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Three-Dimensional Manifolds Defined by Coloring a Simple Polytope

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Abstract—In the present paper we introduce and study a class of three-dimensional manifolds endowed with the action of the group $\mathbb{Z}_2^3$ whose orbit space is a simple convex polytope. These manifolds originate from three-dimensional polytopes whose faces allow a coloring into three colors with the help of the construction used for studying quasitoric manifolds. For such manifolds we prove the existence of an equivariant embedding into Euclidean space $\mathbb{R}^4$. We also describe the action on the set of operations of the equivariant connected sum and the equivariant Dehn surgery. We prove that any such manifold can be obtained from a finitely many three-dimensional tori with the canonical action of the group $\mathbb{Z}_2^3$ by using these operations.

Key words: *three-dimensional manifold, convex polytope, coloring into s colors, equivariant embedding, equivariant connected sum, equivariant Dehn surgery.*

1. INTRODUCTION AND MAIN DEFINITIONS

Suppose P is a simple convex polytope with m hyperfaces $F_1, \dots, F_m$. Let G be any of the groups $\mathbb{Z}_2$ or $\mathbb{S}^1$. By G_i denote the ith coordinate subgroup of the group G^m.

Definition 1 (see [1, 2]). Consider the following equivalence relation on $G^m \times P$:

$$(g, p) \sim (g', p') \iff p' = p, \quad g'g^{-1} \in \bigoplus_{F_i \ni p} G_i. \tag{1}$$

Then the quotient space

$$\mathcal{Z}_P = (G^m \times P)/\sim \tag{2}$$

is called the *manifold defined by the polytope* P and is denoted by $\mathcal{Z}_P$.

To check that the space $\mathcal{Z}_P$ is a manifold, one can consider the local structure of the space (2). The formula $h[g,p] = [hg,p]$ defines a natural G^m-action on $\mathcal{Z}_P$. By construction, the orbit space of this action is identified with the polytope P.

A number of manifolds arises as the result of the following construction. Suppose there exists a monomorphism $\iota\colon G^k \to G^m$ such that the corresponding action $h[g,p] = [\iota(h)g,p]$ of the group G^k on the manifold $\mathcal{Z}_P$ is free. Then the homogeneous space $\mathcal{Z}_P/G^k$ is a manifold endowed with the action of the group $G^{m-k} \cong G^m/\iota(G^k)$ and with the polytope P as the orbit space. Let us denote for $k^*(P)$ the maximal k such that there exists a free G^k-action on $\mathcal{Z}_P$ of the form described above. V. M. Bukhshtaber proposed to consider the number $k^*(P)$ as a new combinatorial invariant of the simple polytope P. It can be shown that $1 \le k^*(P) \le m-n$, where n stands for the dimension of the polytope P. Especially interesting are the polytopes P with $k^*(P) = m-n$. In this case for $G = \mathbb{S}^1$ the *toric manifold* $M^{2n} = \mathcal{Z}_P/\mathbb{T}^{m-n}$ arises. Toric manifolds possess numerous remarkable properties (see [1]).

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We are interested in the special case of the construction described in the previous paragraph. Consider a surjective map $\chi\colon \{1,\dots,m\}\to\{1,\dots,s\}$. It defines an epimorphism $\widetilde{\chi}\colon G^m\to G^s$ satisfying the condition $\widetilde{\chi}(G_i)=G_{\chi(i)}$. Take any monomorphism $G^{m-s}\to G^m$ with the image equal to the kernel of $\widetilde{\chi}$. It is easily proved that the G^{m-s}-action on $\mathcal{Z}_P$ defined by ι is free whenever $F_i\cap F_j\neq\varnothing$ implies $\chi(i)\neq\chi(j)$ for any two different hyperfaces of P. If this condition holds, then the corresponding map $F_i\mapsto\chi(i)$ is called a *coloring into s colors* of the set of the polytope's hyperfaces. Thus we can speak of the manifold $\mathcal{Z}(P,\chi)$ defined by a coloring. This method allows us to obtain a lower bound for k^* in the following way.

Let us denote by $\gamma(K)$ the *vertex chromatic number* of a graph K as the minimal number of colors needed for a coloring of the graph's vertices under the condition that any two adjacent vertices have different colors. Then we have

$$k^*(P)\geq m-\gamma(K_1(P)),$$

where $K_1(P)$ is the adjacency graph of the polytope P.

The manifold $\mathcal{Z}(P,\chi)$ can be defined more explicitly as the quotient space of the Cartesian product $G^s\times P$ by the equivalence relation (1), where the condition $g'g^{-1}\in\bigoplus_{F_i\ni p}G_i$ is replaced by $g'g^{-1}\in\widetilde{\chi}(\bigoplus_{F_i\ni p}G_i)$. Obviously, the latter is equivalent to $g'g^{-1}\in\bigoplus_{F_i\ni p}G_{\chi(i)}$. Thus we arrive to the following definition.

Definition 2. Let P be a simple convex polytope. Suppose $\chi\colon\{1,\dots,m\}\to\{1,\dots,s\}$ is a surjective map defining a coloring of the hyperfaces of P into s colors. The quotient space of the Cartesian product $G^s\times P$ by the equivalence relation

$$(g,p)\sim(g',p')\iff p'=p,\quad g'g^{-1}\in H(p),\qquad \text{where}\quad H(p)=\bigoplus_{F_i\ni p}G_{\chi(i)}, \tag{3}$$

is called the *manifold defined by the coloring* χ and is denoted by $\mathcal{Z}(P,\chi)$.

In the case $G=\mathbb{Z}_2$, one can imagine the manifold $\mathcal{Z}(P,\chi)$ as 2^s copies of the polytope P glued together along some pairs of equally colored hyperfaces.

In this paper we deal with the case $G=\mathbb{Z}_2$ and $\dim P=3$; thus $\mathcal{Z}(P,\chi)=3$. Furthermore, we assume that the hyperfaces of P are colored in three colors. We will prove that there exists an equivariant embedding of such a manifold into the Euclidean space $\mathbb{R}^4$ with a canonical $\mathbb{Z}_2^3$-action. Also, an equivariant Dehn surgery will be defined and it will be shown that the class of all these manifolds is generated by equivariant Dehn surgeries and connected sums over the set of three-dimensional tori with a canonical $\mathbb{Z}_2^3$-action.

2. EQUIVARIANT EMBEDDINGS

In the sequel, by a polytope we mean a simple convex polytope and, if the contrary is not stated, its dimension equals to 3. A polytope allows a coloring into three colors iff the following condition holds (see [3, Problem 42]).

Theorem 1. *The faces of the polytope P can be colored into three colors if and only if each face of P contains an even number of edges. In other words, the adjoint graph of P must be bichromatic. Moreover, the coloring is unique up to a change of colors* (*i.e., a permutation of the set* $\{1,2,3\}$).

Therefore, the manifolds defined by different colorings of the given polytope are homeomorphic to each other. The homeomorphism is equivariant with respect to an automorphism of the group $\mathbb{Z}_2^3$ transposing the summands. Thus we can consider the manifold $\mathcal{Z}_3(P)$ as defined up to an automorphism of the action.

Example. The manifold defined by the cube is the three-dimensional torus $(\mathbb{S}^1)^3$ with the following $\mathbb{Z}_2^3$-action: each generator acts on the corresponding circle by the reflection leaving the points of the other circles immovable.

Consider the reflections with respect to three of the four coordinate hyperplanes in $\mathbb{R}^4$. Putting them into correspondence with the generators of the group $\mathbb{Z}_2^3$, we obtain a $\mathbb{Z}_2^3$-action on $\mathbb{R}^4$.

Theorem 2. *For any polytope P with bichromatic adjoint graph the manifold $\mathcal{Z}_3(P)$ is equivariantly embeddable into the Euclidean space $\mathbb{R}^4$ with the action described above.*

Proof. The orbit space $\mathbb{R}^4/\mathbb{Z}_2^3$ can be identified with the Cartesian product $\mathbb{R}^3_+ \times \mathbb{R}$, where $\mathbb{R}^3_+$ is the three-dimensional octant. Furthermore, if we color the faces of $\mathbb{R}^3_+ \times \mathbb{R}$ (which are the products of the octant's faces and the line $\mathbb{R}$) into three colors, then we can reconstruct the space $\mathbb{R}^4$ as the quotient space of the product $\mathbb{Z}_2^3 \times (\mathbb{R}^3_+ \times \mathbb{R})$ by the equivalence relation (3), where the groups $H(P)$ are defined by the coloring in the same way as above.

Thus if we embed the polytope P into $\mathbb{R}^3_+ \times \mathbb{R}$ in such a way that any of its faces goes to the face of the same color, then applying the reconstruction described above, we obtain an equivariant embedding $\mathcal{Z}_3(P) \to \mathbb{R}^4$.

We will construct the "colored embedding" $P \to \mathbb{R}^3_+ \times \mathbb{R}$ beginning from the adjoint graph P^1 of the polytope P. A coloring of the faces induces the coloring of the edges by the following rule: the color of an edge is opposite to the color of the adjacent faces. It is clear that all the edges of equal color must be embedded into one of the half-planes that are the products of the octant's semiaxes and the line $\mathbb{R}$. Thus the graph P^1 must be embedded into the "three-page book" $(\mathbb{R}_+ \vee \mathbb{R}_+ \vee \mathbb{R}_+) \times \mathbb{R}$, the vertices go to the axis $\{0\} \times \mathbb{R}$.

Lemma 1. *There exists an embedding of the graph P^1 with the edges colored as described above into the three-page book such that any edge is mapped to a page of the same color.*

Proof. It is readily seen that the statement of the lemma is equivalent to the following: the vertices of the graph P^1 can be placed on the line so that any two pairs of vertices connected by edges of equal color do not separate each other. Actually, there is no difference if we put the vertices on a circle. We will prove the lemma in this latter formulation.

Consider the surface of the polytope as a sphere $\mathbb{S}^2$ with the graph P^1 embedded in it. To simplify further constructions, we may assume that the graph's edges are represented by arcs of big circles. This can be achieved by the central projection from a point inside the polytope onto a sphere with center at that point. Our goal is to construct a simple closed curve C on the sphere passing through all the vertices of P^1 such that all the edges of equal color lie on one side of C. It is clear that if we consider the curve C as a circle, then the order of the vertices on it satisfies the required condition.

Let $\{F_i\}_{i=1}^k$ be all the faces of color 3. For any F_i construct a disk $D_i \subset \mathbb{S}^2$ so that the vertices of F_i are on the boundary of D_i, and the interiors of the edges are in the interior of D_i. We may assume that the disks are nonintersecting and that their boundaries are composed of arcs of circles in $\mathbb{S}^2$. Consider the graph with vertex set $\{F_i\}$ and edges corresponding to the edges of P with colors 1 and 2. It is obtained by collapsing all the faces F_i, and so multiple edges can arise. Further, choose a maximal tree T in this graph. For any edge of T, consider the corresponding edge e of P^1 and those two of the disks D_i that are joined by e. Draw a path between these disks sufficiently close to e and take a small thickening of the former. It is easily shown that our construction can be performed in such a way that the union of the disks D_i with the thickenings described above will be homeomorphic to a disk and, moreover, will contain all the edges of color 3 and only these edges. The boundary of the disk obtained is a simple closed curve which separates all the edges of color 3 from the edges of the other colors. Hence the goal of the previous paragraph is attained and this completes the proof of the lemma. $\square$

Now we need to extend the embedding $P^1 \to \mathbb{R}^3_+ \times \mathbb{R}$ to the faces and the interior of the polytope P. Draw a path between these disks sufficiently close to e and take a small thickening of the former. All the faces of equal color must be embedded in the space $\mathbb{R}^2_+ \times \mathbb{R}$ bounded by two of the three book's pages. Since the boundaries of the faces are already embedded in the boundary of $R^2_+ \times R$, the extension is obtained trivially. Finally, the extension to the interior of P is also realized without problems. Namely, the surface of the polytope is embedded into $\partial(\mathbb{R}^3_+) \times \mathbb{R}$, and we take the cone over it with vertex at an interior point of $\mathbb{R}^3_+ \times \mathbb{R}$. This completes the proof of Theorem 2. □

Corollary 1. *Any* 3*-connected* 3*-regular bichromatic graph* $\Gamma \subset \mathbb{S}^2$ *can be represented as the intersection of the sphere with the three-page book for some embedding* $\mathbb{S}^2 \to \mathbb{R}^3$*.*

(A graph is called 3-*connected* if it remains connected after removing of any two of its vertices (with all adjacent edges); a graph is 3-regular if any its vertex has exactly 3 adjacent edges.)

Proof. By the Steinitz theorem (see [4, Chap. 2, Sec. 15]), any 3-connected planar graph is the adjoint graph of a polytope. Since the graph Γ is bichromatic, we see that it defines a polytope with faces colored into three colors. Hence the required embedding has been constructed in the course of the proof of Theorem 2. □

Note that the assumption that Γ is a 3-connected graph is not necessary for the corollary to be true. In this case the proof follows the arguments of Lemma 1 and subsequent arguments.

3. EQUIVARIANT SURGERY

Here we define two operations on the set of polytopes of the form considered. The first is the equivariant sum and is defined for two simple convex polytopes of equal dimension.

Definition 3 (see also [5]). Let P_1 and P_2 be simple convex polytopes of equal dimension. Suppose that the vertices $v_1 \in P_1$ and $v_2 \in P_2$ are chosen and a one-to-one correspondence between the hyperfaces containing v_1 and the hyperfaces containing v_2 is established. The *connected sum* $P_1 \# P_2$ with respect to these data is a polytope combinatorially equivalent to the result of the gluing P_1 and P_2 with small neighborhoods of v_1 and v_2 removed. The corresponding hyperfaces must be glued together.

This definition can be generalized to the wider class of manifolds with faces. The fact that the connected sum of polytopes is combinatorially equivalent to a polytope is proved in [5]. In the case of three-dimensional polytopes, we can apply the Steinitz theorem: here it is sufficient to note that the graph obtained from the adjoint graphs of the polytopes P_1 and P_2 is 3-connected and planar.

Let us note that the connected sum of three-dimensional polytopes with bichromatic adjoint graphs has also the same property. Additionally, if 3-colorings of the summands are given, then there is a natural bijection between the faces adjacent to the vertices at which the gluing is performed and there is a natural coloring of the connected sum. This allows us to define the connected sum $(P_1, \chi_1) \#_{v_1, v_2} (P_2, \chi_2)$ of three-dimensional polytopes with 3-colorings.

The second operation associates to a polytope of the class considered another polytope of the same class. It operates in a neighborhood of an edge as is described in Fig. 1.

Here the transformation of the adjoint graph is represented. The inverse transformation will also be considered. Due to the Steinitz theorem, if the 3-connectedness of the graph is not destroyed, we have a transformation of the polytope (obviously, 3-coloring is respected).

Now our goal is to determine the operations on the set of manifolds defined by 3-colorings of polytopes corresponding to the operations described above. The operation corresponding to the connected sum of polytopes is the equivariant connected sum of the manifolds. It is defined in [1] in a more general case. Note that in the manifold $\mathcal{Z}(P, \chi)$, a small neighborhood of a point taken

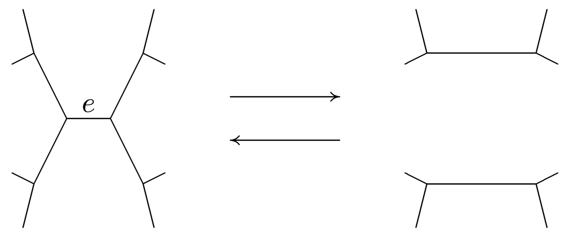

FIG. 1

by the orbit map to a vertex of P is equivariantly homeomorphic to Euclidean space $\mathbb{R}^3$ with the standard $\mathbb{Z}_2^3$-action.

Definition 4. The *equivariant connected sum* $\mathcal{Z}(P_1, \chi_1)\#_{v_1,v_2}\mathcal{Z}(P_2, \chi_2)$ *of the two manifolds* $\mathcal{Z}(P_1, \chi_1)$ *and* $\mathcal{Z}(P_2, \chi_2)$ *with respect to the vertices* $v_1 \in P_1$ *and* $v_2 \in P_2$ is defined as follows. We remove small invariant spherical neighborhoods of the points corresponding to the vertices v_1 and v_2 and glue the summands equivariantly along the boundaries.

The following proposition is obvious.

Lemma 2. *For any polytopes* P_1 *and* P_2 *with colorings* χ_1 *and* χ_2 *there exists an equivariant homeomorphism*

$$\mathcal{Z}(P_1, \chi_1)\#_{v_1,v_2}\mathcal{Z}(P_2, \chi_2) \cong \mathcal{Z}((P_1, \chi_1)\#_{v_1,v_2}(P_2, \chi_2)).$$

It is remarkable that topologically the equivariant connected sum does not depend on the choice of vertices at which the summation is performed. But homeomorphic manifolds obtained in this way can be not equivariantly homeomorphic since their orbit spaces can be different.

Further, let us describe the operation which corresponds to the transformation described by Fig. 1.

Definition 5 (see [6]). *Dehn surgery of a 3-dimensional manifold* consist of removing a solid torus and gluing it back by a homeomorphism of the boundary transposing a meridian and a parallel.

For the complete description of Dehn surgery, one should indicate a circle in the manifold which is the soul of the solid torus to be removed.

Lemma 3. *Suppose* e *is any edge of the polytope* P*; then two copies of* e *glue together in a circle in the manifold* $\mathcal{Z}(P, \chi)$*. The Dehn surgery on the manifold* $\mathcal{Z}(P, \chi)$ *around this circle corresponds to the polytope's transformation* (*from the left to the right*) *shown in Fig.* 1.

The proof is simple. Represent the transformation of the polytope as deleting a quarter of a cylinder with a subsequent gluing of a half-cylinder in its place. This process is pictured in Fig. 2.

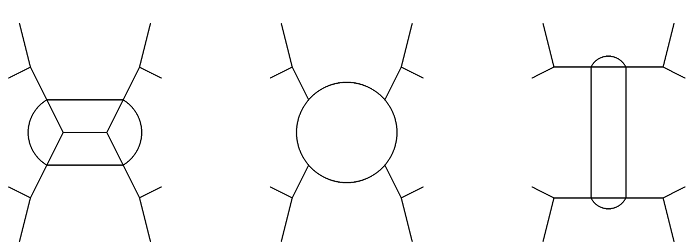

FIG. 2

Consider the two submanifolds in $\mathcal{Z}(P,\chi)$ that correspond to the deleted and to the glued-on part. It is clear that both are solid tori, moreover, the meridians and the parallels of the former are the parallels and meridians of the latter. Thus the arrows in both directions in Fig. 2 induce two inverse Dehn surgeries of the manifold defined by a coloring. Since the action on the transformed manifold is well defined, both operations can be called equivariant Dehn surgeries on manifolds of the class considered. We will refer to the transformations of the polytope pictured in Fig. 1 as to Dehn surgeries of a polytope.

As shown in [7], any three-dimensional manifold can be obtained from the 3-sphere by a finite number of Dehn surgeries. The main result of this section is a similar theorem about generating the manifolds considered by equivariant connected sum and equivariant Dehn surgery.

Theorem 3. *Any manifold of the form $\mathcal{Z}(P,\chi)$ can be obtained from a finite set of three-dimensional tori with the canonical $\mathbb{Z}_2^3$-action by using a finite number of operations of equivariant connected sum and equivariant Dehn surgery.*

Proof. It is sufficient to show that any polytope with bichromatic adjoint graph can be obtained from a finite set of cubes using the operations on the polytopes described above. Equivalently, one could prove that any polytope can be reduced to a number of cubes by decompositions into connected sum and Dehn surgeries of polytopes.

The proof of the following criteria is straightforward.

Lemma 4. 1) *A 3-regular graph $\Gamma\subset\mathbb{S}^2$ fails to be 3-connected iff there exists two nonadjacent edges of Γ such that cutting them makes the graph Γ disconnected.*

2) *A polytope with bichromatic adjoint graph Γ can be decomposed into the connected sum of polytopes iff there exists three nonadjacent edges of Γ such that cutting them makes the graph Γ disconnected. If the decomposition exists, then the adjoint graphs of the summands are bichromatic.*

Proof. The proof is simple. It is left to the reader. □

Let us prove that we can reduce any polytope to a collection of prisms by using Dehn surgeries and decompositions into connected sums. Let $\Gamma\subset\mathbb{S}^2$ be the adjoint graph of the polytope. The graph Γ decomposes $\mathbb{S}^2$ into domains that can be identified with the faces of the polytope. Consider the faces containing at least 6 edges. We call them *big faces.* If there are no adjacent big faces, then the polytope is a prism. Hence we can assume that there exist adjacent big faces. Note also that by virtue of the Euler characteristics not all the faces are big. This implies that a vertex can be chosen incident to two big faces A and B and to one quadrangular face C.

Performing the operation pictured in Fig. 1 in a neighborhood of the edge common to the faces A and B, we obtain a new graph Γ'. If Γ' is 3-connected, then it is the adjoint graph of a polytope. Thus we have performed a Dehn surgery decreasing the number of the polytope's faces by 1. Otherwise, if 3-connectedness is destroyed, then we will prove that the original polytope can be decomposed into a connected sum.

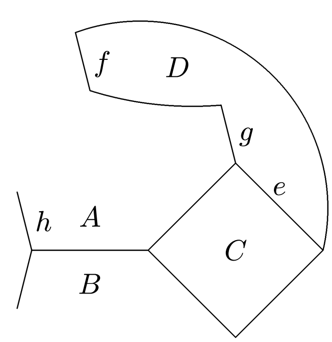

FIG. 3

By item 1) of Lemma 4, it follows that some two faces of Γ' have two edges in common. Obviously, one of them is the face arising from the two faces that are common adjacent to both A and B. Denote the other face by D. Let e and f be the two common edges of the faces considered. Then (see Fig. 3) cutting of the edges f, g and h makes the graph Γ disconnected. Since the face A is big, the edges f, g and h are not adjacent. Consequently by item 2) of Lemma 4 the original polytope splits into a connected sum. Continuing this process, at each step we obtain polytopes with the least number of faces. Hence the process will end at a collection of prisms.

Now consider a prism with $2n$-gon, $n \geq 3$, as the base. We can glue together the midpoints of two edges (the arrow to the left in Fig. 1) in the base such that there are two edges of the base between them. Then the polytope obtained can be decomposed into the connected sum of a cube with a $(2n-2)$-gonal prism. Proceeding in this way, we arrive at a collection of cubes. This completes the proof of Theorem 3. $\square$

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