Extension of colorings

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Abstract

Let K be a combinatorial $(d-1)$-sphere with vertices colored in n colors, $n \geq d+1$. We prove that K bounds a n-colored combinatorial ball. This theorem generalizes previously known facts for $d = 2$ and 3. A further generalization is obtained. Namely, let L be a simplicial complex of dimension d and K be a subcomplex of L. Then any vertex coloring of K in $n \geq d+1$ colors extends to some subdivision of L relative to K. Besides, in both cases the extension can be required to use only $d+1$ of n colors in the complement to K.

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The following nice theorem has appeared independently in works of several mathematicians in the late 70's [1], [2], [3]:

Theorem 1 *Let a triangulation of S^2 be given. Then its vertices can be colored in 4 colors iff it can be extended to an even triangulation of the ball B^3. Here a triangulation of B^3 is called even iff every inner edge is surrounded by an even number of tetrahedra.*

Recently Michael Joswig and the author have rediscovered this fact, making use of it in [4].

First, note that a 4-colorable triangulation of B^3 is even. Thus the "only if" part of Theorem 1 is an implication of the following: Any 4-colored triangulation of S^2 extends to a 4-colored triangulation of B^3. In the present note we generalize Theorem 1 in two ways. Theorem 2 asserts that the same is true for any n-coloring of a combinatorial $(d-1)$-sphere, when $n \geq d+1$. Besides, one can choose in advance from the given n colors any $d+1$ and use only these in the interior of the ball. Theorem 3 yields an ultimate generalization saying

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that any colored triangulation of a subcomplex can be extended to a colored triangulation of the complex after subdividing the latter (but not affecting the former). The same restrictions on the use of colors outside the subcomplex can be imposed. The proof of Theorem 3 goes by extension of the coloring from the $(i-1)$-skeleton to the i-skeleton, using Theorem 2 as a lemma. This imitates the extension technique from PL-topology.

We mention two other results similar to ours. Steve Fisk showed in [2, Lemma 57] that any $(d+1)$-coloring of the boundary of a triangulated d-manifold can be extended to the interior after suitable subdivision. Recently, Nikolaus Witte [6] obtained independently a result close to Theorem 2. He proved that an n-coloring of a $(d-1)$-sphere can be extended to an n-coloring of a d-ball, provided $n \geq d+1$.

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1 Extension from a sphere to a ball

Denote by $\mathrm{Vert}(K)$ the set of vertices of a simplicial complex K. A map $f : \mathrm{Vert}(K) \to \{1, \ldots, n\}$ is called a *coloring* of K in n colors iff any two vertices of K joined by an edge have different colors, that is different images under f.

A simplicial complex is called a combinatorial ball (resp. a combinatorial sphere) iff it is piecewise linearly homeomorphic to a simplex (resp. to the boundary of a simplex). A combinatorial manifold with boundary is a simplicial complex K such that the link of any vertex is a combinatorial sphere or a combinatorial ball. Although it is not obvious, the notion of combinatorial manifold is PL invariant (see [5, Theorem II.3]) and hence any combinatorial sphere or combinatorial ball is a combinatorial manifold as well.

In dimensions up to 3 these matters are simpler, namely, any triangulation of a topological manifold is a combinatorial manifold. Thus, in the following theorem for $d \leq 4$ instead of "a combinatorial (sphere, ball)" one can read "a triangulation of a (sphere, ball)".

Theorem 2 *Let* $f : \mathrm{Vert}(K) \to \{1, \ldots, n\}$ *be a coloring of a combinatorial* $(d-1)$*-sphere* K, $n \geq d+1$. *Then there is a combinatorial ball* L *with a coloring* $g : \mathrm{Vert}(L) \to \{1, \ldots, n\}$ *such that* $\partial L = K$ *and* $g|_{\mathrm{Vert}(K)} = f$ *and also* $g(\mathrm{Vert}(L) \setminus \mathrm{Vert}(K)) \subset \{1, \ldots, d+1\}$.

Of course, instead of $\{1, \ldots, d+1\}$ any other $d+1$ colors can be chosen.

Proof The theorem will be proved by induction on d.

Induction base. For $d = 0$ and $d = 1$ this is trivial.

Induction step. Firstly, if K has no vertex of color $d + 1$, then take the cone over K and assign to the apex the color $d+1$. Otherwise choose in K a vertex p of color $d+1$. The link of p in K is a colored combinatorial $(d-2)$-sphere and is by the induction hypothesis the boundary of a colored combinatorial $(d-1)$-ball N such that the interior vertices of N have colors from the set $\{1, \dots, d\}$. Consider a colored simplicial complex K', obtained from K by replacing the star of p with N. Then K' is also a combinatorial sphere. Apply now an embedded induction argument on the number of vertices of color $d+1$. Indeed, K' has one vertex of that sort less than K, and if, by induction hypothesis, L' is a colored combinatorial ball which bounds K', then we put $L = L' \cup pN$, where pN is the cone over N with apex p. It can be shown (see [5, Corollary II.16 and Theorem II.17]) that L is again a combinatorial ball. The base of this induction argument is the zero number of vertices of color $d + 1$ and was already treated. □

2 General case

A simplicial pair (L, K) consists of a simplicial complex L and a subcomplex K. We will consider subdivisions of L which preserve K as a subcomplex. If L' is a subdivision with this property, then we say that the simplicial pair (L', K) is a subdivision of the simplicial pair (L, K).

Theorem 3 *Let (L, K) be a simplicial pair, $d = \dim L$. Let a coloring $f : \mathrm{Vert}(K) \to \{1, \dots, n\}$ be given with $n \geq d+1$. Then there exists a subdivision (N, K) of (L, K) and a coloring $g : \mathrm{Vert}(N) \to \{1, \dots, n\}$ with $g|_{\mathrm{Vert}(K)} = f$ and $g(\mathrm{Vert}(L) \setminus \mathrm{Vert}(K)) \subset \{1, \dots, d + 1\}$.*

Proof Denote by L_i the i-th relative skeleton of the pair (L, K), that is $L_i = K \cup \{\sigma \in L |\ \dim\ \sigma \leq i\}$. Let us subdivide and color the complex $L = L_d$ recursively.

As the starting point take $N_{-1} = L_{-1} = K$ and $g_{-1} = f$. Suppose we have constructed a subdivision (N_{i-1}, K) of (L_{i-1}, K) and a coloring $g_{i-1} : \mathrm{Vert}(N_{i-1}) \to \{1, \dots, n\}$ extending f. Then for each simplex $\sigma \in L \setminus K$ with $\dim\ \sigma = i$ the boundary $\partial\sigma$ is subdivided by N_{i-1} and colored by g_{i-1}. Thus by Theorem 2 we can extend g_{i-1} onto some subdivision of σ using only colors from $\{1, \dots, i + 1\}$. Having done this for all i-dimensional simplices of $L \setminus K$, we get a colored subdivision N_i of L_i. □

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