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Free groups III

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Contents

Abstract

This is the continuation of the chapter about free groups. Former talks concerning this subject were given by Jan Wey (Free Groups I) and Vjosa Bakiu (Free Groups II).
In this last section the following subjects are going to be presented:

- Presentation of a group
- Word problem
- Universal property of a group given by generators and relations
- Automorphism group of a free group
- Serre's property FA

There are two types of notations used to denote a free group. If it is important to know which set S generates a certain free group, we denote the free group over S by $F(S)$. If we are only interested in the number of elements our group is being generated by, we rather use the notation $\mathbb{F}_n$ to describe the free group on n generators.

Chapter 1

Presentation of a group

To define the presentation of a group we first need to define a generated normal subgroup of this group.

Definition 1.0.1. Let G be a group and let $R \subseteq G$ a subset. The *normal subgroup of* G *generated by* R is denoted by $\langle R \rangle_G^{\triangleleft}$. It is the smallest normal subgroup of G containing the subset R.

Remark 1.0.2. $\langle R \rangle_G^{\triangleleft} = \bigcap \{N \mid N \triangleleft G, R \subseteq N\}$.

Proof. Why is $\langle R \rangle_G^{\triangleleft} = \bigcap \{N \mid N \triangleleft G, R \subseteq N\}$ a normal subgroup? Let $n \in \bigcap \{N \mid N \triangleleft G, R \subseteq N\}$. $\forall$ N with $N \triangleleft G$ and $R \subseteq N$, $gng^{-1} \in N$ $\forall g \in G \Rightarrow gng^{-1} \in \bigcap \{N \mid N \triangleleft G, R \subseteq N\}$.
It is clear that the implications $\subseteq$ and $\supseteq$ are fullfilled. $\square$

Example 1.0.3.

1. Let $R = \{x^n\}$ for some $n \in \mathbb{N}$, let $G = \{x^k \mid k \in \mathbb{Z}\}$.
   Let $\langle R \rangle_G := \{x^{tn} \mid t \in \mathbb{Z}\}$. Now let $x^{tn} \in \langle R \rangle_G$ and let $x^k \in G$. $x^k x^{tn} x^k = x^{k+tn-k} = x^{tn} \in \langle R \rangle_G \Rightarrow \langle R \rangle_G = \langle R \rangle_G^{\triangleleft}$. This is also true for all other abelian groups.

2. Let $G = S_3$ and let $\tau_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \Rightarrow \langle \tau_1 \rangle_{S_3} = \{\tau_1, e\}$. Let $\tau_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$. We have $\tau_2 \tau_1 \tau_2 = \tau_3$, where $\tau_3 := \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$. Note that $\tau_2^{-1} = \tau_2$. Similarly we have $\tau_3 \tau_1 \tau_3 = \tau_2$
   Because τ_1, τ_2, τ_3 generate S_3 we have $\langle \tau_1 \rangle_{S_3}^{\triangleleft} = S_3$

In the following, elements of the free group $F(S)$ over a set S are denoted by words in $(S \cup S^{-1})^*$.

Definition 1.0.4. Let S be a set and let $R \subseteq (S \cup S^{-1})^*$ a subset. Let $F(S)$ be the free group generated by S. Then $\langle S \mid R \rangle := F(S) / \langle R \rangle_{F(S)}^{\triangleleft}$ is *the group generated by* S *with relations* R. For any group G with $G \cong \langle S \mid R \rangle$ we say $\langle S \mid R \rangle$ is a *presentation for* G.

Remark 1.0.5. Any element of the relations R corresponds to the neutral element in the group $\langle S \mid R \rangle$.

Example 1.0.6. 3. Take $\langle x \mid x^n \rangle$. So $S = \{x\}$ and $R = \{x^n\}$ for some $n \in \mathbb{N}$. Note that $F(S)$ is the group G in example 1. Example 1 shows that $\langle x^n \rangle^{\triangleleft}_{F(S)} = \{x^{tn} \mid t \in \mathbb{Z}\}$. It is easily seen that $\langle x \mid x^n \rangle = \{x^m \langle x^n \rangle^{\triangleleft}_{F(S)} \mid m = 0, 1, ..., n-1\} \cong \mathbb{Z}/n\mathbb{Z}$.

4. Take $\langle x, y \mid xyx^{-1} = y^2, yxy^{-1} = x^2 \rangle$. Let $\bar{x}, \bar{y}$ be the images of x, y under the canonical map $\pi : \{x, y\} \to \langle x, y \mid xyx^{-1} = y^2, yxy^{-1} = x^2 \rangle$. $\bar{x} = \bar{x}\bar{y}\bar{x}^{-1}\bar{x}\bar{y}^{-1} = \bar{y}^2\bar{x}\bar{y}^{-1} = \bar{y}\bar{x}^2 \Rightarrow \bar{x} = \bar{y}^{-1}$ $\bar{y}^{-2} = \bar{x}^2 = \bar{y}\bar{x}\bar{y}^{-1} = \bar{y}\bar{y}^{-1}\bar{y}^{-1} = \bar{y}^{-1} \Rightarrow \bar{y} = e = \bar{x}$ So our group $\langle x, y \mid xyx^{-1} = y^2, yxy^{-1} = x^2 \rangle$ is generated by the neutral element e and therefore the group equals to the trivial group.

Remark 1.0.7. Example 4 leads us to the so called **word problem**. This problem consists of deciding whether a group $\langle S \mid R \rangle$ given by generators S and relations R is the trivial group or not. More generally the word problem treats the question whether an element $x \in \langle S \mid R \rangle$ equals to the neutral element in this group. This problem is undecidable.

Before leaving the first chapter we introduce the universal property for groups given by generators and relations.

Theorem 1.0.8. *Let S be a set and $R \subseteq (S \cup S^{-1})^*$, let $\pi : S \to \langle S \mid R \rangle$ be the canonical map. For any group H and any map $\varphi : S \to H$ with $\varphi^*(r) = e$ $\forall r \in R$ $\exists!$ homomorphism $\bar{\varphi} : \langle S \mid R \rangle \to H$ such that $\bar{\varphi} \circ \pi = \varphi$.*

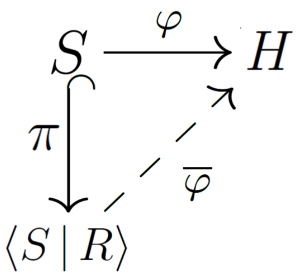

Figure 1.1: Universal property for groups given by generators and relations

Remark 1.0.9. The map $\varphi^* : (S \cup S^{-1})^* \to H$ in the above theorem is a natural extension of the map $\varphi : S \to H$. Let for example $s, s_1, s_2 \in S$. Then $\varphi^*(s^{-1}) = \varphi(s)^{-1}$ and $\varphi^*(s_1 s_2) = \varphi(s_1)\varphi(s_2)$.

Proof. Now we prove the universal property for groups given by generators and relations:
Let $w\langle R \rangle^{\triangleleft}_{F(S)} \in \langle S \mid R \rangle$. We define $\bar{\varphi}\left(w\langle R \rangle^{\triangleleft}_{F(S)}\right) := \varphi^*(w)$. This defines $\bar{\varphi}$ uniquely and makes sure that $\bar{\varphi} \circ \pi = \varphi$. Now we need to check whether

1. $\bar{\varphi}$ is well-defined and

2. whether $\bar{\varphi}$ is a homomorphism.

1. Let $w, \bar{w} \in F(S)$ be such that $w\langle R\rangle^{\triangleleft}_{F(S)} = \bar{w}\langle R\rangle^{\triangleleft}_{F(S)} \Rightarrow \exists r \in \langle R\rangle^{\triangleleft}_{F(S)}$ such that $w = \bar{w}r$. We have to show $\varphi^*(w) = \varphi^*(\bar{w})$. But $\varphi^*(w) = \varphi^*(\bar{w}r) = \varphi^*(\bar{w})\varphi^*(r) = \varphi^*(\bar{w})e = \varphi^*(\bar{w})$. So $\bar{\varphi}$ is well-defined.

2. Let $w, \bar{w} \in F(S)$. $\bar{\varphi}\Big(w\langle R\rangle^{\triangleleft}_{F(S)}\bar{w}\langle R\rangle^{\triangleleft}_{F(S)}\Big) = \bar{\varphi}\Big(w\bar{w}\langle R\rangle^{\triangleleft}_{F(S)}\Big) = \varphi^*(w\bar{w}) = \varphi^*(w)\varphi^*(\bar{w}) = \bar{\varphi}\Big(w\langle R\rangle^{\triangleleft}_{F(S)}\Big)\bar{\varphi}\Big(\bar{w}\langle R\rangle^{\triangleleft}_{F(S)}\Big)$. So $\bar{\varphi}$ is a homomorphism.

$\square$

Chapter 2

Free amalgamated products

Definition 2.0.1. Let A, G_1, G_2 be groups and let $\alpha_1 : A \to G_1$, $\alpha_2 : A \to G_2$ be group homomorphisms. A group G together with two group homomorphisms $\beta_1 : G_1 \to G$, $\beta_2 : G_2 \to G$ such that $\beta_1 \circ \alpha_1 = \beta_2 \circ \alpha_2$ is called a *free amalgamated product* if for any group H with two group homomorphisms $\varphi_1 : G_1 \to H$, $\varphi_2 : G_2 \to H$ there is exactly one homomorphism $\varphi : G \to H$ such that $\varphi \circ \beta_1 = \varphi_1$ and $\varphi \circ \beta_2 = \varphi_2$.

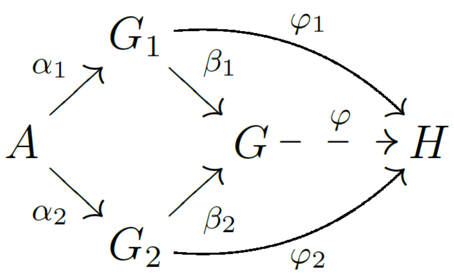

Figure 2.1: Free amalgamated product

We write the free amalgamated product by $G_1 *_A G_2$. If $A = \{e\}$, we write $G_1 * G_2 := G_1 *_{\{e\}} G_2$ which is just the *free product* defined in the previous talks on free groups. So we obtain a more general definition as the free product.

Example 2.0.2. 5. $SL_2(\mathbb{Z}) \cong \mathbb{Z}/6\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/4\mathbb{Z}$.

6. $Isom(\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$.

Theorem 2.0.3. *Every free amalgamated product exists and it is unique up to isomorphism.*

Proof. Let A, G_1, G_2 and α_1, α_2 as in the definition.

1. Uniqueness: Let G and $\tilde{G}$ be two free amalgamated products for the same groups A, G_1, G_2 with given homomorphisms $\alpha_1 : A \to G_1$, $\alpha_2 : A \to G_2$. Let $\beta_1 : G_1 \to G$, $\beta_2 : G_2 \to G$ be the homomorphisms of G and let $\tilde{\beta}_1 : G_1 \to \tilde{G}$, $\tilde{\beta}_2 : G_2 \to \tilde{G}$ be the homomorphisms

of $\tilde{G}$. Using the universal property of the free amalgamated product G for $H := \tilde{G}$ with homomorphisms $\varphi_1 := \tilde{\beta}_1$ and $\varphi_2 := \tilde{\beta}_2$ we obtain that there is a unique homomorphism $\varphi : G \to \tilde{G}$ with

$$\varphi \circ \beta_i = \tilde{\beta}_i, i = 1, 2. \tag{2.1}$$

The same argument shows that there is one unique homomorphism $\tilde{\varphi} : \tilde{G} \to G$ (exchange G by $\tilde{G}$ and β_i by $\tilde{\beta}_i$ for $i = 1, 2$) with

$$\tilde{\varphi} \circ \tilde{\beta}_i = \beta_i, i = 1, 2. \tag{2.2}$$

To show that $G \cong \tilde{G}$ it suffices to show $\tilde{\varphi} \circ \varphi = id_G$ and $\varphi \circ \tilde{\varphi} = id_{\tilde{G}}$. We show $\tilde{\varphi} \circ \varphi = id_G$. The other equality is proved equivalently. First notice that there is exactly one homomorphism $\psi : G \to G$ that fulfills

$$\psi \circ \beta_i = \beta_i, i = 1, 2. \tag{2.3}$$

That becomes clear by considering the universal property of the free amalgamated product and choosing $H = G$ and β_1, β_2 as homomorphisms, then

$$\beta_i \overset{(2.2)}{=} \tilde{\varphi} \circ \tilde{\beta}_i \overset{(2.1)}{=} \tilde{\varphi} \circ \varphi \circ \beta_i.$$

So $\tilde{\varphi} \circ \varphi$ fulfills (2.3) but as id_G does fulfill (2.3) as well and by uniqueness of this homomorphism we get $\tilde{\varphi} \circ \varphi = id_G$. So the free amalgamated product is unique up to isomorphism. That is, if it exists which is shown in part 2 of the proof.

2. Existence: First we define $S := \{x_g \mid g \in G_1\} \sqcup \{x_g \mid g \in G_2\}$ and $R := \{x_{\alpha_1(a)} x_{\alpha_2(a)}^{-1} \mid a \in A\} \cup R_{G_1} \cup R_{G_2}$ where $R_{G_i} := \{x_g x_h x_k^{-1} \mid g, h, k \in G_i, gh = k \text{ in } G_i\}$ for $i = 1, 2$. Now define $G := \langle S \mid R \rangle$. We want to prove that G is the free amalgamated product. In order to do so we need to check that

   (a) G is a group.
   (b) $\exists \beta_1 : G_1 \to G, \beta_2 : G_2 \to G$ such that $\beta_1 \circ \alpha_1 = \beta_2 \circ \alpha_2$.
   (c) For every group H with two homomorphisms $\varphi_1 : G_1 \to H, \varphi_2 : G_2 \to H$ there is exactly one homomorphism $\varphi : G \to H$ such that $\varphi \circ \beta_1 = \varphi_1$ and $\varphi \circ \beta_2 = \varphi_2$.

   (a) This is clear by the definition of G.
   (b) We define $\beta_1 : G_1 \to G, g \mapsto x_g$ and $\beta_2 : G_2 \to G, g \mapsto x_g$. Are these maps homomorphisms? Let $g, h, k \in G_i$ be such that $gh = k$.

   $$\beta_i(gh) = \beta_i(k) = x_k \overset{(*)}{=} x_g x_h = \beta_i(g)\beta_i(h)$$

   The step indicated by $(*)$ uses the relations $x_g x_h x_k^{-1} = e \Leftrightarrow x_g x_h = x_k$. So β_1, β_2 are homomorphisms. But is $\beta_1 \circ \alpha_1 = \beta_2 \circ \alpha_2$? Let $a \in A$ be arbitrary. Then

   $$(\beta_i \circ \alpha_i)(a) = \beta_i(\alpha_i(a)) = x_{\alpha_i(a)},$$

   again by the relations $x_{\alpha_1(a)} x_{\alpha_2(a)}^{-1} = e \Leftrightarrow x_{\alpha 1}(a) = x_{\alpha 2}(a)$. So $\beta_1 \circ \alpha_1 = \beta_2 \circ \alpha_2$.

(c) Let H be a group and let $\varphi_1 : G_1 \to H, \varphi_2 : G_2 \to H$ be two group homomorphisms with $\varphi_1 \circ \alpha_1 = \varphi_2 \circ \alpha_2$. We want to use the universal property for groups given by generators and relations (Theorem 1.0.8.) to show that there is a unique homomorphism $\varphi : G \to H$ such that $\varphi \circ \beta_1 = \varphi_1$ and $\varphi \circ \beta_2 = \varphi_2 (**)$. Let $\pi : S \to G = \langle S \mid R \rangle$ be the canonical map. We define $\psi : S \to H$

$$\psi(x_g) = \begin{cases} \varphi_1(g) & g \in G_1 \\ \varphi_2(g) & g \in G_2 \end{cases}.$$

If we want to use Theorem 1.0.8. we have to show $\psi^*(r) = e \; \forall r \in R$. Here ψ^* denotes the extension of ψ described in remark 1.0.9. Let $r \in R$:

i. Let r be of the form $r = x_{\alpha_1(a)} x^{-1}_{\alpha_2(a)}$ for some $a \in A$, then

$$\psi^*(r) = \psi^*(x_{\alpha_1(a)} x^{-1}_{\alpha_2(a)}) = \varphi_1(\alpha_1(a))\varphi_2(\alpha_2(a))^{-1} \overset{(**)}{=} e.$$

ii. Let r be of the form $x_g x_h x_k^{-1}$ where $g, h, k \in G_i$ such that $gh = k$. Then

$$\psi^*(r) = \psi^*(x_g x_h x_k^{-1}) = \varphi_i(g)\varphi_i(h)\varphi_i(k)^{-1} = \varphi_i(ghk^{-1}) = \varphi_i(e) = e.$$

This is true because φ_i is a homomorphism.

So we are allowed to use the universal property of groups given by generators and relations. Thus, there is a unique homomorphism $\varphi : G = \langle S \mid R \rangle \to H$ such that $\varphi \circ \pi = \psi$. It is obvious that $\varphi \circ \beta_1 = \varphi_1$ and $\varphi \circ \beta_2 = \varphi_2$ by construction of ψ and φ. The last thing that must be shown is that φ is unique with this property. But as S is a mere copy of G_1 and G_2 we can think of π as β_1 and β_2 all in one ("$\pi = \beta_1 \sqcup \beta_2$"). If we then think of ψ as a φ_1 and φ_2 all in one ("$\psi = \varphi_1 \sqcup \varphi_2$") we get the uniqueness of φ in the universal property for the free amalgamated product out of the uniqueness in the universal property for groups given by generators and relations.

□

Chapter 3

Serre's property FA and Aut($\mathbb{F}_n$)

In this last chapter we introduce Serre's property FA and give some examples of groups with this property. These examples are related to free groups and $Aut(\mathbb{F}_n)$. First of all let's have a closer look on $Aut(\mathbb{F}_n)$.

Definition 3.0.1. Let $\mathbb{F}_n$ be a free group. A group homomorphism $\varphi : \mathbb{F}_n \to \mathbb{F}_n$ is an *automorphism of* $\mathbb{F}_n$ if φ is bijective. We define the *automorphism group* of $\mathbb{F}_n$: $Aut(\mathbb{F}_n) := \{\varphi : \mathbb{F}_n \to \mathbb{F}_n \mid \varphi \text{ automorphism of } \mathbb{F}_n\}$.

The following theorem gives some characterization for $Aut(\mathbb{F}_n)$.

Theorem 3.0.2. *$Aut(\mathbb{F}_n)$ is generated by the elementary automorphisms.*

Let $S = \{s_1, ..., s_n\}$ a basis for the free group $\mathbb{F}_n$. By definition, the elementary automorphisms are the following three types of automorphisms:

1. Permutation automorphisms
   Let σ a permutation of $\{1, ..., n\}$. We define $\tilde{\sigma}$: $\tilde{\sigma}(s_i) := s_{\sigma(i)}$. We extend this definition on all elements of $\mathbb{F}_n$ such that $\tilde{\sigma}$ is an automorphism. More precisely let $s \in \mathbb{F}_n$, then $s = \prod_{i_j=0}^{m} s_{i_j}^{\lambda_{i_j}}$, where $s_{i_j} \in S$ and $\lambda_{i_j} \in \{-1, 1\}$. Then $\tilde{\sigma}(s) = \tilde{\sigma}(\prod_{i_j=0}^{m} s_{i_j}^{\lambda_{i_j}}) := \prod_{i_j=0}^{m} \tilde{\sigma}(s_{i_j})^{\lambda_{i_j}}$. Such a $\tilde{\sigma}$ is clearly an automorphism and is called a permutation automorphism.

2. Inversion automorphisms
   A permutation automorphism is an automorphism of the form
   $$\iota_i(s_j) := \begin{cases} \mathrm{s}_i^{-1} & j = i \\ s_j & j \neq i \end{cases}$$
   We can define such a ι_i for every $i \in \{1, ..., n\}$ and we extend any ι_i to $\mathbb{F}_n$ as we did it for the permutation automorphisms. Clearly ι_i is an automorphism of $\mathbb{F}_n$.

3. The Nielsen automorphisms are of the form

$$\rho_{ij}(s_k) := \begin{cases} s_i s_j & k = i \\ s_k & k \neq i \end{cases}$$

We can define such a ρ_{ij} for every $i \in \{1, ..., n\}, j \in \{1, ..., n\}$ and we extend any ρ_{ij} to $\mathbb{F}_n$ as we did it for the permutation automorphisms. Clearly ρ_{ij} is an automorphism of $\mathbb{F}_n$.

Now we want to define Serre's property FA. In order to do this we first need some more basic definition.

Definition 3.0.3. Let X be any tree and G some group acting on the tree X. For every $g \in G$ we define $Fix(g) := \{x \in X \mid gx = x\}$ the *fixed subgroup of g*. For any subgroup $H \subseteq G$ the *fixed subgroup of H* is defined as $Fix(H) := \bigcap_{h \in H} Fix(h)$.

Now we can define Serre's property FA:

Definition 3.0.4. Let G some group. We say *G has Serre's property FA* if for every tree (finite and infinite) that G is acting on, we have $Fix(G) \neq \emptyset$.

Remark 3.0.5. Note that the free group $\mathbb{F}_n$ does not have Serre's property FA. For this consider the talk Free Groups II: Every free group acts freely on some tree.

So if $\mathbb{F}_n$ does not have Serre's property FA what does this property have to do with free groups? The answer is quite obvious given the title of this chapter and it is truly no coincidence that we introduced $Aut(\mathbb{F}_n)$ earlier in this chapter.

Theorem 3.0.6. *For $n \geq 3$, $Aut(\mathbb{F}_n)$ has Serre's property FA.*

The proof of this theorem uses the fact that $Aut(\mathbb{F}_n)$ is generated by the elementary automorphisms.

To conclude this chapter let's have a look at some other groups with Serre's property FA.

Proposition 3.0.7. *If some group G has Serre's property FA then every quotient group of G has Serre's property FA as well.*

Proof. Let $G/N = H$. We assume H does not have Serre's property FA. There is some action $\alpha : H \to Aut(X)$ of H on X, such that $Fix(H) = \emptyset$ for the action α on some tree X. Let $\pi : G \to H$ be the canonical map. $\alpha \circ \pi : G \to Aut(X)$ is an action of G on X. It is clear by the definition of the action of G on X that $Fix(G) = \emptyset$ on X. □

Corollary 3.0.8. *$Gl_n(\mathbb{Z})$ has Serre's property FA.*

Proof. Let $\mathbb{F}_n$ a free group and let N be the commutator group of $\mathbb{F}_n$, i.e. $N = \langle xyx^{-1}y^{-1} \mid x, y \in \mathbb{F}_n \rangle$. It is well known that the commutator subgroup N is a normal subgroup of $\mathbb{F}_n$. Moreover, we claim that N is a normal subgroup of $Aut(\mathbb{F}_n)$. Indeed, firstly let us verify that N is a subgroup of $Aut(\mathbb{F}_n)$: for any element $n \in N$ we define the automorphism $\varphi_n : \mathbb{F}_n \to \mathbb{F}_n$ by $x \in \mathbb{F}_n \mapsto \varphi_n(x) := nxn^{-1}$. It is immediate to verify that φ_n is a bijective homomorphism having its inverse $\varphi_{n^{-1}}$, thus $\varphi_n \in Aut(\mathbb{F}_n)$, for every $n \in N$. Because $\varphi_n = \varphi_{n'}$ implies that $n = n'$ we can see N as a subgroup of $Aut(\mathbb{F}_n)$. Let us verify now that N is also normal in $Aut(\mathbb{F}_n)$. Let $\theta \in Aut(\mathbb{F}_n)$ and we need to prove that for every $n \in N$ we have that $\theta \circ \varphi_n \circ \theta^{-1} = \varphi_{n'}$, for some $n' \in N$. First notice that for every $\theta \in Aut(\mathbb{F}_n)$ and by the definition of N we have that $\theta(N) = N$. Then $(\theta \circ \varphi_n \circ \theta^{-1})(x) = \theta(n\theta^{-1}(x)n^{-1}) = \theta(n)\theta(\theta^{-1}(x))\theta^{-1}(n) = \theta(n)x\theta^{-1}(n)$, thus we take $n' := \theta(n) \in N$. Then our claim follows.

To conclude the proof we use the known fact that $Aut(\mathbb{F}_n)/N \approx GL_n(\mathbb{Z})$ and by Proposition 3.0.7. we obtain that $GL_n(\mathbb{Z})$ has Serre's property FA. □

Example 3.0.9. 7. $Sl_n(\mathbb{Z})$ has Serre's property FA for $n \geq 3$.

8. Nonexample: $Sl_2(\mathbb{Z})$ does not have Serre's property FA.

Bibliography

[1] Clara Löh, Geometric group theory, an introduction, 2011. http://www.mathematik.uni-regensburg.de/loeh/teaching/ggt_ws1011/lecture_notes.pdf.

[2] John Meier, Groups, Graphs and Trees, London Mathematical Society, 2008.

[3] Jan Wey, Free groups I, Pro-seminar manuscript, University of Fribourg, 2017.

[4] Vjosa Bakiu, Free groups II, Pro-seminar manuscript, University of Fribourg, 2017.