Problems in differential geometry

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1. Consider the curve

$$\gamma(t) = (\cos t + \log \tan \frac{t}{2}, \sin t), \quad t \in (0, \pi)$$

   (a) Find the angle between the tangent to γ at time t_0 and the x-axis.

   (b) Show that the segment between the point of tangency and the intersection with the x-axis has the same length for all tangents. (Because of that, this curve is called the *tractrix* or the donkey curve.)

   (c) Compute the curvature of γ at time t_0. *Hint:* In (a) you computed the angular velocity of the tangent vector and the linear velocity of the point on the curve.

   (d) Show that the evolute of the tractrix is the curve $y = \cosh x$.

2. Let C_1 and C_2 be two smooth convex closed curves of equal length and with coincident Steiner points. Show that they intersect at least four times.

   Additional question: Is this true if we assume that not the Steiner points but the barycenters of C_1 and C_2 coincide? (I guess not.)

3. A *loxodrome* is a curve on the sphere that intersects each meridian at a given angle α.

   (a) Compute the length of the loxodrome.

   (b) Compute the curvature and the torsion of the loxodrome.

4. Show that the Mercator parametrization of the sphere

$$\sigma(u, v) = \left(\frac{\cos v}{\cosh u}, \frac{\sin v}{\cosh u}, \tanh u \right)$$

   is conformal. What is the image of the loxodrome in the (u, v)-plane?

5. Let $\gamma \colon [a, b] \to \mathbb{R}^3$ be a unit-speed space curve. Find the area of the boundary of the tubular ϵ-neighborhood of γ.

   Hint: in order to parametrize the boundary of the tube, use a moving orthonormal frame along γ. Since γ is not necessarily a Frenet curve, you might take an arbitrary frame (e_1, e_2, e_3) with $e_1 = \dot{\gamma}$.

6. Compute the first and the second fundamental forms of the graph $z = F(x, y)$. Compute the principal curvatures of the hyperbolic paraboloid $z = x^2 - y^2$.

7. A triply orthogonal system of surfaces consists of three families of surfaces such that the surfaces from different families are orthogonal to each other. Show that the intersection lines of two surfaces from a triply orthogonal system are the curvature lines of each of the surfaces.

8. Let $M \subset \mathbb{R}^3$ be a minimal surface, and $\Gamma\colon M \to \mathbb{S}^2$ the Gauss map. Show that Γ is conformal, and Γ^{-1} is harmonic (that is, each component of $\Gamma^{-1}\colon \mathbb{S}^2 \to M \subset \mathbb{R}^3$ is harmonic with respect to the sphere Laplacian). Is Γ harmonic?

9. Assume that a surface patch $\sigma(s, t)$ has the first fundamental form $dt^2 + f^2(t)d\theta^2$. Does the image of σ necessarily lie on a surface of revolution? Is the image of σ necessarily isometric to a surface of revolution?