Kokotsakis polyhedra and elliptic functions

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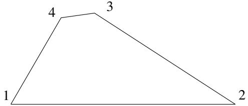
FU Berlin

Algebraic Topology and Abelian Functions
Moscow, June 19, 2013

Dynamics of the quadrilateral folding

Take a quadrilateral in $\mathbb{R}^2$.



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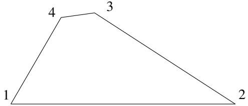

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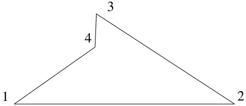

$F_{13}(Q)$

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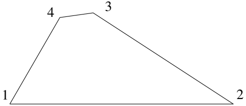

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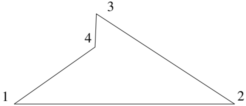

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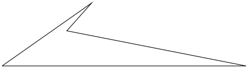
Now fold along 24.



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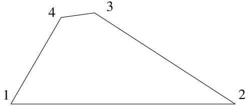

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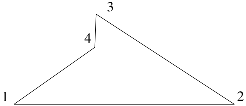

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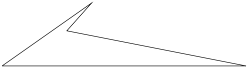

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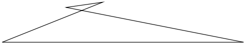
Repeat.



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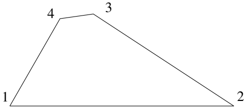

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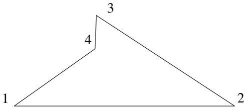

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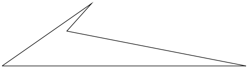

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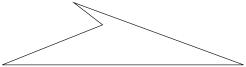

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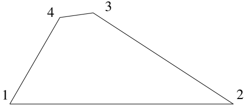

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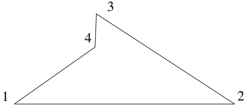

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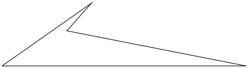

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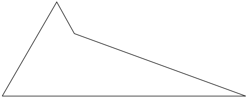

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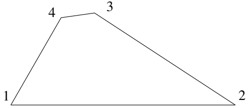

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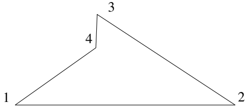

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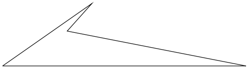

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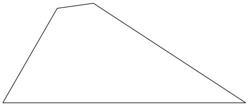

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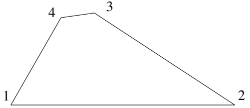

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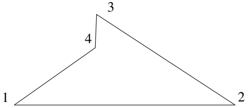

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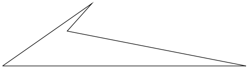

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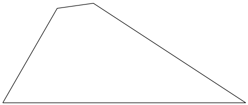


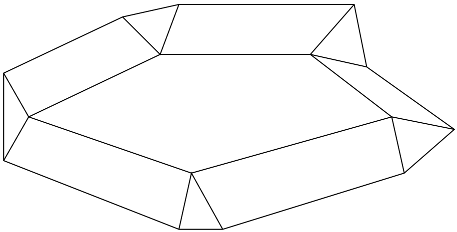
$(F_{24} \circ F_{13})^3(Q)$

Theorem (Darboux, 1879)

If $(F_2 \circ F_1)^n(Q) = Q$ for some n, then $(F_2 \circ F_1)^n(Q') = Q'$ for all Q' with the same edge lengths as Q.

Kokotsakis polyhedra

A Kokotsakis polyhedron consists of an *n*-gon, *n* quadrilaterals at its edges, and *n* triangles between the quadrilaterals.



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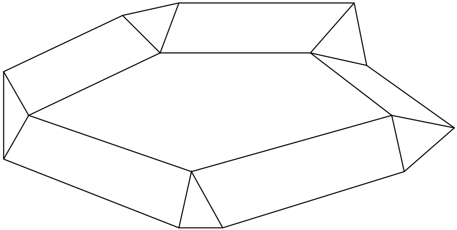

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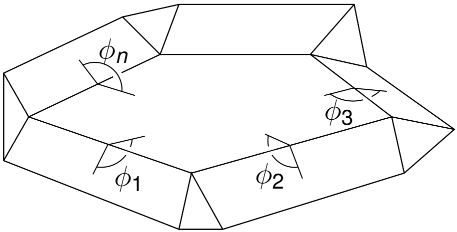

Generically, a Kokotsakis polyhedron is rigid. Adjacent dihedral angles depend on each other:

$$\phi_2 = f_1(\phi_1), \quad \phi_3 = f_2(\phi_2), \quad \ldots, \quad \phi_1 = f_n(\phi_n)$$

Thus, a polyhedron is flexible $\Leftrightarrow f_n \circ \cdots \circ f_1 = \mathrm{id}$.

History

Kokotsakis (1932): characterization of infinitesimal flexibility,
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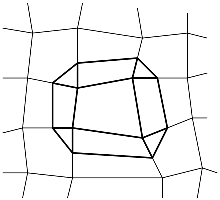
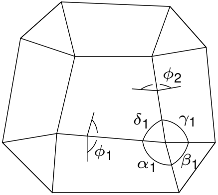
For $n = 4$ the Kokotsakis polyhedron is a piece of a quad-surface.
The surface is flexible $\Leftrightarrow$ every Kokotsakis subpolyhedron is flexible.



Discrete Voss surfaces



If $\alpha_1 = \gamma_1$, $\beta_1 = \delta_1$, then the dependence between ϕ_1 and ϕ_2 has the form

$$\tan \frac{\phi_1}{2} \tan \frac{\phi_2}{2} = c_1^{\pm},$$

for some constants c_1^+ and c_1^-.

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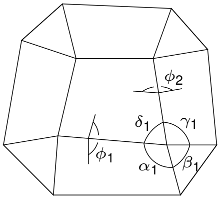

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Hence if $\alpha_i = \gamma_i$ and $\beta_i = \delta_i$ for all i, and $c_1^{\pm} c_3^{\pm} = c_2^{\pm} c_4^{\pm}$, then the polyhedron is flexible.

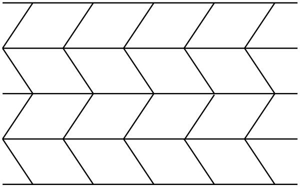
A quad-surface made of such polyhedra is called discrete Voss surface (Graf and Sauer, 1931). A smooth Voss surface (1888) has a conjugate net of geodesics.

Origami

If $\alpha_i + \gamma_i = \pi = \beta_i + \delta_i$, then the situation is similar, the relation being

$$\frac{\tan \frac{\phi_i}{2}}{\tan \frac{\phi_{i+1}}{2}} = c_i^{\pm}$$

Thus $c_1^{\pm} c_2^{\pm} c_3^{\pm} c_4^{\pm} = 1 \Leftrightarrow$ flexibility.



Miura-Ori.

You might also want to google for the “distorted Miura-ori”.

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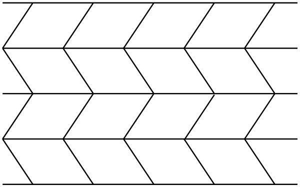

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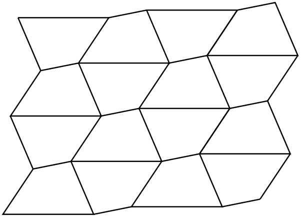
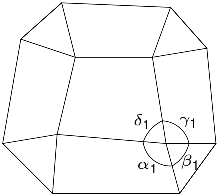
Another surface that can be folded from a piece of paper was found by Kokotsakis.



Reduction to a spherical linkage

Planar angles at a vertex = side lengths of a spherical quadrilateral.
Dihedral angles = angles of the spherical quadrilateral.



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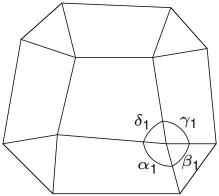

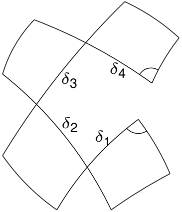

Each pair of adjacent vertices has a common dihedral angle.
Have a chain of 4 quadrilaterals with $\delta_1 + \delta_2 + \delta_3 + \delta_4 = 2\pi$.

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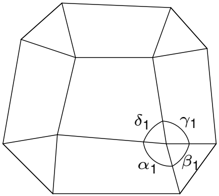

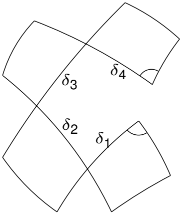

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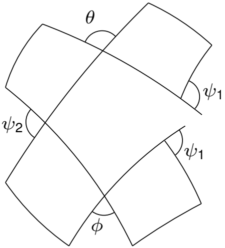
A Kokotsakis polyhedron is flexible $\Leftrightarrow$ the associated spherical linkage flexes so that the marked angles remain equal.

Reduction to an algebraic problem



Change the variables:

$$z = \tan\frac{\phi}{2}, \qquad u = \tan\frac{\theta}{2},$$
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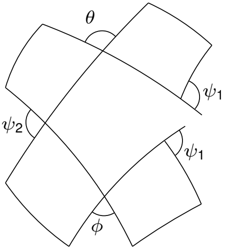

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Relations between adjacent angles become polynomial:

$$P_1(z, w_1) = 0, \quad P_2(z, w_2) = 0, \quad P_3(u, w_1) = 0, \quad P_4(u, w_2) = 0$$

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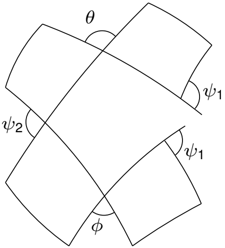

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A Kokotsakis polyhedron is flexible $\Leftrightarrow$ the system has a one-parameter set of solutions.

Approaches to the algebraic problem

1) Elimination method:
$R_{12}(w_1, w_2) :=$ the resultant of $P_1(z, w_1)$ and $P_2(z, w_2)$
$R_{34}(w_1, w_2) :=$ the resultant of $P_3(u, w_1)$ and $P_4(u, w_2)$
A polyhedron is flexible $\Leftrightarrow$ R_{12} and R_{34} have a common factor.
Nawratil, Stachel 2010-2012: the case of a reducible R_{12}

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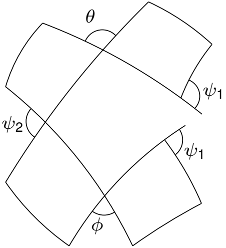
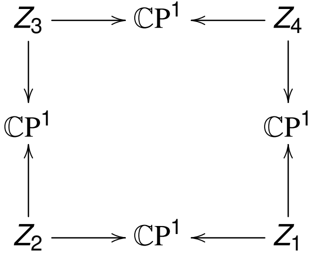
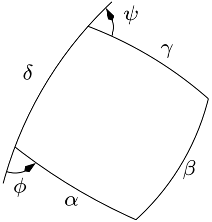
2) Study of a diagram of branched covers:





Compare with Ritt's 1922 solution of the decomposability $f = p_n \circ \cdots \circ p_1$ and commutation $f \circ g = g \circ f$ problems for polynomials.

The configuration space of a spherical quadrilateral



Recall: $z = \tan \frac{\phi}{2}, \; w = \tan \frac{\psi}{2}$.

Then $P(z, w) = 0$ has the form

$$c_{22} z^2 w^2 + c_{20} z^2 + c_{02} w^2 + 2 c_{11} z w + c_{00} = 0$$

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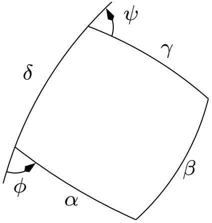

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Actually, this is a curve in $(\mathbb{RP}^1)^2$:

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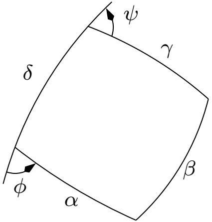

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Actually, this is a curve in $(\mathbb{RP}^1)^2$:

$$c_{22}z_1^2w_1^2 + c_{20}z_1^2w_0^2 + c_{02}z_0^2w_1^2 + 2c_{11}z_1w_1z_0w_0 + c_{00}z_0^2w_0^2 = 0$$

The shape of the configuration space depends on the number (and the type) of solutions of the congruence

$$\alpha \pm \beta \pm \gamma \pm \delta \equiv 0 (\mathrm{mod}\, 2\pi)$$

The coefficients

$$c_{22}z^2w^2 + c_{20}z^2 + c_{02}w^2 + 2c_{11}zw + c_{00} = 0$$

$$\begin{aligned}
c_{22} &= \sin\frac{\alpha+\beta+\gamma-\delta}{2}\sin\frac{\alpha-\beta+\gamma-\delta}{2}\\
c_{20} &= \sin\frac{\alpha-\beta-\gamma-\delta}{2}\sin\frac{\alpha+\beta-\gamma-\delta}{2}\\
c_{02} &= \sin\frac{\alpha+\beta-\gamma+\delta}{2}\sin\frac{\alpha-\beta-\gamma+\delta}{2}\\
c_{11} &= -\sin\alpha\sin\gamma\\
c_{00} &= \sin\frac{\alpha-\beta+\gamma+\delta}{2}\sin\frac{\alpha+\beta+\gamma+\delta}{2}
\end{aligned}$$

The coefficients

$$c_{22} z^2 w^2 + c_{20} z^2 + c_{02} w^2 + 2 c_{11} z w + c_{00} = 0$$

$$c_{22} = \sin \frac{\alpha + \beta + \gamma - \delta}{2} \sin \frac{\alpha - \beta + \gamma - \delta}{2} = \sin \overline{\delta} \sin(\sigma - \beta - \delta)$$

$$c_{20} = \sin \frac{\alpha - \beta - \gamma - \delta}{2} \sin \frac{\alpha + \beta - \gamma - \delta}{2} = \sin \overline{\alpha} \sin(\sigma - \beta - \alpha)$$

$$c_{02} = \sin \frac{\alpha + \beta - \gamma + \delta}{2} \sin \frac{\alpha - \beta - \gamma + \delta}{2} = \sin \overline{\gamma} \sin(\sigma - \beta - \gamma)$$

$$c_{11} = - \sin \alpha \sin \gamma$$

$$c_{00} = \sin \frac{\alpha - \beta + \gamma + \delta}{2} \sin \frac{\alpha + \beta + \gamma + \delta}{2} = \sin \overline{\beta} \sin \sigma$$

Here $\sigma = \frac{\alpha+\beta+\gamma+\delta}{2}$, $\overline{\alpha} = \sigma - \alpha = \frac{-\alpha+\beta+\gamma+\delta}{2}$ etc.

Some trigonometric identities

For any $\alpha, \beta, \gamma, \delta$ and $\sigma = \frac{\alpha+\beta+\gamma+\delta}{2}$, $\overline{\alpha} = \sigma - \alpha = \frac{-\alpha+\beta+\gamma+\delta}{2}$ etc. the following holds.

$$\overline{\alpha} + \overline{\beta} = \gamma + \delta$$

$$\sin\overline{\alpha}\sin\overline{\beta} - \sin\alpha\sin\beta = \sin\sigma\sin(\sigma-\alpha-\beta) = \sin\gamma\sin\delta - \sin\overline{\gamma}\sin\overline{\delta}$$

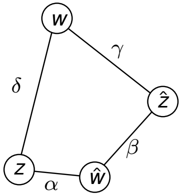
$$\sin\alpha\sin\beta - \sin\overline{\gamma}\sin\overline{\delta} = \sin(\sigma-\alpha-\gamma)\sin(\sigma-\beta-\gamma)$$
$$= \sin\overline{\alpha}\sin\overline{\beta} - \sin\gamma\sin\delta$$

$$\sin\sigma\sin(\sigma-\alpha-\beta)\sin(\sigma-\beta-\gamma)\sin(\sigma-\alpha-\gamma)$$
$$= \sin\alpha\sin\beta\sin\gamma\sin\delta - \sin\overline{\alpha}\sin\overline{\beta}\sin\overline{\gamma}\sin\overline{\delta}$$

Branched covers

Let $Z \subset (\mathbb{C}\mathrm{P}^1)^2$ be the complexified configuration space.

Pick an irreducible component Z^0 of Z.



$$\mathbb{C}\mathrm{P}^1 \xleftarrow{f} Z^0 \xrightarrow{g} \mathbb{C}\mathrm{P}^1$$
$$z \leftarrow\!\shortmid (z, w) \mapsto w$$

The map f is either two-fold or isomorphism.
Branch points of f correspond to $\hat{z} \in \{0, \infty\}$.

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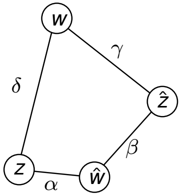

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$$z \leftarrow\!\shortmid (z, w) \mapsto w$$

The map f is either two-fold or isomorphism. Branch points of f correspond to $\hat{z} \in \{0, \infty\}$.

- If $\alpha \pm \beta \pm \gamma \pm \delta \neq 0 (\mathrm{mod}\, 2\pi)$, then both f and g are two-fold with 4 branch points. Hence Z is an elliptic curve.
- If the congruence has a unique solution, then both f and g are two-fold with 2 branch points (and one removable singularity).
- If $\alpha = \delta, \beta = \gamma$, then f is two-fold, and g isomorphism. Similarly for $\alpha + \delta = \pi = \beta + \gamma$. (There is also a trivial component $z = \infty$ or $z = 0$.)
- If $\alpha = \gamma, \beta = \delta$, then $Z = \{zw = \kappa_1\} \cup \{zw = \kappa_2\}$.

Tetragonometry

If $\alpha + \beta + \gamma + \delta = 2\pi$ is the unique solution, then

$$z^{-1} = p \sin t, \quad w^{-1} = q \sin(t + t_0),$$

$$p = \sqrt{\frac{\sin\alpha \sin\delta}{\sin\beta \sin\gamma} - 1}, \quad q = \sqrt{\frac{\sin\gamma \sin\delta}{\sin\alpha \sin\beta} - 1}, \quad \tan t_0 = i\sqrt{\frac{\sin\beta \sin\delta}{\sin\alpha \sin\gamma}}$$

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If $\alpha \pm \beta \pm \gamma \pm \delta \neq 0 (\bmod 2\pi)$, then

$$z = p \,\mathrm{sn}\, t, \; w = q \,\mathrm{sn}(t + t_0), \quad \text{if } \Pi < \overline{\Pi}$$

$$z = p \,\mathrm{cn}\, t, \; w = q \,\mathrm{cn}(t + t_0), \quad \text{if } \Pi > \overline{\Pi}$$

$$p = \sqrt{\frac{\sin\alpha \sin\delta}{\sin\overline{\alpha} \sin\overline{\delta}} - 1}, \quad q = \sqrt{\frac{\sin\gamma \sin\delta}{\sin\overline{\gamma} \sin\overline{\delta}} - 1}$$

Tetragonometry, continued

If $\alpha \pm \beta \pm \gamma \pm \delta \neq 0 (\mathrm{mod}\, 2\pi)$, then

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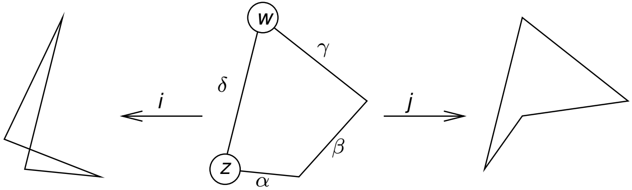
Here $\alpha = \sigma - \alpha = \frac{-\alpha+\beta+\gamma+\delta}{2}$, etc.

$$\Pi = \sin\alpha \sin\beta \sin\gamma \sin\delta, \quad \overline{\Pi} = \sin\overline{\alpha} \sin\overline{\beta} \sin\overline{\gamma} \sin\overline{\delta}$$

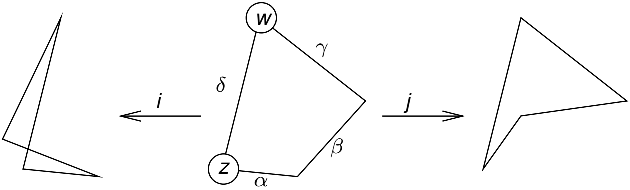
$$k = \begin{cases} \sqrt{1 - \Pi/\overline{\Pi}}, & \text{if } \Pi < \overline{\Pi}, \\ \sqrt{1 - \overline{\Pi}/\Pi}, & \text{if } \Pi > \overline{\Pi} \end{cases} \qquad \mathrm{dn}\, t_0 = \begin{cases} i\sqrt{\frac{\sin\alpha \sin\gamma}{\sin\overline{\alpha} \sin\overline{\gamma}}}, & \text{if } \Pi < \overline{\Pi}, \\ i\sqrt{\frac{\sin\overline{\alpha} \sin\overline{\gamma}}{\sin\alpha \sin\gamma}}, & \text{if } \Pi > \overline{\Pi} \end{cases}$$

Quadrilateral folding, revisited



Folding along a diagonal is the deck transformation of $Z \to \mathbb{CP}^1$.

Quadrilateral folding, revisited

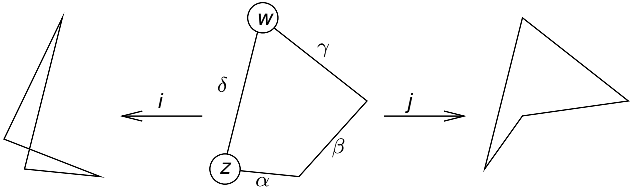


Folding along a diagonal is the deck transformation of $Z \to \mathbb{C}\mathrm{P}^1$.

$$z = pF(t), \; w = qF(t + t_0) \;\Rightarrow\; i \circ j(t) = t + 2t_0$$

Hence, if $(i \circ j)^n$ has a fixed point, then $(i \circ j)^n = \mathrm{id}$.
This proves the Darboux theorem from the first slide (since for Euclidean quadrilaterals there are similar formulas).

Quadrilateral folding, revisited



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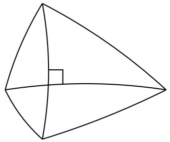
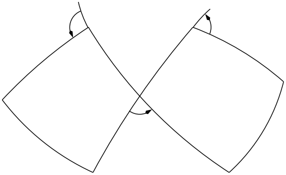


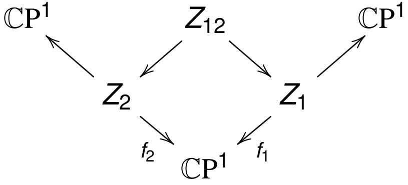
A special class of quadrilaterals:

$$\cos\alpha \cos\gamma = \cos\beta \cos\delta \Leftrightarrow \text{ orthodiagonal}$$

$$\Leftrightarrow (i \circ j)^2 = \mathrm{id} \Leftrightarrow t_0 \text{ is a quarter-period}$$

The configuration space of two coupled quadrilaterals





$$Z_{12} = \{(w_1, z, w_2) \mid P_1(z, w_1) = 0, P_2(z, w_2) = 0\}$$
$$= \{(x_1, x_2) \in Z_1 \times Z_2 \mid f_1(x_1) = f_2(x_2)\}$$

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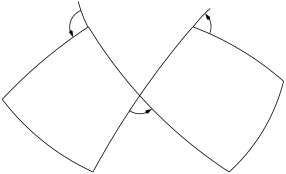

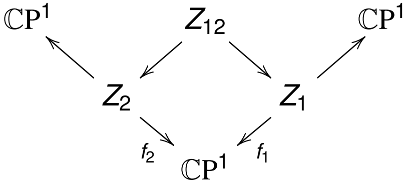

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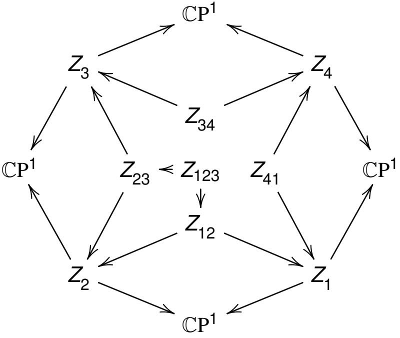
Z_{12} is reducible $\Leftrightarrow \mathrm{Br}(f_1) = \mathrm{Br}(f_2)$

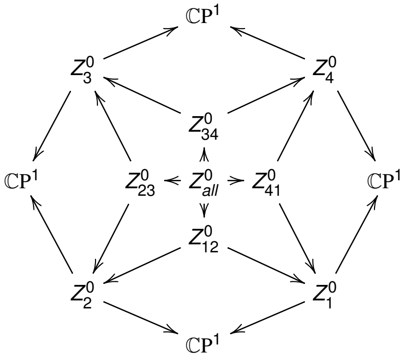
Parametrizations of Z_1 and Z_2 help to describe all reducible couplings.

The big diagram



$$Z_{123} = \{(w_1, z, w_2, u) \mid P_1 = 0, P_2 = 0, P_3 = 0\}$$

The big diagram



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All Z_{ijk} have a common irreducible component Z_{all}^0.

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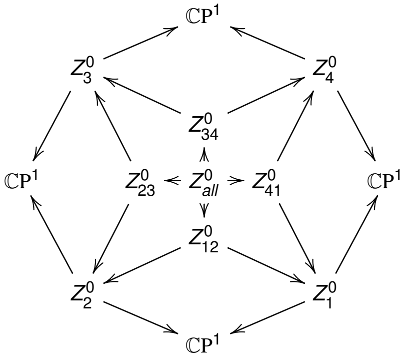

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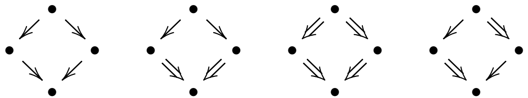
All Z_{ijk} have a common irreducible component Z_{all}^0.

Every map is either two-fold or an isomorphism; all squares are pullback squares

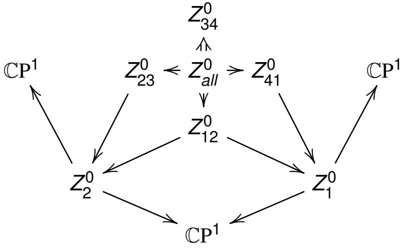


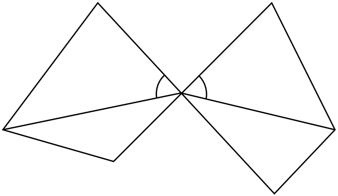
If some $Z^0_{all} \to Z^0_{ij}$ is two-fold...

If $Z^0_{all} \to Z^0_{34}$ is two-fold, then its deck transformation preserves w_1, u, w_2 while changing z.

This descends to an involution on Z^0_{12}:

$$(w_1, z, w_2) \mapsto (w_1, z', w_2)$$





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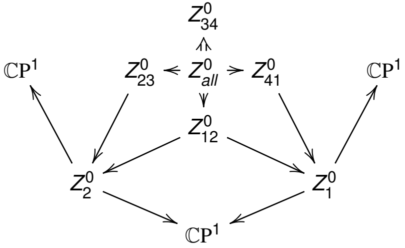

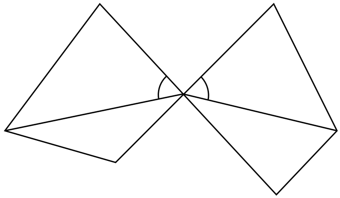

Lemma

If Z^0_{12} carries an involution $(w_1, z, w_2) \mapsto (w_1, z', w_2)$, then either both quadrilaterals are orthodiagonal, or the coupling is reducible.

If no $Z^0_{all} \to Z^0_{ij}$ is two-fold...

... then $Z^0_{12} \to Z_1$ and $Z^0_{41} \to Z_1$ are equivalent

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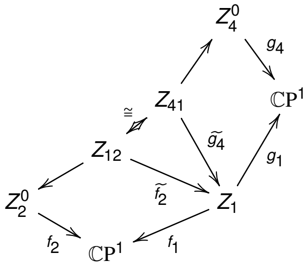

Assume all maps are two-fold. Then $\mathrm{Br}(\widetilde{f}_2) = \mathrm{Br}(\widetilde{g}_4) \Rightarrow$

$$f_1^{-1}(\mathrm{Br}(f_2)) = g_1^{-1}(\mathrm{Br}(g_4)) =: C$$

Hence $i_1(C) = C = j_1(C)$ (and also $-C = C$).

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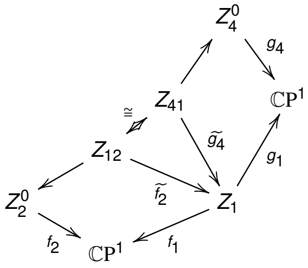

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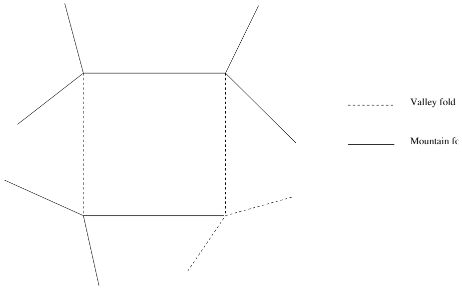
Hence $i_1(C) = C = j_1(C)$ (and also $-C = C$).

On the other hand, C consists of not more than 8 points.

For handcrafters

1) Build your own flexible Kokotsakis polyhedron:



2) Draw a euclidean quadriateral with the folding period 3. For this, choose side lengths satisfying

$$a^2c^2 - b^2d^2 = bd(-a^2 + b^2 - c^2 + d^2)$$

(the position of one of the sides a or c remains fixed). Examples:

$$(1, 6, 12, 8) \qquad (1, 3, 3\sqrt{5}, 5)$$