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Definition and properties of free groups

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Chapter 1

Introduction

In this document I will give you an introduction to free groups. First of all, we will see the formal definition of a free group over a set and afterwards we will have a quick look into some of its properties. The highlight of this report will be the so-called "Table-Tennis Lemma". It helps to prove that a subgroup is free and it is often used.

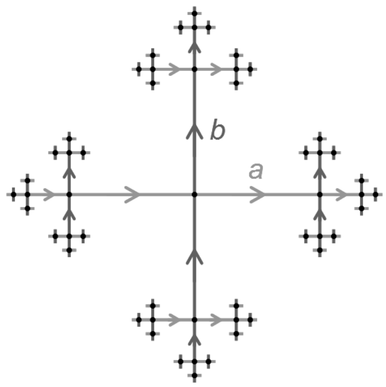

Figure 1.1: Cayley graph for the free group on two generators

For more information about the relation between free groups and Cayley graphs see the document concerning the next presentation "Free groups as groups acting on trees" of Vjosa Bakiu.

Chapter 2

Definition and properties of free groups

In order to understand the notion of a free group, we will first have to introduce the free product of groups.

Definition 2.0.1. Let A be a set. We define the set of words over A (or free monoid on A) as the set of all finite sequences of elements of A. It is denoted by W(A). The multiplication is given by juxtaposition. The neutral element is the empty word. A element ω in W(A) is of the form $\omega = a_1a_2a_3...a_n$ with a_j is an element of A for all j = 1,2,..,n. The integer n is then the length of ω. It is 0 if ω is the empty word and so the length is well defined.

Definition 2.0.2. Let $(\Gamma_i)_{i\in I}$ be a family of groups. Take $A = \bigsqcup_{i\in I} \Gamma_i$. The free product of the groups $(\Gamma_i)_{i\in I}$ is defined to be $*_{i\in I}\Gamma_i := \frac{W(A)}{\sim}$, where $\sim$ is the equivalence relation generated by

$$\omega e_i \omega' \sim \omega\omega' \text{ if } e_i \text{ is the unit of } \Gamma_i \text{ for some i}$$
$$\omega ab\omega' \sim \omega c\omega' \text{ if } a, b, c \in \Gamma_i \text{ for some i and } ab = c \text{ in } \Gamma_i$$

for all $\omega, \omega' \in W(A)$.
It's easy to check that $*_{i\in I}\Gamma_i$ is a group. The multiplication is again given by juxtaposition and the unit is the class represented by the empty word. The inverse of the class represented by $\omega = a_1a_2a_3...a_n$ is given by the class represented by $\omega^{-1} = a_n^{-1}a_{n-1}^{-1}...a_1^{-1}$.

Definition 2.0.3. Let $(\Gamma_i)_{i\in I}$ be a family of groups and $A = \bigsqcup_{i\in I} \Gamma_i$ as above. A word $\omega = a_1a_2a_3...a_n \in W(A)$ for a_j element of Γ_{i_j} is called reduced if and only if $i_j \neq i_{j+1}$ for all $j = 1, 2, ..., n-1$ and a_j is not the neutral element in Γ_{i_j} for any $j = 1, ..., n$.

Example 2.0.4. Take $\Gamma_1 = \{e_1, a, b\}$ and $\Gamma_2 = \{e_2, x, y, z\}$. Then

- the words a, y and z are reduced in $\Gamma_1 * \Gamma_2$.
- axaybz is reduced in $\Gamma_1 * \Gamma_2$.
- axayba is not reduced in $\Gamma_1 * \Gamma_2$.

Proposition 2.0.5. *Let $(\Gamma_i)_{i\in I}$ be a family of groups and $A = \bigsqcup_{i\in I} \Gamma_i$. Every element (equivalence class) of $*_{i\in I}\Gamma_i$ is represented by exactly one reduced word.*

Proof. Existence: By definition $*_{i\in I}\Gamma_i := \frac{W(A)}{\sim}$. So take $\omega = a_1a_2...a_n \in W(A)$, with $a_j \in \Gamma_{i_j}\backslash\{e_{i_j}\}$ for all $j \in \{1,...,n\}$. For this ω we need to prove that there exists ω' that is reduced and such that $\omega \sim \omega'$. Indeed, take $\omega_1 := a_1$. By definition, this is reduced. Take a_1a_2. If $i_1 \neq i_2$ then $\omega_2 := a_1a_2$ is reduced by definition. If $i_1 = i_2$ then $\omega_2 := a_1a_2 \in \Gamma_{i_1}$ is an element of Γ_{i_1} so reduced. Now by induction on n, on each step we add a_j to ω_{j-1}. In finite number of steps we obtain $\omega' := \omega_n$ reduced and such that $\omega \sim \omega'$.
Denote by Red(A) the set of all reduced words in the alphabet A.
Uniqueness: For each $a \in A$ we define T(a) : Red (A) $\to$ Red(A) by $\omega \in Red(A) \mapsto \omega'_a \sim a\omega$, where ω'_a is the reduced word given by the existence step given above.
For $\omega = a_1a_2...a_n \in W(A)$ reduced or not, we define $T(\omega) = T(a_1)T(a_2)...T(a_n)$. We claim that $T(a)T(b) = T(c)$ if $a, b, c \in \Gamma_{i_j}$ and $c = ab \in \Gamma_{i_j}$. Indeed, let $\omega \in Red(A)$. Then $T(a)T(b)\omega = ab\omega = c\omega = T(c)\omega$, since the image set is Red(A). Therefore, since ω is chosen arbitrarily, we get that $T(a)T(b) = T(c)$. This implies that if $\omega_1 \sim \omega_2$ then $T(\omega_1) = T(\omega_2)$.
Take now $\omega \in Red(A)$ and let ω_0 be the empty word in W(A). Then we obtain that $T(\omega)\omega_0 = \omega$.
Let now $\omega_1, \omega_2 \in Red(A)$ such that $\omega_1 \sim \omega_2$, then $T(\omega_1) = T(\omega_2)$ by the claims above. Thus $\omega_1 = T(\omega_1)\omega_0 = T(\omega_2)\omega_0 = \omega_2$. Thus the conclusion. □

Definition 2.0.6. Let X be a set. We define the free group over X as

$$F(X) := *_{x\in X}\mathbb{Z}_x$$

where every $\mathbb{Z}_x \cong \mathbb{Z}$ and every $x \in X$ is taken as the generator "+1" in the corresponding copy of $\mathbb{Z}$, hence X is a subset of F(X). We define by $|X|$ the rank of F(X). Thus, by Proposition 2.0.5, F(X) can also be identified with the set of reduced words in $X \cup X^{-1}$.
A subset B of a group G is called a free subset if the inclusion $B \hookrightarrow F(B)$ extends to a isomorphism of $< B >$ onto F(B) i.e., if $< B >$ is a free group.

Universal property 2.0.7. Let Γ be a group and let $\{\Gamma_i\}_{i\in I}$ be a family of groups. Let $\{h_i : \Gamma_i \to \Gamma\}_{i\in I}$ be a family of homomorphisms. Then there exist exactly one homomorphism $h : *_{i\in I}\Gamma_i \to \Gamma$ such that the following diagram commutes for every i_0 in I.

$$\begin{array}{ccc} \Gamma_{\iota_0} & & \\ \downarrow & \searrow^{h_{\iota_0}} & \\ *_{\iota\in I}\Gamma_\iota & \xrightarrow{h} & \Gamma \end{array}$$

In particular, for a set X, a group Γ and a map $\varphi : X \to \Gamma$ there exist exactly one homomorphism $\Phi : F(X) \to \Gamma$ such that $\varphi(x) = \Phi(x)$ for all $x \in X$.

Proof. We just give an outline of the proof.
Take $\omega = a_1a_2...a_n$ a reduced word, $\omega \in *_{i\in I}\Gamma_i$, with $a_j \in \Gamma_{i_j}\backslash\{e_{i_j}\}$ for $j \in \{1, ..., n\}$ and $i_{j+1} \neq i_j$ for $j \in \{1, ..., n-1\}$. Define

$$h(\omega) := h_{i_1}(a_1)h_{i_2}(a_2)...h_{i_n}(a_n) \in \Gamma$$

and this defines h uniquely in terms of the h_i's, since if there are $a, b, c \in \Gamma_i$ with $ab = c$ for some $i \in I$ then $h(ab) = h_i(a)h_i(b) = h_i(ab) = h_i(c) = h(c)$ since h_i is a homomorphism.
The second statement is a special case of the universal property, since $\Gamma_i = \mathbb{Z}_x \cong \mathbb{Z}$ for every $x \in X$.
We construct Φ for the given map φ. First note that Φ sends the empty word to identity of Γ and it has to agree with φ on the elements of X i.e, $\Phi(x) := \varphi(x)$ for every $x \in X$. For the remaining words, consisting of more than one symbol, Φ can be uniquely extended by defining $\Phi(ab) := \Phi(a)\Phi(b)$. □

Corollary 2.0.8. *Every group G is a quotient of a free group.*

Proof. Take $X = G$ and $h = id : X \to G$. Then, by the universal property there exists an unique homomorphism $\Phi : F(X) \to G$ with $\Phi(x) = h(x)$ for all $x \in X$. Now it follows that

$$G \cong \frac{F(X)}{ker\Phi}.$$

□

Suppose we want to prove that a group Γ is the free product of some groups Γ_1 and Γ_2, we can use the universal property to prove that they are isomorphic, meaning that the kernel of the map from $\Gamma_1 * \Gamma_2$ to Γ is just the neutral element.

Chapter 3

Table-Tennis Lemma and examples

This chapter is built around the so-called "Table-Tennis Lemma". This was a often used by F.Klein, and it is now a standard tool to construct free groups. It is also known as "Ping-Pong Lemma" or "Schottky Lemma". This chapter contains the Lemma and some examples in order to understand how to use it.

Table-Tennis Lemma 3.0.1. Let G be a group which acts on a set X (see the definitions in the document concerning the presentation of Vjosa Bakiu). Let

- $X_1, X_2 \subset X$ be non empty subsets of X with X_2 not included in X_1;
- $\Gamma_1, \Gamma_2 \subset G$ be subgroups with $|\Gamma_1| \geq 3$ and $|\Gamma_2| \geq 2$;
- Γ be the subgroup of G generated by Γ_1 and Γ_2 i.e., $\Gamma := < \Gamma_1, \Gamma_2 >$.

Suppose that $\gamma(X_2) \subset X_1$ for all $\gamma \in \Gamma_1$ with $\gamma \neq 1$ and $\gamma(X_1) \subset X_2$ for all $\gamma \in \Gamma_2$ with $\gamma \neq 1$. Then Γ is isomorphic to the free product of Γ_1 and Γ_2 i.e., $\Gamma \cong \Gamma_1 * \Gamma_2$.

Proof. Take $\tilde{\omega} = a_1 a_2 ... a_n$ a reduced word in $\Gamma_1 * \Gamma_2$ of length ≥ 1 with letters $a_i \in \Gamma_1 \backslash \{1\}$ or $a_i \in \Gamma_2 \backslash \{1\}$.
By the universal property there is a unique map $\varphi : \Gamma_1 * \Gamma_2 \to \Gamma$. Let $\omega = \varphi(\tilde{\omega}) \in \Gamma$. We want to prove that $ker(\varphi) = \{id\}$, therefore it is enough to show that $\omega \neq 1$.
If $\omega = a_1 b_1 a_2 ... b_{n-1} a_n$ where $a_j \in \Gamma_1 \backslash \{1\}$ and $b_k \in \Gamma_2 \backslash \{1\}$. Then $\omega(X_2) = a_1 b_1 a_2 ... b_{n-1} a_n(X_2) \subset a_1 b_1 a_2 ... b_{n-1}(X_1) \subset ...... \subset a_1(X_2) \subset X_1$. Since X_2 is not included in X_1, it follows that $\omega \neq 1$.
If $\omega = b_1 a_1 b_2 ... a_{n-1} b_n$, take $a \in \Gamma_1 \backslash \{1\}$. Then from what we did above, we obtain $a\omega a^{-1} \neq 1$. This implies that $\omega \neq 1$. If $\omega = a_1 b_1 a_2 ... a_n b_n$, take $a \in \Gamma_1 \backslash \{1, a_1^{-1}\}$ and argue analogously with $a\omega a^{-1}$. If $\omega = b_1 a_1 b_2 ... b_{n-1} a_n$, take $a \in \Gamma_1 \backslash \{1, a_n^{-1}\}$ and argue analogously with $a\omega a^{-1}$. □

Example 3.0.2. The subgroup generated by $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ in $SL(2, \mathbb{Z})$ is free and its rank is 2.

Proof. We will prove this assertion by using the Table-Tennis Lemma.

First, we define $\Gamma_1 = \left\{ \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix} \middle| n \in \mathbb{Z} \right\}$, the subgroup generated by $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

$\Gamma_2 = \left\{ \begin{pmatrix} 1 & 0 \\ 2n & 1 \end{pmatrix} \middle| n \in \mathbb{Z} \right\}$, the subgroup generated by $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$

$$X_1 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| |x| > |y| \right\}$$

$$X_2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| |x| < |y| \right\}.$$

We check that all condition of the Table-Tennis Lemma are fulfilled.

- X_1 and X_2 are clearly non-empty sets with $X_2 \not\subset X_1$.
- We have that $|\Gamma_1| \geq 3$ and $|\Gamma_2| \geq 2$.
- We check now that $\gamma(X_2) \subset X_1$ for all $\gamma \in \Gamma_1$ with $\gamma \neq 1$.
  Let $\gamma \in \Gamma_1$ with $\gamma \neq 1$. Hence
  $$\gamma = \begin{pmatrix} 1 & 2k \\ 0 & 1 \end{pmatrix} \text{ for some } k \in \mathbb{Z}\backslash\{0\}.$$
  Let $z = (x_z, y_z) \in X_2$. We have that $|x_z| < |y_z|$ and
  $$\gamma(z) = \begin{pmatrix} 1 & 2k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_z \\ y_z \end{pmatrix} = \begin{pmatrix} x_z + 2ky_z \\ y_z \end{pmatrix}.$$
  Since $|x_z| < |y_z|$ we obtain that $|x_z + 2ky_z| > |(2k-1)y_z|$. In addition, we have $|(2k-1)y_z| = |(2k-1)||y_z| \geq |y_z|$ for all $k \neq 0$.
  So we get that $\gamma(z) \in X_1$ and therefore $\gamma(X_2) \subset X_1$.
- To check $\gamma(X_1) \subset X_2$ for all $\gamma \in \Gamma_2$ with $\gamma \neq 1$, one can argue analogously.

So we obtain that

$$\left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle = <\Gamma_1, \Gamma_2> = \Gamma_1 * \Gamma_2.$$

□

There are also some applications of the Table-Tennis Lemma regarding the automorphisms of the line.

Example 3.0.3. We want to find a free subgroup of $Homeo_+(\mathbb{R})$.
Let $f : [0,1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 4t & 0 \leq t \leq \frac{1}{5} \\ \frac{4}{5} + \frac{1}{4}(t - \frac{1}{5}) & \frac{1}{5} \leq t \leq 1. \end{cases}$$

We define

- $\gamma_1 : \mathbb{R} \to \mathbb{R}$, $t \mapsto [t] + f(\{t\})$ where $[t]$ is the integer part and $\{t\}$ the fractional part of t;
- $T : \mathbb{R} \to \mathbb{R}$, $t \mapsto t - \frac{1}{2}$;
- $\gamma_2 : \mathbb{R} \to \mathbb{R}$ given by $\gamma_2 := T\gamma_1 T^{-1}$.

Then γ_1 and γ_2 generate in $Homeo_+(\mathbb{R})$ a free subgroup of rank 2.

Proof. We use again the Table-Tennis Lemma.

Define $X_1 := \bigcup_{k\in\mathbb{Z}} [k - \frac{1}{5}, k + \frac{1}{5}]$

$X_2 := \bigcup_{k\in\mathbb{Z}} [k + \frac{1}{2} - \frac{1}{5}, k + \frac{1}{2} + \frac{1}{5}]$

$\Gamma_1 :=< \gamma_1 >$ and $\Gamma_2 :=< \gamma_2 >$.

In addition, it is easy to check that

$$T(X_2) = T^{-1}(X_2) = X_1 \text{ resp. } T(X_1) = T^{-1}(X_1) = X_2$$

and that

$$f^n([\tfrac{1}{5}, 1]) \subset [\tfrac{4}{5}, 1] \ \forall n \geq 1 \text{ and } f^n([0, \tfrac{4}{5}]) \subset [0, \tfrac{1}{5}] \ \forall n \leq -1$$

where f^n is the n^{th} iterate of f.
From this it follows that $\gamma_1^n(X_2) \subset X_1$ for all $n \in \mathbb{Z}\backslash\{0\}$ and hence $\gamma(X_2) \subset X_1$ for all $\gamma \in \Gamma_1$ with $\gamma \neq 1$ since γ_1 generates Γ_1.
But now it follows that $\gamma_2^n(X_1) = T\gamma_1 T^{-1}(X_1) = T\gamma_1 T^{-1}(T(X_2)) = T\gamma_1^n(X_2) = T(X_1) \subset X_2$ for all $n \in \mathbb{Z}\backslash\{0\}$.
And since γ_2 generates Γ_2, the assertion follows by applying the Table-Tennis Lemma. □

The Table-Tennis Lemma introduced as in 3.0.1 has some limitations and hence is often just useful in simple cases like creating easily/proving the existence of free groups with rank 2. Fortunately, there exist a more general version of the Lemma, called the generalized Table-Tennis Lemma.

Generalized Table-Tennis Lemma 3.0.4. Let G be a group acting on a set X and let $\Gamma_1, \Gamma_2, ..., \Gamma_k$, $k \geq 2$, be subgroups of G with order greater than 2. Let $X_1, X_2, ..., X_k$ be subsets of X which are pairwise disjoint and nonempty such that

$$\gamma(X_s) \subset X_i \text{ for any } i \neq s \text{ and for any } \gamma \in \Gamma_i,\ \gamma \neq 1.$$

Then $< \Gamma_1, \Gamma_2, ..., \Gamma_k >= \Gamma_1 * \Gamma_2 * ... * \Gamma_k = *_{i\in\{1,...,k\}}\Gamma_i$.

In mathematics, the Tits alternative, named for Jacques Tits, is an important theorem about the structure of finitely generated linear groups.

Tits' alternative 3.0.5. Let G be a finitely generated subgroup of $GL_n(\mathbb{K})$ for a integer $n \geq 1$ and a field $\mathbb{K}$ of characteristic zero. Then one and only one of the two following possibilities occur:

- G has a solvable subgroup of finite index.
- G contains a non-abelian free group.

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