ \mathrm{n2}^3\ \mathrm{n3}^3}\right\}$$

```
µx5P1A3 = CoefficientList[Expand[-((a1 + l) z)^5] n1^-3 n2^-3 n3^-3 N01^-1 N12^-1 z^-5, l]
```

$$\left\{-\frac{\mathrm{a1}^5}{\mathrm{N01\ n1}^3\ \mathrm{N12\ n2}^3\ \mathrm{n3}^3},\ -\frac{5\ \mathrm{a1}^4}{\mathrm{N01\ n1}^3\ \mathrm{N12\ n2}^3\ \mathrm{n3}^3},\ -\frac{10\ \mathrm{a1}^3}{\mathrm{N01\ n1}^3\ \mathrm{N12\ n2}^3\ \mathrm{n3}^3},\ -\frac{10\ \mathrm{a1}^2}{\mathrm{N01\ n1}^3\ \mathrm{N12\ n2}^3\ \mathrm{n3}^3},\ -\frac{5\ \mathrm{a1}}{\mathrm{N01\ n1}^3\ \mathrm{N12\ n2}^3\ \mathrm{n3}^3},\ -\frac{1}{\mathrm{N01\ n1}^3\ \mathrm{N12\ n2}^3\ \mathrm{n3}^3}\right\}$$

```
µx5P2A3 = CoefficientList[Expand[-((a2 + l) z)^5] n1^-3 n2^-3 n3^-3 N02^-1 N12^-1 z^-5, l]
```

$$\left\{-\frac{\mathrm{a2}^5}{\mathrm{N02\ n1}^3\ \mathrm{N12\ n2}^3\ \mathrm{n3}^3},\ -\frac{5\ \mathrm{a2}^4}{\mathrm{N02\ n1}^3\ \mathrm{N12\ n2}^3\ \mathrm{n3}^3},\ -\frac{10\ \mathrm{a2}^3}{\mathrm{N02\ n1}^3\ \mathrm{N12\ n2}^3\ \mathrm{n3}^3},\ -\frac{10\ \mathrm{a2}^2}{\mathrm{N02\ n1}^3\ \mathrm{N12\ n2}^3\ \mathrm{n3}^3},\ -\frac{5\ \mathrm{a2}}{\mathrm{N02\ n1}^3\ \mathrm{N12\ n2}^3\ \mathrm{n3}^3},\ -\frac{1}{\mathrm{N02\ n1}^3\ \mathrm{N12\ n2}^3\ \mathrm{n3}^3}\right\}$$

```
(******************************
 NOTATIONS
 µs1x3YA3 denotes the local datum of p1(X)x^3 at Y
 µs1x3PiA3 denotes the local datum of p1(X)x^3 at Pi
  ******************************)
```

```
μs1x3YA3 =
 Factor[CoefficientList[
    Coefficient[
     Expand[pY1 t^2 + (cYn1 xY t + n1 z)^2 + (cYn2 xY t + n2 z)^2 + (cYn3 xY t + n3 z)^2]
      (rY xY t + 1 z)^3 Series[(cYn1 xY t + n1 z)^-1, {xY, 0, 2}]
      Series[(cYn2 xY t + n2 z)^-1, {xY, 0, 2}] Series[(cYn3 xY t + n3 z)^-1, {xY, 0, 2}],
     t, 2], 1]] /. xY^2 → 1
```

$$\Big\{0,\ \frac{3\left(n1^2+n2^2+n3^2\right) rY^2}{n1\ n2\ n3},$$
$$-\frac{1}{n1^2\ n2^2\ n3^2}\,3\left(cYn3\ n1^3\ n2+cYn3\ n1\ n2^3+cYn2\ n1^3\ n3-cYn1\ n1^2\ n2\ n3-cYn2\ n1\ n2^2\ n3+cYn1\ n2^3\ n3-cYn3\ n1\ n2\ n3^2+cYn2\ n1\ n3^3+cYn1\ n2\ n3^3\right) rY,$$
$$\frac{1}{n1^3\ n2^3\ n3^3}\left(cYn3^2\ n1^4\ n2^2+cYn3^2\ n1^2\ n2^4+cYn2\ cYn3\ n1^4\ n2\ n3-cYn1\ cYn3\ n1^3\ n2^2\ n3-cYn2\ cYn3\ n1^2\ n2^3\ n3+cYn1\ cYn3\ n1\ n2^4\ n3+cYn2^2\ n1^4\ n3^2-cYn1\ cYn2\ n1^3\ n2\ n3^2-cYn1\ cYn2\ n1\ n2^3\ n3^2+cYn1^2\ n2^4\ n3^2-cYn2\ cYn3\ n1^2\ n2\ n3^3-cYn1\ cYn3\ n1\ n2^2\ n3^3+cYn2^2\ n1^2\ n3^4+cYn1\ cYn2\ n1\ n2\ n3^4+cYn1^2\ n2^2\ n3^4+n1^2\ n2^2\ n3^2\ pY1\right)\Big\}$$

```
μs1x3P0A3 = Factor[CoefficientList[Sn0 z^2 ((a0 + 1) z)^3 n1^-3 n2^-3 n3^-3 N01^-1 N02^-1 z^-5, 1]]
```

$$\left\{\frac{a0^3\ Sn0}{N01\ N02\ n1^3\ n2^3\ n3^3},\ \frac{3\ a0^2\ Sn0}{N01\ N02\ n1^3\ n2^3\ n3^3},\ \frac{3\ a0\ Sn0}{N01\ N02\ n1^3\ n2^3\ n3^3},\ \frac{Sn0}{N01\ N02\ n1^3\ n2^3\ n3^3}\right\}$$

```
μs1x3P1A3 = Factor[CoefficientList[- Sn1 z^2 ((a1 + 1) z)^3 n1^-3 n2^-3 n3^-3 N01^-1 N12^-1 z^-5, 1]]
```

$$\left\{-\frac{a1^3\ Sn1}{N01\ n1^3\ N12\ n2^3\ n3^3},\ -\frac{3\ a1^2\ Sn1}{N01\ n1^3\ N12\ n2^3\ n3^3},\ -\frac{3\ a1\ Sn1}{N01\ n1^3\ N12\ n2^3\ n3^3},\ -\frac{Sn1}{N01\ n1^3\ N12\ n2^3\ n3^3}\right\}$$

```
μs1x3P2A3 = Factor[CoefficientList[- Sn2 z^2 ((a2 + 1) z)^3 n1^-3 n2^-3 n3^-3 N02^-1 N12^-1 z^-5, 1]]
```

$$\left\{-\frac{a2^3\ Sn2}{N02\ n1^3\ N12\ n2^3\ n3^3},\ -\frac{3\ a2^2\ Sn2}{N02\ n1^3\ N12\ n2^3\ n3^3},\ -\frac{3\ a2\ Sn2}{N02\ n1^3\ N12\ n2^3\ n3^3},\ -\frac{Sn2}{N02\ n1^3\ N12\ n2^3\ n3^3}\right\}$$

```
μx5YA3 + μx5P0A3 + μx5P1A3 + μx5P2A3
```

$$\Big\{\frac{a0^5}{N01\ N02\ n1^3\ n2^3\ n3^3}-\frac{a1^5}{N01\ n1^3\ N12\ n2^3\ n3^3}-\frac{a2^5}{N02\ n1^3\ N12\ n2^3\ n3^3},$$
$$\frac{5\ a0^4}{N01\ N02\ n1^3\ n2^3\ n3^3}-\frac{5\ a1^4}{N01\ n1^3\ N12\ n2^3\ n3^3}-\frac{5\ a2^4}{N02\ n1^3\ N12\ n2^3\ n3^3},$$
$$\frac{10\ a0^3}{N01\ N02\ n1^3\ n2^3\ n3^3}-\frac{10\ a1^3}{N01\ n1^3\ N12\ n2^3\ n3^3}-\frac{10\ a2^3}{N02\ n1^3\ N12\ n2^3\ n3^3},$$
$$\frac{10\ a0^2}{N01\ N02\ n1^3\ n2^3\ n3^3}-\frac{10\ a1^2}{N01\ n1^3\ N12\ n2^3\ n3^3}-\frac{10\ a2^2}{N02\ n1^3\ N12\ n2^3\ n3^3}+\frac{10\ rY^2}{n1\ n2\ n3},$$
$$\frac{5\ a0}{N01\ N02\ n1^3\ n2^3\ n3^3}-\frac{5\ a1}{N01\ n1^3\ N12\ n2^3\ n3^3}-\frac{5\ a2}{N02\ n1^3\ N12\ n2^3\ n3^3}-\frac{5\left(cYn3\ n1\ n2+cYn2\ n1\ n3+cYn1\ n2\ n3\right) rY}{n1^2\ n2^2\ n3^2},$$
$$\frac{1}{N01\ N02\ n1^3\ n2^3\ n3^3}-\frac{1}{N01\ n1^3\ N12\ n2^3\ n3^3}-\frac{1}{N02\ n1^3\ N12\ n2^3\ n3^3}+\frac{1}{n1^3\ n2^3\ n3^3}\left(cYn3^2\ n1^2\ n2^2+cYn2\ cYn3\ n1^2\ n2\ n3+cYn1\ cYn3\ n1\ n2^2\ n3+cYn2^2\ n1^2\ n3^2+cYn1\ cYn2\ n1\ n2\ n3^2+cYn1^2\ n2^2\ n3^2\right)\Big\}$$

```
Factor[μs1x3YA3 + μs1x3P0A3 + μs1x3P1A3 + μs1x3P2A3]
```

$$\left\{\frac{a0^3\,N12\,Sn0 - a1^3\,N02\,Sn1 - a2^3\,N01\,Sn2}{N01\,N02\,n1^3\,N12\,n2^3\,n3^3},\right.$$

$$\frac{1}{N01\,N02\,n1^3\,N12\,n2^3\,n3^3}\,3\,\big(N01\,N02\,n1^4\,N12\,n2^2\,n3^2\,rY^2 + N01\,N02\,n1^2\,N12\,n2^4\,n3^2\,rY^2 + N01\,N02\,n1^2\,N12\,n2^2\,n3^4\,rY^2 + a0^2\,N12\,Sn0 - a1^2\,N02\,Sn1 - a2^2\,N01\,Sn2\big),$$

$$-\frac{1}{N01\,N02\,n1^3\,N12\,n2^3\,n3^3}\,3\,\big(cYn3\,N01\,N02\,n1^4\,N12\,n2^2\,n3\,rY + cYn3\,N01\,N02\,n1^2\,N12\,n2^4\,n3\,rY + cYn2\,N01\,N02\,n1^4\,N12\,n2\,n3^2\,rY - cYn1\,N01\,N02\,n1^3\,N12\,n2^2\,n3^2\,rY - cYn2\,N01\,N02\,n1^2\,N12\,n2^3\,n3^2\,rY + cYn1\,N01\,N02\,n1\,N12\,n2^4\,n3^2\,rY - cYn3\,N01\,N02\,n1^2\,N12\,n2^2\,n3^3\,rY + cYn2\,N01\,N02\,n1^2\,N12\,n2\,n3^4\,rY + cYn1\,N01\,N02\,n1\,N12\,n2^2\,n3^4\,rY - a0\,N12\,Sn0 + a1\,N02\,Sn1 + a2\,N01\,Sn2\big),$$

$$\frac{1}{N01\,N02\,n1^3\,N12\,n2^3\,n3^3}\,\big(cYn3^2\,N01\,N02\,n1^4\,N12\,n2^2 + cYn3^2\,N01\,N02\,n1^2\,N12\,n2^4 + cYn2\,cYn3\,N01\,N02\,n1^4\,N12\,n2\,n3 - cYn1\,cYn3\,N01\,N02\,n1^3\,N12\,n2^2\,n3 - cYn2\,cYn3\,N01\,N02\,n1^2\,N12\,n2^3\,n3 + cYn1\,cYn3\,N01\,N02\,n1\,N12\,n2^4\,n3 + cYn2^2\,N01\,N02\,n1^4\,N12\,n3^2 - cYn1\,cYn2\,N01\,N02\,n1^3\,N12\,n2\,n3^2 - cYn1\,cYn2\,N01\,N02\,n1\,N12\,n2^3\,n3^2 + cYn1^2\,N01\,N02\,N12\,n2^4\,n3^2 - cYn2\,cYn3\,N01\,N02\,n1^2\,N12\,n2\,n3^3 - cYn1\,cYn3\,N01\,N02\,n1\,N12\,n2^2\,n3^3 + cYn2^2\,N01\,N02\,n1^2\,N12\,n3^4 + cYn1\,cYn2\,N01\,N02\,n1\,N12\,n2\,n3^4 + cYn1^2\,N01\,N02\,N12\,n2^2\,n3^4 + N01\,N02\,n1^2\,N12\,n2^2\,n3^2\,pY1 + N12\,Sn0 - N02\,Sn1 - N01\,Sn2\big)\Big\}$$

```
Factor[Solve[{
  d == a0^5/(N01 N02 n1^3 n2^3 n3^3) - a1^5/(N01 n1^3 N12 n2^3 n3^3) - a2^5/(N02 n1^3 N12 n2^3 n3^3),
  0 == 5 a0^4/(N01 N02 n1^3 n2^3 n3^3) - 5 a1^4/(N01 n1^3 N12 n2^3 n3^3) - 5 a2^4/(N02 n1^3 N12 n2^3 n3^3),
  0 == 10 a0^3/(N01 N02 n1^3 n2^3 n3^3) - 10 a1^3/(N01 n1^3 N12 n2^3 n3^3) - 10 a2^3/(N02 n1^3 N12 n2^3 n3^3),
  0 == 10 a0^2/(N01 N02 n1^3 n2^3 n3^3) - 10 a1^2/(N01 n1^3 N12 n2^3 n3^3) - 10 a2^2/(N02 n1^3 N12 n2^3 n3^3) + 10 rY^2/(n1 n2 n3),
  0 == 5 a0/(N01 N02 n1^3 n2^3 n3^3) - 5 a1/(N01 n1^3 N12 n2^3 n3^3) - 5 a2/(N02 n1^3 N12 n2^3 n3^3) -
    5 (cYn3 n1 n2 + cYn2 n1 n3 + cYn1 n2 n3) rY/(n1^2 n2^2 n3^2),
  0 == 1/(N01 N02 n1^3 n2^3 n3^3) - 1/(N01 n1^3 N12 n2^3 n3^3) - 1/(N02 n1^3 N12 n2^3 n3^3) +
    1/(n1^3 n2^3 n3^3) (cYn3^2 n1^2 n2^2 + cYn2 cYn3 n1^2 n2 n3 + cYn1 cYn3 n1 n2^2 n3 +
      cYn2^2 n1^2 n3^2 + cYn1 cYn2 n1 n2 n3^2 + cYn1^2 n2^2 n3^2),
  p d == (a0^3 N12 Sn0 - a1^3 N02 Sn1 - a2^3 N01 Sn2)/(N01 N02 n1^3 N12 n2^3 n3^3),
  0 == 1/(N01 N02 n1^3 N12 n2^3 n3^3)
    3 (N01 N02 n1^4 N12 n2^2 n3^2 rY^2 + N01 N02 n1^2 N12 n2^4 n3^2 rY^2 + N01 N02 n1^2 N12 n2^2 n3^4 rY^2 +
      a0^2 N12 Sn0 - a1^2 N02 Sn1 - a2^2 N01 Sn2),
  0 == -1/(N01 N02 n1^3 N12 n2^3 n3^3) 3
    (cYn3 N01 N02 n1^4 N12 n2^2 n3 rY + cYn3 N01 N02 n1^2 N12 n2^4 n3 rY +
      cYn2 N01 N02 n1^4 N12 n2 n3^2 rY - cYn1 N01 N02 n1^3 N12 n2^2 n3^2 rY -
      cYn2 N01 N02 n1^2 N12 n2^3 n3^2 rY + cYn1 N01 N02 n1 N12 n2^4 n3^2 rY -
      cYn3 N01 N02 n1^2 N12 n2^2 n3^3 rY + cYn2 N01 N02 n1^2 N12 n2 n3^4 rY +
      cYn1 N01 N02 n1 N12 n2^2 n3^4 rY - a0 N12 Sn0 + a1 N02 Sn1 + a2 N01 Sn2),
  0 == 1/(N01 N02 n1^3 N12 n2^3 n3^3)
    (cYn3^2 N01 N02 n1^4 N12 n2^2 + cYn3^2 N01 N02 n1^2 N12 n2^4 + cYn2 cYn3 N01 N02 n1^4 N12 n2 n3 -
      cYn1 cYn3 N01 N02 n1^3 N12 n2^2 n3 - cYn2 cYn3 N01 N02 n1^2 N12 n2^3 n3 +
      cYn1 cYn3 N01 N02 n1 N12 n2^4 n3 + cYn2^2 N01 N02 n1^4 N12 n3^2 -
      cYn1 cYn2 N01 N02 n1^3 N12 n2 n3^2 - cYn1 cYn2 N01 N02 n1 N12 n2^3 n3^2 +
      cYn1^2 N01 N02 N12 n2^4 n3^2 - cYn2 cYn3 N01 N02 n1^2 N12 n2 n3^3 -
      cYn1 cYn3 N01 N02 n1 N12 n2^2 n3^3 + cYn2^2 N01 N02 n1^2 N12 n3^4 +
      cYn1 cYn2 N01 N02 n1 N12 n2 n3^4 + cYn1^2 N01 N02 N12 n2^2 n3^4 +
      N01 N02 n1^2 N12 n2^2 n3^2 pY1 + N12 Sn0 - N02 Sn1 - N01 Sn2),
  pY1 == 3
 }, {p, d, N01, N02, N12, Sn0, Sn1, Sn2, pY1, n1, n2}]]
```

$$\Big\{\Big\{p \to \frac{3 + cYn1^2 + cYn2^2 + cYn3^2}{rY^2},$$

$$d \to -\frac{1}{cYn1\ cYn2\ n3^3} rY^2 \left(a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY + a1\ a2\ cYn3\ n3\ rY + a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2\right),$$

$$N01 \to -\frac{1}{a2^3\ cYn1\ cYn2\ n3^3\ rY} (a0 - a1) \left(a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY + a1\ a2\ cYn3\ n3\ rY + a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2\right),$$

$$N02 \to \frac{1}{a1^3\ cYn1\ cYn2\ n3^3\ rY} (a0 - a2) \left(a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY +\right.$$
$$\left. a0\ a2\ cYn3\ n3\ rY + a1\ a2\ cYn3\ n3\ rY + a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2\right),$$
$$N12 \to \frac{1}{a0^3\ cYn1\ cYn2\ n3^3\ rY} (a1 - a2) \left(a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY +\right.$$
$$\left. a0\ a2\ cYn3\ n3\ rY + a1\ a2\ cYn3\ n3\ rY + a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2\right),$$
$$Sn0 \to \left(6\ a0^4\ a1^2\ a2^2\ cYn1\ cYn3^4 + 2\ a0^4\ a1^2\ a2^2\ cYn1^3\ cYn3^4 + 2\ a0^4\ a1^2\ a2^2\ cYn1\ cYn2^2\ cYn3^4 +\right.$$
$$2\ a0^4\ a1^2\ a2^2\ cYn1\ cYn3^6 + 12\ a0^4\ a1^2\ a2\ cYn1\ cYn3^3\ n3\ rY + 12\ a0^4\ a1\ a2^2\ cYn1\ cYn3^3\ n3\ rY +$$
$$12\ a0^3\ a1^2\ a2^2\ cYn1\ cYn3^3\ n3\ rY + 4\ a0^4\ a1^2\ a2\ cYn1^3\ cYn3^3\ n3\ rY +$$
$$4\ a0^4\ a1\ a2^2\ cYn1^3\ cYn3^3\ n3\ rY + 2\ a0^3\ a1^2\ a2^2\ cYn1^3\ cYn3^3\ n3\ rY +$$
$$4\ a0^4\ a1^2\ a2\ cYn1\ cYn2^2\ cYn3^3\ n3\ rY + 4\ a0^4\ a1\ a2^2\ cYn1\ cYn2^2\ cYn3^3\ n3\ rY +$$
$$2\ a0^3\ a1^2\ a2^2\ cYn1\ cYn2^2\ cYn3^3\ n3\ rY + 4\ a0^4\ a1^2\ a2\ cYn1\ cYn3^5\ n3\ rY +$$
$$4\ a0^4\ a1\ a2^2\ cYn1\ cYn3^5\ n3\ rY + 8\ a0^3\ a1^2\ a2^2\ cYn1\ cYn3^5\ n3\ rY +$$
$$6\ a0^4\ a1^2\ cYn1\ cYn3^2\ n3^2\ rY^2 + 24\ a0^4\ a1\ a2\ cYn1\ cYn3^2\ n3^2\ rY^2 +$$
$$24\ a0^3\ a1^2\ a2\ cYn1\ cYn3^2\ n3^2\ rY^2 + 6\ a0^4\ a2^2\ cYn1\ cYn3^2\ n3^2\ rY^2 +$$
$$24\ a0^3\ a1\ a2^2\ cYn1\ cYn3^2\ n3^2\ rY^2 + 6\ a0^2\ a1^2\ a2^2\ cYn1\ cYn3^2\ n3^2\ rY^2 +$$
$$2\ a0^4\ a1^2\ cYn1^3\ cYn3^2\ n3^2\ rY^2 + 8\ a0^4\ a1\ a2\ cYn1^3\ cYn3^2\ n3^2\ rY^2 +$$
$$4\ a0^3\ a1^2\ a2\ cYn1^3\ cYn3^2\ n3^2\ rY^2 + 2\ a0^4\ a2^2\ cYn1^3\ cYn3^2\ n3^2\ rY^2 +$$
$$4\ a0^3\ a1\ a2^2\ cYn1^3\ cYn3^2\ n3^2\ rY^2 - 3\ a0^2\ a1^2\ a2^2\ cYn1^3\ cYn3^2\ n3^2\ rY^2 +$$
$$2\ a0^4\ a1^2\ cYn1\ cYn2^2\ cYn3^2\ n3^2\ rY^2 + 8\ a0^4\ a1\ a2\ cYn1\ cYn2^2\ cYn3^2\ n3^2\ rY^2 +$$
$$4\ a0^3\ a1^2\ a2\ cYn1\ cYn2^2\ cYn3^2\ n3^2\ rY^2 + 2\ a0^4\ a2^2\ cYn1\ cYn2^2\ cYn3^2\ n3^2\ rY^2 +$$
$$4\ a0^3\ a1\ a2^2\ cYn1\ cYn2^2\ cYn3^2\ n3^2\ rY^2 - 3\ a0^2\ a1^2\ a2^2\ cYn1\ cYn2^2\ cYn3^2\ n3^2\ rY^2 +$$
$$2\ a0^4\ a1^2\ cYn1\ cYn3^4\ n3^2\ rY^2 + 8\ a0^4\ a1\ a2\ cYn1\ cYn3^4\ n3^2\ rY^2 +$$
$$16\ a0^3\ a1^2\ a2\ cYn1\ cYn3^4\ n3^2\ rY^2 + 2\ a0^4\ a2^2\ cYn1\ cYn3^4\ n3^2\ rY^2 +$$
$$16\ a0^3\ a1\ a2^2\ cYn1\ cYn3^4\ n3^2\ rY^2 + 12\ a0^2\ a1^2\ a2^2\ cYn1\ cYn3^4\ n3^2\ rY^2 +$$
$$12\ a0^4\ a1\ cYn1\ cYn3\ n3^3\ rY^3 + 12\ a0^3\ a1^2\ cYn1\ cYn3\ n3^3\ rY^3 + 12\ a0^4\ a2\ cYn1\ cYn3\ n3^3\ rY^3 +$$
$$36\ a0^3\ a1\ a2\ cYn1\ cYn3\ n3^3\ rY^3 + 12\ a0^2\ a1^2\ a2\ cYn1\ cYn3\ n3^3\ rY^3 +$$
$$12\ a0^3\ a2^2\ cYn1\ cYn3\ n3^3\ rY^3 + 12\ a0^2\ a1\ a2^2\ cYn1\ cYn3\ n3^3\ rY^3 +$$
$$4\ a0^4\ a1\ cYn1^3\ cYn3\ n3^3\ rY^3 + 2\ a0^3\ a1^2\ cYn1^3\ cYn3\ n3^3\ rY^3 + 4\ a0^4\ a2\ cYn1^3\ cYn3\ n3^3\ rY^3 +$$
$$6\ a0^3\ a1\ a2\ cYn1^3\ cYn3\ n3^3\ rY^3 - 2\ a0^2\ a1^2\ a2\ cYn1^3\ cYn3\ n3^3\ rY^3 +$$
$$2\ a0^3\ a2^2\ cYn1^3\ cYn3\ n3^3\ rY^3 - 2\ a0^2\ a1\ a2^2\ cYn1^3\ cYn3\ n3^3\ rY^3 -$$
$$2\ a0\ a1^2\ a2^2\ cYn1^3\ cYn3\ n3^3\ rY^3 + 4\ a0^4\ a1\ cYn1\ cYn2^2\ cYn3\ n3^3\ rY^3 +$$
$$2\ a0^3\ a1^2\ cYn1\ cYn2^2\ cYn3\ n3^3\ rY^3 + 4\ a0^4\ a2\ cYn1\ cYn2^2\ cYn3\ n3^3\ rY^3 +$$
$$6\ a0^3\ a1\ a2\ cYn1\ cYn2^2\ cYn3\ n3^3\ rY^3 - 2\ a0^2\ a1^2\ a2\ cYn1\ cYn2^2\ cYn3\ n3^3\ rY^3 +$$
$$2\ a0^3\ a2^2\ cYn1\ cYn2^2\ cYn3\ n3^3\ rY^3 - 2\ a0^2\ a1\ a2^2\ cYn1\ cYn2^2\ cYn3\ n3^3\ rY^3 -$$
$$2\ a0\ a1^2\ a2^2\ cYn1\ cYn2^2\ cYn3\ n3^3\ rY^3 + 4\ a0^4\ a1\ cYn1\ cYn3^3\ n3^3\ rY^3 +$$
$$8\ a0^3\ a1^2\ cYn1\ cYn3^3\ n3^3\ rY^3 + 4\ a0^4\ a2\ cYn1\ cYn3^3\ n3^3\ rY^3 + 28\ a0^3\ a1\ a2\ cYn1\ cYn3^3\ n3^3\ rY^3 +$$
$$24\ a0^2\ a1^2\ a2\ cYn1\ cYn3^3\ n3^3\ rY^3 + 8\ a0^3\ a2^2\ cYn1\ cYn3^3\ n3^3\ rY^3 +$$
$$24\ a0^2\ a1\ a2^2\ cYn1\ cYn3^3\ n3^3\ rY^3 + 8\ a0\ a1^2\ a2^2\ cYn1\ cYn3^3\ n3^3\ rY^3 + 6\ a0^4\ cYn1\ n3^4\ rY^4 +$$
$$12\ a0^3\ a1\ cYn1\ n3^4\ rY^4 + 6\ a0^2\ a1^2\ cYn1\ n3^4\ rY^4 + 12\ a0^3\ a2\ cYn1\ n3^4\ rY^4 +$$
$$12\ a0^2\ a1\ a2\ cYn1\ n3^4\ rY^4 + 6\ a0^2\ a2^2\ cYn1\ n3^4\ rY^4 + 2\ a0^4\ cYn1^3\ n3^4\ rY^4 +$$
$$2\ a0^3\ a1\ cYn1^3\ n3^4\ rY^4 + a0^2\ a1^2\ cYn1^3\ n3^4\ rY^4 + 2\ a0^3\ a2\ cYn1^3\ n3^4\ rY^4 -$$
$$2\ a0^2\ a1\ a2\ cYn1^3\ n3^4\ rY^4 - 2\ a0\ a1^2\ a2\ cYn1^3\ n3^4\ rY^4 + a0^2\ a2^2\ cYn1^3\ n3^4\ rY^4 -$$
$$2\ a0\ a1\ a2^2\ cYn1^3\ n3^4\ rY^4 + a1^2\ a2^2\ cYn1^3\ n3^4\ rY^4 + 2\ a0^4\ cYn1\ cYn2^2\ n3^4\ rY^4 +$$
$$2\ a0^3\ a1\ cYn1\ cYn2^2\ n3^4\ rY^4 + a0^2\ a1^2\ cYn1\ cYn2^2\ n3^4\ rY^4 + 2\ a0^3\ a2\ cYn1\ cYn2^2\ n3^4\ rY^4 -$$
$$2\ a0^2\ a1\ a2\ cYn1\ cYn2^2\ n3^4\ rY^4 - 2\ a0\ a1^2\ a2\ cYn1\ cYn2^2\ n3^4\ rY^4 + a0^2\ a2^2\ cYn1\ cYn2^2\ n3^4\ rY^4 -$$
$$2\ a0\ a1\ a2^2\ cYn1\ cYn2^2\ n3^4\ rY^4 + a1^2\ a2^2\ cYn1\ cYn2^2\ n3^4\ rY^4 + 2\ a0^4\ cYn1\ cYn3^2\ n3^4\ rY^4 +$$
$$12\ a0^3\ a1\ cYn1\ cYn3^2\ n3^4\ rY^4 + 12\ a0^2\ a1^2\ cYn1\ cYn3^2\ n3^4\ rY^4 + 12\ a0^3\ a2\ cYn1\ cYn3^2\ n3^4\ rY^4 +$$
$$36\ a0^2\ a1\ a2\ cYn1\ cYn3^2\ n3^4\ rY^4 + 16\ a0\ a1^2\ a2\ cYn1\ cYn3^2\ n3^4\ rY^4 +$$
$$12\ a0^2\ a2^2\ cYn1\ cYn3^2\ n3^4\ rY^4 + 16\ a0\ a1\ a2^2\ cYn1\ cYn3^2\ n3^4\ rY^4 +$$
$$2\ a1^2\ a2^2\ cYn1\ cYn3^2\ n3^4\ rY^4 + 4\ a0^3\ cYn1\ cYn3\ n3^5\ rY^5 + 12\ a0^2\ a1\ cYn1\ cYn3\ n3^5\ rY^5 +$$
$$8\ a0\ a1^2\ cYn1\ cYn3\ n3^5\ rY^5 + 12\ a0^2\ a2\ cYn1\ cYn3\ n3^5\ rY^5 + 20\ a0\ a1\ a2\ cYn1\ cYn3\ n3^5\ rY^5 +$$
$$4\ a1^2\ a2\ cYn1\ cYn3\ n3^5\ rY^5 + 8\ a0\ a2^2\ cYn1\ cYn3\ n3^5\ rY^5 + 4\ a1\ a2^2\ cYn1\ cYn3\ n3^5\ rY^5 +$$
$$2\ a0^2\ cYn1\ n3^6\ rY^6 + 4\ a0\ a1\ cYn1\ n3^6\ rY^6 + 2\ a1^2\ cYn1\ n3^6\ rY^6 + 4\ a0\ a2\ cYn1\ n3^6\ rY^6 +$$
$$4\ a1\ a2\ cYn1\ n3^6\ rY^6 + 2\ a2^2\ cYn1\ n3^6\ rY^6 - 2\ a0^2\ a1\ a2\ cYn1^2\ cYn3^2\ rY$$
$$\sqrt{}\ \left(\left(a0\ a1\ a2\ cYn1\ cYn3\ n3 + a0\ a1\ cYn1\ n3^2\ rY + a0\ a2\ cYn1\ n3^2\ rY + a1\ a2\ cYn1\ n3^2\ rY\right)^2 -\right.$$

$$4 a0 a1 a2 cYn1^2 n3^2 \big(a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY +$$
$$a1 a2 cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2\big)\big) + 2 a0^2 a1 a2 cYn2^2 cYn3^2$$
$$rY \surd\big(\big(a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY\big)^2 -$$
$$4 a0 a1 a2 cYn1^2 n3^2 \big(a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY + a1 a2$$
$$cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2\big)\big) - 2 a0^2 a1 cYn1^2 cYn3 n3 rY^2$$
$$\surd\big(\big(a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY\big)^2 -$$
$$4 a0 a1 a2 cYn1^2 n3^2 \big(a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY + a1 a2$$
$$cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2\big)\big) - 2 a0^2 a2 cYn1^2 cYn3 n3 rY^2$$
$$\surd\big(\big(a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY\big)^2 -$$
$$4 a0 a1 a2 cYn1^2 n3^2 \big(a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY + a1 a2$$
$$cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2\big)\big) - a0 a1 a2 cYn1^2 cYn3 n3 rY^2$$
$$\surd\big(\big(a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY\big)^2 -$$
$$4 a0 a1 a2 cYn1^2 n3^2 \big(a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY + a1 a2$$
$$cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2\big)\big) + 2 a0^2 a1 cYn2^2 cYn3 n3 rY^2$$
$$\surd\big(\big(a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY\big)^2 -$$
$$4 a0 a1 a2 cYn1^2 n3^2 \big(a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY + a1 a2$$
$$cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2\big)\big) + 2 a0^2 a2 cYn2^2 cYn3 n3 rY^2$$
$$\surd\big(\big(a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY\big)^2 -$$
$$4 a0 a1 a2 cYn1^2 n3^2 \big(a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY + a1 a2$$
$$cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2\big)\big) + a0 a1 a2 cYn2^2 cYn3 n3 rY^2$$
$$\surd\big(\big(a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY\big)^2 -$$
$$4 a0 a1 a2 cYn1^2 n3^2 \big(a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY +$$
$$a1 a2 cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2\big)\big) - 2 a0^2 cYn1^2 n3^2 rY^3$$
$$\surd\big(\big(a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY\big)^2 -$$
$$4 a0 a1 a2 cYn1^2 n3^2 \big(a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY +$$
$$a1 a2 cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2\big)\big) - a0 a1 cYn1^2 n3^2 rY^3$$
$$\surd\big(\big(a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY\big)^2 -$$
$$4 a0 a1 a2 cYn1^2 n3^2 \big(a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY +$$
$$a1 a2 cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2\big)\big) - a0 a2 cYn1^2 n3^2 rY^3$$
$$\surd\big(\big(a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY\big)^2 -$$
$$4 a0 a1 a2 cYn1^2 n3^2 \big(a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY +$$
$$a1 a2 cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2\big)\big) + a1 a2 cYn1^2 n3^2 rY^3$$
$$\surd\big(\big(a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY\big)^2 -$$
$$4 a0 a1 a2 cYn1^2 n3^2 \big(a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY +$$
$$a1 a2 cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2\big)\big) + 2 a0^2 cYn2^2 n3^2 rY^3$$
$$\surd\big(\big(a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY\big)^2 -$$
$$4 a0 a1 a2 cYn1^2 n3^2 \big(a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY +$$
$$a1 a2 cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2\big)\big) + a0 a1 cYn2^2 n3^2 rY^3$$
$$\surd\big(\big(a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY\big)^2 -$$
$$4 a0 a1 a2 cYn1^2 n3^2 \big(a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY +$$
$$a1 a2 cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2\big)\big) + a0 a2 cYn2^2 n3^2 rY^3$$
$$\surd\big(\big(a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY\big)^2 -$$
$$4 a0 a1 a2 cYn1^2 n3^2 \big(a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY +$$
$$a1 a2 cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2\big)\big) - a1 a2 cYn2^2 n3^2 rY^3$$
$$\surd\big(\big(a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY\big)^2 -$$
$$4 a0 a1 a2 cYn1^2 n3^2 \big(a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY +$$
$$a1 a2 cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2\big)\big)\big) \big/$$
$$\big(2 cYn1 rY^2 \big(a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY + a1 a2 cYn3 n3 rY +$$
$$a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2\big)^2\big),$$

$$
\begin{aligned}
&\mathrm{Sn1} \to \big(6\, a0^2\, a1^4\, a2^2\, cYn1\, cYn3^4 + 2\, a0^2\, a1^4\, a2^2\, cYn1^3\, cYn3^4 + 2\, a0^2\, a1^4\, a2^2\, cYn1\, cYn2^2\, cYn3^4 + \\
&\quad 2\, a0^2\, a1^4\, a2^2\, cYn1\, cYn3^6 + 12\, a0^2\, a1^4\, a2\, cYn1\, cYn3^3\, n3\, rY + \\
&\quad 12\, a0^2\, a1^3\, a2^2\, cYn1\, cYn3^3\, n3\, rY + 12\, a0\, a1^4\, a2^2\, cYn1\, cYn3^3\, n3\, rY + \\
&\quad 4\, a0^2\, a1^4\, a2\, cYn1^3\, cYn3^3\, n3\, rY + 2\, a0^2\, a1^3\, a2^2\, cYn1^3\, cYn3^3\, n3\, rY + \\
&\quad 4\, a0\, a1^4\, a2^2\, cYn1^3\, cYn3^3\, n3\, rY + 4\, a0^2\, a1^4\, a2\, cYn1\, cYn2^2\, cYn3^3\, n3\, rY + \\
&\quad 2\, a0^2\, a1^3\, a2^2\, cYn1\, cYn2^2\, cYn3^3\, n3\, rY + 4\, a0\, a1^4\, a2^2\, cYn1\, cYn2^2\, cYn3^3\, n3\, rY + \\
&\quad 4\, a0^2\, a1^4\, a2\, cYn1\, cYn3^5\, n3\, rY + 8\, a0^2\, a1^3\, a2^2\, cYn1\, cYn3^5\, n3\, rY + \\
&\quad 4\, a0\, a1^4\, a2^2\, cYn1\, cYn3^5\, n3\, rY + 6\, a0^2\, a1^4\, cYn1\, cYn3^2\, n3^2\, rY^2 + \\
&\quad 24\, a0^2\, a1^3\, a2\, cYn1\, cYn3^2\, n3^2\, rY^2 + 24\, a0\, a1^4\, a2\, cYn1\, cYn3^2\, n3^2\, rY^2 + \\
&\quad 6\, a0^2\, a1^2\, a2^2\, cYn1\, cYn3^2\, n3^2\, rY^2 + 24\, a0\, a1^3\, a2^2\, cYn1\, cYn3^2\, n3^2\, rY^2 + \\
&\quad 6\, a1^4\, a2^2\, cYn1\, cYn3^2\, n3^2\, rY^2 + 2\, a0^2\, a1^4\, cYn1^3\, cYn3^2\, n3^2\, rY^2 + \\
&\quad 4\, a0^2\, a1^3\, a2\, cYn1^3\, cYn3^2\, n3^2\, rY^2 + 8\, a0\, a1^4\, a2\, cYn1^3\, cYn3^2\, n3^2\, rY^2 - \\
&\quad 3\, a0^2\, a1^2\, a2^2\, cYn1^3\, cYn3^2\, n3^2\, rY^2 + 4\, a0\, a1^3\, a2^2\, cYn1^3\, cYn3^2\, n3^2\, rY^2 + \\
&\quad 2\, a1^4\, a2^2\, cYn1^3\, cYn3^2\, n3^2\, rY^2 + 2\, a0^2\, a1^4\, cYn1\, cYn2^2\, cYn3^2\, n3^2\, rY^2 + \\
&\quad 4\, a0^2\, a1^3\, a2\, cYn1\, cYn2^2\, cYn3^2\, n3^2\, rY^2 + 8\, a0\, a1^4\, a2\, cYn1\, cYn2^2\, cYn3^2\, n3^2\, rY^2 - \\
&\quad 3\, a0^2\, a1^2\, a2^2\, cYn1\, cYn2^2\, cYn3^2\, n3^2\, rY^2 + 4\, a0\, a1^3\, a2^2\, cYn1\, cYn2^2\, cYn3^2\, n3^2\, rY^2 + \\
&\quad 2\, a1^4\, a2^2\, cYn1\, cYn2^2\, cYn3^2\, n3^2\, rY^2 + 2\, a0^2\, a1^4\, cYn1\, cYn3^4\, n3^2\, rY^2 + \\
&\quad 16\, a0^2\, a1^3\, a2\, cYn1\, cYn3^4\, n3^2\, rY^2 + 8\, a0\, a1^4\, a2\, cYn1\, cYn3^4\, n3^2\, rY^2 + \\
&\quad 12\, a0^2\, a1^2\, a2^2\, cYn1\, cYn3^4\, n3^2\, rY^2 + 16\, a0\, a1^3\, a2^2\, cYn1\, cYn3^4\, n3^2\, rY^2 + \\
&\quad 2\, a1^4\, a2^2\, cYn1\, cYn3^4\, n3^2\, rY^2 + 12\, a0^2\, a1^3\, cYn1\, cYn3\, n3^3\, rY^3 + \\
&\quad 12\, a0\, a1^4\, cYn1\, cYn3\, n3^3\, rY^3 + 12\, a0^2\, a1^2\, a2\, cYn1\, cYn3\, n3^3\, rY^3 + \\
&\quad 36\, a0\, a1^3\, a2\, cYn1\, cYn3\, n3^3\, rY^3 + 12\, a1^4\, a2\, cYn1\, cYn3\, n3^3\, rY^3 + \\
&\quad 12\, a0\, a1^2\, a2^2\, cYn1\, cYn3\, n3^3\, rY^3 + 12\, a1^3\, a2^2\, cYn1\, cYn3\, n3^3\, rY^3 + \\
&\quad 2\, a0^2\, a1^3\, cYn1^3\, cYn3\, n3^3\, rY^3 + 4\, a0\, a1^4\, cYn1^3\, cYn3\, n3^3\, rY^3 - \\
&\quad 2\, a0^2\, a1^2\, a2\, cYn1^3\, cYn3\, n3^3\, rY^3 + 6\, a0\, a1^3\, a2\, cYn1^3\, cYn3\, n3^3\, rY^3 + \\
&\quad 4\, a1^4\, a2\, cYn1^3\, cYn3\, n3^3\, rY^3 - 2\, a0^2\, a1\, a2^2\, cYn1^3\, cYn3\, n3^3\, rY^3 - \\
&\quad 2\, a0\, a1^2\, a2^2\, cYn1^3\, cYn3\, n3^3\, rY^3 + 2\, a1^3\, a2^2\, cYn1^3\, cYn3\, n3^3\, rY^3 + \\
&\quad 2\, a0^2\, a1^3\, cYn1\, cYn2^2\, cYn3\, n3^3\, rY^3 + 4\, a0\, a1^4\, cYn1\, cYn2^2\, cYn3\, n3^3\, rY^3 - \\
&\quad 2\, a0^2\, a1^2\, a2\, cYn1\, cYn2^2\, cYn3\, n3^3\, rY^3 + 6\, a0\, a1^3\, a2\, cYn1\, cYn2^2\, cYn3\, n3^3\, rY^3 + \\
&\quad 4\, a1^4\, a2\, cYn1\, cYn2^2\, cYn3\, n3^3\, rY^3 - 2\, a0^2\, a1\, a2^2\, cYn1\, cYn2^2\, cYn3\, n3^3\, rY^3 - \\
&\quad 2\, a0\, a1^2\, a2^2\, cYn1\, cYn2^2\, cYn3\, n3^3\, rY^3 + 2\, a1^3\, a2^2\, cYn1\, cYn2^2\, cYn3\, n3^3\, rY^3 + \\
&\quad 8\, a0^2\, a1^3\, cYn1\, cYn3^3\, n3^3\, rY^3 + 4\, a0\, a1^4\, cYn1\, cYn3^3\, n3^3\, rY^3 + \\
&\quad 24\, a0^2\, a1^2\, a2\, cYn1\, cYn3^3\, n3^3\, rY^3 + 28\, a0\, a1^3\, a2\, cYn1\, cYn3^3\, n3^3\, rY^3 + \\
&\quad 4\, a1^4\, a2\, cYn1\, cYn3^3\, n3^3\, rY^3 + 8\, a0^2\, a1\, a2^2\, cYn1\, cYn3^3\, n3^3\, rY^3 + \\
&\quad 24\, a0\, a1^2\, a2^2\, cYn1\, cYn3^3\, n3^3\, rY^3 + 8\, a1^3\, a2^2\, cYn1\, cYn3^3\, n3^3\, rY^3 + \\
&\quad 6\, a0^2\, a1^2\, cYn1\, n3^4\, rY^4 + 12\, a0\, a1^3\, cYn1\, n3^4\, rY^4 + 6\, a1^4\, cYn1\, n3^4\, rY^4 + \\
&\quad 12\, a0\, a1^2\, a2\, cYn1\, n3^4\, rY^4 + 12\, a1^3\, a2\, cYn1\, n3^4\, rY^4 + 6\, a1^2\, a2^2\, cYn1\, n3^4\, rY^4 + \\
&\quad a0^2\, a1^2\, cYn1^3\, n3^4\, rY^4 + 2\, a0\, a1^3\, cYn1^3\, n3^4\, rY^4 + 2\, a1^4\, cYn1^3\, n3^4\, rY^4 - \\
&\quad 2\, a0^2\, a1\, a2\, cYn1^3\, n3^4\, rY^4 - 2\, a0\, a1^2\, a2\, cYn1^3\, n3^4\, rY^4 + 2\, a1^3\, a2\, cYn1^3\, n3^4\, rY^4 + \\
&\quad a0^2\, a2^2\, cYn1^3\, n3^4\, rY^4 - 2\, a0\, a1\, a2^2\, cYn1^3\, n3^4\, rY^4 + a1^2\, a2^2\, cYn1^3\, n3^4\, rY^4 + \\
&\quad a0^2\, a1^2\, cYn1\, cYn2^2\, n3^4\, rY^4 + 2\, a0\, a1^3\, cYn1\, cYn2^2\, n3^4\, rY^4 + \\
&\quad 2\, a1^4\, cYn1\, cYn2^2\, n3^4\, rY^4 - 2\, a0^2\, a1\, a2\, cYn1\, cYn2^2\, n3^4\, rY^4 - \\
&\quad 2\, a0\, a1^2\, a2\, cYn1\, cYn2^2\, n3^4\, rY^4 + 2\, a1^3\, a2\, cYn1\, cYn2^2\, n3^4\, rY^4 + \\
&\quad a0^2\, a2^2\, cYn1\, cYn2^2\, n3^4\, rY^4 - 2\, a0\, a1\, a2^2\, cYn1\, cYn2^2\, n3^4\, rY^4 + \\
&\quad a1^2\, a2^2\, cYn1\, cYn2^2\, n3^4\, rY^4 + 12\, a0^2\, a1^2\, cYn1\, cYn3^2\, n3^4\, rY^4 + \\
&\quad 12\, a0\, a1^3\, cYn1\, cYn3^2\, n3^4\, rY^4 + 2\, a1^4\, cYn1\, cYn3^2\, n3^4\, rY^4 + \\
&\quad 16\, a0^2\, a1\, a2\, cYn1\, cYn3^2\, n3^4\, rY^4 + 36\, a0\, a1^2\, a2\, cYn1\, cYn3^2\, n3^4\, rY^4 + \\
&\quad 12\, a1^3\, a2\, cYn1\, cYn3^2\, n3^4\, rY^4 + 2\, a0^2\, a2^2\, cYn1\, cYn3^2\, n3^4\, rY^4 + \\
&\quad 16\, a0\, a1\, a2^2\, cYn1\, cYn3^2\, n3^4\, rY^4 + 12\, a1^2\, a2^2\, cYn1\, cYn3^2\, n3^4\, rY^4 + \\
&\quad 8\, a0^2\, a1\, cYn1\, cYn3\, n3^5\, rY^5 + 12\, a0\, a1^2\, cYn1\, cYn3\, n3^5\, rY^5 + \\
&\quad 4\, a1^3\, cYn1\, cYn3\, n3^5\, rY^5 + 4\, a0^2\, a2\, cYn1\, cYn3\, n3^5\, rY^5 + \\
&\quad 20\, a0\, a1\, a2\, cYn1\, cYn3\, n3^5\, rY^5 + 12\, a1^2\, a2\, cYn1\, cYn3\, n3^5\, rY^5 + \\
&\quad 4\, a0\, a2^2\, cYn1\, cYn3\, n3^5\, rY^5 + 8\, a1\, a2^2\, cYn1\, cYn3\, n3^5\, rY^5 + 2\, a0^2\, cYn1\, n3^6\, rY^6 + \\
&\quad 4\, a0\, a1\, cYn1\, n3^6\, rY^6 + 2\, a1^2\, cYn1\, n3^6\, rY^6 + 4\, a0\, a2\, cYn1\, n3^6\, rY^6 + \\
&\quad 4\, a1\, a2\, cYn1\, n3^6\, rY^6 + 2\, a2^2\, cYn1\, n3^6\, rY^6 - 2\, a0\, a1^2\, a2\, cYn1^2\, cYn3^2\, rY \\
&\qquad \surd\Big(\big(\big(a0\, a1\, a2\, cYn1\, cYn3\, n3 + a0\, a1\, cYn1\, n3^2\, rY + a0\, a2\, cYn1\, n3^2\, rY + a1\, a2\, cYn1\, n3^2\, rY\big)^2 - \\
&\qquad\quad 4\, a0\, a1\, a2\, cYn1^2\, n3^2\, \big(a0\, a1\, a2\, cYn3^2 + a0\, a1\, cYn3\, n3\, rY + a0\, a2\, cYn3\, n3\, rY +
\end{aligned}
$$

$$
\begin{aligned}
&a1\ a2\ cYn3\ n3\ rY + a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2))) + 2\ a0\ a1^2\ a2\ cYn2^2\ cYn3^2\\
&rY \surd((a0\ a1\ a2\ cYn1\ cYn3\ n3 + a0\ a1\ cYn1\ n3^2\ rY + a0\ a2\ cYn1\ n3^2\ rY + a1\ a2\ cYn1\ n3^2\ rY)^2 -\\
&4\ a0\ a1\ a2\ cYn1^2\ n3^2\ (a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY + a1\ a2\\
&cYn3\ n3\ rY + a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2)) - 2\ a0\ a1^2\ cYn1^2\ cYn3\ n3\ rY^2\\
&\surd((a0\ a1\ a2\ cYn1\ cYn3\ n3 + a0\ a1\ cYn1\ n3^2\ rY + a0\ a2\ cYn1\ n3^2\ rY + a1\ a2\ cYn1\ n3^2\ rY)^2 -\\
&4\ a0\ a1\ a2\ cYn1^2\ n3^2\ (a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY + a1\ a2\\
&cYn3\ n3\ rY + a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2)) - a0\ a1\ a2\ cYn1^2\ cYn3\ n3\ rY^2\\
&\surd((a0\ a1\ a2\ cYn1\ cYn3\ n3 + a0\ a1\ cYn1\ n3^2\ rY + a0\ a2\ cYn1\ n3^2\ rY + a1\ a2\ cYn1\ n3^2\ rY)^2 -\\
&4\ a0\ a1\ a2\ cYn1^2\ n3^2\ (a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY + a1\ a2\\
&cYn3\ n3\ rY + a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2)) - 2\ a1^2\ a2\ cYn1^2\ cYn3\ n3\ rY^2\\
&\surd((a0\ a1\ a2\ cYn1\ cYn3\ n3 + a0\ a1\ cYn1\ n3^2\ rY + a0\ a2\ cYn1\ n3^2\ rY + a1\ a2\ cYn1\ n3^2\ rY)^2 -\\
&4\ a0\ a1\ a2\ cYn1^2\ n3^2\ (a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY + a1\ a2\\
&cYn3\ n3\ rY + a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2)) + 2\ a0\ a1^2\ cYn2^2\ cYn3\ n3\ rY^2\\
&\surd((a0\ a1\ a2\ cYn1\ cYn3\ n3 + a0\ a1\ cYn1\ n3^2\ rY + a0\ a2\ cYn1\ n3^2\ rY + a1\ a2\ cYn1\ n3^2\ rY)^2 -\\
&4\ a0\ a1\ a2\ cYn1^2\ n3^2\ (a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY + a1\ a2\\
&cYn3\ n3\ rY + a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2)) + a0\ a1\ a2\ cYn2^2\ cYn3\ n3\ rY^2\\
&\surd((a0\ a1\ a2\ cYn1\ cYn3\ n3 + a0\ a1\ cYn1\ n3^2\ rY + a0\ a2\ cYn1\ n3^2\ rY + a1\ a2\ cYn1\ n3^2\ rY)^2 -\\
&4\ a0\ a1\ a2\ cYn1^2\ n3^2\ (a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY + a1\ a2\\
&cYn3\ n3\ rY + a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2)) + 2\ a1^2\ a2\ cYn2^2\ cYn3\ n3\ rY^2\\
&\surd((a0\ a1\ a2\ cYn1\ cYn3\ n3 + a0\ a1\ cYn1\ n3^2\ rY + a0\ a2\ cYn1\ n3^2\ rY + a1\ a2\ cYn1\ n3^2\ rY)^2 -\\
&4\ a0\ a1\ a2\ cYn1^2\ n3^2\ (a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY +\\
&a1\ a2\ cYn3\ n3\ rY + a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2)) - a0\ a1\ cYn1^2\ n3^2\ rY^3\\
&\surd((a0\ a1\ a2\ cYn1\ cYn3\ n3 + a0\ a1\ cYn1\ n3^2\ rY + a0\ a2\ cYn1\ n3^2\ rY + a1\ a2\ cYn1\ n3^2\ rY)^2 -\\
&4\ a0\ a1\ a2\ cYn1^2\ n3^2\ (a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY +\\
&a1\ a2\ cYn3\ n3\ rY + a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2)) - 2\ a1^2\ cYn1^2\ n3^2\ rY^3\\
&\surd((a0\ a1\ a2\ cYn1\ cYn3\ n3 + a0\ a1\ cYn1\ n3^2\ rY + a0\ a2\ cYn1\ n3^2\ rY + a1\ a2\ cYn1\ n3^2\ rY)^2 -\\
&4\ a0\ a1\ a2\ cYn1^2\ n3^2\ (a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY +\\
&a1\ a2\ cYn3\ n3\ rY + a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2)) + a0\ a2\ cYn1^2\ n3^2\ rY^3\\
&\surd((a0\ a1\ a2\ cYn1\ cYn3\ n3 + a0\ a1\ cYn1\ n3^2\ rY + a0\ a2\ cYn1\ n3^2\ rY + a1\ a2\ cYn1\ n3^2\ rY)^2 -\\
&4\ a0\ a1\ a2\ cYn1^2\ n3^2\ (a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY +\\
&a1\ a2\ cYn3\ n3\ rY + a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2)) - a1\ a2\ cYn1^2\ n3^2\ rY^3\\
&\surd((a0\ a1\ a2\ cYn1\ cYn3\ n3 + a0\ a1\ cYn1\ n3^2\ rY + a0\ a2\ cYn1\ n3^2\ rY + a1\ a2\ cYn1\ n3^2\ rY)^2 -\\
&4\ a0\ a1\ a2\ cYn1^2\ n3^2\ (a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY +\\
&a1\ a2\ cYn3\ n3\ rY + a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2)) + a0\ a1\ cYn2^2\ n3^2\ rY^3\\
&\surd((a0\ a1\ a2\ cYn1\ cYn3\ n3 + a0\ a1\ cYn1\ n3^2\ rY + a0\ a2\ cYn1\ n3^2\ rY + a1\ a2\ cYn1\ n3^2\ rY)^2 -\\
&4\ a0\ a1\ a2\ cYn1^2\ n3^2\ (a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY +\\
&a1\ a2\ cYn3\ n3\ rY + a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2)) + 2\ a1^2\ cYn2^2\ n3^2\ rY^3\\
&\surd((a0\ a1\ a2\ cYn1\ cYn3\ n3 + a0\ a1\ cYn1\ n3^2\ rY + a0\ a2\ cYn1\ n3^2\ rY + a1\ a2\ cYn1\ n3^2\ rY)^2 -\\
&4\ a0\ a1\ a2\ cYn1^2\ n3^2\ (a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY +\\
&a1\ a2\ cYn3\ n3\ rY + a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2)) - a0\ a2\ cYn2^2\ n3^2\ rY^3\\
&\surd((a0\ a1\ a2\ cYn1\ cYn3\ n3 + a0\ a1\ cYn1\ n3^2\ rY + a0\ a2\ cYn1\ n3^2\ rY + a1\ a2\ cYn1\ n3^2\ rY)^2 -\\
&4\ a0\ a1\ a2\ cYn1^2\ n3^2\ (a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY +\\
&a1\ a2\ cYn3\ n3\ rY + a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2)) + a1\ a2\ cYn2^2\ n3^2\ rY^3\\
&\surd((a0\ a1\ a2\ cYn1\ cYn3\ n3 + a0\ a1\ cYn1\ n3^2\ rY + a0\ a2\ cYn1\ n3^2\ rY + a1\ a2\ cYn1\ n3^2\ rY)^2 -\\
&4\ a0\ a1\ a2\ cYn1^2\ n3^2\ (a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY +\\
&a1\ a2\ cYn3\ n3\ rY + a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2)))\ /\\
&(2\ cYn1\ rY^2\ (a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY + a1\ a2\ cYn3\ n3\ rY +\\
&a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2)^2),\\
&Sn2 \to (6\ a0^2\ a1^2\ a2^4\ cYn1\ cYn3^4 + 2\ a0^2\ a1^2\ a2^4\ cYn1^3\ cYn3^4 + 2\ a0^2\ a1^2\ a2^4\ cYn1\ cYn2^2\ cYn3^4 +
\end{aligned}
$$

$2\ a0^2\ a1^2\ a2^4\ cYn1\ cYn3^6 + 12\ a0^2\ a1^2\ a2^3\ cYn1\ cYn3^3\ n3\ rY +$
$12\ a0^2\ a1\ a2^4\ cYn1\ cYn3^3\ n3\ rY + 12\ a0\ a1^2\ a2^4\ cYn1\ cYn3^3\ n3\ rY +$
$2\ a0^2\ a1^2\ a2^3\ cYn1^3\ cYn3^3\ n3\ rY + 4\ a0^2\ a1\ a2^4\ cYn1^3\ cYn3^3\ n3\ rY +$
$4\ a0\ a1^2\ a2^4\ cYn1^3\ cYn3^3\ n3\ rY + 2\ a0^2\ a1^2\ a2^3\ cYn1\ cYn2^2\ cYn3^3\ n3\ rY +$
$4\ a0^2\ a1\ a2^4\ cYn1\ cYn2^2\ cYn3^3\ n3\ rY +$
$4\ a0\ a1^2\ a2^4\ cYn1\ cYn2^2\ cYn3^3\ n3\ rY + 8\ a0^2\ a1^2\ a2^3\ cYn1\ cYn3^5\ n3\ rY +$
$4\ a0^2\ a1\ a2^4\ cYn1\ cYn3^5\ n3\ rY + 4\ a0\ a1^2\ a2^4\ cYn1\ cYn3^5\ n3\ rY +$
$6\ a0^2\ a1^2\ a2^2\ cYn1\ cYn3^2\ n3^2\ rY^2 + 24\ a0^2\ a1\ a2^3\ cYn1\ cYn3^2\ n3^2\ rY^2 +$
$24\ a0\ a1^2\ a2^3\ cYn1\ cYn3^2\ n3^2\ rY^2 + 6\ a0^2\ a2^4\ cYn1\ cYn3^2\ n3^2\ rY^2 +$
$24\ a0\ a1\ a2^4\ cYn1\ cYn3^2\ n3^2\ rY^2 + 6\ a1^2\ a2^4\ cYn1\ cYn3^2\ n3^2\ rY^2 -$
$3\ a0^2\ a1^2\ a2^2\ cYn1^3\ cYn3^2\ n3^2\ rY^2 + 4\ a0^2\ a1\ a2^3\ cYn1^3\ cYn3^2\ n3^2\ rY^2 +$
$4\ a0\ a1^2\ a2^3\ cYn1^3\ cYn3^2\ n3^2\ rY^2 + 2\ a0^2\ a2^4\ cYn1^3\ cYn3^2\ n3^2\ rY^2 +$
$8\ a0\ a1\ a2^4\ cYn1^3\ cYn3^2\ n3^2\ rY^2 + 2\ a1^2\ a2^4\ cYn1^3\ cYn3^2\ n3^2\ rY^2 -$
$3\ a0^2\ a1^2\ a2^2\ cYn1\ cYn2^2\ cYn3^2\ n3^2\ rY^2 +$
$4\ a0^2\ a1\ a2^3\ cYn1\ cYn2^2\ cYn3^2\ n3^2\ rY^2 +$
$4\ a0\ a1^2\ a2^3\ cYn1\ cYn2^2\ cYn3^2\ n3^2\ rY^2 + 2\ a0^2\ a2^4\ cYn1\ cYn2^2\ cYn3^2\ n3^2\ rY^2 +$
$8\ a0\ a1\ a2^4\ cYn1\ cYn2^2\ cYn3^2\ n3^2\ rY^2 + 2\ a1^2\ a2^4\ cYn1\ cYn2^2\ cYn3^2\ n3^2\ rY^2 +$
$12\ a0^2\ a1^2\ a2^2\ cYn1\ cYn3^4\ n3^2\ rY^2 + 16\ a0^2\ a1\ a2^3\ cYn1\ cYn3^4\ n3^2\ rY^2 +$
$16\ a0\ a1^2\ a2^3\ cYn1\ cYn3^4\ n3^2\ rY^2 + 2\ a0^2\ a2^4\ cYn1\ cYn3^4\ n3^2\ rY^2 +$
$8\ a0\ a1\ a2^4\ cYn1\ cYn3^4\ n3^2\ rY^2 + 2\ a1^2\ a2^4\ cYn1\ cYn3^4\ n3^2\ rY^2 +$
$12\ a0^2\ a1\ a2^2\ cYn1\ cYn3\ n3^3\ rY^3 + 12\ a0\ a1^2\ a2^2\ cYn1\ cYn3\ n3^3\ rY^3 +$
$12\ a0^2\ a2^3\ cYn1\ cYn3\ n3^3\ rY^3 + 36\ a0\ a1\ a2^3\ cYn1\ cYn3\ n3^3\ rY^3 +$
$12\ a1^2\ a2^3\ cYn1\ cYn3\ n3^3\ rY^3 + 12\ a0\ a2^4\ cYn1\ cYn3\ n3^3\ rY^3 +$
$12\ a1\ a2^4\ cYn1\ cYn3\ n3^3\ rY^3 - 2\ a0^2\ a1^2\ a2\ cYn1^3\ cYn3\ n3^3\ rY^3 -$
$2\ a0^2\ a1\ a2^2\ cYn1^3\ cYn3\ n3^3\ rY^3 - 2\ a0\ a1^2\ a2^2\ cYn1^3\ cYn3\ n3^3\ rY^3 +$
$2\ a0^2\ a2^3\ cYn1^3\ cYn3\ n3^3\ rY^3 + 6\ a0\ a1\ a2^3\ cYn1^3\ cYn3\ n3^3\ rY^3 +$
$2\ a1^2\ a2^3\ cYn1^3\ cYn3\ n3^3\ rY^3 + 4\ a0\ a2^4\ cYn1^3\ cYn3\ n3^3\ rY^3 +$
$4\ a1\ a2^4\ cYn1^3\ cYn3\ n3^3\ rY^3 - 2\ a0^2\ a1^2\ a2\ cYn1\ cYn2^2\ cYn3\ n3^3\ rY^3 -$
$2\ a0^2\ a1\ a2^2\ cYn1\ cYn2^2\ cYn3\ n3^3\ rY^3 - 2\ a0\ a1^2\ a2^2\ cYn1\ cYn2^2\ cYn3\ n3^3\ rY^3 +$
$2\ a0^2\ a2^3\ cYn1\ cYn2^2\ cYn3\ n3^3\ rY^3 + 6\ a0\ a1\ a2^3\ cYn1\ cYn2^2\ cYn3\ n3^3\ rY^3 +$
$2\ a1^2\ a2^3\ cYn1\ cYn2^2\ cYn3\ n3^3\ rY^3 + 4\ a0\ a2^4\ cYn1\ cYn2^2\ cYn3\ n3^3\ rY^3 +$
$4\ a1\ a2^4\ cYn1\ cYn2^2\ cYn3\ n3^3\ rY^3 + 8\ a0^2\ a1^2\ a2\ cYn1\ cYn3^3\ n3^3\ rY^3 +$
$24\ a0^2\ a1\ a2^2\ cYn1\ cYn3^3\ n3^3\ rY^3 + 24\ a0\ a1^2\ a2^2\ cYn1\ cYn3^3\ n3^3\ rY^3 +$
$8\ a0^2\ a2^3\ cYn1\ cYn3^3\ n3^3\ rY^3 + 28\ a0\ a1\ a2^3\ cYn1\ cYn3^3\ n3^3\ rY^3 +$
$8\ a1^2\ a2^3\ cYn1\ cYn3^3\ n3^3\ rY^3 + 4\ a0\ a2^4\ cYn1\ cYn3^3\ n3^3\ rY^3 +$
$4\ a1\ a2^4\ cYn1\ cYn3^3\ n3^3\ rY^3 + 6\ a0^2\ a2^2\ cYn1\ n3^4\ rY^4 +$
$12\ a0\ a1\ a2^2\ cYn1\ n3^4\ rY^4 + 6\ a1^2\ a2^2\ cYn1\ n3^4\ rY^4 +$
$12\ a0\ a2^3\ cYn1\ n3^4\ rY^4 + 12\ a1\ a2^3\ cYn1\ n3^4\ rY^4 + 6\ a2^4\ cYn1\ n3^4\ rY^4 +$
$a0^2\ a1^2\ cYn1^3\ n3^4\ rY^4 - 2\ a0^2\ a1\ a2\ cYn1^3\ n3^4\ rY^4 -$
$2\ a0\ a1^2\ a2\ cYn1^3\ n3^4\ rY^4 + a0^2\ a2^2\ cYn1^3\ n3^4\ rY^4 -$
$2\ a0\ a1\ a2^2\ cYn1^3\ n3^4\ rY^4 + a1^2\ a2^2\ cYn1^3\ n3^4\ rY^4 +$
$2\ a0\ a2^3\ cYn1^3\ n3^4\ rY^4 + 2\ a1\ a2^3\ cYn1^3\ n3^4\ rY^4 + 2\ a2^4\ cYn1^3\ n3^4\ rY^4 +$
$a0^2\ a1^2\ cYn1\ cYn2^2\ n3^4\ rY^4 - 2\ a0^2\ a1\ a2\ cYn1\ cYn2^2\ n3^4\ rY^4 -$
$2\ a0\ a1^2\ a2\ cYn1\ cYn2^2\ n3^4\ rY^4 + a0^2\ a2^2\ cYn1\ cYn2^2\ n3^4\ rY^4 -$
$2\ a0\ a1\ a2^2\ cYn1\ cYn2^2\ n3^4\ rY^4 + a1^2\ a2^2\ cYn1\ cYn2^2\ n3^4\ rY^4 +$
$2\ a0\ a2^3\ cYn1\ cYn2^2\ n3^4\ rY^4 + 2\ a1\ a2^3\ cYn1\ cYn2^2\ n3^4\ rY^4 +$
$2\ a2^4\ cYn1\ cYn2^2\ n3^4\ rY^4 + 2\ a0^2\ a1^2\ cYn1\ cYn3^2\ n3^4\ rY^4 +$
$16\ a0^2\ a1\ a2\ cYn1\ cYn3^2\ n3^4\ rY^4 + 16\ a0\ a1^2\ a2\ cYn1\ cYn3^2\ n3^4\ rY^4 +$
$12\ a0^2\ a2^2\ cYn1\ cYn3^2\ n3^4\ rY^4 + 36\ a0\ a1\ a2^2\ cYn1\ cYn3^2\ n3^4\ rY^4 +$
$12\ a1^2\ a2^2\ cYn1\ cYn3^2\ n3^4\ rY^4 + 12\ a0\ a2^3\ cYn1\ cYn3^2\ n3^4\ rY^4 +$
$12\ a1\ a2^3\ cYn1\ cYn3^2\ n3^4\ rY^4 + 2\ a2^4\ cYn1\ cYn3^2\ n3^4\ rY^4 +$
$4\ a0^2\ a1\ cYn1\ cYn3\ n3^5\ rY^5 + 4\ a0\ a1^2\ cYn1\ cYn3\ n3^5\ rY^5 +$
$8\ a0^2\ a2\ cYn1\ cYn3\ n3^5\ rY^5 + 20\ a0\ a1\ a2\ cYn1\ cYn3\ n3^5\ rY^5 +$
$8\ a1^2\ a2\ cYn1\ cYn3\ n3^5\ rY^5 + 12\ a0\ a2^2\ cYn1\ cYn3\ n3^5\ rY^5 +$
$12\ a1\ a2^2\ cYn1\ cYn3\ n3^5\ rY^5 + 4\ a2^3\ cYn1\ cYn3\ n3^5\ rY^5 +$
$2\ a0^2\ cYn1\ n3^6\ rY^6 + 4\ a0\ a1\ cYn1\ n3^6\ rY^6 + 2\ a1^2\ cYn1\ n3^6\ rY^6 +$
$4\ a0\ a2\ cYn1\ n3^6\ rY^6 + 4\ a1\ a2\ cYn1\ n3^6\ rY^6 +$

$$2\,a2^2\,cYn1\,n3^6\,rY^6 - 2\,a0\,a1\,a2^2\,cYn1^2\,cYn3^2\,rY\sqrt{\left(\left(a0\,a1\,a2\,cYn1\,cYn3\,n3 + a0\,a1\,cYn1\,n3^2\,rY + a0\,a2\,cYn1\,n3^2\,rY + a1\,a2\,cYn1\,n3^2\,rY\right)^2 - 4\,a0\,a1\,a2\,cYn1^2\,n3^2\left(a0\,a1\,a2\,cYn3^2 + a0\,a1\,cYn3\,n3\,rY + a0\,a2\,cYn3\,n3\,rY + a1\,a2\,cYn3\,n3\,rY + a0\,n3^2\,rY^2 + a1\,n3^2\,rY^2 + a2\,n3^2\,rY^2\right)\right)}$$

$$+ 2\,a0\,a1\,a2^2\,cYn2^2\,cYn3^2\,rY\sqrt{\left(\left(a0\,a1\,a2\,cYn1\,cYn3\,n3 + a0\,a1\,cYn1\,n3^2\,rY + a0\,a2\,cYn1\,n3^2\,rY + a1\,a2\,cYn1\,n3^2\,rY\right)^2 - 4\,a0\,a1\,a2\,cYn1^2\,n3^2\left(a0\,a1\,a2\,cYn3^2 + a0\,a1\,cYn3\,n3\,rY + a0\,a2\,cYn3\,n3\,rY + a1\,a2\,cYn3\,n3\,rY + a0\,n3^2\,rY^2 + a1\,n3^2\,rY^2 + a2\,n3^2\,rY^2\right)\right)}$$

$$- a0\,a1\,a2\,cYn1^2\,cYn3\,n3\,rY^2\sqrt{\left(\left(a0\,a1\,a2\,cYn1\,cYn3\,n3 + a0\,a1\,cYn1\,n3^2\,rY + a0\,a2\,cYn1\,n3^2\,rY + a1\,a2\,cYn1\,n3^2\,rY\right)^2 - 4\,a0\,a1\,a2\,cYn1^2\,n3^2\left(a0\,a1\,a2\,cYn3^2 + a0\,a1\,cYn3\,n3\,rY + a0\,a2\,cYn3\,n3\,rY + a1\,a2\,cYn3\,n3\,rY + a0\,n3^2\,rY^2 + a1\,n3^2\,rY^2 + a2\,n3^2\,rY^2\right)\right)}$$

$$- 2\,a0\,a2^2\,cYn1^2\,cYn3\,n3\,rY^2\sqrt{\left(\left(a0\,a1\,a2\,cYn1\,cYn3\,n3 + a0\,a1\,cYn1\,n3^2\,rY + a0\,a2\,cYn1\,n3^2\,rY + a1\,a2\,cYn1\,n3^2\,rY\right)^2 - 4\,a0\,a1\,a2\,cYn1^2\,n3^2\left(a0\,a1\,a2\,cYn3^2 + a0\,a1\,cYn3\,n3\,rY + a0\,a2\,cYn3\,n3\,rY + a1\,a2\,cYn3\,n3\,rY + a0\,n3^2\,rY^2 + a1\,n3^2\,rY^2 + a2\,n3^2\,rY^2\right)\right)}$$

$$- 2\,a1\,a2^2\,cYn1^2\,cYn3\,n3\,rY^2\sqrt{\left(\left(a0\,a1\,a2\,cYn1\,cYn3\,n3 + a0\,a1\,cYn1\,n3^2\,rY + a0\,a2\,cYn1\,n3^2\,rY + a1\,a2\,cYn1\,n3^2\,rY\right)^2 - 4\,a0\,a1\,a2\,cYn1^2\,n3^2\left(a0\,a1\,a2\,cYn3^2 + a0\,a1\,cYn3\,n3\,rY + a0\,a2\,cYn3\,n3\,rY + a1\,a2\,cYn3\,n3\,rY + a0\,n3^2\,rY^2 + a1\,n3^2\,rY^2 + a2\,n3^2\,rY^2\right)\right)}$$

$$+ a0\,a1\,a2\,cYn2^2\,cYn3\,n3\,rY^2\sqrt{\left(\left(a0\,a1\,a2\,cYn1\,cYn3\,n3 + a0\,a1\,cYn1\,n3^2\,rY + a0\,a2\,cYn1\,n3^2\,rY + a1\,a2\,cYn1\,n3^2\,rY\right)^2 - 4\,a0\,a1\,a2\,cYn1^2\,n3^2\left(a0\,a1\,a2\,cYn3^2 + a0\,a1\,cYn3\,n3\,rY + a0\,a2\,cYn3\,n3\,rY + a1\,a2\,cYn3\,n3\,rY + a0\,n3^2\,rY^2 + a1\,n3^2\,rY^2 + a2\,n3^2\,rY^2\right)\right)}$$

$$+ 2\,a0\,a2^2\,cYn2^2\,cYn3\,n3\,rY^2\sqrt{\left(\left(a0\,a1\,a2\,cYn1\,cYn3\,n3 + a0\,a1\,cYn1\,n3^2\,rY + a0\,a2\,cYn1\,n3^2\,rY + a1\,a2\,cYn1\,n3^2\,rY\right)^2 - 4\,a0\,a1\,a2\,cYn1^2\,n3^2\left(a0\,a1\,a2\,cYn3^2 + a0\,a1\,cYn3\,n3\,rY + a0\,a2\,cYn3\,n3\,rY + a1\,a2\,cYn3\,n3\,rY + a0\,n3^2\,rY^2 + a1\,n3^2\,rY^2 + a2\,n3^2\,rY^2\right)\right)}$$

$$+ 2\,a1\,a2^2\,cYn2^2\,cYn3\,n3\,rY^2\sqrt{\left(\left(a0\,a1\,a2\,cYn1\,cYn3\,n3 + a0\,a1\,cYn1\,n3^2\,rY + a0\,a2\,cYn1\,n3^2\,rY + a1\,a2\,cYn1\,n3^2\,rY\right)^2 - 4\,a0\,a1\,a2\,cYn1^2\,n3^2\left(a0\,a1\,a2\,cYn3^2 + a0\,a1\,cYn3\,n3\,rY + a0\,a2\,cYn3\,n3\,rY + a1\,a2\,cYn3\,n3\,rY + a0\,n3^2\,rY^2 + a1\,n3^2\,rY^2 + a2\,n3^2\,rY^2\right)\right)}$$

$$+ a0\,a1\,cYn1^2\,n3^2\,rY^3\sqrt{\left(\left(a0\,a1\,a2\,cYn1\,cYn3\,n3 + a0\,a1\,cYn1\,n3^2\,rY + a0\,a2\,cYn1\,n3^2\,rY + a1\,a2\,cYn1\,n3^2\,rY\right)^2 - 4\,a0\,a1\,a2\,cYn1^2\,n3^2\left(a0\,a1\,a2\,cYn3^2 + a0\,a1\,cYn3\,n3\,rY + a0\,a2\,cYn3\,n3\,rY + a1\,a2\,cYn3\,n3\,rY + a0\,n3^2\,rY^2 + a1\,n3^2\,rY^2 + a2\,n3^2\,rY^2\right)\right)}$$

$$- a0\,a2\,cYn1^2\,n3^2\,rY^3\sqrt{\left(\left(a0\,a1\,a2\,cYn1\,cYn3\,n3 + a0\,a1\,cYn1\,n3^2\,rY + a0\,a2\,cYn1\,n3^2\,rY + a1\,a2\,cYn1\,n3^2\,rY\right)^2 - 4\,a0\,a1\,a2\,cYn1^2\,n3^2\left(a0\,a1\,a2\,cYn3^2 + a0\,a1\,cYn3\,n3\,rY + a0\,a2\,cYn3\,n3\,rY + a1\,a2\,cYn3\,n3\,rY + a0\,n3^2\,rY^2 + a1\,n3^2\,rY^2 + a2\,n3^2\,rY^2\right)\right)}$$

$$- a1\,a2\,cYn1^2\,n3^2\,rY^3\sqrt{\left(\left(a0\,a1\,a2\,cYn1\,cYn3\,n3 + a0\,a1\,cYn1\,n3^2\,rY + a0\,a2\,cYn1\,n3^2\,rY + a1\,a2\,cYn1\,n3^2\,rY\right)^2 - 4\,a0\,a1\,a2\,cYn1^2\,n3^2\left(a0\,a1\,a2\,cYn3^2 + a0\,a1\,cYn3\,n3\,rY + a0\,a2\,cYn3\,n3\,rY + a1\,a2\,cYn3\,n3\,rY + a0\,n3^2\,rY^2 + a1\,n3^2\,rY^2 + a2\,n3^2\,rY^2\right)\right)}$$

$$- 2\,a2^2\,cYn1^2\,n3^2\,rY^3\sqrt{\left(\left(a0\,a1\,a2\,cYn1\,cYn3\,n3 + a0\,a1\,cYn1\,n3^2\,rY + a0\,a2\,cYn1\,n3^2\,rY + a1\,a2\,cYn1\,n3^2\,rY\right)^2 - 4\,a0\,a1\,a2\,cYn1^2\,n3^2\left(a0\,a1\,a2\,cYn3^2 + a0\,a1\,cYn3\,n3\,rY + a0\,a2\,cYn3\,n3\,rY + a1\,a2\,cYn3\,n3\,rY + a0\,n3^2\,rY^2 + a1\,n3^2\,rY^2 + a2\,n3^2\,rY^2\right)\right)}$$

$$- a0\,a1\,cYn2^2\,n3^2\,rY^3\sqrt{\left(\left(a0\,a1\,a2\,cYn1\,cYn3\,n3 + a0\,a1\,cYn1\,n3^2\,rY + a0\,a2\,cYn1\,n3^2\,rY + a1\,a2\,cYn1\,n3^2\,rY\right)^2 - 4\,a0\,a1\,a2\,cYn1^2\,n3^2\left(a0\,a1\,a2\,cYn3^2 + a0\,a1\,cYn3\,n3\,rY + a0\,a2\,cYn3\,n3\,rY + a1\,a2\,cYn3\,n3\,rY + a0\,n3^2\,rY^2 + a1\,n3^2\,rY^2 + a2\,n3^2\,rY^2\right)\right)}$$

$$+ a0\,a2\,cYn2^2\,n3^2\,rY^3\sqrt{\left(\left(a0\,a1\,a2\,cYn1\,cYn3\,n3 + a0\,a1\,cYn1\,n3^2\,rY + a0\,a2\,cYn1\,n3^2\,rY + a1\,a2\,cYn1\,n3^2\,rY\right)^2 - 4\,a0\,a1\,a2\,cYn1^2\,n3^2\left(a0\,a1\,a2\,cYn3^2 + a0\,a1\,cYn3\,n3\,rY + a0\,a2\,cYn3\,n3\,rY + a1\,a2\,cYn3\,n3\,rY + a0\,n3^2\,rY^2 + a1\,n3^2\,rY^2 + a2\,n3^2\,rY^2\right)\right)}$$

$$+ a1\,a2\,cYn2^2\,n3^2\,rY^3\sqrt{\left(\left(a0\,a1\,a2\,cYn1\,cYn3\,n3 + a0\,a1\,cYn1\,n3^2\,rY + a0\,a2\,cYn1\,n3^2\,rY + a1\,a2\,cYn1\,n3^2\,rY\right)^2 - 4\,a0\,a1\,a2\,cYn1^2\,n3^2\left(a0\,a1\,a2\,cYn3^2 + a0\,a1\,cYn3\,n3\,rY + a0\,a2\,cYn3\,n3\,rY + a1\,a2\,cYn3\,n3\,rY + a0\,n3^2\,rY^2 + a1\,n3^2\,rY^2 + a2\,n3^2\,rY^2\right)\right)}$$

$$+ 2\,a2^2\,cYn2^2\,n3^2\,rY^3\sqrt{\left(\left(a0\,a1\,a2\,cYn1\,cYn3\,n3 + a0\,a1\,cYn1\,n3^2\,rY + a0\,a2\,cYn1\,n3^2\,rY + a1\,a2\,cYn1\,n3^2\,rY\right)^2 - 4\,a0\,a1\,a2\,cYn1^2\,n3^2\left(a0\,a1\,a2\,cYn3^2 + a0\,a1\,cYn3\,n3\,rY + a0\,a2\,cYn3\,n3\,rY + a1\,a2\,cYn3\,n3\,rY + a0\,n3^2\,rY^2 + a1\,n3^2\,rY^2 + a2\,n3^2\,rY^2\right)\right)}\Bigg) \Big/$$

$$\left(2\,\text{cYn1}\,\text{rY}^2\,\left(\text{a0 a1 a2 cYn3}^2+\text{a0 a1 cYn3 n3 rY}+\text{a0 a2 cYn3 n3 rY}+\text{a1 a2 cYn3 n3 rY}+\text{a0 n3}^2\,\text{rY}^2+\text{a1 n3}^2\,\text{rY}^2+\text{a2 n3}^2\,\text{rY}^2\right)^2\right),\ \text{pY1}\to 3,$$

$$\text{n1}\to-\Big(\Big(\text{a0 a1 a2 cYn1 cYn3 n3}+\text{a0 a1 cYn1 n3}^2\,\text{rY}+\text{a0 a2 cYn1 n3}^2\,\text{rY}+\text{a1 a2 cYn1 n3}^2\,\text{rY}+\sqrt{\Big(\left(\text{a0 a1 a2 cYn1 cYn3 n3}+\text{a0 a1 cYn1 n3}^2\,\text{rY}+\text{a0 a2 cYn1 n3}^2\,\text{rY}+\text{a1 a2 cYn1 n3}^2\,\text{rY}\right)^2-4\,\text{a0 a1 a2 cYn1}^2\,\text{n3}^2\,\left(\text{a0 a1 a2 cYn3}^2+\text{a0 a1 cYn3 n3 rY}+\text{a0 a2 cYn3 n3 rY}+\text{a1 a2 cYn3 n3 rY}+\text{a0 n3}^2\,\text{rY}^2+\text{a1 n3}^2\,\text{rY}^2+\text{a2 n3}^2\,\text{rY}^2\right)\Big)}\Big)\Big/\left(2\,\left(\text{a0 a1 a2 cYn3}^2+\text{a0 a1 cYn3 n3 rY}+\text{a0 a2 cYn3 n3 rY}+\text{a1 a2 cYn3 n3 rY}+\text{a0 n3}^2\,\text{rY}^2+\text{a1 n3}^2\,\text{rY}^2+\text{a2 n3}^2\,\text{rY}^2\right)\right)\Big),$$

$$\text{n2}\to-\Big(\Big(\text{cYn2}\,\Big(\text{a0 a1 a2 cYn1 cYn3 n3}+\text{a0 a1 cYn1 n3}^2\,\text{rY}+\text{a0 a2 cYn1 n3}^2\,\text{rY}+\text{a1 a2 cYn1 n3}^2\,\text{rY}-\sqrt{\Big(\left(\text{a0 a1 a2 cYn1 cYn3 n3}+\text{a0 a1 cYn1 n3}^2\,\text{rY}+\text{a0 a2 cYn1 n3}^2\,\text{rY}+\text{a1 a2 cYn1 n3}^2\,\text{rY}\right)^2-4\,\text{a0 a1 a2 cYn1}^2\,\text{n3}^2\,\left(\text{a0 a1 a2 cYn3}^2+\text{a0 a1 cYn3 n3 rY}+\text{a0 a2 cYn3 n3 rY}+\text{a1 a2 cYn3 n3 rY}+\text{a0 n3}^2\,\text{rY}^2+\text{a1 n3}^2\,\text{rY}^2+\text{a2 n3}^2\,\text{rY}^2\right)\Big)}\Big)\Big)\Big/\left(2\,\text{cYn1}\,\left(\text{a0 a1 a2 cYn3}^2+\text{a0 a1 cYn3 n3 rY}+\text{a0 a2 cYn3 n3 rY}+\text{a1 a2 cYn3 n3 rY}+\text{a0 n3}^2\,\text{rY}^2+\text{a1 n3}^2\,\text{rY}^2+\text{a2 n3}^2\,\text{rY}^2\right)\right)\Big)\Big\},$$

$$\Big\{\text{p}\to\frac{3+\text{cYn1}^2+\text{cYn2}^2+\text{cYn3}^2}{\text{rY}^2},\ \text{d}\to-\frac{1}{\text{cYn1 cYn2 n3}^3\,\text{rY}^2}\left(\text{a0 a1 a2 cYn3}^2+\text{a0 a1 cYn3 n3 rY}+\text{a0 a2 cYn3 n3 rY}+\text{a1 a2 cYn3 n3 rY}+\text{a0 n3}^2\,\text{rY}^2+\text{a1 n3}^2\,\text{rY}^2+\text{a2 n3}^2\,\text{rY}^2\right),$$

$$\text{N01}\to\frac{1}{\text{a2}^3\,\text{cYn1 cYn2 n3}^3\,\text{rY}}(\text{a0}-\text{a1})\left(\text{a0 a1 a2 cYn3}^2+\text{a0 a1 cYn3 n3 rY}+\text{a0 a2 cYn3 n3 rY}+\text{a1 a2 cYn3 n3 rY}+\text{a0 n3}^2\,\text{rY}^2+\text{a1 n3}^2\,\text{rY}^2+\text{a2 n3}^2\,\text{rY}^2\right),$$

$$\text{N02}\to-\frac{1}{\text{a1}^3\,\text{cYn1 cYn2 n3}^3\,\text{rY}}(\text{a0}-\text{a2})\left(\text{a0 a1 a2 cYn3}^2+\text{a0 a1 cYn3 n3 rY}+\text{a0 a2 cYn3 n3 rY}+\text{a1 a2 cYn3 n3 rY}+\text{a0 n3}^2\,\text{rY}^2+\text{a1 n3}^2\,\text{rY}^2+\text{a2 n3}^2\,\text{rY}^2\right),$$

$$\text{N12}\to-\frac{1}{\text{a0}^3\,\text{cYn1 cYn2 n3}^3\,\text{rY}}(\text{a1}-\text{a2})\left(\text{a0 a1 a2 cYn3}^2+\text{a0 a1 cYn3 n3 rY}+\text{a0 a2 cYn3 n3 rY}+\text{a1 a2 cYn3 n3 rY}+\text{a0 n3}^2\,\text{rY}^2+\text{a1 n3}^2\,\text{rY}^2+\text{a2 n3}^2\,\text{rY}^2\right),$$

$$\begin{aligned}\text{Sn0}\to\Big(&6\,\text{a0}^4\,\text{a1}^2\,\text{a2}^2\,\text{cYn1 cYn3}^4+2\,\text{a0}^4\,\text{a1}^2\,\text{a2}^2\,\text{cYn1}^3\,\text{cYn3}^4+\\
&2\,\text{a0}^4\,\text{a1}^2\,\text{a2}^2\,\text{cYn1 cYn2}^2\,\text{cYn3}^4+2\,\text{a0}^4\,\text{a1}^2\,\text{a2}^2\,\text{cYn1 cYn3}^6+\\
&12\,\text{a0}^4\,\text{a1}^2\,\text{a2 cYn1 cYn3}^3\,\text{n3 rY}+12\,\text{a0}^4\,\text{a1 a2}^2\,\text{cYn1 cYn3}^3\,\text{n3 rY}+\\
&12\,\text{a0}^3\,\text{a1}^2\,\text{a2}^2\,\text{cYn1 cYn3}^3\,\text{n3 rY}+4\,\text{a0}^4\,\text{a1}^2\,\text{a2 cYn1}^3\,\text{cYn3}^3\,\text{n3 rY}+\\
&4\,\text{a0}^4\,\text{a1 a2}^2\,\text{cYn1}^3\,\text{cYn3}^3\,\text{n3 rY}+2\,\text{a0}^3\,\text{a1}^2\,\text{a2}^2\,\text{cYn1}^3\,\text{cYn3}^3\,\text{n3 rY}+\\
&4\,\text{a0}^4\,\text{a1}^2\,\text{a2 cYn1 cYn2}^2\,\text{cYn3}^3\,\text{n3 rY}+\\
&4\,\text{a0}^4\,\text{a1 a2}^2\,\text{cYn1 cYn2}^2\,\text{cYn3}^3\,\text{n3 rY}+\\
&2\,\text{a0}^3\,\text{a1}^2\,\text{a2}^2\,\text{cYn1 cYn2}^2\,\text{cYn3}^3\,\text{n3 rY}+\\
&4\,\text{a0}^4\,\text{a1}^2\,\text{a2 cYn1 cYn3}^5\,\text{n3 rY}+4\,\text{a0}^4\,\text{a1 a2}^2\,\text{cYn1 cYn3}^5\,\text{n3 rY}+\\
&8\,\text{a0}^3\,\text{a1}^2\,\text{a2}^2\,\text{cYn1 cYn3}^5\,\text{n3 rY}+6\,\text{a0}^4\,\text{a1}^2\,\text{cYn1 cYn3}^2\,\text{n3}^2\,\text{rY}^2+\\
&24\,\text{a0}^4\,\text{a1 a2 cYn1 cYn3}^2\,\text{n3}^2\,\text{rY}^2+24\,\text{a0}^3\,\text{a1}^2\,\text{a2 cYn1 cYn3}^2\,\text{n3}^2\,\text{rY}^2+\\
&6\,\text{a0}^4\,\text{a2}^2\,\text{cYn1 cYn3}^2\,\text{n3}^2\,\text{rY}^2+24\,\text{a0}^3\,\text{a1 a2}^2\,\text{cYn1 cYn3}^2\,\text{n3}^2\,\text{rY}^2+\\
&6\,\text{a0}^2\,\text{a1}^2\,\text{a2}^2\,\text{cYn1 cYn3}^2\,\text{n3}^2\,\text{rY}^2+2\,\text{a0}^4\,\text{a1}^2\,\text{cYn1}^3\,\text{cYn3}^2\,\text{n3}^2\,\text{rY}^2+\\
&8\,\text{a0}^4\,\text{a1 a2 cYn1}^3\,\text{cYn3}^2\,\text{n3}^2\,\text{rY}^2+4\,\text{a0}^3\,\text{a1}^2\,\text{a2 cYn1}^3\,\text{cYn3}^2\,\text{n3}^2\,\text{rY}^2+\\
&2\,\text{a0}^4\,\text{a2}^2\,\text{cYn1}^3\,\text{cYn3}^2\,\text{n3}^2\,\text{rY}^2+4\,\text{a0}^3\,\text{a1 a2}^2\,\text{cYn1}^3\,\text{cYn3}^2\,\text{n3}^2\,\text{rY}^2-\\
&3\,\text{a0}^2\,\text{a1}^2\,\text{a2}^2\,\text{cYn1}^3\,\text{cYn3}^2\,\text{n3}^2\,\text{rY}^2+2\,\text{a0}^4\,\text{a1}^2\,\text{cYn1 cYn2}^2\,\text{cYn3}^2\,\text{n3}^2\,\text{rY}^2+\\
&8\,\text{a0}^4\,\text{a1 a2 cYn1 cYn2}^2\,\text{cYn3}^2\,\text{n3}^2\,\text{rY}^2+\\
&4\,\text{a0}^3\,\text{a1}^2\,\text{a2 cYn1 cYn2}^2\,\text{cYn3}^2\,\text{n3}^2\,\text{rY}^2+\\
&2\,\text{a0}^4\,\text{a2}^2\,\text{cYn1 cYn2}^2\,\text{cYn3}^2\,\text{n3}^2\,\text{rY}^2+\\
&4\,\text{a0}^3\,\text{a1 a2}^2\,\text{cYn1 cYn2}^2\,\text{cYn3}^2\,\text{n3}^2\,\text{rY}^2-\\
&3\,\text{a0}^2\,\text{a1}^2\,\text{a2}^2\,\text{cYn1 cYn2}^2\,\text{cYn3}^2\,\text{n3}^2\,\text{rY}^2+\\
&2\,\text{a0}^4\,\text{a1}^2\,\text{cYn1 cYn3}^4\,\text{n3}^2\,\text{rY}^2+8\,\text{a0}^4\,\text{a1 a2 cYn1 cYn3}^4\,\text{n3}^2\,\text{rY}^2+\\
&16\,\text{a0}^3\,\text{a1}^2\,\text{a2 cYn1 cYn3}^4\,\text{n3}^2\,\text{rY}^2+2\,\text{a0}^4\,\text{a2}^2\,\text{cYn1 cYn3}^4\,\text{n3}^2\,\text{rY}^2+\\
&16\,\text{a0}^3\,\text{a1 a2}^2\,\text{cYn1 cYn3}^4\,\text{n3}^2\,\text{rY}^2+12\,\text{a0}^2\,\text{a1}^2\,\text{a2}^2\,\text{cYn1 cYn3}^4\,\text{n3}^2\,\text{rY}^2+\end{aligned}$$

$12\ a0^4\ a1\ cYn1\ cYn3\ n3^3\ rY^3 + 12\ a0^3\ a1^2\ cYn1\ cYn3\ n3^3\ rY^3 +$
$12\ a0^4\ a2\ cYn1\ cYn3\ n3^3\ rY^3 + 36\ a0^3\ a1\ a2\ cYn1\ cYn3\ n3^3\ rY^3 +$
$12\ a0^2\ a1^2\ a2\ cYn1\ cYn3\ n3^3\ rY^3 + 12\ a0^3\ a2^2\ cYn1\ cYn3\ n3^3\ rY^3 +$
$12\ a0^2\ a1\ a2^2\ cYn1\ cYn3\ n3^3\ rY^3 + 4\ a0^4\ a1\ cYn1^3\ cYn3\ n3^3\ rY^3 +$
$2\ a0^3\ a1^2\ cYn1^3\ cYn3\ n3^3\ rY^3 + 4\ a0^4\ a2\ cYn1^3\ cYn3\ n3^3\ rY^3 +$
$6\ a0^3\ a1\ a2\ cYn1^3\ cYn3\ n3^3\ rY^3 - 2\ a0^2\ a1^2\ a2\ cYn1^3\ cYn3\ n3^3\ rY^3 +$
$2\ a0^3\ a2^2\ cYn1^3\ cYn3\ n3^3\ rY^3 - 2\ a0^2\ a1\ a2^2\ cYn1^3\ cYn3\ n3^3\ rY^3 -$
$2\ a0\ a1^2\ a2^2\ cYn1^3\ cYn3\ n3^3\ rY^3 + 4\ a0^4\ a1\ cYn1\ cYn2^2\ cYn3\ n3^3\ rY^3 +$
$2\ a0^3\ a1^2\ cYn1\ cYn2^2\ cYn3\ n3^3\ rY^3 + 4\ a0^4\ a2\ cYn1\ cYn2^2\ cYn3\ n3^3\ rY^3 +$
$6\ a0^3\ a1\ a2\ cYn1\ cYn2^2\ cYn3\ n3^3\ rY^3 - 2\ a0^2\ a1^2\ a2\ cYn1\ cYn2^2\ cYn3\ n3^3\ rY^3 +$
$2\ a0^3\ a2^2\ cYn1\ cYn2^2\ cYn3\ n3^3\ rY^3 - 2\ a0^2\ a1\ a2^2\ cYn1\ cYn2^2\ cYn3\ n3^3\ rY^3 -$
$2\ a0\ a1^2\ a2^2\ cYn1\ cYn2^2\ cYn3\ n3^3\ rY^3 + 4\ a0^4\ a1\ cYn1\ cYn3^3\ n3^3\ rY^3 +$
$8\ a0^3\ a1^2\ cYn1\ cYn3^3\ n3^3\ rY^3 + 4\ a0^4\ a2\ cYn1\ cYn3^3\ n3^3\ rY^3 +$
$28\ a0^3\ a1\ a2\ cYn1\ cYn3^3\ n3^3\ rY^3 + 24\ a0^2\ a1^2\ a2\ cYn1\ cYn3^3\ n3^3\ rY^3 +$
$8\ a0^3\ a2^2\ cYn1\ cYn3^3\ n3^3\ rY^3 + 24\ a0^2\ a1\ a2^2\ cYn1\ cYn3^3\ n3^3\ rY^3 +$
$8\ a0\ a1^2\ a2^2\ cYn1\ cYn3^3\ n3^3\ rY^3 + 6\ a0^4\ cYn1\ n3^4\ rY^4 +$
$12\ a0^3\ a1\ cYn1\ n3^4\ rY^4 + 6\ a0^2\ a1^2\ cYn1\ n3^4\ rY^4 +$
$12\ a0^3\ a2\ cYn1\ n3^4\ rY^4 + 12\ a0^2\ a1\ a2\ cYn1\ n3^4\ rY^4 +$
$6\ a0^2\ a2^2\ cYn1\ n3^4\ rY^4 + 2\ a0^4\ cYn1^3\ n3^4\ rY^4 +$
$2\ a0^3\ a1\ cYn1^3\ n3^4\ rY^4 + a0^2\ a1^2\ cYn1^3\ n3^4\ rY^4 +$
$2\ a0^3\ a2\ cYn1^3\ n3^4\ rY^4 - 2\ a0^2\ a1\ a2\ cYn1^3\ n3^4\ rY^4 -$
$2\ a0\ a1^2\ a2\ cYn1^3\ n3^4\ rY^4 + a0^2\ a2^2\ cYn1^3\ n3^4\ rY^4 -$
$2\ a0\ a1\ a2^2\ cYn1^3\ n3^4\ rY^4 + a1^2\ a2^2\ cYn1^3\ n3^4\ rY^4 +$
$2\ a0^4\ cYn1\ cYn2^2\ n3^4\ rY^4 + 2\ a0^3\ a1\ cYn1\ cYn2^2\ n3^4\ rY^4 +$
$a0^2\ a1^2\ cYn1\ cYn2^2\ n3^4\ rY^4 + 2\ a0^3\ a2\ cYn1\ cYn2^2\ n3^4\ rY^4 -$
$2\ a0^2\ a1\ a2\ cYn1\ cYn2^2\ n3^4\ rY^4 - 2\ a0\ a1^2\ a2\ cYn1\ cYn2^2\ n3^4\ rY^4 +$
$a0^2\ a2^2\ cYn1\ cYn2^2\ n3^4\ rY^4 - 2\ a0\ a1\ a2^2\ cYn1\ cYn2^2\ n3^4\ rY^4 +$
$a1^2\ a2^2\ cYn1\ cYn2^2\ n3^4\ rY^4 + 2\ a0^4\ cYn1\ cYn3^2\ n3^4\ rY^4 +$
$12\ a0^3\ a1\ cYn1\ cYn3^2\ n3^4\ rY^4 + 12\ a0^2\ a1^2\ cYn1\ cYn3^2\ n3^4\ rY^4 +$
$12\ a0^3\ a2\ cYn1\ cYn3^2\ n3^4\ rY^4 + 36\ a0^2\ a1\ a2\ cYn1\ cYn3^2\ n3^4\ rY^4 +$
$16\ a0\ a1^2\ a2\ cYn1\ cYn3^2\ n3^4\ rY^4 + 12\ a0^2\ a2^2\ cYn1\ cYn3^2\ n3^4\ rY^4 +$
$16\ a0\ a1\ a2^2\ cYn1\ cYn3^2\ n3^4\ rY^4 + 2\ a1^2\ a2^2\ cYn1\ cYn3^2\ n3^4\ rY^4 +$
$4\ a0^3\ cYn1\ cYn3\ n3^5\ rY^5 + 12\ a0^2\ a1\ cYn1\ cYn3\ n3^5\ rY^5 +$
$8\ a0\ a1^2\ cYn1\ cYn3\ n3^5\ rY^5 + 12\ a0^2\ a2\ cYn1\ cYn3\ n3^5\ rY^5 +$
$20\ a0\ a1\ a2\ cYn1\ cYn3\ n3^5\ rY^5 + 4\ a1^2\ a2\ cYn1\ cYn3\ n3^5\ rY^5 +$
$8\ a0\ a2^2\ cYn1\ cYn3\ n3^5\ rY^5 + 4\ a1\ a2^2\ cYn1\ cYn3\ n3^5\ rY^5 +$
$2\ a0^2\ cYn1\ n3^6\ rY^6 + 4\ a0\ a1\ cYn1\ n3^6\ rY^6 + 2\ a1^2\ cYn1\ n3^6\ rY^6 +$
$4\ a0\ a2\ cYn1\ n3^6\ rY^6 + 4\ a1\ a2\ cYn1\ n3^6\ rY^6 +$
$2\ a2^2\ cYn1\ n3^6\ rY^6 - 2\ a0^2\ a1\ a2\ cYn1^2\ cYn3^2\ rY$
$\surd\big(\big(a0\ a1\ a2\ cYn1\ cYn3\ n3 + a0\ a1\ cYn1\ n3^2\ rY + a0\ a2\ cYn1\ n3^2\ rY + a1\ a2\ cYn1\ n3^2\ rY\big)^2 -$
$4\ a0\ a1\ a2\ cYn1^2\ n3^2\ \big(a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY +$
$a1\ a2\ cYn3\ n3\ rY + a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2\big)\big) + 2\ a0^2\ a1\ a2\ cYn2^2\ cYn3^2$
$rY\ \surd\big(\big(a0\ a1\ a2\ cYn1\ cYn3\ n3 + a0\ a1\ cYn1\ n3^2\ rY + a0\ a2\ cYn1\ n3^2\ rY + a1\ a2\ cYn1\ n3^2\ rY\big)^2 -$
$4\ a0\ a1\ a2\ cYn1^2\ n3^2\ \big(a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY + a1\ a2$
$cYn3\ n3\ rY + a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2\big)\big) - 2\ a0^2\ a1\ cYn1^2\ cYn3\ n3\ rY^2$
$\surd\big(\big(a0\ a1\ a2\ cYn1\ cYn3\ n3 + a0\ a1\ cYn1\ n3^2\ rY + a0\ a2\ cYn1\ n3^2\ rY + a1\ a2\ cYn1\ n3^2\ rY\big)^2 -$
$4\ a0\ a1\ a2\ cYn1^2\ n3^2\ \big(a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY + a1\ a2$
$cYn3\ n3\ rY + a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2\big)\big) - 2\ a0^2\ a2\ cYn1^2\ cYn3\ n3\ rY^2$
$\surd\big(\big(a0\ a1\ a2\ cYn1\ cYn3\ n3 + a0\ a1\ cYn1\ n3^2\ rY + a0\ a2\ cYn1\ n3^2\ rY + a1\ a2\ cYn1\ n3^2\ rY\big)^2 -$
$4\ a0\ a1\ a2\ cYn1^2\ n3^2\ \big(a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY + a1\ a2$
$cYn3\ n3\ rY + a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2\big)\big) - a0\ a1\ a2\ cYn1^2\ cYn3\ n3\ rY^2$
$\surd\big(\big(a0\ a1\ a2\ cYn1\ cYn3\ n3 + a0\ a1\ cYn1\ n3^2\ rY + a0\ a2\ cYn1\ n3^2\ rY + a1\ a2\ cYn1\ n3^2\ rY\big)^2 -$
$4\ a0\ a1\ a2\ cYn1^2\ n3^2\ \big(a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY + a1\ a2$
$cYn3\ n3\ rY + a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2\big)\big) + 2\ a0^2\ a1\ cYn2^2\ cYn3\ n3\ rY^2$
$\surd\big(\big(a0\ a1\ a2\ cYn1\ cYn3\ n3 + a0\ a1\ cYn1\ n3^2\ rY + a0\ a2\ cYn1\ n3^2\ rY + a1\ a2\ cYn1\ n3^2\ rY\big)^2 -$

$4 a0 a1 a2 cYn1^{2} n3^{2} \left(a0 a1 a2 cYn3^{2} + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY + a1 a2\right.$
$\left.cYn3 n3 rY + a0 n3^{2} rY^{2} + a1 n3^{2} rY^{2} + a2 n3^{2} rY^{2}\right)\Big) + 2 a0^{2} a2 cYn2^{2} cYn3 n3 rY^{2}$
$\sqrt{}\Big(\left(a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^{2} rY + a0 a2 cYn1 n3^{2} rY + a1 a2 cYn1 n3^{2} rY\right)^{2} -$
$4 a0 a1 a2 cYn1^{2} n3^{2} \left(a0 a1 a2 cYn3^{2} + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY + a1 a2\right.$
$\left.cYn3 n3 rY + a0 n3^{2} rY^{2} + a1 n3^{2} rY^{2} + a2 n3^{2} rY^{2}\right)\Big) + a0 a1 a2 cYn2^{2} cYn3 n3 rY^{2}$
$\sqrt{}\Big(\left(a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^{2} rY + a0 a2 cYn1 n3^{2} rY + a1 a2 cYn1 n3^{2} rY\right)^{2} -$
$4 a0 a1 a2 cYn1^{2} n3^{2} \left(a0 a1 a2 cYn3^{2} + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY +\right.$
$\left.a1 a2 cYn3 n3 rY + a0 n3^{2} rY^{2} + a1 n3^{2} rY^{2} + a2 n3^{2} rY^{2}\right)\Big) - 2 a0^{2} cYn1^{2} n3^{2} rY^{3}$
$\sqrt{}\Big(\left(a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^{2} rY + a0 a2 cYn1 n3^{2} rY + a1 a2 cYn1 n3^{2} rY\right)^{2} -$
$4 a0 a1 a2 cYn1^{2} n3^{2} \left(a0 a1 a2 cYn3^{2} + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY +\right.$
$\left.a1 a2 cYn3 n3 rY + a0 n3^{2} rY^{2} + a1 n3^{2} rY^{2} + a2 n3^{2} rY^{2}\right)\Big) - a0 a1 cYn1^{2} n3^{2} rY^{3}$
$\sqrt{}\Big(\left(a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^{2} rY + a0 a2 cYn1 n3^{2} rY + a1 a2 cYn1 n3^{2} rY\right)^{2} -$
$4 a0 a1 a2 cYn1^{2} n3^{2} \left(a0 a1 a2 cYn3^{2} + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY +\right.$
$\left.a1 a2 cYn3 n3 rY + a0 n3^{2} rY^{2} + a1 n3^{2} rY^{2} + a2 n3^{2} rY^{2}\right)\Big) - a0 a2 cYn1^{2} n3^{2} rY^{3}$
$\sqrt{}\Big(\left(a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^{2} rY + a0 a2 cYn1 n3^{2} rY + a1 a2 cYn1 n3^{2} rY\right)^{2} -$
$4 a0 a1 a2 cYn1^{2} n3^{2} \left(a0 a1 a2 cYn3^{2} + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY +\right.$
$\left.a1 a2 cYn3 n3 rY + a0 n3^{2} rY^{2} + a1 n3^{2} rY^{2} + a2 n3^{2} rY^{2}\right)\Big) + a1 a2 cYn1^{2} n3^{2} rY^{3}$
$\sqrt{}\Big(\left(a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^{2} rY + a0 a2 cYn1 n3^{2} rY + a1 a2 cYn1 n3^{2} rY\right)^{2} -$
$4 a0 a1 a2 cYn1^{2} n3^{2} \left(a0 a1 a2 cYn3^{2} + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY +\right.$
$\left.a1 a2 cYn3 n3 rY + a0 n3^{2} rY^{2} + a1 n3^{2} rY^{2} + a2 n3^{2} rY^{2}\right)\Big) + 2 a0^{2} cYn2^{2} n3^{2} rY^{3}$
$\sqrt{}\Big(\left(a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^{2} rY + a0 a2 cYn1 n3^{2} rY + a1 a2 cYn1 n3^{2} rY\right)^{2} -$
$4 a0 a1 a2 cYn1^{2} n3^{2} \left(a0 a1 a2 cYn3^{2} + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY +\right.$
$\left.a1 a2 cYn3 n3 rY + a0 n3^{2} rY^{2} + a1 n3^{2} rY^{2} + a2 n3^{2} rY^{2}\right)\Big) + a0 a1 cYn2^{2} n3^{2} rY^{3}$
$\sqrt{}\Big(\left(a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^{2} rY + a0 a2 cYn1 n3^{2} rY + a1 a2 cYn1 n3^{2} rY\right)^{2} -$
$4 a0 a1 a2 cYn1^{2} n3^{2} \left(a0 a1 a2 cYn3^{2} + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY +\right.$
$\left.a1 a2 cYn3 n3 rY + a0 n3^{2} rY^{2} + a1 n3^{2} rY^{2} + a2 n3^{2} rY^{2}\right)\Big) + a0 a2 cYn2^{2} n3^{2} rY^{3}$
$\sqrt{}\Big(\left(a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^{2} rY + a0 a2 cYn1 n3^{2} rY + a1 a2 cYn1 n3^{2} rY\right)^{2} -$
$4 a0 a1 a2 cYn1^{2} n3^{2} \left(a0 a1 a2 cYn3^{2} + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY +\right.$
$\left.a1 a2 cYn3 n3 rY + a0 n3^{2} rY^{2} + a1 n3^{2} rY^{2} + a2 n3^{2} rY^{2}\right)\Big) - a1 a2 cYn2^{2} n3^{2} rY^{3}$
$\sqrt{}\Big(\left(a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^{2} rY + a0 a2 cYn1 n3^{2} rY + a1 a2 cYn1 n3^{2} rY\right)^{2} -$
$4 a0 a1 a2 cYn1^{2} n3^{2} \left(a0 a1 a2 cYn3^{2} + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY +\right.$
$\left.a1 a2 cYn3 n3 rY + a0 n3^{2} rY^{2} + a1 n3^{2} rY^{2} + a2 n3^{2} rY^{2}\right)\Big)\Big)\Big) /$
$\Big(2 cYn1 rY^{2} \left(a0 a1 a2 cYn3^{2} + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY + a1 a2 cYn3 n3 rY +\right.$
$\left.a0 n3^{2} rY^{2} + a1 n3^{2} rY^{2} + a2 n3^{2} rY^{2}\right)^{2}\Big),$
$Sn1 \to \Big(6 a0^{2} a1^{4} a2^{2} cYn1 cYn3^{4} + 2 a0^{2} a1^{4} a2^{2} cYn1^{3} cYn3^{4} + 2 a0^{2} a1^{4} a2^{2} cYn1 cYn2^{2} cYn3^{4} +$
$2 a0^{2} a1^{4} a2^{2} cYn1 cYn3^{6} + 12 a0^{2} a1^{4} a2 cYn1 cYn3^{3} n3 rY +$
$12 a0^{2} a1^{3} a2^{2} cYn1 cYn3^{3} n3 rY + 12 a0 a1^{4} a2^{2} cYn1 cYn3^{3} n3 rY +$
$4 a0^{2} a1^{4} a2 cYn1^{3} cYn3^{3} n3 rY + 2 a0^{2} a1^{3} a2^{2} cYn1^{3} cYn3^{3} n3 rY +$
$4 a0 a1^{4} a2^{2} cYn1^{3} cYn3^{3} n3 rY + 4 a0^{2} a1^{4} a2 cYn1 cYn2^{2} cYn3^{3} n3 rY +$
$2 a0^{2} a1^{3} a2^{2} cYn1 cYn2^{2} cYn3^{3} n3 rY +$
$4 a0 a1^{4} a2^{2} cYn1 cYn2^{2} cYn3^{3} n3 rY +$
$4 a0^{2} a1^{4} a2 cYn1 cYn3^{5} n3 rY + 8 a0^{2} a1^{3} a2^{2} cYn1 cYn3^{5} n3 rY +$
$4 a0 a1^{4} a2^{2} cYn1 cYn3^{5} n3 rY + 6 a0^{2} a1^{4} cYn1 cYn3^{2} n3^{2} rY^{2} +$
$24 a0^{2} a1^{3} a2 cYn1 cYn3^{2} n3^{2} rY^{2} + 24 a0 a1^{4} a2 cYn1 cYn3^{2} n3^{2} rY^{2} +$
$6 a0^{2} a1^{2} a2^{2} cYn1 cYn3^{2} n3^{2} rY^{2} + 24 a0 a1^{3} a2^{2} cYn1 cYn3^{2} n3^{2} rY^{2} +$
$6 a1^{4} a2^{2} cYn1 cYn3^{2} n3^{2} rY^{2} + 2 a0^{2} a1^{4} cYn1^{3} cYn3^{2} n3^{2} rY^{2} +$
$4 a0^{2} a1^{3} a2 cYn1^{3} cYn3^{2} n3^{2} rY^{2} + 8 a0 a1^{4} a2 cYn1^{3} cYn3^{2} n3^{2} rY^{2} -$
$3 a0^{2} a1^{2} a2^{2} cYn1^{3} cYn3^{2} n3^{2} rY^{2} + 4 a0 a1^{3} a2^{2} cYn1^{3} cYn3^{2} n3^{2} rY^{2} +$
$2 a1^{4} a2^{2} cYn1^{3} cYn3^{2} n3^{2} rY^{2} + 2 a0^{2} a1^{4} cYn1 cYn2^{2} cYn3^{2} n3^{2} rY^{2} +$
$4 a0^{2} a1^{3} a2 cYn1 cYn2^{2} cYn3^{2} n3^{2} rY^{2} +$
$8 a0 a1^{4} a2 cYn1 cYn2^{2} cYn3^{2} n3^{2} rY^{2} -$
$3 a0^{2} a1^{2} a2^{2} cYn1 cYn2^{2} cYn3^{2} n3^{2} rY^{2} +$

$4\ a0\ a1^3\ a2^2\ cYn1\ cYn2^2\ cYn3^2\ n3^2\ rY^2 +$
$2\ a1^4\ a2^2\ cYn1\ cYn2^2\ cYn3^2\ n3^2\ rY^2 + 2\ a0^2\ a1^4\ cYn1\ cYn3^4\ n3^2\ rY^2 +$
$16\ a0^2\ a1^3\ a2\ cYn1\ cYn3^4\ n3^2\ rY^2 + 8\ a0\ a1^4\ a2\ cYn1\ cYn3^4\ n3^2\ rY^2 +$
$12\ a0^2\ a1^2\ a2^2\ cYn1\ cYn3^4\ n3^2\ rY^2 + 16\ a0\ a1^3\ a2^2\ cYn1\ cYn3^4\ n3^2\ rY^2 +$
$2\ a1^4\ a2^2\ cYn1\ cYn3^4\ n3^2\ rY^2 + 12\ a0^2\ a1^3\ cYn1\ cYn3\ n3^3\ rY^3 +$
$12\ a0\ a1^4\ cYn1\ cYn3\ n3^3\ rY^3 + 12\ a0^2\ a1^2\ a2\ cYn1\ cYn3\ n3^3\ rY^3 +$
$36\ a0\ a1^3\ a2\ cYn1\ cYn3\ n3^3\ rY^3 + 12\ a1^4\ a2\ cYn1\ cYn3\ n3^3\ rY^3 +$
$12\ a0\ a1^2\ a2^2\ cYn1\ cYn3\ n3^3\ rY^3 + 12\ a1^3\ a2^2\ cYn1\ cYn3\ n3^3\ rY^3 +$
$2\ a0^2\ a1^3\ cYn1^3\ cYn3\ n3^3\ rY^3 + 4\ a0\ a1^4\ cYn1^3\ cYn3\ n3^3\ rY^3 -$
$2\ a0^2\ a1^2\ a2\ cYn1^3\ cYn3\ n3^3\ rY^3 + 6\ a0\ a1^3\ a2\ cYn1^3\ cYn3\ n3^3\ rY^3 +$
$4\ a1^4\ a2\ cYn1^3\ cYn3\ n3^3\ rY^3 - 2\ a0^2\ a1\ a2^2\ cYn1^3\ cYn3\ n3^3\ rY^3 -$
$2\ a0\ a1^2\ a2^2\ cYn1^3\ cYn3\ n3^3\ rY^3 + 2\ a1^3\ a2^2\ cYn1^3\ cYn3\ n3^3\ rY^3 +$
$2\ a0^2\ a1^3\ cYn1\ cYn2^2\ cYn3\ n3^3\ rY^3 + 4\ a0\ a1^4\ cYn1\ cYn2^2\ cYn3\ n3^3\ rY^3 -$
$2\ a0^2\ a1^2\ a2\ cYn1\ cYn2^2\ cYn3\ n3^3\ rY^3 +$
$6\ a0\ a1^3\ a2\ cYn1\ cYn2^2\ cYn3\ n3^3\ rY^3 + 4\ a1^4\ a2\ cYn1\ cYn2^2\ cYn3\ n3^3\ rY^3 -$
$2\ a0^2\ a1\ a2^2\ cYn1\ cYn2^2\ cYn3\ n3^3\ rY^3 -$
$2\ a0\ a1^2\ a2^2\ cYn1\ cYn2^2\ cYn3\ n3^3\ rY^3 +$
$2\ a1^3\ a2^2\ cYn1\ cYn2^2\ cYn3\ n3^3\ rY^3 + 8\ a0^2\ a1^3\ cYn1\ cYn3^3\ n3^3\ rY^3 +$
$4\ a0\ a1^4\ cYn1\ cYn3^3\ n3^3\ rY^3 + 24\ a0^2\ a1^2\ a2\ cYn1\ cYn3^3\ n3^3\ rY^3 +$
$28\ a0\ a1^3\ a2\ cYn1\ cYn3^3\ n3^3\ rY^3 + 4\ a1^4\ a2\ cYn1\ cYn3^3\ n3^3\ rY^3 +$
$8\ a0^2\ a1\ a2^2\ cYn1\ cYn3^3\ n3^3\ rY^3 + 24\ a0\ a1^2\ a2^2\ cYn1\ cYn3^3\ n3^3\ rY^3 +$
$8\ a1^3\ a2^2\ cYn1\ cYn3^3\ n3^3\ rY^3 + 6\ a0^2\ a1^2\ cYn1\ n3^4\ rY^4 +$
$12\ a0\ a1^3\ cYn1\ n3^4\ rY^4 + 6\ a1^4\ cYn1\ n3^4\ rY^4 +$
$12\ a0\ a1^2\ a2\ cYn1\ n3^4\ rY^4 + 12\ a1^3\ a2\ cYn1\ n3^4\ rY^4 +$
$6\ a1^2\ a2^2\ cYn1\ n3^4\ rY^4 + a0^2\ a1^2\ cYn1^3\ n3^4\ rY^4 +$
$2\ a0\ a1^3\ cYn1^3\ n3^4\ rY^4 + 2\ a1^4\ cYn1^3\ n3^4\ rY^4 -$
$2\ a0^2\ a1\ a2\ cYn1^3\ n3^4\ rY^4 - 2\ a0\ a1^2\ a2\ cYn1^3\ n3^4\ rY^4 +$
$2\ a1^3\ a2\ cYn1^3\ n3^4\ rY^4 + a0^2\ a2^2\ cYn1^3\ n3^4\ rY^4 -$
$2\ a0\ a1\ a2^2\ cYn1^3\ n3^4\ rY^4 + a1^2\ a2^2\ cYn1^3\ n3^4\ rY^4 +$
$a0^2\ a1^2\ cYn1\ cYn2^2\ n3^4\ rY^4 + 2\ a0\ a1^3\ cYn1\ cYn2^2\ n3^4\ rY^4 +$
$2\ a1^4\ cYn1\ cYn2^2\ n3^4\ rY^4 - 2\ a0^2\ a1\ a2\ cYn1\ cYn2^2\ n3^4\ rY^4 -$
$2\ a0\ a1^2\ a2\ cYn1\ cYn2^2\ n3^4\ rY^4 + 2\ a1^3\ a2\ cYn1\ cYn2^2\ n3^4\ rY^4 +$
$a0^2\ a2^2\ cYn1\ cYn2^2\ n3^4\ rY^4 - 2\ a0\ a1\ a2^2\ cYn1\ cYn2^2\ n3^4\ rY^4 +$
$a1^2\ a2^2\ cYn1\ cYn2^2\ n3^4\ rY^4 + 12\ a0^2\ a1^2\ cYn1\ cYn3^2\ n3^4\ rY^4 +$
$12\ a0\ a1^3\ cYn1\ cYn3^2\ n3^4\ rY^4 + 2\ a1^4\ cYn1\ cYn3^2\ n3^4\ rY^4 +$
$16\ a0^2\ a1\ a2\ cYn1\ cYn3^2\ n3^4\ rY^4 + 36\ a0\ a1^2\ a2\ cYn1\ cYn3^2\ n3^4\ rY^4 +$
$12\ a1^3\ a2\ cYn1\ cYn3^2\ n3^4\ rY^4 + 2\ a0^2\ a2^2\ cYn1\ cYn3^2\ n3^4\ rY^4 +$
$16\ a0\ a1\ a2^2\ cYn1\ cYn3^2\ n3^4\ rY^4 + 12\ a1^2\ a2^2\ cYn1\ cYn3^2\ n3^4\ rY^4 +$
$8\ a0^2\ a1\ cYn1\ cYn3\ n3^5\ rY^5 + 12\ a0\ a1^2\ cYn1\ cYn3\ n3^5\ rY^5 +$
$4\ a1^3\ cYn1\ cYn3\ n3^5\ rY^5 + 4\ a0^2\ a2\ cYn1\ cYn3\ n3^5\ rY^5 +$
$20\ a0\ a1\ a2\ cYn1\ cYn3\ n3^5\ rY^5 + 12\ a1^2\ a2\ cYn1\ cYn3\ n3^5\ rY^5 +$
$4\ a0\ a2^2\ cYn1\ cYn3\ n3^5\ rY^5 + 8\ a1\ a2^2\ cYn1\ cYn3\ n3^5\ rY^5 +$
$2\ a0^2\ cYn1\ n3^6\ rY^6 + 4\ a0\ a1\ cYn1\ n3^6\ rY^6 + 2\ a1^2\ cYn1\ n3^6\ rY^6 +$
$4\ a0\ a2\ cYn1\ n3^6\ rY^6 + 4\ a1\ a2\ cYn1\ n3^6\ rY^6 +$
$2\ a2^2\ cYn1\ n3^6\ rY^6 - 2\ a0\ a1^2\ a2\ cYn1^2\ cYn3^2\ rY$
$\sqrt{}\Big(\big(a0\ a1\ a2\ cYn1\ cYn3\ n3 + a0\ a1\ cYn1\ n3^2\ rY + a0\ a2\ cYn1\ n3^2\ rY + a1\ a2\ cYn1\ n3^2\ rY\big)^2 -$
$4\ a0\ a1\ a2\ cYn1^2\ n3^2\ \big(a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY +$
$a1\ a2\ cYn3\ n3\ rY + a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2\big)\Big) + 2\ a0\ a1^2\ a2\ cYn2^2\ cYn3^2$
$rY\ \sqrt{}\Big(\big(a0\ a1\ a2\ cYn1\ cYn3\ n3 + a0\ a1\ cYn1\ n3^2\ rY + a0\ a2\ cYn1\ n3^2\ rY + a1\ a2\ cYn1\ n3^2\ rY\big)^2 -$
$4\ a0\ a1\ a2\ cYn1^2\ n3^2\ \big(a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY + a1\ a2$
$cYn3\ n3\ rY + a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2\big)\Big) - 2\ a0\ a1^2\ cYn1^2\ cYn3\ n3\ rY^2$
$\sqrt{}\Big(\big(a0\ a1\ a2\ cYn1\ cYn3\ n3 + a0\ a1\ cYn1\ n3^2\ rY + a0\ a2\ cYn1\ n3^2\ rY + a1\ a2\ cYn1\ n3^2\ rY\big)^2 -$
$4\ a0\ a1\ a2\ cYn1^2\ n3^2\ \big(a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY + a1\ a2$
$cYn3\ n3\ rY + a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2\big)\Big) - a0\ a1\ a2\ cYn1^2\ cYn3\ n3\ rY^2$
$\sqrt{}\Big(\big(a0\ a1\ a2\ cYn1\ cYn3\ n3 + a0\ a1\ cYn1\ n3^2\ rY + a0\ a2\ cYn1\ n3^2\ rY + a1\ a2\ cYn1\ n3^2\ rY\big)^2 -$
$4\ a0\ a1\ a2\ cYn1^2\ n3^2\ \big(a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY + a1\ a2$

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        cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2)) - 2 a1^2 a2 cYn1^2 cYn3 n3 rY^2
    √((a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY)^2 -
      4 a0 a1 a2 cYn1^2 n3^2 (a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY + a1 a2
        cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2)) + 2 a0 a1^2 cYn2^2 cYn3 n3 rY^2
    √((a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY)^2 -
      4 a0 a1 a2 cYn1^2 n3^2 (a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY + a1 a2
        cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2)) + a0 a1 a2 cYn2^2 cYn3 n3 rY^2
    √((a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY)^2 -
      4 a0 a1 a2 cYn1^2 n3^2 (a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY + a1 a2
        cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2)) + 2 a1^2 a2 cYn2^2 cYn3 n3 rY^2
    √((a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY)^2 -
      4 a0 a1 a2 cYn1^2 n3^2 (a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY +
        a1 a2 cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2)) - a0 a1 cYn1^2 n3^2 rY^3
    √((a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY)^2 -
      4 a0 a1 a2 cYn1^2 n3^2 (a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY +
        a1 a2 cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2)) - 2 a1^2 cYn1^2 n3^2 rY^3
    √((a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY)^2 -
      4 a0 a1 a2 cYn1^2 n3^2 (a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY +
        a1 a2 cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2)) + a0 a2 cYn1^2 n3^2 rY^3
    √((a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY)^2 -
      4 a0 a1 a2 cYn1^2 n3^2 (a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY +
        a1 a2 cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2)) - a1 a2 cYn1^2 n3^2 rY^3
    √((a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY)^2 -
      4 a0 a1 a2 cYn1^2 n3^2 (a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY +
        a1 a2 cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2)) + a0 a1 cYn2^2 n3^2 rY^3
    √((a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY)^2 -
      4 a0 a1 a2 cYn1^2 n3^2 (a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY +
        a1 a2 cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2)) + 2 a1^2 cYn2^2 n3^2 rY^3
    √((a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY)^2 -
      4 a0 a1 a2 cYn1^2 n3^2 (a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY +
        a1 a2 cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2)) - a0 a2 cYn2^2 n3^2 rY^3
    √((a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY)^2 -
      4 a0 a1 a2 cYn1^2 n3^2 (a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY +
        a1 a2 cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2)) + a1 a2 cYn2^2 n3^2 rY^3
    √((a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY)^2 -
      4 a0 a1 a2 cYn1^2 n3^2 (a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY +
        a1 a2 cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2)))) /
  (2 cYn1 rY^2 (a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY + a1 a2 cYn3 n3 rY +
      a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2)^2),
Sn2 → (6 a0^2 a1^2 a2^4 cYn1 cYn3^4 + 2 a0^2 a1^2 a2^4 cYn1^3 cYn3^4 + 2 a0^2 a1^2 a2^4 cYn1 cYn2^2 cYn3^4 +
    2 a0^2 a1^2 a2^4 cYn1 cYn3^6 + 12 a0^2 a1^2 a2^3 cYn1 cYn3^3 n3 rY +
    12 a0^2 a1 a2^4 cYn1 cYn3^3 n3 rY + 12 a0 a1^2 a2^4 cYn1 cYn3^3 n3 rY +
    2 a0^2 a1^2 a2^3 cYn1^3 cYn3^3 n3 rY +
    4 a0^2 a1 a2^4 cYn1^3 cYn3^3 n3 rY + 4 a0 a1^2 a2^4 cYn1^3 cYn3^3 n3 rY +
    2 a0^2 a1^2 a2^3 cYn1 cYn2^2 cYn3^3 n3 rY +
    4 a0^2 a1 a2^4 cYn1 cYn2^2 cYn3^3 n3 rY +
    4 a0 a1^2 a2^4 cYn1 cYn2^2 cYn3^3 n3 rY +
    8 a0^2 a1^2 a2^3 cYn1 cYn3^5 n3 rY + 4 a0^2 a1 a2^4 cYn1 cYn3^5 n3 rY +
    4 a0 a1^2 a2^4 cYn1 cYn3^5 n3 rY + 6 a0^2 a1^2 a2^2 cYn1 cYn3^2 n3^2 rY^2 +
    24 a0^2 a1 a2^3 cYn1 cYn3^2 n3^2 rY^2 +
    24 a0 a1^2 a2^3 cYn1 cYn3^2 n3^2 rY^2 + 6 a0^2 a2^4 cYn1 cYn3^2 n3^2 rY^2 +
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$24 a0 a1 a2^4 cYn1 cYn3^2 n3^2 rY^2 + 6 a1^2 a2^4 cYn1 cYn3^2 n3^2 rY^2 -$
$3 a0^2 a1^2 a2^2 cYn1^3 cYn3^2 n3^2 rY^2 + 4 a0^2 a1 a2^3 cYn1^3 cYn3^2 n3^2 rY^2 +$
$4 a0 a1^2 a2^3 cYn1^3 cYn3^2 n3^2 rY^2 + 2 a0^2 a2^4 cYn1^3 cYn3^2 n3^2 rY^2 +$
$8 a0 a1 a2^4 cYn1^3 cYn3^2 n3^2 rY^2 + 2 a1^2 a2^4 cYn1^3 cYn3^2 n3^2 rY^2 -$
$3 a0^2 a1^2 a2^2 cYn1 cYn2^2 cYn3^2 n3^2 rY^2 +$
$4 a0^2 a1 a2^3 cYn1 cYn2^2 cYn3^2 n3^2 rY^2 +$
$4 a0 a1^2 a2^3 cYn1 cYn2^2 cYn3^2 n3^2 rY^2 +$
$2 a0^2 a2^4 cYn1 cYn2^2 cYn3^2 n3^2 rY^2 +$
$8 a0 a1 a2^4 cYn1 cYn2^2 cYn3^2 n3^2 rY^2 +$
$2 a1^2 a2^4 cYn1 cYn2^2 cYn3^2 n3^2 rY^2 +$
$12 a0^2 a1^2 a2^2 cYn1 cYn3^4 n3^2 rY^2 +$
$16 a0^2 a1 a2^3 cYn1 cYn3^4 n3^2 rY^2 + 16 a0 a1^2 a2^3 cYn1 cYn3^4 n3^2 rY^2 +$
$2 a0^2 a2^4 cYn1 cYn3^4 n3^2 rY^2 + 8 a0 a1 a2^4 cYn1 cYn3^4 n3^2 rY^2 +$
$2 a1^2 a2^4 cYn1 cYn3^4 n3^2 rY^2 + 12 a0^2 a1 a2^2 cYn1 cYn3 n3^3 rY^3 +$
$12 a0 a1^2 a2^2 cYn1 cYn3 n3^3 rY^3 + 12 a0^2 a2^3 cYn1 cYn3 n3^3 rY^3 +$
$36 a0 a1 a2^3 cYn1 cYn3 n3^3 rY^3 + 12 a1^2 a2^3 cYn1 cYn3 n3^3 rY^3 +$
$12 a0 a2^4 cYn1 cYn3 n3^3 rY^3 + 12 a1 a2^4 cYn1 cYn3 n3^3 rY^3 -$
$2 a0^2 a1^2 a2 cYn1^3 cYn3 n3^3 rY^3 - 2 a0^2 a1 a2^2 cYn1^3 cYn3 n3^3 rY^3 -$
$2 a0 a1^2 a2^2 cYn1^3 cYn3 n3^3 rY^3 + 2 a0^2 a2^3 cYn1^3 cYn3 n3^3 rY^3 +$
$6 a0 a1 a2^3 cYn1^3 cYn3 n3^3 rY^3 + 2 a1^2 a2^3 cYn1^3 cYn3 n3^3 rY^3 +$
$4 a0 a2^4 cYn1^3 cYn3 n3^3 rY^3 + 4 a1 a2^4 cYn1^3 cYn3 n3^3 rY^3 -$
$2 a0^2 a1^2 a2 cYn1 cYn2^2 cYn3 n3^3 rY^3 -$
$2 a0^2 a1 a2^2 cYn1 cYn2^2 cYn3 n3^3 rY^3 -$
$2 a0 a1^2 a2^2 cYn1 cYn2^2 cYn3 n3^3 rY^3 +$
$2 a0^2 a2^3 cYn1 cYn2^2 cYn3 n3^3 rY^3 +$
$6 a0 a1 a2^3 cYn1 cYn2^2 cYn3 n3^3 rY^3 +$
$2 a1^2 a2^3 cYn1 cYn2^2 cYn3 n3^3 rY^3 + 4 a0 a2^4 cYn1 cYn2^2 cYn3 n3^3 rY^3 +$
$4 a1 a2^4 cYn1 cYn2^2 cYn3 n3^3 rY^3 + 8 a0^2 a1^2 a2 cYn1 cYn3^3 n3^3 rY^3 +$
$24 a0^2 a1 a2^2 cYn1 cYn3^3 n3^3 rY^3 + 24 a0 a1^2 a2^2 cYn1 cYn3^3 n3^3 rY^3 +$
$8 a0^2 a2^3 cYn1 cYn3^3 n3^3 rY^3 + 28 a0 a1 a2^3 cYn1 cYn3^3 n3^3 rY^3 +$
$8 a1^2 a2^3 cYn1 cYn3^3 n3^3 rY^3 + 4 a0 a2^4 cYn1 cYn3^3 n3^3 rY^3 +$
$4 a1 a2^4 cYn1 cYn3^3 n3^3 rY^3 + 6 a0^2 a2^2 cYn1 n3^4 rY^4 +$
$12 a0 a1 a2^2 cYn1 n3^4 rY^4 + 6 a1^2 a2^2 cYn1 n3^4 rY^4 +$
$12 a0 a2^3 cYn1 n3^4 rY^4 + 12 a1 a2^3 cYn1 n3^4 rY^4 +$
$6 a2^4 cYn1 n3^4 rY^4 + a0^2 a1^2 cYn1^3 n3^4 rY^4 -$
$2 a0^2 a1 a2 cYn1^3 n3^4 rY^4 - 2 a0 a1^2 a2 cYn1^3 n3^4 rY^4 +$
$a0^2 a2^2 cYn1^3 n3^4 rY^4 - 2 a0 a1 a2^2 cYn1^3 n3^4 rY^4 +$
$a1^2 a2^2 cYn1^3 n3^4 rY^4 + 2 a0 a2^3 cYn1^3 n3^4 rY^4 +$
$2 a1 a2^3 cYn1^3 n3^4 rY^4 + 2 a2^4 cYn1^3 n3^4 rY^4 +$
$a0^2 a1^2 cYn1 cYn2^2 n3^4 rY^4 - 2 a0^2 a1 a2 cYn1 cYn2^2 n3^4 rY^4 -$
$2 a0 a1^2 a2 cYn1 cYn2^2 n3^4 rY^4 + a0^2 a2^2 cYn1 cYn2^2 n3^4 rY^4 -$
$2 a0 a1 a2^2 cYn1 cYn2^2 n3^4 rY^4 + a1^2 a2^2 cYn1 cYn2^2 n3^4 rY^4 +$
$2 a0 a2^3 cYn1 cYn2^2 n3^4 rY^4 + 2 a1 a2^3 cYn1 cYn2^2 n3^4 rY^4 +$
$2 a2^4 cYn1 cYn2^2 n3^4 rY^4 + 2 a0^2 a1^2 cYn1 cYn3^2 n3^4 rY^4 +$
$16 a0^2 a1 a2 cYn1 cYn3^2 n3^4 rY^4 + 16 a0 a1^2 a2 cYn1 cYn3^2 n3^4 rY^4 +$
$12 a0^2 a2^2 cYn1 cYn3^2 n3^4 rY^4 + 36 a0 a1 a2^2 cYn1 cYn3^2 n3^4 rY^4 +$
$12 a1^2 a2^2 cYn1 cYn3^2 n3^4 rY^4 + 12 a0 a2^3 cYn1 cYn3^2 n3^4 rY^4 +$
$12 a1 a2^3 cYn1 cYn3^2 n3^4 rY^4 + 2 a2^4 cYn1 cYn3^2 n3^4 rY^4 +$
$4 a0^2 a1 cYn1 cYn3 n3^5 rY^5 + 4 a0 a1^2 cYn1 cYn3 n3^5 rY^5 +$
$8 a0^2 a2 cYn1 cYn3 n3^5 rY^5 + 20 a0 a1 a2 cYn1 cYn3 n3^5 rY^5 +$
$8 a1^2 a2 cYn1 cYn3 n3^5 rY^5 + 12 a0 a2^2 cYn1 cYn3 n3^5 rY^5 +$
$12 a1 a2^2 cYn1 cYn3 n3^5 rY^5 + 4 a2^3 cYn1 cYn3 n3^5 rY^5 +$
$2 a0^2 cYn1 n3^6 rY^6 + 4 a0 a1 cYn1 n3^6 rY^6 + 2 a1^2 cYn1 n3^6 rY^6 +$
$4 a0 a2 cYn1 n3^6 rY^6 + 4 a1 a2 cYn1 n3^6 rY^6 +$
$2 a2^2 cYn1 n3^6 rY^6 - 2 a0 a1 a2^2 cYn1^2 cYn3^2 rY$
$\sqrt{}\Big(\big(a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY\big)^2 -$
$4 a0 a1 a2 cYn1^2 n3^2 \big(a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY +$

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        a1 a2 cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2)) + 2 a0 a1 a2^2 cYn2^2 cYn3^2
    rY √((a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY)^2 -
      4 a0 a1 a2 cYn1^2 n3^2 (a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY + a1 a2
         cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2)) - a0 a1 a2 cYn1^2 cYn3 n3 rY^2
    √((a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY)^2 -
      4 a0 a1 a2 cYn1^2 n3^2 (a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY + a1 a2
         cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2)) - 2 a0 a2^2 cYn1^2 cYn3 n3 rY^2
    √((a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY)^2 -
      4 a0 a1 a2 cYn1^2 n3^2 (a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY + a1 a2
         cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2)) - 2 a1 a2^2 cYn1^2 cYn3 n3 rY^2
    √((a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY)^2 -
      4 a0 a1 a2 cYn1^2 n3^2 (a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY + a1 a2
         cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2)) + a0 a1 a2 cYn2^2 cYn3 n3 rY^2
    √((a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY)^2 -
      4 a0 a1 a2 cYn1^2 n3^2 (a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY + a1 a2
         cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2)) + 2 a0 a2^2 cYn2^2 cYn3 n3 rY^2
    √((a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY)^2 -
      4 a0 a1 a2 cYn1^2 n3^2 (a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY + a1 a2
         cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2)) + 2 a1 a2^2 cYn2^2 cYn3 n3 rY^2
    √((a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY)^2 -
      4 a0 a1 a2 cYn1^2 n3^2 (a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY +
        a1 a2 cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2)) + a0 a1 cYn1^2 n3^2 rY^3
    √((a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY)^2 -
      4 a0 a1 a2 cYn1^2 n3^2 (a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY +
        a1 a2 cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2)) - a0 a2 cYn1^2 n3^2 rY^3
    √((a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY)^2 -
      4 a0 a1 a2 cYn1^2 n3^2 (a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY +
        a1 a2 cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2)) - a1 a2 cYn1^2 n3^2 rY^3
    √((a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY)^2 -
      4 a0 a1 a2 cYn1^2 n3^2 (a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY +
        a1 a2 cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2)) - 2 a2^2 cYn1^2 n3^2 rY^3
    √((a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY)^2 -
      4 a0 a1 a2 cYn1^2 n3^2 (a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY +
        a1 a2 cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2)) - a0 a1 cYn2^2 n3^2 rY^3
    √((a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY)^2 -
      4 a0 a1 a2 cYn1^2 n3^2 (a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY +
        a1 a2 cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2)) + a0 a2 cYn2^2 n3^2 rY^3
    √((a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY)^2 -
      4 a0 a1 a2 cYn1^2 n3^2 (a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY +
        a1 a2 cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2)) + a1 a2 cYn2^2 n3^2 rY^3
    √((a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY)^2 -
      4 a0 a1 a2 cYn1^2 n3^2 (a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY +
        a1 a2 cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2)) + 2 a2^2 cYn2^2 n3^2 rY^3
    √((a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY)^2 -
      4 a0 a1 a2 cYn1^2 n3^2 (a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY +
        a1 a2 cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2))) /
  (2 cYn1 rY^2 (a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY + a1 a2 cYn3 n3 rY +
      a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2)^2), pY1 → 3,
n1 → -((a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY +
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$$\sqrt{\left(\left(\text{a0 a1 a2 cYn1 cYn3 n3} + \text{a0 a1 cYn1 n3}^2\,\text{rY} + \text{a0 a2 cYn1 n3}^2\,\text{rY} + \text{a1 a2 cYn1 n3}^2\,\text{rY}\right)^2 - 4\,\text{a0 a1 a2 cYn1}^2\,\text{n3}^2\left(\text{a0 a1 a2 cYn3}^2 + \text{a0 a1 cYn3 n3 rY} + \text{a0 a2 cYn3 n3 rY} + \text{a1 a2 cYn3 n3 rY} + \text{a0 n3}^2\,\text{rY}^2 + \text{a1 n3}^2\,\text{rY}^2 + \text{a2 n3}^2\,\text{rY}^2\right)\right)}\Big/$$
$$\left(2\left(\text{a0 a1 a2 cYn3}^2 + \text{a0 a1 cYn3 n3 rY} + \text{a0 a2 cYn3 n3 rY} + \text{a1 a2 cYn3 n3 rY} + \text{a0 n3}^2\,\text{rY}^2 + \text{a1 n3}^2\,\text{rY}^2 + \text{a2 n3}^2\,\text{rY}^2\right)\right)\Big),$$

$$\text{n2} \to -\Big(\Big(\text{cYn2}\Big(\text{a0 a1 a2 cYn1 cYn3 n3} + \text{a0 a1 cYn1 n3}^2\,\text{rY} + \text{a0 a2 cYn1 n3}^2\,\text{rY} + \text{a1 a2 cYn1 n3}^2\,\text{rY} - \sqrt{\left(\left(\text{a0 a1 a2 cYn1 cYn3 n3} + \text{a0 a1 cYn1 n3}^2\,\text{rY} + \text{a0 a2 cYn1 n3}^2\,\text{rY} + \text{a1 a2 cYn1 n3}^2\,\text{rY}\right)^2 - 4\,\text{a0 a1 a2 cYn1}^2\,\text{n3}^2\left(\text{a0 a1 a2 cYn3}^2 + \text{a0 a1 cYn3 n3 rY} + \text{a0 a2 cYn3 n3 rY} + \text{a1 a2 cYn3 n3 rY} + \text{a0 n3}^2\,\text{rY}^2 + \text{a1 n3}^2\,\text{rY}^2 + \text{a2 n3}^2\,\text{rY}^2\right)\right)}\Big)\Big)\Big/$$
$$\left(2\,\text{cYn1}\left(\text{a0 a1 a2 cYn3}^2 + \text{a0 a1 cYn3 n3 rY} + \text{a0 a2 cYn3 n3 rY} + \text{a1 a2 cYn3 n3 rY} + \text{a0 n3}^2\,\text{rY}^2 + \text{a1 n3}^2\,\text{rY}^2 + \text{a2 n3}^2\,\text{rY}^2\right)\right)\Big)\Big\},$$

$$\Big\{\text{p} \to \frac{3 + \text{cYn1}^2 + \text{cYn2}^2 + \text{cYn3}^2}{\text{rY}^2},\ \text{d} \to -\frac{1}{\text{cYn1 cYn2 n3}^3}\,\text{rY}^2\left(\text{a0 a1 a2 cYn3}^2 + \text{a0 a1 cYn3 n3 rY} + \text{a0 a2 cYn3 n3 rY} + \text{a1 a2 cYn3 n3 rY} + \text{a0 n3}^2\,\text{rY}^2 + \text{a1 n3}^2\,\text{rY}^2 + \text{a2 n3}^2\,\text{rY}^2\right),$$

$$\text{N01} \to -\frac{1}{\text{a2}^3\,\text{cYn1 cYn2 n3}^3\,\text{rY}}(\text{a0} - \text{a1})\left(\text{a0 a1 a2 cYn3}^2 + \text{a0 a1 cYn3 n3 rY} + \text{a0 a2 cYn3 n3 rY} + \text{a1 a2 cYn3 n3 rY} + \text{a0 n3}^2\,\text{rY}^2 + \text{a1 n3}^2\,\text{rY}^2 + \text{a2 n3}^2\,\text{rY}^2\right),$$

$$\text{N02} \to \frac{1}{\text{a1}^3\,\text{cYn1 cYn2 n3}^3\,\text{rY}}(\text{a0} - \text{a2})\left(\text{a0 a1 a2 cYn3}^2 + \text{a0 a1 cYn3 n3 rY} + \text{a0 a2 cYn3 n3 rY} + \text{a1 a2 cYn3 n3 rY} + \text{a0 n3}^2\,\text{rY}^2 + \text{a1 n3}^2\,\text{rY}^2 + \text{a2 n3}^2\,\text{rY}^2\right),$$

$$\text{N12} \to \frac{1}{\text{a0}^3\,\text{cYn1 cYn2 n3}^3\,\text{rY}}(\text{a1} - \text{a2})\left(\text{a0 a1 a2 cYn3}^2 + \text{a0 a1 cYn3 n3 rY} + \text{a0 a2 cYn3 n3 rY} + \text{a1 a2 cYn3 n3 rY} + \text{a0 n3}^2\,\text{rY}^2 + \text{a1 n3}^2\,\text{rY}^2 + \text{a2 n3}^2\,\text{rY}^2\right),$$

$$\begin{aligned}\text{Sn0} \to \Big(&6\,\text{a0}^4\,\text{a1}^2\,\text{a2}^2\,\text{cYn1 cYn3}^4 + 2\,\text{a0}^4\,\text{a1}^2\,\text{a2}^2\,\text{cYn1}^3\,\text{cYn3}^4 +\\
&2\,\text{a0}^4\,\text{a1}^2\,\text{a2}^2\,\text{cYn1 cYn2}^2\,\text{cYn3}^4 + 2\,\text{a0}^4\,\text{a1}^2\,\text{a2}^2\,\text{cYn1 cYn3}^6 +\\
&12\,\text{a0}^4\,\text{a1}^2\,\text{a2 cYn1 cYn3}^3\,\text{n3 rY} + 12\,\text{a0}^4\,\text{a1 a2}^2\,\text{cYn1 cYn3}^3\,\text{n3 rY} +\\
&12\,\text{a0}^3\,\text{a1}^2\,\text{a2}^2\,\text{cYn1 cYn3}^3\,\text{n3 rY} + 4\,\text{a0}^4\,\text{a1}^2\,\text{a2 cYn1}^3\,\text{cYn3}^3\,\text{n3 rY} +\\
&4\,\text{a0}^4\,\text{a1 a2}^2\,\text{cYn1}^3\,\text{cYn3}^3\,\text{n3 rY} + 2\,\text{a0}^3\,\text{a1}^2\,\text{a2}^2\,\text{cYn1}^3\,\text{cYn3}^3\,\text{n3 rY} +\\
&4\,\text{a0}^4\,\text{a1}^2\,\text{a2 cYn1 cYn2}^2\,\text{cYn3}^3\,\text{n3 rY} +\\
&4\,\text{a0}^4\,\text{a1 a2}^2\,\text{cYn1 cYn2}^2\,\text{cYn3}^3\,\text{n3 rY} +\\
&2\,\text{a0}^3\,\text{a1}^2\,\text{a2}^2\,\text{cYn1 cYn2}^2\,\text{cYn3}^3\,\text{n3 rY} +\\
&4\,\text{a0}^4\,\text{a1}^2\,\text{a2 cYn1 cYn3}^5\,\text{n3 rY} + 4\,\text{a0}^4\,\text{a1 a2}^2\,\text{cYn1 cYn3}^5\,\text{n3 rY} +\\
&8\,\text{a0}^3\,\text{a1}^2\,\text{a2}^2\,\text{cYn1 cYn3}^5\,\text{n3 rY} + 6\,\text{a0}^4\,\text{a1}^2\,\text{cYn1 cYn3}^2\,\text{n3}^2\,\text{rY}^2 +\\
&24\,\text{a0}^4\,\text{a1 a2 cYn1 cYn3}^2\,\text{n3}^2\,\text{rY}^2 + 24\,\text{a0}^3\,\text{a1}^2\,\text{a2 cYn1 cYn3}^2\,\text{n3}^2\,\text{rY}^2 +\\
&6\,\text{a0}^4\,\text{a2}^2\,\text{cYn1 cYn3}^2\,\text{n3}^2\,\text{rY}^2 + 24\,\text{a0}^3\,\text{a1 a2}^2\,\text{cYn1 cYn3}^2\,\text{n3}^2\,\text{rY}^2 +\\
&6\,\text{a0}^2\,\text{a1}^2\,\text{a2}^2\,\text{cYn1 cYn3}^2\,\text{n3}^2\,\text{rY}^2 + 2\,\text{a0}^4\,\text{a1}^2\,\text{cYn1}^3\,\text{cYn3}^2\,\text{n3}^2\,\text{rY}^2 +\\
&8\,\text{a0}^4\,\text{a1 a2 cYn1}^3\,\text{cYn3}^2\,\text{n3}^2\,\text{rY}^2 + 4\,\text{a0}^3\,\text{a1}^2\,\text{a2 cYn1}^3\,\text{cYn3}^2\,\text{n3}^2\,\text{rY}^2 +\\
&2\,\text{a0}^4\,\text{a2}^2\,\text{cYn1}^3\,\text{cYn3}^2\,\text{n3}^2\,\text{rY}^2 + 4\,\text{a0}^3\,\text{a1 a2}^2\,\text{cYn1}^3\,\text{cYn3}^2\,\text{n3}^2\,\text{rY}^2 -\\
&3\,\text{a0}^2\,\text{a1}^2\,\text{a2}^2\,\text{cYn1}^3\,\text{cYn3}^2\,\text{n3}^2\,\text{rY}^2 +\\
&2\,\text{a0}^4\,\text{a1}^2\,\text{cYn1 cYn2}^2\,\text{cYn3}^2\,\text{n3}^2\,\text{rY}^2 +\\
&8\,\text{a0}^4\,\text{a1 a2 cYn1 cYn2}^2\,\text{cYn3}^2\,\text{n3}^2\,\text{rY}^2 +\\
&4\,\text{a0}^3\,\text{a1}^2\,\text{a2 cYn1 cYn2}^2\,\text{cYn3}^2\,\text{n3}^2\,\text{rY}^2 +\\
&2\,\text{a0}^4\,\text{a2}^2\,\text{cYn1 cYn2}^2\,\text{cYn3}^2\,\text{n3}^2\,\text{rY}^2 +\\
&4\,\text{a0}^3\,\text{a1 a2}^2\,\text{cYn1 cYn2}^2\,\text{cYn3}^2\,\text{n3}^2\,\text{rY}^2 -\\
&3\,\text{a0}^2\,\text{a1}^2\,\text{a2}^2\,\text{cYn1 cYn2}^2\,\text{cYn3}^2\,\text{n3}^2\,\text{rY}^2 +\\
&2\,\text{a0}^4\,\text{a1}^2\,\text{cYn1 cYn3}^4\,\text{n3}^2\,\text{rY}^2 + 8\,\text{a0}^4\,\text{a1 a2 cYn1 cYn3}^4\,\text{n3}^2\,\text{rY}^2 +\\
&16\,\text{a0}^3\,\text{a1}^2\,\text{a2 cYn1 cYn3}^4\,\text{n3}^2\,\text{rY}^2 + 2\,\text{a0}^4\,\text{a2}^2\,\text{cYn1 cYn3}^4\,\text{n3}^2\,\text{rY}^2 +\\
&16\,\text{a0}^3\,\text{a1 a2}^2\,\text{cYn1 cYn3}^4\,\text{n3}^2\,\text{rY}^2 + 12\,\text{a0}^2\,\text{a1}^2\,\text{a2}^2\,\text{cYn1 cYn3}^4\,\text{n3}^2\,\text{rY}^2 +\\
&12\,\text{a0}^4\,\text{a1 cYn1 cYn3 n3}^3\,\text{rY}^3 + 12\,\text{a0}^3\,\text{a1}^2\,\text{cYn1 cYn3 n3}^3\,\text{rY}^3 +\\
&12\,\text{a0}^4\,\text{a2 cYn1 cYn3 n3}^3\,\text{rY}^3 + 36\,\text{a0}^3\,\text{a1 a2 cYn1 cYn3 n3}^3\,\text{rY}^3 +\end{aligned}$$

$12\,a0^{2}\,a1^{2}\,a2\,cYn1\,cYn3\,n3^{3}\,rY^{3} + 12\,a0^{3}\,a2^{2}\,cYn1\,cYn3\,n3^{3}\,rY^{3} +$
$12\,a0^{2}\,a1\,a2^{2}\,cYn1\,cYn3\,n3^{3}\,rY^{3} + 4\,a0^{4}\,a1\,cYn1^{3}\,cYn3\,n3^{3}\,rY^{3} +$
$2\,a0^{3}\,a1^{2}\,cYn1^{3}\,cYn3\,n3^{3}\,rY^{3} + 4\,a0^{4}\,a2\,cYn1^{3}\,cYn3\,n3^{3}\,rY^{3} +$
$6\,a0^{3}\,a1\,a2\,cYn1^{3}\,cYn3\,n3^{3}\,rY^{3} - 2\,a0^{2}\,a1^{2}\,a2\,cYn1^{3}\,cYn3\,n3^{3}\,rY^{3} +$
$2\,a0^{3}\,a2^{2}\,cYn1^{3}\,cYn3\,n3^{3}\,rY^{3} - 2\,a0^{2}\,a1\,a2^{2}\,cYn1^{3}\,cYn3\,n3^{3}\,rY^{3} -$
$2\,a0\,a1^{2}\,a2^{2}\,cYn1^{3}\,cYn3\,n3^{3}\,rY^{3} + 4\,a0^{4}\,a1\,cYn1\,cYn2^{2}\,cYn3\,n3^{3}\,rY^{3} +$
$2\,a0^{3}\,a1^{2}\,cYn1\,cYn2^{2}\,cYn3\,n3^{3}\,rY^{3} + 4\,a0^{4}\,a2\,cYn1\,cYn2^{2}\,cYn3\,n3^{3}\,rY^{3} +$
$6\,a0^{3}\,a1\,a2\,cYn1\,cYn2^{2}\,cYn3\,n3^{3}\,rY^{3} -$
$2\,a0^{2}\,a1^{2}\,a2\,cYn1\,cYn2^{2}\,cYn3\,n3^{3}\,rY^{3} +$
$2\,a0^{3}\,a2^{2}\,cYn1\,cYn2^{2}\,cYn3\,n3^{3}\,rY^{3} -$
$2\,a0^{2}\,a1\,a2^{2}\,cYn1\,cYn2^{2}\,cYn3\,n3^{3}\,rY^{3} -$
$2\,a0\,a1^{2}\,a2^{2}\,cYn1\,cYn2^{2}\,cYn3\,n3^{3}\,rY^{3} + 4\,a0^{4}\,a1\,cYn1\,cYn3^{3}\,n3^{3}\,rY^{3} +$
$8\,a0^{3}\,a1^{2}\,cYn1\,cYn3^{3}\,n3^{3}\,rY^{3} + 4\,a0^{4}\,a2\,cYn1\,cYn3^{3}\,n3^{3}\,rY^{3} +$
$28\,a0^{3}\,a1\,a2\,cYn1\,cYn3^{3}\,n3^{3}\,rY^{3} + 24\,a0^{2}\,a1^{2}\,a2\,cYn1\,cYn3^{3}\,n3^{3}\,rY^{3} +$
$8\,a0^{3}\,a2^{2}\,cYn1\,cYn3^{3}\,n3^{3}\,rY^{3} + 24\,a0^{2}\,a1\,a2^{2}\,cYn1\,cYn3^{3}\,n3^{3}\,rY^{3} +$
$8\,a0\,a1^{2}\,a2^{2}\,cYn1\,cYn3^{3}\,n3^{3}\,rY^{3} + 6\,a0^{4}\,cYn1\,n3^{4}\,rY^{4} +$
$12\,a0^{3}\,a1\,cYn1\,n3^{4}\,rY^{4} + 6\,a0^{2}\,a1^{2}\,cYn1\,n3^{4}\,rY^{4} +$
$12\,a0^{3}\,a2\,cYn1\,n3^{4}\,rY^{4} + 12\,a0^{2}\,a1\,a2\,cYn1\,n3^{4}\,rY^{4} +$
$6\,a0^{2}\,a2^{2}\,cYn1\,n3^{4}\,rY^{4} + 2\,a0^{4}\,cYn1^{3}\,n3^{4}\,rY^{4} +$
$2\,a0^{3}\,a1\,cYn1^{3}\,n3^{4}\,rY^{4} + a0^{2}\,a1^{2}\,cYn1^{3}\,n3^{4}\,rY^{4} +$
$2\,a0^{3}\,a2\,cYn1^{3}\,n3^{4}\,rY^{4} - 2\,a0^{2}\,a1\,a2\,cYn1^{3}\,n3^{4}\,rY^{4} -$
$2\,a0\,a1^{2}\,a2\,cYn1^{3}\,n3^{4}\,rY^{4} + a0^{2}\,a2^{2}\,cYn1^{3}\,n3^{4}\,rY^{4} -$
$2\,a0\,a1\,a2^{2}\,cYn1^{3}\,n3^{4}\,rY^{4} + a1^{2}\,a2^{2}\,cYn1^{3}\,n3^{4}\,rY^{4} +$
$2\,a0^{4}\,cYn1\,cYn2^{2}\,n3^{4}\,rY^{4} + 2\,a0^{3}\,a1\,cYn1\,cYn2^{2}\,n3^{4}\,rY^{4} +$
$a0^{2}\,a1^{2}\,cYn1\,cYn2^{2}\,n3^{4}\,rY^{4} + 2\,a0^{3}\,a2\,cYn1\,cYn2^{2}\,n3^{4}\,rY^{4} -$
$2\,a0^{2}\,a1\,a2\,cYn1\,cYn2^{2}\,n3^{4}\,rY^{4} - 2\,a0\,a1^{2}\,a2\,cYn1\,cYn2^{2}\,n3^{4}\,rY^{4} +$
$a0^{2}\,a2^{2}\,cYn1\,cYn2^{2}\,n3^{4}\,rY^{4} - 2\,a0\,a1\,a2^{2}\,cYn1\,cYn2^{2}\,n3^{4}\,rY^{4} +$
$a1^{2}\,a2^{2}\,cYn1\,cYn2^{2}\,n3^{4}\,rY^{4} + 2\,a0^{4}\,cYn1\,cYn3^{2}\,n3^{4}\,rY^{4} +$
$12\,a0^{3}\,a1\,cYn1\,cYn3^{2}\,n3^{4}\,rY^{4} + 12\,a0^{2}\,a1^{2}\,cYn1\,cYn3^{2}\,n3^{4}\,rY^{4} +$
$12\,a0^{3}\,a2\,cYn1\,cYn3^{2}\,n3^{4}\,rY^{4} + 36\,a0^{2}\,a1\,a2\,cYn1\,cYn3^{2}\,n3^{4}\,rY^{4} +$
$16\,a0\,a1^{2}\,a2\,cYn1\,cYn3^{2}\,n3^{4}\,rY^{4} + 12\,a0^{2}\,a2^{2}\,cYn1\,cYn3^{2}\,n3^{4}\,rY^{4} +$
$16\,a0\,a1\,a2^{2}\,cYn1\,cYn3^{2}\,n3^{4}\,rY^{4} + 2\,a1^{2}\,a2^{2}\,cYn1\,cYn3^{2}\,n3^{4}\,rY^{4} +$
$4\,a0^{3}\,cYn1\,cYn3\,n3^{5}\,rY^{5} + 12\,a0^{2}\,a1\,cYn1\,cYn3\,n3^{5}\,rY^{5} +$
$8\,a0\,a1^{2}\,cYn1\,cYn3\,n3^{5}\,rY^{5} + 12\,a0^{2}\,a2\,cYn1\,cYn3\,n3^{5}\,rY^{5} +$
$20\,a0\,a1\,a2\,cYn1\,cYn3\,n3^{5}\,rY^{5} + 4\,a1^{2}\,a2\,cYn1\,cYn3\,n3^{5}\,rY^{5} +$
$8\,a0\,a2^{2}\,cYn1\,cYn3\,n3^{5}\,rY^{5} + 4\,a1\,a2^{2}\,cYn1\,cYn3\,n3^{5}\,rY^{5} +$
$2\,a0^{2}\,cYn1\,n3^{6}\,rY^{6} + 4\,a0\,a1\,cYn1\,n3^{6}\,rY^{6} + 2\,a1^{2}\,cYn1\,n3^{6}\,rY^{6} +$
$4\,a0\,a2\,cYn1\,n3^{6}\,rY^{6} + 4\,a1\,a2\,cYn1\,n3^{6}\,rY^{6} +$
$2\,a2^{2}\,cYn1\,n3^{6}\,rY^{6} + 2\,a0^{2}\,a1\,a2\,cYn1^{2}\,cYn3^{2}\,rY$
$\sqrt{}\,\big(\big(a0\,a1\,a2\,cYn1\,cYn3\,n3 + a0\,a1\,cYn1\,n3^{2}\,rY + a0\,a2\,cYn1\,n3^{2}\,rY + a1\,a2\,cYn1\,n3^{2}\,rY\big)^{2} -$
$4\,a0\,a1\,a2\,cYn1^{2}\,n3^{2}\,\big(a0\,a1\,a2\,cYn3^{2} + a0\,a1\,cYn3\,n3\,rY + a0\,a2\,cYn3\,n3\,rY +$
$a1\,a2\,cYn3\,n3\,rY + a0\,n3^{2}\,rY^{2} + a1\,n3^{2}\,rY^{2} + a2\,n3^{2}\,rY^{2}\big)\big) - 2\,a0^{2}\,a1\,a2\,cYn2^{2}\,cYn3^{2}$
$rY\,\sqrt{}\,\big(\big(a0\,a1\,a2\,cYn1\,cYn3\,n3 + a0\,a1\,cYn1\,n3^{2}\,rY + a0\,a2\,cYn1\,n3^{2}\,rY + a1\,a2\,cYn1\,n3^{2}\,rY\big)^{2} -$
$4\,a0\,a1\,a2\,cYn1^{2}\,n3^{2}\,\big(a0\,a1\,a2\,cYn3^{2} + a0\,a1\,cYn3\,n3\,rY + a0\,a2\,cYn3\,n3\,rY + a1\,a2$
$cYn3\,n3\,rY + a0\,n3^{2}\,rY^{2} + a1\,n3^{2}\,rY^{2} + a2\,n3^{2}\,rY^{2}\big)\big) + 2\,a0^{2}\,a1\,cYn1^{2}\,cYn3\,n3\,rY^{2}$
$\sqrt{}\,\big(\big(a0\,a1\,a2\,cYn1\,cYn3\,n3 + a0\,a1\,cYn1\,n3^{2}\,rY + a0\,a2\,cYn1\,n3^{2}\,rY + a1\,a2\,cYn1\,n3^{2}\,rY\big)^{2} -$
$4\,a0\,a1\,a2\,cYn1^{2}\,n3^{2}\,\big(a0\,a1\,a2\,cYn3^{2} + a0\,a1\,cYn3\,n3\,rY + a0\,a2\,cYn3\,n3\,rY + a1\,a2$
$cYn3\,n3\,rY + a0\,n3^{2}\,rY^{2} + a1\,n3^{2}\,rY^{2} + a2\,n3^{2}\,rY^{2}\big)\big) + 2\,a0^{2}\,a2\,cYn1^{2}\,cYn3\,n3\,rY^{2}$
$\sqrt{}\,\big(\big(a0\,a1\,a2\,cYn1\,cYn3\,n3 + a0\,a1\,cYn1\,n3^{2}\,rY + a0\,a2\,cYn1\,n3^{2}\,rY + a1\,a2\,cYn1\,n3^{2}\,rY\big)^{2} -$
$4\,a0\,a1\,a2\,cYn1^{2}\,n3^{2}\,\big(a0\,a1\,a2\,cYn3^{2} + a0\,a1\,cYn3\,n3\,rY + a0\,a2\,cYn3\,n3\,rY + a1\,a2$
$cYn3\,n3\,rY + a0\,n3^{2}\,rY^{2} + a1\,n3^{2}\,rY^{2} + a2\,n3^{2}\,rY^{2}\big)\big) + a0\,a1\,a2\,cYn1^{2}\,cYn3\,n3\,rY^{2}$
$\sqrt{}\,\big(\big(a0\,a1\,a2\,cYn1\,cYn3\,n3 + a0\,a1\,cYn1\,n3^{2}\,rY + a0\,a2\,cYn1\,n3^{2}\,rY + a1\,a2\,cYn1\,n3^{2}\,rY\big)^{2} -$
$4\,a0\,a1\,a2\,cYn1^{2}\,n3^{2}\,\big(a0\,a1\,a2\,cYn3^{2} + a0\,a1\,cYn3\,n3\,rY + a0\,a2\,cYn3\,n3\,rY + a1\,a2$
$cYn3\,n3\,rY + a0\,n3^{2}\,rY^{2} + a1\,n3^{2}\,rY^{2} + a2\,n3^{2}\,rY^{2}\big)\big) - 2\,a0^{2}\,a1\,cYn2^{2}\,cYn3\,n3\,rY^{2}$
$\sqrt{}\,\big(\big(a0\,a1\,a2\,cYn1\,cYn3\,n3 + a0\,a1\,cYn1\,n3^{2}\,rY + a0\,a2\,cYn1\,n3^{2}\,rY + a1\,a2\,cYn1\,n3^{2}\,rY\big)^{2} -$

$$4\,a0\,a1\,a2\,cYn1^2\,n3^2\,\big(a0\,a1\,a2\,cYn3^2 + a0\,a1\,cYn3\,n3\,rY + a0\,a2\,cYn3\,n3\,rY + a1\,a2$$
$$cYn3\,n3\,rY + a0\,n3^2\,rY^2 + a1\,n3^2\,rY^2 + a2\,n3^2\,rY^2\big)\big) - 2\,a0^2\,a2\,cYn2^2\,cYn3\,n3\,rY^2$$
$$\surd\big(\big(a0\,a1\,a2\,cYn1\,cYn3\,n3 + a0\,a1\,cYn1\,n3^2\,rY + a0\,a2\,cYn1\,n3^2\,rY + a1\,a2\,cYn1\,n3^2\,rY\big)^2 -$$
$$4\,a0\,a1\,a2\,cYn1^2\,n3^2\,\big(a0\,a1\,a2\,cYn3^2 + a0\,a1\,cYn3\,n3\,rY + a0\,a2\,cYn3\,n3\,rY + a1\,a2$$
$$cYn3\,n3\,rY + a0\,n3^2\,rY^2 + a1\,n3^2\,rY^2 + a2\,n3^2\,rY^2\big)\big) - a0\,a1\,a2\,cYn2^2\,cYn3\,n3\,rY^2$$
$$\surd\big(\big(a0\,a1\,a2\,cYn1\,cYn3\,n3 + a0\,a1\,cYn1\,n3^2\,rY + a0\,a2\,cYn1\,n3^2\,rY + a1\,a2\,cYn1\,n3^2\,rY\big)^2 -$$
$$4\,a0\,a1\,a2\,cYn1^2\,n3^2\,\big(a0\,a1\,a2\,cYn3^2 + a0\,a1\,cYn3\,n3\,rY + a0\,a2\,cYn3\,n3\,rY +$$
$$a1\,a2\,cYn3\,n3\,rY + a0\,n3^2\,rY^2 + a1\,n3^2\,rY^2 + a2\,n3^2\,rY^2\big)\big) + 2\,a0^2\,cYn1^2\,n3^2\,rY^3$$
$$\surd\big(\big(a0\,a1\,a2\,cYn1\,cYn3\,n3 + a0\,a1\,cYn1\,n3^2\,rY + a0\,a2\,cYn1\,n3^2\,rY + a1\,a2\,cYn1\,n3^2\,rY\big)^2 -$$
$$4\,a0\,a1\,a2\,cYn1^2\,n3^2\,\big(a0\,a1\,a2\,cYn3^2 + a0\,a1\,cYn3\,n3\,rY + a0\,a2\,cYn3\,n3\,rY +$$
$$a1\,a2\,cYn3\,n3\,rY + a0\,n3^2\,rY^2 + a1\,n3^2\,rY^2 + a2\,n3^2\,rY^2\big)\big) + a0\,a1\,cYn1^2\,n3^2\,rY^3$$
$$\surd\big(\big(a0\,a1\,a2\,cYn1\,cYn3\,n3 + a0\,a1\,cYn1\,n3^2\,rY + a0\,a2\,cYn1\,n3^2\,rY + a1\,a2\,cYn1\,n3^2\,rY\big)^2 -$$
$$4\,a0\,a1\,a2\,cYn1^2\,n3^2\,\big(a0\,a1\,a2\,cYn3^2 + a0\,a1\,cYn3\,n3\,rY + a0\,a2\,cYn3\,n3\,rY +$$
$$a1\,a2\,cYn3\,n3\,rY + a0\,n3^2\,rY^2 + a1\,n3^2\,rY^2 + a2\,n3^2\,rY^2\big)\big) + a0\,a2\,cYn1^2\,n3^2\,rY^3$$
$$\surd\big(\big(a0\,a1\,a2\,cYn1\,cYn3\,n3 + a0\,a1\,cYn1\,n3^2\,rY + a0\,a2\,cYn1\,n3^2\,rY + a1\,a2\,cYn1\,n3^2\,rY\big)^2 -$$
$$4\,a0\,a1\,a2\,cYn1^2\,n3^2\,\big(a0\,a1\,a2\,cYn3^2 + a0\,a1\,cYn3\,n3\,rY + a0\,a2\,cYn3\,n3\,rY +$$
$$a1\,a2\,cYn3\,n3\,rY + a0\,n3^2\,rY^2 + a1\,n3^2\,rY^2 + a2\,n3^2\,rY^2\big)\big) - a1\,a2\,cYn1^2\,n3^2\,rY^3$$
$$\surd\big(\big(a0\,a1\,a2\,cYn1\,cYn3\,n3 + a0\,a1\,cYn1\,n3^2\,rY + a0\,a2\,cYn1\,n3^2\,rY + a1\,a2\,cYn1\,n3^2\,rY\big)^2 -$$
$$4\,a0\,a1\,a2\,cYn1^2\,n3^2\,\big(a0\,a1\,a2\,cYn3^2 + a0\,a1\,cYn3\,n3\,rY + a0\,a2\,cYn3\,n3\,rY +$$
$$a1\,a2\,cYn3\,n3\,rY + a0\,n3^2\,rY^2 + a1\,n3^2\,rY^2 + a2\,n3^2\,rY^2\big)\big) - 2\,a0^2\,cYn2^2\,n3^2\,rY^3$$
$$\surd\big(\big(a0\,a1\,a2\,cYn1\,cYn3\,n3 + a0\,a1\,cYn1\,n3^2\,rY + a0\,a2\,cYn1\,n3^2\,rY + a1\,a2\,cYn1\,n3^2\,rY\big)^2 -$$
$$4\,a0\,a1\,a2\,cYn1^2\,n3^2\,\big(a0\,a1\,a2\,cYn3^2 + a0\,a1\,cYn3\,n3\,rY + a0\,a2\,cYn3\,n3\,rY +$$
$$a1\,a2\,cYn3\,n3\,rY + a0\,n3^2\,rY^2 + a1\,n3^2\,rY^2 + a2\,n3^2\,rY^2\big)\big) - a0\,a1\,cYn2^2\,n3^2\,rY^3$$
$$\surd\big(\big(a0\,a1\,a2\,cYn1\,cYn3\,n3 + a0\,a1\,cYn1\,n3^2\,rY + a0\,a2\,cYn1\,n3^2\,rY + a1\,a2\,cYn1\,n3^2\,rY\big)^2 -$$
$$4\,a0\,a1\,a2\,cYn1^2\,n3^2\,\big(a0\,a1\,a2\,cYn3^2 + a0\,a1\,cYn3\,n3\,rY + a0\,a2\,cYn3\,n3\,rY +$$
$$a1\,a2\,cYn3\,n3\,rY + a0\,n3^2\,rY^2 + a1\,n3^2\,rY^2 + a2\,n3^2\,rY^2\big)\big) - a0\,a2\,cYn2^2\,n3^2\,rY^3$$
$$\surd\big(\big(a0\,a1\,a2\,cYn1\,cYn3\,n3 + a0\,a1\,cYn1\,n3^2\,rY + a0\,a2\,cYn1\,n3^2\,rY + a1\,a2\,cYn1\,n3^2\,rY\big)^2 -$$
$$4\,a0\,a1\,a2\,cYn1^2\,n3^2\,\big(a0\,a1\,a2\,cYn3^2 + a0\,a1\,cYn3\,n3\,rY + a0\,a2\,cYn3\,n3\,rY +$$
$$a1\,a2\,cYn3\,n3\,rY + a0\,n3^2\,rY^2 + a1\,n3^2\,rY^2 + a2\,n3^2\,rY^2\big)\big) + a1\,a2\,cYn2^2\,n3^2\,rY^3$$
$$\surd\big(\big(a0\,a1\,a2\,cYn1\,cYn3\,n3 + a0\,a1\,cYn1\,n3^2\,rY + a0\,a2\,cYn1\,n3^2\,rY + a1\,a2\,cYn1\,n3^2\,rY\big)^2 -$$
$$4\,a0\,a1\,a2\,cYn1^2\,n3^2\,\big(a0\,a1\,a2\,cYn3^2 + a0\,a1\,cYn3\,n3\,rY + a0\,a2\,cYn3\,n3\,rY +$$
$$a1\,a2\,cYn3\,n3\,rY + a0\,n3^2\,rY^2 + a1\,n3^2\,rY^2 + a2\,n3^2\,rY^2\big)\big)\big)\big) \big/$$
$$\big(2\,cYn1\,rY^2\,\big(a0\,a1\,a2\,cYn3^2 + a0\,a1\,cYn3\,n3\,rY + a0\,a2\,cYn3\,n3\,rY + a1\,a2\,cYn3\,n3\,rY +$$
$$a0\,n3^2\,rY^2 + a1\,n3^2\,rY^2 + a2\,n3^2\,rY^2\big)^2\big),$$
$$Sn1 \to \big(6\,a0^2\,a1^4\,a2^2\,cYn1\,cYn3^4 + 2\,a0^2\,a1^4\,a2^2\,cYn1^3\,cYn3^4 + 2\,a0^2\,a1^4\,a2^2\,cYn1\,cYn2^2\,cYn3^4 +$$
$$2\,a0^2\,a1^4\,a2^2\,cYn1\,cYn3^6 + 12\,a0^2\,a1^4\,a2\,cYn1\,cYn3^3\,n3\,rY +$$
$$12\,a0^2\,a1^3\,a2^2\,cYn1\,cYn3^3\,n3\,rY +$$
$$12\,a0\,a1^4\,a2^2\,cYn1\,cYn3^3\,n3\,rY + 4\,a0^2\,a1^4\,a2\,cYn1^3\,cYn3^3\,n3\,rY +$$
$$2\,a0^2\,a1^3\,a2^2\,cYn1^3\,cYn3^3\,n3\,rY + 4\,a0\,a1^4\,a2^2\,cYn1^3\,cYn3^3\,n3\,rY +$$
$$4\,a0^2\,a1^4\,a2\,cYn1\,cYn2^2\,cYn3^3\,n3\,rY +$$
$$2\,a0^2\,a1^3\,a2^2\,cYn1\,cYn2^2\,cYn3^3\,n3\,rY +$$
$$4\,a0\,a1^4\,a2^2\,cYn1\,cYn2^2\,cYn3^3\,n3\,rY +$$
$$4\,a0^2\,a1^4\,a2\,cYn1\,cYn3^5\,n3\,rY + 8\,a0^2\,a1^3\,a2^2\,cYn1\,cYn3^5\,n3\,rY +$$
$$4\,a0\,a1^4\,a2^2\,cYn1\,cYn3^5\,n3\,rY + 6\,a0^2\,a1^4\,cYn1\,cYn3^2\,n3^2\,rY^2 +$$
$$24\,a0^2\,a1^3\,a2\,cYn1\,cYn3^2\,n3^2\,rY^2 + 24\,a0\,a1^4\,a2\,cYn1\,cYn3^2\,n3^2\,rY^2 +$$
$$6\,a0^2\,a1^2\,a2^2\,cYn1\,cYn3^2\,n3^2\,rY^2 + 24\,a0\,a1^3\,a2^2\,cYn1\,cYn3^2\,n3^2\,rY^2 +$$
$$6\,a1^4\,a2^2\,cYn1\,cYn3^2\,n3^2\,rY^2 + 2\,a0^2\,a1^4\,cYn1^3\,cYn3^2\,n3^2\,rY^2 +$$
$$4\,a0^2\,a1^3\,a2\,cYn1^3\,cYn3^2\,n3^2\,rY^2 + 8\,a0\,a1^4\,a2\,cYn1^3\,cYn3^2\,n3^2\,rY^2 -$$
$$3\,a0^2\,a1^2\,a2^2\,cYn1^3\,cYn3^2\,n3^2\,rY^2 +$$
$$4\,a0\,a1^3\,a2^2\,cYn1^3\,cYn3^2\,n3^2\,rY^2 + 2\,a1^4\,a2^2\,cYn1^3\,cYn3^2\,n3^2\,rY^2 +$$
$$2\,a0^2\,a1^4\,cYn1\,cYn2^2\,cYn3^2\,n3^2\,rY^2 +$$
$$4\,a0^2\,a1^3\,a2\,cYn1\,cYn2^2\,cYn3^2\,n3^2\,rY^2 +$$

$8\ a0\ a1^{4}\ a2\ cYn1\ cYn2^{2}\ cYn3^{2}\ n3^{2}\ rY^{2}\ -$
$3\ a0^{2}\ a1^{2}\ a2^{2}\ cYn1\ cYn2^{2}\ cYn3^{2}\ n3^{2}\ rY^{2}\ +$
$4\ a0\ a1^{3}\ a2^{2}\ cYn1\ cYn2^{2}\ cYn3^{2}\ n3^{2}\ rY^{2}\ +$
$2\ a1^{4}\ a2^{2}\ cYn1\ cYn2^{2}\ cYn3^{2}\ n3^{2}\ rY^{2}\ +$
$2\ a0^{2}\ a1^{4}\ cYn1\ cYn3^{4}\ n3^{2}\ rY^{2} + 16\ a0^{2}\ a1^{3}\ a2\ cYn1\ cYn3^{4}\ n3^{2}\ rY^{2}\ +$
$8\ a0\ a1^{4}\ a2\ cYn1\ cYn3^{4}\ n3^{2}\ rY^{2} + 12\ a0^{2}\ a1^{2}\ a2^{2}\ cYn1\ cYn3^{4}\ n3^{2}\ rY^{2}\ +$
$16\ a0\ a1^{3}\ a2^{2}\ cYn1\ cYn3^{4}\ n3^{2}\ rY^{2} + 2\ a1^{4}\ a2^{2}\ cYn1\ cYn3^{4}\ n3^{2}\ rY^{2}\ +$
$12\ a0^{2}\ a1^{3}\ cYn1\ cYn3\ n3^{3}\ rY^{3} + 12\ a0\ a1^{4}\ cYn1\ cYn3\ n3^{3}\ rY^{3}\ +$
$12\ a0^{2}\ a1^{2}\ a2\ cYn1\ cYn3\ n3^{3}\ rY^{3} + 36\ a0\ a1^{3}\ a2\ cYn1\ cYn3\ n3^{3}\ rY^{3}\ +$
$12\ a1^{4}\ a2\ cYn1\ cYn3\ n3^{3}\ rY^{3} + 12\ a0\ a1^{2}\ a2^{2}\ cYn1\ cYn3\ n3^{3}\ rY^{3}\ +$
$12\ a1^{3}\ a2^{2}\ cYn1\ cYn3\ n3^{3}\ rY^{3} + 2\ a0^{2}\ a1^{3}\ cYn1^{3}\ cYn3\ n3^{3}\ rY^{3}\ +$
$4\ a0\ a1^{4}\ cYn1^{3}\ cYn3\ n3^{3}\ rY^{3} - 2\ a0^{2}\ a1^{2}\ a2\ cYn1^{3}\ cYn3\ n3^{3}\ rY^{3}\ +$
$6\ a0\ a1^{3}\ a2\ cYn1^{3}\ cYn3\ n3^{3}\ rY^{3} + 4\ a1^{4}\ a2\ cYn1^{3}\ cYn3\ n3^{3}\ rY^{3}\ -$
$2\ a0^{2}\ a1\ a2^{2}\ cYn1^{3}\ cYn3\ n3^{3}\ rY^{3} - 2\ a0\ a1^{2}\ a2^{2}\ cYn1^{3}\ cYn3\ n3^{3}\ rY^{3}\ +$
$2\ a1^{3}\ a2^{2}\ cYn1^{3}\ cYn3\ n3^{3}\ rY^{3} + 2\ a0^{2}\ a1^{3}\ cYn1\ cYn2^{2}\ cYn3\ n3^{3}\ rY^{3}\ +$
$4\ a0\ a1^{4}\ cYn1\ cYn2^{2}\ cYn3\ n3^{3}\ rY^{3}\ -$
$2\ a0^{2}\ a1^{2}\ a2\ cYn1\ cYn2^{2}\ cYn3\ n3^{3}\ rY^{3}\ +$
$6\ a0\ a1^{3}\ a2\ cYn1\ cYn2^{2}\ cYn3\ n3^{3}\ rY^{3}\ +$
$4\ a1^{4}\ a2\ cYn1\ cYn2^{2}\ cYn3\ n3^{3}\ rY^{3}\ -$
$2\ a0^{2}\ a1\ a2^{2}\ cYn1\ cYn2^{2}\ cYn3\ n3^{3}\ rY^{3}\ -$
$2\ a0\ a1^{2}\ a2^{2}\ cYn1\ cYn2^{2}\ cYn3\ n3^{3}\ rY^{3}\ +$
$2\ a1^{3}\ a2^{2}\ cYn1\ cYn2^{2}\ cYn3\ n3^{3}\ rY^{3}\ +$
$8\ a0^{2}\ a1^{3}\ cYn1\ cYn3^{3}\ n3^{3}\ rY^{3} + 4\ a0\ a1^{4}\ cYn1\ cYn3^{3}\ n3^{3}\ rY^{3}\ +$
$24\ a0^{2}\ a1^{2}\ a2\ cYn1\ cYn3^{3}\ n3^{3}\ rY^{3} + 28\ a0\ a1^{3}\ a2\ cYn1\ cYn3^{3}\ n3^{3}\ rY^{3}\ +$
$4\ a1^{4}\ a2\ cYn1\ cYn3^{3}\ n3^{3}\ rY^{3} + 8\ a0^{2}\ a1\ a2^{2}\ cYn1\ cYn3^{3}\ n3^{3}\ rY^{3}\ +$
$24\ a0\ a1^{2}\ a2^{2}\ cYn1\ cYn3^{3}\ n3^{3}\ rY^{3} + 8\ a1^{3}\ a2^{2}\ cYn1\ cYn3^{3}\ n3^{3}\ rY^{3}\ +$
$6\ a0^{2}\ a1^{2}\ cYn1\ n3^{4}\ rY^{4} + 12\ a0\ a1^{3}\ cYn1\ n3^{4}\ rY^{4}\ +$
$6\ a1^{4}\ cYn1\ n3^{4}\ rY^{4} + 12\ a0\ a1^{2}\ a2\ cYn1\ n3^{4}\ rY^{4}\ +$
$12\ a1^{3}\ a2\ cYn1\ n3^{4}\ rY^{4} + 6\ a1^{2}\ a2^{2}\ cYn1\ n3^{4}\ rY^{4}\ +$
$a0^{2}\ a1^{2}\ cYn1^{3}\ n3^{4}\ rY^{4} + 2\ a0\ a1^{3}\ cYn1^{3}\ n3^{4}\ rY^{4}\ +$
$2\ a1^{4}\ cYn1^{3}\ n3^{4}\ rY^{4} - 2\ a0^{2}\ a1\ a2\ cYn1^{3}\ n3^{4}\ rY^{4}\ -$
$2\ a0\ a1^{2}\ a2\ cYn1^{3}\ n3^{4}\ rY^{4} + 2\ a1^{3}\ a2\ cYn1^{3}\ n3^{4}\ rY^{4}\ +$
$a0^{2}\ a2^{2}\ cYn1^{3}\ n3^{4}\ rY^{4} - 2\ a0\ a1\ a2^{2}\ cYn1^{3}\ n3^{4}\ rY^{4}\ +$
$a1^{2}\ a2^{2}\ cYn1^{3}\ n3^{4}\ rY^{4} + a0^{2}\ a1^{2}\ cYn1\ cYn2^{2}\ n3^{4}\ rY^{4}\ +$
$2\ a0\ a1^{3}\ cYn1\ cYn2^{2}\ n3^{4}\ rY^{4} + 2\ a1^{4}\ cYn1\ cYn2^{2}\ n3^{4}\ rY^{4}\ -$
$2\ a0^{2}\ a1\ a2\ cYn1\ cYn2^{2}\ n3^{4}\ rY^{4} - 2\ a0\ a1^{2}\ a2\ cYn1\ cYn2^{2}\ n3^{4}\ rY^{4}\ +$
$2\ a1^{3}\ a2\ cYn1\ cYn2^{2}\ n3^{4}\ rY^{4} + a0^{2}\ a2^{2}\ cYn1\ cYn2^{2}\ n3^{4}\ rY^{4}\ -$
$2\ a0\ a1\ a2^{2}\ cYn1\ cYn2^{2}\ n3^{4}\ rY^{4} + a1^{2}\ a2^{2}\ cYn1\ cYn2^{2}\ n3^{4}\ rY^{4}\ +$
$12\ a0^{2}\ a1^{2}\ cYn1\ cYn3^{2}\ n3^{4}\ rY^{4} + 12\ a0\ a1^{3}\ cYn1\ cYn3^{2}\ n3^{4}\ rY^{4}\ +$
$2\ a1^{4}\ cYn1\ cYn3^{2}\ n3^{4}\ rY^{4} + 16\ a0^{2}\ a1\ a2\ cYn1\ cYn3^{2}\ n3^{4}\ rY^{4}\ +$
$36\ a0\ a1^{2}\ a2\ cYn1\ cYn3^{2}\ n3^{4}\ rY^{4} + 12\ a1^{3}\ a2\ cYn1\ cYn3^{2}\ n3^{4}\ rY^{4}\ +$
$2\ a0^{2}\ a2^{2}\ cYn1\ cYn3^{2}\ n3^{4}\ rY^{4} + 16\ a0\ a1\ a2^{2}\ cYn1\ cYn3^{2}\ n3^{4}\ rY^{4}\ +$
$12\ a1^{2}\ a2^{2}\ cYn1\ cYn3^{2}\ n3^{4}\ rY^{4} + 8\ a0^{2}\ a1\ cYn1\ cYn3\ n3^{5}\ rY^{5}\ +$
$12\ a0\ a1^{2}\ cYn1\ cYn3\ n3^{5}\ rY^{5} + 4\ a1^{3}\ cYn1\ cYn3\ n3^{5}\ rY^{5}\ +$
$4\ a0^{2}\ a2\ cYn1\ cYn3\ n3^{5}\ rY^{5} + 20\ a0\ a1\ a2\ cYn1\ cYn3\ n3^{5}\ rY^{5}\ +$
$12\ a1^{2}\ a2\ cYn1\ cYn3\ n3^{5}\ rY^{5} + 4\ a0\ a2^{2}\ cYn1\ cYn3\ n3^{5}\ rY^{5}\ +$
$8\ a1\ a2^{2}\ cYn1\ cYn3\ n3^{5}\ rY^{5} + 2\ a0^{2}\ cYn1\ n3^{6}\ rY^{6}\ +$
$4\ a0\ a1\ cYn1\ n3^{6}\ rY^{6} + 2\ a1^{2}\ cYn1\ n3^{6}\ rY^{6}\ +$
$4\ a0\ a2\ cYn1\ n3^{6}\ rY^{6} + 4\ a1\ a2\ cYn1\ n3^{6}\ rY^{6}\ +$
$2\ a2^{2}\ cYn1\ n3^{6}\ rY^{6} + 2\ a0\ a1^{2}\ a2\ cYn1^{2}\ cYn3^{2}\ rY$
$\surd\ (\ (a0\ a1\ a2\ cYn1\ cYn3\ n3 + a0\ a1\ cYn1\ n3^{2}\ rY + a0\ a2\ cYn1\ n3^{2}\ rY + a1\ a2\ cYn1\ n3^{2}\ rY)^{2}\ -$
$4\ a0\ a1\ a2\ cYn1^{2}\ n3^{2}\ (a0\ a1\ a2\ cYn3^{2} + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY\ +$
$a1\ a2\ cYn3\ n3\ rY + a0\ n3^{2}\ rY^{2} + a1\ n3^{2}\ rY^{2} + a2\ n3^{2}\ rY^{2})\ ) - 2\ a0\ a1^{2}\ a2\ cYn2^{2}\ cYn3^{2}$
$rY\ \surd\ (\ (a0\ a1\ a2\ cYn1\ cYn3\ n3 + a0\ a1\ cYn1\ n3^{2}\ rY + a0\ a2\ cYn1\ n3^{2}\ rY + a1\ a2\ cYn1\ n3^{2}\ rY)^{2}\ -$
$4\ a0\ a1\ a2\ cYn1^{2}\ n3^{2}\ (a0\ a1\ a2\ cYn3^{2} + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY + a1\ a2$
$cYn3\ n3\ rY + a0\ n3^{2}\ rY^{2} + a1\ n3^{2}\ rY^{2} + a2\ n3^{2}\ rY^{2})\ ) + 2\ a0\ a1^{2}\ cYn1^{2}\ cYn3\ n3\ rY^{2}$

$$\sqrt{}\,((a0\ a1\ a2\ cYn1\ cYn3\ n3 + a0\ a1\ cYn1\ n3^2\ rY + a0\ a2\ cYn1\ n3^2\ rY + a1\ a2\ cYn1\ n3^2\ rY)^2 -$$
$$4\ a0\ a1\ a2\ cYn1^2\ n3^2\ (a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY + a1\ a2$$
$$cYn3\ n3\ rY + a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2)) + a0\ a1\ a2\ cYn1^2\ cYn3\ n3\ rY^2$$
$$\sqrt{}\,((a0\ a1\ a2\ cYn1\ cYn3\ n3 + a0\ a1\ cYn1\ n3^2\ rY + a0\ a2\ cYn1\ n3^2\ rY + a1\ a2\ cYn1\ n3^2\ rY)^2 -$$
$$4\ a0\ a1\ a2\ cYn1^2\ n3^2\ (a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY + a1\ a2$$
$$cYn3\ n3\ rY + a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2)) + 2\ a1^2\ a2\ cYn1^2\ cYn3\ n3\ rY^2$$
$$\sqrt{}\,((a0\ a1\ a2\ cYn1\ cYn3\ n3 + a0\ a1\ cYn1\ n3^2\ rY + a0\ a2\ cYn1\ n3^2\ rY + a1\ a2\ cYn1\ n3^2\ rY)^2 -$$
$$4\ a0\ a1\ a2\ cYn1^2\ n3^2\ (a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY + a1\ a2$$
$$cYn3\ n3\ rY + a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2)) - 2\ a0\ a1^2\ cYn2^2\ cYn3\ n3\ rY^2$$
$$\sqrt{}\,((a0\ a1\ a2\ cYn1\ cYn3\ n3 + a0\ a1\ cYn1\ n3^2\ rY + a0\ a2\ cYn1\ n3^2\ rY + a1\ a2\ cYn1\ n3^2\ rY)^2 -$$
$$4\ a0\ a1\ a2\ cYn1^2\ n3^2\ (a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY + a1\ a2$$
$$cYn3\ n3\ rY + a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2)) - a0\ a1\ a2\ cYn2^2\ cYn3\ n3\ rY^2$$
$$\sqrt{}\,((a0\ a1\ a2\ cYn1\ cYn3\ n3 + a0\ a1\ cYn1\ n3^2\ rY + a0\ a2\ cYn1\ n3^2\ rY + a1\ a2\ cYn1\ n3^2\ rY)^2 -$$
$$4\ a0\ a1\ a2\ cYn1^2\ n3^2\ (a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY + a1\ a2$$
$$cYn3\ n3\ rY + a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2)) - 2\ a1^2\ a2\ cYn2^2\ cYn3\ n3\ rY^2$$
$$\sqrt{}\,((a0\ a1\ a2\ cYn1\ cYn3\ n3 + a0\ a1\ cYn1\ n3^2\ rY + a0\ a2\ cYn1\ n3^2\ rY + a1\ a2\ cYn1\ n3^2\ rY)^2 -$$
$$4\ a0\ a1\ a2\ cYn1^2\ n3^2\ (a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY +$$
$$a1\ a2\ cYn3\ n3\ rY + a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2)) + a0\ a1\ cYn1^2\ n3^2\ rY^3$$
$$\sqrt{}\,((a0\ a1\ a2\ cYn1\ cYn3\ n3 + a0\ a1\ cYn1\ n3^2\ rY + a0\ a2\ cYn1\ n3^2\ rY + a1\ a2\ cYn1\ n3^2\ rY)^2 -$$
$$4\ a0\ a1\ a2\ cYn1^2\ n3^2\ (a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY +$$
$$a1\ a2\ cYn3\ n3\ rY + a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2)) + 2\ a1^2\ cYn1^2\ n3^2\ rY^3$$
$$\sqrt{}\,((a0\ a1\ a2\ cYn1\ cYn3\ n3 + a0\ a1\ cYn1\ n3^2\ rY + a0\ a2\ cYn1\ n3^2\ rY + a1\ a2\ cYn1\ n3^2\ rY)^2 -$$
$$4\ a0\ a1\ a2\ cYn1^2\ n3^2\ (a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY +$$
$$a1\ a2\ cYn3\ n3\ rY + a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2)) - a0\ a2\ cYn1^2\ n3^2\ rY^3$$
$$\sqrt{}\,((a0\ a1\ a2\ cYn1\ cYn3\ n3 + a0\ a1\ cYn1\ n3^2\ rY + a0\ a2\ cYn1\ n3^2\ rY + a1\ a2\ cYn1\ n3^2\ rY)^2 -$$
$$4\ a0\ a1\ a2\ cYn1^2\ n3^2\ (a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY +$$
$$a1\ a2\ cYn3\ n3\ rY + a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2)) + a1\ a2\ cYn1^2\ n3^2\ rY^3$$
$$\sqrt{}\,((a0\ a1\ a2\ cYn1\ cYn3\ n3 + a0\ a1\ cYn1\ n3^2\ rY + a0\ a2\ cYn1\ n3^2\ rY + a1\ a2\ cYn1\ n3^2\ rY)^2 -$$
$$4\ a0\ a1\ a2\ cYn1^2\ n3^2\ (a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY +$$
$$a1\ a2\ cYn3\ n3\ rY + a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2)) - a0\ a1\ cYn2^2\ n3^2\ rY^3$$
$$\sqrt{}\,((a0\ a1\ a2\ cYn1\ cYn3\ n3 + a0\ a1\ cYn1\ n3^2\ rY + a0\ a2\ cYn1\ n3^2\ rY + a1\ a2\ cYn1\ n3^2\ rY)^2 -$$
$$4\ a0\ a1\ a2\ cYn1^2\ n3^2\ (a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY +$$
$$a1\ a2\ cYn3\ n3\ rY + a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2)) - 2\ a1^2\ cYn2^2\ n3^2\ rY^3$$
$$\sqrt{}\,((a0\ a1\ a2\ cYn1\ cYn3\ n3 + a0\ a1\ cYn1\ n3^2\ rY + a0\ a2\ cYn1\ n3^2\ rY + a1\ a2\ cYn1\ n3^2\ rY)^2 -$$
$$4\ a0\ a1\ a2\ cYn1^2\ n3^2\ (a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY +$$
$$a1\ a2\ cYn3\ n3\ rY + a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2)) + a0\ a2\ cYn2^2\ n3^2\ rY^3$$
$$\sqrt{}\,((a0\ a1\ a2\ cYn1\ cYn3\ n3 + a0\ a1\ cYn1\ n3^2\ rY + a0\ a2\ cYn1\ n3^2\ rY + a1\ a2\ cYn1\ n3^2\ rY)^2 -$$
$$4\ a0\ a1\ a2\ cYn1^2\ n3^2\ (a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY +$$
$$a1\ a2\ cYn3\ n3\ rY + a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2)) - a1\ a2\ cYn2^2\ n3^2\ rY^3$$
$$\sqrt{}\,((a0\ a1\ a2\ cYn1\ cYn3\ n3 + a0\ a1\ cYn1\ n3^2\ rY + a0\ a2\ cYn1\ n3^2\ rY + a1\ a2\ cYn1\ n3^2\ rY)^2 -$$
$$4\ a0\ a1\ a2\ cYn1^2\ n3^2\ (a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY +$$
$$a1\ a2\ cYn3\ n3\ rY + a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2)))) /$$
$$(2\ cYn1\ rY^2\ (a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY + a1\ a2\ cYn3\ n3\ rY +$$
$$a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2)^2),$$
$$Sn2 \to (6\ a0^2\ a1^2\ a2^4\ cYn1\ cYn3^4 + 2\ a0^2\ a1^2\ a2^4\ cYn1^3\ cYn3^4 + 2\ a0^2\ a1^2\ a2^4\ cYn1\ cYn2^2\ cYn3^4 +$$
$$2\ a0^2\ a1^2\ a2^4\ cYn1\ cYn3^6 + 12\ a0^2\ a1^2\ a2^3\ cYn1\ cYn3^3\ n3\ rY +$$
$$12\ a0^2\ a1\ a2^4\ cYn1\ cYn3^3\ n3\ rY + 12\ a0\ a1^2\ a2^4\ cYn1\ cYn3^3\ n3\ rY +$$
$$2\ a0^2\ a1^2\ a2^3\ cYn1^3\ cYn3^3\ n3\ rY +$$
$$4\ a0^2\ a1\ a2^4\ cYn1^3\ cYn3^3\ n3\ rY + 4\ a0\ a1^2\ a2^4\ cYn1^3\ cYn3^3\ n3\ rY +$$
$$2\ a0^2\ a1^2\ a2^3\ cYn1\ cYn2^2\ cYn3^3\ n3\ rY +$$

$4 a0^2 a1 a2^4 cYn1 cYn2^2 cYn3^3 n3 rY +$
$4 a0 a1^2 a2^4 cYn1 cYn2^2 cYn3^3 n3 rY +$
$8 a0^2 a1^2 a2^3 cYn1 cYn3^5 n3 rY + 4 a0^2 a1 a2^4 cYn1 cYn3^5 n3 rY +$
$4 a0 a1^2 a2^4 cYn1 cYn3^5 n3 rY + 6 a0^2 a1^2 a2^2 cYn1 cYn3^2 n3^2 rY^2 +$
$24 a0^2 a1 a2^3 cYn1 cYn3^2 n3^2 rY^2 +$
$24 a0 a1^2 a2^3 cYn1 cYn3^2 n3^2 rY^2 + 6 a0^2 a2^4 cYn1 cYn3^2 n3^2 rY^2 +$
$24 a0 a1 a2^4 cYn1 cYn3^2 n3^2 rY^2 + 6 a1^2 a2^4 cYn1 cYn3^2 n3^2 rY^2 -$
$3 a0^2 a1^2 a2^2 cYn1^3 cYn3^2 n3^2 rY^2 + 4 a0^2 a1 a2^3 cYn1^3 cYn3^2 n3^2 rY^2 +$
$4 a0 a1^2 a2^3 cYn1^3 cYn3^2 n3^2 rY^2 + 2 a0^2 a2^4 cYn1^3 cYn3^2 n3^2 rY^2 +$
$8 a0 a1 a2^4 cYn1^3 cYn3^2 n3^2 rY^2 + 2 a1^2 a2^4 cYn1^3 cYn3^2 n3^2 rY^2 -$
$3 a0^2 a1^2 a2^2 cYn1 cYn2^2 cYn3^2 n3^2 rY^2 +$
$4 a0^2 a1 a2^3 cYn1 cYn2^2 cYn3^2 n3^2 rY^2 +$
$4 a0 a1^2 a2^3 cYn1 cYn2^2 cYn3^2 n3^2 rY^2 +$
$2 a0^2 a2^4 cYn1 cYn2^2 cYn3^2 n3^2 rY^2 +$
$8 a0 a1 a2^4 cYn1 cYn2^2 cYn3^2 n3^2 rY^2 +$
$2 a1^2 a2^4 cYn1 cYn2^2 cYn3^2 n3^2 rY^2 +$
$12 a0^2 a1^2 a2^2 cYn1 cYn3^4 n3^2 rY^2 +$
$16 a0^2 a1 a2^3 cYn1 cYn3^4 n3^2 rY^2 + 16 a0 a1^2 a2^3 cYn1 cYn3^4 n3^2 rY^2 +$
$2 a0^2 a2^4 cYn1 cYn3^4 n3^2 rY^2 + 8 a0 a1 a2^4 cYn1 cYn3^4 n3^2 rY^2 +$
$2 a1^2 a2^4 cYn1 cYn3^4 n3^2 rY^2 + 12 a0^2 a1 a2^2 cYn1 cYn3 n3^3 rY^3 +$
$12 a0 a1^2 a2^2 cYn1 cYn3 n3^3 rY^3 + 12 a0^2 a2^3 cYn1 cYn3 n3^3 rY^3 +$
$36 a0 a1 a2^3 cYn1 cYn3 n3^3 rY^3 + 12 a1^2 a2^3 cYn1 cYn3 n3^3 rY^3 +$
$12 a0 a2^4 cYn1 cYn3 n3^3 rY^3 + 12 a1 a2^4 cYn1 cYn3 n3^3 rY^3 -$
$2 a0^2 a1^2 a2 cYn1^3 cYn3 n3^3 rY^3 - 2 a0^2 a1 a2^2 cYn1^3 cYn3 n3^3 rY^3 -$
$2 a0 a1^2 a2^2 cYn1^3 cYn3 n3^3 rY^3 + 2 a0^2 a2^3 cYn1^3 cYn3 n3^3 rY^3 +$
$6 a0 a1 a2^3 cYn1^3 cYn3 n3^3 rY^3 + 2 a1^2 a2^3 cYn1^3 cYn3 n3^3 rY^3 +$
$4 a0 a2^4 cYn1^3 cYn3 n3^3 rY^3 + 4 a1 a2^4 cYn1^3 cYn3 n3^3 rY^3 -$
$2 a0^2 a1^2 a2 cYn1 cYn2^2 cYn3 n3^3 rY^3 -$
$2 a0^2 a1 a2^2 cYn1 cYn2^2 cYn3 n3^3 rY^3 -$
$2 a0 a1^2 a2^2 cYn1 cYn2^2 cYn3 n3^3 rY^3 +$
$2 a0^2 a2^3 cYn1 cYn2^2 cYn3 n3^3 rY^3 +$
$6 a0 a1 a2^3 cYn1 cYn2^2 cYn3 n3^3 rY^3 +$
$2 a1^2 a2^3 cYn1 cYn2^2 cYn3 n3^3 rY^3 +$
$4 a0 a2^4 cYn1 cYn2^2 cYn3 n3^3 rY^3 + 4 a1 a2^4 cYn1 cYn2^2 cYn3 n3^3 rY^3 +$
$8 a0^2 a1^2 a2 cYn1 cYn3^3 n3^3 rY^3 + 24 a0^2 a1 a2^2 cYn1 cYn3^3 n3^3 rY^3 +$
$24 a0 a1^2 a2^2 cYn1 cYn3^3 n3^3 rY^3 + 8 a0^2 a2^3 cYn1 cYn3^3 n3^3 rY^3 +$
$28 a0 a1 a2^3 cYn1 cYn3^3 n3^3 rY^3 + 8 a1^2 a2^3 cYn1 cYn3^3 n3^3 rY^3 +$
$4 a0 a2^4 cYn1 cYn3^3 n3^3 rY^3 + 4 a1 a2^4 cYn1 cYn3^3 n3^3 rY^3 +$
$6 a0^2 a2^2 cYn1 n3^4 rY^4 + 12 a0 a1 a2^2 cYn1 n3^4 rY^4 +$
$6 a1^2 a2^2 cYn1 n3^4 rY^4 + 12 a0 a2^3 cYn1 n3^4 rY^4 +$
$12 a1 a2^3 cYn1 n3^4 rY^4 + 6 a2^4 cYn1 n3^4 rY^4 +$
$a0^2 a1^2 cYn1^3 n3^4 rY^4 - 2 a0^2 a1 a2 cYn1^3 n3^4 rY^4 -$
$2 a0 a1^2 a2 cYn1^3 n3^4 rY^4 + a0^2 a2^2 cYn1^3 n3^4 rY^4 -$
$2 a0 a1 a2^2 cYn1^3 n3^4 rY^4 + a1^2 a2^2 cYn1^3 n3^4 rY^4 +$
$2 a0 a2^3 cYn1^3 n3^4 rY^4 + 2 a1 a2^3 cYn1^3 n3^4 rY^4 +$
$2 a2^4 cYn1^3 n3^4 rY^4 + a0^2 a1^2 cYn1 cYn2^2 n3^4 rY^4 -$
$2 a0^2 a1 a2 cYn1 cYn2^2 n3^4 rY^4 - 2 a0 a1^2 a2 cYn1 cYn2^2 n3^4 rY^4 +$
$a0^2 a2^2 cYn1 cYn2^2 n3^4 rY^4 - 2 a0 a1 a2^2 cYn1 cYn2^2 n3^4 rY^4 +$
$a1^2 a2^2 cYn1 cYn2^2 n3^4 rY^4 + 2 a0 a2^3 cYn1 cYn2^2 n3^4 rY^4 +$
$2 a1 a2^3 cYn1 cYn2^2 n3^4 rY^4 + 2 a2^4 cYn1 cYn2^2 n3^4 rY^4 +$
$2 a0^2 a1^2 cYn1 cYn3^2 n3^4 rY^4 + 16 a0^2 a1 a2 cYn1 cYn3^2 n3^4 rY^4 +$
$16 a0 a1^2 a2 cYn1 cYn3^2 n3^4 rY^4 + 12 a0^2 a2^2 cYn1 cYn3^2 n3^4 rY^4 +$
$36 a0 a1 a2^2 cYn1 cYn3^2 n3^4 rY^4 + 12 a1^2 a2^2 cYn1 cYn3^2 n3^4 rY^4 +$
$12 a0 a2^3 cYn1 cYn3^2 n3^4 rY^4 + 12 a1 a2^3 cYn1 cYn3^2 n3^4 rY^4 +$
$2 a2^4 cYn1 cYn3^2 n3^4 rY^4 + 4 a0^2 a1 cYn1 cYn3 n3^5 rY^5 +$
$4 a0 a1^2 cYn1 cYn3 n3^5 rY^5 + 8 a0^2 a2 cYn1 cYn3 n3^5 rY^5 +$
$20 a0 a1 a2 cYn1 cYn3 n3^5 rY^5 + 8 a1^2 a2 cYn1 cYn3 n3^5 rY^5 +$

$$12\,a0\,a2^2\,cYn1\,cYn3\,n3^5\,rY^5 + 12\,a1\,a2^2\,cYn1\,cYn3\,n3^5\,rY^5 +$$
$$4\,a2^3\,cYn1\,cYn3\,n3^5\,rY^5 + 2\,a0^2\,cYn1\,n3^6\,rY^6 +$$
$$4\,a0\,a1\,cYn1\,n3^6\,rY^6 + 2\,a1^2\,cYn1\,n3^6\,rY^6 +$$
$$4\,a0\,a2\,cYn1\,n3^6\,rY^6 + 4\,a1\,a2\,cYn1\,n3^6\,rY^6 +$$
$$2\,a2^2\,cYn1\,n3^6\,rY^6 + 2\,a0\,a1\,a2^2\,cYn1^2\,cYn3^2\,rY$$
$$\sqrt{\left(\left(a0\,a1\,a2\,cYn1\,cYn3\,n3 + a0\,a1\,cYn1\,n3^2\,rY + a0\,a2\,cYn1\,n3^2\,rY + a1\,a2\,cYn1\,n3^2\,rY\right)^2 - 4\,a0\,a1\,a2\,cYn1^2\,n3^2\left(a0\,a1\,a2\,cYn3^2 + a0\,a1\,cYn3\,n3\,rY + a0\,a2\,cYn3\,n3\,rY + a1\,a2\,cYn3\,n3\,rY + a0\,n3^2\,rY^2 + a1\,n3^2\,rY^2 + a2\,n3^2\,rY^2\right)\right)} - 2\,a0\,a1\,a2^2\,cYn2^2\,cYn3^2$$
$$rY\sqrt{\left(\left(a0\,a1\,a2\,cYn1\,cYn3\,n3 + a0\,a1\,cYn1\,n3^2\,rY + a0\,a2\,cYn1\,n3^2\,rY + a1\,a2\,cYn1\,n3^2\,rY\right)^2 - 4\,a0\,a1\,a2\,cYn1^2\,n3^2\left(a0\,a1\,a2\,cYn3^2 + a0\,a1\,cYn3\,n3\,rY + a0\,a2\,cYn3\,n3\,rY + a1\,a2\,cYn3\,n3\,rY + a0\,n3^2\,rY^2 + a1\,n3^2\,rY^2 + a2\,n3^2\,rY^2\right)\right)} + a0\,a1\,a2\,cYn1^2\,cYn3\,n3\,rY^2$$
$$\sqrt{\left(\left(a0\,a1\,a2\,cYn1\,cYn3\,n3 + a0\,a1\,cYn1\,n3^2\,rY + a0\,a2\,cYn1\,n3^2\,rY + a1\,a2\,cYn1\,n3^2\,rY\right)^2 - 4\,a0\,a1\,a2\,cYn1^2\,n3^2\left(a0\,a1\,a2\,cYn3^2 + a0\,a1\,cYn3\,n3\,rY + a0\,a2\,cYn3\,n3\,rY + a1\,a2\,cYn3\,n3\,rY + a0\,n3^2\,rY^2 + a1\,n3^2\,rY^2 + a2\,n3^2\,rY^2\right)\right)} + 2\,a0\,a2^2\,cYn1^2\,cYn3\,n3\,rY^2$$
$$\sqrt{\left(\left(a0\,a1\,a2\,cYn1\,cYn3\,n3 + a0\,a1\,cYn1\,n3^2\,rY + a0\,a2\,cYn1\,n3^2\,rY + a1\,a2\,cYn1\,n3^2\,rY\right)^2 - 4\,a0\,a1\,a2\,cYn1^2\,n3^2\left(a0\,a1\,a2\,cYn3^2 + a0\,a1\,cYn3\,n3\,rY + a0\,a2\,cYn3\,n3\,rY + a1\,a2\,cYn3\,n3\,rY + a0\,n3^2\,rY^2 + a1\,n3^2\,rY^2 + a2\,n3^2\,rY^2\right)\right)} + 2\,a1\,a2^2\,cYn1^2\,cYn3\,n3\,rY^2$$
$$\sqrt{\left(\left(a0\,a1\,a2\,cYn1\,cYn3\,n3 + a0\,a1\,cYn1\,n3^2\,rY + a0\,a2\,cYn1\,n3^2\,rY + a1\,a2\,cYn1\,n3^2\,rY\right)^2 - 4\,a0\,a1\,a2\,cYn1^2\,n3^2\left(a0\,a1\,a2\,cYn3^2 + a0\,a1\,cYn3\,n3\,rY + a0\,a2\,cYn3\,n3\,rY + a1\,a2\,cYn3\,n3\,rY + a0\,n3^2\,rY^2 + a1\,n3^2\,rY^2 + a2\,n3^2\,rY^2\right)\right)} - a0\,a1\,a2\,cYn2^2\,cYn3\,n3\,rY^2$$
$$\sqrt{\left(\left(a0\,a1\,a2\,cYn1\,cYn3\,n3 + a0\,a1\,cYn1\,n3^2\,rY + a0\,a2\,cYn1\,n3^2\,rY + a1\,a2\,cYn1\,n3^2\,rY\right)^2 - 4\,a0\,a1\,a2\,cYn1^2\,n3^2\left(a0\,a1\,a2\,cYn3^2 + a0\,a1\,cYn3\,n3\,rY + a0\,a2\,cYn3\,n3\,rY + a1\,a2\,cYn3\,n3\,rY + a0\,n3^2\,rY^2 + a1\,n3^2\,rY^2 + a2\,n3^2\,rY^2\right)\right)} - 2\,a0\,a2^2\,cYn2^2\,cYn3\,n3\,rY^2$$
$$\sqrt{\left(\left(a0\,a1\,a2\,cYn1\,cYn3\,n3 + a0\,a1\,cYn1\,n3^2\,rY + a0\,a2\,cYn1\,n3^2\,rY + a1\,a2\,cYn1\,n3^2\,rY\right)^2 - 4\,a0\,a1\,a2\,cYn1^2\,n3^2\left(a0\,a1\,a2\,cYn3^2 + a0\,a1\,cYn3\,n3\,rY + a0\,a2\,cYn3\,n3\,rY + a1\,a2\,cYn3\,n3\,rY + a0\,n3^2\,rY^2 + a1\,n3^2\,rY^2 + a2\,n3^2\,rY^2\right)\right)} - 2\,a1\,a2^2\,cYn2^2\,cYn3\,n3\,rY^2$$
$$\sqrt{\left(\left(a0\,a1\,a2\,cYn1\,cYn3\,n3 + a0\,a1\,cYn1\,n3^2\,rY + a0\,a2\,cYn1\,n3^2\,rY + a1\,a2\,cYn1\,n3^2\,rY\right)^2 - 4\,a0\,a1\,a2\,cYn1^2\,n3^2\left(a0\,a1\,a2\,cYn3^2 + a0\,a1\,cYn3\,n3\,rY + a0\,a2\,cYn3\,n3\,rY + a1\,a2\,cYn3\,n3\,rY + a0\,n3^2\,rY^2 + a1\,n3^2\,rY^2 + a2\,n3^2\,rY^2\right)\right)} - a0\,a1\,cYn1^2\,n3^2\,rY^3$$
$$\sqrt{\left(\left(a0\,a1\,a2\,cYn1\,cYn3\,n3 + a0\,a1\,cYn1\,n3^2\,rY + a0\,a2\,cYn1\,n3^2\,rY + a1\,a2\,cYn1\,n3^2\,rY\right)^2 - 4\,a0\,a1\,a2\,cYn1^2\,n3^2\left(a0\,a1\,a2\,cYn3^2 + a0\,a1\,cYn3\,n3\,rY + a0\,a2\,cYn3\,n3\,rY + a1\,a2\,cYn3\,n3\,rY + a0\,n3^2\,rY^2 + a1\,n3^2\,rY^2 + a2\,n3^2\,rY^2\right)\right)} + a0\,a2\,cYn1^2\,n3^2\,rY^3$$
$$\sqrt{\left(\left(a0\,a1\,a2\,cYn1\,cYn3\,n3 + a0\,a1\,cYn1\,n3^2\,rY + a0\,a2\,cYn1\,n3^2\,rY + a1\,a2\,cYn1\,n3^2\,rY\right)^2 - 4\,a0\,a1\,a2\,cYn1^2\,n3^2\left(a0\,a1\,a2\,cYn3^2 + a0\,a1\,cYn3\,n3\,rY + a0\,a2\,cYn3\,n3\,rY + a1\,a2\,cYn3\,n3\,rY + a0\,n3^2\,rY^2 + a1\,n3^2\,rY^2 + a2\,n3^2\,rY^2\right)\right)} + a1\,a2\,cYn1^2\,n3^2\,rY^3$$
$$\sqrt{\left(\left(a0\,a1\,a2\,cYn1\,cYn3\,n3 + a0\,a1\,cYn1\,n3^2\,rY + a0\,a2\,cYn1\,n3^2\,rY + a1\,a2\,cYn1\,n3^2\,rY\right)^2 - 4\,a0\,a1\,a2\,cYn1^2\,n3^2\left(a0\,a1\,a2\,cYn3^2 + a0\,a1\,cYn3\,n3\,rY + a0\,a2\,cYn3\,n3\,rY + a1\,a2\,cYn3\,n3\,rY + a0\,n3^2\,rY^2 + a1\,n3^2\,rY^2 + a2\,n3^2\,rY^2\right)\right)} + 2\,a2^2\,cYn1^2\,n3^2\,rY^3$$
$$\sqrt{\left(\left(a0\,a1\,a2\,cYn1\,cYn3\,n3 + a0\,a1\,cYn1\,n3^2\,rY + a0\,a2\,cYn1\,n3^2\,rY + a1\,a2\,cYn1\,n3^2\,rY\right)^2 - 4\,a0\,a1\,a2\,cYn1^2\,n3^2\left(a0\,a1\,a2\,cYn3^2 + a0\,a1\,cYn3\,n3\,rY + a0\,a2\,cYn3\,n3\,rY + a1\,a2\,cYn3\,n3\,rY + a0\,n3^2\,rY^2 + a1\,n3^2\,rY^2 + a2\,n3^2\,rY^2\right)\right)} + a0\,a1\,cYn2^2\,n3^2\,rY^3$$
$$\sqrt{\left(\left(a0\,a1\,a2\,cYn1\,cYn3\,n3 + a0\,a1\,cYn1\,n3^2\,rY + a0\,a2\,cYn1\,n3^2\,rY + a1\,a2\,cYn1\,n3^2\,rY\right)^2 - 4\,a0\,a1\,a2\,cYn1^2\,n3^2\left(a0\,a1\,a2\,cYn3^2 + a0\,a1\,cYn3\,n3\,rY + a0\,a2\,cYn3\,n3\,rY + a1\,a2\,cYn3\,n3\,rY + a0\,n3^2\,rY^2 + a1\,n3^2\,rY^2 + a2\,n3^2\,rY^2\right)\right)} - a0\,a2\,cYn2^2\,n3^2\,rY^3$$
$$\sqrt{\left(\left(a0\,a1\,a2\,cYn1\,cYn3\,n3 + a0\,a1\,cYn1\,n3^2\,rY + a0\,a2\,cYn1\,n3^2\,rY + a1\,a2\,cYn1\,n3^2\,rY\right)^2 - 4\,a0\,a1\,a2\,cYn1^2\,n3^2\left(a0\,a1\,a2\,cYn3^2 + a0\,a1\,cYn3\,n3\,rY + a0\,a2\,cYn3\,n3\,rY + a1\,a2\,cYn3\,n3\,rY + a0\,n3^2\,rY^2 + a1\,n3^2\,rY^2 + a2\,n3^2\,rY^2\right)\right)} - a1\,a2\,cYn2^2\,n3^2\,rY^3$$
$$\sqrt{\left(\left(a0\,a1\,a2\,cYn1\,cYn3\,n3 + a0\,a1\,cYn1\,n3^2\,rY + a0\,a2\,cYn1\,n3^2\,rY + a1\,a2\,cYn1\,n3^2\,rY\right)^2 - 4\,a0\,a1\,a2\,cYn1^2\,n3^2\left(a0\,a1\,a2\,cYn3^2 + a0\,a1\,cYn3\,n3\,rY + a0\,a2\,cYn3\,n3\,rY + a1\,a2\,cYn3\,n3\,rY + a0\,n3^2\,rY^2 + a1\,n3^2\,rY^2 + a2\,n3^2\,rY^2\right)\right)} - 2\,a2^2\,cYn2^2\,n3^2\,rY^3$$

$\sqrt{}\,\Big(\big(\text{a0 a1 a2 cYn1 cYn3 n3} + \text{a0 a1 cYn1 n3}^2\,\text{rY} + \text{a0 a2 cYn1 n3}^2\,\text{rY} + \text{a1 a2 cYn1 n3}^2\,\text{rY}\big)^2 -$

$\quad 4\,\text{a0 a1 a2 cYn1}^2\,\text{n3}^2\,\big(\text{a0 a1 a2 cYn3}^2 + \text{a0 a1 cYn3 n3 rY} + \text{a0 a2 cYn3 n3 rY} +$

$\qquad \text{a1 a2 cYn3 n3 rY} + \text{a0 n3}^2\,\text{rY}^2 + \text{a1 n3}^2\,\text{rY}^2 + \text{a2 n3}^2\,\text{rY}^2\big)\Big)\Big)\Big/$

$\Big(2\,\text{cYn1 rY}^2\,\big(\text{a0 a1 a2 cYn3}^2 + \text{a0 a1 cYn3 n3 rY} + \text{a0 a2 cYn3 n3 rY} + \text{a1 a2 cYn3 n3 rY} +$

$\quad \text{a0 n3}^2\,\text{rY}^2 + \text{a1 n3}^2\,\text{rY}^2 + \text{a2 n3}^2\,\text{rY}^2\big)^2\Big)$, $\text{pY1} \to 3$,

$\text{n1} \to -\Big(\Big(\text{a0 a1 a2 cYn1 cYn3 n3} + \text{a0 a1 cYn1 n3}^2\,\text{rY} + \text{a0 a2 cYn1 n3}^2\,\text{rY} + \text{a1 a2 cYn1 n3}^2\,\text{rY} -$

$\quad \sqrt{}\,\Big(\big(\text{a0 a1 a2 cYn1 cYn3 n3} + \text{a0 a1 cYn1 n3}^2\,\text{rY} + \text{a0 a2 cYn1 n3}^2\,\text{rY} + \text{a1 a2 cYn1 n3}^2\,\text{rY}\big)^2 -$

$\qquad 4\,\text{a0 a1 a2 cYn1}^2\,\text{n3}^2\,\big(\text{a0 a1 a2 cYn3}^2 + \text{a0 a1 cYn3 n3 rY} + \text{a0 a2 cYn3 n3 rY} +$

$\qquad\quad \text{a1 a2 cYn3 n3 rY} + \text{a0 n3}^2\,\text{rY}^2 + \text{a1 n3}^2\,\text{rY}^2 + \text{a2 n3}^2\,\text{rY}^2\big)\Big)\Big)\Big/$

$\quad \Big(2\,\big(\text{a0 a1 a2 cYn3}^2 + \text{a0 a1 cYn3 n3 rY} + \text{a0 a2 cYn3 n3 rY} + \text{a1 a2 cYn3 n3 rY} +$

$\qquad \text{a0 n3}^2\,\text{rY}^2 + \text{a1 n3}^2\,\text{rY}^2 + \text{a2 n3}^2\,\text{rY}^2\big)\Big)\Big)$,

$\text{n2} \to -\Big(\Big(\text{cYn2}\,\big(\text{a0 a1 a2 cYn1 cYn3 n3} + \text{a0 a1 cYn1 n3}^2\,\text{rY} + \text{a0 a2 cYn1 n3}^2\,\text{rY} +$

$\qquad \text{a1 a2 cYn1 n3}^2\,\text{rY} + \sqrt{}\,\Big(\big(\text{a0 a1 a2 cYn1 cYn3 n3} + \text{a0 a1 cYn1 n3}^2\,\text{rY} + \text{a0 a2 cYn1 n3}^2\,\text{rY} +$

$\qquad\quad \text{a1 a2 cYn1 n3}^2\,\text{rY}\big)^2 - 4\,\text{a0 a1 a2 cYn1}^2\,\text{n3}^2\,\big(\text{a0 a1 a2 cYn3}^2 + \text{a0 a1 cYn3 n3 rY} +$

$\qquad\quad \text{a0 a2 cYn3 n3 rY} + \text{a1 a2 cYn3 n3 rY} + \text{a0 n3}^2\,\text{rY}^2 + \text{a1 n3}^2\,\text{rY}^2 + \text{a2 n3}^2\,\text{rY}^2\big)\Big)\big)\Big)\Big/$

$\quad \Big(2\,\text{cYn1}\,\big(\text{a0 a1 a2 cYn3}^2 + \text{a0 a1 cYn3 n3 rY} + \text{a0 a2 cYn3 n3 rY} + \text{a1 a2 cYn3 n3 rY} +$

$\qquad \text{a0 n3}^2\,\text{rY}^2 + \text{a1 n3}^2\,\text{rY}^2 + \text{a2 n3}^2\,\text{rY}^2\big)\Big)\Big)\Big\}$,

$\Big\{\text{p} \to \dfrac{3 + \text{cYn1}^2 + \text{cYn2}^2 + \text{cYn3}^2}{\text{rY}^2}$, $\text{d} \to -\dfrac{1}{\text{cYn1 cYn2 n3}^3}$

$\quad \text{rY}^2$

$\quad \big(\text{a0 a1 a2 cYn3}^2 + \text{a0 a1 cYn3 n3 rY} + \text{a0 a2 cYn3 n3 rY} +$

$\qquad \text{a1 a2 cYn3 n3 rY} + \text{a0 n3}^2\,\text{rY}^2 + \text{a1 n3}^2\,\text{rY}^2 + \text{a2 n3}^2\,\text{rY}^2\big)$,

$\text{N01} \to \dfrac{1}{\text{a2}^3\,\text{cYn1 cYn2 n3}^3\,\text{rY}}\,(\text{a0} - \text{a1})\,\big(\text{a0 a1 a2 cYn3}^2 + \text{a0 a1 cYn3 n3 rY} +$

$\quad \text{a0 a2 cYn3 n3 rY} + \text{a1 a2 cYn3 n3 rY} +$

$\quad \text{a0 n3}^2\,\text{rY}^2 + \text{a1 n3}^2\,\text{rY}^2 + \text{a2 n3}^2\,\text{rY}^2\big)$,

$\text{N02} \to -\dfrac{1}{\text{a1}^3\,\text{cYn1 cYn2 n3}^3\,\text{rY}}\,(\text{a0} - \text{a2})\,\big(\text{a0 a1 a2 cYn3}^2 + \text{a0 a1 cYn3 n3 rY} +$

$\quad \text{a0 a2 cYn3 n3 rY} + \text{a1 a2 cYn3 n3 rY} + \text{a0 n3}^2\,\text{rY}^2 + \text{a1 n3}^2\,\text{rY}^2 + \text{a2 n3}^2\,\text{rY}^2\big)$,

$\text{N12} \to -\dfrac{1}{\text{a0}^3\,\text{cYn1 cYn2 n3}^3\,\text{rY}}\,(\text{a1} - \text{a2})\,\big(\text{a0 a1 a2 cYn3}^2 + \text{a0 a1 cYn3 n3 rY} +$

$\quad \text{a0 a2 cYn3 n3 rY} + \text{a1 a2 cYn3 n3 rY} + \text{a0 n3}^2\,\text{rY}^2 + \text{a1 n3}^2\,\text{rY}^2 + \text{a2 n3}^2\,\text{rY}^2\big)$,

$\text{Sn0} \to \big(6\,\text{a0}^4\,\text{a1}^2\,\text{a2}^2\,\text{cYn1 cYn3}^4 + 2\,\text{a0}^4\,\text{a1}^2\,\text{a2}^2\,\text{cYn1}^3\,\text{cYn3}^4 +$

$\quad 2\,\text{a0}^4\,\text{a1}^2\,\text{a2}^2\,\text{cYn1 cYn2}^2\,\text{cYn3}^4 + 2\,\text{a0}^4\,\text{a1}^2\,\text{a2}^2\,\text{cYn1 cYn3}^6 +$

$\quad 12\,\text{a0}^4\,\text{a1}^2\,\text{a2 cYn1 cYn3}^3\,\text{n3 rY} + 12\,\text{a0}^4\,\text{a1 a2}^2\,\text{cYn1 cYn3}^3\,\text{n3 rY} +$

$\quad 12\,\text{a0}^3\,\text{a1}^2\,\text{a2}^2\,\text{cYn1 cYn3}^3\,\text{n3 rY} + 4\,\text{a0}^4\,\text{a1}^2\,\text{a2 cYn1}^3\,\text{cYn3}^3\,\text{n3 rY} +$

$\quad 4\,\text{a0}^4\,\text{a1 a2}^2\,\text{cYn1}^3\,\text{cYn3}^3\,\text{n3 rY} + 2\,\text{a0}^3\,\text{a1}^2\,\text{a2}^2\,\text{cYn1}^3\,\text{cYn3}^3\,\text{n3 rY} +$

$\quad 4\,\text{a0}^4\,\text{a1}^2\,\text{a2 cYn1 cYn2}^2\,\text{cYn3}^3\,\text{n3 rY} +$

$\quad 4\,\text{a0}^4\,\text{a1 a2}^2\,\text{cYn1 cYn2}^2\,\text{cYn3}^3\,\text{n3 rY} +$

$\quad 2\,\text{a0}^3\,\text{a1}^2\,\text{a2}^2\,\text{cYn1 cYn2}^2\,\text{cYn3}^3\,\text{n3 rY} +$

$\quad 4\,\text{a0}^4\,\text{a1}^2\,\text{a2 cYn1 cYn3}^5\,\text{n3 rY} + 4\,\text{a0}^4\,\text{a1 a2}^2\,\text{cYn1 cYn3}^5\,\text{n3 rY} +$

$\quad 8\,\text{a0}^3\,\text{a1}^2\,\text{a2}^2\,\text{cYn1 cYn3}^5\,\text{n3 rY} + 6\,\text{a0}^4\,\text{a1}^2\,\text{cYn1 cYn3}^2\,\text{n3}^2\,\text{rY}^2 +$

$\quad 24\,\text{a0}^4\,\text{a1 a2 cYn1 cYn3}^2\,\text{n3}^2\,\text{rY}^2 + 24\,\text{a0}^3\,\text{a1}^2\,\text{a2 cYn1 cYn3}^2\,\text{n3}^2\,\text{rY}^2 +$

$\quad 6\,\text{a0}^4\,\text{a2}^2\,\text{cYn1 cYn3}^2\,\text{n3}^2\,\text{rY}^2 + 24\,\text{a0}^3\,\text{a1 a2}^2\,\text{cYn1 cYn3}^2\,\text{n3}^2\,\text{rY}^2 +$

$\quad 6\,\text{a0}^2\,\text{a1}^2\,\text{a2}^2\,\text{cYn1 cYn3}^2\,\text{n3}^2\,\text{rY}^2 + 2\,\text{a0}^4\,\text{a1}^2\,\text{cYn1}^3\,\text{cYn3}^2\,\text{n3}^2\,\text{rY}^2 +$

$\quad 8\,\text{a0}^4\,\text{a1 a2 cYn1}^3\,\text{cYn3}^2\,\text{n3}^2\,\text{rY}^2 + 4\,\text{a0}^3\,\text{a1}^2\,\text{a2 cYn1}^3\,\text{cYn3}^2\,\text{n3}^2\,\text{rY}^2 +$

$\quad 2\,\text{a0}^4\,\text{a2}^2\,\text{cYn1}^3\,\text{cYn3}^2\,\text{n3}^2\,\text{rY}^2 + 4\,\text{a0}^3\,\text{a1 a2}^2\,\text{cYn1}^3\,\text{cYn3}^2\,\text{n3}^2\,\text{rY}^2 -$

$\quad 3\,\text{a0}^2\,\text{a1}^2\,\text{a2}^2\,\text{cYn1}^3\,\text{cYn3}^2\,\text{n3}^2\,\text{rY}^2 +$

$\quad 2\,\text{a0}^4\,\text{a1}^2\,\text{cYn1 cYn2}^2\,\text{cYn3}^2\,\text{n3}^2\,\text{rY}^2 +$

$\quad 8\,\text{a0}^4\,\text{a1 a2 cYn1 cYn2}^2\,\text{cYn3}^2\,\text{n3}^2\,\text{rY}^2 +$

$\quad 4\,\text{a0}^3\,\text{a1}^2\,\text{a2 cYn1 cYn2}^2\,\text{cYn3}^2\,\text{n3}^2\,\text{rY}^2 +$

$\quad 2\,\text{a0}^4\,\text{a2}^2\,\text{cYn1 cYn2}^2\,\text{cYn3}^2\,\text{n3}^2\,\text{rY}^2 +$

$\quad 4\,\text{a0}^3\,\text{a1 a2}^2\,\text{cYn1 cYn2}^2\,\text{cYn3}^2\,\text{n3}^2\,\text{rY}^2 -$

$3\ a0^2\ a1^2\ a2^2\ cYn1\ cYn2^2\ cYn3^2\ n3^2\ rY^2 +$
$2\ a0^4\ a1^2\ cYn1\ cYn3^4\ n3^2\ rY^2 + 8\ a0^4\ a1\ a2\ cYn1\ cYn3^4\ n3^2\ rY^2 +$
$16\ a0^3\ a1^2\ a2\ cYn1\ cYn3^4\ n3^2\ rY^2 +$
$2\ a0^4\ a2^2\ cYn1\ cYn3^4\ n3^2\ rY^2 + 16\ a0^3\ a1\ a2^2\ cYn1\ cYn3^4\ n3^2\ rY^2 +$
$12\ a0^2\ a1^2\ a2^2\ cYn1\ cYn3^4\ n3^2\ rY^2 +$
$12\ a0^4\ a1\ cYn1\ cYn3\ n3^3\ rY^3 + 12\ a0^3\ a1^2\ cYn1\ cYn3\ n3^3\ rY^3 +$
$12\ a0^4\ a2\ cYn1\ cYn3\ n3^3\ rY^3 + 36\ a0^3\ a1\ a2\ cYn1\ cYn3\ n3^3\ rY^3 +$
$12\ a0^2\ a1^2\ a2\ cYn1\ cYn3\ n3^3\ rY^3 + 12\ a0^3\ a2^2\ cYn1\ cYn3\ n3^3\ rY^3 +$
$12\ a0^2\ a1\ a2^2\ cYn1\ cYn3\ n3^3\ rY^3 + 4\ a0^4\ a1\ cYn1^3\ cYn3\ n3^3\ rY^3 +$
$2\ a0^3\ a1^2\ cYn1^3\ cYn3\ n3^3\ rY^3 + 4\ a0^4\ a2\ cYn1^3\ cYn3\ n3^3\ rY^3 +$
$6\ a0^3\ a1\ a2\ cYn1^3\ cYn3\ n3^3\ rY^3 - 2\ a0^2\ a1^2\ a2\ cYn1^3\ cYn3\ n3^3\ rY^3 +$
$2\ a0^3\ a2^2\ cYn1^3\ cYn3\ n3^3\ rY^3 - 2\ a0^2\ a1\ a2^2\ cYn1^3\ cYn3\ n3^3\ rY^3 -$
$2\ a0\ a1^2\ a2^2\ cYn1^3\ cYn3\ n3^3\ rY^3 + 4\ a0^4\ a1\ cYn1\ cYn2^2\ cYn3\ n3^3\ rY^3 +$
$2\ a0^3\ a1^2\ cYn1\ cYn2^2\ cYn3\ n3^3\ rY^3 +$
$4\ a0^4\ a2\ cYn1\ cYn2^2\ cYn3\ n3^3\ rY^3 +$
$6\ a0^3\ a1\ a2\ cYn1\ cYn2^2\ cYn3\ n3^3\ rY^3 -$
$2\ a0^2\ a1^2\ a2\ cYn1\ cYn2^2\ cYn3\ n3^3\ rY^3 +$
$2\ a0^3\ a2^2\ cYn1\ cYn2^2\ cYn3\ n3^3\ rY^3 -$
$2\ a0^2\ a1\ a2^2\ cYn1\ cYn2^2\ cYn3\ n3^3\ rY^3 -$
$2\ a0\ a1^2\ a2^2\ cYn1\ cYn2^2\ cYn3\ n3^3\ rY^3 +$
$4\ a0^4\ a1\ cYn1\ cYn3^3\ n3^3\ rY^3 + 8\ a0^3\ a1^2\ cYn1\ cYn3^3\ n3^3\ rY^3 +$
$4\ a0^4\ a2\ cYn1\ cYn3^3\ n3^3\ rY^3 + 28\ a0^3\ a1\ a2\ cYn1\ cYn3^3\ n3^3\ rY^3 +$
$24\ a0^2\ a1^2\ a2\ cYn1\ cYn3^3\ n3^3\ rY^3 + 8\ a0^3\ a2^2\ cYn1\ cYn3^3\ n3^3\ rY^3 +$
$24\ a0^2\ a1\ a2^2\ cYn1\ cYn3^3\ n3^3\ rY^3 + 8\ a0\ a1^2\ a2^2\ cYn1\ cYn3^3\ n3^3\ rY^3 +$
$6\ a0^4\ cYn1\ n3^4\ rY^4 + 12\ a0^3\ a1\ cYn1\ n3^4\ rY^4 +$
$6\ a0^2\ a1^2\ cYn1\ n3^4\ rY^4 + 12\ a0^3\ a2\ cYn1\ n3^4\ rY^4 +$
$12\ a0^2\ a1\ a2\ cYn1\ n3^4\ rY^4 + 6\ a0^2\ a2^2\ cYn1\ n3^4\ rY^4 +$
$2\ a0^4\ cYn1^3\ n3^4\ rY^4 + 2\ a0^3\ a1\ cYn1^3\ n3^4\ rY^4 +$
$a0^2\ a1^2\ cYn1^3\ n3^4\ rY^4 + 2\ a0^3\ a2\ cYn1^3\ n3^4\ rY^4 -$
$2\ a0^2\ a1\ a2\ cYn1^3\ n3^4\ rY^4 - 2\ a0\ a1^2\ a2\ cYn1^3\ n3^4\ rY^4 +$
$a0^2\ a2^2\ cYn1^3\ n3^4\ rY^4 - 2\ a0\ a1\ a2^2\ cYn1^3\ n3^4\ rY^4 +$
$a1^2\ a2^2\ cYn1^3\ n3^4\ rY^4 + 2\ a0^4\ cYn1\ cYn2^2\ n3^4\ rY^4 +$
$2\ a0^3\ a1\ cYn1\ cYn2^2\ n3^4\ rY^4 + a0^2\ a1^2\ cYn1\ cYn2^2\ n3^4\ rY^4 +$
$2\ a0^3\ a2\ cYn1\ cYn2^2\ n3^4\ rY^4 - 2\ a0^2\ a1\ a2\ cYn1\ cYn2^2\ n3^4\ rY^4 -$
$2\ a0\ a1^2\ a2\ cYn1\ cYn2^2\ n3^4\ rY^4 + a0^2\ a2^2\ cYn1\ cYn2^2\ n3^4\ rY^4 -$
$2\ a0\ a1\ a2^2\ cYn1\ cYn2^2\ n3^4\ rY^4 + a1^2\ a2^2\ cYn1\ cYn2^2\ n3^4\ rY^4 +$
$2\ a0^4\ cYn1\ cYn3^2\ n3^4\ rY^4 + 12\ a0^3\ a1\ cYn1\ cYn3^2\ n3^4\ rY^4 +$
$12\ a0^2\ a1^2\ cYn1\ cYn3^2\ n3^4\ rY^4 + 12\ a0^3\ a2\ cYn1\ cYn3^2\ n3^4\ rY^4 +$
$36\ a0^2\ a1\ a2\ cYn1\ cYn3^2\ n3^4\ rY^4 + 16\ a0\ a1^2\ a2\ cYn1\ cYn3^2\ n3^4\ rY^4 +$
$12\ a0^2\ a2^2\ cYn1\ cYn3^2\ n3^4\ rY^4 + 16\ a0\ a1\ a2^2\ cYn1\ cYn3^2\ n3^4\ rY^4 +$
$2\ a1^2\ a2^2\ cYn1\ cYn3^2\ n3^4\ rY^4 + 4\ a0^3\ cYn1\ cYn3\ n3^5\ rY^5 +$
$12\ a0^2\ a1\ cYn1\ cYn3\ n3^5\ rY^5 + 8\ a0\ a1^2\ cYn1\ cYn3\ n3^5\ rY^5 +$
$12\ a0^2\ a2\ cYn1\ cYn3\ n3^5\ rY^5 + 20\ a0\ a1\ a2\ cYn1\ cYn3\ n3^5\ rY^5 +$
$4\ a1^2\ a2\ cYn1\ cYn3\ n3^5\ rY^5 + 8\ a0\ a2^2\ cYn1\ cYn3\ n3^5\ rY^5 +$
$4\ a1\ a2^2\ cYn1\ cYn3\ n3^5\ rY^5 + 2\ a0^2\ cYn1\ n3^6\ rY^6 +$
$4\ a0\ a1\ cYn1\ n3^6\ rY^6 + 2\ a1^2\ cYn1\ n3^6\ rY^6 +$
$4\ a0\ a2\ cYn1\ n3^6\ rY^6 + 4\ a1\ a2\ cYn1\ n3^6\ rY^6 +$
$2\ a2^2\ cYn1\ n3^6\ rY^6 + 2\ a0^2\ a1\ a2\ cYn1^2\ cYn3^2\ rY$
$\sqrt{}\ \big(\big(a0\ a1\ a2\ cYn1\ cYn3\ n3 + a0\ a1\ cYn1\ n3^2\ rY + a0\ a2\ cYn1\ n3^2\ rY + a1\ a2\ cYn1\ n3^2\ rY\big)^2 -$
$4\ a0\ a1\ a2\ cYn1^2\ n3^2\ \big(a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY +$
$a1\ a2\ cYn3\ n3\ rY + a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2\big)\big) - 2\ a0^2\ a1\ a2\ cYn2^2\ cYn3^2$
$rY\ \sqrt{}\ \big(\big(a0\ a1\ a2\ cYn1\ cYn3\ n3 + a0\ a1\ cYn1\ n3^2\ rY + a0\ a2\ cYn1\ n3^2\ rY + a1\ a2\ cYn1\ n3^2\ rY\big)^2 -$
$4\ a0\ a1\ a2\ cYn1^2\ n3^2\ \big(a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY + a1\ a2$
$cYn3\ n3\ rY + a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2\big)\big) + 2\ a0^2\ a1\ cYn1^2\ cYn3\ n3\ rY^2$
$\sqrt{}\ \big(\big(a0\ a1\ a2\ cYn1\ cYn3\ n3 + a0\ a1\ cYn1\ n3^2\ rY + a0\ a2\ cYn1\ n3^2\ rY + a1\ a2\ cYn1\ n3^2\ rY\big)^2 -$
$4\ a0\ a1\ a2\ cYn1^2\ n3^2\ \big(a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY + a1\ a2$

$cYn3\ n3\ rY + a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2)) + 2\ a0^2\ a2\ cYn1^2\ cYn3\ n3\ rY^2$
$\surd((a0\ a1\ a2\ cYn1\ cYn3\ n3 + a0\ a1\ cYn1\ n3^2\ rY + a0\ a2\ cYn1\ n3^2\ rY + a1\ a2\ cYn1\ n3^2\ rY)^2 -$
$4\ a0\ a1\ a2\ cYn1^2\ n3^2\ (a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY + a1\ a2$
$cYn3\ n3\ rY + a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2)) + a0\ a1\ a2\ cYn1^2\ cYn3\ n3\ rY^2$
$\surd((a0\ a1\ a2\ cYn1\ cYn3\ n3 + a0\ a1\ cYn1\ n3^2\ rY + a0\ a2\ cYn1\ n3^2\ rY + a1\ a2\ cYn1\ n3^2\ rY)^2 -$
$4\ a0\ a1\ a2\ cYn1^2\ n3^2\ (a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY + a1\ a2$
$cYn3\ n3\ rY + a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2)) - 2\ a0^2\ a1\ cYn2^2\ cYn3\ n3\ rY^2$
$\surd((a0\ a1\ a2\ cYn1\ cYn3\ n3 + a0\ a1\ cYn1\ n3^2\ rY + a0\ a2\ cYn1\ n3^2\ rY + a1\ a2\ cYn1\ n3^2\ rY)^2 -$
$4\ a0\ a1\ a2\ cYn1^2\ n3^2\ (a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY + a1\ a2$
$cYn3\ n3\ rY + a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2)) - 2\ a0^2\ a2\ cYn2^2\ cYn3\ n3\ rY^2$
$\surd((a0\ a1\ a2\ cYn1\ cYn3\ n3 + a0\ a1\ cYn1\ n3^2\ rY + a0\ a2\ cYn1\ n3^2\ rY + a1\ a2\ cYn1\ n3^2\ rY)^2 -$
$4\ a0\ a1\ a2\ cYn1^2\ n3^2\ (a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY + a1\ a2$
$cYn3\ n3\ rY + a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2)) - a0\ a1\ a2\ cYn2^2\ cYn3\ n3\ rY^2$
$\surd((a0\ a1\ a2\ cYn1\ cYn3\ n3 + a0\ a1\ cYn1\ n3^2\ rY + a0\ a2\ cYn1\ n3^2\ rY + a1\ a2\ cYn1\ n3^2\ rY)^2 -$
$4\ a0\ a1\ a2\ cYn1^2\ n3^2\ (a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY +$
$a1\ a2\ cYn3\ n3\ rY + a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2)) + 2\ a0^2\ cYn1^2\ n3^2\ rY^3$
$\surd((a0\ a1\ a2\ cYn1\ cYn3\ n3 + a0\ a1\ cYn1\ n3^2\ rY + a0\ a2\ cYn1\ n3^2\ rY + a1\ a2\ cYn1\ n3^2\ rY)^2 -$
$4\ a0\ a1\ a2\ cYn1^2\ n3^2\ (a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY +$
$a1\ a2\ cYn3\ n3\ rY + a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2)) + a0\ a1\ cYn1^2\ n3^2\ rY^3$
$\surd((a0\ a1\ a2\ cYn1\ cYn3\ n3 + a0\ a1\ cYn1\ n3^2\ rY + a0\ a2\ cYn1\ n3^2\ rY + a1\ a2\ cYn1\ n3^2\ rY)^2 -$
$4\ a0\ a1\ a2\ cYn1^2\ n3^2\ (a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY +$
$a1\ a2\ cYn3\ n3\ rY + a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2)) + a0\ a2\ cYn1^2\ n3^2\ rY^3$
$\surd((a0\ a1\ a2\ cYn1\ cYn3\ n3 + a0\ a1\ cYn1\ n3^2\ rY + a0\ a2\ cYn1\ n3^2\ rY + a1\ a2\ cYn1\ n3^2\ rY)^2 -$
$4\ a0\ a1\ a2\ cYn1^2\ n3^2\ (a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY +$
$a1\ a2\ cYn3\ n3\ rY + a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2)) - a1\ a2\ cYn1^2\ n3^2\ rY^3$
$\surd((a0\ a1\ a2\ cYn1\ cYn3\ n3 + a0\ a1\ cYn1\ n3^2\ rY + a0\ a2\ cYn1\ n3^2\ rY + a1\ a2\ cYn1\ n3^2\ rY)^2 -$
$4\ a0\ a1\ a2\ cYn1^2\ n3^2\ (a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY +$
$a1\ a2\ cYn3\ n3\ rY + a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2)) - 2\ a0^2\ cYn2^2\ n3^2\ rY^3$
$\surd((a0\ a1\ a2\ cYn1\ cYn3\ n3 + a0\ a1\ cYn1\ n3^2\ rY + a0\ a2\ cYn1\ n3^2\ rY + a1\ a2\ cYn1\ n3^2\ rY)^2 -$
$4\ a0\ a1\ a2\ cYn1^2\ n3^2\ (a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY +$
$a1\ a2\ cYn3\ n3\ rY + a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2)) - a0\ a1\ cYn2^2\ n3^2\ rY^3$
$\surd((a0\ a1\ a2\ cYn1\ cYn3\ n3 + a0\ a1\ cYn1\ n3^2\ rY + a0\ a2\ cYn1\ n3^2\ rY + a1\ a2\ cYn1\ n3^2\ rY)^2 -$
$4\ a0\ a1\ a2\ cYn1^2\ n3^2\ (a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY +$
$a1\ a2\ cYn3\ n3\ rY + a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2)) - a0\ a2\ cYn2^2\ n3^2\ rY^3$
$\surd((a0\ a1\ a2\ cYn1\ cYn3\ n3 + a0\ a1\ cYn1\ n3^2\ rY + a0\ a2\ cYn1\ n3^2\ rY + a1\ a2\ cYn1\ n3^2\ rY)^2 -$
$4\ a0\ a1\ a2\ cYn1^2\ n3^2\ (a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY +$
$a1\ a2\ cYn3\ n3\ rY + a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2)) + a1\ a2\ cYn2^2\ n3^2\ rY^3$
$\surd((a0\ a1\ a2\ cYn1\ cYn3\ n3 + a0\ a1\ cYn1\ n3^2\ rY + a0\ a2\ cYn1\ n3^2\ rY + a1\ a2\ cYn1\ n3^2\ rY)^2 -$
$4\ a0\ a1\ a2\ cYn1^2\ n3^2\ (a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY +$
$a1\ a2\ cYn3\ n3\ rY + a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2)))\ /$
$(2\ cYn1\ rY^2\ (a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY + a1\ a2\ cYn3\ n3\ rY +$
$a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2)^2),$
$Sn1 \to (6\ a0^2\ a1^4\ a2^2\ cYn1\ cYn3^4 + 2\ a0^2\ a1^4\ a2^2\ cYn1^3\ cYn3^4 + 2\ a0^2\ a1^4\ a2^2\ cYn1\ cYn2^2\ cYn3^4 +$
$2\ a0^2\ a1^4\ a2^2\ cYn1\ cYn3^6 + 12\ a0^2\ a1^4\ a2\ cYn1\ cYn3^3\ n3\ rY +$
$12\ a0^2\ a1^3\ a2^2\ cYn1\ cYn3^3\ n3\ rY +$
$12\ a0\ a1^4\ a2^2\ cYn1\ cYn3^3\ n3\ rY + 4\ a0^2\ a1^4\ a2\ cYn1^3\ cYn3^3\ n3\ rY +$
$2\ a0^2\ a1^3\ a2^2\ cYn1^3\ cYn3^3\ n3\ rY + 4\ a0\ a1^4\ a2^2\ cYn1^3\ cYn3^3\ n3\ rY +$
$4\ a0^2\ a1^4\ a2\ cYn1\ cYn2^2\ cYn3^3\ n3\ rY +$
$2\ a0^2\ a1^3\ a2^2\ cYn1\ cYn2^2\ cYn3^3\ n3\ rY +$
$4\ a0\ a1^4\ a2^2\ cYn1\ cYn2^2\ cYn3^3\ n3\ rY +$

$4\ a0^2\ a1^4\ a2\ cYn1\ cYn3^5\ n3\ rY + 8\ a0^2\ a1^3\ a2^2\ cYn1\ cYn3^5\ n3\ rY +$
$4\ a0\ a1^4\ a2^2\ cYn1\ cYn3^5\ n3\ rY + 6\ a0^2\ a1^4\ cYn1\ cYn3^2\ n3^2\ rY^2 +$
$24\ a0^2\ a1^3\ a2\ cYn1\ cYn3^2\ n3^2\ rY^2 +$
$24\ a0\ a1^4\ a2\ cYn1\ cYn3^2\ n3^2\ rY^2 + 6\ a0^2\ a1^2\ a2^2\ cYn1\ cYn3^2\ n3^2\ rY^2 +$
$24\ a0\ a1^3\ a2^2\ cYn1\ cYn3^2\ n3^2\ rY^2 +$
$6\ a1^4\ a2^2\ cYn1\ cYn3^2\ n3^2\ rY^2 + 2\ a0^2\ a1^4\ cYn1^3\ cYn3^2\ n3^2\ rY^2 +$
$4\ a0^2\ a1^3\ a2\ cYn1^3\ cYn3^2\ n3^2\ rY^2 + 8\ a0\ a1^4\ a2\ cYn1^3\ cYn3^2\ n3^2\ rY^2 -$
$3\ a0^2\ a1^2\ a2^2\ cYn1^3\ cYn3^2\ n3^2\ rY^2 +$
$4\ a0\ a1^3\ a2^2\ cYn1^3\ cYn3^2\ n3^2\ rY^2 + 2\ a1^4\ a2^2\ cYn1^3\ cYn3^2\ n3^2\ rY^2 +$
$2\ a0^2\ a1^4\ cYn1\ cYn2^2\ cYn3^2\ n3^2\ rY^2 +$
$4\ a0^2\ a1^3\ a2\ cYn1\ cYn2^2\ cYn3^2\ n3^2\ rY^2 +$
$8\ a0\ a1^4\ a2\ cYn1\ cYn2^2\ cYn3^2\ n3^2\ rY^2 -$
$3\ a0^2\ a1^2\ a2^2\ cYn1\ cYn2^2\ cYn3^2\ n3^2\ rY^2 +$
$4\ a0\ a1^3\ a2^2\ cYn1\ cYn2^2\ cYn3^2\ n3^2\ rY^2 +$
$2\ a1^4\ a2^2\ cYn1\ cYn2^2\ cYn3^2\ n3^2\ rY^2 +$
$2\ a0^2\ a1^4\ cYn1\ cYn3^4\ n3^2\ rY^2 + 16\ a0^2\ a1^3\ a2\ cYn1\ cYn3^4\ n3^2\ rY^2 +$
$8\ a0\ a1^4\ a2\ cYn1\ cYn3^4\ n3^2\ rY^2 + 12\ a0^2\ a1^2\ a2^2\ cYn1\ cYn3^4\ n3^2\ rY^2 +$
$16\ a0\ a1^3\ a2^2\ cYn1\ cYn3^4\ n3^2\ rY^2 +$
$2\ a1^4\ a2^2\ cYn1\ cYn3^4\ n3^2\ rY^2 + 12\ a0^2\ a1^3\ cYn1\ cYn3\ n3^3\ rY^3 +$
$12\ a0\ a1^4\ cYn1\ cYn3\ n3^3\ rY^3 + 12\ a0^2\ a1^2\ a2\ cYn1\ cYn3\ n3^3\ rY^3 +$
$36\ a0\ a1^3\ a2\ cYn1\ cYn3\ n3^3\ rY^3 + 12\ a1^4\ a2\ cYn1\ cYn3\ n3^3\ rY^3 +$
$12\ a0\ a1^2\ a2^2\ cYn1\ cYn3\ n3^3\ rY^3 + 12\ a1^3\ a2^2\ cYn1\ cYn3\ n3^3\ rY^3 +$
$2\ a0^2\ a1^3\ cYn1^3\ cYn3\ n3^3\ rY^3 + 4\ a0\ a1^4\ cYn1^3\ cYn3\ n3^3\ rY^3 -$
$2\ a0^2\ a1^2\ a2\ cYn1^3\ cYn3\ n3^3\ rY^3 + 6\ a0\ a1^3\ a2\ cYn1^3\ cYn3\ n3^3\ rY^3 +$
$4\ a1^4\ a2\ cYn1^3\ cYn3\ n3^3\ rY^3 - 2\ a0^2\ a1\ a2^2\ cYn1^3\ cYn3\ n3^3\ rY^3 -$
$2\ a0\ a1^2\ a2^2\ cYn1^3\ cYn3\ n3^3\ rY^3 + 2\ a1^3\ a2^2\ cYn1^3\ cYn3\ n3^3\ rY^3 +$
$2\ a0^2\ a1^3\ cYn1\ cYn2^2\ cYn3\ n3^3\ rY^3 +$
$4\ a0\ a1^4\ cYn1\ cYn2^2\ cYn3\ n3^3\ rY^3 -$
$2\ a0^2\ a1^2\ a2\ cYn1\ cYn2^2\ cYn3\ n3^3\ rY^3 +$
$6\ a0\ a1^3\ a2\ cYn1\ cYn2^2\ cYn3\ n3^3\ rY^3 +$
$4\ a1^4\ a2\ cYn1\ cYn2^2\ cYn3\ n3^3\ rY^3 -$
$2\ a0^2\ a1\ a2^2\ cYn1\ cYn2^2\ cYn3\ n3^3\ rY^3 -$
$2\ a0\ a1^2\ a2^2\ cYn1\ cYn2^2\ cYn3\ n3^3\ rY^3 +$
$2\ a1^3\ a2^2\ cYn1\ cYn2^2\ cYn3\ n3^3\ rY^3 +$
$8\ a0^2\ a1^3\ cYn1\ cYn3^3\ n3^3\ rY^3 + 4\ a0\ a1^4\ cYn1\ cYn3^3\ n3^3\ rY^3 +$
$24\ a0^2\ a1^2\ a2\ cYn1\ cYn3^3\ n3^3\ rY^3 + 28\ a0\ a1^3\ a2\ cYn1\ cYn3^3\ n3^3\ rY^3 +$
$4\ a1^4\ a2\ cYn1\ cYn3^3\ n3^3\ rY^3 + 8\ a0^2\ a1\ a2^2\ cYn1\ cYn3^3\ n3^3\ rY^3 +$
$24\ a0\ a1^2\ a2^2\ cYn1\ cYn3^3\ n3^3\ rY^3 + 8\ a1^3\ a2^2\ cYn1\ cYn3^3\ n3^3\ rY^3 +$
$6\ a0^2\ a1^2\ cYn1\ n3^4\ rY^4 + 12\ a0\ a1^3\ cYn1\ n3^4\ rY^4 +$
$6\ a1^4\ cYn1\ n3^4\ rY^4 + 12\ a0\ a1^2\ a2\ cYn1\ n3^4\ rY^4 +$
$12\ a1^3\ a2\ cYn1\ n3^4\ rY^4 + 6\ a1^2\ a2^2\ cYn1\ n3^4\ rY^4 +$
$a0^2\ a1^2\ cYn1^3\ n3^4\ rY^4 + 2\ a0\ a1^3\ cYn1^3\ n3^4\ rY^4 +$
$2\ a1^4\ cYn1^3\ n3^4\ rY^4 - 2\ a0^2\ a1\ a2\ cYn1^3\ n3^4\ rY^4 -$
$2\ a0\ a1^2\ a2\ cYn1^3\ n3^4\ rY^4 + 2\ a1^3\ a2\ cYn1^3\ n3^4\ rY^4 +$
$a0^2\ a2^2\ cYn1^3\ n3^4\ rY^4 - 2\ a0\ a1\ a2^2\ cYn1^3\ n3^4\ rY^4 +$
$a1^2\ a2^2\ cYn1^3\ n3^4\ rY^4 + a0^2\ a1^2\ cYn1\ cYn2^2\ n3^4\ rY^4 +$
$2\ a0\ a1^3\ cYn1\ cYn2^2\ n3^4\ rY^4 + 2\ a1^4\ cYn1\ cYn2^2\ n3^4\ rY^4 -$
$2\ a0^2\ a1\ a2\ cYn1\ cYn2^2\ n3^4\ rY^4 - 2\ a0\ a1^2\ a2\ cYn1\ cYn2^2\ n3^4\ rY^4 +$
$2\ a1^3\ a2\ cYn1\ cYn2^2\ n3^4\ rY^4 + a0^2\ a2^2\ cYn1\ cYn2^2\ n3^4\ rY^4 -$
$2\ a0\ a1\ a2^2\ cYn1\ cYn2^2\ n3^4\ rY^4 + a1^2\ a2^2\ cYn1\ cYn2^2\ n3^4\ rY^4 +$
$12\ a0^2\ a1^2\ cYn1\ cYn3^2\ n3^4\ rY^4 + 12\ a0\ a1^3\ cYn1\ cYn3^2\ n3^4\ rY^4 +$
$2\ a1^4\ cYn1\ cYn3^2\ n3^4\ rY^4 + 16\ a0^2\ a1\ a2\ cYn1\ cYn3^2\ n3^4\ rY^4 +$
$36\ a0\ a1^2\ a2\ cYn1\ cYn3^2\ n3^4\ rY^4 + 12\ a1^3\ a2\ cYn1\ cYn3^2\ n3^4\ rY^4 +$
$2\ a0^2\ a2^2\ cYn1\ cYn3^2\ n3^4\ rY^4 + 16\ a0\ a1\ a2^2\ cYn1\ cYn3^2\ n3^4\ rY^4 +$
$12\ a1^2\ a2^2\ cYn1\ cYn3^2\ n3^4\ rY^4 + 8\ a0^2\ a1\ cYn1\ cYn3\ n3^5\ rY^5 +$
$12\ a0\ a1^2\ cYn1\ cYn3\ n3^5\ rY^5 + 4\ a1^3\ cYn1\ cYn3\ n3^5\ rY^5 +$
$4\ a0^2\ a2\ cYn1\ cYn3\ n3^5\ rY^5 + 20\ a0\ a1\ a2\ cYn1\ cYn3\ n3^5\ rY^5 +$

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12 a1^2 a2 cYn1 cYn3 n3^5 rY^5 + 4 a0 a2^2 cYn1 cYn3 n3^5 rY^5 +
8 a1 a2^2 cYn1 cYn3 n3^5 rY^5 + 2 a0^2 cYn1 n3^6 rY^6 +
4 a0 a1 cYn1 n3^6 rY^6 + 2 a1^2 cYn1 n3^6 rY^6 +
4 a0 a2 cYn1 n3^6 rY^6 + 4 a1 a2 cYn1 n3^6 rY^6 +
2 a2^2 cYn1 n3^6 rY^6 + 2 a0 a1^2 a2 cYn1^2 cYn3^2 rY
 √((a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY)^2 -
   4 a0 a1 a2 cYn1^2 n3^2 (a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY +
     a1 a2 cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2)) - 2 a0 a1^2 a2 cYn2^2 cYn3^2
 rY √((a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY)^2 -
   4 a0 a1 a2 cYn1^2 n3^2 (a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY + a1 a2
     cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2)) + 2 a0 a1^2 cYn1^2 cYn3 n3 rY^2
 √((a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY)^2 -
   4 a0 a1 a2 cYn1^2 n3^2 (a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY + a1 a2
     cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2)) + a0 a1 a2 cYn1^2 cYn3 n3 rY^2
 √((a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY)^2 -
   4 a0 a1 a2 cYn1^2 n3^2 (a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY + a1 a2
     cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2)) + 2 a1^2 a2 cYn1^2 cYn3 n3 rY^2
 √((a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY)^2 -
   4 a0 a1 a2 cYn1^2 n3^2 (a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY + a1 a2
     cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2)) - 2 a0 a1^2 cYn2^2 cYn3 n3 rY^2
 √((a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY)^2 -
   4 a0 a1 a2 cYn1^2 n3^2 (a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY + a1 a2
     cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2)) - a0 a1 a2 cYn2^2 cYn3 n3 rY^2
 √((a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY)^2 -
   4 a0 a1 a2 cYn1^2 n3^2 (a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY + a1 a2
     cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2)) - 2 a1^2 a2 cYn2^2 cYn3 n3 rY^2
 √((a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY)^2 -
   4 a0 a1 a2 cYn1^2 n3^2 (a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY +
     a1 a2 cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2)) + a0 a1 cYn1^2 n3^2 rY^3
 √((a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY)^2 -
   4 a0 a1 a2 cYn1^2 n3^2 (a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY +
     a1 a2 cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2)) + 2 a1^2 cYn1^2 n3^2 rY^3
 √((a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY)^2 -
   4 a0 a1 a2 cYn1^2 n3^2 (a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY +
     a1 a2 cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2)) - a0 a2 cYn1^2 n3^2 rY^3
 √((a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY)^2 -
   4 a0 a1 a2 cYn1^2 n3^2 (a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY +
     a1 a2 cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2)) + a1 a2 cYn1^2 n3^2 rY^3
 √((a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY)^2 -
   4 a0 a1 a2 cYn1^2 n3^2 (a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY +
     a1 a2 cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2)) - a0 a1 cYn2^2 n3^2 rY^3
 √((a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY)^2 -
   4 a0 a1 a2 cYn1^2 n3^2 (a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY +
     a1 a2 cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2)) - 2 a1^2 cYn2^2 n3^2 rY^3
 √((a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY)^2 -
   4 a0 a1 a2 cYn1^2 n3^2 (a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY +
     a1 a2 cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2)) + a0 a2 cYn2^2 n3^2 rY^3
 √((a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY)^2 -
   4 a0 a1 a2 cYn1^2 n3^2 (a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY +
     a1 a2 cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2)) - a1 a2 cYn2^2 n3^2 rY^3
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$$\sqrt{\left(\left(a0\ a1\ a2\ cYn1\ cYn3\ n3 + a0\ a1\ cYn1\ n3^2\ rY + a0\ a2\ cYn1\ n3^2\ rY + a1\ a2\ cYn1\ n3^2\ rY\right)^2 -\right.}$$
$$4\ a0\ a1\ a2\ cYn1^2\ n3^2\ \left(a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY +\right.$$
$$\left.\left. a1\ a2\ cYn3\ n3\ rY + a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2\right)\right)\Big) \Big/$$
$$\Big(2\ cYn1\ rY^2\ \left(a0\ a1\ a2\ cYn3^2 + a0\ a1\ cYn3\ n3\ rY + a0\ a2\ cYn3\ n3\ rY + a1\ a2\ cYn3\ n3\ rY +\right.$$
$$\left. a0\ n3^2\ rY^2 + a1\ n3^2\ rY^2 + a2\ n3^2\ rY^2\right)^2\Big),$$
$$Sn2 \to \Big(6\ a0^2\ a1^2\ a2^4\ cYn1\ cYn3^4 + 2\ a0^2\ a1^2\ a2^4\ cYn1^3\ cYn3^4 + 2\ a0^2\ a1^2\ a2^4\ cYn1\ cYn2^2\ cYn3^4 +$$
$$2\ a0^2\ a1^2\ a2^4\ cYn1\ cYn3^6 + 12\ a0^2\ a1^2\ a2^3\ cYn1\ cYn3^3\ n3\ rY +$$
$$12\ a0^2\ a1\ a2^4\ cYn1\ cYn3^3\ n3\ rY + 12\ a0\ a1^2\ a2^4\ cYn1\ cYn3^3\ n3\ rY +$$
$$2\ a0^2\ a1^2\ a2^3\ cYn1^3\ cYn3^3\ n3\ rY +$$
$$4\ a0^2\ a1\ a2^4\ cYn1^3\ cYn3^3\ n3\ rY + 4\ a0\ a1^2\ a2^4\ cYn1^3\ cYn3^3\ n3\ rY +$$
$$2\ a0^2\ a1^2\ a2^3\ cYn1\ cYn2^2\ cYn3^3\ n3\ rY +$$
$$4\ a0^2\ a1\ a2^4\ cYn1\ cYn2^2\ cYn3^3\ n3\ rY +$$
$$4\ a0\ a1^2\ a2^4\ cYn1\ cYn2^2\ cYn3^3\ n3\ rY +$$
$$8\ a0^2\ a1^2\ a2^3\ cYn1\ cYn3^5\ n3\ rY + 4\ a0^2\ a1\ a2^4\ cYn1\ cYn3^5\ n3\ rY +$$
$$4\ a0\ a1^2\ a2^4\ cYn1\ cYn3^5\ n3\ rY + 6\ a0^2\ a1^2\ a2^2\ cYn1\ cYn3^2\ n3^2\ rY^2 +$$
$$24\ a0^2\ a1\ a2^3\ cYn1\ cYn3^2\ n3^2\ rY^2 +$$
$$24\ a0\ a1^2\ a2^3\ cYn1\ cYn3^2\ n3^2\ rY^2 + 6\ a0^2\ a2^4\ cYn1\ cYn3^2\ n3^2\ rY^2 +$$
$$24\ a0\ a1\ a2^4\ cYn1\ cYn3^2\ n3^2\ rY^2 + 6\ a1^2\ a2^4\ cYn1\ cYn3^2\ n3^2\ rY^2 -$$
$$3\ a0^2\ a1^2\ a2^2\ cYn1^3\ cYn3^2\ n3^2\ rY^2 +$$
$$4\ a0^2\ a1\ a2^3\ cYn1^3\ cYn3^2\ n3^2\ rY^2 +$$
$$4\ a0\ a1^2\ a2^3\ cYn1^3\ cYn3^2\ n3^2\ rY^2 + 2\ a0^2\ a2^4\ cYn1^3\ cYn3^2\ n3^2\ rY^2 +$$
$$8\ a0\ a1\ a2^4\ cYn1^3\ cYn3^2\ n3^2\ rY^2 + 2\ a1^2\ a2^4\ cYn1^3\ cYn3^2\ n3^2\ rY^2 -$$
$$3\ a0^2\ a1^2\ a2^2\ cYn1\ cYn2^2\ cYn3^2\ n3^2\ rY^2 +$$
$$4\ a0^2\ a1\ a2^3\ cYn1\ cYn2^2\ cYn3^2\ n3^2\ rY^2 +$$
$$4\ a0\ a1^2\ a2^3\ cYn1\ cYn2^2\ cYn3^2\ n3^2\ rY^2 +$$
$$2\ a0^2\ a2^4\ cYn1\ cYn2^2\ cYn3^2\ n3^2\ rY^2 +$$
$$8\ a0\ a1\ a2^4\ cYn1\ cYn2^2\ cYn3^2\ n3^2\ rY^2 +$$
$$2\ a1^2\ a2^4\ cYn1\ cYn2^2\ cYn3^2\ n3^2\ rY^2 +$$
$$12\ a0^2\ a1^2\ a2^2\ cYn1\ cYn3^4\ n3^2\ rY^2 +$$
$$16\ a0^2\ a1\ a2^3\ cYn1\ cYn3^4\ n3^2\ rY^2 +$$
$$16\ a0\ a1^2\ a2^3\ cYn1\ cYn3^4\ n3^2\ rY^2 +$$
$$2\ a0^2\ a2^4\ cYn1\ cYn3^4\ n3^2\ rY^2 + 8\ a0\ a1\ a2^4\ cYn1\ cYn3^4\ n3^2\ rY^2 +$$
$$2\ a1^2\ a2^4\ cYn1\ cYn3^4\ n3^2\ rY^2 + 12\ a0^2\ a1\ a2^2\ cYn1\ cYn3\ n3^3\ rY^3 +$$
$$12\ a0\ a1^2\ a2^2\ cYn1\ cYn3\ n3^3\ rY^3 + 12\ a0^2\ a2^3\ cYn1\ cYn3\ n3^3\ rY^3 +$$
$$36\ a0\ a1\ a2^3\ cYn1\ cYn3\ n3^3\ rY^3 + 12\ a1^2\ a2^3\ cYn1\ cYn3\ n3^3\ rY^3 +$$
$$12\ a0\ a2^4\ cYn1\ cYn3\ n3^3\ rY^3 + 12\ a1\ a2^4\ cYn1\ cYn3\ n3^3\ rY^3 -$$
$$2\ a0^2\ a1^2\ a2\ cYn1^3\ cYn3\ n3^3\ rY^3 - 2\ a0^2\ a1\ a2^2\ cYn1^3\ cYn3\ n3^3\ rY^3 -$$
$$2\ a0\ a1^2\ a2^2\ cYn1^3\ cYn3\ n3^3\ rY^3 + 2\ a0^2\ a2^3\ cYn1^3\ cYn3\ n3^3\ rY^3 +$$
$$6\ a0\ a1\ a2^3\ cYn1^3\ cYn3\ n3^3\ rY^3 + 2\ a1^2\ a2^3\ cYn1^3\ cYn3\ n3^3\ rY^3 +$$
$$4\ a0\ a2^4\ cYn1^3\ cYn3\ n3^3\ rY^3 + 4\ a1\ a2^4\ cYn1^3\ cYn3\ n3^3\ rY^3 -$$
$$2\ a0^2\ a1^2\ a2\ cYn1\ cYn2^2\ cYn3\ n3^3\ rY^3 -$$
$$2\ a0^2\ a1\ a2^2\ cYn1\ cYn2^2\ cYn3\ n3^3\ rY^3 -$$
$$2\ a0\ a1^2\ a2^2\ cYn1\ cYn2^2\ cYn3\ n3^3\ rY^3 +$$
$$2\ a0^2\ a2^3\ cYn1\ cYn2^2\ cYn3\ n3^3\ rY^3 +$$
$$6\ a0\ a1\ a2^3\ cYn1\ cYn2^2\ cYn3\ n3^3\ rY^3 +$$
$$2\ a1^2\ a2^3\ cYn1\ cYn2^2\ cYn3\ n3^3\ rY^3 +$$
$$4\ a0\ a2^4\ cYn1\ cYn2^2\ cYn3\ n3^3\ rY^3 +$$
$$4\ a1\ a2^4\ cYn1\ cYn2^2\ cYn3\ n3^3\ rY^3 +$$
$$8\ a0^2\ a1^2\ a2\ cYn1\ cYn3^3\ n3^3\ rY^3 + 24\ a0^2\ a1\ a2^2\ cYn1\ cYn3^3\ n3^3\ rY^3 +$$
$$24\ a0\ a1^2\ a2^2\ cYn1\ cYn3^3\ n3^3\ rY^3 + 8\ a0^2\ a2^3\ cYn1\ cYn3^3\ n3^3\ rY^3 +$$
$$28\ a0\ a1\ a2^3\ cYn1\ cYn3^3\ n3^3\ rY^3 + 8\ a1^2\ a2^3\ cYn1\ cYn3^3\ n3^3\ rY^3 +$$
$$4\ a0\ a2^4\ cYn1\ cYn3^3\ n3^3\ rY^3 + 4\ a1\ a2^4\ cYn1\ cYn3^3\ n3^3\ rY^3 +$$
$$6\ a0^2\ a2^2\ cYn1\ n3^4\ rY^4 + 12\ a0\ a1\ a2^2\ cYn1\ n3^4\ rY^4 +$$
$$6\ a1^2\ a2^2\ cYn1\ n3^4\ rY^4 + 12\ a0\ a2^3\ cYn1\ n3^4\ rY^4 +$$
$$12\ a1\ a2^3\ cYn1\ n3^4\ rY^4 + 6\ a2^4\ cYn1\ n3^4\ rY^4 +$$
$$a0^2\ a1^2\ cYn1^3\ n3^4\ rY^4 - 2\ a0^2\ a1\ a2\ cYn1^3\ n3^4\ rY^4 -$$

$$2\,a0\,a1^{2}\,a2\,cYn1^{3}\,n3^{4}\,rY^{4}+a0^{2}\,a2^{2}\,cYn1^{3}\,n3^{4}\,rY^{4}-$$
$$2\,a0\,a1\,a2^{2}\,cYn1^{3}\,n3^{4}\,rY^{4}+a1^{2}\,a2^{2}\,cYn1^{3}\,n3^{4}\,rY^{4}+$$
$$2\,a0\,a2^{3}\,cYn1^{3}\,n3^{4}\,rY^{4}+2\,a1\,a2^{3}\,cYn1^{3}\,n3^{4}\,rY^{4}+$$
$$2\,a2^{4}\,cYn1^{3}\,n3^{4}\,rY^{4}+a0^{2}\,a1^{2}\,cYn1\,cYn2^{2}\,n3^{4}\,rY^{4}-$$
$$2\,a0^{2}\,a1\,a2\,cYn1\,cYn2^{2}\,n3^{4}\,rY^{4}-2\,a0\,a1^{2}\,a2\,cYn1\,cYn2^{2}\,n3^{4}\,rY^{4}+$$
$$a0^{2}\,a2^{2}\,cYn1\,cYn2^{2}\,n3^{4}\,rY^{4}-2\,a0\,a1\,a2^{2}\,cYn1\,cYn2^{2}\,n3^{4}\,rY^{4}+$$
$$a1^{2}\,a2^{2}\,cYn1\,cYn2^{2}\,n3^{4}\,rY^{4}+2\,a0\,a2^{3}\,cYn1\,cYn2^{2}\,n3^{4}\,rY^{4}+$$
$$2\,a1\,a2^{3}\,cYn1\,cYn2^{2}\,n3^{4}\,rY^{4}+2\,a2^{4}\,cYn1\,cYn2^{2}\,n3^{4}\,rY^{4}+$$
$$2\,a0^{2}\,a1^{2}\,cYn1\,cYn3^{2}\,n3^{4}\,rY^{4}+16\,a0^{2}\,a1\,a2\,cYn1\,cYn3^{2}\,n3^{4}\,rY^{4}+$$
$$16\,a0\,a1^{2}\,a2\,cYn1\,cYn3^{2}\,n3^{4}\,rY^{4}+12\,a0^{2}\,a2^{2}\,cYn1\,cYn3^{2}\,n3^{4}\,rY^{4}+$$
$$36\,a0\,a1\,a2^{2}\,cYn1\,cYn3^{2}\,n3^{4}\,rY^{4}+12\,a1^{2}\,a2^{2}\,cYn1\,cYn3^{2}\,n3^{4}\,rY^{4}+$$
$$12\,a0\,a2^{3}\,cYn1\,cYn3^{2}\,n3^{4}\,rY^{4}+12\,a1\,a2^{3}\,cYn1\,cYn3^{2}\,n3^{4}\,rY^{4}+$$
$$2\,a2^{4}\,cYn1\,cYn3^{2}\,n3^{4}\,rY^{4}+4\,a0^{2}\,a1\,cYn1\,cYn3\,n3^{5}\,rY^{5}+$$
$$4\,a0\,a1^{2}\,cYn1\,cYn3\,n3^{5}\,rY^{5}+8\,a0^{2}\,a2\,cYn1\,cYn3\,n3^{5}\,rY^{5}+$$
$$20\,a0\,a1\,a2\,cYn1\,cYn3\,n3^{5}\,rY^{5}+8\,a1^{2}\,a2\,cYn1\,cYn3\,n3^{5}\,rY^{5}+$$
$$12\,a0\,a2^{2}\,cYn1\,cYn3\,n3^{5}\,rY^{5}+12\,a1\,a2^{2}\,cYn1\,cYn3\,n3^{5}\,rY^{5}+$$
$$4\,a2^{3}\,cYn1\,cYn3\,n3^{5}\,rY^{5}+2\,a0^{2}\,cYn1\,n3^{6}\,rY^{6}+$$
$$4\,a0\,a1\,cYn1\,n3^{6}\,rY^{6}+2\,a1^{2}\,cYn1\,n3^{6}\,rY^{6}+$$
$$4\,a0\,a2\,cYn1\,n3^{6}\,rY^{6}+4\,a1\,a2\,cYn1\,n3^{6}\,rY^{6}+$$
$$2\,a2^{2}\,cYn1\,n3^{6}\,rY^{6}+2\,a0\,a1\,a2^{2}\,cYn1^{2}\,cYn3^{2}\,rY$$
$$\surd\big(\big(a0\,a1\,a2\,cYn1\,cYn3\,n3+a0\,a1\,cYn1\,n3^{2}\,rY+a0\,a2\,cYn1\,n3^{2}\,rY+a1\,a2\,cYn1\,n3^{2}\,rY\big)^{2}-$$
$$4\,a0\,a1\,a2\,cYn1^{2}\,n3^{2}\,\big(a0\,a1\,a2\,cYn3^{2}+a0\,a1\,cYn3\,n3\,rY+a0\,a2\,cYn3\,n3\,rY+$$
$$a1\,a2\,cYn3\,n3\,rY+a0\,n3^{2}\,rY^{2}+a1\,n3^{2}\,rY^{2}+a2\,n3^{2}\,rY^{2}\big)\big)-2\,a0\,a1\,a2^{2}\,cYn2^{2}\,cYn3^{2}$$
$$rY\,\surd\big(\big(a0\,a1\,a2\,cYn1\,cYn3\,n3+a0\,a1\,cYn1\,n3^{2}\,rY+a0\,a2\,cYn1\,n3^{2}\,rY+a1\,a2\,cYn1\,n3^{2}\,rY\big)^{2}-$$
$$4\,a0\,a1\,a2\,cYn1^{2}\,n3^{2}\,\big(a0\,a1\,a2\,cYn3^{2}+a0\,a1\,cYn3\,n3\,rY+a0\,a2\,cYn3\,n3\,rY+a1\,a2$$
$$cYn3\,n3\,rY+a0\,n3^{2}\,rY^{2}+a1\,n3^{2}\,rY^{2}+a2\,n3^{2}\,rY^{2}\big)\big)+a0\,a1\,a2\,cYn1^{2}\,cYn3\,n3\,rY^{2}$$
$$\surd\big(\big(a0\,a1\,a2\,cYn1\,cYn3\,n3+a0\,a1\,cYn1\,n3^{2}\,rY+a0\,a2\,cYn1\,n3^{2}\,rY+a1\,a2\,cYn1\,n3^{2}\,rY\big)^{2}-$$
$$4\,a0\,a1\,a2\,cYn1^{2}\,n3^{2}\,\big(a0\,a1\,a2\,cYn3^{2}+a0\,a1\,cYn3\,n3\,rY+a0\,a2\,cYn3\,n3\,rY+a1\,a2$$
$$cYn3\,n3\,rY+a0\,n3^{2}\,rY^{2}+a1\,n3^{2}\,rY^{2}+a2\,n3^{2}\,rY^{2}\big)\big)+2\,a0\,a2^{2}\,cYn1^{2}\,cYn3\,n3\,rY^{2}$$
$$\surd\big(\big(a0\,a1\,a2\,cYn1\,cYn3\,n3+a0\,a1\,cYn1\,n3^{2}\,rY+a0\,a2\,cYn1\,n3^{2}\,rY+a1\,a2\,cYn1\,n3^{2}\,rY\big)^{2}-$$
$$4\,a0\,a1\,a2\,cYn1^{2}\,n3^{2}\,\big(a0\,a1\,a2\,cYn3^{2}+a0\,a1\,cYn3\,n3\,rY+a0\,a2\,cYn3\,n3\,rY+a1\,a2$$
$$cYn3\,n3\,rY+a0\,n3^{2}\,rY^{2}+a1\,n3^{2}\,rY^{2}+a2\,n3^{2}\,rY^{2}\big)\big)+2\,a1\,a2^{2}\,cYn1^{2}\,cYn3\,n3\,rY^{2}$$
$$\surd\big(\big(a0\,a1\,a2\,cYn1\,cYn3\,n3+a0\,a1\,cYn1\,n3^{2}\,rY+a0\,a2\,cYn1\,n3^{2}\,rY+a1\,a2\,cYn1\,n3^{2}\,rY\big)^{2}-$$
$$4\,a0\,a1\,a2\,cYn1^{2}\,n3^{2}\,\big(a0\,a1\,a2\,cYn3^{2}+a0\,a1\,cYn3\,n3\,rY+a0\,a2\,cYn3\,n3\,rY+a1\,a2$$
$$cYn3\,n3\,rY+a0\,n3^{2}\,rY^{2}+a1\,n3^{2}\,rY^{2}+a2\,n3^{2}\,rY^{2}\big)\big)-a0\,a1\,a2\,cYn2^{2}\,cYn3\,n3\,rY^{2}$$
$$\surd\big(\big(a0\,a1\,a2\,cYn1\,cYn3\,n3+a0\,a1\,cYn1\,n3^{2}\,rY+a0\,a2\,cYn1\,n3^{2}\,rY+a1\,a2\,cYn1\,n3^{2}\,rY\big)^{2}-$$
$$4\,a0\,a1\,a2\,cYn1^{2}\,n3^{2}\,\big(a0\,a1\,a2\,cYn3^{2}+a0\,a1\,cYn3\,n3\,rY+a0\,a2\,cYn3\,n3\,rY+a1\,a2$$
$$cYn3\,n3\,rY+a0\,n3^{2}\,rY^{2}+a1\,n3^{2}\,rY^{2}+a2\,n3^{2}\,rY^{2}\big)\big)-2\,a0\,a2^{2}\,cYn2^{2}\,cYn3\,n3\,rY^{2}$$
$$\surd\big(\big(a0\,a1\,a2\,cYn1\,cYn3\,n3+a0\,a1\,cYn1\,n3^{2}\,rY+a0\,a2\,cYn1\,n3^{2}\,rY+a1\,a2\,cYn1\,n3^{2}\,rY\big)^{2}-$$
$$4\,a0\,a1\,a2\,cYn1^{2}\,n3^{2}\,\big(a0\,a1\,a2\,cYn3^{2}+a0\,a1\,cYn3\,n3\,rY+a0\,a2\,cYn3\,n3\,rY+a1\,a2$$
$$cYn3\,n3\,rY+a0\,n3^{2}\,rY^{2}+a1\,n3^{2}\,rY^{2}+a2\,n3^{2}\,rY^{2}\big)\big)-2\,a1\,a2^{2}\,cYn2^{2}\,cYn3\,n3\,rY^{2}$$
$$\surd\big(\big(a0\,a1\,a2\,cYn1\,cYn3\,n3+a0\,a1\,cYn1\,n3^{2}\,rY+a0\,a2\,cYn1\,n3^{2}\,rY+a1\,a2\,cYn1\,n3^{2}\,rY\big)^{2}-$$
$$4\,a0\,a1\,a2\,cYn1^{2}\,n3^{2}\,\big(a0\,a1\,a2\,cYn3^{2}+a0\,a1\,cYn3\,n3\,rY+a0\,a2\,cYn3\,n3\,rY+$$
$$a1\,a2\,cYn3\,n3\,rY+a0\,n3^{2}\,rY^{2}+a1\,n3^{2}\,rY^{2}+a2\,n3^{2}\,rY^{2}\big)\big)-a0\,a1\,cYn1^{2}\,n3^{2}\,rY^{3}$$
$$\surd\big(\big(a0\,a1\,a2\,cYn1\,cYn3\,n3+a0\,a1\,cYn1\,n3^{2}\,rY+a0\,a2\,cYn1\,n3^{2}\,rY+a1\,a2\,cYn1\,n3^{2}\,rY\big)^{2}-$$
$$4\,a0\,a1\,a2\,cYn1^{2}\,n3^{2}\,\big(a0\,a1\,a2\,cYn3^{2}+a0\,a1\,cYn3\,n3\,rY+a0\,a2\,cYn3\,n3\,rY+$$
$$a1\,a2\,cYn3\,n3\,rY+a0\,n3^{2}\,rY^{2}+a1\,n3^{2}\,rY^{2}+a2\,n3^{2}\,rY^{2}\big)\big)+a0\,a2\,cYn1^{2}\,n3^{2}\,rY^{3}$$
$$\surd\big(\big(a0\,a1\,a2\,cYn1\,cYn3\,n3+a0\,a1\,cYn1\,n3^{2}\,rY+a0\,a2\,cYn1\,n3^{2}\,rY+a1\,a2\,cYn1\,n3^{2}\,rY\big)^{2}-$$
$$4\,a0\,a1\,a2\,cYn1^{2}\,n3^{2}\,\big(a0\,a1\,a2\,cYn3^{2}+a0\,a1\,cYn3\,n3\,rY+a0\,a2\,cYn3\,n3\,rY+$$
$$a1\,a2\,cYn3\,n3\,rY+a0\,n3^{2}\,rY^{2}+a1\,n3^{2}\,rY^{2}+a2\,n3^{2}\,rY^{2}\big)\big)+a1\,a2\,cYn1^{2}\,n3^{2}\,rY^{3}$$
$$\surd\big(\big(a0\,a1\,a2\,cYn1\,cYn3\,n3+a0\,a1\,cYn1\,n3^{2}\,rY+a0\,a2\,cYn1\,n3^{2}\,rY+a1\,a2\,cYn1\,n3^{2}\,rY\big)^{2}-$$
$$4\,a0\,a1\,a2\,cYn1^{2}\,n3^{2}\,\big(a0\,a1\,a2\,cYn3^{2}+a0\,a1\,cYn3\,n3\,rY+a0\,a2\,cYn3\,n3\,rY+$$

```
        a1 a2 cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2)) + 2 a2^2 cYn1^2 n3^2 rY^3
    √((a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY)^2 -
      4 a0 a1 a2 cYn1^2 n3^2 (a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY +
        a1 a2 cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2)) + a0 a1 cYn2^2 n3^2 rY^3
    √((a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY)^2 -
      4 a0 a1 a2 cYn1^2 n3^2 (a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY +
        a1 a2 cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2)) - a0 a2 cYn2^2 n3^2 rY^3
    √((a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY)^2 -
      4 a0 a1 a2 cYn1^2 n3^2 (a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY +
        a1 a2 cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2)) - a1 a2 cYn2^2 n3^2 rY^3
    √((a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY)^2 -
      4 a0 a1 a2 cYn1^2 n3^2 (a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY +
        a1 a2 cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2)) - 2 a2^2 cYn2^2 n3^2 rY^3
    √((a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY)^2 -
      4 a0 a1 a2 cYn1^2 n3^2 (a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY +
        a1 a2 cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2)))) /
  (2 cYn1 rY^2 (a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY + a1 a2 cYn3 n3 rY +
      a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2)^2), pY1 → 3,
n1 → -((a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY -
      √((a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY + a1 a2 cYn1 n3^2 rY)^2 -
        4 a0 a1 a2 cYn1^2 n3^2 (a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY +
          a1 a2 cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2))) /
    (2 (a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY + a1 a2 cYn3 n3 rY +
        a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2))),
n2 → -((cYn2 (a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY +
        a1 a2 cYn1 n3^2 rY + √((a0 a1 a2 cYn1 cYn3 n3 + a0 a1 cYn1 n3^2 rY + a0 a2 cYn1 n3^2 rY +
              a1 a2 cYn1 n3^2 rY)^2 - 4 a0 a1 a2 cYn1^2 n3^2 (a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY +
              a0 a2 cYn3 n3 rY + a1 a2 cYn3 n3 rY + a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2)))) /
    (2 cYn1 (a0 a1 a2 cYn3^2 + a0 a1 cYn3 n3 rY + a0 a2 cYn3 n3 rY + a1 a2 cYn3 n3 rY +
        a0 n3^2 rY^2 + a1 n3^2 rY^2 + a2 n3^2 rY^2)))}}
```

Computations for Lemma 6.20.
Case B.1 where there exists N^6 s.t. Fix(N) = $Y^4 \cup P_0$.

```
(******************************
  NOTATIONS
  rY xY denotes the restriction of x to Y
  ai denotes the Hopf weight at Pi
  Nij denotes the product of the normal weights shared between Pi and Pj
  Sni is the sum of the square of the normal weights at Pi
  cYn0 xY denotes the first Chern class of the normal bundle of Y in the
  direction of N
  cYj xY^j denotes the j-th Chern class of the normal bundle of Y in the
  remaining directions
  μx5YB1 denotes the local datum of x^5 at Y
  μx5PiB1 denotes the local datum of x^5 at Pi
  μs1x3YB1 denotes the local datum of p1(X)x^3 at Y
  μs1x3PiB1 denotes the local datum of p1(X)x^3 at Pi

  We use the variable t to extract the coefficient of order 2 of xY. We
  then evaluate xY^2 to 1.
  The orientation of the point P0 is 1 while those from P1 and P2 are -
 1. This explains the signs at their local datum
  Once evaluated, we extract all the coefficients in l into a list.
  ******************************)
```

```
μx5YB1 =
 Factor[CoefficientList[
    Coefficient[Expand[(rY xY t + l z)^5] Series[(cYn0 xY t + n0 z)^-1, {xY, 0, 2}]
      Series[(cY2 xY^2 t^2 + cY1 xY t z + z^2)^-1, {xY, 0, 2}], t, 2], l]] /. xY^2 → 1
```

$$\left\{0,\ 0,\ 0,\ \frac{10\ rY^2}{n0},\ -\frac{5\ (cYn0 + cY1\ n0)\ rY}{n0^2},\ \frac{cYn0^2 + cY1\ cYn0\ n0 + cY1^2\ n0^2 - cY2\ n0^2}{n0^3}\right\}$$

```
μx5P0B1 = CoefficientList[Expand[((a0 + l) z)^5] (n0 z)^-1 (n0 z)^-1 (n0 z)^-1 (N01 z)^-1 (N02 z)^-1,
  l]
```

$$\left\{\frac{a0^5}{n0^3\ N01\ N02},\ \frac{5\ a0^4}{n0^3\ N01\ N02},\ \frac{10\ a0^3}{n0^3\ N01\ N02},\ \frac{10\ a0^2}{n0^3\ N01\ N02},\ \frac{5\ a0}{n0^3\ N01\ N02},\ \frac{1}{n0^3\ N01\ N02}\right\}$$

```
μx5P1B1 = CoefficientList[-Expand[((a1 + l) z)^5] N01^-1 N12^-1 z^-5, l]
```

$$\left\{-\frac{a1^5}{N01\ N12},\ -\frac{5\ a1^4}{N01\ N12},\ -\frac{10\ a1^3}{N01\ N12},\ -\frac{10\ a1^2}{N01\ N12},\ -\frac{5\ a1}{N01\ N12},\ -\frac{1}{N01\ N12}\right\}$$

```
μx5P2B1 = CoefficientList[-Expand[((a2 + l) z)^5] N02^-1 N12^-1 z^-5, l]
```

$$\left\{-\frac{a2^5}{N02\ N12},\ -\frac{5\ a2^4}{N02\ N12},\ -\frac{10\ a2^3}{N02\ N12},\ -\frac{10\ a2^2}{N02\ N12},\ -\frac{5\ a2}{N02\ N12},\ -\frac{1}{N02\ N12}\right\}$$

```
SymmetricReduction[Expand[(yY1 t + z)^2 + (yY2 t + z)^2], {yY1, yY2}, {cY1 xY, cY2 xY^2}]
```

$$\left\{cY1^2\ t^2\ xY^2 - 2\ cY2\ t^2\ xY^2 + 2\ cY1\ t\ xY\ z + 2\ z^2,\ 0\right\}$$

```
μs1x3YB1 =
 Factor[CoefficientList[
    Coefficient[(pY1 t^2 + (cYn0 xY t + n0 z)^2 + cY1^2 t^2 xY^2 - 2 cY2 t^2 xY^2 + 2 cY1 t xY z + 2 z^2)
      Expand[(rY xY t + l z)^3] Series[(cYn0 xY t + n0 z)^-1, {xY, 0, 2}]
      Series[(cY2 xY^2 t^2 + cY1 xY t z + z^2)^-1, {xY, 0, 2}], t, 2], l]] /. xY^2 → 1
```

$$\left\{0,\ \frac{3\ \left(2 + n0^2\right)\ rY^2}{n0},\ \frac{3\ \left(-2\ cYn0 + cYn0\ n0^2 - cY1\ n0^3\right)\ rY}{n0^2},\ \frac{2\ cYn0^2 + cY1^2\ n0^2 - 4\ cY2\ n0^2 - cY1\ cYn0\ n0^3 + cY1^2\ n0^4 - cY2\ n0^4 + n0^2\ pY1}{n0^3}\right\}$$

```
μs1x3P0B1 =
 CoefficientList[Sn0 z^2 Expand[((a0 + 1) z)^3] (n0 z)^-1 (n0 z)^-1 (n0 z)^-1 (N01 z)^-1 (N02 z)^-1,
  1]
```

$$\left\{\frac{a0^3\,Sn0}{n0^3\,N01\,N02},\frac{3\,a0^2\,Sn0}{n0^3\,N01\,N02},\frac{3\,a0\,Sn0}{n0^3\,N01\,N02},\frac{Sn0}{n0^3\,N01\,N02}\right\}$$

```
μs1x3P1B1 = CoefficientList[-Sn1 z^2 Expand[((a1 + 1) z)^3] N01^-1 N12^-1 z^-5, 1]
```

$$\left\{-\frac{a1^3\,Sn1}{N01\,N12},-\frac{3\,a1^2\,Sn1}{N01\,N12},-\frac{3\,a1\,Sn1}{N01\,N12},-\frac{Sn1}{N01\,N12}\right\}$$

```
μs1x3P2B1 = CoefficientList[-Sn2 z^2 Expand[((a2 + 1) z)^3] N02^-1 N12^-1 z^-5, 1]
```

$$\left\{-\frac{a2^3\,Sn2}{N02\,N12},-\frac{3\,a2^2\,Sn2}{N02\,N12},-\frac{3\,a2\,Sn2}{N02\,N12},-\frac{Sn2}{N02\,N12}\right\}$$

```
μx5YB1 + μx5P0B1 + μx5P1B1 + μx5P2B1
```

$$\left\{\frac{a0^5}{n0^3\,N01\,N02}-\frac{a1^5}{N01\,N12}-\frac{a2^5}{N02\,N12},\frac{5\,a0^4}{n0^3\,N01\,N02}-\frac{5\,a1^4}{N01\,N12}-\frac{5\,a2^4}{N02\,N12},\right.$$
$$\frac{10\,a0^3}{n0^3\,N01\,N02}-\frac{10\,a1^3}{N01\,N12}-\frac{10\,a2^3}{N02\,N12},\frac{10\,a0^2}{n0^3\,N01\,N02}-\frac{10\,a1^2}{N01\,N12}-\frac{10\,a2^2}{N02\,N12}+\frac{10\,rY^2}{n0},$$
$$\frac{5\,a0}{n0^3\,N01\,N02}-\frac{5\,a1}{N01\,N12}-\frac{5\,a2}{N02\,N12}-\frac{5\,(cYn0+cY1\,n0)\,rY}{n0^2},$$
$$\left.\frac{cYn0^2+cY1\,cYn0\,n0+cY1^2\,n0^2-cY2\,n0^2}{n0^3}+\frac{1}{n0^3\,N01\,N02}-\frac{1}{N01\,N12}-\frac{1}{N02\,N12}\right\}$$

```
μs1x3YB1 + μs1x3P0B1 + μs1x3P1B1 + μs1x3P2B1
```

$$\left\{\frac{a0^3\,Sn0}{n0^3\,N01\,N02}-\frac{a1^3\,Sn1}{N01\,N12}-\frac{a2^3\,Sn2}{N02\,N12},\frac{3\,(2+n0^2)\,rY^2}{n0}+\frac{3\,a0^2\,Sn0}{n0^3\,N01\,N02}-\frac{3\,a1^2\,Sn1}{N01\,N12}-\frac{3\,a2^2\,Sn2}{N02\,N12},\right.$$
$$\frac{3\,(-2\,cYn0+cYn0\,n0^2-cY1\,n0^3)\,rY}{n0^2}+\frac{3\,a0\,Sn0}{n0^3\,N01\,N02}-\frac{3\,a1\,Sn1}{N01\,N12}-\frac{3\,a2\,Sn2}{N02\,N12},$$
$$\frac{2\,cYn0^2+cY1^2\,n0^2-4\,cY2\,n0^2-cY1\,cYn0\,n0^3+cY1^2\,n0^4-cY2\,n0^4+n0^2\,pY1}{n0^3}+$$
$$\left.\frac{Sn0}{n0^3\,N01\,N02}-\frac{Sn1}{N01\,N12}-\frac{Sn2}{N02\,N12}\right\}$$

```
Solve[{
  (*******************************
  The following equations are given by Lemma 6.13
   *******************************)
  0 == -3 a0^2/n0^3 + 3 rY^2/n0, 0 == -3 a0/n0^3 - 3 cYn0 rY/n0^2, 0 == -1/n0^3 + cYn0^2/n0^3
 }, {rY, cYn0}]
```

$$\left\{\left\{rY\to-\frac{a0}{n0},\ cYn0\to 1\right\},\left\{rY\to\frac{a0}{n0},\ cYn0\to -1\right\}\right\}$$

$$
\begin{aligned}
&\mathtt{Factor}\Big[\mathtt{Solve}\Big[\Big\{\\
&d == \frac{a0^5}{n0^3\,N01\,N02} - \frac{a1^5}{N01\,N12} - \frac{a2^5}{N02\,N12},\\
&0 == \frac{5\,a0^4}{n0^3\,N01\,N02} - \frac{5\,a1^4}{N01\,N12} - \frac{5\,a2^4}{N02\,N12},\\
&0 == \frac{10\,a0^3}{n0^3\,N01\,N02} - \frac{10\,a1^3}{N01\,N12} - \frac{10\,a2^3}{N02\,N12},\\
&0 == \frac{10\,a0^2}{n0^3\,N01\,N02} - \frac{10\,a1^2}{N01\,N12} - \frac{10\,a2^2}{N02\,N12} + \frac{10\,rY^2}{n0},\\
&0 == \frac{5\,a0}{n0^3\,N01\,N02} - \frac{5\,a1}{N01\,N12} - \frac{5\,a2}{N02\,N12} - \frac{5\,(cYn0 + cY1\,n0)\,rY}{n0^2},\\
&0 == \frac{cYn0^2 + cY1\,cYn0\,n0 + cY1^2\,n0^2 - cY2\,n0^2}{n0^3} + \frac{1}{n0^3\,N01\,N02} - \frac{1}{N01\,N12} - \frac{1}{N02\,N12},\\
&p\,d == \frac{a0^3\,Sn0}{n0^3\,N01\,N02} - \frac{a1^3\,Sn1}{N01\,N12} - \frac{a2^3\,Sn2}{N02\,N12},\\
&0 == \frac{3\,(2 + n0^2)\,rY^2}{n0} + \frac{3\,a0^2\,Sn0}{n0^3\,N01\,N02} - \frac{3\,a1^2\,Sn1}{N01\,N12} - \frac{3\,a2^2\,Sn2}{N02\,N12},\\
&0 == \frac{3\,(-2\,cYn0 + cYn0\,n0^2 - cY1\,n0^3)\,rY}{n0^2} + \frac{3\,a0\,Sn0}{n0^3\,N01\,N02} - \frac{3\,a1\,Sn1}{N01\,N12} - \frac{3\,a2\,Sn2}{N02\,N12},\\
&0 == \frac{2\,cYn0^2 + cY1^2\,n0^2 - 4\,cY2\,n0^2 - cY1\,cYn0\,n0^3 + cY1^2\,n0^4 - cY2\,n0^4 + n0^2\,pY1}{n0^3} +\\
&\quad \frac{Sn0}{n0^3\,N01\,N02} - \frac{Sn1}{N01\,N12} - \frac{Sn2}{N02\,N12},\\
&pY1 == 3,
\end{aligned}
$$

```
(******************************
 The following equations are given by Lemma 6.13
 ******************************)
```

$$
\begin{aligned}
&0 == -\frac{3\,a0^2}{n0^3} + \frac{3\,rY^2}{n0},\; 0 == -\frac{3\,a0}{n0^3} - \frac{3\,cYn0\,rY}{n0^2},\; 0 == -\frac{1}{n0^3} + \frac{cYn0^2}{n0^3}\\
&\Big\},\ \{p, d, n0, N01, N02, N12, Sn0, Sn1, Sn2, pY1, cY1, cY2, cYn0\}\Big]\Big]
\end{aligned}
$$

$$\Big\{\Big\{p \to \frac{4\, a1^2\, a2^2 + a1^2\, rY^2 + a2^2\, rY^2}{a1^2\, a2^2\, rY^2},\ d \to a1\, a2\, rY^3,\ n0 \to -\frac{a0}{rY},$$
$$N01 \to -\frac{(a0 - a1)\, a1}{a2^2},\ N02 \to \frac{(a0 - a2)\, a2}{a1^2},\ N12 \to -\frac{a1\, (a1 - a2)\, a2}{rY^3},$$
$$Sn0 \to \frac{3\, a0^2\, a1^2\, a2^2 + a0^2\, a1^2\, rY^2 - 2\, a0\, a1^2\, a2\, rY^2 + a0^2\, a2^2\, rY^2 - 2\, a0\, a1\, a2^2\, rY^2 + 2\, a1^2\, a2^2\, rY^2}{a1^2\, a2^2\, rY^2},$$
$$Sn1 \to \frac{a0^2\, a2^2 - 2\, a0\, a1\, a2^2 + 4\, a1^2\, a2^2 + a1^2\, rY^2 - 2\, a1\, a2\, rY^2 + a2^2\, rY^2}{a2^2\, rY^2},$$
$$Sn2 \to \frac{a0^2\, a1^2 - 2\, a0\, a1^2\, a2 + 4\, a1^2\, a2^2 + a1^2\, rY^2 - 2\, a1\, a2\, rY^2 + a2^2\, rY^2}{a1^2\, rY^2},$$
$$pY1 \to 3,\ cY1 \to -\frac{(a1 + a2)\, rY}{a1\, a2},\ cY2 \to \frac{rY^2}{a1\, a2},\ cYn0 \to 1\Big\},$$

$$\Big\{p \to \frac{4\, a1^2\, a2^2 + a1^2\, rY^2 + a2^2\, rY^2}{a1^2\, a2^2\, rY^2},\ d \to a1\, a2\, rY^3,\ n0 \to -\frac{a0}{rY},$$
$$N01 \to \frac{(a0 - a1)\, a1}{a2^2},\ N02 \to -\frac{(a0 - a2)\, a2}{a1^2},\ N12 \to \frac{a1\, (a1 - a2)\, a2}{rY^3},$$
$$Sn0 \to \frac{3\, a0^2\, a1^2\, a2^2 + a0^2\, a1^2\, rY^2 - 2\, a0\, a1^2\, a2\, rY^2 + a0^2\, a2^2\, rY^2 - 2\, a0\, a1\, a2^2\, rY^2 + 2\, a1^2\, a2^2\, rY^2}{a1^2\, a2^2\, rY^2},$$
$$Sn1 \to \frac{a0^2\, a2^2 - 2\, a0\, a1\, a2^2 + 4\, a1^2\, a2^2 + a1^2\, rY^2 - 2\, a1\, a2\, rY^2 + a2^2\, rY^2}{a2^2\, rY^2},$$
$$Sn2 \to \frac{a0^2\, a1^2 - 2\, a0\, a1^2\, a2 + 4\, a1^2\, a2^2 + a1^2\, rY^2 - 2\, a1\, a2\, rY^2 + a2^2\, rY^2}{a1^2\, rY^2},$$
$$pY1 \to 3,\ cY1 \to -\frac{(a1 + a2)\, rY}{a1\, a2},\ cY2 \to \frac{rY^2}{a1\, a2},\ cYn0 \to 1\Big\},$$

$$\Big\{p \to \frac{4\, a1^2\, a2^2 + a1^2\, rY^2 + a2^2\, rY^2}{a1^2\, a2^2\, rY^2},\ d \to -a1\, a2\, rY^3,\ n0 \to \frac{a0}{rY},$$
$$N01 \to -\frac{(a0 - a1)\, a1}{a2^2},\ N02 \to \frac{(a0 - a2)\, a2}{a1^2},\ N12 \to \frac{a1\, (a1 - a2)\, a2}{rY^3},$$
$$Sn0 \to \frac{3\, a0^2\, a1^2\, a2^2 + a0^2\, a1^2\, rY^2 - 2\, a0\, a1^2\, a2\, rY^2 + a0^2\, a2^2\, rY^2 - 2\, a0\, a1\, a2^2\, rY^2 + 2\, a1^2\, a2^2\, rY^2}{a1^2\, a2^2\, rY^2},$$
$$Sn1 \to \frac{a0^2\, a2^2 - 2\, a0\, a1\, a2^2 + 4\, a1^2\, a2^2 + a1^2\, rY^2 - 2\, a1\, a2\, rY^2 + a2^2\, rY^2}{a2^2\, rY^2},$$
$$Sn2 \to \frac{a0^2\, a1^2 - 2\, a0\, a1^2\, a2 + 4\, a1^2\, a2^2 + a1^2\, rY^2 - 2\, a1\, a2\, rY^2 + a2^2\, rY^2}{a1^2\, rY^2},$$
$$pY1 \to 3,\ cY1 \to -\frac{(a1 + a2)\, rY}{a1\, a2},\ cY2 \to \frac{rY^2}{a1\, a2},\ cYn0 \to -1\Big\},$$

$$\Big\{p \to \frac{4\, a1^2\, a2^2 + a1^2\, rY^2 + a2^2\, rY^2}{a1^2\, a2^2\, rY^2},\ d \to -a1\, a2\, rY^3,\ n0 \to \frac{a0}{rY},$$
$$N01 \to \frac{(a0 - a1)\, a1}{a2^2},\ N02 \to -\frac{(a0 - a2)\, a2}{a1^2},\ N12 \to -\frac{a1\, (a1 - a2)\, a2}{rY^3},$$
$$Sn0 \to \frac{3\, a0^2\, a1^2\, a2^2 + a0^2\, a1^2\, rY^2 - 2\, a0\, a1^2\, a2\, rY^2 + a0^2\, a2^2\, rY^2 - 2\, a0\, a1\, a2^2\, rY^2 + 2\, a1^2\, a2^2\, rY^2}{a1^2\, a2^2\, rY^2},$$
$$Sn1 \to \frac{a0^2\, a2^2 - 2\, a0\, a1\, a2^2 + 4\, a1^2\, a2^2 + a1^2\, rY^2 - 2\, a1\, a2\, rY^2 + a2^2\, rY^2}{a2^2\, rY^2},$$
$$Sn2 \to \frac{a0^2\, a1^2 - 2\, a0\, a1^2\, a2 + 4\, a1^2\, a2^2 + a1^2\, rY^2 - 2\, a1\, a2\, rY^2 + a2^2\, rY^2}{a1^2\, rY^2},$$
$$pY1 \to 3,\ cY1 \to -\frac{(a1 + a2)\, rY}{a1\, a2},\ cY2 \to \frac{rY^2}{a1\, a2},\ cYn0 \to -1\Big\}\Big\}$$

Computations for Lemma 6.20.

Case B.2 where there exists N^6 s.t. Fix(N) = $Y^4 \cup P_1$.

```
(******************************
  NOTATIONS
  rY xY denotes the restriction of x to Y
  ai denotes the Hopf weight at Pi
  Nij denotes the product of the normal weights shared between Pi and Pj
  Sni is the sum of the square of the normal weights at Pi
  cYn1 xY denotes the first Chern class of the normal bundle of Y in the
  direction of N
  cYj xY^j denotes the j-th Chern class of the normal bundle of Y in the
  remaining directions
  μx5YB2 denotes the local datum of x^5 at Y
  μx5PiB2 denotes the local datum of x^5 at Pi
  μs1x3YB2 denotes the local datum of p1(X)x^3 at Y
  μs1x3PiB2 denotes the local datum of p1(X)x^3 at Pi

  We use the variable t to extract the coefficient of order 2 of xY. We
  then evaluate xY^2 to 1.
  The orientation of the point P0 is 1 while those from P1 and P2 are -
 1. This explains the signs at their local datum
  Once evaluated, we extract all the coefficients in l into a list.
  ******************************)
```

```
μx5YB2 =
 Factor[CoefficientList[
    Coefficient[Expand[(rY xY t + l z)^5] Series[(cYn1 xY t + n1 z)^-1, {xY, 0, 2}]
      Series[(cY2 xY^2 t^2 + cY1 xY t z + z^2)^-1, {xY, 0, 2}], t, 2], l]] /. xY^2 → 1
```

$$\left\{0,\ 0,\ 0,\ \frac{10\ rY^2}{n1},\ -\frac{5\ (cYn1 + cY1\ n1)\ rY}{n1^2},\ \frac{cYn1^2 + cY1\ cYn1\ n1 + cY1^2\ n1^2 - cY2\ n1^2}{n1^3}\right\}$$

```
μx5P0B2 = CoefficientList[Expand[((a0 + l) z)^5] N01^-1 N02^-1 z^-5, l]
```

$$\left\{\frac{a0^5}{N01\ N02},\ \frac{5\ a0^4}{N01\ N02},\ \frac{10\ a0^3}{N01\ N02},\ \frac{10\ a0^2}{N01\ N02},\ \frac{5\ a0}{N01\ N02},\ \frac{1}{N01\ N02}\right\}$$

```
μx5P1B2 =
 CoefficientList[-Expand[((a1 + l) z)^5] (n1 z)^-1 (n1 z)^-1 (n1 z)^-1 (N01 z)^-1 (N12 z)^-1, l]
```

$$\left\{-\frac{a1^5}{N01\ n1^3\ N12},\ -\frac{5\ a1^4}{N01\ n1^3\ N12},\ -\frac{10\ a1^3}{N01\ n1^3\ N12},\ -\frac{10\ a1^2}{N01\ n1^3\ N12},\ -\frac{5\ a1}{N01\ n1^3\ N12},\ -\frac{1}{N01\ n1^3\ N12}\right\}$$

```
μx5P2B2 = CoefficientList[-Expand[((a2 + l) z)^5] N02^-1 N12^-1 z^-5, l]
```

$$\left\{-\frac{a2^5}{N02\ N12},\ -\frac{5\ a2^4}{N02\ N12},\ -\frac{10\ a2^3}{N02\ N12},\ -\frac{10\ a2^2}{N02\ N12},\ -\frac{5\ a2}{N02\ N12},\ -\frac{1}{N02\ N12}\right\}$$

```
SymmetricReduction[Expand[(yY1 t + z)^2 + (yY2 t + z)^2], {yY1, yY2}, {cY1 xY, cY2 xY^2}]
```

$$\left\{cY1^2\ t^2\ xY^2 - 2\ cY2\ t^2\ xY^2 + 2\ cY1\ t\ xY\ z + 2\ z^2,\ 0\right\}$$

```
μs1x3YB2 =
 Factor[CoefficientList[
    Coefficient[(pY1 t^2 + (cYn1 xY t + n1 z)^2 + cY1^2 t^2 xY^2 - 2 cY2 t^2 xY^2 + 2 cY1 t xY z + 2 z^2)
      Expand[(rY xY t + 1 z)^3] Series[(cYn1 xY t + n1 z)^-1, {xY, 0, 2}]
      Series[(cY2 xY^2 t^2 + cY1 xY t z + z^2)^-1, {xY, 0, 2}], t, 2], 1]] /. xY^2 → 1
```

$$\left\{0,\ \frac{3\left(2+n1^2\right) rY^2}{n1},\ \frac{3\left(-2\, cYn1+cYn1\, n1^2-cY1\, n1^3\right) rY}{n1^2},\ \frac{2\, cYn1^2+cY1^2\, n1^2-4\, cY2\, n1^2-cY1\, cYn1\, n1^3+cY1^2\, n1^4-cY2\, n1^4+n1^2\, pY1}{n1^3}\right\}$$

```
μs1x3P0B2 = CoefficientList[Sn0 z^2 Expand[((a0 + 1) z)^3] N01^-1 N02^-1 z^-5, 1]
```

$$\left\{\frac{a0^3\, Sn0}{N01\, N02},\ \frac{3\, a0^2\, Sn0}{N01\, N02},\ \frac{3\, a0\, Sn0}{N01\, N02},\ \frac{Sn0}{N01\, N02}\right\}$$

```
μs1x3P1B2 =
 CoefficientList[-Sn1 z^2 Expand[((a1 + 1) z)^3] (n1 z)^-1 (n1 z)^-1 (n1 z)^-1 (N01 z)^-1 (N12 z)^-1,
  1]
```

$$\left\{-\frac{a1^3\, Sn1}{N01\, n1^3\, N12},\ -\frac{3\, a1^2\, Sn1}{N01\, n1^3\, N12},\ -\frac{3\, a1\, Sn1}{N01\, n1^3\, N12},\ -\frac{Sn1}{N01\, n1^3\, N12}\right\}$$

```
μs1x3P2B2 = CoefficientList[-Sn2 z^2 Expand[((a2 + 1) z)^3] N02^-1 N12^-1 z^-5, 1]
```

$$\left\{-\frac{a2^3\, Sn2}{N02\, N12},\ -\frac{3\, a2^2\, Sn2}{N02\, N12},\ -\frac{3\, a2\, Sn2}{N02\, N12},\ -\frac{Sn2}{N02\, N12}\right\}$$

```
μx5YB2 + μx5P0B2 + μx5P1B2 + μx5P2B2
```

$$\left\{\frac{a0^5}{N01\, N02}-\frac{a2^5}{N02\, N12}-\frac{a1^5}{N01\, n1^3\, N12},\ \frac{5\, a0^4}{N01\, N02}-\frac{5\, a2^4}{N02\, N12}-\frac{5\, a1^4}{N01\, n1^3\, N12},\right.$$
$$\frac{10\, a0^3}{N01\, N02}-\frac{10\, a2^3}{N02\, N12}-\frac{10\, a1^3}{N01\, n1^3\, N12},\ \frac{10\, a0^2}{N01\, N02}-\frac{10\, a2^2}{N02\, N12}-\frac{10\, a1^2}{N01\, n1^3\, N12}+\frac{10\, rY^2}{n1},$$
$$\frac{5\, a0}{N01\, N02}-\frac{5\, a2}{N02\, N12}-\frac{5\, a1}{N01\, n1^3\, N12}-\frac{5\left(cYn1+cY1\, n1\right) rY}{n1^2},$$
$$\left.\frac{1}{N01\, N02}+\frac{cYn1^2+cY1\, cYn1\, n1+cY1^2\, n1^2-cY2\, n1^2}{n1^3}-\frac{1}{N02\, N12}-\frac{1}{N01\, n1^3\, N12}\right\}$$

```
μs1x3YB2 + μs1x3P0B2 + μs1x3P1B2 + μs1x3P2B2
```

$$\left\{\frac{a0^3\, Sn0}{N01\, N02}-\frac{a1^3\, Sn1}{N01\, n1^3\, N12}-\frac{a2^3\, Sn2}{N02\, N12},\ \frac{3\left(2+n1^2\right) rY^2}{n1}+\frac{3\, a0^2\, Sn0}{N01\, N02}-\frac{3\, a1^2\, Sn1}{N01\, n1^3\, N12}-\frac{3\, a2^2\, Sn2}{N02\, N12},\right.$$
$$\frac{3\left(-2\, cYn1+cYn1\, n1^2-cY1\, n1^3\right) rY}{n1^2}+\frac{3\, a0\, Sn0}{N01\, N02}-\frac{3\, a1\, Sn1}{N01\, n1^3\, N12}-\frac{3\, a2\, Sn2}{N02\, N12},$$
$$\frac{2\, cYn1^2+cY1^2\, n1^2-4\, cY2\, n1^2-cY1\, cYn1\, n1^3+cY1^2\, n1^4-cY2\, n1^4+n1^2\, pY1}{n1^3}+$$
$$\left.\frac{Sn0}{N01\, N02}-\frac{Sn1}{N01\, n1^3\, N12}-\frac{Sn2}{N02\, N12}\right\}$$

$$
\begin{aligned}
&\mathtt{Factor}\Big[\mathtt{Solve}\Big[\Big\{\\
&\quad d == \frac{a0^5}{N01\,N02} - \frac{a2^5}{N02\,N12} - \frac{a1^5}{N01\,n1^3\,N12},\\
&\quad 0 == \frac{5\,a0^4}{N01\,N02} - \frac{5\,a2^4}{N02\,N12} - \frac{5\,a1^4}{N01\,n1^3\,N12},\\
&\quad 0 == \frac{10\,a0^3}{N01\,N02} - \frac{10\,a2^3}{N02\,N12} - \frac{10\,a1^3}{N01\,n1^3\,N12},\\
&\quad 0 == \frac{10\,a0^2}{N01\,N02} - \frac{10\,a2^2}{N02\,N12} - \frac{10\,a1^2}{N01\,n1^3\,N12} + \frac{10\,rY^2}{n1},\\
&\quad 0 == \frac{5\,a0}{N01\,N02} - \frac{5\,a2}{N02\,N12} - \frac{5\,a1}{N01\,n1^3\,N12} - \frac{5\,(cYn1 + cY1\,n1)\,rY}{n1^2},\\
&\quad 0 == \frac{1}{N01\,N02} + \frac{cYn1^2 + cY1\,cYn1\,n1 + cY1^2\,n1^2 - cY2\,n1^2}{n1^3} - \frac{1}{N02\,N12} - \frac{1}{N01\,n1^3\,N12},\\
&\quad p\,d == \frac{a0^3\,Sn0}{N01\,N02} - \frac{a1^3\,Sn1}{N01\,n1^3\,N12} - \frac{a2^3\,Sn2}{N02\,N12},\\
&\quad 0 == \frac{3\,(2 + n1^2)\,rY^2}{n1} + \frac{3\,a0^2\,Sn0}{N01\,N02} - \frac{3\,a1^2\,Sn1}{N01\,n1^3\,N12} - \frac{3\,a2^2\,Sn2}{N02\,N12},\\
&\quad 0 == \frac{3\,(-2\,cYn1 + cYn1\,n1^2 - cY1\,n1^3)\,rY}{n1^2} + \frac{3\,a0\,Sn0}{N01\,N02} - \frac{3\,a1\,Sn1}{N01\,n1^3\,N12} - \frac{3\,a2\,Sn2}{N02\,N12},\\
&\quad 0 == \frac{2\,cYn1^2 + cY1^2\,n1^2 - 4\,cY2\,n1^2 - cY1\,cYn1\,n1^3 + cY1^2\,n1^4 - cY2\,n1^4 + n1^2\,pY1}{n1^3} +\\
&\qquad \frac{Sn0}{N01\,N02} - \frac{Sn1}{N01\,n1^3\,N12} - \frac{Sn2}{N02\,N12},\\
&\quad pY1 == 3,\\
&\quad (\text{******************************}\\
&\quad\ \text{The following equations are given by Lemma 6.13}\\
&\quad\ \ \text{******************************})\\
&\quad 0 == -\frac{3\,a1^2}{n1^3} + \frac{3\,rY^2}{n1},\ 0 == -\frac{3\,a1}{n1^3} - \frac{3\,cYn1\,rY}{n1^2},\ 0 == -\frac{1}{n1^3} + \frac{cYn1^2}{n1^3}\\
&\Big\},\ \{p,\ d,\ n1,\ N01,\ N02,\ N12,\ Sn0,\ Sn1,\ Sn2,\ cY1,\ cY2,\ pY1,\ cYn1\}\Big]\Big]
\end{aligned}
$$

$$
\begin{aligned}
&\Big\{\Big\{p \to \frac{4\,a0^2\,a2^2 + a0^2\,rY^2 + a2^2\,rY^2}{a0^2\,a2^2\,rY^2},\ d \to a0\,a2\,rY^3,\ n1 \to -\frac{a1}{rY},\\
&N01 \to -\frac{a0\,(a0 - a1)}{a2^2},\ N02 \to -\frac{a0\,(a0 - a2)\,a2}{rY^3},\ N12 \to \frac{(a1 - a2)\,a2}{a0^2},\\
&Sn0 \to \frac{4\,a0^2\,a2^2 - 2\,a0\,a1\,a2^2 + a1^2\,a2^2 + a0^2\,rY^2 - 2\,a0\,a2\,rY^2 + a2^2\,rY^2}{a2^2\,rY^2},\\
&Sn1 \to \frac{3\,a0^2\,a1^2\,a2^2 + a0^2\,a1^2\,rY^2 - 2\,a0^2\,a1\,a2\,rY^2 + 2\,a0^2\,a2^2\,rY^2 - 2\,a0\,a1\,a2^2\,rY^2 + a1^2\,a2^2\,rY^2}{a0^2\,a2^2\,rY^2},\\
&Sn2 \to \frac{a0^2\,a1^2 - 2\,a0^2\,a1\,a2 + 4\,a0^2\,a2^2 + a0^2\,rY^2 - 2\,a0\,a2\,rY^2 + a2^2\,rY^2}{a0^2\,rY^2},\\
&cY1 \to -\frac{(a0 + a2)\,rY}{a0\,a2},\ cY2 \to \frac{rY^2}{a0\,a2},\ pY1 \to 3,\ cYn1 \to 1\Big\},
\end{aligned}
$$

$$
\begin{aligned}
&\Big\{p \to \frac{4\,a0^2\,a2^2 + a0^2\,rY^2 + a2^2\,rY^2}{a0^2\,a2^2\,rY^2},\ d \to a0\,a2\,rY^3,\ n1 \to -\frac{a1}{rY},\\
&N01 \to \frac{a0\,(a0 - a1)}{a2^2},\ N02 \to \frac{a0\,(a0 - a2)\,a2}{rY^3},\ N12 \to -\frac{(a1 - a2)\,a2}{a0^2},\\
&Sn0 \to \frac{4\,a0^2\,a2^2 - 2\,a0\,a1\,a2^2 + a1^2\,a2^2 + a0^2\,rY^2 - 2\,a0\,a2\,rY^2 + a2^2\,rY^2}{a2^2\,rY^2},\\
&Sn1 \to \frac{3\,a0^2\,a1^2\,a2^2 + a0^2\,a1^2\,rY^2 - 2\,a0^2\,a1\,a2\,rY^2 + 2\,a0^2\,a2^2\,rY^2 - 2\,a0\,a1\,a2^2\,rY^2 + a1^2\,a2^2\,rY^2}{a0^2\,a2^2\,rY^2},\\
&Sn2 \to \frac{a0^2\,a1^2 - 2\,a0^2\,a1\,a2 + 4\,a0^2\,a2^2 + a0^2\,rY^2 - 2\,a0\,a2\,rY^2 + a2^2\,rY^2}{a0^2\,rY^2},\\
&cY1 \to -\frac{(a0 + a2)\,rY}{a0\,a2},\ cY2 \to \frac{rY^2}{a0\,a2},\ pY1 \to 3,\ cYn1 \to 1\Big\},
\end{aligned}
$$

$$
\begin{aligned}
&\Big\{p \to \frac{4\,a0^2\,a2^2 + a0^2\,rY^2 + a2^2\,rY^2}{a0^2\,a2^2\,rY^2},\ d \to -a0\,a2\,rY^3,\ n1 \to \frac{a1}{rY},\\
&N01 \to -\frac{a0\,(a0 - a1)}{a2^2},\ N02 \to \frac{a0\,(a0 - a2)\,a2}{rY^3},\ N12 \to \frac{(a1 - a2)\,a2}{a0^2},\\
&Sn0 \to \frac{4\,a0^2\,a2^2 - 2\,a0\,a1\,a2^2 + a1^2\,a2^2 + a0^2\,rY^2 - 2\,a0\,a2\,rY^2 + a2^2\,rY^2}{a2^2\,rY^2},\\
&Sn1 \to \frac{3\,a0^2\,a1^2\,a2^2 + a0^2\,a1^2\,rY^2 - 2\,a0^2\,a1\,a2\,rY^2 + 2\,a0^2\,a2^2\,rY^2 - 2\,a0\,a1\,a2^2\,rY^2 + a1^2\,a2^2\,rY^2}{a0^2\,a2^2\,rY^2},\\
&Sn2 \to \frac{a0^2\,a1^2 - 2\,a0^2\,a1\,a2 + 4\,a0^2\,a2^2 + a0^2\,rY^2 - 2\,a0\,a2\,rY^2 + a2^2\,rY^2}{a0^2\,rY^2},\\
&cY1 \to -\frac{(a0 + a2)\,rY}{a0\,a2},\ cY2 \to \frac{rY^2}{a0\,a2},\ pY1 \to 3,\ cYn1 \to -1\Big\},
\end{aligned}
$$

$$
\begin{aligned}
&\Big\{p \to \frac{4\,a0^2\,a2^2 + a0^2\,rY^2 + a2^2\,rY^2}{a0^2\,a2^2\,rY^2},\ d \to -a0\,a2\,rY^3,\ n1 \to \frac{a1}{rY},\\
&N01 \to \frac{a0\,(a0 - a1)}{a2^2},\ N02 \to -\frac{a0\,(a0 - a2)\,a2}{rY^3},\ N12 \to -\frac{(a1 - a2)\,a2}{a0^2},\\
&Sn0 \to \frac{4\,a0^2\,a2^2 - 2\,a0\,a1\,a2^2 + a1^2\,a2^2 + a0^2\,rY^2 - 2\,a0\,a2\,rY^2 + a2^2\,rY^2}{a2^2\,rY^2},\\
&Sn1 \to \frac{3\,a0^2\,a1^2\,a2^2 + a0^2\,a1^2\,rY^2 - 2\,a0^2\,a1\,a2\,rY^2 + 2\,a0^2\,a2^2\,rY^2 - 2\,a0\,a1\,a2^2\,rY^2 + a1^2\,a2^2\,rY^2}{a0^2\,a2^2\,rY^2},\\
&Sn2 \to \frac{a0^2\,a1^2 - 2\,a0^2\,a1\,a2 + 4\,a0^2\,a2^2 + a0^2\,rY^2 - 2\,a0\,a2\,rY^2 + a2^2\,rY^2}{a0^2\,rY^2},\\
&cY1 \to -\frac{(a0 + a2)\,rY}{a0\,a2},\ cY2 \to \frac{rY^2}{a0\,a2},\ pY1 \to 3,\ cYn1 \to -1\Big\}\Big\}
\end{aligned}
$$

Computations for Lemma 6.20.

Case B.3 where there exists N^6 s.t. Fix(N) = $Y^4 \cup P_0$ and M^6 Fix(M) = $Y^4 \cup P_1$.

```
(******************************
  NOTATIONS
  rY xY denotes the restriction of x to Y
  ai denotes the Hopf weight at Pi
  Nij denotes the product of the normal weights shared between Pi and Pj
  Sni is the sum of the square of the normal weights at Pi
  cYn0 xY denotes the first Chern class of the normal bundle of Y in the
  direction of N
  cYn1 xY denotes the first Chern class of the normal bundle of Y in the
  direction of M
  cYj xY^j denotes the j-th Chern class of the normal bundle of Y in the
  remaining directions
  μx5YB3 denotes the local datum of x^5 at Y
  μx5PiB3 denotes the local datum of x^5 at Pi
  μs1x3YB3 denotes the local datum of p1(X)x^3 at Y
  μs1x3PiB3 denotes the local datum of p1(X)x^3 at Pi

  We use the variable t to extract the coefficient of order 2 of xY. We
  then evaluate xY^2 to 1.
  The orientation of the point P0 is 1 while those from P1 and P2 are -
 1. This explains the signs at their local datum
  Once evaluated, we extract all the coefficients in 1 into a list.
  ******************************)
```

```
μx5YB3 =
 Factor[CoefficientList[
    Coefficient[Expand[(rY xY t + 1 z)^5] Series[(cYn0 xY t + n0 z)^-1, {xY, 0, 2}]
      Series[(cYn1 xY t + n1 z)^-1, {xY, 0, 2}] Series[(cY1 xY t + z)^-1, {xY, 0, 2}], t, 2],
    1]] /. xY^2 → 1
```

$$\left\{0,\ 0,\ 0,\ \frac{10\ \text{rY}^2}{\text{n0 n1}},\ -\frac{5\ (\text{cYn1 n0} + \text{cYn0 n1} + \text{cY1 n0 n1})\ \text{rY}}{\text{n0}^2\ \text{n1}^2},\ \frac{\text{cYn1}^2\ \text{n0}^2 + \text{cYn0 cYn1 n0 n1} + \text{cY1 cYn1 n0}^2\ \text{n1} + \text{cYn0}^2\ \text{n1}^2 + \text{cY1 cYn0 n0 n1}^2 + \text{cY1}^2\ \text{n0}^2\ \text{n1}^2}{\text{n0}^3\ \text{n1}^3}\right\}$$

```
μx5P0B3 = CoefficientList[Expand[((a0 + 1) z)^5] (n0 z)^-1 (n0 z)^-1 (n0 z)^-1 (N01 z)^-1 (N02 z)^-1,
  1]
```

$$\left\{\frac{\text{a0}^5}{\text{n0}^3\ \text{N01 N02}},\ \frac{5\ \text{a0}^4}{\text{n0}^3\ \text{N01 N02}},\ \frac{10\ \text{a0}^3}{\text{n0}^3\ \text{N01 N02}},\ \frac{10\ \text{a0}^2}{\text{n0}^3\ \text{N01 N02}},\ \frac{5\ \text{a0}}{\text{n0}^3\ \text{N01 N02}},\ \frac{1}{\text{n0}^3\ \text{N01 N02}}\right\}$$

```
μx5P1B3 =
 CoefficientList[-Expand[((a1 + 1) z)^5] (n1 z)^-1 (n1 z)^-1 (n1 z)^-1 (N01 z)^-1 (N12 z)^-1, 1]
```

$$\left\{-\frac{\text{a1}^5}{\text{N01 n1}^3\ \text{N12}},\ -\frac{5\ \text{a1}^4}{\text{N01 n1}^3\ \text{N12}},\ -\frac{10\ \text{a1}^3}{\text{N01 n1}^3\ \text{N12}},\ -\frac{10\ \text{a1}^2}{\text{N01 n1}^3\ \text{N12}},\ -\frac{5\ \text{a1}}{\text{N01 n1}^3\ \text{N12}},\ -\frac{1}{\text{N01 n1}^3\ \text{N12}}\right\}$$

```
μx5P2B3 = CoefficientList[-Expand[((a2 + 1) z)^5] N02^-1 N12^-1 z^-5, 1]
```

$$\left\{-\frac{\text{a2}^5}{\text{N02 N12}},\ -\frac{5\ \text{a2}^4}{\text{N02 N12}},\ -\frac{10\ \text{a2}^3}{\text{N02 N12}},\ -\frac{10\ \text{a2}^2}{\text{N02 N12}},\ -\frac{5\ \text{a2}}{\text{N02 N12}},\ -\frac{1}{\text{N02 N12}}\right\}$$

```
μs1x3YB3 =
 Factor[CoefficientList[
    Coefficient[(pY1 t^2 + (cYn0 xY t + n0 z)^2 + (cYn1 xY t + n1 z)^2 + (cY1 xY t + z)^2)
      Expand[(rY xY t + l z)^3] Series[(cYn0 xY t + n0 z)^-1, {xY, 0, 2}]
      Series[(cYn1 xY t + n1 z)^-1, {xY, 0, 2}] Series[(cY1 xY t + z)^-1, {xY, 0, 2}], t, 2],
    l]] /. xY^2 → 1
```

$$\left\{0, \frac{3\left(1+n0^2+n1^2\right) rY^2}{n0\, n1}, -\frac{1}{n0^2\, n1^2}3\left(cYn1\, n0+cYn1\, n0^3+cYn0\, n1-cY1\, n0\, n1-cYn0\, n0^2\, n1+cY1\, n0^3\, n1-cYn1\, n0\, n1^2+cYn0\, n1^3+cY1\, n0\, n1^3\right) rY, \frac{1}{n0^3\, n1^3}\left(cYn1^2\, n0^2+cYn1^2\, n0^4+cYn0\, cYn1\, n0\, n1-cY1\, cYn1\, n0^2\, n1-cYn0\, cYn1\, n0^3\, n1+cY1\, cYn1\, n0^4\, n1+cYn0^2\, n1^2-cY1\, cYn0\, n0\, n1^2-cY1\, cYn0\, n0^3\, n1^2+cY1^2\, n0^4\, n1^2-cYn0\, cYn1\, n0\, n1^3-cY1\, cYn1\, n0^2\, n1^3+cYn0^2\, n1^4+cY1\, cYn0\, n0\, n1^4+cY1^2\, n0^2\, n1^4+n0^2\, n1^2\, pY1\right)\right\}$$

```
μs1x3P0B3 =
 CoefficientList[Sn0 z^2 Expand[((a0 + l) z)^3] (n0 z)^-1 (n0 z)^-1 (n0 z)^-1 (N01 z)^-1 (N02 z)^-1,
  l]
```

$$\left\{\frac{a0^3\, Sn0}{n0^3\, N01\, N02}, \frac{3\, a0^2\, Sn0}{n0^3\, N01\, N02}, \frac{3\, a0\, Sn0}{n0^3\, N01\, N02}, \frac{Sn0}{n0^3\, N01\, N02}\right\}$$

```
μs1x3P1B3 =
 CoefficientList[-Sn1 z^2 Expand[((a1 + l) z)^3] (n1 z)^-1 (n1 z)^-1 (n1 z)^-1 (N01 z)^-1 (N12 z)^-1,
  l]
```

$$\left\{-\frac{a1^3\, Sn1}{N01\, n1^3\, N12}, -\frac{3\, a1^2\, Sn1}{N01\, n1^3\, N12}, -\frac{3\, a1\, Sn1}{N01\, n1^3\, N12}, -\frac{Sn1}{N01\, n1^3\, N12}\right\}$$

```
μs1x3P2B3 = CoefficientList[-Sn2 z^2 Expand[((a2 + l) z)^3] N02^-1 N12^-1 z^-5, l]
```

$$\left\{-\frac{a2^3\, Sn2}{N02\, N12}, -\frac{3\, a2^2\, Sn2}{N02\, N12}, -\frac{3\, a2\, Sn2}{N02\, N12}, -\frac{Sn2}{N02\, N12}\right\}$$

```
μx5YB3 + μx5P0B3 + μx5P1B3 + μx5P2B3
```

$$\left\{\frac{a0^5}{n0^3\, N01\, N02}-\frac{a2^5}{N02\, N12}-\frac{a1^5}{N01\, n1^3\, N12}, \frac{5\, a0^4}{n0^3\, N01\, N02}-\frac{5\, a2^4}{N02\, N12}-\frac{5\, a1^4}{N01\, n1^3\, N12}, \frac{10\, a0^3}{n0^3\, N01\, N02}-\frac{10\, a2^3}{N02\, N12}-\frac{10\, a1^3}{N01\, n1^3\, N12}, \frac{10\, a0^2}{n0^3\, N01\, N02}-\frac{10\, a2^2}{N02\, N12}-\frac{10\, a1^2}{N01\, n1^3\, N12}+\frac{10\, rY^2}{n0\, n1}, \frac{5\, a0}{n0^3\, N01\, N02}-\frac{5\, a2}{N02\, N12}-\frac{5\, a1}{N01\, n1^3\, N12}-\frac{5\left(cYn1\, n0+cYn0\, n1+cY1\, n0\, n1\right) rY}{n0^2\, n1^2}, \frac{1}{n0^3\, N01\, N02}+\frac{cYn1^2\, n0^2+cYn0\, cYn1\, n0\, n1+cY1\, cYn1\, n0^2\, n1+cYn0^2\, n1^2+cY1\, cYn0\, n0\, n1^2+cY1^2\, n0^2\, n1^2}{n0^3\, n1^3}-\frac{1}{N02\, N12}-\frac{1}{N01\, n1^3\, N12}\right\}$$

μs1x3YB3 + μs1x3P0B3 + μs1x3P1B3 + μs1x3P2B3

$$\left\{\frac{a0^3\,Sn0}{n0^3\,N01\,N02}-\frac{a1^3\,Sn1}{N01\,n1^3\,N12}-\frac{a2^3\,Sn2}{N02\,N12},\right.$$
$$\frac{3\left(1+n0^2+n1^2\right)rY^2}{n0\,n1}+\frac{3\,a0^2\,Sn0}{n0^3\,N01\,N02}-\frac{3\,a1^2\,Sn1}{N01\,n1^3\,N12}-\frac{3\,a2^2\,Sn2}{N02\,N12},$$
$$-\frac{1}{n0^2\,n1^2}3\left(cYn1\,n0+cYn1\,n0^3+cYn0\,n1-cY1\,n0\,n1-cYn0\,n0^2\,n1+cY1\,n0^3\,n1-cYn1\,n0\,n1^2+cYn0\,n1^3+cY1\,n0\,n1^3\right)rY+\frac{3\,a0\,Sn0}{n0^3\,N01\,N02}-\frac{3\,a1\,Sn1}{N01\,n1^3\,N12}-\frac{3\,a2\,Sn2}{N02\,N12},$$
$$\frac{1}{n0^3\,n1^3}\left(cYn1^2\,n0^2+cYn1^2\,n0^4+cYn0\,cYn1\,n0\,n1-cY1\,cYn1\,n0^2\,n1-cYn0\,cYn1\,n0^3\,n1+cY1\,cYn1\,n0^4\,n1+cYn0^2\,n1^2-cY1\,cYn0\,n0\,n1^2-cY1\,cYn0\,n0^3\,n1^2+cY1^2\,n0^4\,n1^2-cYn0\,cYn1\,n0\,n1^3-cY1\,cYn1\,n0^2\,n1^3+cYn0^2\,n1^4+cY1\,cYn0\,n0\,n1^4+cY1^2\,n0^2\,n1^4+n0^2\,n1^2\,pY1\right)+\frac{Sn0}{n0^3\,N01\,N02}-\frac{Sn1}{N01\,n1^3\,N12}-\left.\frac{Sn2}{N02\,N12}\right\}$$

$$\text{Factor}\Big[\text{Solve}\Big[\Big\{$$

$$d == \frac{a0^5}{n0^3\ N01\ N02} - \frac{a2^5}{N02\ N12} - \frac{a1^5}{N01\ n1^3\ N12},$$

$$0 == \frac{5\ a0^4}{n0^3\ N01\ N02} - \frac{5\ a2^4}{N02\ N12} - \frac{5\ a1^4}{N01\ n1^3\ N12},$$

$$0 == \frac{10\ a0^3}{n0^3\ N01\ N02} - \frac{10\ a2^3}{N02\ N12} - \frac{10\ a1^3}{N01\ n1^3\ N12},$$

$$0 == \frac{10\ a0^2}{n0^3\ N01\ N02} - \frac{10\ a2^2}{N02\ N12} - \frac{10\ a1^2}{N01\ n1^3\ N12} + \frac{10\ rY^2}{n0\ n1},$$

$$0 == \frac{5\ a0}{n0^3\ N01\ N02} - \frac{5\ a2}{N02\ N12} - \frac{5\ a1}{N01\ n1^3\ N12} - \frac{5\ (cYn1\ n0 + cYn0\ n1 + cY1\ n0\ n1)\ rY}{n0^2\ n1^2},$$

$$0 == \frac{1}{n0^3\ N01\ N02} + \frac{cYn1^2\ n0^2 + cYn0\ cYn1\ n0\ n1 + cY1\ cYn1\ n0^2\ n1 + cYn0^2\ n1^2 + cY1\ cYn0\ n0\ n1^2 + cY1^2\ n0^2\ n1^2}{n0^3\ n1^3} - \frac{1}{N02\ N12} - \frac{1}{N01\ n1^3\ N12},$$

$$p\ d == \frac{a0^3\ Sn0}{n0^3\ N01\ N02} - \frac{a1^3\ Sn1}{N01\ n1^3\ N12} - \frac{a2^3\ Sn2}{N02\ N12},$$

$$0 == \frac{3\ \left(1 + n0^2 + n1^2\right)\ rY^2}{n0\ n1} + \frac{3\ a0^2\ Sn0}{n0^3\ N01\ N02} - \frac{3\ a1^2\ Sn1}{N01\ n1^3\ N12} - \frac{3\ a2^2\ Sn2}{N02\ N12},$$

$$0 == -\frac{1}{n0^2\ n1^2} 3\ \left(cYn1\ n0 + cYn1\ n0^3 + cYn0\ n1 - cY1\ n0\ n1 - cYn0\ n0^2\ n1 + cY1\ n0^3\ n1 - cYn1\ n0\ n1^2 + cYn0\ n1^3 + cY1\ n0\ n1^3\right)\ rY + \frac{3\ a0\ Sn0}{n0^3\ N01\ N02} - \frac{3\ a1\ Sn1}{N01\ n1^3\ N12} - \frac{3\ a2\ Sn2}{N02\ N12},$$

$$0 == \frac{1}{n0^3\ n1^3} \left(cYn1^2\ n0^2 + cYn1^2\ n0^4 + cYn0\ cYn1\ n0\ n1 - cY1\ cYn1\ n0^2\ n1 - cYn0\ cYn1\ n0^3\ n1 + cY1\ cYn1\ n0^4\ n1 + cYn0^2\ n1^2 - cY1\ cYn0\ n0\ n1^2 - cY1\ cYn0\ n0^3\ n1^2 + cY1^2\ n0^4\ n1^2 - cYn0\ cYn1\ n0\ n1^3 - cY1\ cYn1\ n0^2\ n1^3 + cYn0^2\ n1^4 + cY1\ cYn0\ n0\ n1^4 + cY1^2\ n0^2\ n1^4 + n0^2\ n1^2\ pY1\right) + \frac{Sn0}{n0^3\ N01\ N02} - \frac{Sn1}{N01\ n1^3\ N12} - \frac{Sn2}{N02\ N12},$$

$$pY1 == 3,$$

(******************************
The following equations are given by Lemma 6.13
******************************)

$$0 == -\frac{3\ a0^2}{n0^3} + \frac{3\ rY^2}{n0},\quad 0 == -\frac{3\ a0}{n0^3} - \frac{3\ cYn0\ rY}{n0^2},\quad 0 == -\frac{1}{n0^3} + \frac{cYn0^2}{n0^3},$$

$$0 == -\frac{3\ a1^2}{n1^3} + \frac{3\ rY^2}{n1},\quad 0 == -\frac{3\ a1}{n1^3} - \frac{3\ cYn1\ rY}{n1^2},\quad 0 == -\frac{1}{n1^3} + \frac{cYn1^2}{n1^3}$$

$$\Big\},\ \{p, d, n0, n1, N01, N02, N12, pY1, cY1, cYn0, cYn1, Sn0, Sn1, Sn2\}\Big]\Big]$$

$$\Big\{\Big\{p \to \frac{5\ a2^2 + rY^2}{a2^2\ rY^2},\ d \to -a2\ rY^4,\ n0 \to -\frac{a0}{rY},\ n1 \to -\frac{a1}{rY},\ N01 \to -\frac{(a0 - a1)\ rY}{a2^2},$$

$$N02 \to -\frac{(a0 - a2)\ a2}{rY^2},\ N12 \to -\frac{(a1 - a2)\ a2}{rY^2},\ pY1 \to 3,\ cY1 \to -\frac{rY}{a2},\ cYn0 \to 1,$$

$$cYn1 \to 1,\ Sn0 \to \frac{4\ a0^2\ a2^2 - 2\ a0\ a1\ a2^2 + a1^2\ a2^2 + a0^2\ rY^2 - 2\ a0\ a2\ rY^2 + a2^2\ rY^2}{a2^2\ rY^2},$$

$$\text{Sn1} \to \frac{\text{a0}^2\,\text{a2}^2 - 2\,\text{a0}\,\text{a1}\,\text{a2}^2 + 4\,\text{a1}^2\,\text{a2}^2 + \text{a1}^2\,\text{rY}^2 - 2\,\text{a1}\,\text{a2}\,\text{rY}^2 + \text{a2}^2\,\text{rY}^2}{\text{a2}^2\,\text{rY}^2},$$
$$\text{Sn2} \to \frac{\text{a0}^2 + \text{a1}^2 - 2\,\text{a0}\,\text{a2} - 2\,\text{a1}\,\text{a2} + 5\,\text{a2}^2}{\text{rY}^2}\Big\},$$

$$\Big\{\text{p} \to \frac{5\,\text{a2}^2 + \text{rY}^2}{\text{a2}^2\,\text{rY}^2},\ \text{d} \to -\text{a2}\,\text{rY}^4,\ \text{n0} \to -\frac{\text{a0}}{\text{rY}},\ \text{n1} \to -\frac{\text{a1}}{\text{rY}},\ \text{N01} \to \frac{(\text{a0} - \text{a1})\,\text{rY}}{\text{a2}^2},$$
$$\text{N02} \to \frac{(\text{a0} - \text{a2})\,\text{a2}}{\text{rY}^2},\ \text{N12} \to \frac{(\text{a1} - \text{a2})\,\text{a2}}{\text{rY}^2},\ \text{pY1} \to 3,\ \text{cY1} \to -\frac{\text{rY}}{\text{a2}},\ \text{cYn0} \to 1,$$
$$\text{cYn1} \to 1,\ \text{Sn0} \to \frac{4\,\text{a0}^2\,\text{a2}^2 - 2\,\text{a0}\,\text{a1}\,\text{a2}^2 + \text{a1}^2\,\text{a2}^2 + \text{a0}^2\,\text{rY}^2 - 2\,\text{a0}\,\text{a2}\,\text{rY}^2 + \text{a2}^2\,\text{rY}^2}{\text{a2}^2\,\text{rY}^2},$$
$$\text{Sn1} \to \frac{\text{a0}^2\,\text{a2}^2 - 2\,\text{a0}\,\text{a1}\,\text{a2}^2 + 4\,\text{a1}^2\,\text{a2}^2 + \text{a1}^2\,\text{rY}^2 - 2\,\text{a1}\,\text{a2}\,\text{rY}^2 + \text{a2}^2\,\text{rY}^2}{\text{a2}^2\,\text{rY}^2},$$
$$\text{Sn2} \to \frac{\text{a0}^2 + \text{a1}^2 - 2\,\text{a0}\,\text{a2} - 2\,\text{a1}\,\text{a2} + 5\,\text{a2}^2}{\text{rY}^2}\Big\},$$

$$\Big\{\text{p} \to \frac{5\,\text{a2}^2 + \text{rY}^2}{\text{a2}^2\,\text{rY}^2},\ \text{d} \to \text{a2}\,\text{rY}^4,\ \text{n0} \to -\frac{\text{a0}}{\text{rY}},\ \text{n1} \to \frac{\text{a1}}{\text{rY}},\ \text{N01} \to -\frac{(\text{a0} - \text{a1})\,\text{rY}}{\text{a2}^2},$$
$$\text{N02} \to \frac{(\text{a0} - \text{a2})\,\text{a2}}{\text{rY}^2},\ \text{N12} \to -\frac{(\text{a1} - \text{a2})\,\text{a2}}{\text{rY}^2},\ \text{pY1} \to 3,\ \text{cY1} \to -\frac{\text{rY}}{\text{a2}},\ \text{cYn0} \to 1,$$
$$\text{cYn1} \to -1,\ \text{Sn0} \to \frac{4\,\text{a0}^2\,\text{a2}^2 - 2\,\text{a0}\,\text{a1}\,\text{a2}^2 + \text{a1}^2\,\text{a2}^2 + \text{a0}^2\,\text{rY}^2 - 2\,\text{a0}\,\text{a2}\,\text{rY}^2 + \text{a2}^2\,\text{rY}^2}{\text{a2}^2\,\text{rY}^2},$$
$$\text{Sn1} \to \frac{\text{a0}^2\,\text{a2}^2 - 2\,\text{a0}\,\text{a1}\,\text{a2}^2 + 4\,\text{a1}^2\,\text{a2}^2 + \text{a1}^2\,\text{rY}^2 - 2\,\text{a1}\,\text{a2}\,\text{rY}^2 + \text{a2}^2\,\text{rY}^2}{\text{a2}^2\,\text{rY}^2},$$
$$\text{Sn2} \to \frac{\text{a0}^2 + \text{a1}^2 - 2\,\text{a0}\,\text{a2} - 2\,\text{a1}\,\text{a2} + 5\,\text{a2}^2}{\text{rY}^2}\Big\},$$

$$\Big\{\text{p} \to \frac{5\,\text{a2}^2 + \text{rY}^2}{\text{a2}^2\,\text{rY}^2},\ \text{d} \to \text{a2}\,\text{rY}^4,\ \text{n0} \to -\frac{\text{a0}}{\text{rY}},\ \text{n1} \to \frac{\text{a1}}{\text{rY}},\ \text{N01} \to \frac{(\text{a0} - \text{a1})\,\text{rY}}{\text{a2}^2},$$
$$\text{N02} \to -\frac{(\text{a0} - \text{a2})\,\text{a2}}{\text{rY}^2},\ \text{N12} \to \frac{(\text{a1} - \text{a2})\,\text{a2}}{\text{rY}^2},\ \text{pY1} \to 3,\ \text{cY1} \to -\frac{\text{rY}}{\text{a2}},\ \text{cYn0} \to 1,$$
$$\text{cYn1} \to -1,\ \text{Sn0} \to \frac{4\,\text{a0}^2\,\text{a2}^2 - 2\,\text{a0}\,\text{a1}\,\text{a2}^2 + \text{a1}^2\,\text{a2}^2 + \text{a0}^2\,\text{rY}^2 - 2\,\text{a0}\,\text{a2}\,\text{rY}^2 + \text{a2}^2\,\text{rY}^2}{\text{a2}^2\,\text{rY}^2},$$
$$\text{Sn1} \to \frac{\text{a0}^2\,\text{a2}^2 - 2\,\text{a0}\,\text{a1}\,\text{a2}^2 + 4\,\text{a1}^2\,\text{a2}^2 + \text{a1}^2\,\text{rY}^2 - 2\,\text{a1}\,\text{a2}\,\text{rY}^2 + \text{a2}^2\,\text{rY}^2}{\text{a2}^2\,\text{rY}^2},$$
$$\text{Sn2} \to \frac{\text{a0}^2 + \text{a1}^2 - 2\,\text{a0}\,\text{a2} - 2\,\text{a1}\,\text{a2} + 5\,\text{a2}^2}{\text{rY}^2}\Big\},$$

$$\Big\{\text{p} \to \frac{5\,\text{a2}^2 + \text{rY}^2}{\text{a2}^2\,\text{rY}^2},\ \text{d} \to \text{a2}\,\text{rY}^4,\ \text{n0} \to \frac{\text{a0}}{\text{rY}},\ \text{n1} \to -\frac{\text{a1}}{\text{rY}},\ \text{N01} \to -\frac{(\text{a0} - \text{a1})\,\text{rY}}{\text{a2}^2},$$
$$\text{N02} \to -\frac{(\text{a0} - \text{a2})\,\text{a2}}{\text{rY}^2},\ \text{N12} \to \frac{(\text{a1} - \text{a2})\,\text{a2}}{\text{rY}^2},\ \text{pY1} \to 3,\ \text{cY1} \to -\frac{\text{rY}}{\text{a2}},\ \text{cYn0} \to -1,$$
$$\text{cYn1} \to 1,\ \text{Sn0} \to \frac{4\,\text{a0}^2\,\text{a2}^2 - 2\,\text{a0}\,\text{a1}\,\text{a2}^2 + \text{a1}^2\,\text{a2}^2 + \text{a0}^2\,\text{rY}^2 - 2\,\text{a0}\,\text{a2}\,\text{rY}^2 + \text{a2}^2\,\text{rY}^2}{\text{a2}^2\,\text{rY}^2},$$
$$\text{Sn1} \to \frac{\text{a0}^2\,\text{a2}^2 - 2\,\text{a0}\,\text{a1}\,\text{a2}^2 + 4\,\text{a1}^2\,\text{a2}^2 + \text{a1}^2\,\text{rY}^2 - 2\,\text{a1}\,\text{a2}\,\text{rY}^2 + \text{a2}^2\,\text{rY}^2}{\text{a2}^2\,\text{rY}^2},$$
$$\text{Sn2} \to \frac{\text{a0}^2 + \text{a1}^2 - 2\,\text{a0}\,\text{a2} - 2\,\text{a1}\,\text{a2} + 5\,\text{a2}^2}{\text{rY}^2}\Big\},$$

$$\Big\{\text{p} \to \frac{5\,\text{a2}^2 + \text{rY}^2}{\text{a2}^2\,\text{rY}^2},\ \text{d} \to \text{a2}\,\text{rY}^4,\ \text{n0} \to \frac{\text{a0}}{\text{rY}},\ \text{n1} \to -\frac{\text{a1}}{\text{rY}},\ \text{N01} \to \frac{(\text{a0} - \text{a1})\,\text{rY}}{\text{a2}^2},$$
$$\text{N02} \to \frac{(\text{a0} - \text{a2})\,\text{a2}}{\text{rY}^2},\ \text{N12} \to -\frac{(\text{a1} - \text{a2})\,\text{a2}}{\text{rY}^2},\ \text{pY1} \to 3,\ \text{cY1} \to -\frac{\text{rY}}{\text{a2}},\ \text{cYn0} \to -1,$$
$$\text{cYn1} \to 1,\ \text{Sn0} \to \frac{4\,\text{a0}^2\,\text{a2}^2 - 2\,\text{a0}\,\text{a1}\,\text{a2}^2 + \text{a1}^2\,\text{a2}^2 + \text{a0}^2\,\text{rY}^2 - 2\,\text{a0}\,\text{a2}\,\text{rY}^2 + \text{a2}^2\,\text{rY}^2}{\text{a2}^2\,\text{rY}^2},$$
$$\text{Sn1} \to \frac{\text{a0}^2\,\text{a2}^2 - 2\,\text{a0}\,\text{a1}\,\text{a2}^2 + 4\,\text{a1}^2\,\text{a2}^2 + \text{a1}^2\,\text{rY}^2 - 2\,\text{a1}\,\text{a2}\,\text{rY}^2 + \text{a2}^2\,\text{rY}^2}{\text{a2}^2\,\text{rY}^2},$$

$$\text{Sn2} \to \frac{\text{a0}^2 + \text{a1}^2 - 2\ \text{a0}\ \text{a2} - 2\ \text{a1}\ \text{a2} + 5\ \text{a2}^2}{\text{rY}^2}\Big\},$$

$$\Big\{\text{p} \to \frac{5\ \text{a2}^2 + \text{rY}^2}{\text{a2}^2\ \text{rY}^2},\ \text{d} \to -\text{a2}\ \text{rY}^4,\ \text{n0} \to \frac{\text{a0}}{\text{rY}},\ \text{n1} \to \frac{\text{a1}}{\text{rY}},\ \text{N01} \to -\frac{(\text{a0} - \text{a1})\ \text{rY}}{\text{a2}^2},$$

$$\text{N02} \to \frac{(\text{a0} - \text{a2})\ \text{a2}}{\text{rY}^2},\ \text{N12} \to \frac{(\text{a1} - \text{a2})\ \text{a2}}{\text{rY}^2},\ \text{pY1} \to 3,\ \text{cY1} \to -\frac{\text{rY}}{\text{a2}},\ \text{cYn0} \to -1,$$

$$\text{cYn1} \to -1,\ \text{Sn0} \to \frac{4\ \text{a0}^2\ \text{a2}^2 - 2\ \text{a0}\ \text{a1}\ \text{a2}^2 + \text{a1}^2\ \text{a2}^2 + \text{a0}^2\ \text{rY}^2 - 2\ \text{a0}\ \text{a2}\ \text{rY}^2 + \text{a2}^2\ \text{rY}^2}{\text{a2}^2\ \text{rY}^2},$$

$$\text{Sn1} \to \frac{\text{a0}^2\ \text{a2}^2 - 2\ \text{a0}\ \text{a1}\ \text{a2}^2 + 4\ \text{a1}^2\ \text{a2}^2 + \text{a1}^2\ \text{rY}^2 - 2\ \text{a1}\ \text{a2}\ \text{rY}^2 + \text{a2}^2\ \text{rY}^2}{\text{a2}^2\ \text{rY}^2},$$

$$\text{Sn2} \to \frac{\text{a0}^2 + \text{a1}^2 - 2\ \text{a0}\ \text{a2} - 2\ \text{a1}\ \text{a2} + 5\ \text{a2}^2}{\text{rY}^2}\Big\},$$

$$\Big\{\text{p} \to \frac{5\ \text{a2}^2 + \text{rY}^2}{\text{a2}^2\ \text{rY}^2},\ \text{d} \to -\text{a2}\ \text{rY}^4,\ \text{n0} \to \frac{\text{a0}}{\text{rY}},\ \text{n1} \to \frac{\text{a1}}{\text{rY}},\ \text{N01} \to \frac{(\text{a0} - \text{a1})\ \text{rY}}{\text{a2}^2},$$

$$\text{N02} \to -\frac{(\text{a0} - \text{a2})\ \text{a2}}{\text{rY}^2},\ \text{N12} \to -\frac{(\text{a1} - \text{a2})\ \text{a2}}{\text{rY}^2},\ \text{pY1} \to 3,\ \text{cY1} \to -\frac{\text{rY}}{\text{a2}},\ \text{cYn0} \to -1,$$

$$\text{cYn1} \to -1,\ \text{Sn0} \to \frac{4\ \text{a0}^2\ \text{a2}^2 - 2\ \text{a0}\ \text{a1}\ \text{a2}^2 + \text{a1}^2\ \text{a2}^2 + \text{a0}^2\ \text{rY}^2 - 2\ \text{a0}\ \text{a2}\ \text{rY}^2 + \text{a2}^2\ \text{rY}^2}{\text{a2}^2\ \text{rY}^2},$$

$$\text{Sn1} \to \frac{\text{a0}^2\ \text{a2}^2 - 2\ \text{a0}\ \text{a1}\ \text{a2}^2 + 4\ \text{a1}^2\ \text{a2}^2 + \text{a1}^2\ \text{rY}^2 - 2\ \text{a1}\ \text{a2}\ \text{rY}^2 + \text{a2}^2\ \text{rY}^2}{\text{a2}^2\ \text{rY}^2},$$

$$\text{Sn2} \to \frac{\text{a0}^2 + \text{a1}^2 - 2\ \text{a0}\ \text{a2} - 2\ \text{a1}\ \text{a2} + 5\ \text{a2}^2}{\text{rY}^2}\Big\}\Big\}$$

Computations for Lemma 6.20. Case B.4 where there exists N^6 s.t. Fix(N) = $Y^4 \cup P_1$ and M^6 Fix(M) = $Y^4 \cup P_2$.

```
(******************************
 NOTATIONS
 rY xY denotes the restriction of x to Y
 ai denotes the Hopf weight at Pi
 Nij denotes the product of the normal weights shared between Pi and Pj
 Sni is the sum of the square of the normal weights at Pi
 cYn1 xY denotes the first Chern class of the normal bundle of Y in the
 direction of N
 cYn2 xY denotes the first Chern class of the normal bundle of Y in the
 direction of M
 cYj xY^j denotes the j-th Chern class of the normal bundle of Y in the
 remaining directions
 μx5YB4 denotes the local datum of x^5 at Y
 μx5PiB4 denotes the local datum of x^5 at Pi
 μs1x3YB4 denotes the local datum of p1(X)x^3 at Y
 μs1x3PiB4 denotes the local datum of p1(X)x^3 at Pi

 We use the variable t to extract the coefficient of order 2 of xY. We
 then evaluate xY^2 to 1.
 The orientation of the point P0 is 1 while those from P1 and P2 are -
1. This explains the signs at their local datum
 Once evaluated, we extract all the coefficients in l into a list.
 ******************************)
```

```
μx5YB4 =
 Factor[CoefficientList[
   Coefficient[Expand[(rY xY t + l z)^5] Series[(cY1 xY t + z)^-1, {xY, 0, 2}]
     Series[(cYn1 xY t + n1 z)^-1, {xY, 0, 2}] Series[(cYn2 xY t + n2 z)^-1, {xY, 0, 2}],
    t, 2], l]] /. xY^2 → 1
```

$$\left\{0,\ 0,\ 0,\ \frac{10\ rY^2}{n1\ n2},\ -\frac{5\ (cYn2\ n1 + cYn1\ n2 + cY1\ n1\ n2)\ rY}{n1^2\ n2^2},\ \frac{cYn2^2\ n1^2 + cYn1\ cYn2\ n1\ n2 + cY1\ cYn2\ n1^2\ n2 + cYn1^2\ n2^2 + cY1\ cYn1\ n1\ n2^2 + cY1^2\ n1^2\ n2^2}{n1^3\ n2^3}\right\}$$

```
μx5P0B4 = CoefficientList[Expand[((a0 + l) z)^5] N01^-1 N02^-1 z^-5, l]
```

$$\left\{\frac{a0^5}{N01\ N02},\ \frac{5\ a0^4}{N01\ N02},\ \frac{10\ a0^3}{N01\ N02},\ \frac{10\ a0^2}{N01\ N02},\ \frac{5\ a0}{N01\ N02},\ \frac{1}{N01\ N02}\right\}$$

```
μx5P1B4 =
 CoefficientList[-Expand[((a1 + l) z)^5] (n1 z)^-1 (n1 z)^-1 (n1 z)^-1 (N01 z)^-1 (N12 z)^-1, l]
```

$$\left\{-\frac{a1^5}{N01\ n1^3\ N12},\ -\frac{5\ a1^4}{N01\ n1^3\ N12},\ -\frac{10\ a1^3}{N01\ n1^3\ N12},\ -\frac{10\ a1^2}{N01\ n1^3\ N12},\ -\frac{5\ a1}{N01\ n1^3\ N12},\ -\frac{1}{N01\ n1^3\ N12}\right\}$$

```
μx5P2B4 =
 CoefficientList[-Expand[((a2 + l) z)^5] (n2 z)^-1 (n2 z)^-1 (n2 z)^-1 (N02 z)^-1 (N12 z)^-1, l]
```

$$\left\{-\frac{a2^5}{N02\ N12\ n2^3},\ -\frac{5\ a2^4}{N02\ N12\ n2^3},\ -\frac{10\ a2^3}{N02\ N12\ n2^3},\ -\frac{10\ a2^2}{N02\ N12\ n2^3},\ -\frac{5\ a2}{N02\ N12\ n2^3},\ -\frac{1}{N02\ N12\ n2^3}\right\}$$

```
μs1x3YB4 =
 Factor[CoefficientList[
   Coefficient[(pY1 t^2 + (cY1 xY t + z)^2 + (cYn1 xY t + n1 z)^2 + (cYn2 xY t + n2 z)^2)
     Expand[(rY xY t + l z)^3] Series[(cY1 xY t + z)^-1, {xY, 0, 2}]
     Series[(cYn1 xY t + n1 z)^-1, {xY, 0, 2}] Series[(cYn2 xY t + n2 z)^-1, {xY, 0, 2}],
    t, 2], l]] /. xY^2 → 1
```

$$\left\{0,\ \frac{3\ \left(1 + n1^2 + n2^2\right)\ rY^2}{n1\ n2},\right.$$
$$-\frac{1}{n1^2\ n2^2}3\ \left(cYn2\ n1 + cYn2\ n1^3 + cYn1\ n2 - cY1\ n1\ n2 - cYn1\ n1^2\ n2 + cY1\ n1^3\ n2 - cYn2\ n1\ n2^2 + cYn1\ n2^3 + cY1\ n1\ n2^3\right)\ rY,$$
$$\frac{1}{n1^3\ n2^3}\ \left(cYn2^2\ n1^2 + cYn2^2\ n1^4 + cYn1\ cYn2\ n1\ n2 - cY1\ cYn2\ n1^2\ n2 - cYn1\ cYn2\ n1^3\ n2 + cY1\ cYn2\ n1^4\ n2 + cYn1^2\ n2^2 - cY1\ cYn1\ n1\ n2^2 - cY1\ cYn1\ n1^3\ n2^2 + cY1^2\ n1^4\ n2^2 - cYn1\ cYn2\ n1\ n2^3 - cY1\ cYn2\ n1^2\ n2^3 + cYn1^2\ n2^4 + cY1\ cYn1\ n1\ n2^4 + cY1^2\ n1^2\ n2^4 + n1^2\ n2^2\ pY1\right)\left.\right\}$$

```
μs1x3P0B4 = CoefficientList[Sn0 z^2 Expand[((a0 + l) z)^3] N01^-1 N02^-1 z^-5, l]
```

$$\left\{\frac{a0^3\ Sn0}{N01\ N02},\ \frac{3\ a0^2\ Sn0}{N01\ N02},\ \frac{3\ a0\ Sn0}{N01\ N02},\ \frac{Sn0}{N01\ N02}\right\}$$

```
μs1x3P1B4 =
 CoefficientList[-Sn1 z^2 Expand[((a1 + l) z)^3] (n1 z)^-1 (n1 z)^-1 (n1 z)^-1 (N01 z)^-1 (N12 z)^-1,
  l]
```

$$\left\{-\frac{a1^3\ Sn1}{N01\ n1^3\ N12},\ -\frac{3\ a1^2\ Sn1}{N01\ n1^3\ N12},\ -\frac{3\ a1\ Sn1}{N01\ n1^3\ N12},\ -\frac{Sn1}{N01\ n1^3\ N12}\right\}$$

μs1x3P2B4 =
CoefficientList[-Sn2 z^2 Expand[((a2 + 1) z)^3] (n2 z)^-1 (n2 z)^-1 (n2 z)^-1 (N02 z)^-1 (N12 z)^-1, 1]

$$\left\{-\frac{a2^3\,Sn2}{N02\,N12\,n2^3},\;-\frac{3\,a2^2\,Sn2}{N02\,N12\,n2^3},\;-\frac{3\,a2\,Sn2}{N02\,N12\,n2^3},\;-\frac{Sn2}{N02\,N12\,n2^3}\right\}$$

μx5YB4 + μx5P0B4 + μx5P1B4 + μx5P2B4

$$\left\{\frac{a0^5}{N01\,N02}-\frac{a1^5}{N01\,n1^3\,N12}-\frac{a2^5}{N02\,N12\,n2^3},\;\frac{5\,a0^4}{N01\,N02}-\frac{5\,a1^4}{N01\,n1^3\,N12}-\frac{5\,a2^4}{N02\,N12\,n2^3},\right.$$
$$\frac{10\,a0^3}{N01\,N02}-\frac{10\,a1^3}{N01\,n1^3\,N12}-\frac{10\,a2^3}{N02\,N12\,n2^3},\;\frac{10\,a0^2}{N01\,N02}-\frac{10\,a1^2}{N01\,n1^3\,N12}-\frac{10\,a2^2}{N02\,N12\,n2^3}+\frac{10\,rY^2}{n1\,n2},$$
$$\frac{5\,a0}{N01\,N02}-\frac{5\,a1}{N01\,n1^3\,N12}-\frac{5\,a2}{N02\,N12\,n2^3}-\frac{5\,(cYn2\,n1+cYn1\,n2+cY1\,n1\,n2)\,rY}{n1^2\,n2^2},$$
$$\frac{1}{N01\,N02}-\frac{1}{N01\,n1^3\,N12}-\frac{1}{N02\,N12\,n2^3}+$$
$$\left.\frac{cYn2^2\,n1^2+cYn1\,cYn2\,n1\,n2+cY1\,cYn2\,n1^2\,n2+cYn1^2\,n2^2+cY1\,cYn1\,n1\,n2^2+cY1^2\,n1^2\,n2^2}{n1^3\,n2^3}\right\}$$

μs1x3YB4 + μs1x3P0B4 + μs1x3P1B4 + μs1x3P2B4

$$\left\{\frac{a0^3\,Sn0}{N01\,N02}-\frac{a1^3\,Sn1}{N01\,n1^3\,N12}-\frac{a2^3\,Sn2}{N02\,N12\,n2^3},\right.$$
$$\frac{3\,(1+n1^2+n2^2)\,rY^2}{n1\,n2}+\frac{3\,a0^2\,Sn0}{N01\,N02}-\frac{3\,a1^2\,Sn1}{N01\,n1^3\,N12}-\frac{3\,a2^2\,Sn2}{N02\,N12\,n2^3},$$
$$-\frac{1}{n1^2\,n2^2}3\,(cYn2\,n1+cYn2\,n1^3+cYn1\,n2-cY1\,n1\,n2-cYn1\,n1^2\,n2+cY1\,n1^3\,n2-$$
$$cYn2\,n1\,n2^2+cYn1\,n2^3+cY1\,n1\,n2^3)\,rY+\frac{3\,a0\,Sn0}{N01\,N02}-\frac{3\,a1\,Sn1}{N01\,n1^3\,N12}-\frac{3\,a2\,Sn2}{N02\,N12\,n2^3},$$
$$\frac{1}{n1^3\,n2^3}\,(cYn2^2\,n1^2+cYn2^2\,n1^4+cYn1\,cYn2\,n1\,n2-cY1\,cYn2\,n1^2\,n2-cYn1\,cYn2\,n1^3\,n2+$$
$$cY1\,cYn2\,n1^4\,n2+cYn1^2\,n2^2-cY1\,cYn1\,n1\,n2^2-cY1\,cYn1\,n1^3\,n2^2+cY1^2\,n1^4\,n2^2-$$
$$cYn1\,cYn2\,n1\,n2^3-cY1\,cYn2\,n1^2\,n2^3+cYn1^2\,n2^4+cY1\,cYn1\,n1\,n2^4+$$
$$\left.cY1^2\,n1^2\,n2^4+n1^2\,n2^2\,pY1)+\frac{Sn0}{N01\,N02}-\frac{Sn1}{N01\,n1^3\,N12}-\frac{Sn2}{N02\,N12\,n2^3}\right\}$$

Factor[Solve[{

$$d == \frac{a0^5}{N01\,N02} - \frac{a1^5}{N01\,n1^3\,N12} - \frac{a2^5}{N02\,N12\,n2^3},$$

$$0 == \frac{5\,a0^4}{N01\,N02} - \frac{5\,a1^4}{N01\,n1^3\,N12} - \frac{5\,a2^4}{N02\,N12\,n2^3},$$

$$0 == \frac{10\,a0^3}{N01\,N02} - \frac{10\,a1^3}{N01\,n1^3\,N12} - \frac{10\,a2^3}{N02\,N12\,n2^3},$$

$$0 == \frac{10\,a0^2}{N01\,N02} - \frac{10\,a1^2}{N01\,n1^3\,N12} - \frac{10\,a2^2}{N02\,N12\,n2^3} + \frac{10\,rY^2}{n1\,n2},$$

$$0 == \frac{5\,a0}{N01\,N02} - \frac{5\,a1}{N01\,n1^3\,N12} - \frac{5\,a2}{N02\,N12\,n2^3} - \frac{5\,(cYn2\,n1 + cYn1\,n2 + cY1\,n1\,n2)\,rY}{n1^2\,n2^2},$$

$$0 == \frac{1}{N01\,N02} - \frac{1}{N01\,n1^3\,N12} - \frac{1}{N02\,N12\,n2^3} + \frac{cYn2^2\,n1^2 + cYn1\,cYn2\,n1\,n2 + cY1\,cYn2\,n1^2\,n2 + cYn1^2\,n2^2 + cY1\,cYn1\,n1\,n2^2 + cY1^2\,n1^2\,n2^2}{n1^3\,n2^3},$$

$$p\,d == \frac{a0^3\,Sn0}{N01\,N02} - \frac{a1^3\,Sn1}{N01\,n1^3\,N12} - \frac{a2^3\,Sn2}{N02\,N12\,n2^3},$$

$$0 == \frac{3\,(1 + n1^2 + n2^2)\,rY^2}{n1\,n2} + \frac{3\,a0^2\,Sn0}{N01\,N02} - \frac{3\,a1^2\,Sn1}{N01\,n1^3\,N12} - \frac{3\,a2^2\,Sn2}{N02\,N12\,n2^3},$$

$$0 == -\frac{1}{n1^2\,n2^2}\,3\,(cYn2\,n1 + cYn2\,n1^3 + cYn1\,n2 - cY1\,n1\,n2 - cYn1\,n1^2\,n2 + cY1\,n1^3\,n2 - cYn2\,n1\,n2^2 + cYn1\,n2^3 + cY1\,n1\,n2^3)\,rY + \frac{3\,a0\,Sn0}{N01\,N02} - \frac{3\,a1\,Sn1}{N01\,n1^3\,N12} - \frac{3\,a2\,Sn2}{N02\,N12\,n2^3},$$

$$0 == \frac{1}{n1^3\,n2^3}\,(cYn2^2\,n1^2 + cYn2^2\,n1^4 + cYn1\,cYn2\,n1\,n2 - cY1\,cYn2\,n1^2\,n2 - cYn1\,cYn2\,n1^3\,n2 + cY1\,cYn2\,n1^4\,n2 + cYn1^2\,n2^2 - cY1\,cYn1\,n1\,n2^2 - cY1\,cYn1\,n1^3\,n2^2 + cY1^2\,n1^4\,n2^2 - cYn1\,cYn2\,n1\,n2^3 - cY1\,cYn2\,n1^2\,n2^3 + cYn1^2\,n2^4 + cY1\,cYn1\,n1\,n2^4 + cY1^2\,n1^2\,n2^4 + n1^2\,n2^2\,pY1) + \frac{Sn0}{N01\,N02} - \frac{Sn1}{N01\,n1^3\,N12} - \frac{Sn2}{N02\,N12\,n2^3},$$

pY1 == 3, (******************************
The following equations are given by Lemma 6.13
******************************)

$$0 == -\frac{3\,a1^2}{n1^3} + \frac{3\,rY^2}{n1},\quad 0 == -\frac{3\,a1}{n1^3} - \frac{3\,cYn1\,rY}{n1^2},\quad 0 == -\frac{1}{n1^3} + \frac{cYn1^2}{n1^3},$$

$$0 == -\frac{3\,a2^2}{n2^3} + \frac{3\,rY^2}{n2},\quad 0 == -\frac{3\,a2}{n2^3} - \frac{3\,cYn2\,rY}{n2^2},\quad 0 == -\frac{1}{n2^3} + \frac{cYn2^2}{n2^3}$$

}, {p, d, n1, n2, N01, N02, N12, cY1, cYn1, cYn2, Sn0, Sn1, Sn2, pY1}]]

$$\Big\{\Big\{p \to \frac{5\,a0^2 + rY^2}{a0^2\,rY^2},\ d \to -a0\,rY^4,\ n1 \to -\frac{a1}{rY},\ n2 \to -\frac{a2}{rY},$$

$$N01 \to -\frac{a0\,(a0 - a1)}{rY^2},\ N02 \to \frac{a0\,(a0 - a2)}{rY^2},\ N12 \to -\frac{(a1 - a2)\,rY}{a0^2},$$

$$cY1 \to -\frac{rY}{a0},\ cYn1 \to 1,\ cYn2 \to 1,\ Sn0 \to \frac{5\,a0^2 - 2\,a0\,a1 + a1^2 - 2\,a0\,a2 + a2^2}{rY^2},$$

$$Sn1 \to \frac{4\,a0^2\,a1^2 - 2\,a0^2\,a1\,a2 + a0^2\,a2^2 + a0^2\,rY^2 - 2\,a0\,a1\,rY^2 + a1^2\,rY^2}{a0^2\,rY^2},$$

$$Sn2 \to \frac{a0^2\,a1^2 - 2\,a0^2\,a1\,a2 + 4\,a0^2\,a2^2 + a0^2\,rY^2 - 2\,a0\,a2\,rY^2 + a2^2\,rY^2}{a0^2\,rY^2},\ pY1 \to 3\Big\},$$

$$\left\{p \to \frac{5\,a0^2 + rY^2}{a0^2\,rY^2},\ d \to -a0\,rY^4,\ n1 \to -\frac{a1}{rY},\ n2 \to -\frac{a2}{rY},\ N01 \to \frac{a0\,(a0 - a1)}{rY^2},\ N02 \to -\frac{a0\,(a0 - a2)}{rY^2},\right.$$
$$N12 \to \frac{(a1 - a2)\,rY}{a0^2},\ cY1 \to -\frac{rY}{a0},\ cYn1 \to 1,\ cYn2 \to 1,\ Sn0 \to \frac{5\,a0^2 - 2\,a0\,a1 + a1^2 - 2\,a0\,a2 + a2^2}{rY^2},$$
$$Sn1 \to \frac{4\,a0^2\,a1^2 - 2\,a0^2\,a1\,a2 + a0^2\,a2^2 + a0^2\,rY^2 - 2\,a0\,a1\,rY^2 + a1^2\,rY^2}{a0^2\,rY^2},$$
$$\left.Sn2 \to \frac{a0^2\,a1^2 - 2\,a0^2\,a1\,a2 + 4\,a0^2\,a2^2 + a0^2\,rY^2 - 2\,a0\,a2\,rY^2 + a2^2\,rY^2}{a0^2\,rY^2},\ pY1 \to 3\right\},$$

$$\left\{p \to \frac{5\,a0^2 + rY^2}{a0^2\,rY^2},\ d \to a0\,rY^4,\ n1 \to -\frac{a1}{rY},\ n2 \to \frac{a2}{rY},\ N01 \to \frac{a0\,(a0 - a1)}{rY^2},\right.$$
$$N02 \to \frac{a0\,(a0 - a2)}{rY^2},\ N12 \to -\frac{(a1 - a2)\,rY}{a0^2},\ cY1 \to -\frac{rY}{a0},$$
$$cYn1 \to 1,\ cYn2 \to -1,\ Sn0 \to \frac{5\,a0^2 - 2\,a0\,a1 + a1^2 - 2\,a0\,a2 + a2^2}{rY^2},$$
$$Sn1 \to \frac{4\,a0^2\,a1^2 - 2\,a0^2\,a1\,a2 + a0^2\,a2^2 + a0^2\,rY^2 - 2\,a0\,a1\,rY^2 + a1^2\,rY^2}{a0^2\,rY^2},$$
$$\left.Sn2 \to \frac{a0^2\,a1^2 - 2\,a0^2\,a1\,a2 + 4\,a0^2\,a2^2 + a0^2\,rY^2 - 2\,a0\,a2\,rY^2 + a2^2\,rY^2}{a0^2\,rY^2},\ pY1 \to 3\right\},$$

$$\left\{p \to \frac{5\,a0^2 + rY^2}{a0^2\,rY^2},\ d \to a0\,rY^4,\ n1 \to -\frac{a1}{rY},\ n2 \to \frac{a2}{rY},\ N01 \to -\frac{a0\,(a0 - a1)}{rY^2},\right.$$
$$N02 \to -\frac{a0\,(a0 - a2)}{rY^2},\ N12 \to \frac{(a1 - a2)\,rY}{a0^2},\ cY1 \to -\frac{rY}{a0},$$
$$cYn1 \to 1,\ cYn2 \to -1,\ Sn0 \to \frac{5\,a0^2 - 2\,a0\,a1 + a1^2 - 2\,a0\,a2 + a2^2}{rY^2},$$
$$Sn1 \to \frac{4\,a0^2\,a1^2 - 2\,a0^2\,a1\,a2 + a0^2\,a2^2 + a0^2\,rY^2 - 2\,a0\,a1\,rY^2 + a1^2\,rY^2}{a0^2\,rY^2},$$
$$\left.Sn2 \to \frac{a0^2\,a1^2 - 2\,a0^2\,a1\,a2 + 4\,a0^2\,a2^2 + a0^2\,rY^2 - 2\,a0\,a2\,rY^2 + a2^2\,rY^2}{a0^2\,rY^2},\ pY1 \to 3\right\},$$

$$\left\{p \to \frac{5\,a0^2 + rY^2}{a0^2\,rY^2},\ d \to a0\,rY^4,\ n1 \to \frac{a1}{rY},\ n2 \to -\frac{a2}{rY},\ N01 \to -\frac{a0\,(a0 - a1)}{rY^2},\right.$$
$$N02 \to -\frac{a0\,(a0 - a2)}{rY^2},\ N12 \to -\frac{(a1 - a2)\,rY}{a0^2},\ cY1 \to -\frac{rY}{a0},$$
$$cYn1 \to -1,\ cYn2 \to 1,\ Sn0 \to \frac{5\,a0^2 - 2\,a0\,a1 + a1^2 - 2\,a0\,a2 + a2^2}{rY^2},$$
$$Sn1 \to \frac{4\,a0^2\,a1^2 - 2\,a0^2\,a1\,a2 + a0^2\,a2^2 + a0^2\,rY^2 - 2\,a0\,a1\,rY^2 + a1^2\,rY^2}{a0^2\,rY^2},$$
$$\left.Sn2 \to \frac{a0^2\,a1^2 - 2\,a0^2\,a1\,a2 + 4\,a0^2\,a2^2 + a0^2\,rY^2 - 2\,a0\,a2\,rY^2 + a2^2\,rY^2}{a0^2\,rY^2},\ pY1 \to 3\right\},$$

$$\left\{p \to \frac{5\,a0^2 + rY^2}{a0^2\,rY^2},\ d \to a0\,rY^4,\ n1 \to \frac{a1}{rY},\ n2 \to -\frac{a2}{rY},\ N01 \to \frac{a0\,(a0 - a1)}{rY^2},\right.$$
$$N02 \to \frac{a0\,(a0 - a2)}{rY^2},\ N12 \to \frac{(a1 - a2)\,rY}{a0^2},\ cY1 \to -\frac{rY}{a0},$$
$$cYn1 \to -1,\ cYn2 \to 1,\ Sn0 \to \frac{5\,a0^2 - 2\,a0\,a1 + a1^2 - 2\,a0\,a2 + a2^2}{rY^2},$$
$$Sn1 \to \frac{4\,a0^2\,a1^2 - 2\,a0^2\,a1\,a2 + a0^2\,a2^2 + a0^2\,rY^2 - 2\,a0\,a1\,rY^2 + a1^2\,rY^2}{a0^2\,rY^2},$$
$$\left.Sn2 \to \frac{a0^2\,a1^2 - 2\,a0^2\,a1\,a2 + 4\,a0^2\,a2^2 + a0^2\,rY^2 - 2\,a0\,a2\,rY^2 + a2^2\,rY^2}{a0^2\,rY^2},\ pY1 \to 3\right\},$$

$$\left\{p \to \frac{5\,a0^2 + rY^2}{a0^2\,rY^2},\ d \to -a0\,rY^4,\ n1 \to \frac{a1}{rY},\ n2 \to \frac{a2}{rY},\ N01 \to \frac{a0\,(a0 - a1)}{rY^2},\right.$$
$$N02 \to -\frac{a0\,(a0 - a2)}{rY^2},\ N12 \to -\frac{(a1 - a2)\,rY}{a0^2},\ cY1 \to -\frac{rY}{a0},$$

$$\text{cYn1} \to -1,\ \text{cYn2} \to -1,\ \text{Sn0} \to \frac{5\ \text{a0}^2 - 2\ \text{a0 a1} + \text{a1}^2 - 2\ \text{a0 a2} + \text{a2}^2}{\text{rY}^2},$$

$$\text{Sn1} \to \frac{4\ \text{a0}^2\ \text{a1}^2 - 2\ \text{a0}^2\ \text{a1 a2} + \text{a0}^2\ \text{a2}^2 + \text{a0}^2\ \text{rY}^2 - 2\ \text{a0 a1 rY}^2 + \text{a1}^2\ \text{rY}^2}{\text{a0}^2\ \text{rY}^2},$$

$$\text{Sn2} \to \frac{\text{a0}^2\ \text{a1}^2 - 2\ \text{a0}^2\ \text{a1 a2} + 4\ \text{a0}^2\ \text{a2}^2 + \text{a0}^2\ \text{rY}^2 - 2\ \text{a0 a2 rY}^2 + \text{a2}^2\ \text{rY}^2}{\text{a0}^2\ \text{rY}^2},\ \text{pY1} \to 3\Big\},$$

$$\Big\{\text{p} \to \frac{5\ \text{a0}^2 + \text{rY}^2}{\text{a0}^2\ \text{rY}^2},\ \text{d} \to -\text{a0 rY}^4,\ \text{n1} \to \frac{\text{a1}}{\text{rY}},\ \text{n2} \to \frac{\text{a2}}{\text{rY}},\ \text{N01} \to -\frac{\text{a0 (a0} - \text{a1)}}{\text{rY}^2},$$

$$\text{N02} \to \frac{\text{a0 (a0} - \text{a2)}}{\text{rY}^2},\ \text{N12} \to \frac{\text{(a1} - \text{a2) rY}}{\text{a0}^2},\ \text{cY1} \to -\frac{\text{rY}}{\text{a0}},\ \text{cYn1} \to -1,$$

$$\text{cYn2} \to -1,\ \text{Sn0} \to \frac{5\ \text{a0}^2 - 2\ \text{a0 a1} + \text{a1}^2 - 2\ \text{a0 a2} + \text{a2}^2}{\text{rY}^2},$$

$$\text{Sn1} \to \frac{4\ \text{a0}^2\ \text{a1}^2 - 2\ \text{a0}^2\ \text{a1 a2} + \text{a0}^2\ \text{a2}^2 + \text{a0}^2\ \text{rY}^2 - 2\ \text{a0 a1 rY}^2 + \text{a1}^2\ \text{rY}^2}{\text{a0}^2\ \text{rY}^2},$$

$$\text{Sn2} \to \frac{\text{a0}^2\ \text{a1}^2 - 2\ \text{a0}^2\ \text{a1 a2} + 4\ \text{a0}^2\ \text{a2}^2 + \text{a0}^2\ \text{rY}^2 - 2\ \text{a0 a2 rY}^2 + \text{a2}^2\ \text{rY}^2}{\text{a0}^2\ \text{rY}^2},\ \text{pY1} \to 3\Big\}\Big\}$$

Computations for Lemma 6.20. Case B.5 where there exists N_i^6 s.t. Fix(N_i) = $Y^4 \cup P_i$ for every P_i.

```
(******************************
  NOTATIONS
  rY xY denotes the restriction of x to Y
  ai denotes the Hopf weight at Pi
  Nij denotes the product of the normal weights shared between Pi and Pj
  Sni is the sum of the square of the normal weights at Pi
  cYni xY denotes the first Chern class of the normal bundle of Y in the
  direction of Ni
  µx5YB5 denotes the local datum of x^5 at Y
  µx5PiB5 denotes the local datum of x^5 at Pi
  µs1x3YB5 denotes the local datum of p1(X)x^3 at Y
  µs1x3PiB5 denotes the local datum of p1(X)x^3 at Pi

  We use the variable t to extract the coefficient of order 2 of xY. We
  then evaluate xY^2 to 1.
  The orientation of the point P0 is 1 while those from P1 and P2 are -
 1. This explains the signs at their local datum
  Once evaluated, we extract all the coefficients in l into a list.
  ******************************)

µx5YB5 =
 Factor[CoefficientList[
    Coefficient[Expand[(rY xY t + l z)^5] Series[(cYn0 xY t + n0 z)^-1, {xY, 0, 2}]
      Series[(cYn1 xY t + n1 z)^-1, {xY, 0, 2}] Series[(cYn2 xY t + n2 z)^-1, {xY, 0, 2}],
     t, 2], l]] /. xY^2 → 1
```

$$\Big\{0,\ 0,\ 0,\ \frac{10\ \text{rY}^2}{\text{n0 n1 n2}},\ -\frac{5\ \text{(cYn2 n0 n1} + \text{cYn1 n0 n2} + \text{cYn0 n1 n2) rY}}{\text{n0}^2\ \text{n1}^2\ \text{n2}^2},$$

$$\frac{1}{\text{n0}^3\ \text{n1}^3\ \text{n2}^3}\big(\text{cYn2}^2\ \text{n0}^2\ \text{n1}^2 + \text{cYn1 cYn2 n0}^2\ \text{n1 n2} + \text{cYn0 cYn2 n0 n1}^2\ \text{n2} + \text{cYn1}^2\ \text{n0}^2\ \text{n2}^2 + \text{cYn0 cYn1 n0 n1 n2}^2 + \text{cYn0}^2\ \text{n1}^2\ \text{n2}^2\big)\Big\}$$

```
μx5P0B5 = CoefficientList[Expand[((a0 + l) z)^5] (n0 z)^-1 (n0 z)^-1 (n0 z)^-1 (N01 z)^-1 (N02 z)^-1,
  l]
```

$$\left\{\frac{a0^5}{n0^3\ N01\ N02}, \frac{5\ a0^4}{n0^3\ N01\ N02}, \frac{10\ a0^3}{n0^3\ N01\ N02}, \frac{10\ a0^2}{n0^3\ N01\ N02}, \frac{5\ a0}{n0^3\ N01\ N02}, \frac{1}{n0^3\ N01\ N02}\right\}$$

```
μx5P1B5 =
 CoefficientList[-Expand[((a1 + l) z)^5] (n1 z)^-1 (n1 z)^-1 (n1 z)^-1 (N01 z)^-1 (N12 z)^-1, l]
```

$$\left\{-\frac{a1^5}{N01\ n1^3\ N12}, -\frac{5\ a1^4}{N01\ n1^3\ N12}, -\frac{10\ a1^3}{N01\ n1^3\ N12}, -\frac{10\ a1^2}{N01\ n1^3\ N12}, -\frac{5\ a1}{N01\ n1^3\ N12}, -\frac{1}{N01\ n1^3\ N12}\right\}$$

```
μx5P2B5 =
 CoefficientList[-Expand[((a2 + l) z)^5] (n2 z)^-1 (n2 z)^-1 (n2 z)^-1 (N02 z)^-1 (N12 z)^-1, l]
```

$$\left\{-\frac{a2^5}{N02\ N12\ n2^3}, -\frac{5\ a2^4}{N02\ N12\ n2^3}, -\frac{10\ a2^3}{N02\ N12\ n2^3}, -\frac{10\ a2^2}{N02\ N12\ n2^3}, -\frac{5\ a2}{N02\ N12\ n2^3}, -\frac{1}{N02\ N12\ n2^3}\right\}$$

```
μs1x3YB5 =
 Factor[CoefficientList[
    Coefficient[(pY1 t^2 + (cYn0 xY t + n0 z)^2 + (cYn1 xY t + n1 z)^2 + (cYn2 xY t + n2 z)^2)
      Expand[(rY xY t + l z)^3] Series[(cYn0 xY t + n0 z)^-1, {xY, 0, 2}]
      Series[(cYn1 xY t + n1 z)^-1, {xY, 0, 2}] Series[(cYn2 xY t + n2 z)^-1, {xY, 0, 2}],
     t, 2], l]] /. xY^2 → 1
```

$$\left\{0, \frac{3\left(n0^2+n1^2+n2^2\right) rY^2}{n0\ n1\ n2},\right.$$
$$-\frac{1}{n0^2\ n1^2\ n2^2}3\left(cYn2\ n0^3\ n1+cYn2\ n0\ n1^3+cYn1\ n0^3\ n2-cYn0\ n0^2\ n1\ n2-\right.$$
$$\left.cYn1\ n0\ n1^2\ n2+cYn0\ n1^3\ n2-cYn2\ n0\ n1\ n2^2+cYn1\ n0\ n2^3+cYn0\ n1\ n2^3\right) rY,$$
$$\frac{1}{n0^3\ n1^3\ n2^3}\left(cYn2^2\ n0^4\ n1^2+cYn2^2\ n0^2\ n1^4+cYn1\ cYn2\ n0^4\ n1\ n2-cYn0\ cYn2\ n0^3\ n1^2\ n2-\right.$$
$$cYn1\ cYn2\ n0^2\ n1^3\ n2+cYn0\ cYn2\ n0\ n1^4\ n2+cYn1^2\ n0^4\ n2^2-cYn0\ cYn1\ n0^3\ n1\ n2^2-$$
$$cYn0\ cYn1\ n0\ n1^3\ n2^2+cYn0^2\ n1^4\ n2^2-cYn1\ cYn2\ n0^2\ n1\ n2^3-cYn0\ cYn2\ n0\ n1^2\ n2^3+$$
$$\left.\left.cYn1^2\ n0^2\ n2^4+cYn0\ cYn1\ n0\ n1\ n2^4+cYn0^2\ n1^2\ n2^4+n0^2\ n1^2\ n2^2\ pY1\right)\right\}$$

```
μs1x3P0B5 =
 CoefficientList[Sn0 z^2 Expand[((a0 + l) z)^3] (n0 z)^-1 (n0 z)^-1 (n0 z)^-1 (N01 z)^-1 (N02 z)^-1,
  l]
```

$$\left\{\frac{a0^3\ Sn0}{n0^3\ N01\ N02}, \frac{3\ a0^2\ Sn0}{n0^3\ N01\ N02}, \frac{3\ a0\ Sn0}{n0^3\ N01\ N02}, \frac{Sn0}{n0^3\ N01\ N02}\right\}$$

```
μs1x3P1B5 =
 CoefficientList[-Sn1 z^2 Expand[((a1 + l) z)^3] (n1 z)^-1 (n1 z)^-1 (n1 z)^-1 (N01 z)^-1 (N12 z)^-1,
  l]
```

$$\left\{-\frac{a1^3\ Sn1}{N01\ n1^3\ N12}, -\frac{3\ a1^2\ Sn1}{N01\ n1^3\ N12}, -\frac{3\ a1\ Sn1}{N01\ n1^3\ N12}, -\frac{Sn1}{N01\ n1^3\ N12}\right\}$$

```
μs1x3P2B5 =
 CoefficientList[-Sn2 z ^2 Expand[((a2 + l) z)^3] (n2 z)^-1 (n2 z)^-1 (n2 z)^-1 (N02 z)^-1 (N12 z)^-1,
  l]
```

$$\left\{-\frac{a2^3\ Sn2}{N02\ N12\ n2^3}, -\frac{3\ a2^2\ Sn2}{N02\ N12\ n2^3}, -\frac{3\ a2\ Sn2}{N02\ N12\ n2^3}, -\frac{Sn2}{N02\ N12\ n2^3}\right\}$$

μx5YB5 + μx5P0B5 + μx5P1B5 + μx5P2B5

$$\left\{\frac{a0^5}{n0^3\,N01\,N02}-\frac{a1^5}{N01\,n1^3\,N12}-\frac{a2^5}{N02\,N12\,n2^3},\ \frac{5\,a0^4}{n0^3\,N01\,N02}-\frac{5\,a1^4}{N01\,n1^3\,N12}-\frac{5\,a2^4}{N02\,N12\,n2^3},\right.$$
$$\frac{10\,a0^3}{n0^3\,N01\,N02}-\frac{10\,a1^3}{N01\,n1^3\,N12}-\frac{10\,a2^3}{N02\,N12\,n2^3},\ \frac{10\,a0^2}{n0^3\,N01\,N02}-\frac{10\,a1^2}{N01\,n1^3\,N12}-\frac{10\,a2^2}{N02\,N12\,n2^3}+\frac{10\,rY^2}{n0\,n1\,n2},$$
$$\frac{5\,a0}{n0^3\,N01\,N02}-\frac{5\,a1}{N01\,n1^3\,N12}-\frac{5\,a2}{N02\,N12\,n2^3}-\frac{5\,(cYn2\,n0\,n1+cYn1\,n0\,n2+cYn0\,n1\,n2)\,rY}{n0^2\,n1^2\,n2^2},$$
$$\frac{1}{n0^3\,N01\,N02}-\frac{1}{N01\,n1^3\,N12}-\frac{1}{N02\,N12\,n2^3}+\frac{1}{n0^3\,n1^3\,n2^3}\left(cYn2^2\,n0^2\,n1^2+cYn1\,cYn2\,n0^2\,n1\,n2+\right.$$
$$\left.\left.cYn0\,cYn2\,n0\,n1^2\,n2+cYn1^2\,n0^2\,n2^2+cYn0\,cYn1\,n0\,n1\,n2^2+cYn0^2\,n1^2\,n2^2\right)\right\}$$

μs1x3YB5 + μs1x3P0B5 + μs1x3P1B5 + μs1x3P2B5

$$\left\{\frac{a0^3\,Sn0}{n0^3\,N01\,N02}-\frac{a1^3\,Sn1}{N01\,n1^3\,N12}-\frac{a2^3\,Sn2}{N02\,N12\,n2^3},\right.$$
$$\frac{3\,\left(n0^2+n1^2+n2^2\right)\,rY^2}{n0\,n1\,n2}+\frac{3\,a0^2\,Sn0}{n0^3\,N01\,N02}-\frac{3\,a1^2\,Sn1}{N01\,n1^3\,N12}-\frac{3\,a2^2\,Sn2}{N02\,N12\,n2^3},$$
$$-\frac{1}{n0^2\,n1^2\,n2^2}3\,\left(cYn2\,n0^3\,n1+cYn2\,n0\,n1^3+cYn1\,n0^3\,n2-cYn0\,n0^2\,n1\,n2-cYn1\,n0\,n1^2\,n2+\right.$$
$$\left.cYn0\,n1^3\,n2-cYn2\,n0\,n1\,n2^2+cYn1\,n0\,n2^3+cYn0\,n1\,n2^3\right)\,rY+\frac{3\,a0\,Sn0}{n0^3\,N01\,N02}-\frac{3\,a1\,Sn1}{N01\,n1^3\,N12}-$$
$$\frac{3\,a2\,Sn2}{N02\,N12\,n2^3},\ \frac{1}{n0^3\,n1^3\,n2^3}\left(cYn2^2\,n0^4\,n1^2+cYn2^2\,n0^2\,n1^4+cYn1\,cYn2\,n0^4\,n1\,n2-\right.$$
$$cYn0\,cYn2\,n0^3\,n1^2\,n2-cYn1\,cYn2\,n0^2\,n1^3\,n2+cYn0\,cYn2\,n0\,n1^4\,n2+cYn1^2\,n0^4\,n2^2-$$
$$cYn0\,cYn1\,n0^3\,n1\,n2^2-cYn0\,cYn1\,n0\,n1^3\,n2^2+cYn0^2\,n1^4\,n2^2-cYn1\,cYn2\,n0^2\,n1\,n2^3-$$
$$\left.cYn0\,cYn2\,n0\,n1^2\,n2^3+cYn1^2\,n0^2\,n2^4+cYn0\,cYn1\,n0\,n1\,n2^4+cYn0^2\,n1^2\,n2^4+n0^2\,n1^2\,n2^2\,pY1\right)+$$
$$\left.\frac{Sn0}{n0^3\,N01\,N02}-\frac{Sn1}{N01\,n1^3\,N12}-\frac{Sn2}{N02\,N12\,n2^3}\right\}$$

$$\mathtt{Factor}\Big[\mathtt{Solve}\Big[\Big\{$$

$$d == \frac{a0^5}{n0^3\ N01\ N02} - \frac{a1^5}{N01\ n1^3\ N12} - \frac{a2^5}{N02\ N12\ n2^3},$$

$$0 == \frac{5\ a0^4}{n0^3\ N01\ N02} - \frac{5\ a1^4}{N01\ n1^3\ N12} - \frac{5\ a2^4}{N02\ N12\ n2^3},$$

$$0 == \frac{10\ a0^3}{n0^3\ N01\ N02} - \frac{10\ a1^3}{N01\ n1^3\ N12} - \frac{10\ a2^3}{N02\ N12\ n2^3},$$

$$0 == \frac{10\ a0^2}{n0^3\ N01\ N02} - \frac{10\ a1^2}{N01\ n1^3\ N12} - \frac{10\ a2^2}{N02\ N12\ n2^3} + \frac{10\ rY^2}{n0\ n1\ n2},$$

$$0 == \frac{5\ a0}{n0^3\ N01\ N02} - \frac{5\ a1}{N01\ n1^3\ N12} - \frac{5\ a2}{N02\ N12\ n2^3} - \frac{5\ (cYn2\ n0\ n1 + cYn1\ n0\ n2 + cYn0\ n1\ n2)\ rY}{n0^2\ n1^2\ n2^2},$$

$$0 == \frac{1}{n0^3\ N01\ N02} - \frac{1}{N01\ n1^3\ N12} - \frac{1}{N02\ N12\ n2^3} + \frac{1}{n0^3\ n1^3\ n2^3}\left(cYn2^2\ n0^2\ n1^2 + cYn1\ cYn2\ n0^2\ n1\ n2 + cYn0\ cYn2\ n0\ n1^2\ n2 + cYn1^2\ n0^2\ n2^2 + cYn0\ cYn1\ n0\ n1\ n2^2 + cYn0^2\ n1^2\ n2^2\right),$$

$$p\ d == \frac{a0^3\ Sn0}{n0^3\ N01\ N02} - \frac{a1^3\ Sn1}{N01\ n1^3\ N12} - \frac{a2^3\ Sn2}{N02\ N12\ n2^3},$$

$$0 == \frac{3\ (n0^2 + n1^2 + n2^2)\ rY^2}{n0\ n1\ n2} + \frac{3\ a0^2\ Sn0}{n0^3\ N01\ N02} - \frac{3\ a1^2\ Sn1}{N01\ n1^3\ N12} - \frac{3\ a2^2\ Sn2}{N02\ N12\ n2^3},$$

$$0 == -\frac{1}{n0^2\ n1^2\ n2^2} 3\left(cYn2\ n0^3\ n1 + cYn2\ n0\ n1^3 + cYn1\ n0^3\ n2 - cYn0\ n0^2\ n1\ n2 - cYn1\ n0\ n1^2\ n2 + cYn0\ n1^3\ n2 - cYn2\ n0\ n1\ n2^2 + cYn1\ n0\ n2^3 + cYn0\ n1\ n2^3\right) rY + \frac{3\ a0\ Sn0}{n0^3\ N01\ N02} - \frac{3\ a1\ Sn1}{N01\ n1^3\ N12} - \frac{3\ a2\ Sn2}{N02\ N12\ n2^3},$$

$$0 == \frac{1}{n0^3\ n1^3\ n2^3}\left(cYn2^2\ n0^4\ n1^2 + cYn2^2\ n0^2\ n1^4 + cYn1\ cYn2\ n0^4\ n1\ n2 - cYn0\ cYn2\ n0^3\ n1^2\ n2 - cYn1\ cYn2\ n0^2\ n1^3\ n2 + cYn0\ cYn2\ n0\ n1^4\ n2 + cYn1^2\ n0^4\ n2^2 - cYn0\ cYn1\ n0^3\ n1\ n2^2 - cYn0\ cYn1\ n0\ n1^3\ n2^2 + cYn0^2\ n1^4\ n2^2 - cYn1\ cYn2\ n0^2\ n1\ n2^3 - cYn0\ cYn2\ n0\ n1^2\ n2^3 + cYn1^2\ n0^2\ n2^4 + cYn0\ cYn1\ n0\ n1\ n2^4 + cYn0^2\ n1^2\ n2^4 + n0^2\ n1^2\ n2^2\ pY1\right) + \frac{Sn0}{n0^3\ N01\ N02} - \frac{Sn1}{N01\ n1^3\ N12} - \frac{Sn2}{N02\ N12\ n2^3},$$

$$pY1 == 3,$$

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(*******************************
 The following equations are given by Lemma 6.13
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$$0 == -\frac{3\ a0^2}{n0^3} + \frac{3\ rY^2}{n0},\quad 0 == -\frac{3\ a0}{n0^3} - \frac{3\ cYn0\ rY}{n0^2},\quad 0 == -\frac{1}{n0^3} + \frac{cYn0^2}{n0^3},$$

$$0 == -\frac{3\ a1^2}{n1^3} + \frac{3\ rY^2}{n1},\quad 0 == -\frac{3\ a1}{n1^3} - \frac{3\ cYn1\ rY}{n1^2},\quad 0 == -\frac{1}{n1^3} + \frac{cYn1^2}{n1^3},$$

$$0 == -\frac{3\ a2^2}{n2^3} + \frac{3\ rY^2}{n2},\quad 0 == -\frac{3\ a2}{n2^3} - \frac{3\ cYn2\ rY}{n2^2},\quad 0 == -\frac{1}{n2^3} + \frac{cYn2^2}{n2^3}$$

$$\Big\}, \{p, d, n0, n1, n2, N01, N02, N12, cYn0, cYn1, cYn2, pY1, Sn0, Sn1, Sn2\}\Big]\Big]$$

$$\Big\{\Big\{p \to \frac{6}{rY^2},\ d \to rY^5,\ n0 \to -\frac{a0}{rY},\ n1 \to -\frac{a1}{rY},\ n2 \to -\frac{a2}{rY},$$

$$N01 \to \frac{a0 - a1}{rY},\ N02 \to -\frac{a0 - a2}{rY},\ N12 \to -\frac{a1 - a2}{rY},\ cYn0 \to 1,\ cYn1 \to 1,$$

$$\text{cYn2} \to 1,\ \text{pY1} \to 3,\ \text{Sn0} \to \frac{5\ \text{a0}^2 - 2\ \text{a0}\ \text{a1} + \text{a1}^2 - 2\ \text{a0}\ \text{a2} + \text{a2}^2}{\text{rY}^2},$$

$$\text{Sn1} \to \frac{\text{a0}^2 - 2\ \text{a0}\ \text{a1} + 5\ \text{a1}^2 - 2\ \text{a1}\ \text{a2} + \text{a2}^2}{\text{rY}^2},\ \text{Sn2} \to \frac{\text{a0}^2 + \text{a1}^2 - 2\ \text{a0}\ \text{a2} - 2\ \text{a1}\ \text{a2} + 5\ \text{a2}^2}{\text{rY}^2}\Big\},$$

$$\Big\{\text{p} \to \frac{6}{\text{rY}^2},\ \text{d} \to \text{rY}^5,\ \text{n0} \to -\frac{\text{a0}}{\text{rY}},\ \text{n1} \to -\frac{\text{a1}}{\text{rY}},\ \text{n2} \to -\frac{\text{a2}}{\text{rY}},\ \text{N01} \to -\frac{\text{a0} - \text{a1}}{\text{rY}},$$

$$\text{N02} \to \frac{\text{a0} - \text{a2}}{\text{rY}},\ \text{N12} \to \frac{\text{a1} - \text{a2}}{\text{rY}},\ \text{cYn0} \to 1,\ \text{cYn1} \to 1,\ \text{cYn2} \to 1,\ \text{pY1} \to 3,$$

$$\text{Sn0} \to \frac{5\ \text{a0}^2 - 2\ \text{a0}\ \text{a1} + \text{a1}^2 - 2\ \text{a0}\ \text{a2} + \text{a2}^2}{\text{rY}^2},\ \text{Sn1} \to \frac{\text{a0}^2 - 2\ \text{a0}\ \text{a1} + 5\ \text{a1}^2 - 2\ \text{a1}\ \text{a2} + \text{a2}^2}{\text{rY}^2},$$

$$\text{Sn2} \to \frac{\text{a0}^2 + \text{a1}^2 - 2\ \text{a0}\ \text{a2} - 2\ \text{a1}\ \text{a2} + 5\ \text{a2}^2}{\text{rY}^2}\Big\},\ \Big\{\text{p} \to \frac{6}{\text{rY}^2},\ \text{d} \to -\text{rY}^5,\ \text{n0} \to -\frac{\text{a0}}{\text{rY}},$$

$$\text{n1} \to -\frac{\text{a1}}{\text{rY}},\ \text{n2} \to \frac{\text{a2}}{\text{rY}},\ \text{N01} \to \frac{\text{a0} - \text{a1}}{\text{rY}},\ \text{N02} \to \frac{\text{a0} - \text{a2}}{\text{rY}},\ \text{N12} \to \frac{\text{a1} - \text{a2}}{\text{rY}},\ \text{cYn0} \to 1,$$

$$\text{cYn1} \to 1,\ \text{cYn2} \to -1,\ \text{pY1} \to 3,\ \text{Sn0} \to \frac{5\ \text{a0}^2 - 2\ \text{a0}\ \text{a1} + \text{a1}^2 - 2\ \text{a0}\ \text{a2} + \text{a2}^2}{\text{rY}^2},$$

$$\text{Sn1} \to \frac{\text{a0}^2 - 2\ \text{a0}\ \text{a1} + 5\ \text{a1}^2 - 2\ \text{a1}\ \text{a2} + \text{a2}^2}{\text{rY}^2},\ \text{Sn2} \to \frac{\text{a0}^2 + \text{a1}^2 - 2\ \text{a0}\ \text{a2} - 2\ \text{a1}\ \text{a2} + 5\ \text{a2}^2}{\text{rY}^2}\Big\},$$

$$\Big\{\text{p} \to \frac{6}{\text{rY}^2},\ \text{d} \to -\text{rY}^5,\ \text{n0} \to -\frac{\text{a0}}{\text{rY}},\ \text{n1} \to -\frac{\text{a1}}{\text{rY}},\ \text{n2} \to \frac{\text{a2}}{\text{rY}},\ \text{N01} \to -\frac{\text{a0} - \text{a1}}{\text{rY}},$$

$$\text{N02} \to -\frac{\text{a0} - \text{a2}}{\text{rY}},\ \text{N12} \to -\frac{\text{a1} - \text{a2}}{\text{rY}},\ \text{cYn0} \to 1,\ \text{cYn1} \to 1,\ \text{cYn2} \to -1,\ \text{pY1} \to 3,$$

$$\text{Sn0} \to \frac{5\ \text{a0}^2 - 2\ \text{a0}\ \text{a1} + \text{a1}^2 - 2\ \text{a0}\ \text{a2} + \text{a2}^2}{\text{rY}^2},\ \text{Sn1} \to \frac{\text{a0}^2 - 2\ \text{a0}\ \text{a1} + 5\ \text{a1}^2 - 2\ \text{a1}\ \text{a2} + \text{a2}^2}{\text{rY}^2},$$

$$\text{Sn2} \to \frac{\text{a0}^2 + \text{a1}^2 - 2\ \text{a0}\ \text{a2} - 2\ \text{a1}\ \text{a2} + 5\ \text{a2}^2}{\text{rY}^2}\Big\},\ \Big\{\text{p} \to \frac{6}{\text{rY}^2},\ \text{d} \to -\text{rY}^5,\ \text{n0} \to -\frac{\text{a0}}{\text{rY}},$$

$$\text{n1} \to \frac{\text{a1}}{\text{rY}},\ \text{n2} \to -\frac{\text{a2}}{\text{rY}},\ \text{N01} \to \frac{\text{a0} - \text{a1}}{\text{rY}},\ \text{N02} \to \frac{\text{a0} - \text{a2}}{\text{rY}},\ \text{N12} \to -\frac{\text{a1} - \text{a2}}{\text{rY}},\ \text{cYn0} \to 1,$$

$$\text{cYn1} \to -1,\ \text{cYn2} \to 1,\ \text{pY1} \to 3,\ \text{Sn0} \to \frac{5\ \text{a0}^2 - 2\ \text{a0}\ \text{a1} + \text{a1}^2 - 2\ \text{a0}\ \text{a2} + \text{a2}^2}{\text{rY}^2},$$

$$\text{Sn1} \to \frac{\text{a0}^2 - 2\ \text{a0}\ \text{a1} + 5\ \text{a1}^2 - 2\ \text{a1}\ \text{a2} + \text{a2}^2}{\text{rY}^2},\ \text{Sn2} \to \frac{\text{a0}^2 + \text{a1}^2 - 2\ \text{a0}\ \text{a2} - 2\ \text{a1}\ \text{a2} + 5\ \text{a2}^2}{\text{rY}^2}\Big\},$$

$$\Big\{\text{p} \to \frac{6}{\text{rY}^2},\ \text{d} \to -\text{rY}^5,\ \text{n0} \to -\frac{\text{a0}}{\text{rY}},\ \text{n1} \to \frac{\text{a1}}{\text{rY}},\ \text{n2} \to -\frac{\text{a2}}{\text{rY}},\ \text{N01} \to -\frac{\text{a0} - \text{a1}}{\text{rY}},$$

$$\text{N02} \to -\frac{\text{a0} - \text{a2}}{\text{rY}},\ \text{N12} \to \frac{\text{a1} - \text{a2}}{\text{rY}},\ \text{cYn0} \to 1,\ \text{cYn1} \to -1,\ \text{cYn2} \to 1,\ \text{pY1} \to 3,$$

$$\text{Sn0} \to \frac{5\ \text{a0}^2 - 2\ \text{a0}\ \text{a1} + \text{a1}^2 - 2\ \text{a0}\ \text{a2} + \text{a2}^2}{\text{rY}^2},\ \text{Sn1} \to \frac{\text{a0}^2 - 2\ \text{a0}\ \text{a1} + 5\ \text{a1}^2 - 2\ \text{a1}\ \text{a2} + \text{a2}^2}{\text{rY}^2},$$

$$\text{Sn2} \to \frac{\text{a0}^2 + \text{a1}^2 - 2\ \text{a0}\ \text{a2} - 2\ \text{a1}\ \text{a2} + 5\ \text{a2}^2}{\text{rY}^2}\Big\},\ \Big\{\text{p} \to \frac{6}{\text{rY}^2},\ \text{d} \to \text{rY}^5,\ \text{n0} \to -\frac{\text{a0}}{\text{rY}},$$

$$\text{n1} \to \frac{\text{a1}}{\text{rY}},\ \text{n2} \to \frac{\text{a2}}{\text{rY}},\ \text{N01} \to \frac{\text{a0} - \text{a1}}{\text{rY}},\ \text{N02} \to -\frac{\text{a0} - \text{a2}}{\text{rY}},\ \text{N12} \to \frac{\text{a1} - \text{a2}}{\text{rY}},\ \text{cYn0} \to 1,$$

$$\text{cYn1} \to -1,\ \text{cYn2} \to -1,\ \text{pY1} \to 3,\ \text{Sn0} \to \frac{5\ \text{a0}^2 - 2\ \text{a0}\ \text{a1} + \text{a1}^2 - 2\ \text{a0}\ \text{a2} + \text{a2}^2}{\text{rY}^2},$$

$$\text{Sn1} \to \frac{\text{a0}^2 - 2\ \text{a0}\ \text{a1} + 5\ \text{a1}^2 - 2\ \text{a1}\ \text{a2} + \text{a2}^2}{\text{rY}^2},\ \text{Sn2} \to \frac{\text{a0}^2 + \text{a1}^2 - 2\ \text{a0}\ \text{a2} - 2\ \text{a1}\ \text{a2} + 5\ \text{a2}^2}{\text{rY}^2}\Big\},$$

$$\Big\{\text{p} \to \frac{6}{\text{rY}^2},\ \text{d} \to \text{rY}^5,\ \text{n0} \to -\frac{\text{a0}}{\text{rY}},\ \text{n1} \to \frac{\text{a1}}{\text{rY}},\ \text{n2} \to \frac{\text{a2}}{\text{rY}},\ \text{N01} \to -\frac{\text{a0} - \text{a1}}{\text{rY}},$$

$$\text{N02} \to \frac{\text{a0} - \text{a2}}{\text{rY}},\ \text{N12} \to -\frac{\text{a1} - \text{a2}}{\text{rY}},\ \text{cYn0} \to 1,\ \text{cYn1} \to -1,\ \text{cYn2} \to -1,\ \text{pY1} \to 3,$$

$$\text{Sn0} \to \frac{5\ \text{a0}^2 - 2\ \text{a0}\ \text{a1} + \text{a1}^2 - 2\ \text{a0}\ \text{a2} + \text{a2}^2}{\text{rY}^2},\ \text{Sn1} \to \frac{\text{a0}^2 - 2\ \text{a0}\ \text{a1} + 5\ \text{a1}^2 - 2\ \text{a1}\ \text{a2} + \text{a2}^2}{\text{rY}^2},$$

$$\text{Sn2} \to \frac{\text{a0}^2 + \text{a1}^2 - 2\ \text{a0}\ \text{a2} - 2\ \text{a1}\ \text{a2} + 5\ \text{a2}^2}{\text{rY}^2}\Big\},\ \Big\{\text{p} \to \frac{6}{\text{rY}^2},\ \text{d} \to -\text{rY}^5,\ \text{n0} \to \frac{\text{a0}}{\text{rY}},$$

$$n1 \to -\frac{a1}{rY},\ n2 \to -\frac{a2}{rY},\ N01 \to \frac{a0-a1}{rY},\ N02 \to -\frac{a0-a2}{rY},\ N12 \to \frac{a1-a2}{rY},\ cYn0 \to -1,$$
$$cYn1 \to 1,\ cYn2 \to 1,\ pY1 \to 3,\ Sn0 \to \frac{5\,a0^2-2\,a0\,a1+a1^2-2\,a0\,a2+a2^2}{rY^2},$$
$$Sn1 \to \frac{a0^2-2\,a0\,a1+5\,a1^2-2\,a1\,a2+a2^2}{rY^2},\ Sn2 \to \frac{a0^2+a1^2-2\,a0\,a2-2\,a1\,a2+5\,a2^2}{rY^2}\Big\},$$
$$\Big\{p \to \frac{6}{rY^2},\ d \to -rY^5,\ n0 \to \frac{a0}{rY},\ n1 \to -\frac{a1}{rY},\ n2 \to -\frac{a2}{rY},\ N01 \to -\frac{a0-a1}{rY},$$
$$N02 \to \frac{a0-a2}{rY},\ N12 \to -\frac{a1-a2}{rY},\ cYn0 \to -1,\ cYn1 \to 1,\ cYn2 \to 1,\ pY1 \to 3,$$
$$Sn0 \to \frac{5\,a0^2-2\,a0\,a1+a1^2-2\,a0\,a2+a2^2}{rY^2},\ Sn1 \to \frac{a0^2-2\,a0\,a1+5\,a1^2-2\,a1\,a2+a2^2}{rY^2},$$
$$Sn2 \to \frac{a0^2+a1^2-2\,a0\,a2-2\,a1\,a2+5\,a2^2}{rY^2}\Big\},\ \Big\{p \to \frac{6}{rY^2},\ d \to rY^5,\ n0 \to \frac{a0}{rY},$$
$$n1 \to -\frac{a1}{rY},\ n2 \to \frac{a2}{rY},\ N01 \to \frac{a0-a1}{rY},\ N02 \to \frac{a0-a2}{rY},\ N12 \to -\frac{a1-a2}{rY},\ cYn0 \to -1,$$
$$cYn1 \to 1,\ cYn2 \to -1,\ pY1 \to 3,\ Sn0 \to \frac{5\,a0^2-2\,a0\,a1+a1^2-2\,a0\,a2+a2^2}{rY^2},$$
$$Sn1 \to \frac{a0^2-2\,a0\,a1+5\,a1^2-2\,a1\,a2+a2^2}{rY^2},\ Sn2 \to \frac{a0^2+a1^2-2\,a0\,a2-2\,a1\,a2+5\,a2^2}{rY^2}\Big\},$$
$$\Big\{p \to \frac{6}{rY^2},\ d \to rY^5,\ n0 \to \frac{a0}{rY},\ n1 \to -\frac{a1}{rY},\ n2 \to \frac{a2}{rY},\ N01 \to -\frac{a0-a1}{rY},$$
$$N02 \to -\frac{a0-a2}{rY},\ N12 \to \frac{a1-a2}{rY},\ cYn0 \to -1,\ cYn1 \to 1,$$
$$cYn2 \to -1,\ pY1 \to 3,\ Sn0 \to \frac{5\,a0^2-2\,a0\,a1+a1^2-2\,a0\,a2+a2^2}{rY^2},$$
$$Sn1 \to \frac{a0^2-2\,a0\,a1+5\,a1^2-2\,a1\,a2+a2^2}{rY^2},\ Sn2 \to \frac{a0^2+a1^2-2\,a0\,a2-2\,a1\,a2+5\,a2^2}{rY^2}\Big\},$$
$$\Big\{p \to \frac{6}{rY^2},\ d \to rY^5,\ n0 \to \frac{a0}{rY},\ n1 \to \frac{a1}{rY},\ n2 \to -\frac{a2}{rY},\ N01 \to \frac{a0-a1}{rY},\ N02 \to \frac{a0-a2}{rY},\ N12 \to \frac{a1-a2}{rY},$$
$$cYn0 \to -1,\ cYn1 \to -1,\ cYn2 \to 1,\ pY1 \to 3,\ Sn0 \to \frac{5\,a0^2-2\,a0\,a1+a1^2-2\,a0\,a2+a2^2}{rY^2},$$
$$Sn1 \to \frac{a0^2-2\,a0\,a1+5\,a1^2-2\,a1\,a2+a2^2}{rY^2},\ Sn2 \to \frac{a0^2+a1^2-2\,a0\,a2-2\,a1\,a2+5\,a2^2}{rY^2}\Big\},$$
$$\Big\{p \to \frac{6}{rY^2},\ d \to rY^5,\ n0 \to \frac{a0}{rY},\ n1 \to \frac{a1}{rY},\ n2 \to -\frac{a2}{rY},\ N01 \to -\frac{a0-a1}{rY},$$
$$N02 \to -\frac{a0-a2}{rY},\ N12 \to -\frac{a1-a2}{rY},\ cYn0 \to -1,\ cYn1 \to -1,$$
$$cYn2 \to 1,\ pY1 \to 3,\ Sn0 \to \frac{5\,a0^2-2\,a0\,a1+a1^2-2\,a0\,a2+a2^2}{rY^2},$$
$$Sn1 \to \frac{a0^2-2\,a0\,a1+5\,a1^2-2\,a1\,a2+a2^2}{rY^2},\ Sn2 \to \frac{a0^2+a1^2-2\,a0\,a2-2\,a1\,a2+5\,a2^2}{rY^2}\Big\},$$
$$\Big\{p \to \frac{6}{rY^2},\ d \to -rY^5,\ n0 \to \frac{a0}{rY},\ n1 \to \frac{a1}{rY},\ n2 \to \frac{a2}{rY},\ N01 \to \frac{a0-a1}{rY},\ N02 \to -\frac{a0-a2}{rY},$$
$$N12 \to -\frac{a1-a2}{rY},\ cYn0 \to -1,\ cYn1 \to -1,\ cYn2 \to -1,\ pY1 \to 3,$$
$$Sn0 \to \frac{5\,a0^2-2\,a0\,a1+a1^2-2\,a0\,a2+a2^2}{rY^2},\ Sn1 \to \frac{a0^2-2\,a0\,a1+5\,a1^2-2\,a1\,a2+a2^2}{rY^2},$$
$$Sn2 \to \frac{a0^2+a1^2-2\,a0\,a2-2\,a1\,a2+5\,a2^2}{rY^2}\Big\},\ \Big\{p \to \frac{6}{rY^2},\ d \to -rY^5,\ n0 \to \frac{a0}{rY},$$
$$n1 \to \frac{a1}{rY},\ n2 \to \frac{a2}{rY},\ N01 \to -\frac{a0-a1}{rY},\ N02 \to \frac{a0-a2}{rY},\ N12 \to \frac{a1-a2}{rY},\ cYn0 \to -1,$$
$$cYn1 \to -1,\ cYn2 \to -1,\ pY1 \to 3,\ Sn0 \to \frac{5\,a0^2-2\,a0\,a1+a1^2-2\,a0\,a2+a2^2}{rY^2},$$

$$\text{Sn1} \to \frac{\text{a0}^2 - 2\,\text{a0 a1} + 5\,\text{a1}^2 - 2\,\text{a1 a2} + \text{a2}^2}{\text{rY}^2},\ \text{Sn2} \to \frac{\text{a0}^2 + \text{a1}^2 - 2\,\text{a0 a2} - 2\,\text{a1 a2} + 5\,\text{a2}^2}{\text{rY}^2}\}\}$$

Computations for Lemma 6.21.
Case C. where the action is semi-free around Y.

```
(*******************************
  NOTATIONS
  rY xY denotes the restriction of x to Y
  ai denotes the Hopf weight at Pi
  Nij denotes the product of the normal weights shared between Pi and Pj
  Sni is the sum of the square of the normal weights at Pi
  cYj xY^j denotes the j-th Chern class of the normal bundle of Y in the
  remaining directions
  μx5YC denotes the local datum of x^5 at Y
  μx5PiC denotes the local datum of x^5 at Pi
  μs1x3YC denotes the local datum of p1(X)x^3 at Y
  μs1x3PiC denotes the local datum of p1(X)x^3 at Pi

  We use the variable t to extract the coefficient of order 2 of xY. We
  then evaluate xY^2 to 1.
  The orientation of the point P0 is 1 while those from P1 and P2 are -
 1. This explains the signs at their local datum
  Once evaluated, we extract all the coefficients in l into a list.
  *******************************)
```

```
μx5YC =
 Factor[CoefficientList[
    Coefficient[Expand[(rY xY t + l z)^5]
      Series[(cY2 xY^2 t^2 z + cY1 xY t z^2 + z^3)^-1, {xY, 0, 2}], t, 2], l]] /. xY^2 → 1
```

$$\{0,\ 0,\ 0,\ 10\,\text{rY}^2,\ -5\,\text{cY1 rY},\ \text{cY1}^2 - \text{cY2}\}$$

```
μx5P0C = CoefficientList[Coefficient[Expand[((a0 + l) z)^5] N01^-1 N02^-1 z^-5, t, 0], l]
```

$$\left\{\frac{\text{a0}^5}{\text{N01 N02}},\ \frac{5\,\text{a0}^4}{\text{N01 N02}},\ \frac{10\,\text{a0}^3}{\text{N01 N02}},\ \frac{10\,\text{a0}^2}{\text{N01 N02}},\ \frac{5\,\text{a0}}{\text{N01 N02}},\ \frac{1}{\text{N01 N02}}\right\}$$

```
μx5P1C = CoefficientList[Coefficient[-Expand[((a1 + l) z)^5] N01^-1 N12^-1 z^-5, t, 0], l]
```

$$\left\{-\frac{\text{a1}^5}{\text{N01 N12}},\ -\frac{5\,\text{a1}^4}{\text{N01 N12}},\ -\frac{10\,\text{a1}^3}{\text{N01 N12}},\ -\frac{10\,\text{a1}^2}{\text{N01 N12}},\ -\frac{5\,\text{a1}}{\text{N01 N12}},\ -\frac{1}{\text{N01 N12}}\right\}$$

```
μx5P2C = CoefficientList[Coefficient[-Expand[((a2 + l) z)^5] N02^-1 N12^-1 z^-5, t, 0], l]
```

$$\left\{-\frac{\text{a2}^5}{\text{N02 N12}},\ -\frac{5\,\text{a2}^4}{\text{N02 N12}},\ -\frac{10\,\text{a2}^3}{\text{N02 N12}},\ -\frac{10\,\text{a2}^2}{\text{N02 N12}},\ -\frac{5\,\text{a2}}{\text{N02 N12}},\ -\frac{1}{\text{N02 N12}}\right\}$$

```
SymmetricReduction[Expand[(yY1 t + z)^2 + (yY2 t + z)^2], {yY1, yY2}, {cY1 xY, cY2 xY^2}]
```

$$\{\text{cY1}^2\,t^2\,\text{xY}^2 - 2\,\text{cY2}\,t^2\,\text{xY}^2 + 2\,\text{cY1}\,t\,\text{xY}\,z + 2\,z^2,\ 0\}$$

```
μs1x3YC =
 Factor[CoefficientList[
    Coefficient[(pY1 t^2 + cY1^2 t^2 xY^2 - 2 cY2 t^2 xY^2 + 2 cY1 t xY z + 3 z^2)
      Expand[(rY xY t + 1 z)^3] Series[(cY2 xY^2 t^2 z + cY1 xY t z^2 + z^3)^-1, {xY, 0, 2}], t, 2],
    1]] /. xY^2 → 1
```

$$\left\{0,\ 9\,\mathrm{rY}^2,\ -3\,\mathrm{cY1}\,\mathrm{rY},\ 2\,\mathrm{cY1}^2-5\,\mathrm{cY2}+\mathrm{pY1}\right\}$$

```
μs1x3P0C =
 Simplify[CoefficientList[Coefficient[Sn0 z^2 Expand[((a0 + 1) z)^3] N01^-1 N02^-1 z^-5, t, 0],
   1]]
```

$$\left\{\frac{\mathrm{a0}^3\,\mathrm{Sn0}}{\mathrm{N01}\,\mathrm{N02}},\ \frac{3\,\mathrm{a0}^2\,\mathrm{Sn0}}{\mathrm{N01}\,\mathrm{N02}},\ \frac{3\,\mathrm{a0}\,\mathrm{Sn0}}{\mathrm{N01}\,\mathrm{N02}},\ \frac{\mathrm{Sn0}}{\mathrm{N01}\,\mathrm{N02}}\right\}$$

```
μs1x3P1C =
 Simplify[CoefficientList[Coefficient[-Sn1 z^2 Expand[((a1 + 1) z)^3] N01^-1 N12^-1 z^-5,
    t, 0], 1]]
```

$$\left\{-\frac{\mathrm{a1}^3\,\mathrm{Sn1}}{\mathrm{N01}\,\mathrm{N12}},\ -\frac{3\,\mathrm{a1}^2\,\mathrm{Sn1}}{\mathrm{N01}\,\mathrm{N12}},\ -\frac{3\,\mathrm{a1}\,\mathrm{Sn1}}{\mathrm{N01}\,\mathrm{N12}},\ -\frac{\mathrm{Sn1}}{\mathrm{N01}\,\mathrm{N12}}\right\}$$

```
μs1x3P2C =
 Simplify[CoefficientList[Coefficient[-Sn2 z^2 Expand[((a2 + 1) z)^3] N02^-1 N12^-1 z^-5,
    t, 0], 1]]
```

$$\left\{-\frac{\mathrm{a2}^3\,\mathrm{Sn2}}{\mathrm{N02}\,\mathrm{N12}},\ -\frac{3\,\mathrm{a2}^2\,\mathrm{Sn2}}{\mathrm{N02}\,\mathrm{N12}},\ -\frac{3\,\mathrm{a2}\,\mathrm{Sn2}}{\mathrm{N02}\,\mathrm{N12}},\ -\frac{\mathrm{Sn2}}{\mathrm{N02}\,\mathrm{N12}}\right\}$$

```
μx5YC + μx5P0C + μx5P1C + μx5P2C
```

$$\left\{\frac{\mathrm{a0}^5}{\mathrm{N01}\,\mathrm{N02}}-\frac{\mathrm{a1}^5}{\mathrm{N01}\,\mathrm{N12}}-\frac{\mathrm{a2}^5}{\mathrm{N02}\,\mathrm{N12}},\ \frac{5\,\mathrm{a0}^4}{\mathrm{N01}\,\mathrm{N02}}-\frac{5\,\mathrm{a1}^4}{\mathrm{N01}\,\mathrm{N12}}-\frac{5\,\mathrm{a2}^4}{\mathrm{N02}\,\mathrm{N12}},\right.$$
$$\frac{10\,\mathrm{a0}^3}{\mathrm{N01}\,\mathrm{N02}}-\frac{10\,\mathrm{a1}^3}{\mathrm{N01}\,\mathrm{N12}}-\frac{10\,\mathrm{a2}^3}{\mathrm{N02}\,\mathrm{N12}},\ \frac{10\,\mathrm{a0}^2}{\mathrm{N01}\,\mathrm{N02}}-\frac{10\,\mathrm{a1}^2}{\mathrm{N01}\,\mathrm{N12}}-\frac{10\,\mathrm{a2}^2}{\mathrm{N02}\,\mathrm{N12}}+10\,\mathrm{rY}^2,$$
$$\left.\frac{5\,\mathrm{a0}}{\mathrm{N01}\,\mathrm{N02}}-\frac{5\,\mathrm{a1}}{\mathrm{N01}\,\mathrm{N12}}-\frac{5\,\mathrm{a2}}{\mathrm{N02}\,\mathrm{N12}}-5\,\mathrm{cY1}\,\mathrm{rY},\ \mathrm{cY1}^2-\mathrm{cY2}+\frac{1}{\mathrm{N01}\,\mathrm{N02}}-\frac{1}{\mathrm{N01}\,\mathrm{N12}}-\frac{1}{\mathrm{N02}\,\mathrm{N12}}\right\}$$

```
μs1x3YC + μs1x3P0C + μs1x3P1C + μs1x3P2C
```

$$\left\{\frac{\mathrm{a0}^3\,\mathrm{Sn0}}{\mathrm{N01}\,\mathrm{N02}}-\frac{\mathrm{a1}^3\,\mathrm{Sn1}}{\mathrm{N01}\,\mathrm{N12}}-\frac{\mathrm{a2}^3\,\mathrm{Sn2}}{\mathrm{N02}\,\mathrm{N12}},\ 9\,\mathrm{rY}^2+\frac{3\,\mathrm{a0}^2\,\mathrm{Sn0}}{\mathrm{N01}\,\mathrm{N02}}-\frac{3\,\mathrm{a1}^2\,\mathrm{Sn1}}{\mathrm{N01}\,\mathrm{N12}}-\frac{3\,\mathrm{a2}^2\,\mathrm{Sn2}}{\mathrm{N02}\,\mathrm{N12}},\right.$$
$$\left.-3\,\mathrm{cY1}\,\mathrm{rY}+\frac{3\,\mathrm{a0}\,\mathrm{Sn0}}{\mathrm{N01}\,\mathrm{N02}}-\frac{3\,\mathrm{a1}\,\mathrm{Sn1}}{\mathrm{N01}\,\mathrm{N12}}-\frac{3\,\mathrm{a2}\,\mathrm{Sn2}}{\mathrm{N02}\,\mathrm{N12}},\ 2\,\mathrm{cY1}^2-5\,\mathrm{cY2}+\mathrm{pY1}+\frac{\mathrm{Sn0}}{\mathrm{N01}\,\mathrm{N02}}-\frac{\mathrm{Sn1}}{\mathrm{N01}\,\mathrm{N12}}-\frac{\mathrm{Sn2}}{\mathrm{N02}\,\mathrm{N12}}\right\}$$

$$\text{Solve}\Big[\Big\{ d == \frac{a0^5}{N01\,N02} - \frac{a1^5}{N01\,N12} - \frac{a2^5}{N02\,N12},\ 0 == \frac{5\,a0^4}{N01\,N02} - \frac{5\,a1^4}{N01\,N12} - \frac{5\,a2^4}{N02\,N12},\ 0 == \frac{10\,a0^3}{N01\,N02} - \frac{10\,a1^3}{N01\,N12} - \frac{10\,a2^3}{N02\,N12},\ 0 == \frac{10\,a0^2}{N01\,N02} - \frac{10\,a1^2}{N01\,N12} - \frac{10\,a2^2}{N02\,N12} + 10\,rY^2 \Big\}, \{d, N01, N02, N12\}\Big]$$

$$\Big\{\Big\{d \to -a0\,a1\,a2\,rY^2,\ N01 \to -\frac{a0\,(a0-a1)\,a1}{a2^2\,rY},\ N02 \to \frac{a0\,(a0-a2)\,a2}{a1^2\,rY},\ N12 \to \frac{a1\,(a1-a2)\,a2}{a0^2\,rY}\Big\},$$
$$\Big\{d \to -a0\,a1\,a2\,rY^2,\ N01 \to \frac{a0\,(a0-a1)\,a1}{a2^2\,rY},\ N02 \to -\frac{a0\,(a0-a2)\,a2}{a1^2\,rY},\ N12 \to -\frac{a1\,(a1-a2)\,a2}{a0^2\,rY}\Big\}\Big\}$$

```
Solve[{
  d == a0^5/(NO1 NO2) - a1^5/(NO1 N12) - a2^5/(NO2 N12),
  0 == 5 a0^4/(NO1 NO2) - 5 a1^4/(NO1 N12) - 5 a2^4/(NO2 N12),
  0 == 10 a0^3/(NO1 NO2) - 10 a1^3/(NO1 N12) - 10 a2^3/(NO2 N12),
  0 == 10 a0^2/(NO1 NO2) - 10 a1^2/(NO1 N12) - 10 a2^2/(NO2 N12) + 10 rY^2,
  0 == 5 a0/(NO1 NO2) - 5 a1/(NO1 N12) - 5 a2/(NO2 N12) - 5 cY1 rY,
  0 == cY1^2 - cY2 + 1/(NO1 NO2) - 1/(NO1 N12) - 1/(NO2 N12),
  p d == a0^3 Sn0/(NO1 NO2) - a1^3 Sn1/(NO1 N12) - a2^3 Sn2/(NO2 N12),
  0 == 9 rY^2 + 3 a0^2 Sn0/(NO1 NO2) - 3 a1^2 Sn1/(NO1 N12) - 3 a2^2 Sn2/(NO2 N12),
  0 == -3 cY1 rY + 3 a0 Sn0/(NO1 NO2) - 3 a1 Sn1/(NO1 N12) - 3 a2 Sn2/(NO2 N12),
  0 == 2 cY1^2 - 5 cY2 + pY1 + Sn0/(NO1 NO2) - Sn1/(NO1 N12) - Sn2/(NO2 N12),
  pY1 == 3
 }, {p, d, NO1, NO2, N12, cY1, cY2, pY1, Sn0, Sn1, Sn2}]
```

$$\Big\{\Big\{p\to\frac{3\,a0^2\,a1^2\,a2^2+a0^2\,a1^2\,rY^2+a0^2\,a2^2\,rY^2+a1^2\,a2^2\,rY^2}{a0^2\,a1^2\,a2^2\,rY^2},\ d\to-a0\,a1\,a2\,rY^2,$$

$$NO1\to-\frac{a0\,(a0-a1)\,a1}{a2^2\,rY},\ NO2\to\frac{a0\,(a0-a2)\,a2}{a1^2\,rY},\ N12\to\frac{a1\,(a1-a2)\,a2}{a0^2\,rY},$$

$$cY1\to\frac{-a0\,a1\,rY-a0\,a2\,rY-a1\,a2\,rY}{a0\,a1\,a2},\ cY2\to\frac{(a0+a1+a2)\,rY^2}{a0\,a1\,a2},\ pY1\to3,$$

$$Sn0\to\frac{3\,a0^2\,a1^2\,a2^2+a0^2\,a1^2\,rY^2-2\,a0\,a1^2\,a2\,rY^2+a0^2\,a2^2\,rY^2-2\,a0\,a1\,a2^2\,rY^2+2\,a1^2\,a2^2\,rY^2}{a1^2\,a2^2\,rY^2},$$

$$Sn1\to\frac{3\,a0^2\,a1^2\,a2^2+a0^2\,a1^2\,rY^2-2\,a0^2\,a1\,a2\,rY^2+2\,a0^2\,a2^2\,rY^2-2\,a0\,a1\,a2^2\,rY^2+a1^2\,a2^2\,rY^2}{a0^2\,a2^2\,rY^2},$$

$$Sn2\to\frac{3\,a0^2\,a1^2\,a2^2+2\,a0^2\,a1^2\,rY^2-2\,a0^2\,a1\,a2\,rY^2-2\,a0\,a1^2\,a2\,rY^2+a0^2\,a2^2\,rY^2+a1^2\,a2^2\,rY^2}{a0^2\,a1^2\,rY^2}\Big\},$$

$$\Big\{p\to\frac{3\,a0^2\,a1^2\,a2^2+a0^2\,a1^2\,rY^2+a0^2\,a2^2\,rY^2+a1^2\,a2^2\,rY^2}{a0^2\,a1^2\,a2^2\,rY^2},\ d\to-a0\,a1\,a2\,rY^2,$$

$$NO1\to\frac{a0\,(a0-a1)\,a1}{a2^2\,rY},\ NO2\to-\frac{a0\,(a0-a2)\,a2}{a1^2\,rY},\ N12\to-\frac{a1\,(a1-a2)\,a2}{a0^2\,rY},$$

$$cY1\to\frac{-a0\,a1\,rY-a0\,a2\,rY-a1\,a2\,rY}{a0\,a1\,a2},\ cY2\to\frac{(a0+a1+a2)\,rY^2}{a0\,a1\,a2},\ pY1\to3,$$

$$Sn0\to\frac{3\,a0^2\,a1^2\,a2^2+a0^2\,a1^2\,rY^2-2\,a0\,a1^2\,a2\,rY^2+a0^2\,a2^2\,rY^2-2\,a0\,a1\,a2^2\,rY^2+2\,a1^2\,a2^2\,rY^2}{a1^2\,a2^2\,rY^2},$$

$$Sn1\to\frac{3\,a0^2\,a1^2\,a2^2+a0^2\,a1^2\,rY^2-2\,a0^2\,a1\,a2\,rY^2+2\,a0^2\,a2^2\,rY^2-2\,a0\,a1\,a2^2\,rY^2+a1^2\,a2^2\,rY^2}{a0^2\,a2^2\,rY^2},$$

$$Sn2\to\frac{3\,a0^2\,a1^2\,a2^2+2\,a0^2\,a1^2\,rY^2-2\,a0^2\,a1\,a2\,rY^2-2\,a0\,a1^2\,a2\,rY^2+a0^2\,a2^2\,rY^2+a1^2\,a2^2\,rY^2}{a0^2\,a1^2\,rY^2}\Big\}\Big\}$$

BIBLIOGRAPHY

[1] C. Allday and V. Puppe. *Cohomological methods in transformation groups*, volume 32 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1993. ISBN 0-521-35022-0. doi: 10.1017/CBO9780511526275. URL http://dx.doi.org/10.1017/CBO9780511526275.

[2] L. Astey, S. Gitler, E. Micha, and G. Pastor. On the homotopy type of complete intersections. *Topology*, 44(1):249–260, 2005. ISSN 0040-9383. doi: 10.1016/j.top.2004.04.001. URL http://dx.doi.org/10.1016/j.top.2004.04.001.

[3] M. F. Atiyah and R. Bott. A Lefschetz fixed point formula for elliptic complexes. I. *Ann. of Math. (2)*, 86:374–407, 1967. ISSN 0003-486X.

[4] M. F. Atiyah and R. Bott. The moment map and equivariant cohomology. *Topology*, 23(1):1–28, 1984. ISSN 0040-9383. doi: 10.1016/0040-9383(84)90021-1. URL http://dx.doi.org/10.1016/0040-9383(84)90021-1.

[5] M. F. Atiyah and I. M. Singer. The index of elliptic operators. III. *Ann. of Math. (2)*, 87:546–604, 1968. ISSN 0003-486X.

[6] I. K. Babenko. Real homotopy properties of complete intersections. *Izv. Akad. Nauk SSSR Ser. Mat.*, 43(5):1004–1024, 1197, 1979. ISSN 0373-2436.

[7] Nicole Berline and Michèle Vergne. Classes caractéristiques équivariantes. Formule de localisation en cohomologie équivariante. *C. R. Acad. Sci. Paris Sér. I Math.*, 295(9):539–541, 1982. ISSN 0249-6321.

[8] André Blanchard. Sur les variétés analytiques complexes. *Ann. Sci. Ecole Norm. Sup. (3)*, 73:157–202, 1956. ISSN 0012-9593.

[9] Raoul Bott. On a theorem of Lefschetz. *Michigan Math. J.*, 6: 211–216, 1959. ISSN 0026-2285.

[10] Raoul Bott and Loring W. Tu. *Differential forms in algebraic topology*, volume 82 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1982. ISBN 0-387-90613-4.

[11] Glen E. Bredon. The cohomology ring structure of a fixed point set. *Ann. of Math. (2)*, 80:524–537, 1964. ISSN 0003-486X.

[12] Glen E. Bredon. *Introduction to compact transformation groups*. Academic Press, New York, 1972. Pure and Applied Mathematics, Vol. 46.

[13] Theodor Bröcker and Tammo tom Dieck. *Representations of compact Lie groups*, volume 98 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. ISBN 0-387-13678-9. Translated from the German manuscript, Corrected reprint of the 1985 translation.

[14] W. Browder and W. C. Hsiang. Some problems on homotopy theory manifolds and transformation groups. In *Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 2*, Proc. Sympos. Pure Math., XXXII, pages 251–267. Amer. Math. Soc., Providence, R.I., 1978.

[15] P. E. Conner and E. E. Floyd. Orbit spaces of circle groups of transformations. *Ann. of Math. (2)*, 67:90–98, 1958. ISSN 0003-486X.

[16] A. Dessai and M. Wiemeler. Complete intersections with s^1- action. August 2011. URL http://arxiv.org/abs/1108.5327.

[17] Wolfgang Ebeling. An example of two homeomorphic, nondiffeomorphic complete intersection surfaces. *Invent. Math.*, 99(3): 651–654, 1990. ISSN 0020-9910. doi: 10.1007/BF01234435. URL http://dx.doi.org/10.1007/BF01234435.

[18] John Ewing and Suresh Moolgavkar. Euler characteristics of complete intersections. *Proc. Amer. Math. Soc.*, 56:390–391, 1976. ISSN 0002-9939.

[19] Fuquan Fang. Topology of complete intersections. *Comment. Math. Helv.*, 72(3):466–480, 1997. ISSN 0010-2571. doi: 10.1007/s000140050028. URL http://dx.doi.org/10.1007/s000140050028.

[20] Fuquan Fang and Stephan Klaus. Topological classification of 4-dimensional complete intersections. *Manuscripta Math.*, 90(2): 139–147, 1996. ISSN 0025-2611. doi: 10.1007/BF02568299. URL http://dx.doi.org/10.1007/BF02568299.

[21] Fuquan Fang and Peng Shao. Complete intersections with metrics of positive scalar curvature. *C. R. Math. Acad. Sci. Paris*, 347 (13-14):797–800, 2009. ISSN 1631-073X. doi: 10.1016/j.crma.2009.03.033. URL http://dx.doi.org/10.1016/j.crma.2009.03.033.

[22] Fuquan Fang and Jianbo Wang. Homeomorphism classification of complex projective complete intersections of dimensions 5, 6 and 7. *Math. Z.*, 266(3):719–746, 2010. ISSN 0025-5874. doi:

10.1007/s00209-009-0597-5. URL http://dx.doi.org/10.1007/s00209-009-0597-5.

[23] E. E. Floyd. Fixed point sets of compact abelian Lie groups of transformations. *Ann. of Math. (2)*, 66:30–35, 1957. ISSN 0003-486X.

[24] Michael Hartley Freedman. The topology of four-dimensional manifolds. *J. Differential Geom.*, 17(3):357–453, 1982. ISSN 0022-040X. URL http://projecteuclid.org/euclid.jdg/1214437136.

[25] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002. ISBN 0-521-79160-X; 0-521-79540-0.

[26] Akio Hattori and Tomoyoshi Yoshida. Lifting compact group actions in fiber bundles. *Japan. J. Math. (N.S.)*, 2(1):13–25, 1976.

[27] Friedrich Hirzebruch. Der Satz von Riemann-Roch in Faisceau-theoretischer Formulierung: einige Anwendungen und offene Fragen. In *Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III*, pages 457–473. Erven P. Noordhoff N.V., Groningen, 1956.

[28] Friedrich Hirzebruch, Thomas Berger, and Rainer Jung. *Manifolds and modular forms*. Aspects of Mathematics, E20. Friedr. Vieweg & Sohn, Braunschweig, 1992. ISBN 3-528-06414-5. With appendices by Nils-Peter Skoruppa and by Paul Baum.

[29] Wu-yi Hsiang. *Cohomology theory of topological transformation groups*. Springer-Verlag, New York-Heidelberg, 1975. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 85.

[30] P. E. Jupp. Classification of certain 6-manifolds. *Proc. Cambridge Philos. Soc.*, 73:293–300, 1973.

[31] Shoshichi Kobayashi. *Transformation groups in differential geometry*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. ISBN 3-540-58659-8. Reprint of the 1972 edition.

[32] János Kollár, Karen E. Smith, and Alessio Corti. *Rational and nearly rational varieties*, volume 92 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2004. ISBN 0-521-83207-1. doi: 10.1017/CBO9780511734991. URL http://dx.doi.org/10.1017/CBO9780511734991.

[33] Matthias Kreck. Surgery and duality. *Ann. of Math. (2)*, 149(3):707–754, 1999. ISSN 0003-486X. doi: 10.2307/121071. URL http://dx.doi.org/10.2307/121071.

[34] Ravindra S. Kulkarni and John W. Wood. Topology of nonsingular complex hypersurfaces. *Adv. in Math.*, 35(3):239–263, 1980. ISSN 0001-8708. doi: 10.1016/0001-8708(80)90051-1. URL http://dx.doi.org/10.1016/0001-8708(80)90051-1.

[35] A. Libgober. Some properties of the signature of complete intersections. *Proc. Amer. Math. Soc.*, 79(3):373–375, 1980. ISSN 0002-9939. doi: 10.2307/2043069. URL http://dx.doi.org/10.2307/2043069.

[36] A. Libgober, J. Wood, and D. Zagier. Congruences modulo powers of 2 for the signature of complete intersections. *Quart. J. Math. Oxford Ser. (2)*, 31(122):209–218, 1980. ISSN 0033-5606. doi: 10.1093/qmath/31.2.209. URL http://dx.doi.org/10.1093/qmath/31.2.209.

[37] A. S. Libgober and J. W. Wood. On the topological structure of even-dimensional complete intersections. *Trans. Amer. Math. Soc.*, 267(2):637–660, 1981. ISSN 0002-9947. doi: 10.2307/1998676. URL http://dx.doi.org/10.2307/1998676.

[38] Anatoly S. Libgober and John W. Wood. Diffeomorphic complete intersections with different multidegrees. *Bull. Amer. Math. Soc. (N.S.)*, 2(3):459–461, 1980. ISSN 0273-0979. doi: 10.1090/S0273-0979-1980-14767-9. URL http://dx.doi.org/10.1090/S0273-0979-1980-14767-9.

[39] Anatoly S. Libgober and John W. Wood. Differentiable structures on complete intersections. I. *Topology*, 21(4):469–482, 1982. ISSN 0040-9383. doi: 10.1016/0040-9383(82)90024-6. URL http://dx.doi.org/10.1016/0040-9383(82)90024-6.

[40] Anatoly S. Libgober and John W. Wood. Differentiable structures on complete intersections. II. In *Singularities, Part 2 (Arcata, Calif., 1981)*, volume 40 of *Proc. Sympos. Pure Math.*, pages 123–133. Amer. Math. Soc., Providence, RI, 1983.

[41] Yu. I. Manin. *Cubic forms*, volume 4 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, second edition, 1986. ISBN 0-444-87823-8. Algebra, geometry, arithmetic, Translated from the Russian by M. Hazewinkel.

[42] Mikiya Masuda. S^1-actions on twisted $\mathbf{C}P^3$. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 31(1):1–31, 1984. ISSN 0040-8980.

[43] Willi Meier. Some topological properties of Kähler manifolds and homogeneous spaces. *Math. Z.*, 183(4):473–481, 1983. ISSN 0025-5874. doi: 10.1007/BF01173924. URL http://dx.doi.org/10.1007/BF01173924.

[44] John W. Milnor and James D. Stasheff. *Characteristic classes*. Princeton University Press, Princeton, N. J., 1974. Annals of Mathematics Studies, No. 76.

[45] Ignasi Mundet i Riera. Lifts of smooth group actions to line bundles. *Bull. London Math. Soc.*, 33(3):351–361, 2001. ISSN 0024-6093. doi: 10.1017/S0024609301007937. URL http://dx.doi.org/10.1017/S0024609301007937.

[46] Joseph Neisendorfer. The rational homotopy groups of complete intersections. *Illinois J. Math.*, 23(2):175–182, 1979. ISSN 0019-2082. URL http://projecteuclid.org/euclid.ijm/1256048231.

[47] Ted Petrie. Smooth S^1 actions on homotopy complex projective spaces and related topics. *Bull. Amer. Math. Soc.*, 78:105–153, 1972. ISSN 0002-9904.

[48] Volker Puppe. Do manifolds have little symmetry? *J. Fixed Point Theory Appl.*, 2(1):85–96, 2007. ISSN 1661-7738. doi: 10.1007/s11784-007-0021-x. URL http://dx.doi.org/10.1007/s11784-007-0021-x.

[49] P. A. Smith. Transformations of finite period. *Ann. of Math. (2)*, 39(1):127–164, 1938. ISSN 0003-486X. doi: 10.2307/1968718. URL http://dx.doi.org/10.2307/1968718.

[50] Claudia Traving. Klassifikation vollständiger Durchschnitte. Master's thesis, Mainz University, 1985.

[51] C. T. C. Wall. Classification problems in differential topology. V. On certain 6-manifolds. *Invent. Math. 1 (1966), 355-374; corrigendum, ibid*, 2:306, 1966. ISSN 0020-9910.

DECLARATION

By my signature, I hereby testify that I am the sole author of the written work *Circle actions on complete intersections*. I have compiled it in my own words on the basis of a personal work with no illicit help. Any ideas used in support of this work which are not originally my own are cited and referenced accordingly.

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