Exercises in Riemannian Geometry

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1 Smooth manifolds

1. Give an example of two non-commuting vector fields in $\mathbb{R}^2$.
2. Compute the commutator of the following vector fields in $\mathbb{R}^3$.

   (a)
   $$R_1(x) = \begin{pmatrix} 0 \\ x^3 \\ -x^2 \end{pmatrix}, \quad R_2(x) = \begin{pmatrix} -x^3 \\ 0 \\ x^1 \end{pmatrix}$$
   where (x^1, x^2, x^3) are coordinates in $\mathbb{R}^3$.

   (b) Linear vector fields $X(x) = Ax$ and $Y(x) = Bx$, where A and B are $n \times n$ matrices with constant coefficients.

   (c) The rotation vector fields $R_v(x) = x \times v$, $R_w(x) = x \times w$, where $v, w \in \mathbb{R}^3$, and $\times$ is the vector product.
3. Give an example of a 1-form on $\mathbb{R}^2$ that is not the differential of a function.
4. Compute the differential of the form
   $$\omega = \sum_{i=1}^{n} f_i \, dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \in \Omega^{n-1}(\mathbb{R}^n)$$
5. Give an example of a 1-form $\omega \in \Omega^1(\mathbb{R}^2)$ such that $d\omega = 0$, but $\omega \neq df$ for $f \in C^\infty(\mathbb{R}^2 \setminus \{0\})$.
6. Let M and N be smooth manifolds, $\dim M = m \leq n = \dim N$. For a smooth map $F \colon M \to N$ and a differential form $\omega \in \Omega^m(N)$ define the integral of ω along F as
   $$\int_F \omega = \int_M F^* \omega$$
   Show that for any diffeomorphism $\Phi \colon M \to M$ we have $\int_{F \circ \Phi} \omega = \int_F \omega$.

7. Assume that $\omega \in \Omega^1(M)$ is such that for all closed paths $\gamma\colon [0,1] \to M$ we have $\int_\gamma \omega = 0$. Show that $d\omega = 0$.

8*. Is the boundary of a non-orientable manifold always orientable?

9. Show that the tensors

$$e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes \eta^{j_1} \otimes \cdots \otimes \eta^{j_s} \in T^r_s(V)$$

with all indices ranging independently from 1 to n form a basis of $T^r_s(V)$. In particular, $\dim T^r_s(V) = n^{r+s}$.

10. Describe the natural isomorphism $T^1_2(V) \cong \mathrm{Hom}(V^*, V^* \otimes V^*)$.

11. For $\alpha, \beta, \gamma \in \mathrm{Hom}(V,V)$ write $\mathrm{tr}(\alpha \circ \beta \circ \gamma)$ in terms of $\otimes$ and tr. Show that $\mathrm{tr}(\alpha\circ\beta\circ\gamma) = \mathrm{tr}(\beta\circ\gamma\circ\alpha)$ but in general $\mathrm{tr}(\alpha\circ\beta\circ\gamma) \neq \mathrm{tr}(\alpha\circ\gamma\circ\beta)$.

12. Show that every $(0,3)$-tensor that is symmetric in one pair of arguments and antisymmetric in another pair is identically zero.

13. The divergence of a vector field X with respect to a volume form ω is defined by

$$d(i_X\omega) = (\operatorname{div} X)\omega.$$

Check that for $\omega = dx^1 \wedge \cdots \wedge dx^n$ and $X = \sum_i X^i \frac{\partial}{\partial x^i}$ we have $\operatorname{div} X = \sum_{i=1}^n \frac{\partial X^i}{\partial x^i}$.

14. Compute the divergence (with respect to the area form $dx \wedge dy = r dr \wedge d\varphi$) of a radially symmetric vector field $f(r)\partial_r$. For what functions f is this field divergence-free?

15. Since $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, every differential n-form $\omega \in \Omega^n(\mathbb{R}^{n+1})$ restricts to an n-form on $\mathbb{S}^n$ (the pullback along the inclusion map). Show that the form

$$\omega = \sum_{i=0}^{n} (-1)^i x^i dx^0 \wedge \cdots \hat{i} \cdots \wedge dx^n$$

restricts to a nowhere vanishing form on $\mathbb{S}^n$. (Hint: $\omega = i_X(dx^0 \wedge \cdots \wedge dx^n)$ for an appropriate vector field X on $\mathbb{R}^n$.)

16. (a) Give an example of two vector fields X and Y such that $X(p) = 0$ but $(L_X Y)(p) \neq 0$.
    (b) Show that if $X(p) = 0$ and $Y(p) = 0$, then $(L_X Y)(p) = 0$.

17*. Show that $[X,Y] = 0$ everywhere on M if and only if the flows of X and Y commute: $\varphi_t \circ \psi_s = \psi_s \circ \varphi_t$ for all t, s.

18. Let $\nabla^1, \ldots, \nabla^n$ be covariant derivatives. Show that their linear combination $\sum_{i=1}^n \lambda_i \nabla^i$ is a covariant derivative if $\sum_{i=1}^n \lambda_i = 1$.

19. Show that in any coordinate system the components of the torsion tensor are related to the Christoffel symbols via

$$T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj}.$$

(In particular, although Γ^i_{ij} don't form a tensor, their antisymmetrization does.)

20. Compute the length of the logarithmic spiral given in the polar coordinates by $r = e^{\varphi}$, $r \leq 1$.

21. Show that the covariant derivatives of $(0,1)$-tensor fields and of $(0,2)$-tensor fields satisfy the derivation property

$$\nabla_X(f\alpha) = X(f)\alpha + f \cdot \nabla_X \alpha.$$

22. Fill in the details in the proof of the existence of the Levi-Civita connection.

23. Let $M \subset N$ be an immersed submanifold, and let $\widetilde{\nabla}$ be a covariant derivative on N. Let $Y\colon M \to TN$ be a vector field along M (that is, $Y(p) \in T_pN$, but $Y(p)$ is defined only for $p \in M$). Show that for every $X \in T_pM$ the covariant derivative $\widetilde{\nabla}_X Y$ is well-defined, that is independent of a choice of an extension $\tilde{Y}$ of the vector field Y to N.

(Hint: use the local immersion theorem and a coordinate representation of $\widetilde{\nabla}$.)

24. Let $F\colon M \to N$ be a smooth map, and let X, Y be vector fields on M. Show that

$$dF([X,Y]) = [dF(X), dF(Y)].$$

In particular, the right hand side is independent of a choice of extensions of the vector fields $dF(X)$ and $dF(Y)$.